A Uniqueness Theorem for C*-algebras of Hausdorff Étale Groupoids

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Abstract

In this thesis we study the ideal intersection property for inclusions of C*-algebras $C^*_r(H_\alpha) \hookrightarrow C^*_r(G)$ induced from a family of open subgroupoids $\{H_\alpha\}$ of a locally compact Hausdorff étale groupoid G. For such a family of open subgroupoids we define the notion of relative topological principality and we show that if G is relatively topologically principal with respect to $\{H_\alpha\}$ then a representation of $C^*_r(G)$ is faithful if and only if the restriction of the representation to each of the subalgebras $C^*_r(H_\alpha)$ is faithful. This gives a new method of verifying injectivity of representations of reduced groupoid C*-algebras. As an application of our result we prove a uniqueness theorem for C*-algebras of left cancellative small categories which generalizes a theorem of Marcelo Laca and Camila Sehnem for Toeplitz algebras of group embeddable monoids.

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1 Introduction

Gelfand duality states that the commutative C*-algebras are, up to isomorphism, the algebras of continuous functions vanishing at infinity on locally compact Hausdorff (LCH) topological spaces and that any such space can be recovered, up to homeomorphism, as the spectrum of the corresponding C*-algebra. This imbues the study of C*-algebras with a topological flavor and many topological concepts such as the quotient of a space by a group action, topological K-theory, and vector bundles find natural generalizations to noncommutative C*-algebras. On the other hand, C*-algebras being, up to isomorphism, algebras of linear operators they are well suited to representing symmetry. For example to a group G we can associate a C*-algebra $C^*(G)$ with the property that any unitary representation of G factors through $C^*(G)$. Topology, algebra, and operator theory combine in the study of the crossed product C*-algebras $C_0(X) \rtimes G$ associated to a group acting by homeomorphisms on a LCH space. In this thesis we study generalizations of crossed products called groupoid C*-algebras which make possible the study of *partial symmetry* using the techniques of operator algebras. Let us explain what we mean by partial symmetry.

In all major areas of mathematics the notion of symmetry is reasoned about using groups and group actions. Groups are incredibly useful for both quantifying and describing the qualitative structure of symmetries of a mathematical object; however, they are limited to describing symmetries that are global in nature. In some cases one is interested in understanding symmetries between parts of a structure such as a fractal that exhibits high degrees of self similarity that may not be satisfactorily captured by global symmetries. In this thesis we will encounter two distinct generalizations of a group that enable a formalized study of partial symmetry: inverse semigroups and groupoids.

Having indicated our objects of study let us explain our line of research. Given a morphism of C*-algebras $\varphi : A \to C$ one often wants to know whether φ is injective. If B is a C*-subalgebra of A we say that B *detects ideals* in A, or has *the ideal intersection property*, if the only (closed 2-sided) ideal J of A such that $B \cap J = \{0\}$ is the zero ideal. If B detects ideals in A, then we can verify injectivity of φ by checking that it is injective on the subalgebra B. An important early result in this direction is the work of Kawamura and Tomiyama [11] in which they show that if the diagonal subalgebra $C(X)$ in the crossed product $C(X) \rtimes G$ by a discrete group detects ideals in $C(X) \rtimes G$, then the interior of the set of fixed points of X for each nontrivial element of G is empty. Moreover when G is amenable they prove the converse. In the paper [1] by Archbold and Spielberg this result is generalized to the setting of discrete C*-dynamical systems where $C(X)$ is replaced by a not necessarily commutative C*-algebra A. They introduce the notion of a *topologically free* action, which is the appropriate notion of freeness in this setting, and show that topological freeness of the action implies that any ideal $I \subseteq A \rtimes G$ that intersects trivially with A must be contained in the kernel of the canonical quotient map $\lambda : A \rtimes G \to A \rtimes_r G$ where $A \rtimes_r G$ is the reduced crossed product. In particular if G is amenable, so λ is injective, and the action is topologically free, one may verify injectivity of a $*$-homomorphism out of $A \rtimes G$ on the subalgebra A. It is often the case that results that can be proved for crossed products with amenable groups still hold for either the full or reduced crossed product. The result of Archbold and Spielberg indicates that, if one wants to concretely verify injectivity of maps out of a crossed product then one should look at the reduced crossed product so that one need not be concerned with the ideal $\operatorname{Ker}\lambda$. Ostensibly armed with this intuition Exel, Laca, and Quigg [10] generalized the result of Kawamura and Tomiyama on a different axis than that of Archbold and Spielberg. They show that if I is an ideal in the *reduced partial crossed product* $C_0(X) \rtimes_r G$ associated to a topologically free *partial action* of a discrete group then $I \cap C_0(X) = \{0\}$ implies $I = \{0\}$. This result generalizes nicely to C*-algebras of Hausdorff étale groupoids [8]. In the étale groupoid setting the concept of topological freeness splits into several non-equivalent definitions; however they all agree in the case that the groupoid is second countable and Hausdorff. The most commonly used condition in the Hausdorff second countable setting is the property of being *topologically principal* which says that the unit space agrees with the interior of the isotropy. In the same paper Exel gives an example of a non Hausdorff topologically principal groupoid such that the C*-subalgebra of C_0 functions on the unit space does not have the ideal intersection property in $C^*_r(G)$. Recently, however, it has become apparent that the correct notion of "reduced" C*-algebra for a non Hausdorff étale groupoid is a quotient of $C^*_r(G)$ called the essential groupoid C*-algebra. In the preprint [13] Kennedy, Kim, Li, Raum, and Ursu show that $C_0(G^{(0)})$ has the ideal intersection property in the essential groupoid C*-algebra of a non Hausdorff étale groupoid provided the groupoid satisfies an appropriate topological freeness condition [13, Definition 6.3.]. Finally one would be remiss to not mention the extensive work of Meyer and Kwaśniewski in the setting of actions of inverse semigroups by Hilbert bimodules [14, 15, 16, 17, 18].

One thing all of the above examples have in common is that they primarily study the ideal intersection property for an inclusion $A \subseteq B$ where A is the unit fiber of a Fell bundle that satisfies some sort of topological freeness condition. In [3, Theorem 3.1.] Brown, Nagy, Reznikoff, Sims, and Williams show that if G is a second countable Hausdorff étale groupoid, which they do not require to be topologically principal, i.e. one may have $G^{(0)} \subsetneq \mathrm{Iso}(G)^\circ$, then the reduced C*-algebra of the interior of the isotropy subgroupoid detects ideals in the reduced groupoid C*-algebra. This reduces to the previous cases in the event G is topologically principal.

The above theorem represents a conceptual step forward for understanding ideals in $C^*_r(G)$ in the abstract, but to use it in concrete situations is not necessarily easy. Moreover, there are situations where one has an open subgroupoid H of an étale groupoid G such that H is neither contained in $\mathrm{Iso}(G)^\circ$ nor does it contain $\mathrm{Iso}(G)^\circ$

yet one can prove $C^*_r(H)$ detects ideals in $C^*_r(G)$. For example such is the case of the uniqueness theorem for semigroup C*-algebras proved by Laca and Sehnem in [19]. For a monoid P embedded in a group G the semigroup C*-algebra $C^*_r(P)$ is isomorphic to the C*-algebra of a partial transformation groupoid $G \ltimes \Omega$. The group of units P^* of P is a subgroup of G and the transformation groupoid $P^* \ltimes \Omega$ embeds as a clopen subgroupoid of $G \ltimes \Omega$. Laca and Sehnem show that $C^*_r(P^* \ltimes \Omega)$ detects ideals in $C^*_r(G \ltimes \Omega)$ [19, Theorem 5.1.].

In this thesis we address the following question. Is there a condition on an open subgroupoid that guarantees that the reduced C*-algebra of the subgroupoid detects ideals in the reduced C*-algebra of the groupoid and moreover such that $\mathrm{Iso}(G)^\circ$ always satisfies this condition with respect to G? We answer this question affirmatively by introducing the notion of a groupoid being *relatively topologically principal* with respect to a family of open subgroupoids and prove that if G is relatively topologically principal with respect to a family of open subgroupoids $\mathcal{H}$ then a representation of $C^*_r(G)$ is faithful if and only if it is faithful on $C^*_r(H)$ for all $H \in \mathcal{H}$. We then use this theorem to prove a uniqueness theorem for the C*-algebras of left cancellative small categories, as introduced by Spielberg [29], which reduces to the theorem of Laca and Sehnem if the category is a group embeddable monoid.

Let us now explain the structure of this thesis. Chapter 2 consists of preliminaries on inverse semigroups, étale groupoids, and the C*-algebras associated to these objects. We make no claim to originality in this chapter. For groupoids our main reference is the notes by Sims [27] and for inverse semigroups we follow the book by Lawson [21]. We include almost all the proofs and several examples so that a reader familiar with C*-algebras but not inverse semigroups or groupoids has at their disposal a short, but not spare, introduction to the subject and will be able to follow the arguments in chapter 3. In section 3.1 we introduce the notion of a topologically principal groupoid and we prove, following [8] with little modification, that $C_0(G^{(0)})$ detects ideals in $C_r^*(G)$ when G is a topologically principal Hausdorff and second countable étale groupoid. This serves as motivation for section 3.2. in which we define relative topological principality of a second countable Hausdorff étale groupoid with respect to a family of open subgroupoids and we prove the main theorem of this thesis, Theorem 3.15. In chapter 3 we recall the basics of C*-algebras associated to left cancellative small categories as defined by Jack Spielberg [29] and we apply our main result to prove a uniqueness theorem for these C*-algebras, which reduces to [19, Theorem 5.1.] when the left cancellative small category is a group embeddable monoid.

2 Preliminaries

2.1 Inverse Semigroups

Definition 2.1. Recall that a *semigroup* is a set with an associative binary operation and a *monoid* is a semigroup with an identity element. A semigroup (resp. monoid) S is said to be *regular* if for every $s \in S$ there exists an element $t \in S$ such that $sts = s$ and $tst = t$. A regular semigroup (resp. regular monoid) is an *inverse semigroup* (resp. *inverse monoid*) if the element t is unique. In this case t is said to be the inverse of s and is denoted by s^{-1}.

Example 2.2. Any group is an inverse monoid.

Example 2.3. Let X be a set. The *symmetric monoid* on X is the monoid $\mathcal{I}(X)$ of bijective functions $f : Y \to Z$ where $Y, Z \subseteq X$. We call elements of $\mathcal{I}(X)$ *partial bijections* on X. The binary operation is function composition on the largest domain where this makes sense. Thus, for $f : \mathrm{dom}(f) \to \mathrm{im}(f)$ and $g : \mathrm{dom}(g) \to \mathrm{im}(g)$ then $f \circ g$ is the partial bijection

$$g^{-1}(\mathrm{dom}(f) \cap \mathrm{im}(g)) \to \mathrm{im}(f) \cap f(\mathrm{im}(g))$$
$$x \mapsto f(g(x)).$$

With this definition one verifies that the $\mathcal{I}(X)$ is an inverse monoid where the inverse f^{-1} of f is the function $\mathrm{im}(f) \to \mathrm{dom}(f)$ given by $f(x) \mapsto x$.

The following lemma shows that a semigroup homomorphism between inverse semigroups automatically preserves inverses.

Lemma 2.4. *Let S and T be inverse semigroups and $\phi : S \to T$ a semigroup homomorphism. then $\phi(s^{-1}) = \phi(s)^{-1}$ for all $s \in S$.*

Proof. We have $\phi(s)\phi(s^{-1})\phi(s) = \phi(s)$ and $\phi(s^{-1})\phi(s)\phi(s^{-1}) = \phi(s^{-1})$. □

If S is an inverse semigroup then $E(S)$ (or just E) denotes the set $\{e \in S \mid e = e^2\}$ of all *idempotent* elements in S. One verifies that for each $s \in S$ the elements $\mathbf{d}(s) := s^{-1}s$ and $\mathbf{r}(s) := ss^{-1}$ are idempotents. The maps $\mathbf{d}, \mathbf{r} : S \to E$ are called the *domain* and *range* maps respectively. Given $s \in S$ uniqueness of the element s^{-1} implies that any idempotent is its own inverse hence any idempotent e may be written as $e = ss^{-1} = t^{-1}t$ for some $s, t \in S$ i.e. both $\mathbf{d}$ and $\mathbf{r}$ are surjective. The following result is fundamental to the theory of inverse semigroups.

Proposition 2.5. *([21, Theorem 1.1.3.]) Let S be a regular semigroup. Then S is an inverse semigroup if and only if the idempotents commute.*

Proof. First suppose that S is a regular semigroup in which idempotents commute. Let $s \in S$ and suppose that both $t, r \in S$ are inverses of S. Then $rs, ts, sr,$ and st are idempotents hence

$$r = rsr = r(sts)r = (rs)(ts)r = (ts)(rs)r = tsr, \text{ and}$$

$$t = tst = t(srs)t = t(sr)(st) = t(st)(sr) = tsr$$

so $t = r$.

We now prove the converse. To do so we show first that in an inverse semigroup the inverse of a product ef of two idempotents e and f is itself idempotent. Let h be the inverse of ef. Then $(fhe)^2 = f(hefh)e = fhe$ so fhe is idempotent. Moreover $fhe = (ef)^{-1}$ since $(fhe)ef(fhe) = (fhe)^2 = fhe$ and $(ef)fhe(ef) = (ef)h(ef) = ef$.

Now suppose S is an inverse semigroup. We claim that $ef = fe$ for any idempotents $e, f \in S$. Let $(ef)^{-1}$ be the inverse of ef. By the result from the previous paragraph $f(ef)^{-1}e$ is an idempotent inverse of ef hence $(ef)^{-1} = f(ef)^{-1}e$ by uniqueness of inverses. It follows that $(ef)^{-1}$ is idempotent, hence self inverse, hence $ef = (ef)^{-1}$ and ef is idempotent. Having deduced that $E(S)$ is closed under multiplication we have that fe is idempotent as well, hence $ef(fe)ef = ef$ and $fe(ef)fe = fe$ which implies that both ef and fe are inverses of ef. The result follows by uniqueness of inverses. □

One of the most useful properties of inverse semigroups is the existence of a partial order, called the *natural partial order*. Given $s, t \in S$ we say $s \leq t$ if there is an idempotent $e \in E(S)$ such that $s = te$. The following two results will be used frequently without mention; the proofs are elementary and may be found in [21, Lemma 1.4.6] and [21, Proposition 1.4.7] respectively.

Lemma 2.6. *Let S be an inverse semigroup. The following are equivalent:*

(i) $s \leq t$.

(ii) $s = ft$ *for some idempotent* f.

(iii) $s^{-1} \leq t^{-1}$.

(iv) $s = ss^{-1}t$.

(v) $s = ts^{-1}s$.

Definition 2.7. Let E be a partially ordered set. An *order ideal* in E is a nonempty subset X of E such that $e \in X$ implies $f \in X$ for all $f \leq e$.

