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Abstract: A review of the literature demonstrates that the Fox–Wright function is not only a mathematical puzzle, but its role is naturally to represent basic physical phenomena. Motivated by this fact, we studied a new representation of this function in terms of complex delta functions. This representation was useful to compute its Laplace transform with respect to the third parameter γ for which it also generalizes the one and two-parameter Mittag-Leffler functions. New identities involving the Fox–Wright function were discussed and used to simplify the results. Different fractional transforms were evaluated and the solution of a fractional kinetic equation was obtained by using its new representation. Several new properties of this function were discussed as a distribution.

Keywords: Fox–Wright function; Mittag-Leffler function; fractional images; *H*-function; kinetic equation

MSC: 44A20; 26A33

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1. Introduction and Motivation

Developments in environmental sciences are significantly influenced by contemporary gas theories and astrophysics. Differential equations are important to model the evolution of stars such as the sun, playing a significant role in studies on global warming [1]. The entire inner structure of stars is formed of gases, which are described by three properties: mass, temperature and pressure. In actuality, a cloud needs a rather more powerful gravitational force, as compared to its internal pressure, in order to become a star. Nuclear fusion takes place in the cloud, producing light, as a protostar develops as the result. Equations of state, translucence, and nuclear energy production rate serve as the foundation for mathematical models and constructions. Nuclear processes in such stars are the source of energy. Therefore, the reaction rate for each type of generation and devastation describes the way in which the kinetic equation characterizes the change in the chemical composition of stars. In order to investigate this composition $\mathcal{E}(t)$ using the production $p(\mathcal{E})$ and destruction $d(\mathcal{E})$ rate, Haubold and Mathai proposed the following basic kinetic equation [1]:

$$\frac{d\mathcal{E}}{dt} = -d(\mathcal{E}_t) + p(\mathcal{E}_t),$$

where $\mathcal{E}_t$ is defined by $\mathcal{E}_t(t^*) = \mathcal{E}(t - t^*), t^* > 0$. By neglecting the spatial variation and inhomogeneity in $\mathcal{E}(t)$ with respect to species concentration, the following are obtained: $\mathcal{E}_j(t = 0) = \mathcal{E}_0$, and

$$\frac{d\mathcal{E}_j}{dt} = -c_j\mathcal{E}_j(t).$$

Next, performing integration on this equation and ignoring subscript j leads to the following:

$$\mathcal{E}(t) - \mathcal{E}_0 = -c I_{0+}^{-1} \mathcal{E}(t).$$

By means of the Riemann–Liouville (R–L) fractional integral $I_{0+}^{\delta}, \delta > 0$, we can obtain the following non-integer-order kinetic equation:

$$\mathcal{E}(t) - \mathcal{E}_0 = -c^{\delta} I_{0+}^{\delta} \mathcal{E}(t),$$

where c is a constant. Following this, we have the following fractional kinetic equation [1–3] involving a general integrable function f(t):

$$\mathcal{E}(t) - f(t)\mathcal{E}_0 = -d^{\delta} I_{0+}^{\delta} \mathcal{E}(t).$$

A review of the literature reveals that there is no single equation addressing the integration with respect to the third parameter γ of a Fox–Wright function ${}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right]$. More recently, Giusti et al. [4] beautifully described the key results and applications emerging from the three-parameter generalization of the Mittag-Leffler function in connection with a special Fox–Wright function. For example, this function is important to model and study anomalous relaxation in dielectrics [5], linear viscoelasticity [6], renewal processes [7], stochastic processes and diffusion [8]. Hence, there is a natural need for the fractional calculus of such functions.

Taking motivation from these facts, we present this research article, which is organized as follows. Basic preliminaries and required definitions are given in the next section, Section 2. Distributional representation of a Fox–Wright function and its application to the fractional kinetic equation are presented in Section 3. New fractional calculus formulae or identities involving a Fox–Wright function ${}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right]$ are given in Sections 3.1 and 3.2. Existence of a new representation and its distributional properties are discussed in Section 4. Further applications of the new representation are discussed in Section 5. The conclusion and future directions are included in the last section, Section 6. Throughout the article, $\Re$ stands for the real portion of any complex number, whereas $\mathbb{C}$ stands for complex numbers and $\mathbb{R}$ denotes the reals. $\mathbb{Z}_0$ is a set of negative integers that contains 0, while $\mathbb{R}^+$ is a set of positive reals.

2. Preliminaries

2.1. Special Functions and Fractional Integral Transforms

The gamma function is the generalization of a factorial and is considered as a basic special function, defined as [9]

$$\Gamma(s) = \int_0^{\infty} t^{\omega-1} e^{-t} dt; (s \in \mathbb{C}; \Re(s) > 0). \tag{1}$$

This is a well-studied special function with tremendous applicability and representations; the basic Pochhammer symbols $(\gamma)_r$ are defined by

$$(\gamma)_r = \frac{\Gamma(\gamma + r)}{\Gamma(\gamma)} = \begin{cases} 1 \ (r = 0,) \\ \gamma(\gamma+1)\dots(\gamma+n-1) \ (r \in \mathbb{C} \setminus \{0\}; r = n \in \mathbb{N}; \gamma \in \mathbb{C}). \end{cases} \tag{2}$$

Magnus Gustaf Mittag-Leffler [10] suggested a function defined by

$$E_{\alpha}(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(\alpha k + 1)}, \alpha \in \mathbb{C}, \Re(\alpha) > 0, \tag{3}$$

which appears to be a natural replacement for the exponential function. The Mittag-Leffler function has numerous generalizations. For instance, the definition of the two-parameter Mittag-Leffler functions is

$$E_{\alpha,\beta}(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta \epsilon \mathbb{C}, \Re(\alpha) > 0 \tag{4}$$

and three variables provide a definition of the generalized Mittag-Leffler function or, in fact, a special Fox–Wright function ${}_1\Psi_1$ given by

$$E^{\gamma}_{\alpha,\beta}(s) = \sum_{k=0}^{\infty} \frac{(\gamma)_k s^k}{k!\Gamma(\alpha k + \beta)} = \frac{1}{\Gamma(\gamma)} {}_1\Psi_1\left[\begin{matrix}(\gamma, 1) \\ (\beta, \alpha)\end{matrix}; s\right], \alpha, \beta, \gamma \epsilon \mathbb{C}, \Re(\alpha) > 0, \tag{5}$$

which was also studied by Prabhakar [11]. Note that $E^{\gamma}_{\alpha,\beta}(\omega)$ is an entire function [12] of order $\rho = 1/\Re(\alpha)$ and type $\sigma = 1$. Furthermore, Fox–Wright and Fox–H functions [13] are related to the generalized Mittag-Leffler function as follows:

$$\begin{gathered} E^{\gamma}_{\alpha,\beta}(s) = \sum_{k=0}^{\infty} \frac{(\gamma)_p s^k}{k!\Gamma(\alpha k+\beta)} = \frac{1}{\Gamma(\gamma)} {}_1\Psi_1\left[\begin{matrix}(\gamma, 1) \\ (\beta, \alpha)\end{matrix}; s\right] = H^{1,1}_{1,2}\left[-s\middle|\begin{matrix}(1-\gamma, 1) \\ (0, 1), (1-\beta, \alpha)\end{matrix}\right]; \\ E^{1}_{\alpha,\beta}(s) = E_{\alpha,\beta}(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(\alpha k+\beta)} = {}_1\Psi_1\left[\begin{matrix}(1, 1) \\ (\beta, \alpha)\end{matrix}; s\right] = H^{1,1}_{1,2}\left[-s\middle|\begin{matrix}(0, 1) \\ (0, 1), (1-\beta, \alpha)\end{matrix}\right]; \\ E^{1}_{\alpha,1}(s) = E_{\alpha}(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(\alpha k+1)} = {}_1\Psi_1\left[\begin{matrix}(1, 1) \\ (1, \alpha)\end{matrix}; s\right] = H^{1,1}_{1,2}\left[-s\middle|\begin{matrix}(0, 1) \\ (0, 1), (0, \alpha)\end{matrix}\right]. \end{gathered} \tag{6}$$

These relations are important in order to study the asymptotic behavior of the generalized Mittag-Leffler function. Here (and following), an H-function [13] is defined by

$$\begin{aligned} H^{m,n}_{p,q}(\omega) &= H^{m,n}_{p,q}\left[s\middle|\begin{matrix}(a_i, A_i) \\ (b_j, B_j)\end{matrix}\right] = H^{m,n}_{p,q}\left[s\middle|\begin{matrix}(a_1, A_1), \ldots, & (a_p, A_p) \\ (b_1, B_1), \ldots, & (b_q, B_q)\end{matrix}\right] \\ &= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_j\mathfrak{s}\right)\prod_{i=1}^{n}\Gamma\left(1-a_j-A_j\mathfrak{s}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-B_j\mathfrak{s}\right)\prod_{i=n+1}^{p}\Gamma\left(a_j+A_j\mathfrak{s}\right)} s^{-\mathfrak{s}} d\mathfrak{s}, \end{aligned} \tag{7}$$

$$\left(\left(1 \leqq m \leqq q; 0 \leqq n \leqq p, A_i > 0 \wedge B_j > 0, a_i \in \mathbb{C} \wedge b_j \in \mathbb{C}(i = 1, \cdots, p \wedge j = 1, \cdots, q)\right)\right),$$

where, $\mathcal{L}$ is a suitable contour of Mellin–Barnes type, which splits up the poles of $\{\Gamma(b_j + B_j\mathfrak{s})\}_{j=1}^{m}$ and $\{\Gamma(1 - a_j - A_j\mathfrak{s})\}_{j=1}^{n}$. Considering $A_p = B_q = 1$ in Equation (7), we obtain Meijer G-function [13]:

$$H^{m,n}_{p,q}\left[s\middle|\begin{matrix}(a_1, 1), \ldots, & (a_p, 1) \\ (b_1, 1), \ldots, & (b_q, 1)\end{matrix}\right] = G^{m,n}_{p,q}\left[s\middle|\begin{matrix}a_1 \ldots, & a_p \\ b_1, \ldots, & b_q\end{matrix}\right]. \tag{8}$$