Proposition 2.8. *Let S be an inverse semigroup and let $s, t, u, v \in S$ and $e, f \in E$.*

(i) *The relation $\leq$ is a partial order on S.*

(ii) $e \leq f \iff e = ef = fe$.

(iii) *If $s \leq t$ and $u \leq v$ then $su \leq tv$.*

(iv) *If $s \leq t$ then $s^{-1}s \leq t^{-1}t$ and $ss^{-1} \leq tt^{-1}$.*

(v) *$E(S)$ is an order ideal of S.*

Corollary 2.9. *The subsemigroup of idempotents E of an inverse semigroup S is a semilattice.*

Proof. The natural partial order restricted to E becomes $e \leq f \iff ef = e$. The meet operation is just multiplication in E which satisfies the three required axioms: associativity, commutativity, and idempotency. $\square$

If E has a top element 1 in the natural partial order then $e1 = 1e = e$ for all idempotents E hence $s^{-1}s1 = s^{-1}s \implies s1 = s$ and similarly $1s = s$ for all $s \in S$. It follows that S is a monoid with identity 1 and we call S an *inverse monoid*. We assume that all homomorphisms between inverse monoids map the identity to the identity. Given an idempotent $e \in E$ the collection S_e of all $s \in S$ such that $s^{-1}s = e = ss^{-1}$ forms a group with identity e. If S is an inverse monoid we give special attention to the group S_1. It is the group of *invertible elements* of S and we denote it by S^*.

Remark 2.10. Sometimes the semilattice of idempotents E of an inverse semigroup S is actually a lattice, potentially a nice one such as a boolean algebra or frame.

Depending on one's aims it is often necessary to take into account this richer order structure when constructing groupoids or C*-algebras from inverse semigroups.

Example 2.11.

- A group is an inverse semigroup with idempotent semilattice $E = \{1\}$. Conversely an inverse semigroup whose idempotent semilattice is a singleton is a group.
- A semilattice is a commutative semigroup in which every element is idempotent. All such semigroups are inverse.
- The semilattice of idempotents E of the symmetric monoid $\mathcal{I}(X)$ consists of identity functions on subsets of X and is isomorphic to the complete atomic boolean algebra $\mathcal{P}(X)$. The group of units $\mathcal{I}(X)^*$ is the symmetric group on X.

2.1.1 Partial Symmetries

Recall that one way to define an action of a group G on an object x of a category is that it is a homomorphism $G \to \mathrm{Aut}(x)$. If the category is concrete in the sense that each of its objects is a set with extra structure and its morphisms are structure preserving functions then we can identify $\mathrm{Aut}(x)$ with a subgroup of the symmetric group $\mathrm{Sym}(x)$ on the underlying set of x. We take a similar approach to defining actions of inverse semigroups; in this case however, it is not immediate what we should use for $\mathrm{Aut}(x)$ when x is an object in an arbitrary category. We skirt around this issue by only defining partial symmetries for objects of certain concrete categories as this will be sufficient for our needs. For a more general approach to partial symmetry see [5].

Let X be a set, Y a topological space, and A a C*-algebra. Recall from Example 2.11 that a partial bijection of X is a bijection between two subsets of X and that $\mathcal{I}(X)$ denotes the symmetric inverse monoid consisting of all partial bijections of X.

Definition 2.12. We define a *partial homeomorphism* of a topological space Y to be a homeomorphism between two open subsets of Y and a *partial automorphism* of a C*-algebra A to be a $*$-isomorphism between two closed two-sided ideals of A.

The inverse subsemigroups $\mathcal{I}_{\mathbf{Top}}(Y) \subseteq \mathcal{I}(Y)$ and $\mathcal{I}_{\mathbf{C^*\text{-}Alg}}(A) \subseteq \mathcal{I}(A)$ are defined to be the inverse semigroups of all partial homeomorphisms of Y and all partial automorphisms of A respectively.

Definition 2.13. Let S be an inverse semigroup.

(1) An *action* of S on a set X is a homomorphism $\theta : S \to \mathcal{I}(X)$ such that $\bigcup_{s \in S} \mathrm{dom}(\theta_s) = X$.

(2) An *action* of S on a topological space Y is a homomorphism $\theta : S \to \mathcal{I}_{\mathbf{Top}}(Y)$ such that $\bigcup_{s \in S} \mathrm{dom}(\theta_s) = Y$.

(3) An *action* of S on a C*-algebra A is a homomorphism $\theta : S \to \mathcal{I}_{\mathbf{C^*\text{-}Alg}}(A)$ such that $\bigcup_{s \in S} \mathrm{dom}(\theta_s) = A$.

Above we have stated that all semigroup homomorphisms between monoids will be assumed to be monoid homomorphisms i.e. they map the identity to the identity. Hence if S is an inverse monoid the conditions on $(1) - (3)$ are always satisfied.

Notation 2.14. Given an action $\theta : S \curvearrowright X$ we set $D^{\theta}_{s^{-1}s} := \mathrm{dom}(\theta_s)$. This notation is suggestive of the fact that $\mathrm{dom}(\theta_s)$ depends only on the domain idempotent $\mathbf{d}(s) = s^{-1}s$ of s. Given $s \in S$ and $x \in D^{\theta}_{s^{-1}s}$ we often denote $\theta_s(x)$ by $s \cdot x$.

The basic concepts concerning group actions carry over to inverse semigroups but we include the definitions below to dispel any confusion.

Definition 2.15. Let θ be an action of S on X.

(1) An element $x \in X$ is a *fixed point* of s if $s \cdot x = x$.

(2) A fixed point x of s is said to be *trivial* if there is an idempotent $e \leq s$ such that $e \cdot x$ is defined.

(3) An action is *free* if the only fixed points of each $s \in S$ are trivial.

(4) The *orbit* $\mathcal{O}_x$ of a point $x \in X$ is the set of all $y \in X$ for which there exists $s \in S$ such that $s \cdot x = y$.

(5) An action is *transitive* if $\mathcal{O}_x = X$ for all $x \in X$.

Example 2.16. Cayley's theorem states that a group acts faithfully on itself by left multiplication and thus every group is a subgroup of a symmetric group. This result may be extended to inverse semigroups and in this context it is called the Wagner-Preston theorem. Let S be an inverse semigroup and for each $s \in S$ let $D_{s^{-1}s} = \{t \in S \mid tt^{-1} \leq s^{-1}s\}$. Define $\lambda : D_{s^{-1}s} \to D_{ss^{-1}}, t \mapsto st$. We call this the Wagner-Preston action. See e.g. [21, Theorem 1.5.1] for the proof that this action is well defined and injective.

We now introduce an action on a space that is intrinsic to a given inverse semigroup.

Definition 2.17. Let E be a semilattice and let $\{0, 1\}$ be the two element boolean algebra. A *semicharacter* is a nonzero homomorphism of meet semilattices $\chi : E \to \{0, 1\}$ which is to say χ is a function $E \to \{0, 1\}$ satisfying $\chi(ef) = \chi(e)\chi(f)$ for all $e, f \in E$ and such that $\chi(e) = 1$ for at least one $e \in E$. The *spectrum* of E, denoted $\widehat{E}$, is the set of semicharacters with the topology of pointwise convergence.

The topology of pointwise convergence coincides with the relative product topology $\widehat{E} \subseteq \{0, 1\}^E$. A basis is given by the compact open sets

$$\widehat{E}(e, F) := \{\chi \mid \chi(e) = 1, \chi(f) = 0 \text{ for all } f \in F\}$$

where $\{e\} \sqcup F$ is a finite subset of E. For $e \in E$ we set $\widehat{E}(e) := \widehat{E}(e, \emptyset)$.

Proposition 2.18. *(cf. [9, Proposition 10.3.].) Let* S *be an inverse semigroup with semilattice of idempotents* E*. For* $s \in S$ *set* $D_{s^{-1}s} := \widehat{E}(s^{-1}s)$ *and* $\theta_s : D_{s^{-1}s} \to D_{ss^{-1}}, \chi \mapsto \chi(s^{-1} \cdot s)$*. Then* $\theta : s \mapsto \theta_s$ *is an action of* S *on* $\widehat{E}$ *by partial homeomorphisms.*

Proof. Let $s, t \in S$, $e, f \in E$, and $\chi \in \widehat{E}$. We need to check that $\theta_s(\chi)$ is a character, $\theta_s : D_{s^{-1}s} \to D_{ss^{-1}}$ is a homeomorphism, $s \mapsto \theta_s$ is a semigroup homomorphism, and $\widehat{E} = \bigcup_{s\in S} D_{s^{-1}s}$. First observe that $(s\cdot\chi)(ef) = \chi(s^{-1}efs) = \chi(s^{-1}ess^{-1}fs) = \chi(s^{-1}es)\chi(s^{-1}fs) = (s\cdot\chi)(e)(s\cdot\chi)(f)$ so $s \cdot \chi$ is a character. Now if $\chi \in D_{s^{-1}s}$ then $\chi(s^{-1}s) = 1$ by definition hence $(s\cdot\chi)(ss^{-1}) = \chi(s^{-1}ss^{-1}s) = \chi(s^{-1}s) = 1$. Since $\theta_{s^{-1}}$ is the inverse map for θ_s it follows that θ_s is a bijection $D_{s^{-1}s} \to D_{ss^{-1}}$. To see that θ_s is continuous and open notice that for every basic open set $\widehat{E}(e, F)$ we have $\widehat{E}(e, F) \cap \widehat{E}(s^{-1}s) = \widehat{E}(s^{-1}ses^{-1}s, F)$ and θ_s maps $\widehat{E}(s^{-1}ses^{-1}s, F)$ to $\widehat{E}(ses^{-1}, F)$. Since $D_{(st)^{-1}(st)} = \theta_{s^{-1}}(D_{t^{-1}t} \cap D_{ss^{-1}})$ and for $\chi \in D_{(st)^{-1}(st)}$ we have $\chi_{(st)^{-1}(st)}(e) = \chi(t^{-1}s^{-1}est) = \theta_s\theta_t(\chi)$, θ is multiplicative. Moreover if S is a monoid then $\theta_1 = 1$. Finally, $\bigcup_{s\in S} D_{s^{-1}s} = \bigcup_{s\in S} \widehat{E}(s^{-1}s) = \widehat{E}$. $\square$

2.2 Groupoids and Groupoid C*-algebras

2.2.1 Groupoids

Recall that a category $\mathcal{C}$ consists of a class $\mathcal{C}^{(0)}$ of objects, a class $\mathcal{C}^{(1)}$ of morphisms, class functions $\mathbf{s}, \mathbf{t} : \mathcal{C}^{(1)} \to \mathcal{C}^{(0)}$ called the *source* and *target* maps, and a binary operation called *composition*. Denote the subclass of morphisms $a \to b$ by $\mathcal{C}_a^b$. For every three objects x, y, z composition is the map $\mathcal{C}_y^z \times \mathcal{C}_x^y \to \mathcal{C}_x^z, (g, f) \mapsto gf$. For any objects and morphisms $w \xrightarrow{g} x \xrightarrow{f} y \xrightarrow{h} z$ composition is subject to the following two axioms:

- **Associativity:** $(hf)g = h(fg)$.
- **Existence of identities**: There is a morphism $\mathrm{id}_x : x \to x$ such that $\mathrm{id}_x\, g = g$ and $f\,\mathrm{id}_x = f$.

Many of the examples of categories one first encounters are large categories of mathematical objects and structure preserving maps between them, for example the categories of sets and functions, topological spaces and continuous maps, or groups and group homomorphisms. Not only are these categories intuitively large but they are usually *large* in a technical sense: at least one of $\mathcal{C}^{(0)}$ or $\mathcal{C}^{(1)}$ is a proper class. If both $\mathcal{C}^{(0)}$ and $\mathcal{C}^{(1)}$ are sets then we say that the category $\mathcal{C}$ is *small*. Small categories may be thought of as a quite general sort of algebraic object. If $\mathcal{C}$ is a small category we write $\mathcal{C}^{(2)}$ for the pullback

$$\mathcal{C}^{(2)} := \mathcal{C}^{(1)}{}_{\mathbf{s}} \times_{\mathbf{t}} \mathcal{C}^{(1)} = \{(f, g) \in \mathcal{C}^{(1)} \times \mathcal{C}^{(1)} \mid \mathbf{s}(f) = \mathbf{t}(g)\}$$

consisting of pairs of composable elements. Composition then gives a map $\mathcal{C}^{(2)} \to \mathcal{C}^{(1)}, (f, g) \mapsto fg$ which we will refer to as either multiplication or composition. For notational simplicity we will denote $\mathcal{C}^{(1)}$ by just $\mathcal{C}$ and we will view the set of objects $\mathcal{C}^{(0)}$ as a subset of $\mathcal{C}^{(1)}$ via the canonical injection $x \mapsto \mathrm{id}_x$. Elements of $\mathcal{C}^{(0)}$ will be referred to as *units*. If $x \xrightarrow{f} y$ is a morphism such that there exists a morphism $y \xrightarrow{f^{-1}} x$ such that $ff^{-1} = \mathrm{id}_y$ and $f^{-1}f = \mathrm{id}_x$ then we say that f and f^{-1} are *invertible* or that they are *isomorphisms*. If $\mathcal{C}$ is a small category we denote by $\mathcal{C}^*$ the subcategory of all invertible elements in $\mathcal{C}$ i.e. $\mathcal{C}^* := \{f \in \mathcal{C} \mid \exists f^{-1} \in \mathcal{C}\}$.

Example 2.19.

- A monoid P may be viewed as a small category with one object. Let $\mathcal{C}^{(0)} = \{*\}$ and $\mathcal{C} := P$. We identify id_* with the identity of the monoid and define composition in the category to be multiplication in P. Conversely any small category with one object is a monoid.

- A group is a monoid where every element is invertible.

Definition 2.20. A *groupoid* is a small category G such that every $\gamma \in G$ is invertible. If a groupoid G is also a topological space we say that G is a *topological groupoid* provided that the multiplication map $G^{(2)} \to G, (\alpha, \beta) \mapsto \alpha\beta$, inversion map $\alpha \mapsto \alpha^{-1}$, and source map $\mathbf{s} : G \to G^{(0)}$ are continuous.

Definition 2.21. A topological groupoid is said to be *étale* if $\mathbf{s}$ and $\mathbf{t}$ are local homeomorphisms.

Given our analytic ends it is necessary to assume at the minimum that G is locally compact and the unit space $G^{(0)}$ is Hausdorff. **Thus all étale groupoids encountered below are assumed to be locally compact and have Hausdorff unit space.** For reasons explored in [8] our main theorem requires us to assume the stronger condition that G is locally compact and that G is itself Hausdorff.

Definition 2.22. A *groupoid homomorphism* ϕ from G to H is a functor $G \to H$, i.e., it is a map $\phi : G \to H$ such that $(\phi \times \phi)(G^{(2)}) \subseteq H^{(2)}$ and $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ for all $(\alpha, \beta) \in G^{(2)}$. If G and H are topological groupoids we require any groupoid homomorphism to be continuous.

Example 2.23. Let G be a topological group with a continuous action on a LCH space X. The *action groupoid* $G \ltimes X$ has underlying space $G \times X$, source and target maps $\mathbf{s}((g, x)) = x$ and $\mathbf{t}((g, x)) = g \cdot x$, multiplication $(g, h \cdot x)(h, x) = (gh, x)$ and inversion $(g, x)^{-1} = (g^{-1}, g \cdot x)$. If G is discrete then for any $(g, x) \in G \ltimes X$ the source and target maps restrict to homeomorphisms $\{g\} \times X \xrightarrow{\cong} X$ so the transformation groupoid is étale in this case. Conversely if $U \subseteq G \times X$ is an open set such that $\mathbf{s}|_U$ is injective then U contains a basic open neighborhood $W \times V$ on which $\mathbf{s}$ is injective. It follows that W is a singleton hence G is discrete.

Definition 2.24. A subset U of a topological groupoid G is called a *bisection* if the restrictions of $\mathbf{s}$ and $\mathbf{t}$ to U are homeomorphisms onto $\mathbf{s}(U)$ and $\mathbf{t}(U)$ respectively. If U is a bisection and also an open subset of G then we say that U is an *open bisection*. We denote by $\mathrm{Bis}(G)$ the collection of open bisections.

Definition 2.25. A topological groupoid is said to be *ample* if its topology admits a basis of compact open bisections.

Remark 2.26. The presence of a basis of open bisections implies that an ample groupoid is étale.

Example 2.27.

- The subcategory $\mathcal{C}^*$ of all the invertible elements in a small category $\mathcal{C}$ is a groupoid.

- A *group bundle* $\mathcal{G}$ is a topological space X and a collection of groups $\{\Gamma_x \mid x \in X\}$. If we identify $x \in X$ with the identity of the group Γ_x and topologize $\mathcal{G} = \bigcup_{x \in X} \Gamma_x$ by pulling back the topology of X along $\mathbf{s} : \mathcal{G} \to X$ then $\mathcal{G}$ is an étale groupoid with unit space $\mathcal{G}^{(0)} = X$.

- If X is a LCH topological space and $R \subseteq X \times X$ is an equivalence relation such that the relative topology on R is locally compact, then R is an étale groupoid upon identifying X with $\Delta = \{(x, x) \mid x \in X\} \subseteq R$ and defining $\mathbf{s}((x, y)) = y$, $\mathbf{t}((x, y)) = x$, $(x, y)^{-1} = (y, x)$, and $(x, y)(y, z) = (x, z)$.

- If X is a LCH space then X may be viewed as an étale groupoid $\mathcal{G}$ where $\mathcal{G} = \mathcal{G}^{(0)} = X$.

2.2.2 Groupoid C*-algebras

We now define a C*-algebra from an étale groupoid G. To the reader wishing to learn more about Hausdorff étale groupoids and their C*-algebras we highly recommend Aidan Sims' notes [27].

Definition 2.28. Let G be a Hausdorff étale groupoid. The *convolution algebra* of G is the $*$-algebra with underlying vector space $C_c(G)$ of compactly supported complex valued functions on G, equipped with convolution product and $*$-operation

$$(f * g)(\alpha) = \sum_{\beta \in G^{t(\alpha)}} f(\beta)g(\beta^{-1}\alpha) \text{ and } f^*(\alpha) = \overline{f(\alpha^{-1})}.$$

We now want to complete the convolution algebra under a suitable norm to obtain a C*-algebra. Similar to group C*-algebras there are two canonical choices: the maximal C*-norm and the reduced norm. The former leads to what is known as the maximal, full, or universal C*-algebra of an étale groupoid and the latter leads to the reduced C*-algebra of an étale groupoid.

Definition 2.29. The universal C*-algebra $C^*(G)$ of a Hausdorff étale groupoid G is the completion of $C_c(G)$ in the norm $\|f\| := \sup\{\pi(f) \mid \pi \text{ is a } *\text{-representation of } C_c(G)\}$.

We refer the reader to [27, Proposition 3.2.1.] for the proof that this is a well defined norm so that $C^*(G)$ is a C*-algebra. There one may also find a proof that this definition agrees with the original one due to Renault [25].

Example 2.30.

- Let $X = \{1, 2, \cdots, n\}$ and let R be the equivalence relation $R = X \times X$. R is a discrete hence étale groupoid. Upon identifying $f \in C_c(R)$ with the matrix $(f(i,j))_{ij}$ convolution

$$(f * g)(i,j) = \sum_{1 \leq k \leq n} f(i,k)g(k,j)$$

  is seen to be matrix multiplication and the $*$-operation on $C_c(R)$ corresponds to the adjoint operation on $M_n(\mathbb{C})$. Hence $C_c(R) \cong M_n(\mathbb{C})$ as $*$-algebras. Moreover since the latter is a C*-algebra we have $C^*(R) = C_c(R) \cong M_n(\mathbb{C})$ by uniqueness of C*-norms.

- If G is a group then $C^*(G)$ is the group C*-algebra.

We now define the reduced C*-algebra. Similar to the group case we will do so using a regular representation.

Definition 2.31. Let G be a Hausdorff étale groupoid and let $x \in G^{(0)}$. The *regular representation* $\pi_x : C_c(G) \to \mathbb{B}(\ell^2(G_x))$ is defined by

$$\pi_x(f)\delta_\beta := \sum_{\alpha \in G_{t(\beta)}} f(\alpha)\delta_{\alpha\beta}.$$

The *reduced C*-algebra* $C_r^*(G)$ of G is defined to be the closure of $(\bigoplus_{x\in G^{(0)}} \pi_x)(C_c(G))$ inside $\bigoplus_{x \in G^{(0)}} \mathbb{B}(\ell^2(G_x))$.

We refer to the norm on $C_r^*(G)$ as the *reduced norm* and denote it by $\|\cdot\|_r$.

Example 2.32.

- Let G be a discrete group. Then G is an étale groupoid with unit space $\{\mathrm{id}_G\}$; the regular representation π_{id_G} is the left regular representation of G hence $C_r^*(G)$ is the reduced C*-algebra of G.

- If X is a LCH space then the convolution product on $C_c(X)$ is the pointwise product of continuous compactly supported functions. The regular representation π_x is the evaluation map $C_c(X) \to \mathbb{C}, f \mapsto f(x)$ hence $\|f\|_r = \sup_{x\in X} |f(x)|$ and $C_r^*(X) = C_0(X)$.

Lemma 2.33. *Let G be a Hausdorff étale groupoid and let Γ be an open bisection. There is an inclusion of vector spaces $\iota : C_c(\Gamma) \hookrightarrow C_c(G)$ which extends to an isometric map of Banach spaces $C_0(\Gamma) \to C_r^*(G)$ and identifies the Banach space $C_0(\Gamma)$ with $\overline{C_c(\Gamma)} \subseteq C_r^*(G)$.*

Proof. If $g \in C_c(\Gamma)$ then extending g by zero to all of G gives the map ι. For $x \in G^{(0)}$ we have

$$(g^* * g)(x) = \sum_{\alpha \in G^x} g^*(\alpha)g(\alpha^{-1}) = \sum_{\alpha\in G^x} |g(\alpha^{-1})|^2 = |g(\gamma)|^2$$

where γ is chosen such that $s(\gamma) = x$ and $\gamma \in \Gamma$ if $\Gamma \cap s^{-1}(x) \neq \emptyset$ and $g(\gamma) = 0$ otherwise. It follows that

$$\|g\|_r^2 = \|g^* * g\|_r = \sup_{x \in G^{(0)}} |(g^* * g)(x)| = \sup_{\gamma \in \Gamma} |g(\gamma)|^2 = \|g\|_\infty^2$$

as desired. □

Lemma 2.34. *([24, Proposition 1.9.]) Let G be a Hausdorff étale groupoid and H an open subgroupoid. Then the inclusion $C_c(H) \hookrightarrow C_c(G)$ extends to an injective $*$-homomorphism $C_r^*(H) \hookrightarrow C_r^*(G)$ which identifies $C_r^*(H)$ with $\overline{C_c(H)} \subseteq C_r^*(G)$.*

Proof. We denote by π_x^G and π_y^H the regular representations of $C_c(G)$ and $C_c(H)$ associated to $x \in G^{(0)}$ and $y \in H^{(0)}$ respectively. To prove the result it suffices to show that, for every $x \in \mathbf{s}(HG)$, $\pi_x^G|_{C_c(H)}$ is a direct sum of representations π_y^H for some $y \in H^{(0)}$. Define a relation on $R \subseteq G_x \times G_x$ by $(\alpha, \beta) \in R \iff \alpha\beta^{-1} \in H$. Let $Y := \{\alpha \in G_x \mid (\alpha, \alpha) \in R\} = G_x^{H^{(0)}}$. Then $R' := R \cap (Y \times Y)$ is an equivalence relation on Y. If E is an equivalence class of R' then $\ell^2(E)$ is a $\pi^G|_{C_c(H)}$-invariant subspace of $\ell^2(G_x)$. Let $\alpha \in E$ and set $y = \mathbf{t}(\alpha)$. The map $H_y \to E$ given by $h \mapsto h\alpha$ is bijective, hence its inverse induces a unitary map $u : \ell^2(E) \to \ell^2(H_y)$ by $u\delta_\beta = \delta_{\beta\alpha^{-1}}$. Then for all $f \in C_c(H)$ we have $u(\pi_x^G|_{C_c(H)})(f)u^* = \pi_y^H(f)$. □

In light of the above lemma we will frequently consider the reduced C*-algebra of an open subgroupoid as a C*-subalgebra of $C_r^*(G)$.

2.2.3 Inverse Semigroups and Étale Groupoids

In Example 2.23 we showed how to associate an étale groupoid $G \ltimes X$ to any action of a discrete group G on a topological space. In this section we will define the transformation groupoid $S \ltimes X$ associated to an action of an inverse semigroup S on a space X. Recall from Definition 2.13 that an element $s \in S$ acts as a homeomorphism

$D_{s^{-1}s} \to D_{ss^{-1}}$ between two open subsets of X. We call $D_{s^{-1}s}$ the domain of s and $D_{ss^{-1}}$ the range of s. In the case that $s = e$ is an idempotent then the domain and range of e agree $D_{e^{-1}e} = D_e = D_{ee^{-1}}$ and moreover e acts as the identity map on D_e. In this way the restriction of the action $S \to \mathcal{I}(X)$ to E represents E as a concrete semilattice of open subsets of X. This topological intuition, where arbitrary elements $s \in S$ are thought of as abstractly representing "functions" and idempotents are thought of as "domains" and "ranges" of elements of S, can be an extremely useful guide when working with inverse semigroups.

Let S be an inverse semigroup with semilattice of idempotents E and suppose $S \curvearrowright X$ is an action of S on a topological space X by partial homeomorphisms. If S is a group then every element of S acts on all of X and it suffices to take $S \times X$ as the underlying space for the transformation groupoid. For an arbitrary inverse semigroup we instead consider the set

$$S * X := \{(s, x) \in S \times X \mid x \in D_{s^{-1}s}\}$$

consisting of pairs (s, x) such that $s \cdot x$ is defined. If we equip $S * X$ with multiplication and inversion in the same manner as we did in the group case we find that the resulting object is not generally a groupoid, at least if we want the unit space to be identified with X, because $(s, x)(s, x)^{-1} = (s, x)(s^{-1}, s \cdot x) = (ss^{-1}, s \cdot x)$ and the projection $E \times X \to X$ is not typically bijective. Thinking of an idempotent e with $x \in D_e$ as being sort of like a neighborhood of x we define the *germ relation* as follows. For $(s, x), (t, y) \in S * X$ we say $(s, x) \sim (t, y) \iff x = y$ and there exists $e \in E$ such that $x \in D_e$ and $se = te$. This is an equivalence relation; the only nontrivial verification is transitivity. Let $(s, x) \sim (t, y)$ and $(t, y) \sim (r, z)$. Then $x = y = z$ and there are idempotents e, f with $x \in D_e \cap D_f = D_{ef}$ such that $se = te$ and $tf = rf$. It follows that $sef = tef = tfe = rfe = ref$ so $(s, x) \sim (r, x) = (r, z)$. Define $S \ltimes X$ to be the quotient $S * X/\sim$ and denote the equivalence class of (s, x) by $[s, x]$. We identify the subset $\{[e, x] \mid e \in E, x \in D_e\}$ with X via $[e, x] \mapsto x$. This map is well defined since for $(e, x), (f, x) \in S * X$ with e, f idempotent we have $(e, x) \sim (ef, x) \sim (f, x)$ and it is clearly bijective.

Remark 2.35. Let $S \curvearrowright X$ be an inverse semigroup action on a space. Recall that the natural partial order on S is defined by $s \leq t \iff s = te$ for some idempotent e. It is then immediate that $te \leq t$ for any idempotent e and element t. If $s, t \in S$ share a common lower bound $r \leq s, t$ then $r = se = tf$ for idempotents e, f so $r = r(ef) = s(ef) = t(ef)$ and we see that the germ relation $(s, x) \sim (t, x)$ is the same as requiring the existence of a lower bound $r \leq s, t$ such that $x \in D_{r^{-1}r}$.

Proposition 2.36. *([9]) Given an action $\theta : S \curvearrowright X$ of an inverse semigroup S on a set X the source and target maps*

$$\boldsymbol{s} : S \ltimes X \to X,\ [s, x] \mapsto x, \quad \boldsymbol{t} : S \ltimes X \to X,\ [s, x] \mapsto s \cdot x$$

and multiplication and inversion maps

$$[s, t \cdot x] \cdot [t, x] := [st, x], \quad [s, x]^{-1} := [s^{-1}, s \cdot x].$$

make $S \ltimes X$ a groupoid.

Proof. One easily verifies that the source and target maps are well defined using the definition of $\sim$. We now check that multiplication and inversion are well defined. Let $([s, t \cdot x], [t, x]), ([u, v \cdot x], [v, x]) \in (S \ltimes X)^{(2)}$ with $([s, t \cdot x], [t, x]) = ([u, v \cdot x], [v, x])$. Then there exist $e, f \in E$ such that $se = ue$, $tf = vf$, $t \cdot x \in D_e$, and $x \in D_f$. Moreover we can choose e such that $e \leq tt^{-1}vv^{-1}$ by replacing e with $e(tt^{-1}vv^{-1})$ if necessary.