However, an H-function [13] has the following connection with a Fox–Wright function ${}_p\Psi_q$:

$$\begin{gathered} {}_p\Psi_q\left[\begin{matrix}(a_i, A_i) \\ (b_j, B_j)\end{matrix}; s\right] = H^{1,p}_{p,q+1}\left[-s\middle|\begin{matrix}(1-a_1, A_1), \ldots, & (1-a_p, A_p) \\ (0, 1), (1-b_1, B_1), \ldots, & ((1-b_q, B_q)\end{matrix}\right] \\ \left(a_i \epsilon \mathbb{R}^{+}(i = 1, \ldots p); B_j \epsilon \mathbb{R}^{+}(j = 1, \ldots q); 1 + \sum_{i=1}^{q} B_i - \sum_{j=1}^{p} A_j > 0\right) \end{gathered} \tag{9}$$

It is further related with the hypergeometric and other special functions [9] as follows:

$${}_p\Psi_q\left[\begin{matrix}(a_i, 1) \\ (b_j, 1)\end{matrix}; s\right] = {}_pF_q\left[\begin{matrix}a_i \\ b_j\end{matrix}; s\right] \cdot \frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_q)} = G^{1,p}_{p,q+1}\left[-s\middle|\begin{matrix}(1-a_1, 1), \ldots, & (1-a_p, 1) \\ 0, (1-b_1, 1), \ldots, & (1-b_q, 1)\end{matrix}\right] \tag{10}$$

The Kiryakova fractional transforms ((multiple) E–K integral operators), as defined in [14] (p. 8, Equation (18)), are

$$I_{(\alpha_k),m}^{(\gamma_k),(\delta_k)} f(z) = \begin{cases} f(z); \quad (\delta_k = 1) \\ \int_0^1 f(z\sigma) H_{m,m}^{m,0} \left[\sigma \middle| \begin{matrix} \left(\gamma_k + \delta_k - \frac{1}{\alpha_k} + 1, \frac{1}{\alpha_k} \right)_1^m \\ \left(\gamma_k - \frac{1}{\alpha_k} + 1, \frac{1}{\alpha_k} \right)_1^m \end{matrix} \right] d\sigma; (\sum_k \delta_k > 0) \\ = z^{-1} \int_0^z f(\xi) H_{m,m}^{m,0} \left[\frac{\xi}{z} \middle| \begin{matrix} \left(\gamma_k + \delta_k - \frac{1}{\alpha_k} + 1, \frac{1}{\alpha_k} \right)_1^m \\ \left(\gamma_k - \frac{1}{\alpha_k} + 1, \frac{1}{\alpha_k} \right)_1^m \end{matrix} \right] d\xi; (\sum_k \delta_k > 0). \end{cases} \tag{11}$$

Order of integration is expressed by δ_k's, and γ_k's are taken as weights, while α_k's are accompanying parameters. Since $H_{m,m}^{m,0}$ becomes zero when $|\sigma| > 1$, the upper limit as infinity is approached becomes meaningless in Equation (11). The corresponding Kiryakova's fractional derivative ((multiple) E–K derivative operators) of Riemann–Liouville (R–L) form, having multi-order $(\delta_m \geq 0, \ldots, \delta_1 \geq 0) = \delta$, is well defined by [14] (p. 9)

$$D_{(\beta_k),m}^{(\gamma_k)_1^m,(\delta_k)}(f(z)) := D_\eta I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(z) = D_\eta \int_0^1 f(z\sigma) H_{m,m}^{m,0} \left[\sigma \middle| \begin{matrix} \left(\gamma_k + \eta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \end{matrix} \right] d\sigma \tag{12}$$

$$D_{(\beta_k),m}^{(\gamma_k)_1^m,(\delta_k)}(f(z)) := D_\eta I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(z) = D_\eta \int_0^1 f(z\sigma) H_{m,m}^{m,0} \left[\sigma \middle| \begin{matrix} \left(\gamma_k + \eta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \end{matrix} \right] d\sigma \tag{13}$$

where D_η, is a polynomial of variable $z\left(\frac{d}{dz}\right)$ of degree $\eta_1 + \ldots + \eta_m$, given by

$$D_\eta = \prod_{r=1}^m \prod_{j=1}^{\eta_r} \frac{1}{\beta_r} z \frac{d}{dz} + \gamma_r + j; \eta_k = \begin{cases} ([\delta_k] + 1; \delta_k \notin \mathbb{Z} \\ \delta_k; \delta_k \in \mathbb{Z} \end{cases} \tag{14}$$

and the corresponding Kiryakova's fractional derivative ((multiple) E–K derivative operators) in Caputo sense is expressed by [14] (p. 9, as well as the related references within)

$$^{*}D_{(\beta_k),m}^{(\gamma_k)_1^m,(\delta_k)}(f(z)) = I_{(\beta_k),m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} D_\eta f(\omega). \tag{15}$$

The following action of Kiryakova's fractional transform [14] (p. 9, Equation (27)) is significant for this research:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\{z^p\} = \prod_{k=1}^m \frac{\Gamma\left(\gamma_k + 1 + \frac{p}{\beta_k}\right)}{\Gamma\left(\gamma_k + \delta_k + 1 + \frac{p}{\beta_k}\right)} z^p; (\delta_k \geq 0; k = 1, \ldots, m \wedge p > [-\beta_k(1 + \gamma_k)]). \tag{16}$$

Relationship of the kernels of different fractional operators and the (multiple) E–K operators is listed in Table 1.

Table 1. Significant special cases of (multiple) E–K operators [14–18].

Cases of Equation (11)	Relationship among the Kernels of Different Fractional Operators [14]
Marichev–Saigo–Maeda (M-S-M) $(m = 3; 1 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha)$	$H_{3,3}^{3,0}\left(\frac{t}{x}\right) = G_{3,3}^{3,0}\left[\frac{t}{x} \middle| \begin{matrix} \gamma_1' + \gamma_2', & \nu - \gamma_1, \nu - \gamma_2 \\ \gamma_1', \gamma_2', & \nu - \gamma_1 - \gamma_2 \end{matrix}\right] = \frac{x^{-\gamma_1}}{\Gamma(\nu)} (x-t)^{\delta-1} t^{-\gamma_1'} F_3\left(\gamma_1, \gamma_1', \gamma_2, \gamma_2', \nu; 1 - \frac{t}{x}; 1 - \frac{x}{t}\right)$
Saigo $(m = 2; \alpha_1 = \alpha_2 = \alpha > 0; \sigma = \frac{t}{x} \wedge \sigma = \frac{x}{t})$	$H_{2,2}^{2,0}\left[\sigma \middle| \begin{matrix} (\gamma_1 + \nu_1 + 1 - \frac{1}{\alpha}, \frac{1}{\alpha}), (\gamma_2 + \nu_2 + 1 - \frac{1}{\alpha}, \frac{1}{\alpha}) \\ (\gamma_1 + 1 - \frac{1}{\alpha}, \frac{1}{\alpha}), (\gamma_2 + 1 - \frac{1}{\alpha}, \frac{1}{\alpha}) \end{matrix}\right] = G_{2,2}^{2,0}\left[\sigma^\alpha \middle| \begin{matrix} \gamma_1 + \nu_1, \gamma_2 + \nu_2 \\ \gamma_1, \gamma_2 \end{matrix}\right] =$ $\alpha \frac{\sigma^{\alpha\gamma_2}(1-\sigma^\alpha)^{\nu_1+\nu_2-1}}{\Gamma(\nu_1+\nu_2)} {}_2F_1(\nu_2 - \gamma_1 + \gamma_2, \nu_1; \nu_1 + \nu_2; 1 - \sigma^\alpha)$
Erdélyi–Kober (E–K) $(m = 1)$	$H_{1,0}^{1,1}\left[\sigma \middle| \begin{matrix} (\gamma + \nu, \frac{1}{\alpha}) \\ (\gamma, \frac{1}{\alpha}) \end{matrix}\right] = \alpha \sigma^{\alpha-1} G_{1,0}^{1,1}\left[\sigma^\alpha \middle| \begin{matrix} \gamma + \nu \\ \gamma \end{matrix}\right] = \alpha \frac{\sigma^{\alpha(\gamma+1)-1}(1-\sigma^\alpha)^{\nu-1}}{\Gamma(\nu)}$
Riemann–Liouville (R–L) $(m = 1 = \alpha; \text{for } \frac{t}{x} = \sigma; \frac{x}{t} = \sigma)$	$H_{1,0}^{1,1}\left[\sigma \middle| \begin{matrix} (\gamma + \nu, 1) \\ (\gamma, 1) \end{matrix}\right] = G_{1,0}^{1,1}\left[\frac{t}{x} \middle| \begin{matrix} \gamma + \nu \\ \gamma \end{matrix}\right] = \frac{(x-t)^{\nu-1} t^\gamma}{\Gamma(\nu)}$

2.2. Special Functions and Theory of Distributions

Generalized functions (also known as distributions) constitute continuous linear functionals on a specific set of test functions, whereas the distribution space is the inverse (or dual) to the space of test functions [19] (Volume I–V) and [20]. Gelfand and Shilov [19] provided a thorough study and explanation of such spaces. The most frequently mentioned test functions are compact support, denoted by $\mathcal{D}$—its dual space is $\mathcal{D}'$. The convergent integral can be used to construct distributions that correspond to a locally integrable function $\vartheta(t)$ and test function $\chi(t)$ as follows:

$$\langle\vartheta(t),\chi(t)\rangle=\int_{-\infty}^{\infty}\vartheta(t)\chi(t)\mathrm{dt}. \tag{17}$$