Let $h := (t^{-1}et)(v^{-1}ev)f$. Then $x \in D_{t^{-1}et} \cap D_{v^{-1}ev} \cap D_f = D_h$ and we have

$$\begin{aligned} sth &= st(t^{-1}et)(v^{-1}ev)f = st(t^{-1}et)(t^{-1}et)(v^{-1}ev)f = set(t^{-1}et)(v^{-1}ev)f \\ &= setf(t^{-1}et)(v^{-1}ev) = uevf(t^{-1}et)(v^{-1}ev) = (uev)f(v^{-1}ev)(t^{-1}et) \\ &= (uvv^{-1}ev)f(v^{-1}ev)(t^{-1}et) = uvf(v^{-1}ev)(t^{-1}et) = uvh. \end{aligned}$$

Hence $[st, x] = [uv, x]$ as desired. To see that inversion is well defined let $x \in X$ and $s, t \in S$ such that $[s, x] = [t, x]$. Find $e \in E$ with $x \in D_e$ and $se = te$. We can choose $e \leq s^{-1}st^{-1}t$ by replacing it with $es^{-1}st^{-1}t$ if necessary. Then $se = te \implies s^{-1}(ses^{-1}) = es^{-1} = et^{-1} = t^{-1}(tet^{-1})$. Since $x \in D_e \subseteq D_{s^{-1}s} \cap D_{t^{-1}t}$ we have $s \cdot x = t \cdot x \in D_{ses^{-1}} \cap D_{tet^{-1}}$ so $[s, x]^{-1} = [t, x]^{-1}$ and this completes the proof. □

Suppose that X is a LCH space and S acts on X. For $s \in S$ and an open subset $U \subseteq D_{s^{-1}s}$ let $[s, U] = \{[s, x] \mid x \in U\}$.

Proposition 2.37. *([9]) Let $S \curvearrowright X$ be an action of an inverse semigroup S on a LCH topological space X.*

(1) The collection $B := \{[s, U] \mid s \in S, U$ is an open subset of $D_{s^{-1}s}\}$ is a basis for a topology $\mathcal{T}$ on $S \ltimes X$.

(2) $S \ltimes X$ is a locally compact étale groupoid with respect to the topology $\mathcal{T}$.

Proof. (1) Let $[r, x] \in [s, U] \cap [t, V]$ where $[s, U], [t, V] \in B$. We claim that there exists $e \in E$ and an open $W \subseteq D_{(re)^{-1}(re)}$ such that

$$[r, x] \in [re, W] \subseteq [s, U] \cap [t, V].$$

Since $[r, x] \in [s, U] \cap [t, V]$ we have $[r, x] = [s, x] = [t, x]$ and there are idempotents h, f such that $x \in D_h \cap D_f$, $rh = sh$ and $rf = tf$. Let $e = hf$ then $re = se = te$. Let $W := U \cap V \cap D_{(re)^{-1}(re)}$. Then $x \in W$ so $[r, x] = [re, x] \in [re, W]$. It remains only to show that $[re, W] \subseteq [s, U] \cap [t, V]$. Let $[re, y] \in [re, W]$. Then $y \in W \subseteq U \cap V$ so we have $[re, y] = [se, y] \in [se, U]$ and similarly $[re, y] = [te, y] \in [te, V]$.

(2) We need to show that the multiplication and inversion maps are continuous and $\mathbf{s}$ is a local homeomorphism. Let $([s, t \cdot x], [t, x]) \in (S \ltimes X)^{(2)}$ and suppose $[st, x]$ lies in a basic open set $[r, V]$. Then $x \in V$ and there exists $e \in E$ such that $ste = re$. Set $U = V \cap D_e \cap D_{t^{-1}t}$. We will show that the product of any

$$([s, t \cdot x'], [t, x']) \in ([s, D_{s^{-1}s}] \times [t, U]) \cap (S \ltimes X)^{(2)}$$

is in $[r, V]$. Now, the product is $[st, x'] = [r, x']$ and since $t \cdot x \in D_{s^{-1}s}$ and $x \in V \subseteq U$ the open set $([s, D_{s^{-1}s}] \times [t, U]) \cap (S \ltimes X)^{(2)}$ is a neighborhood of $([s, t \cdot x], [t, x])$ in $(S \ltimes X)^{(2)}$. The proves continuity of multiplication. Continuity of inversion follows from the fact that $[s, U]^{-1} = [s^{-1}, s \cdot U]$. Finally we want to show that the source map is a local homeomorphism. Let $U \subseteq X$ be open and let F be the set of all $s \in S$ such that $D_{s^{-1}s} \cap U$ is nonempty. Then $\mathbf{s}^{-1}(U) = \{[t, x] \mid t \in F, x \in U \cap D_{t^{-1}t}\} = \bigcup_{s \in F}[s, D_{s^{-1}s} \cap U]$ so $\mathbf{s}$ is continuous. Moreover $\mathbf{s}$ is clearly an open bijection on each $[s, D_{s^{-1}s}]$ so $S \ltimes X$ is an étale groupoid. Finally we show that $S \ltimes X$ is locally compact. Let $[s, x]$ be arbitrary and let U be an open bisection containing $[s, x]$. The source map restricts to a homeomorphism $U \to \mathbf{s}(U)$. Since X is locally compact and Hausdorff we can find a compact neighborhood K of x contained in $\mathbf{s}(U)$ and containing an open neighborhood V of x. Then $\mathbf{s}^{-1}(K)$ is a compact set containing the open neighborhood $\mathbf{s}^{-1}(\mathbf{s}(U))$ of $[s, x]$ as desired.

$\square$

Example 2.38. The collection $\mathrm{Bis}(G)$ of open bisections of an étale groupoid G is an inverse monoid with zero. The product of two open bisections U and V is $UV = \{\alpha\beta \mid \alpha \in U, \beta \in V, \mathbf{s}(\alpha) = \mathbf{t}(\beta)\}$, the inverse is $U^{-1} = \{\alpha^{-1} \mid \alpha \in U\}$, the identity is $G^{(0)}$, and the zero element is the empty set. Each open bisection U determines a partial homeomorphism $\Gamma_U : \mathbf{s}(U) \to \mathbf{t}(U)$ of $G^{(0)}$ by setting $\Gamma_U(x) = (\mathbf{t}|_U \circ \mathbf{s}|_U^{-1})(x)$. The map $U \mapsto \Gamma_U$ is an action of $\mathrm{Bis}(G)$ on $G^{(0)}$. We claim that the associated transformation groupoid $\mathcal{G} := \mathrm{Bis}(G) \ltimes G^{(0)}$ is isomorphic to G as a topological groupoid. Define $\phi : \mathcal{G} \to G$ by letting $\phi[U, x] = \mathbf{s}|_U^{-1}(x)$ be the unique element of U with source x, equivalently it is the unique element of U with target $\Gamma_U(x)$. If $[U, x] = [V, x]$ then there's an open neighborhood W of x in $G^{(0)}$ with $UW = VW$. But $UW \subseteq U$ and $VW \subseteq V$ so

$$\phi[U, x] = \mathbf{s}|_U^{-1}(x) = \mathbf{s}|_{UW}^{-1}(x) = \mathbf{s}|_{VW}^{-1}(x) = \mathbf{s}|_V^{-1}(x) = \phi[V, x]$$

and ϕ is well defined. The map ϕ is injective because $\phi[U, x] = \phi[V, y]$ implies, by definition, that the open bisection $U \cap V$ is nonempty, but $U \cap V$ is a lower bound for U and V so $[U, x] = [U \cap V, x] = [V, x]$. Surjectivity follows from the fact that $\mathrm{Bis}(G)$ covers G. If $U \in \mathrm{Bis}(G)$ and $X \subseteq D_{U^{-1}U} = \mathbf{s}(U)$ then $[U, X] = [UX, X] = [UX, (UX)^{-1}(UX)]$ and we see that every element of the basis

$$B = \{[U, X] \mid U \in \mathrm{Bis}(G),\ X \text{ is an open subset of } D_{U^{-1}U}\}$$

defining the topology on $\mathcal{G}$, is written as $[U, \mathbf{s}(U)]$ for some $U \in \mathrm{Bis}(G)$. Thus $[U, \mathbf{s}(U)] \mapsto \phi([U, \mathbf{s}(U)]) = U$ is bijection between the basis B of $\mathcal{G}$ and the basis $\mathrm{Bis}(G)$ of G. In particular ϕ is a homeomorphism. It remains to show that ϕ, or equivalently its inverse, is a groupoid homomorphism. Let $(\alpha, \beta) \in G^{(2)}$ and choose open bisections U, V with $\alpha \in U$ and $\beta \in V$, then UV is an open bisection containing $\alpha\beta$ and

$$\phi^{-1}(\alpha)\phi^{-1}(\beta) = [U, \mathbf{s}(\alpha)][V, \mathbf{s}(\beta)] = [UV, \mathbf{s}(\beta)] = [UV, \mathbf{s}(\alpha\beta)] = \phi^{-1}(\alpha\beta)$$

as desired.

The situation now established is that, to any action of an inverse semigroup on a space we can associate an étale groupoid, the transformation groupoid. To any étale groupoid we can associate a canonical inverse semigroup action such that, upon taking the transformation groupoid of this action, one gets back the original groupoid. This leads to several immediate questions. Does every inverse semigroup arise as $\mathrm{Bis}(G)$ for some étale groupoid? Can this construction be made functorial? Does it give rise to some sort of equivalence of categories? These questions have been addressed satisfactorily, mostly through work of Mark Lawson, Daniel Lenz, and Pedro Resende [22] [20] [26].

3 Uniqueness Theorems

3.1 Topologically Principal Groupoids and the Ideal Intersection Property

In this section we define principal and topologically principal groupoids, we give examples which demonstrate why these conditions are considered generalizations of free and topologically free group actions and we follow [8] in proving that $C_0(G^{(0)})$ detects ideals in $C^*_r(G)$ for a Hausdorff second countable topologically free étale groupoid G. This motivates the main theorem of the thesis, which appears in section 3.2.

Lemma 3.1. *([27, Lemma 2.1.13.]) Let G be a groupoid. The subset*

$$R = \{(\boldsymbol{t}(\gamma), \boldsymbol{s}(\gamma)) \mid \gamma \in G\} \subseteq G^{(0)} \times G^{(0)}$$

is an equivalence relation on $G^{(0)}$ and $\gamma \mapsto (\boldsymbol{t}(\gamma), \boldsymbol{s}(\gamma))$ is a surjective groupoid homomorphism $\phi : G \to R$.

Proof. R is reflexive because $(x, x) = (\mathbf{t}(x), \mathbf{s}(x)) \in R$ for all $x \in G^{(0)}$. R is symmetric because $(\mathbf{s}(\gamma), \mathbf{t}(\gamma)) = (\mathbf{t}(\gamma^{-1}), \mathbf{s}(\gamma^{-1}))$ for all $\gamma \in G$ and R is transitive because if $(x, y), (y, z) \in R$ then there are $\alpha, \beta \in G$ such that $\mathbf{t}(\alpha) = x, \mathbf{s}(\alpha) = y = \mathbf{t}(\beta)$, and $\mathbf{s}(\beta) = z$. Then $(x, z) = (\mathbf{t}(\alpha\beta), \mathbf{s}(\alpha\beta))$. Finally ϕ is clearly a surjective groupoid homomorphism. □

Definition 3.2. We say a groupoid is *principal* if $\phi : \gamma \mapsto (\mathbf{t}(\gamma), \mathbf{s}(\gamma))$ is injective.

It follows from the above lemma that principal groupoids are, up to algebraic isomorphism, precisely the groupoids which are equivalence relations on the unit space. Note however that, in the topological setting, the requirement that ϕ be injective does not necessarily imply that it is a homeomorphism. Following [27] we will only refer to a principal groupoid as an *equivalence relation* if it has the relative topology in $G^{(0)} \times G^{(0)}$.

Definition 3.3. Let G be a groupoid and let $x \in G$. The *isotropy group at* x is the group $G_x^x := \{\gamma \in G \mid \mathbf{s}(\gamma) = x = \mathbf{t}(\gamma)\}$. The *isotropy group bundle* or just *isotropy* of G is the subgroupoid

$$\mathrm{Iso}(G) := \bigcup_{x \in G^{(0)}} G_x^x.$$

Remark 3.4. An alternative definition of a principal groupoid is that it is a groupoid such that $\mathrm{Iso}(G) = G^{(0)}$.

Example 3.5. Let $G \ltimes X$ be the transformation groupoid of an action of a group G on a space X. The isotropy group at x is canonically isomorphic to the stabilizer subgroup G_x of x, hence $G \ltimes X$ is principal if and only if the action $G \curvearrowright X$ is free.

We now consider a weakening of the notion of a principal groupoid that has importance for groupoid C*-algebras.

Definition 3.6. A topological groupoid G is said to be *topologically principal* if the set of units with trivial isotropy is dense in the unit space:

$$\overline{\{x \in G^{(0)} \mid G_x^x = \{x\}\}} = G^{(0)}.$$

Remark 3.7. An étale groupoid is called *effective* if the interior of its isotropy bundle is equal to the unit space: $\mathrm{Iso}(G)^{\circ} = G^{(0)}$. If G is Hausdorff and has a Baire unit space then G is effective if and only if G is topologically principal.

We will follow [8] in proving the following result.

Theorem 3.8. *([8, Theorem 4.4.]) Let G be a second countable Hausdorff étale groupoid and suppose that G is topologically principal. Then any representation $\pi : C_r^*(G) \to \mathbb{B}(\mathcal{H})$ that is injective on $C_0(G^{(0)})$ is faithful.*

The conclusion of Theorem 3.8 has an equivalent formulation in terms of ideals. If $B \hookrightarrow A$ is an inclusion of C*-algebras then we say that B has the *ideal intersection property* in A if any nonzero ideal in A has a nonzero intersection with B.

Proposition 3.9. *Let $B \hookrightarrow A$ be an inclusion of C*-algebras. The following are equivalent.*

(1) $B \hookrightarrow A$ has the ideal intersection property.

(2) If $\pi : A \to \mathbb{B}(\mathcal{H})$ is a representation such that the restriction of π to B is injective then π is faithful.

Proof. Suppose first that $B \hookrightarrow A$ has the ideal intersection property and let π be a representation of A that is injective on B. Then $\mathrm{Ker}\,\pi \cap B = \{0\} \implies \mathrm{Ker}\,\pi = \{0\}$ so π is faithful. Conversely suppose that if π is a representation on A that is injective on B then π is faithful. Let J be an ideal in A and let π be a representation of A such that $J = \mathrm{Ker}\,\pi$. If π is injective on B then $J \cap B = \{0\} \implies J = \{0\}$ so $B \hookrightarrow A$ has the ideal intersection property. $\square$

Let X be a LCH space and let π be a representation of $C_0(X)$ with $\operatorname{Ker}\pi = C_0(U)$ for some open subset U of X. By the *support* of π we mean the complement $X \setminus U$ of U in X. The proofs of the following two lemmas and the proof of Theorem 3.8 following are included with little modification from [8].

Lemma 3.10. *([8, Lemma 4.2.]) Let π be a representation $C_0(X)$ and let x be in the support of π. Then $|f(x)| \leq \|\pi(f)\|$ for all $f \in C_0(X)$.*

Proof. The kernel of the evaluation character ev_x is $C_0(X \setminus \{x\})$ hence ev_x factors through $C_0(X)/C_0(U) \cong \pi(C_0(X))$. Since $*$-homomorphisms are contractive the result follows. $\square$

Lemma 3.11. *([8, Lemma 4.3.]) Let G be a Hausdorff étale groupoid and let $\pi : C^*(G) \to \mathbb{B}(\mathcal{H})$ be a representation. If $x \in G^{(0)}$ is an element of the unit space such that $G_x^x = \{x\}$ and x lies in the support of $\pi|_{C_0(G^{(0)})}$, then for every $f \in C_c(G)$ we have $|f(x)| \leq \|\pi(f)\|$.*

Proof. Let $\mathcal{V}$ be the collection of open neighborhoods of $x \in G^{(0)}$ partially ordered by reverse inclusion: $v \leq w \iff w \subseteq v$. For each $v \in \mathcal{V}$ let $v_1, v_2 \in \mathcal{V}$ be precompact open neighborhoods such that $\overline{v_2} \subseteq v_1 \subseteq \overline{v_1} \subseteq v$ and use Urysohn's lemma to produce a continuous function $g_v : G^{(0)} \to [0,1]$ such that the restriction of g_v to $\overline{v_2}$ is 1 and such that g_v is zero off of v_1. Since $\operatorname{supp} g_v \subseteq \overline{v_1}$ and the latter set is compact we have $g_v \in C_c(G^{(0)})$.

We claim that there is a $\xi_v \in \mathcal{H}$ such that $\|\xi_v\| = 1$ and $\pi(g_v)\xi_v = \xi_v$. Use Urysohn's lemma again to produce a continuous function $h_v : G^{(0)} \to [0,1]$ such that $h_v(x) \neq 0$ and h_v vanishes off of v_2. The definitions of g_v and h_v then imply that

$$g_v h_v = h_v. \tag{3.1}$$

By assumption x is contained in the support of π so Lemma 3.10 gives $0 < |h_v(x)| \leq \|\pi(h_v)\|$ so that $\pi(h_v) \neq 0$ and we may, therefore, find $\eta_v \in \mathcal{H}$ such that $\|\pi(h_v)\eta_v\| =$

1. Set $\xi_\nu = \pi(h_\nu)\eta_\nu$ then we have

$$\pi(g_\nu)\xi_\nu = \pi(g_\nu)\pi(h_\nu)\eta_\nu = \pi(g_\nu h_\nu)\eta_\nu = \pi(h_\nu)\eta_\nu = \xi_\nu$$

as desired.

Next we claim that

$$\lim_{\nu \in \mathcal{V}} \langle \pi(f)\xi_\nu, \xi_\nu \rangle = f(x) \text{ for all } f \in C_c(G). \tag{3.2}$$

Since an arbitrary $f \in C_c(G)$ can be written as a finite sum of f_i where each f_i is supported on a bisection we assume, without loss of generality, that $K := \operatorname{supp} f \subseteq U$ for some open bisection U. Denote by ϕ_U the partial homeomorphism $\mathbf{s}(U) \to \mathbf{t}(U), \mathbf{s}(\gamma) \mapsto \mathbf{t}(\gamma)$ induced by U. The proof of (3.2) will be broken into the following three cases:

(a) $x \notin \mathbf{s}(U)$,

(b) $x \in \mathbf{s}(U)$ and $\phi_U(x) \neq x$,

(c) $x \in \mathbf{s}(U)$ and $\phi_U(x) = x$.

First assume (a) then there is a $\nu_0 \in \mathcal{V}$, with $\nu_0 \cap \mathbf{s}(K) = \emptyset$. For every $\nu \subseteq \nu_0$, we have

$$(fg_\nu)(\gamma) = f(\gamma)g_\nu(\mathbf{s}(\gamma)) = 0, \text{ for all } \gamma \in G$$

since either $\gamma \notin K$ in which case $f(\gamma) = 0$ or $\mathbf{s}(\gamma) \in \mathbf{s}(K)$ in which case $g_\nu(\mathbf{s}(\gamma)) = 0$. Thus $\pi(f)\xi_\nu = \pi(fg_\nu)\xi_\nu = 0$ so the left hand side of (3.2) vanishes but so does the right side since $x \notin \mathbf{s}(K)$.

Next assume that (b) holds. Let A and B be disjoint open subsets of $G^{(0)}$ with $x \in A$ and $\phi_U(x) \in B$. Set $\nu_0 = A \cap \phi_U^{-1}(B)$ and observe that ν_0 is an open neighborhood of x such that $\nu_0 \cap \phi_U(\nu_0) = \emptyset$. For any $\nu \subseteq \nu_0$ we have

$$(g_\nu^* f g_\nu)(\gamma) = \overline{g_\nu(\mathbf{t}(\gamma))} f(\gamma) g_\nu(\mathbf{s}(\gamma)), \text{ for all } \gamma \in G.$$

If this is nonzero for some γ then $\gamma \in U$ and $\mathbf{s}(\gamma)\mathbf{t}(\gamma) \in \nu$, but this is impossible because $\mathbf{t}(\gamma) = \phi_U(\mathbf{s}(\gamma)) \in \nu \cap \phi_U(\nu) = \emptyset$. Thus $g_\nu^* f g_\nu = 0$ and we have

$$\langle \pi_\nu(f)\xi_\nu, \xi_\nu \rangle = \langle \pi(f)\pi(g_\nu)\xi_\nu, \pi(g_\nu)\xi_\nu \rangle = \langle \pi(g_\nu)^* \pi(f)\pi(g_\nu)\xi_\nu, \xi_\nu \rangle = 0$$

so the right hand side of (3.2) is zero. To see that the left hand side vanishes as well observe that if $x \in U$ then ϕ_U fixes x, which contradicts (b), thus $f(x) = 0$.

Finally assume (c). Then there exists $\gamma \in U \cap G_x^x = \{x\}$ so $\gamma = x$ and $x \in U$. Let $\epsilon > 0$ and let ν_0 be a neighborhood of x contained in $U \cap G^{(0)}$ such that $|f(y) - f(x)| \leq \epsilon$ for all $y \in \nu_0$. For every $\nu \subseteq \nu_0$ we have $(f g_\nu)(\gamma) = f(\gamma) g_\nu(\mathbf{s}(\gamma))$ for all $\gamma \in G$. If this is nonzero for some γ then $\gamma \in U$ and $\mathbf{s}(\gamma) \in \nu \subseteq U$ hence $\gamma = \mathbf{s}(\gamma) \in \nu$ and $f g_\nu$ is supported in $\nu \subseteq G^{(0)}$. For $y \in \nu$ we have

$$|(f g_\nu)(y) - f(x) g_\nu(y)| = |f(y) - f(x)||g_\nu(y)| \leq \epsilon \, \|g_\nu\| = \epsilon$$

which implies $\|f g_\nu - f(x) g_\nu\| \leq \epsilon$. Thus for all $\nu \subseteq \nu_0$

$$\begin{aligned} |\langle \pi(f)\xi_\nu, \xi_\nu \rangle - f(x)| &= |\langle \pi(f)\xi_\nu, \xi_\nu \rangle - \langle f(x)\xi_\nu, \xi_\nu \rangle| \\ &= |\langle \pi(f g_\nu)\xi_\nu, \xi_\nu \rangle - \langle f(x)\pi(g_\nu)\xi_\nu, \xi_\nu \rangle| \\ &= |\langle \pi(f g_\nu - f(x) g_\nu)\xi_\nu, \xi_\nu \rangle| \leq \|f g_\nu - f(x) g_\nu\| \, \|\xi_\nu\|^2 \leq \epsilon. \end{aligned}$$

It follows that (3.2) holds, hence $|f(x)| = \lim_{\nu \in \mathcal{V}} |\langle \pi(f)\xi_\nu, \xi_\nu \rangle| \leq \|\pi(f)\|$ as desired. $\square$

Let us now prove the uniqueness theorem, which states that if G is a topologically principal second countable Hausdorff étale groupoid, then a representation of

$C^*_r(G)$ is faithful if and only if its restriction to $C_0(X)$ is faithful. We follow [8].

Proof of Theorem 3.8. Let $\pi : C^*_r(G) \to \mathbb{B}(\mathcal{H})$ be a representation that is injective on $C_0(G_0)$. The support of π is $G^{(0)}$ hence for every $f \in C_c(G)$ and $x \in G^{(0)}$ such that $G^x_x = \{x\}$ we have

$$|f(x)| \leq \|\pi(f)\| \tag{3.3}$$

by Lemma 3.11. Since G is topologically principal the set of all such x is dense in $G^{(0)}$ and since $G^{(0)}$ is clopen in G, the restriction of f to the unit space is continuous. Thus Equation (3.3) holds for all $x \in G^{(0)}$ and it follows that $\sup_{x\in G^{(0)}} |f(x)| \leq \|\pi(f)\|$. Let $E_r : C^*_r(G) \to C_0(G^{(0)})$ be the faithful conditional expectation extending the restriction map $C_c(G) \to C_c(G^{(0)})$. Then $\|E_r(f)\| = \sup_{x\in G^{(0)}} |f(x)| \leq \|\pi(f)\|$ implies that there is a well defined conditional expectation $F : \pi(C^*_r(G)) \to \pi(C_0(G^{(0)})), \pi(a) \mapsto \pi(E(a))$. Let $a \in \operatorname{Ker}\pi$; then $0 = F(\pi(a^*a)) = \pi(E(a^*a)) \implies E(a^*a) = 0$, but since E is faithful $a = 0$ as desired. □

3.2 Relative Topological Principality

In the previous section we explored the concept of topological principality for an étale groupoid G and showed how the diagonal subalgebra $C_0(G^{(0)})$ has the ideal intersection property in the reduced C*-algebra of a topologically principal groupoid. This leads naturally to the following problem. Let G be a second countable Hausdorff étale groupoid. Is it possible to reduce faithfulness of representations to faithfulness on some (preferably proper) subalgebra $C_0(G^{(0)}) \subseteq B \subseteq C^*_r(G)$? In [3] the authors show that we may always take B to be $C^*_r(\mathrm{Iso}(G)^\circ)$, the reduced C*-algebra of the interior of the isotropy subgroupoid. In practice, there may be other open subgroupoids $H \subseteq G$, perhaps more naturally defined in terms of the particular groupoid or combinatorial data one is working with, such that the inclusion $C^*_r(H) \subseteq C^*_r(G)$ has the ideal intersection property. An example is given by

Marcelo Laca and Camila Sehnem in the context of semigroup C*-algebras in [19, Theorem 5.1.]. This motivates the following definition.

Definition 3.12. Let G be an étale groupoid and let $\mathcal{H}$ be a family of open subgroupoids. We say that G is *topologically principal relative to* $\mathcal{H}$ if the set

$$X_{\mathcal{H}} := \{u \in G^{(0)} : \exists \Gamma \in \mathrm{Bis}(G) \text{ and } H \in \mathcal{H} \text{ s.t. } u \in \Gamma^{-1}\Gamma \text{ and } \Gamma G_u^u \Gamma^{-1} \subseteq H\}$$

is dense in $G^{(0)}$.

In practice the family $\mathcal{H}$ will often be a singleton $\mathcal{H} = \{H\}$ for some open subgroupoid H. In this case we will simply say that G is topologically principal relative to H and write $X_H := X_{\{H\}}$.

Example 3.13. Let G be an étale groupoid and let $H := G^{(0)}$ be the unit space. The set X_H of Definition 3.12 coincides with the set $\{u \in G^{(0)} \mid G_u^u = \{u\}\}$ of units with trivial isotropy because if $\Gamma G_u^u \Gamma^{-1} \subseteq G^{(0)}$ for some open bisection Γ then the group $\Gamma G_u^u \Gamma^{-1}$ is trivial but $\Gamma G_u^u \Gamma^{-1}$ is isomorphic to G_u^u. Thus G is topologically principal if and only if G is topologically principal relative to the unit space.

Example 3.14. Let G be a second countable Hausdorff étale groupoid. In [3] the authors show that the set $\{u \in G^{(0)} \mid G_u^u \subseteq \mathrm{Iso}(G)^\circ\}$ is dense in $G^{(0)}$ hence G is topologically principal relative to $\mathrm{Iso}(G)^\circ$.

The main point of this section is state and prove the following theorem inspired by [3, Theorem 3.1.] and [19, Theorem 5.1.].

Theorem 3.15. *Let G be an second countable Hausdorff étale groupoid and suppose that G is topologically principal relative to a family of open subgroupoids $\mathcal{H}$. Then a representation ρ of $C_r^*(G)$ is faithful if and only if it is faithful on $C_r^*(H) \subseteq C_r^*(G)$ for all $H \in \mathcal{H}$.*

In order to do this we will study the partial inner automorphisms of an étale groupoid and how they interact with C*-algebras. Consider a group G. It is useful in group theory to let G act on itself by conjugation. The inner automorphisms of G are defined to be the automorphisms that arise from this action. Now let G be an étale groupoid. We want to make G act on itself by conjugation. By the very nature of the groupoid an element $\gamma \in G$ will have a restricted domain since groupoids model partial symmetry. Explicitly we can only conjugate $\alpha \in G$ by γ if $\mathbf{s}(\gamma) = \mathbf{t}(\alpha)$ and $\mathbf{s}(\alpha) = \mathbf{t}(\gamma^{-1})$ but then $\mathbf{s}(\alpha) = \mathbf{t}(\alpha)$, so α must be in the isotropy bundle of G. This will not be sufficient for our purposes so we will broaden our understanding of what it means for G act on itself by conjugation to allow an action of $\mathrm{Bis}(G)$ on G. If G is a group then $\mathrm{Bis}(G)$ may essentially be identified with G and we recover the standard notion of an inner automorphism of groups.[1]

Let G be an étale groupoid. By a *partial automorphism* of G we mean an isomorphism $H \to K$ between two subgroupoids of G in the category of topological groupoids. Given an open bisection $\Gamma \in \mathrm{Bis}(G)$ we set

$$D_{\Gamma^{-1}\Gamma} := \{H \text{ is a subgroupoid of } G \mid H^{(0)} \subseteq \mathbf{s}(\Gamma)\}, \text{ and}$$

$$D^{\circ}_{\Gamma^{-1}\Gamma} := \{H \in D_{\Gamma^{-1}\Gamma} \mid H \text{ is open}\}.$$

Once we verify that the map $\mathrm{Ad}_\Gamma : H \to \Gamma H \Gamma^{-1}, h \mapsto \alpha h \beta^{-1}$, where α, β are the unique elements of Γ such that $\mathbf{s}(\alpha) = \mathbf{t}(h)$ and $\mathbf{s}(\beta) = \mathbf{s}(h)$, is an isomorphism of topological groupoids we will be able to deduce that $\Gamma \mapsto \mathrm{Ad}_\Gamma$ is an action of $\mathrm{Bis}(G)$ on the set $\mathrm{Sub}(G)$ of subgroupoids of G that restricts to an action on the set $\mathrm{Sub}^{\circ}(G)$ of open subgroupoids.