These types of distributions are known as regular distributions. Singular functions are significant entities because they are included in the class of generalized functions (or distributions). As a result, their definition and other operations of calculus lead to functions. Furthermore, $(\forall\chi\in\mathcal{D};\mathrm{c}\in\mathbb{R})$; delta function is the most popular distribution, defined by

$$\int_{-\infty}^{\infty}\delta(\mathrm{t}-\mathrm{c})\chi(t)\mathrm{dt}=\chi(c) \tag{18}$$

and $\delta(\mathrm{ct})=\frac{\delta(\mathrm{t})}{|\mathrm{c}|}, c\neq 0$. This is one of the best instances of a singular distribution because it behaves as a continuous linear functional on a set of test functions and cannot be constructed from a locally integrable function. Assuming this is true, then

$$\chi(0)=\int_{-\infty}^{\infty}\delta(t)\chi(t)\mathrm{dt};\forall\chi\in\mathcal{D} \tag{19}$$

But, for $\chi(\mathrm{t})=\exp\left(\frac{1}{\mathrm{c}^2-\mathrm{t}^2}\right)$, the above equation becomes

$$\chi(0)=\int_{-\mathrm{c}}^{c}\delta(t)\exp\left(\frac{1}{c^2-t^2}\right)\mathrm{dt} \tag{20}$$

if we assume $\delta(\mathrm{t})$ is a locally integrable function. However, in that case, the Lebesgue's general theory of convergence demonstrates that Equation (20) converges to zero as $c\to 0$, leading to an inconsistency.

An infinitely differentiable class of test functions that is closed under Fourier transformation includes rapidly decaying functions denoted by $\mathcal{S}$, with $\mathcal{S}'$ as their dual space. In actuality, the Fourier transforms of the functions of the abovementioned dual space $\mathcal{D}'$ are not its elements, but rather they belong to a different space $\mathcal{Z}$ which consists of complex functions. It is significant to note that Fourier transforms of $\mathcal{Z}$ belong to $\mathcal{D}$. Furthermore, an entire function does not vanish in only a particular interval $\omega_1<t<\omega_2$, which provides the following inclusion:

$$\mathcal{Z}\subset\mathcal{S}\subset\mathcal{S}'\subset\mathcal{Z}'\wedge\mathcal{D}\subset\mathcal{S}\subset\mathcal{S}'\subset\mathcal{D}'\wedge\mathcal{Z}\cap\mathcal{D}\equiv 0 \tag{21}$$

And, for $\forall\chi\in\mathcal{Z}$ and $\omega=\mathrm{t}+\iota\mathrm{u}\in\mathbb{C};\mathrm{t},\mathrm{u}\in\mathbb{R}$,

$$|\omega^{\mathrm{p}}\chi(\omega)|\leq\mathrm{C_p}\mathrm{e}^{\eta|u|};(\mathrm{p}=0,1,2,\ldots). \tag{22}$$

This involves the constants η and C_p determined by χ, and u is the imaginary part of ω. Similarly, the derivatives of the delta function

$$\left\langle\delta^{(k)}(\omega),\wp(\omega)\right\rangle=(-1)^k\wp^{(k)}(0) \tag{23}$$

act as a singular distribution. For example, the electromotive force in an electric circuit [20] (p. 164) is given by

$$\vartheta(\omega) = \sum_{k=0}^{\infty} \delta^{(k)}(\omega - k). \tag{24}$$

Fourier transforms of commonly used trigonometric and hyperbolic functions such as $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ are also delta functions (singular distributions) [19] (Volume 1). Fourier transform of the exponential function [19] (Volume 1, p. 169, Equation (8))

$$\mathcal{F}\left[e^{\alpha t};\theta\right] = 2\pi\delta(\theta - \iota\alpha) \tag{25}$$

Is an element of $\mathcal{Z}'$ such that, for $\forall g \in \mathcal{Z}'$ [19] (p. 159, Equation (4)); see also [20] (p. 201, Equation (9)),

$$g(\omega + b) = \sum_{k=0}^{\infty} g^{(k)}(\omega)\frac{b^k}{k!} \qquad (\omega,\ b \in \mathbb{C}) \tag{26}$$

leads to the following expansion, given in [19] (p. 160, Volume I):

$$\delta(\omega + b) = \sum_{k=0}^{\infty} \delta^{(k)}(\omega)\frac{b^k}{k!}. \tag{27}$$

The convolution of delta function with a suitable function yields

$$\delta(t - a) * g(t) = g(t - a); \qquad \delta^{(k)}(t - a) * g(t) = g^{(k)}(t - a) \tag{28}$$

and

$$\begin{aligned} \left(\sum_{i=0}^{\infty} \delta^{(i)}(\omega - v)\right) * \left(\sum_{i=0}^{\infty} \delta(\omega - v)\right) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i} \delta^{(j)}(\omega - v) \\ \left(\sum_{i=0}^{\infty} \delta^{(i)}(\omega - v)\right) * \left(\sum_{i=0}^{\infty} \delta^{(i)}(\omega - v)\right) &= \left(\sum_{j=0}^{\infty} (v+1)\delta^{(j)}(\omega - v)\right) \end{aligned} \tag{29}$$

For this research, we consider [21] (Equation (2.13))

$$\Gamma(t + \iota u)E_{\alpha,\beta}^{t+\iota u}(s) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n s^r}{n!r!\Gamma(\alpha r + \beta)}\delta(u - \iota(t + n + r)). \tag{30}$$

Based on this, we developed many new and novel results. Prior to this, Chaudhry and Qadir [22] obtained the following representation of a gamma function:

$$\Gamma(t + \iota u) = 2\pi\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\delta(u - \iota(t + k)) \tag{31}$$

This can also be obtained by putting $s = 0$ in Equation (30), which is also modified and generalized by Tassaddiq [23–25] as follows:

$$\Gamma(\gamma) = 2\pi\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\delta(\gamma + k). \tag{32}$$

For further similar studies of other special functions, the interested reader is referred to [26,27] and references therein.

Unless otherwise mentioned in this article, the values of the parameters shall be deemed normal, as specified in Section 2.

3. New Representation of a Fox–Wright Function with Application to the Fractional Kinetic Equation

This section contains the distributional representation of a Fox–Wright function as a series involving the complex delta function [19,20]. This is extremely useful in computing the Laplace transform of this function with respect to the third parameter γ, which results in the solution of the new integral equation involving it.

Theorem 1. *A Fox–Wright function has a new representation, given as*

$$ {}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right]=2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\delta(\gamma+n+r). \tag{33}$$

Proof. This follows from the modification of (30), given as

$$\delta(\theta–\iota(\nu+n+r))=\delta\left[\frac{1}{\iota}(\iota\theta+(\nu+n+r))\right]=|\iota|\delta(\nu+\iota\theta+n+r)=\delta(\gamma+n+r). \tag{34}$$

The stated form can thus be obtained by inserting (34) in (30) and using (5). □

Corollary 1. A Fox–Wright function has a new representation, which is given as

$$ {}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right]=2\pi\sum_{n,r,p=0}^{\infty}\frac{(-1)^n s^r(n+r)^p}{n!r!\Gamma(\alpha r+\beta)p!}\delta^{(p)}(\gamma) \tag{35}$$

Proof. By using (27) in (33),

$$\delta(\gamma+n+r)=\sum_{p=0}^{\infty}\frac{(n+r)^p}{p!}\delta^{(p)}(\gamma). \tag{36}$$

Thus, the stated form is obtained. □

The gamma function findings can be retrieved in Equations (33) and (35) by using $s=0$.

As a result, it is clear that the mathematical notions and facts concerning the delta function exist for a Fox–Wright function in relation to the new representation. This sheds new light on more new results in various directions. For example, by using [20] (p. 227) for $L\left\{\delta^{(p)}(\gamma);\omega\right\}=\omega^p$, one can obtain

$$\begin{aligned} L\left({}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right)&=L\left(2\pi\sum_{m,r,p=0}^{\infty}\frac{(-1)^m s^r(m+r)^p}{m!r!\Gamma(\alpha r+\beta)p!}\delta^{(p)}(\gamma);\omega\right)\\ &=2\pi\sum_{m,r,p=0}^{\infty}\frac{(-1)^m s^r(m+r)^p}{m!r!\Gamma(\alpha r+\beta)p!}L\left(\delta^{(p)}(\gamma);\omega\right)\\ &=2\pi\sum_{m,r,p=0}^{\infty}\frac{(-1)^m s^r(m+r)^p}{m!r!\Gamma(\alpha r+\beta)p!}\omega^p\\ &=2\pi\sum_{m,r=0}^{\infty}\frac{(-1)^m s^r}{m!r!\Gamma(\alpha r+\beta)}{}_0\Psi_0\left[\begin{matrix}-\\-\end{matrix}\middle|(m+r)\omega\right]\\ &=2\pi\exp(-e^{\omega})\,{}_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\middle|se^{\omega}\right]\end{aligned} \tag{37}$$

Taking $s=0$, this provides

$$L\{\Gamma(\gamma);\omega\}=2\pi\exp(-e^{\omega}) \tag{38}$$

And, by using the relation (5),

$$\begin{aligned}
L\left(\Gamma(\gamma-c)E^{\gamma-c}_{\alpha,\beta}(s);\omega\right) &= 2\pi\sum_{m,r,p=0}^{\infty}\frac{(-1)^m s^r (m+r)^p}{m!r!\Gamma(\alpha r+\beta)p!}L\left\{\delta^{(r)}(\gamma-c);\omega\right\} \\
&= 2\pi\sum_{m,r,p=0}^{\infty}\frac{(-1)^m s^r (m+r)^p}{m!r!\Gamma(\alpha r+\beta)p!}\omega^p e^{-\omega c} \\
&= 2\pi e^{-\omega c}\exp(-e^{\omega})\,{}_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\middle|se^{\omega}\right].
\end{aligned} \tag{39}$$

Moreover, one can compute that

$$L\{\Gamma(\gamma-c);\omega\} = 2\pi e^{-\omega c}\exp(-e^{\omega}). \tag{40}$$

For fractional calculus, many researchers have made substantial contributions [28]. Prior research on many generic families of fractional kinetic equations has been performed in the literature [29]. In contrast to the many multi-parameter extensions of the Mittag-Leffler function and the Hurwitz–Lerch function, Srivastava examined far more general functions in [30,31]. In particular, kinetic equations of fractional order seem to have obtained interest, due to the recent uncovering of their relationship with the theory of continuous-time random walks [32]. These equations are being explored, with the goal of first determining and then interpreting certain physical effects known to regulate processes that include diffusion in porous media, anomalous propagation, and so on. A review of the literature revealed that no such equation incorporating the integration of the Fox–Wright function with regard to its third parameter has ever been created. The primary objective of this section is to pose and address this issue.