Lemma 3.16. *Let G be an étale groupoid, $\Gamma \in \mathrm{Bis}(G)$, and $H \in D_{\Gamma\Gamma^{-1}}$. Then the map*

$$\mathrm{Ad}_\Gamma : H \to \Gamma H \Gamma^{-1} \textit{ given by } \alpha \mapsto \Gamma\alpha\Gamma^{-1} := (\boldsymbol{s}|_\Gamma^{-1} \circ \boldsymbol{t})(\alpha) \cdot \alpha \cdot (\boldsymbol{t}|_{\Gamma^{-1}}^{-1} \circ \boldsymbol{s})(\alpha)$$

[1]This notion of inner automorphism for étale groupoids is further justified by an equivalence of bicategories of pseudogroups and étale groupoids [20, pp.21].

is an isomorphism of topological groupoids.

Proof. Let $(\alpha, \beta) \in H^{(2)}$. Since Γ is a bisection and $H^{(0)} \subseteq \mathbf{s}(\Gamma)$ there are unique elements $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ such that

$$\mathbf{t}(\alpha) = \mathbf{s}(\gamma_1), \mathbf{s}(\alpha) = \mathbf{t}(\gamma_2^{-1}) = \mathbf{s}(\gamma_2) = \mathbf{t}(\beta), \text{ and } \mathbf{s}(\beta) = \mathbf{t}(\gamma_3^{-1}).$$

Thus $\Gamma\alpha\Gamma^{-1}\Gamma\beta\Gamma^{-1} = \gamma_1\alpha\gamma_2^{-1}\gamma_2\beta\gamma_3^{-1} = \gamma_1\alpha\beta\gamma_3^{-1} = \Gamma\alpha\beta\Gamma^{-1}$ so Ad_Γ is a homomorphism. Continuity follows from continuity of the source, range, and multiplication maps. Since $\mathrm{Ad}_{\Gamma^{-1}}$ is a continuous inverse Ad_Γ is an isomorphism of topological groupoids. □

Definition 3.17. Let G be an étale groupoid and let $\phi : H \to K$ be a partial automorphism of G. We say that ϕ is *inner* if there is an open bisection $\Gamma \in \mathrm{Bis}(G)$ such that $\phi(h) = \Gamma h \Gamma^{-1}$ for all $h \in H$.

Definition 3.18. Let S be an inverse semigroup and let A be a C*-algebra. We denote by $\mathrm{pSym}(A)$ the inverse subsemigroup of $\mathcal{I}(A)$ consisting of $*$-isomorphisms $\varphi : B \to C$ where B and C are C*-subalgebras of A. *An action of* S *on* A *by subalgebra isomorphisms* is a semigroup homomorphism $\alpha : S \to \mathrm{pSym}(A)$.

Remark 3.19. We have chosen to call such an action an action by subalgebra isomorphisms to distinguish it from the more common notion of partial automorphisms given in Definition 2.12. Thus a partial automorphism of C*-algebras is a subalgebra isomorphism where the domain and codomain are 2-sided ideals.

Definition 3.20. Let B and C be C*-subalgebras of a C*-algebra A. We say a subalgebra isomorphism $\phi : B \to C$ of A is *inner* if there is a partial isometry ν in the universal enveloping von Neumann algebra A^{**} of A such that $\phi(b) = \nu b \nu^*$ and $\phi^{-1}(c) = \nu^* c \nu$ for all $b \in B$ and $c \in C$.

It is a standard theme that inner automorphisms are somehow more trivial than outer ones. For us the most important consequence of a subalgebra isomorphism being inner is the following.

Lemma 3.21. *Let* B *and* C *be C*-subalgebras of a C*-algebra* A. *Suppose* $\rho : A \to \mathbb{B}(\mathcal{H})$ *is a representation and* $\phi : B \to C$ *is an inner subalgebra isomorphism of* A. *Then* ρ *is faithful on* B *if and only if* ρ *is faithful on* C.

Proof. Suppose ρ is faithful on B and let $\tilde{\rho}$ denote the unique weak-* continuous extension to A^{**}. Let $c \in C$ be arbitrary and let $b \in B$ satisfy $c = vcv^* = \phi(b)$, where $v \in A^{**}$ is the partial isometry implementing ϕ. Note that $\|b\| = \|c\|$ because ϕ is an isomorphism. Since ρ is a $*$-homomorphism $\|\rho(c)\| \leq \|c\|$. As for the opposite inequality we have

$$\|c\| = \|b\| = \|\rho(b)\| = \|\rho(v^*vbv^*v)\| = \|\tilde{\rho}(v^*)\rho(vbv^*)\tilde{\rho}(v)\| \leq \|\rho(vbv^*)\| = \|\rho(c)\|$$

thus ρ is faithful on C. The result follows by symmetry. □

The next lemma shows that the reduced C*-algebra construction induces an action of $\mathrm{Bis}(G)$ on $C^*_r(G)$ by subalgebra isomorphisms. Note that the following lemma and its proof are basically [13, Lemma 3.7.] with some input from [13, Lemma 3.8.]. We have made two modifications: we do not require $G^{(0)}$ to be compact since we do not use their notion of a G-C*-algebra and also we are interested in the reduced groupoid C*-algebra $C^*_r(G)$ instead of the universal one. We should also mention that the set of partial isometries in a Von Neumann algebra does not typically form an inverse semigroup with the induced multiplication from the Von Neumann algebra as claimed in [13, Lemma 3.7.]. This does not not effect the results as one only needs the image of the map to be an inverse semigroup.

Lemma 3.22. *(cf. [13, Lemma 3.7, Lemma 3.8]) Let* G *be a Hausdorff étale groupoid let* S *denote the collection of partial isometries in the enveloping von Neumann algebra* $C^*_r(G)^{**}$.

Then there is a multiplicative and involutive map $\mathrm{Bis}(G) \to S, \Gamma \mapsto \nu_\Gamma$ *such that for all* $\Gamma \in \mathrm{Bis}(G)$ *and open subgroupoids* $H \in D^\circ_{\Gamma^{-1}\Gamma}$,

(i) The net $(f)_{0 \le f \le \mathbb{1}_\Gamma} \subseteq C_c(\Gamma)$ *converges to* ν_Γ *in the weak-* topology as* f *increases to* $\mathbb{1}_\Gamma$.

(ii) $\nu_\Gamma^* \nu_\Gamma = \mathbb{1}_{\Gamma^{-1}\Gamma}$ *and* $\nu_\Gamma \nu_\Gamma^* = \mathbb{1}_{\Gamma\Gamma^{-1}}$.

(iii) If $g \in C_c(V)$ *for some* $V \in \mathrm{Bis}(H)$ *then* $\nu_\Gamma g = g(\Gamma^{-1}\cdot) \in C_c(\Gamma V)$*, and* $g\nu_\Gamma^* = g(\cdot\Gamma) \in C_c(V\Gamma^{-1})$.

(iv) The isomorphism $\phi : C^*_r(H) \to C^*_r(\Gamma H \Gamma^{-1})$ *induced by the groupoid isomorphism* $H \to \Gamma H \Gamma^{-1}$ *is implemented by conjugation:* $\mathrm{Ad}_{\nu_\Gamma} : a \mapsto \nu_\Gamma a \nu_\Gamma^*$ *and is therefore an inner subalgebra isomorphism.*

Proof. Let $\Gamma \in \mathrm{Bis}(G)$. By Lemma 2.33 there is an isometric isomorphism of Banach spaces $C_0(\Gamma) \xrightarrow{\cong} \overline{C_c(\Gamma)} \subseteq C^*_r(G)$ and hence we have isometric embeddings $B^\infty(\Gamma) \hookrightarrow C_0(\Gamma)^{**} \hookrightarrow C^*_r(G)^{**}$ where $B^\infty(\Gamma)$ is the algebra of bounded Borel functions on Γ and the embedding $B^\infty(\Gamma) \hookrightarrow C_0(\Gamma)^{**}$ is given by sending $f \in B^\infty(\Gamma)$ to the linear functional $C_0(\Gamma)^* \to \mathbb{C}, \mu \mapsto \int_\Gamma f d\mu$. Since the increasing net $(f)_{0 \le f \le \mathbb{1}_\Gamma}$ in $C_c(\Gamma)$ converges pointwise to $\mathbb{1}_\Gamma \in B^\infty(\Gamma)$ the monotone convergence theorem implies $(\int_\Gamma f d\mu)_{0 \le f \le \mathbb{1}_\Gamma} \to \int_\Gamma \mathbb{1}_\Gamma d\mu$ which is to say $(f)_{0 \le f \le \mathbb{1}_\Gamma} \to \mathbb{1}_\Gamma$ in the weak-* topology on $B^\infty(\Gamma)$ and hence in $C^*_r(G)^{**}$ since $C_0(\Gamma)^{**}$ is weak-* closed in the latter. Set $\nu_\Gamma := \mathbb{1}_\Gamma$ and let $g \in C_c(V)$ for an open bisection V of H. Then for all $0 \le f \le \mathbb{1}_\Gamma$ we have

$$(f * g)(x) = \begin{cases} f(y)g(y^{-1}x) & \text{when } y \in \Gamma, y^{-1}x \in V \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi_g : B^\infty(\Gamma) \to B^\infty(\Gamma V), f \mapsto f * g$. Then by definition of the convolution product ψ_g is continuous with respect to the topology of pointwise convergence on

$B^\infty(\Gamma)$ and $B^\infty(\Gamma V)$ hence

$$(v_\Gamma g)(x) = \lim_{0 \leq f \leq \mathbb{1}_\Gamma} (f * g)(x) = g(\Gamma^{-1}x)$$

and similarly

$$(g v_{\Gamma^{-1}})(x) = \lim_{0 \leq f \leq \mathbb{1}_{\Gamma^{-1}}} (g * f)(x) = g(x\Gamma).$$

To see that $\mathrm{Bis}(\Gamma) \to S, \Gamma \mapsto v_\Gamma$ is a semigroup homomorphism observe that

$$v_{\Gamma_1} v_{\Gamma_2} = \lim_{0 \leq f \leq \mathbb{1}_{\Gamma_2}} v_{\Gamma_1} f = \lim_{0 \leq f \leq \mathbb{1}_{\Gamma_2}} f(\Gamma_1^{-1} \cdot) = \lim_{0 \leq f \leq \mathbb{1}_{\Gamma_1 \Gamma_2}} f = v_{\Gamma_1 \Gamma_2},$$

and similarly $v_\Gamma^* = v_{\Gamma^{-1}}$. Moreover

$$v_\Gamma v_\Gamma^* = \lim_{0 \leq f \leq \mathbb{1}_{\Gamma^{-1}}} v_\Gamma f = \lim_{0 \leq f \leq \mathbb{1}_{\Gamma^{-1}}} f(\Gamma^{-1} \cdot) = \mathbb{1}_{\Gamma\Gamma^{-1}}$$

and similarly $v_\Gamma^* v_\Gamma = \mathbb{1}_{\Gamma^{-1}\Gamma}$.

Finally we prove (iv). By (iii) we have $v_\Gamma g v_\Gamma^* = g(\Gamma^{-1} \cdot \Gamma) \in C_c(\Gamma H \Gamma^{-1})$ for any $g \in C_c(H)$. By hypothesis we have $\Gamma^{-1}\Gamma H \Gamma^{-1}\Gamma = H$ and thus $\mathrm{Ad}_{v_\Gamma^*} : C_c(\Gamma H \Gamma^{-1}) \to C_c(H)$ is the inverse of Ad_{v_Γ}. Since $\|a\| = \|v_\Gamma^* v_\Gamma a v_\Gamma^* v_\Gamma\| \leq \|v_\Gamma a v_\Gamma^*\| \leq \|a\|$ the result follows. $\square$

The following lemmas are generalizations of results from [3] to allow for a collection of subalgebras and subgroupoids.

Lemma 3.23 (cf. [3] Theorem 3.2.)**.** *Let $\mathcal{B}$ be a collection of C*-subalgebras of a C*-algebra A. For each $B \in \mathcal{B}$ suppose that S_B is a collection of states on B such that every state $\varphi \in S_B$ has a unique extension $\tilde{\varphi}$ to a state on A. Let $S := \bigcup_{B \in \mathcal{B}} S_B$ and suppose that the sum of GNS representations $\bigoplus_{\varphi \in S} \pi_{\tilde{\varphi}}$ of the extensions of the states in S is faithful on A. Then a representation $\rho : A \to \mathbb{B}(\mathcal{H})$ is faithful if and only if it is faithful on B for all $B \in \mathcal{B}$.*

Proof. Suppose that the restriction of $\rho : A \to \mathbb{B}(\mathcal{H})$ to B is faithful for all $B \in \mathcal{B}$.

Set $J := \operatorname{Ker}\rho$, let $B \in \mathcal{B}$ be arbitrary, and observe that $J \cap B = \emptyset$. The subspace $A_0 := J + B$ is a subalgebra of A by e.g. [7, Corollary 1.8.4.]. Let $\gamma : A_0 \to B$ be the quotient map and let $\varphi \in S_B$ be arbitrary. Every state on A_0 extends to a state on A hence φ extends uniquely to a state $\tilde{\varphi}|_{A_0}$ on A_0. Since $\varphi \circ \gamma$ is one such extension it must be the unique one and we have $\tilde{\varphi}(a) = \varphi(\gamma(a))$ for all $a \in A_0$. Let $x \in J$. Then $a^*x^*xa \in J$ for all $a \in A$ hence $\langle \pi_{\tilde{\varphi}}(x)h, \pi_{\tilde{\varphi}}(x)h\rangle = 0$ for all $h \in \mathcal{H}_{\tilde{\varphi}}$ therefore $\tilde{\varphi}(x) = 0$. Since $\bigoplus_{\varphi \in S_B} \pi_{\tilde{\varphi}}$ is faithful and $B \in \mathcal{B}$ and $\varphi \in S_B$ are arbitrary the element x is zero. $\square$

Lemma 3.24 (cf. [3] Lemma 3.3.). *Let G be a second countable Hausdorff étale groupoid that is topologically principal relative to a family $\mathcal{H}$ of open subgroupoids. Suppose that $u \in X_{\mathcal{H}}$ and $\Gamma \in \operatorname{Bis}(G)$ satisfy $u \in \Gamma^{-1}\Gamma$ and $\Gamma G_u^u \Gamma^{-1} \subseteq H$ for some $H \in \mathcal{H}$. Then for all $f \in C_c(G)$ there exists $b \in C_c((\Gamma^{-1}H\Gamma)^{(0)})^+$ such that $\|b\| = b(u) = 1$ and $bfb \in C_c(\Gamma^{-1}H\Gamma)$.*

Proof. Let $f = \sum_{D \in F} f_D$ where F is a finite collection of precompact open bisections and $\operatorname{supp}(f_D) \subseteq D$. For $D \in F$ choose open neighborhoods V_D as follows.