Theorem 2. *Following non-integer order kinetic equation with respect to the third parameter of a Fox–Wright function,*

$$\mathcal{E}(\gamma) - \mathcal{E}_0\,{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right] = -d^{\delta}I^{\delta}_{0+}\mathcal{E}(\gamma);\sigma\in\mathbb{R};\gamma\in\mathbb{R}^+\wedge\alpha,\beta,d,\delta>0 \tag{41}$$

leads to the following solution:

$$\mathcal{E}(\gamma) = \frac{2\pi\mathcal{E}_0}{\gamma}\sum_{n,r,p=0}^{\infty}\frac{(-1)^n s^r\left(\frac{n+r}{\gamma}\right)^p}{n!r!\Gamma(\alpha r+\beta)p!}E_{\delta,-p}\left(-d^{\delta}\gamma^{\delta}\right). \tag{42}$$

Proof. Let us begin by applying the Laplace transform (see [1,2]) to both sides of (41):

$$L\{\mathcal{E}(\gamma)\} - \mathcal{E}_0 L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right]\right\} = L\left\{-d^{\delta}I^{\delta}_{0+}\mathcal{E}(\gamma)\right\}, \tag{43}$$

in which

$$\mathcal{E}(\omega) = L[\mathcal{E}(t):\omega] = \int_0^{\infty}e^{-\omega t}\mathcal{E}(t)dt, \Re(\omega)>0, \tag{44}$$

$$L\left\{I^{\delta}_{0+}\mathcal{E}(\gamma);\omega\right\} = \omega^{-\delta}\mathcal{E}(\omega). \tag{45}$$

Then, by employing (37),

$$\mathcal{E}(\omega) = 2\pi\mathcal{E}_0\sum_{n,r,p=0}^{\infty}\frac{(-1)^n s^r(n+r)^p}{n!r!\Gamma(\alpha r+\beta)p!}\omega^p - \left(\frac{\omega}{d}\right)^{-\delta}\mathcal{E}(\omega) \tag{46}$$

expresses the above Equation (46) as follows:

$$\mathcal{E}(\omega)\left[1+\left(\frac{\omega}{d}\right)^{-\delta}\right]=2\pi\mathcal{E}_0\sum_{n,r,p=0}^{\infty}\frac{(-1)^n s^r (n+r)^p}{n!r!\Gamma(\alpha r+\beta)p!}\omega^p. \tag{47}$$

One can determine the result after a simple computation, stated as

$$\mathcal{E}(\omega)=2\pi\mathcal{E}_0\sum_{n,r,p=0}^{\infty}\frac{(-1)^n \mathrm{s}^r (n+r)^p}{n!r!\Gamma(\alpha r+\beta)p!}\omega^p\sum_{m=0}^{\infty}\left[-\left(\frac{\omega}{d}\right)^{-\delta}\right]^m. \tag{48}$$

Furthermore, let $\delta\mathrm{m}-\mathrm{p}>0;\delta>0$ and use $L^{-1}\{\omega^{-\delta};\gamma\}=\frac{\gamma^{\delta-1}}{\Gamma(\delta)}$ to calculate L^{-1} (the inverse Laplace transform) of (48), given by

$$\mathcal{E}(\gamma)=2\pi\mathcal{E}_0\sum_{n,r,p=0}^{\infty}\frac{(-1)^n \mathrm{s}^r (n+r)^p}{n!r!\Gamma(\alpha r+\beta)p!}\gamma^{-p-1}\times\sum_{\mathrm{m}=0}^{\infty}\frac{\left(-\mathrm{d}^\delta\gamma^\delta\right)^{\mathrm{m}}}{\Gamma(\delta\mathrm{m}-\mathrm{p})}. \tag{49}$$

Using (4) in (49), we can finally obtain (42). □

Remark 1. It is remarkable that the solution approach is traditional [1,2], and the response rate $\mathcal{E}(\gamma)$ is a function of the fractional parameter δ. Typically, it is described in terms of the Mittag-Leffler function, as seen in the preceding solution. As a result, the sum over the coefficients $\mathrm{C}^{\mathrm{s}}_{\beta,\alpha}(\gamma)$ in (42) is clearly defined and finite:

$$\mathrm{C}^{\mathrm{s}}_{\beta,\alpha}(\gamma)=\sum_{\mathrm{n,r,p}=0}^{\infty}\frac{(-1)^{\mathrm{n}}\left(\frac{\mathrm{n+r}}{\gamma}\right)^{\mathrm{p}}}{\mathrm{n!r!}\Gamma(\alpha\mathrm{r}+\beta)\mathrm{p!}}=\exp\left(-\mathrm{e}^{1/\gamma}\right){}_0\Psi_1\left[\left.\begin{matrix}-\\(\beta,\alpha)\end{matrix}\right|\mathrm{e}^{1/\gamma}\right]. \tag{50}$$

In the same way,

$$\lim_{\gamma\to\infty}\mathrm{C}^{\mathrm{s}}_{\beta,\alpha}(\gamma)=\exp(-1){}_0\Psi_1\left[\left.\begin{matrix}-\\(\beta,\alpha)\end{matrix}\right|1\right];\ \alpha,\beta>0. \tag{51}$$

3.1. New Fractional Image Formulae Involving a Fox–Wright Function

Lemma 1. Using the definition of a Fox–Wright function, we demonstrate that the following identity is accurate:

$$\sum_{n,\mathrm{r}=0}^{\infty}\frac{(-1)^n\mathrm{s}^{\mathrm{r}}}{\mathrm{n!r!}\Gamma(\alpha\mathrm{r}+\beta)}{}_0\Psi_0\left[\left.\begin{matrix}-\\-\end{matrix}\right|(\mathrm{n+r})\omega\right]=\exp(-\mathrm{e}^\omega){}_0\Psi_1\left[\left.\begin{matrix}-\\(\beta,\alpha)\end{matrix}\right|\mathrm{se}^\omega\right]. \tag{52}$$

Proof. Using Equation (37), we obtain the following:

$$\mathrm{L}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\mathrm{s}\right];\omega\right)=2\pi\sum_{\mathrm{n,r,p}=0}^{\infty}\frac{(-1)^{\mathrm{n}}\mathrm{s}^{\mathrm{r}}(\mathrm{n+r})^{\mathrm{p}}}{\mathrm{n!r!}\Gamma(\alpha\mathrm{r}+\beta)\mathrm{p!}}\omega^{\mathrm{p}}=2\pi\sum_{\mathrm{n,r}=0}^{\infty}\frac{(-1)^{\mathrm{n}}\mathrm{s}^{\mathrm{r}}}{\mathrm{n!r!}\Gamma(\alpha\mathrm{r}+\beta)}{}_0\Psi_0\left[\left.\begin{matrix}-\\-\end{matrix}\right|(\mathrm{n+r})\omega\right], \tag{53}$$

then

$$\begin{aligned}\mathrm{L}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\mathrm{s}\right];\omega\right)&=2\pi\sum_{\mathrm{n,r,p}=0}^{\infty}\frac{(-1)^{\mathrm{n}}\mathrm{s}^{\mathrm{r}}(\mathrm{n+r})^{\mathrm{p}}}{\mathrm{n!r!}\Gamma(\alpha\mathrm{r}+\beta)\mathrm{p!}}\omega^{\mathrm{p}}=\sum_{\mathrm{n}=0}^{\infty}\frac{(-\mathrm{e}^\omega)^{\mathrm{n}}}{\mathrm{n!}}{}_0\Psi_1\left[\left.\begin{matrix}-\\(\beta,\alpha)\end{matrix}\right|\mathrm{se}^\omega\right]\\&=\exp(-\mathrm{e}^\omega){}_0\Psi_1\left[\left.\begin{matrix}-\\(\beta,\alpha)\end{matrix}\right|\mathrm{se}^\omega\right].\end{aligned} \tag{54}$$

The necessary result is, therefore, established from both of the aforementioned Equations (53) and (54). □

Remark 2. It should be noted that a general result can be deduced from (52) as follows:

$$\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}{}_p\Psi_q\left[\begin{matrix}(a_i,A_i)\\(b_j,B_j)\end{matrix}\middle|(n+r)\omega\right]=\exp(-e^{\omega})_p\Psi_{q+1}\left[\begin{matrix}- & (a_i,A_i)\\(\beta,\alpha) & (b_j,B_j)\end{matrix}\middle|se^{\omega}\right] \tag{55}$$

Theorem 3. *The Kriyakova's fractional transform involving the Fox–Wright function is computedas*

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\left(\omega^{\chi-1}L\left({}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right)\right)=2\pi\omega^{\chi-1}\exp(-e^{\omega})_m\Psi_{m+1}\left[\begin{matrix}- & \left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\(\beta,\alpha) & \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\middle|se^{\omega}\right] \tag{56}$$

$$[-\beta_i(1+\gamma_i)]<p;\delta_i\geq 0;i=1,\ldots,m$$

Proof. Consider the following:

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\left(\omega^{\chi-1}L\left({}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right)\right)=I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\left(\omega^{\chi-1}2\pi\sum_{m,,p,r=0}^{\infty}\frac{(-1)^m s^r(m+r)^p}{m!r!\Gamma(\alpha r+\beta)p!}\omega^p\right). \tag{57}$$

The summation and integration are then exchanged:

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\left(\omega^{\chi-1}L\left({}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right)\right)=2\pi\sum_{m,,p,r=0}^{\infty}\frac{(-1)^m s^r(m+r)^p}{m!r!\Gamma(\alpha r+\beta)p!}I^{(I),I}_{(I_i),m}\left(\omega^{\chi-1}\omega^p\right), \tag{58}$$

which, after using (16), gives

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\left(\omega^{\chi-1}L\left({}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right)\right)=2\pi\sum_{m,,p,r=0}^{\infty}\frac{(-1)^m s^r(m+r)^p}{m!r!\Gamma(\alpha r+\beta)p!}\prod_{i=1}^{m}\frac{\Gamma\left(\gamma_i+1+\frac{\chi+p-1}{\beta_i}\right)}{\Gamma\left(\gamma_i+\delta_i+1+\frac{\chi+p-1}{\beta_i}\right)}\omega^{p+\chi-1}, \tag{59}$$

and, by making use of Equation (9) in Equation (59), gives the subsequent result

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m}\left(\omega^{\chi-1}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right)\right)=2\pi\omega^{\chi-1}\sum_{m,,p,r=0}^{\infty}\frac{(-1)^m s^r}{m!r!\Gamma(\alpha r+\beta)}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\middle|(m+r)\omega\right]. \tag{60}$$

$$[-\beta_i(1+\gamma_i)]<p;\delta_i\geq 0;i=1,\ldots,m.$$

As a result, applying Remark 2 yields the appropriate simplified form. □

Important special cases of Equation (56) are listed in Table 2.

Table 2. Formulae for fractional integrals containing a Fox–Wright function.

m = 3	Marichev–Saigo–Maeda Fractional Integrals
	$I^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}_{0+}\left(\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}\right)=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}\exp(-e^{\omega})$ ${}_3\Psi_4\left[\begin{matrix}- & (\chi,1) & \left(\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1\right) & \left(\chi+\gamma_2'-\gamma_1',1\right)\\(\beta,\alpha) & \left(\chi+\gamma_2',1\right) & \left(\chi+\delta-\gamma_1-\gamma_1',1\right) & \chi+\delta-\gamma_1'-\gamma_2\end{matrix}\middle|se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C}).$
	$I^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}_{0-}\left(\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}\right)=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1'-1}\exp(-e^{\omega})$ ${}_3\Psi_4\left[\begin{matrix}- & \left(1-\chi-\delta+\gamma_1+\gamma_1',1\right) & \left(1-\chi+\gamma_1+\gamma_2'-\delta,1\right) & 1-\chi-\gamma_1\\(\beta,\alpha) & (1-\chi,1) & \left(1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta,1\right) & 1-\chi+\gamma_1-\gamma_2\end{matrix}\middle|se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C}).$

Table 2. *Cont.*

m = 2	Saigo fractional integrals
	$I_{0+}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}\right)=2\pi\omega^{\chi-\gamma_1-1}\exp(-e^{\omega})\,{}_2\Psi_3\left[\begin{matrix}- & (\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\(\beta,\alpha) & (\chi-\gamma_2,1) & (\chi+\delta+\gamma_2)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
	$I_{-}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}\right)=$ $2\pi\omega^{\chi-\gamma_1-1}\exp(-e^{\omega})\,{}_2\Psi_3\left[\begin{matrix}- & (\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,1)\\(\beta,\alpha) & (1-\chi,1) & ((\gamma_1+\gamma_2+\delta-\chi+1,1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
m = 1	**Erdélyi–Kober fractional integrals**
	$I_{0+}^{\gamma,\delta}\left(\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}\right)=2\pi\omega^{\chi-1}\exp(-e^{\omega})\,{}_1\Psi_2\left[\begin{matrix}- & (\chi+\gamma,1)\\(\beta,\alpha) & (\chi+\gamma+\delta,1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
	$I_{0-}^{\gamma,\delta}\left(\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}\right)=2\pi\omega^{\chi-1}\exp(-e^{\omega})\,{}_1\Psi_2\left[\begin{matrix}- & (\gamma-\chi+1,-1)\\(\beta,\alpha) & (\gamma+\delta-\chi+1,-1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
m = 1	**Riemann–Liouville (R–L) fractional integrals**
	$I_{0+}^{\delta}\left(\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}\right)=2\pi\omega^{\chi+\delta-1}\exp(-e^{\omega})\,{}_1\Psi_2\left[\begin{matrix}- & (\chi,1)\\(\beta,\alpha) & (\delta+\chi,1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
	$I_{-}^{\delta}\left(\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}\right)=2\pi\omega^{\chi+\delta-1}\exp(-e^{\omega})\,{}_1\Psi_2\left[\begin{matrix}- & (1-\delta-\chi,-1)\\(\beta,\alpha) & (1-\chi,-1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.

3.2. Generalized Fractional Derivatives Involving a Fox–Wright Function

We can derive the generalized fractional derivatives involving a Fox–Wright function by applying Theorem 1's technique and the new representation of a Fox–Wright function. Here, we directly derive them using the general result [14] (Theorem 4), stated as

$$D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}\left\{z^c\,{}_p\Psi_q\left[\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix};\lambda z^{\mu}\right]\right\}=z^c\left\{{}_{p+m}\Psi_{q+m}\left[\begin{matrix}(a_i,\alpha_i)_1^p,\left(\gamma_k+\delta_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\(b_j,\beta_j)_1^q,\left(\gamma_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix};\lambda z^{\mu}\right]\right\}. \quad (61)$$

Applying Kriyakova's fractional derivatives (Multiple E–K fractional derivatives) (61) on (37) and then using (52) yields generalized fractional derivatives involving a Fox–Wright function:

$$\left(D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}\right)=2\pi\omega^{\chi-1}\exp(-e^{\omega})\,{}_m\Psi_{m+1}\left[\begin{matrix}- & \left(\gamma_k+\delta_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\(\beta,\alpha) & \left(\gamma_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\middle\vert se^{\omega}\right] \quad (62)$$

Further related cases of Equation (62) are listed in Table 3.

Table 3. Formulae for fractional derivatives containing a Fox–Wright function.

m = 3	Marichev–Saigo–Maeda Fractional Derivative
	$D_{0+}^{\gamma_1,\gamma_1{}',\gamma_2,\gamma_2{}',\delta}\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1{}'-1}\exp(-e^{\omega})$ ${}_3\Psi_4\left[\begin{matrix}- & (\chi,1) & (\chi-\gamma_2+\gamma_1,1) & \left(\chi+\gamma_1+\gamma_1{}'+\gamma_2{}'-\delta,1\right)\\(\beta,\alpha) & (\chi-\gamma_2,1) & \left(\chi-\delta+\gamma_1++\gamma_2{}',1\right) & \left(\chi-\delta+\gamma_1{}'+\gamma_1,1\right)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
	$D_{-}^{\gamma_1,\gamma_1{}',\gamma_2,\gamma_2{}',\delta}\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}=2\pi\omega^{\delta+\chi-\gamma_1-\gamma_1{}'-1}\exp(-e^{\omega})$ ${}_3\Psi_4\left[\begin{matrix}- & \left(1-\chi+\gamma_2{}',1\right) & \left(1+\gamma_2{}'-\chi-\gamma_2+\gamma_1,1\right) & \left(1-\chi-\gamma_1-\gamma_1{}'+\delta,1\right)\\(\beta,\alpha) & (1-\chi,1) & \left(1-\chi-\gamma_1{}'+\gamma_2{}',1\right) & \left(1-\chi+\delta-\gamma_1{}'-\gamma_1-\gamma_2,1\right)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.

Table 3. *Cont.*

m = 2	Saigo fractional derivative
	$D_{0+}^{\gamma_1,\gamma_2,\delta}\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}=2\pi\omega^{\chi-\gamma_1-1}\exp(-e^{\omega})$ ${}_2\Psi_3\left[\begin{matrix}- & (\chi,1) & (\chi+\delta+\gamma_2+\gamma_1,1)\\(\beta,\alpha) & (\chi+\gamma_2,1) & (\chi+\delta,1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
	$D_{-}^{\gamma_1,\gamma_2,\delta}\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}=2\pi\omega^{\chi-\gamma_1-1}\exp(-e^{\omega})$ ${}_2\Psi_3\left[\begin{matrix}- & (1-\chi-\gamma_2,1) & (1-\chi+\delta+\gamma_1,1)\\(\beta,\alpha) & (1-\chi+\delta-\gamma_2,1) & (1-\chi,1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
m = 1	**Erdélyi–Kober fractional derivative**
	$D_{0+}^{\gamma,\delta}\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}=2\pi\omega^{\chi-\gamma_1-1}\exp(-e^{\omega}){}_1\Psi_2\left[\begin{matrix}- & (\gamma+\delta+\chi,1)\\(\beta,\alpha) & (\gamma+\chi,1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
	$D_{-}^{\gamma,\delta}\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}=2\pi\omega^{\chi-1}\exp(-e^{\omega}){}_1\Psi_2\left[\begin{matrix}- & (1-\chi+\gamma+\delta,1)\\(\beta,\alpha) & (1-\chi+\gamma,1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
m = 1	**Riemann–Liouville (R–L) fractional derivative**
	$D_{0+}^{\delta}\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}=2\pi\omega^{\chi-1}\exp(-e^{\omega}){}_1\Psi_2\left[\begin{matrix}- & (\chi,1)\\(\beta,\alpha) & (\chi-\delta,1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.
	$D_{-}^{\delta}\omega^{\chi-1}L\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\omega\right\}=2\pi\omega^{\chi-1}\exp(-e^{\omega}){}_1\Psi_2\left[\begin{matrix}- & (\delta-\chi+1,1)\\(\beta,\alpha) & (1-\chi,1)\end{matrix}\middle\vert se^{\omega}\right]$ $(\alpha,\beta,\gamma\epsilon\mathbb{C},\Re(\alpha),\Re(\gamma)>0;\omega\in\mathbb{C})$.