- If there exists $\alpha \in D \cap G_u^u$ then $D \cap \Gamma^{-1}H\Gamma$ is an open neighborhood of α hence $V_D := \mathbf{s}(D \cap \Gamma^{-1}H\Gamma) \cap \mathbf{t}(D \cap \Gamma^{-1}H\Gamma)$ is an open neighborhood of u and $V_D D V_D \subseteq D \cap \Gamma^{-1}H\Gamma$.

- If there is an $\alpha \in D$ with $r(\alpha) = u \neq s(\alpha)$ or $r(\alpha) \neq u = s(\alpha)$ then take an open neighborhood of $\alpha \in U \subseteq D$ such that $s(U) \cap r(U) = \emptyset$ and set $V_D := r(U)$ so that $V_D D V_D = \emptyset$.

- If $u \notin r(D) \cup s(D)$ then take an open neighborhood V_D of u such that $f_D|_{V_D D V_D} = 0$.

Let $V := \cap_{D \in F} V_D$. Then V is open and contains u. Choose $b \in C_c(V)^+$ such that $b(u) = \|b\| = 1$. By construction $\operatorname{supp}(bfb) \subseteq \Gamma^{-1}H\Gamma$ as desired. $\square$

The following lemma appears in [4]. Note that it does not require the second countability and Hausdorff assumptions but that is the generality we use it in.

Lemma 3.25. *([4, Lemma 1.2.]) Let G be a second countable Haudorff étale groupoid and let $u \in G^{(0)}$. The restriction map $C_c(G) \to C_c(G_u^u)$ extends to a completely positive contraction $\vartheta_u : C_r^*(G) \to C_r^*(G_u^u)$.*

Lemma 3.26 (cf. [3] Lemma 3.5.)**.** *Let G be a second countable Hausdorff étale groupoid which is topologically principal relative to a family $\mathcal{H}$ of open subgroupoids. Suppose that $u \in X_{\mathcal{H}}$ and $\Gamma \in \mathrm{Bis}(G)$ such that $u \in \Gamma^{-1}\Gamma$ and $\Gamma G_u^u \Gamma^{-1} \subseteq H$ for some $H \in \mathcal{H}$. Let $\epsilon > 0$ and $a \in C_r^*(G)$. Then there exists $c \in C_r^*(\Gamma^{-1}H\Gamma)$ and $b \in C_c(G^{(0)})^+$ such that $\|b\| = 1$ and $\varphi(b) = 1$ for all states φ that factor through $C_r^*(G_u^u)$ and such that $\|bab - c\| < \epsilon$.*

Proof. Let $\epsilon > 0$ and $a \in C_r^*(G)$. Let $f \in C_c(G)$ such that $\|a - f\| < \epsilon$. Apply Lemma 3.24 to find $b \in C_c(G^{(0)})^+$ such that $\|b\| = b(u) = 1$ and $bfb \in C_c(\Gamma^{-1}H\Gamma)$. Let $\vartheta_u : C_r^*(\Gamma^{-1}H\Gamma) \to C_r^*(G_u^u)$ be the completely positive contraction induced by the restriction map $C_c(\Gamma^{-1}H\Gamma) \to C_c(G_u^u), f \mapsto f|_{G_u^u}$. Then $\vartheta_u(b) = \delta_u$ is the identity of $C_r^*(G_u^u)$ hence if φ is a state on $C_r^*(\Gamma^{-1}H\Gamma)$ such that $\varphi = \psi \circ \vartheta_u$ for some state ψ on $C_r^*(G_u^u)$ we have $\varphi(b) = \psi(\delta_u) = 1$. Moreover if we set $c = bfb$ we have

$$\|bab - c\| = \|bab - bfb\| < \epsilon$$

as desired. ☐

Before we prove our main theorem we need one final lemma.

Lemma 3.27. *Let G be a second countable LCH étale groupoid and let $x \in G^{(0)}$. Let $\phi : C_r^*(G) \to C_0(G^{(0)})$ be the canonical faithful conditional expectation, let $\vartheta_x : C_r^*(G) \to C_r^*(G_x^x)$ be the completely positive contraction extending the restriction map $C_c(G) \to C_c(G_x^x)$, let $\tau : C_r^*(G_x^x) \to \mathbb{C}$ be the canonical faithful trace $\tau(a) = \langle \delta_x, a\delta_x \rangle$ and let*

$\mathrm{ev}_x : C_0(G^{(0)}) \to \mathbb{C}$ *be the evaluation by* x *map. Then the following diagram commutes.*

$$\begin{tikzcd} C_r^*(G) \arrow[r, "\Phi"] \arrow[d, "\vartheta_x"'] & C_0(G^{(0)}) \arrow[d, "\mathrm{ev}_x"] \\ C_r^*(G_x^x) \arrow[r, "\tau"'] & \mathbb{C} \end{tikzcd}$$

Proof. If $f \in C_c(G)$ then $\tau(\vartheta_x(f)) = \langle \delta_x, f|_{G_x^x}\delta_x \rangle = f(x) = \mathrm{ev}_x \circ \Phi$. The result follows by continuity. □

In the following proof we make use of [2, Theorem 5.4.]. We state it here for convenience.

Theorem 3.28. *([2, Theorem 5.4.]) Let* A *be a C*-algebra and let* B *be a C*-subalgebra of* A*. Suppose that* ϕ *is a state on* B *such that for each* $a \in A$ *and* $\epsilon > 0$*, there exists* $b, c \in B$ *satisfying*

- $|\phi(b)| = \|b\| = 1$,
- $b \geq 0$*, and*
- $\|bab - c\| < \epsilon$.

Then ϕ *extends uniquely to* A*. If* ϕ *is a pure state then so is its extension.*

Proof of Theorem 3.15. For each $u \in X_{\mathcal{H}}$, $H \in \mathcal{H}$, and $\Gamma \in \mathrm{Bis}(G)$ such that $u \in \Gamma^{-1}\Gamma$ and $\Gamma G_u^u \Gamma^{-1} \subseteq H$ let $S_{u,\Gamma,H}$ denote the collection of states on $C_r^*(\Gamma^{-1}H\Gamma)$ that factor through $C_r^*(G_u^u)$. Let $\rho : C_r^*(G) \to \mathbb{B}(\mathcal{H})$ be a representation that is faithful on the family of C*-subalgebras $\{C_r^*(H) \mid H \in \mathcal{H}\}$. Then ρ is faithful on $C_r^*(K) \subseteq C_r^*(H)$ for any open subgroupoid K of H. Let $K := \mathbf{t}(\Gamma)H\mathbf{t}(\Gamma)$. Then $K \subseteq H$ is an open subgroupoid, $K \in D^{\circ}_{\Gamma\Gamma^{-1}}$, and moreover $\Gamma^{-1}K\Gamma = \Gamma^{-1}H\Gamma$. By Lemma 3.22 $\mathrm{Ad}_{v_\Gamma^*}$: $C_r^*(K) \to C_r^*(\Gamma^{-1}H\Gamma)$ is an inner subalgebra isomorphism. Hence by Lemma 3.21 faithfulness of ρ on $C_r^*(H)$ implies faithfulness on $C_r^*(\Gamma^{-1}H\Gamma)$. By Lemma 3.23 the theorem follows from a proof of the following statements:

(1) For each $u \in X_{\mathcal{H}}$, $H \in \mathcal{H}$ and Γ with $H \in D_{\Gamma\Gamma^{-1}}$ such that $G_u^u \subseteq \Gamma^{-1}H\Gamma$, every state $\varphi \in S_{u,\Gamma,H}$ has a unique state extension to $C_r^*(G)$.

(2) The direct sum $\pi := \bigoplus \pi_{\tilde{\varphi}}$ of GNS representations of all the extended states is faithful on $C_r^*(G)$.

Statement (1) follows from Lemma 3.26 and Theorem 3.28. Let $\pi_{S_{u,\Gamma,H}} := \bigoplus_{\varphi \in S_{u,\Gamma,H}} \pi_{\tilde{\varphi}}$ be the direct sum of the GNS representations of the extended states $\tilde{\varphi}$. We now prove (2). Let $\Phi_G : C_r^*(G) \to C_0(G^{(0)})$ be the canonical faithful conditional expectation. We show $\pi_{S_{u,\Gamma,H}}(a) = 0 \implies \Phi_G(a^*a) = 0$ in which case $a = 0$ by faithfulness of Φ_G. Suppose for contradiction $\Phi_G(a^*a) \neq 0$. Then $\Phi_G(a^*a)$ is a positive function on $G^{(0)}$ so by density of $X_{\mathcal{H}}$ in $G^{(0)}$ there is a $u \in X_{\mathcal{H}}$ such that $\Phi_G(a^*a)(u) > 0$. Let $\epsilon = \frac{\Phi_G(a^*a)(u)}{2}$. By Lemma 3.26 we can find $b \in C_c(G^{(0)})^+$ and $c \in C_r^*(\Gamma^{-1}H\Gamma)$ such that $\|ba^*ab - c\| < \epsilon/2$ and $\varphi(b) = 1$ for all states that factor through $C_r^*(G_u^u)$. By assumption $\pi_{S_{u,\Gamma,H}}(a) = 0$ hence $\|\pi_{S_{u,\Gamma,H}}(ba^*ab)\| \leq \|\pi_{S_{u,\Gamma,H}}(a^*a)\| = 0$ hence if $\varphi \in S_{u,\Gamma,H}$ and $\{e_\lambda\}$ is an approximate unit in $C_r^*(G)$ we have

$$\tilde{\varphi}(ba^*ab) = \lim_\lambda \langle \pi_{\tilde{\varphi}}(ba^*ab)(e_\lambda + N_{\tilde{\varphi}}), (e_\lambda + N_{\tilde{\varphi}}) \rangle = 0$$

and therefore

$$|\tilde{\varphi}(c)| = |\tilde{\varphi}(ba^*ab - c)| \leq \|ba^*ab - c\| < \epsilon/2.$$

Let $\vartheta_u : C_r^*(G) \to C_r^*(G_u^u)$ be the completely positive contraction extending the restriction map. Then we have

$$\|\vartheta_u(c)\| = \sup\{|\psi(\vartheta_u(c))| \mid \psi \text{ is a state on } C_r^*(G_u^u)\} \leq \sup\{|\varphi(c)| \mid \varphi \in S_{u,\Gamma,H}\} \leq \epsilon/2.$$

Now let $\Phi_{\Gamma,H} : C_r^*(\Gamma^{-1}H\Gamma) \to C_0((\Gamma^{-1}H\Gamma)^0)$ be the conditional expectation given by restricting functions to the unit space. For $x \in (\Gamma^{-1}H\Gamma)^0$ let ev_x be the evaluation by x character on $C_0((\Gamma^{-1}H\Gamma)^0)$. Then $\mathrm{ev}_x \circ \Phi_{\Gamma,H}$ is a state on $C_r^*(\Gamma^{-1}H\Gamma)$; moreover for each $x \in X_{\Gamma^{-1}H\Gamma} = \{x \in (\Gamma^{-1}H\Gamma)^0 \mid G_x^x \subseteq \Gamma^{-1}H\Gamma\}$ the state $\mathrm{ev}_x \circ \Phi_{\Gamma,H}$ factors through $C_r^*(G_x^x)$ by Lemma 3.27. Let $\tau : C_r^*(G_u^u) \to \mathbb{C}$ be the canonical tracial state. Since $\Phi_G|_{\Gamma^{-1}H\Gamma} = \Phi_{\Gamma,H}$ we have

$$|\mathrm{ev}_u \circ \Phi_G(c)| = |\mathrm{ev}_u \circ \Phi_{\Gamma,H}(c)| = |\tau(\vartheta_u(c))| \leq \|\vartheta_u(c)\| \leq \epsilon/2.$$

Since Φ_G is $C_0(G^{(0)})$-bilinear and ev_u is a character we have

$$\mathrm{ev}_u \circ \Phi_G(ba^*ab) = \mathrm{ev}_u(b)(\mathrm{ev}_u \circ \Phi_G)(a^*a)\,\mathrm{ev}_u(b) = (\mathrm{ev}_u \circ \Phi_G)(a^*a).$$

Thus

$$\begin{aligned}
|\mathrm{ev}_u \circ \Phi_G(a^*a)| &\leq |\mathrm{ev}_u \circ \Phi_G(a^*a) - \mathrm{ev}_u \circ \Phi_G(c) + \mathrm{ev}_u \circ \Phi_G(c)| \\
&\leq |\mathrm{ev}_u \circ \Phi_G(ba^*ab) - \mathrm{ev}_u \circ \Phi_G(c)| + |\mathrm{ev}_u \circ \Phi_G(c)| \\
&\leq \|ba^*ab - c\| + |\mathrm{ev}_u \circ \Phi_G(c)| \\
&< \epsilon.
\end{aligned}$$

But this contradicts our choice of ϵ hence $\Phi_G(a^*a) = 0$ and therefore $a = 0$. It follows that π is injective. □

4 Applications

4.1 Uniqueness for Reduced C*-algebras of Left Cancellative Small Categories

C*-algebras from left cancellative small categories (LCSCs) were defined by Jack Spielberg in [29] generalizing his earlier work on C*-algebras of categories of paths [28]. The latter include C*-algebras of k-graphs and quasi-lattice ordered groups. A left cancellative monoid is just a LCSC with one object; the aim of this section is to prove a generalization of the uniqueness theorem for group embeddable monoids of Laca and Sehnem [19, Theorem 5.1.] in the setting of LCSCs. The inverse semigroup approach to C*-algebras of LCSCs as described in [23] allows us to make easy comparison with semigroup C*-algebras so we will follow that approach here.

4.1.1 Basic Definitions

Definition 4.1. A *left cancellative small category* is a small category $\mathcal{C}$ such that $xy = xz \implies y = z$ for all $x, y, z \in \mathcal{C}$.

Definition 4.2. ([23, Definition 2.3.]) The left inverse hull $\mathcal{I}_l := \mathcal{I}_l(\mathcal{C})$ of a LCSC $\mathcal{C}$ is the smallest inverse subsemigroup of $\mathcal{I}(\mathcal{C})$ such that the left multiplication maps $\mathbf{s}(c)\mathcal{C} \xrightarrow{\cong} c\mathcal{C}, x \mapsto cx$ are in $\mathcal{I}_l$ for all $c \in \mathcal{C}$.

Remark 4.3. Recall from Example 2.11 that the semilattice of idempotents of the symmetric monoid on a set X is canonically isomorphic to the power set $\mathcal{P}(X)$. We will frequently use this isomorphism to identify idempotents in $\mathcal{I}_l(\mathcal{C})$ with subsets of $\mathcal{C}$.

Definition 4.4. Let $\mathcal{C}$ be a LCSC and let E be the semilattice of idempotents of $\mathcal{I}_l$. The *boolean closure* $\overline{E}$ of E in $\mathcal{I}(\mathcal{C})$ is the set of subsets of $\mathcal{C}$ of the form $e \setminus \bigcup_{i=1}^{n} f_i$ where $f_1, \cdots, f_n, e \in E$ and $f_1, \cdots, f_n \leq e$.