4. Convergence of New Series Representation as a Distribution

The new series representation of a Fox–Wright function is established using a delta function, which is significant if it is correctly specified in terms of the distributional concept. As a result, it is exciting to show that the given representation is a distribution (generalized function) on space $\mathcal{Z}$, as stated in the following theorem.

Theorem 4. *A Fox–Wright function acts as a distribution over the space* $\mathcal{Z}$.

Proof. Consider the following combination, by taking $\wp_1(\gamma),\wp_2(\gamma)\epsilon\mathcal{Z}$ and $c_1,c_2\epsilon\mathbb{C}$:

$$\begin{aligned}&\left\langle{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],c_1\wp_1(\gamma)+c_2\wp_2(\gamma)\right\rangle\\&=\left\langle 2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\delta(\gamma+n+r),c_1\wp_1(\gamma)+c_2\wp_2(\gamma)\right\rangle.\end{aligned}\tag{63}$$

$$\Longrightarrow\left\langle{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],c_1\wp_1(\gamma)+c_2\wp_2(\gamma)\right\rangle=c_1\left\langle{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],\wp_1(\gamma)\right\rangle+c_2\left\langle{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],\wp_2(\gamma)\right\rangle.\tag{64}$$

After that, choose any sequence, $\{\wp_\ell\}_{\ell=1}^{\ell=\infty}\longrightarrow 0$ and, making use of $\{\langle\delta(\gamma+n+r)),\wp_\ell\rangle\}_{\ell=1}^{\ell=\infty}\to 0$,

$$\Rightarrow\left\{\left\langle{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],\wp_\ell(\gamma)\right\rangle\right\}_{\ell=1}^{\ell=\infty}=2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\left\{\langle\delta(\gamma+n+r)),\wp_\ell(\gamma)\rangle\right\}_{\ell=1}^{\ell=\infty}\longrightarrow 0.\tag{65}$$

To study the convergence of new representations, consider the following:

$$\begin{aligned}\left\langle{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],\wp(\gamma)\right\rangle&=2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\langle\delta(\gamma+n+r)),\wp(\gamma)\rangle;\ (\forall\wp(\gamma)\epsilon\mathcal{Z})\\&=2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\wp(-(n+r)),\end{aligned}\tag{66}$$

where

$$\text{sum over the coefficients} = \sum_{n,r=0}^{\infty} \frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)} = e^{-1}\,{}_0\Psi_1\left[\begin{matrix} - \\ (\beta,\alpha) \end{matrix}\middle| s\right]. \tag{67}$$

Consequently, Equation (66) displays that $\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],\wp(\gamma)\right\rangle;\forall\wp(\gamma)\epsilon\mathcal{Z}$ is convergent as a product of functions that increase slowly and diminish quickly. The Abel theorem can also be used to confirm this. As a result, a Fox–Wright function behaves as a distribution over $\mathcal{Z}$. □

The following example is used to better understand the preceding subject in the sense of generalized functions [20] by using the shifting property of a delta function.

Example 1. Let $\wp(\gamma) = \tau^{\gamma\xi}(\xi > 0; \gamma \in \mathbb{C})$; then,

$$\begin{aligned}\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],\wp(\gamma)\right\rangle &= \left\langle 2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\delta(\gamma+n+r),\tau^{\gamma\xi}\right\rangle = 2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\tau^{-n\xi-r\xi}\\ &= 2\pi\sum_{n,r=0}^{\infty}\frac{\left(-\tau^{-\xi}\right)^n}{n!}\frac{\left(s\tau^{-\xi}\right)^r}{r!\Gamma(\alpha r+\beta)}\\ &= 2\pi\exp\left(-\tau^{-\xi}\right){}_0\Psi_1\left[\begin{matrix} - \\ (\beta,\alpha)\end{matrix}\middle| s\tau^{-\xi}\right].\end{aligned} \tag{68}$$

For $s = 0$, this leads to

$$\left\langle\Gamma(\gamma),\tau^{\gamma\xi}\right\rangle = 2\pi\sum_{n=0}^{\infty}\frac{\left(-\tau^{-\xi}\right)^n}{n!} = 2\pi{}_0\Psi_0\left[\begin{matrix} - \\ - \end{matrix}\middle| -\tau^{-\xi}\right] = \exp\left(-\tau^{-\xi}\right) \tag{69}$$

These results provide new perspectives on the existence of further similar results; for example, considering $\tau = e^{-1}$ in Equation (68), one can derive the Laplace transform of $\Gamma(\gamma)E^{\gamma}_{\alpha,\beta}(s)$.

4.1. Validity of the New Generalized Representation

The primary goal in this section is to validate the stability of the new identities achieved through novel representation. Taking $u_i = v_i = 1$ in [21] (Equation (2.1)) and using (5), the Fourier transform representation of a Fox–Wright function is given as

$${}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right] = \sqrt{2\pi}\mathcal{F}\left[e^{\Re(\gamma)x}\exp(-e^x){}_0\Psi_1\left[\begin{matrix} - \\ (\beta,\alpha)\end{matrix}\middle| se^x\right];\xi\right]. \tag{70}$$

The Fourier transform preserves the duality property; hence, for any function $u(t)$,

$$\mathcal{F}\left[\sqrt{2\pi}\mathcal{F}[u(t);\theta];\xi\right] = 2\pi u(-\xi). \tag{71}$$

The following result is obtained by applying this characteristic to Equation (70):

$$\begin{aligned}\mathcal{F}\left\{{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right];\xi\right\} &= \mathcal{F}\left[\sqrt{2\pi}\mathcal{F}\left[e^{\Re(\gamma)x}\exp(-e^x){}_0\Psi_1\left[\begin{matrix} - \\ (\beta,\alpha)\end{matrix}\middle| se^x\right]\right];\xi\right]\\ &= f(-\xi) = 2\pi e^{-\Re(\gamma)\xi}\exp\left(-e^{-\xi}\right){}_0\Psi_1\left[\begin{matrix} - \\ (\beta,\alpha)\end{matrix}\middle| se^{-\xi}\right].\end{aligned} \tag{72}$$

The aforementioned identity's corresponding form is given as

$$\int_{-\infty}^{+\infty} e^{i\theta\xi}{}_1\Psi_1\left[\begin{matrix}(\nu+i\theta,1)\\(\beta,\alpha)\end{matrix};s\right]d\theta = 2\pi e^{-\Re(\gamma)\xi}\exp\left(-e^{-\xi}\right){}_0\Psi_1\left[\begin{matrix} - \\ (\beta,\alpha)\end{matrix}\middle| se^{-\xi}\right]; \quad (\gamma = \nu + i\theta\ ). \tag{73}$$

This can also be accomplished as a special instance of the main identity (68) by putting $\tau = e; \gamma = \nu + i\theta$. These particulars demonstrate that the findings of this novel representation are consistent with those produced by traditional methods. Additionally, by taking $\xi = 0$ in (73), the following formula can be obtained:

$$\int_{-\infty}^{+\infty} {}_1\Psi_1\left[\begin{matrix}(\nu + i\theta, 1)\\(\beta, \alpha)\end{matrix}; s\right] d\theta = 2\pi e^{-1} {}_0\Psi_1\left[\begin{matrix}-\\(\beta, \alpha)\end{matrix}\middle| s\right]. \tag{74}$$

This certifies the authenticity of a new representation, resulting in new identities that are unreachable through conventional procedures. However, individual instances of these unexpected consequences are consistent with previous findings. These identities are easily accomplished by employing the Fourier transform, and proving that this is a more efficient method for validating new representations.

4.2. New Properties of a Fox–Wright Function as a Distribution

Following the concepts and approach in [20] (pp. 199–207, Chapter 7), new distribution features are provided here. These properties hold for a Fox–Wright function due to its new formulation in terms of the delta function [19,20].

Theorem 5. *A special Fox–Wright function has the following characteristics as a generalized function (distribution) for an arbitrary test function, in which $\wp(\gamma)\epsilon\mathcal{Z}$, c, c_1 and c_2 are arbitrary real or complex constants:*

(a) *The combined effect of a Fox–Wright function and any distribution f*

$$\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right] + f, \wp(\gamma)\right\rangle = \left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right], \wp(\gamma)\right\rangle + \langle f, \wp(\gamma)\rangle;$$

(b) *A Fox–Wright function multiplied by an arbitrary constant$c_1$gives the following:*

$$\left\langle c_1 {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right], \wp(\gamma)\right\rangle = \left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right], c_1\wp(\gamma)\right\rangle;$$

(c) *An arbitrary complex constant$c$is used to shift a Fox–Wright function:*

$$\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma - c,\ 1)\\(\beta,\alpha)\end{matrix}; s\right], \wp(\gamma)\right\rangle = \left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right], \wp(\gamma + c)\right\rangle;$$

(d) *A Fox–Wright function is transposed as*

$$\left\langle {}_1\Psi_1\left[\begin{matrix}(-\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right], \wp(\gamma)\right\rangle = \left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right], \wp(-\gamma)\right\rangle;$$

(e) *The independent variable multiplied by a positive constantc_1:*

$$\left\langle {}_1\Psi_1\left[\begin{matrix}(c_1\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right], \wp(\gamma)\right\rangle = \left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right], \frac{1}{c_1}\wp\left(\frac{\gamma}{c_1}\right)\right\rangle;$$

(f) *Differentiating a Fox–Wright function as a distribution:*

$$\left\langle \frac{d^m}{d\gamma^m}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right]\right), \wp(\gamma)\right\rangle = \sum_{m,r=0}^{\infty} \frac{(-1)^m s^r}{m!r!\Gamma(\alpha r + \beta)}(-1)^m \wp^m(-n - r);$$

(g) *A special Fox–Wright function's distributional Fourier transform:*

$$\left\langle \mathcal{F}\left[{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right]\right], \wp(\gamma)\right\rangle = \left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix}; s\right], \mathcal{F}[\wp](\gamma)\right\rangle;$$

(h) *The Fourier transform's duality property:*

$$\left\langle \mathcal{F}\left[{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right]\right], \mathcal{F}[\wp(\gamma)]\right\rangle = 2\pi\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right], \wp(-\gamma)\right\rangle;$$

(i) *The Fourier transform and Parseval's identity:*

$$\left\langle \mathcal{F}\left[{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right]\right], \overline{\mathcal{F}[\wp(\gamma)]}\right\rangle = \left\langle \overline{\mathcal{F}\left[{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right]\right]}, \mathcal{F}[\wp(\gamma)]\right\rangle = 2\pi\left\langle {}_1\Psi_1\left[\begin{matrix}(\Re(\gamma),\ 1) \\ (\beta,\alpha)\end{matrix};s\right], \overline{[\wp(\Re(\gamma))]}\right\rangle;$$

(j) *Differentiation characteristics of the Fourier transform:*

$$\left\langle \mathcal{F}\left[\frac{d^m}{d\gamma^m}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right]\right), \wp(\gamma)\right]\right\rangle = \left\langle \left(-\mathrm{it}\right)^{\mathrm{m}}{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right], \mathcal{F}[\wp](\gamma)\right\rangle;$$

(k) *A Fox–Wright function's Taylor series:*

$$\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma + c_1,\ 1) \\ (\beta,\alpha)\end{matrix};s\right], \wp(\gamma)\right\rangle = \left\langle \sum_{\mathrm{n}=0}^{\infty}\frac{(\mathrm{c}_1)^{\mathrm{n}}}{\mathrm{n}!}\frac{d^n}{d\gamma^n}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right]\right), \mathcal{F}[\wp](\gamma)\right\rangle;$$

(l) *A Fox–Wright function has the property of convolution:*

$${}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right] * f(\gamma) = 2\pi\sum_{\mathrm{m,r,p}=0}^{\infty}\frac{(-1)^m\mathrm{s}^{\mathrm{r}}(m+r)^p}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)p!}\frac{d^p}{d\gamma^p}(f(\gamma));$$

(m) *If f is a bounded support distribution, then*

$$\mathcal{F}\left[\mathrm{f}(\gamma) * \exp\left(-\mathrm{e}^{\xi}\right){}_0\Psi_1\left[\begin{matrix}- \\ (\beta,\alpha)\end{matrix}\middle|\mathrm{se}^{\xi}\right]\right] = \mathcal{F}[f(\gamma);\gamma]_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right].$$

Proof. The approach of Theorem 4 and the attributes of the delta function can be used to achieve Results (a)–(e). Similarly, Equation (13) is used to show result (f):

$$\left\langle \frac{d^m}{d\gamma^m}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right]\right), \wp(\gamma)\right\rangle = \sum_{\mathrm{m,r}=0}^{\infty}\frac{(-1)^m\mathrm{s}^{\mathrm{r}}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)}(-1)^{\mathrm{m}}\wp^{\mathrm{m}}(-\mathrm{n}-\mathrm{r}).$$

It is a convergent sum of fast-decaying and slow-growing functions (as claimed and demonstrated in Theorem 4). The demonstration of outcomes (g)–(k) can also be obtained by using the delta function properties for the Fourier transform. As a result, the following is a confirmation of result (g):

$$\left\langle \mathcal{F}\left[{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right]\right], \wp(\gamma)\right\rangle = 2\pi\sum_{\mathrm{m,r}=0}^{\infty}\frac{(-1)^m\mathrm{s}^{\mathrm{r}}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)}\langle\mathcal{F}[\delta(\gamma+\mathrm{n}+\mathrm{r})], \wp(\gamma)\rangle$$

$$= 2\pi\sum_{\mathrm{m,r}=0}^{\infty}\frac{(-1)^m\mathrm{s}^{\mathrm{r}}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)}\langle\delta(\gamma+\mathrm{n}+\mathrm{r}), \mathcal{F}[\wp(\gamma)]\rangle = \left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};\omega\right], \mathcal{F}[\wp(\gamma)]\right\rangle$$

Likewise, Parseval's identity of Fourier transform is established, given as

$$\left\langle \mathcal{F}\left[{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right]\right], \overline{\mathcal{F}[\wp(\gamma)]}\right\rangle = \left\langle \overline{\mathcal{F}\left[{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1) \\ (\beta,\alpha)\end{matrix};s\right]\right]}, \mathcal{F}[\wp(\gamma)]\right\rangle = 2\pi\left\langle \left[{}_1\Psi_1\left[\begin{matrix}(\Re(\gamma),\ 1) \\ (\beta,\alpha)\end{matrix};s\right]\right], \overline{[\Re(\gamma)]}\right\rangle.$$

The result (i) can be proven by considering

$$\left\langle \mathcal{F}\left[\frac{d}{d\gamma}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right]\right)\right],\wp(\gamma)\right\rangle = 2\pi\sum_{\mathrm{m,r}=0}^{\infty}\frac{(-1)^m \mathrm{s^r}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)}\left\langle \mathcal{F}\left[\delta^{(1)}(\gamma+\mathrm{n}+\mathrm{r})\right],\wp(\gamma)\right\rangle$$

$$\left\langle \mathcal{F}\left[\frac{d}{d\gamma}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right]\right)\right],\wp(\gamma)\right\rangle = 2\pi\sum_{\mathrm{m,r}=0}^{\infty}\frac{(-1)^m \mathrm{s^r}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)}\left\langle \mathcal{F}[\delta(\gamma+\mathrm{n}+\mathrm{r})],\wp^{(1)}(\gamma)\right\rangle$$

$$\left\langle \mathcal{F}\left[\frac{d}{d\gamma}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right]\right)\right],\wp(\gamma)\right\rangle = 2\pi\sum_{\mathrm{m,r}=0}^{\infty}\frac{(-1)^m \mathrm{s^r}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)}\left\langle \delta(\gamma+\mathrm{n}+\mathrm{r}),\mathcal{F}[\wp^{(1)}(\gamma)]\right\rangle$$

$$\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right],\mathcal{F}[\wp^{(1)}(\gamma)]\right\rangle = \left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right],(-\mathrm{it})\wp(\gamma)\mathrm{e}^{-\mathrm{i}\gamma\mathrm{t}}\right\rangle$$

$$\left\langle \mathcal{F}\left[\frac{d}{d\gamma}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right]\right)\right],\wp(\gamma)\right\rangle = \left\langle (-\mathrm{it})\mathcal{F}\left[{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right]\right],\wp(\gamma)\right\rangle$$

and so forth; this results in

$$\left\langle \mathcal{F}\left[\frac{d^m}{d\gamma^m}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right]\right)\right],\wp(\gamma)\right\rangle = \left\langle (-\mathrm{it})^{\mathrm{m}}\mathcal{F}\left[{}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right]\right],\wp(\gamma)\right\rangle.$$

Equation (14) allows for the following proof of outcome number (j), which is

$$\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma+c_1,\ 1)\\(\beta,\alpha)\end{matrix};s\right],\wp(\gamma)\right\rangle = 2\pi\sum_{\mathrm{m,r}=0}^{\infty}\frac{(-1)^m \mathrm{s^r}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)}\langle\delta(\gamma+\mathrm{n}+\mathrm{r}+\mathrm{c}_1),\wp(\gamma)\rangle$$

$$= 2\pi\sum_{\mathrm{m,r}=0}^{\infty}\frac{(-1)^m \mathrm{s^r}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)}\langle\delta(\gamma+\mathrm{n}+\mathrm{r}),\wp(\gamma-\mathrm{c}_1)\rangle$$

$$= \lim_{\nu\to\infty}\left\langle 2\pi\sum_{\mathrm{m,r}=0}^{\infty}\frac{(-1)^m \mathrm{s^r}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)}\delta(\gamma+\mathrm{n}+\mathrm{r}),\sum_{\mathrm{m}=0}^{\nu}\frac{(-\mathrm{c}_1)^{\mathrm{n}}}{\mathrm{n!}}\wp^{(\mathrm{m})}(\gamma)\right\rangle$$

$$= \lim_{\nu\to\infty}\left\langle \sum_{\mathrm{m}=0}^{\nu}\frac{(\mathrm{c}_1)^{\mathrm{m}}}{\mathrm{m!}}\frac{d^m}{d\gamma^m}\left({}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right]\right),\wp(\gamma)\right\rangle,$$

This produces the desired outcome. Next, using Equation (16), which is also clarified by the following example [20] (p. 207), result (k) can be shown.

Example 2. Let $f(\gamma) = \exp(a\gamma)$; then,

$${}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};\omega\right]*\exp(a\gamma) = 2\pi\sum_{\mathrm{m,r,p}=0}^{\infty}\frac{(-1)^m \mathrm{s^r}(m+r)^p}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)p!}\delta^{(\mathrm{p})}(\gamma)*\exp(a\gamma); a>0$$

$${}_1\Psi_1\left[\begin{matrix}(\gamma,\ 1)\\(\beta,\alpha)\end{matrix};s\right]*\exp(a\gamma) = 2\pi\sum_{\mathrm{m,r,p}=0}^{\infty}\frac{(-1)^m \mathrm{s^r}(m+r)^p}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)p!}\frac{d^p}{d\gamma^p}(\exp(a\gamma))$$

$${}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\mathrm{s}\right]*\exp(\mathrm{a}\gamma) = 2\pi\exp(\mathrm{a}\gamma)\sum_{\mathrm{m,r,p}=0}^{\infty}\frac{(-1)^{\mathrm{m}}\mathrm{s^r}(\mathrm{a}(\mathrm{m}+\mathrm{r}))^{\mathrm{p}}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)\mathrm{p!}}$$

$$= 2\pi\exp(a\gamma-e^{\mathrm{a}})_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\middle|se^{\mathrm{a}}\right].$$

The definition of sinhaz and cosh az can also be used to further compute the following identities:

$${}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};\mathrm{s}\right]*\sinh\mathrm{a}\gamma = 2\pi\sum_{\mathrm{m,r,p}=0}^{\infty}\frac{(-1)^{\mathrm{m}}\mathrm{s^r}(\mathrm{m}+\mathrm{r})^{\mathrm{p}}}{\mathrm{m!r!}\Gamma(\alpha\mathrm{r}+\beta)\mathrm{p!}}\frac{\mathrm{d^p}}{\mathrm{d}\gamma^{\mathrm{p}}}(\sinh\mathrm{a}\gamma)$$

$$ {}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right]*\cosh a\gamma=2\pi\sum_{m,r,p=0}^{\infty}\frac{(-1)^m s^r(m+r)^p}{m!r!\Gamma(\alpha r+\beta)p!}\frac{d^p}{d\gamma^p}(\cosh a\gamma) $$

Next, result (l) is proven by using the fact that Fourier and inverse Fourier transformations are continuous linear functional from $\mathcal{D}'$ to $\mathcal{Z}'$ [20] (p. 203) and, therefore, $2\pi e^{\Re(\gamma)\xi}exp(-e^{\xi})_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\middle|se^{\xi}\right]\in\mathcal{D}'$, in view of Equation (70). Hereafter, in light of Theorem 7.9.1, as presented and proven in [20] (p. 206), the proof of result (m) is finished. The example that follows helps to further illustrate this.

Example 3. Consider the following distribution $f(\gamma)$ of bounded support:

$$ f(\gamma)=\begin{cases}1 & |\gamma|<1\\ 0 & |\gamma|\geq 1\end{cases} $$

Then, based on the information provided above, we obtain

$$ \mathcal{F}\left[f(\gamma)*e^{\Re(\gamma)\xi}\exp(-e^{\xi})_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\middle|se^{\xi}\right]\right]=\mathcal{F}[f(\gamma)]\mathcal{F}\left[e^{\Re(\gamma)\xi}\exp(-e^{\xi})_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\middle|se^{\xi}\right]\right]=\frac{\sin\xi}{\xi}{}_1\Psi_1\left[\begin{matrix}(\xi,1)\\(\beta,\alpha)\end{matrix};\omega\right], $$

which, because of the novel representation, is an advantageous identity. □

5. Further Applications and Discussion

The convergent behavior of ${}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right]$ with slowly rising functions is the subject of the discussion above, but the sum may converge for a wide range of entities. As a result, the new series representation is convergent for every $\wp(\gamma)\epsilon\mathcal{Z}$; nevertheless, due to the definition of the delta function, this infinite series is well defined over a wider set of functions. Here is an additional debate that serves this goal. Every function is mapped to its value at zero by the linear Dirac delta function. Consequently, using (35), one can compute the following new identities for a real t:

$$ \begin{aligned}\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],\wp(\gamma)\right\rangle&=2\pi\sum_{n,r,p=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\frac{(n+r)^p}{p!}\left\langle\delta^{(p)}(\gamma),\wp(\gamma)\right\rangle,\\&=2\pi\sum_{n,r,p=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\frac{(n+r)^p}{p!}(-1)^p\wp^{(p)}(0).\end{aligned} \quad (75) $$

Example 4. Let $\wp(\gamma)=e^{a\gamma}$; then, if $\wp^{(p)}(0)=a^p$,

$$ \begin{aligned}\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],e^{a\gamma}\right\rangle&=2\pi\sum_{n,r,p=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\frac{(n+r)^p}{p!}(-1)^p a^p\\&=2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}e^{-an-ar}\\&=2\pi\exp(-e^{-a})_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\middle|e^{-a}\right].\end{aligned} \quad (76) $$

Example 5. Let $\wp(\gamma)=\sin a\gamma$; then, for $\wp^{(p)}(0)=(-1)^p(a)^{2p+1}$; $\wp^{(p)}(0)=0; p=0,2,4,\ldots,$

$$\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],\wp(\gamma)\right\rangle = 2\pi\sum_{n,r,p=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\frac{(n+r)^{2p+1}}{(2p+1)!}(-1)^p(a)^{2p+1}$$
$$= 2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\mathrm{sin}a(-n-r)$$
$$= \mathrm{Im}\left(2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}e^{i(a(-r-n))}\right)$$
$$= \mathrm{Im}\left(2\pi\exp\left(-e^{-ia}\right){}_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\middle|e^{-ia}\right]\right), \tag{77}$$

where Im denotes the imaginary part of the complex number. Similarly, if $\wp(\gamma) = cosa\gamma$, then, when $\wp^{(p)}(0) = (-1)^p(a)^{2p}; \wp^{(p)}(0) = 0; p = 1, 3, 5, \ldots$,

$$\left\langle {}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right],\mathrm{cos}a\gamma\right\rangle = \Re\left(2\pi exp\left(-e^{-ia}\right){}_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\middle|se^{-ia}\right]\right) \tag{78}$$

It follows that, for the new representation of a special Fox–Wright function, the mathematical notions and facts related to delta functions actually exist. This sheds new light on more novel outcomes in various directions. In Section 3, a fractional kinetic equation using a Fox–Wright function was already solved using these identities, along with additional fractional formulae that were computed.

It should be noted that (52) and (56) are taken into account while evaluating the succeeding outcomes involving the products of a vast class of special functions:

$$\int_0^1 \xi^{\chi-1}\exp(-e^{\omega}){}_0\Psi_1\left[\begin{matrix}-\\(\beta,\alpha)\end{matrix}\middle|se^{\omega}\right]H^{m,0}_{m,m}\left[\xi\middle|\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right]d\xi$$
$$= 2\pi\omega^{\chi-1}\exp(-e^{\omega})_m\Psi_{m+1}\left[\begin{matrix}- & \left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\(\beta,\alpha) & \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\middle|se^{\omega}\right]. \tag{79}$$

Therefore, new integrals of products of special functions can be computed by using (11) and (33) along with the formulation of Dirac delta function:

$$\int_0^1 {}_1\Psi_1\left[\begin{matrix}(\gamma\xi,1)\\(\beta,\alpha)\end{matrix};s\right]H^{m,0}_{m,m}\left[\gamma\middle|\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right]d\gamma$$
$$= 2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\int_0^1\delta(\gamma\xi+n+r)H^{m,0}_{m,m}\left[\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\middle|\gamma\right]d\gamma$$
$$= 2\pi\xi^{-1}\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}H^{m,0}_{m,m}\left[\begin{matrix}\left(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\middle|-\frac{n+r}{\xi}\right]$$
$$= 2\pi\xi^{-1}\exp\left(-e^{\xi}\right)H^{m,0}_{m,m}\left[\begin{matrix}- & \left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\(\beta,\alpha) & \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\middle|se^{1/\xi}\right]. \tag{80}$$

Moreover, new integrals of special function products are computed using the Fox–H function relation $H^{m,0}_{m,m}\left[\frac{\xi}{\omega}\middle|\begin{matrix}\left(\gamma k+\delta_k+1-\frac{1}{\beta k},\frac{1}{\beta k}\right)_1^m\\\left(\gamma k+1-\frac{1}{\beta k},\frac{1}{\beta k}\right)_1^m\end{matrix}\right]$ with other special functions, such as those specified in Equations (7)–(10) for the G-function, Fox–Wright function, and Mittag-Leffler function. As an example,

$$\begin{aligned}\int_0^1 {}_1\Psi_1\left[\begin{matrix}(\gamma\xi,1)\\(\beta,\alpha)\end{matrix};s\right]G^{m,0}_{m,m}\left[\gamma\left|\begin{matrix}(\gamma_k+\delta_k)_1^m\\(\gamma_k)_1^m\end{matrix}\right.\right]d\gamma &= 2\pi\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}\int_0^1\delta(\gamma\xi+n+r)G^{m,0}_{m,m}\left[\gamma\left|\begin{matrix}- & (\gamma_k+\delta_k)_1^m\\- & (\gamma_k)_1^m\end{matrix}\right.\right]d\gamma\\ &= 2\pi\xi^{-1}\sum_{n,r=0}^{\infty}\frac{(-1)^n s^r}{n!r!\Gamma(\alpha r+\beta)}G^{m,0}_{m,m}\left[-\frac{n+r}{\xi}\left|\begin{matrix}(\gamma_k+\delta_k)_1^m\\(\gamma_k)_1^m\end{matrix}\right.\right] = 2\pi\xi^{-1}\exp(-e^{\xi})G^{m,0}_{m,m}\left[se^{\xi}\left|\begin{matrix}- & (\gamma_k+\delta_k)_1^m\\(\beta,\alpha) & (\gamma_k)_1^m\end{matrix}\right.\right]\end{aligned} \quad (81)$$

6. Conclusions

The various E–K operators of the generalized fractional calculus were used to obtain the new fractional transformations of a Fox–Wright function. As a result, analogous new images were obtained as special instances of the various other famous fractional transforms. This was only conceivable because the distributional representation was used to study the Laplace transform of a Fox–Wright function, whereas ${}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};s\right]$ was used to design and solve a new fractional kinetic equation with respect to the parameter γ. As corollaries, specific examples involving the original Mittag-Leffler function were presented. A freshly derived representation of the generalized Mittag-Leffler function and its Laplace transform was critical in achieving the goal of this research. It is possible to conclude that this discovery is significant in terms of expanding the applicability of the Fox–Wright function beyond its initial scope.

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