The set

$$\overline{\mathcal{I}}_l := \{se \mid s \in \mathcal{I}_l, e \in \overline{E}, e \leq s^{-1}s\}$$

is easily seen to be an inverse semigroup with semilattice of idempotents $\overline{E}$.

Remark 4.5. The zero element in the symmetric monoid $\mathcal{I}(\mathcal{C})$ is the partial bijection whose domain and range are the empty set; it is also the zero of the semilattice of idempotents $E(\mathcal{I}(\mathcal{C})) \cong \mathcal{P}(\mathcal{C})$. We denote this element by 0. Whether $0 \in \mathcal{I}_l$ depends on whether the empty set may be written as an intersection $\bigcap_{s \in F} \text{dom}(s)$ for a finite set of elements $F \subseteq \mathcal{I}_l$.

We abuse notation and identify an element $c \in \mathcal{C}$ with the corresponding left multiplication map $c \in \mathcal{I}_l(\mathcal{C})$. We denote its inverse in $\mathcal{I}_l$ by c^{-1}. The latter map is defined by $c\mathcal{C} \to \mathbf{s}(c)\mathcal{C}, cx \mapsto x$ and in the case where $c \in \mathcal{C}^*$ it coincides with left multiplication by $c^{-1} \in \mathcal{C}$. An arbitrary element $s \in \mathcal{I}_l$ may be written in the form $d_n^{-1}c_n \cdots d_1^{-1}c_1$ where each $d_i, c_i \in \mathcal{C}$.

A representation $T : \mathcal{I}_l \to \mathbb{B}(\ell^2(\mathcal{C}))$ is determined by mapping $c \in \mathcal{C}$ to the partial isometry $T_c : \ell^2(\mathcal{C}) \to \ell^2(\mathcal{C})$ where

$$T_c\delta_x = \begin{cases} \delta_{cx} & \text{if } \mathbf{s}(c) = \mathbf{t}(x) \\ 0 & \text{otherwise,} \end{cases}$$

and extending to all of $\mathcal{I}_l$ by setting $T_{d_n^{-1}c_n \cdots d_1^{-1}c_1} := T^*_{d_n}T_{c_n} \cdots T^*_{d_1}T_{c_1}$.

Definition 4.6. ([23, Definition 2.2]) The left reduced C*-algebra of $\mathcal{C}$ is

$$C^*_r(\mathcal{C}) := C^*(\{T_c \mid c \in \mathcal{C}\}) = \overline{\operatorname{span}}\{T_s \mid s \in \mathcal{I}_l\} \subseteq \mathbb{B}(\ell^2(\mathcal{C})).$$

An important commutative C*-subalgebra of $C^*_r(\mathcal{C})$ is the diagonal.

Definition 4.7. The diagonal subalgebra of $C^*_r(\mathcal{C})$ is

$$D_r(\mathcal{C}) := \overline{\operatorname{span}}\{T_e \mid e \in E(\mathcal{I}_l)\}.$$

Lemma 4.8. *The spectrum of the diagonal is homeomorphic to the subspace*

$$\Omega := \{\chi \in \widehat{E} \mid \chi(f) = \vee_{i=1}^n \chi(e_i), \textit{ whenever } f = \vee_{i=1}^n e_i \in \mathcal{I}_l\}$$

of semicharacters which preserve any finite suprema which exist in $\mathcal{I}_l$.

Proof. The map $T : E \to D_r(\mathcal{C}), e \mapsto T_e$ is a faithful representation of the semigroup E. We denote by $\tilde{\chi}$ the restriction of $\chi \in \operatorname{Spec}(D_r(\mathcal{C}))$ to the semigroup $E \cong \{T_e \mid e \in E\}$.

Then every such semicharacter preserves joins since $e, f \in E$ with $e \vee f \in E$ implies $T_{e \vee f} = T_e + T_f - T_{ef}$. Conversely every $\chi \in \Omega$ preserves the multiplicative and linear relations among the T_e hence extends to a character on $D_r(\mathcal{C}) = \overline{span}\{T_e \mid e \in E\}$. This gives a bijection between $\mathrm{Spec}(D_r(\mathcal{C}))$ and Ω. To show that it is indeed a homeomorphism we note that $D_r(\mathcal{C})$ admits a surjection $C^*(E) \to D_r(\mathcal{C})$ from the universal C*-algebra generated by a representatation of the inverse semigroup E. Explicitly $C^*(E)$ is generated by projections $\{t_e \mid e \in E\}$ subject to the relations $t_e t_f = t_{ef}$ and $t_{e^*} = t_e^*$. This realizes $\mathrm{Spec}(D_r(\mathcal{C}))$ as a subspace of $\mathrm{Spec}(C^*(E))$. On the otherhand the universal property of $C^*(E)$ gives a 1-1 correspondence between semicharacters $E \to \{0, 1\}$ and characters $C^*(E) \to \mathbb{C}$. Since the weak-* topology on $\mathrm{Spec}(C^*(E))$ is the topology of pointwise convergence on $\widehat{E}$ commutativity of the following diagram proves the result.

$$\begin{tikzcd}
\mathrm{Spec}(C^*(E)) \arrow[r, "\chi \mapsto \tilde{\chi}"] & \widehat{E} \\
\mathrm{Spec}(D_r(\mathcal{C})) \arrow[u, hook] \arrow[r, "\chi \mapsto \tilde{\chi}"] & \Omega \arrow[u, hook]
\end{tikzcd}$$

$\square$

Remark 4.9. For an alternate proof of Lemma 4.8 one may use [6, Corollary 5.6.26.].

For each $c \in \mathcal{C}$ let $\chi_c : E \to \{0, 1\}$ be the semicharacter defined by $\chi_c(e) = 1$ if $cc^{-1}\mathcal{C} \leq e$ and $\chi_c(e) = 0$ otherwise.

Lemma 4.10. *([23, Lemma 2.12.]) The semicharacters* $\{\chi_c \mid c \in \mathcal{C}\}$ *are dense in* Ω.

Proof. One easily verifies that each $\chi_c \in \Omega$. Recall that a basic open subset of the spectrum $\widehat{E}$ of the semilattice $E = E(\mathcal{I}_l(\mathcal{C}))$ is of the form $\widehat{E}(e, F) := \{\chi \mid \chi(e) = 1, \chi(f) = 0 \text{ for all } f \in F\}$. Density follows immediately from the observation that if $\widehat{E}(e, F) \neq \emptyset$ then $e \setminus \cup F \neq \emptyset$ so $\chi_c \in \widehat{E}(e, F)$ where $c \in e \setminus \cup F$. $\square$

Recall from Proposition 2.18 that for any inverse semigroup S there is a canonical action $S \curvearrowright \widehat{E}$ on the spectrum of its semilattice of idempotents. The action is defined by letting the domain of s be $D_{s^{-1}s} = \{\chi \mid \chi(s^{-1}s) = 1\}$ and for χ in the domain of s, $s \cdot \chi = \chi(s^{-1} \cdot s)$.

Lemma 4.11. *Ω is an $\mathcal{I}_l$-invariant subset of $\widehat{E}$.*

Proof. Let $\chi \in \Omega$ and suppose $\chi \in D_{s^{-1}s}$ for some $s \in \mathcal{I}_l$. We claim that $s \cdot \chi \in \Omega$. Suppose f is an idempotent which can be written as a join $f = \bigvee_{i=1}^{n} e_i$ of a finite set of idempotents. Then

$$(s \cdot \chi)(\bigvee_{i=1}^{n} e_i) = \chi(s^{-1} \bigvee_{i=1}^{n} e_i s) = \chi(\bigvee_{i=1}^{n} s^{-1} e_i s) = \bigvee_{i=1}^{n} \chi(s^{-1} e_i s) = \bigvee_{i=1}^{n} (s \cdot \chi)(e_i)$$

so $\chi \in \Omega$ as desired. □

It follows that the action of $\mathcal{I}_l$ on $\widehat{E}$ restricts to an action of $\mathcal{I}_l$ on Ω. Now for every $\chi \in \Omega$ we may extend χ to an element of $\widehat{\widehat{E}}$ by setting $\overline{\chi}(e \setminus \vee_{i=1}^{n} e_i) = \chi(e) \setminus \bigvee_{i=1}^{n} \chi(e_i)$. This gives an embedding $\Omega \hookrightarrow \widehat{\widehat{E}}$ and the canonical action of $\overline{\mathcal{I}_l}$ on $\widehat{\widehat{E}}$ restricts to an action on Ω.

Definition 4.12. The transformation groupoid of a LCSC is the transformation groupoid $\overline{\mathcal{I}}_l \ltimes \Omega$.

The following result from [29] allows us to deduce that our Theorem 4.19 is indeed a uniqueness theorem for the reduced C*-algebra of a LCSC.

Theorem 4.13. *[29, Theorem 11.6.] Let $\mathcal{C}$ be a LCSC and suppose that $\overline{\mathcal{I}}_l \ltimes \Omega$ is Hausdorff. Then $C_r^*(\overline{\mathcal{I}}_l \ltimes \Omega) \cong C_r^*(\mathcal{C})$.*

4.1.2 The Uniqueness Theorem

Lemma 4.14. *Let $e \in \mathcal{C}^{(0)}$. Then $[u, \chi_e] \in \mathcal{G}_{\chi_e}^{\chi_e}$ for all $u \in \mathcal{C}^*$ such that $u^{-1}u = e = uu^{-1}$.*

Proof. Let $f \in E(\mathcal{I}_l)$. Then

$$\chi_e(f) = \begin{cases} 1 & \text{if } e \leq f \\ 0 & \text{otherwise,} \end{cases}$$

so we need to show that $e \leq f \iff e \leq u^{-1}fu$. If $e \leq f$ i.e. $ef = e$ then $uu^{-1}f = uu^{-1} \implies u^{-1}uu^{-1}fu = u^{-1}uu^{-1}u = u^{-1}u = e$ that is $e(u^{-1}fu) = e$. Conversely if $e(u^{-1}fu) = e$ then $(u^{-1}u)(u^{-1}fu) = uu^{-1}$ so conjugating by u we get $(uu^{-1})(uu^{-1})f(uu^{-1}) = uu^{-1}$ which is equivalent to $ef = e$ since $e = uu^{-1} = u^{-1}u$ and idempotents commute. □

Lemma 4.15. *If* $s \in \mathcal{I}_l$, $p, q \in \mathcal{C}$, *and* $p \in \mathrm{dom}(s)$ *then* $s(pq) = s(p)q$.

Proof. If $s = d^{-1}c$ for $d, c \in \mathcal{C}$ then $cp = dr$ for some $r \in \mathcal{C}$ hence $cpq = drq$ and it follows that $s(pq) = rq = s(p)q$. We proceed by induction on the word length of $s = d_k^{-1}c_k \cdots d_1^{-1}c_1$. The desired result holds for $k = 1$; assume it holds for all $k \leq \ell$. Let $p, pq \in \mathrm{dom}(s)$ where $s = d_{\ell+1}^{-1}c_{\ell+1} \cdots d_1^{-1}c_1$. Then $d_1^{-1}c_1(p)$ and $d_1^{-1}c_1(pq) = d_1^{-1}c_1(p)q$ are in $\mathrm{dom}(d_{\ell+1}^{-1}c_{\ell+1} \cdots d_2^{-1}c_2)$. By induction we have $d_{\ell+1}^{-1}c_{\ell+1} \cdots d_1^{-1}c_1(pq) = d_{\ell+1}^{-1}c_{\ell+1} \cdots d_2^{-1}c_2(d_1^{-1}c_1(p)q) = d_{\ell+1}^{-1}c_{\ell+1} \cdots d_1^{-1}c_1(p)q$. □

Lemma 4.16. *Let* $s \in \mathcal{I}_l$ *such that* $s^{-1}s = e = ss^{-1}$ *where* $e \in \mathcal{C}^0$. *Then* $s \in \mathcal{C}^*$.

Proof. Let $p \in e\mathcal{C}e$. Then $s(p) = s(ep) = s(e)p$ so s is left multiplication by $s(e)$ and similarly s^{-1} is left multiplication by $s^{-1}(e)$ hence $s^{-1}(e) = s(e)^{-1}$ and $s \in \mathcal{C}^*$. □

Lemma 4.17. *Let* $\mathcal{C}$ *be a left cancellative small category, let* $\mathcal{G} := \mathcal{I}_l \ltimes \Omega$, *and suppose* $c \in \mathcal{C}$. *Then* $\mathcal{G}_{\chi_c}^{\chi_c} = \{[cuc^{-1}, \chi_c] \mid u \in \boldsymbol{s}(c)\mathcal{C}^*\boldsymbol{s}(c)\}$.

Proof. Let $e = \mathbf{s}(c)$. Then

$$\mathcal{G}_{\chi_e}^{\chi_e} = \{[u, \chi_e] \mid u \in \mathcal{I}_l, u^{-1}u = e = uu^{-1}\} = \{[u, \chi_e] \mid u \in \mathcal{C}^*, u^{-1}u = e = uu^{-1}\},$$

Where the first equality is from Lemma 4.14 and the second equality is from Lemma 4.16. We then have $\mathcal{G}_{\chi_c}^{\chi_c} = [c, \chi_e]\mathcal{G}_{\chi_e}^{\chi_e}[c, \chi_e]^{-1} = \{[cuc^{-1}, \chi_c] \mid u \in \mathbf{s}(c)\mathcal{C}^*\mathbf{s}(c)\}$. $\square$

Proposition 4.18. *Let $\mathcal{C}$ be a left cancellative small category such that $\mathcal{G} := \overline{\mathcal{I}_l} \ltimes \Omega$ is second countable and Hausdorff. Then $\mathcal{G}$ is topologically principal relative to $\mathcal{C}^* \ltimes \Omega$.*

Proof. Given $c \in \mathcal{C}$ the subset $[c, \Omega(c^{-1}c)]$ of $\mathcal{G}$ is an open bisection with $\mathbf{s}([c, \Omega(c^{-1}c)]) = \Omega(c^{-1}c)$ and $\mathbf{t}([c, \Omega(c^{-1}c)]) = \Omega(cc^{-1})$. Let $\Gamma := [c, \Omega(c^{-1}c)]^{-1}$. Then $\Gamma\mathcal{G}_{\chi_c}^{\chi_c}\Gamma^{-1} = \mathcal{G}_{\chi_{\mathbf{s}(c)}}^{\chi_{\mathbf{s}(c)}} \subseteq \mathcal{C}^* \ltimes \Omega$ hence $X := \{\chi_c \mid c \in \mathcal{C}\}$ is set of units such that the isotropy at each element of X may be conjugated into $\mathcal{C}^* \ltimes \Omega$ by an open bisection. Since X is dense by Lemma 4.10, $\mathcal{G}$ is topologically principal relative to $\mathcal{C}^* \ltimes \Omega$. $\square$

An application of Theorem 3.15 yields the following.

Theorem 4.19. *A representation $\pi : C_r^*(\overline{\mathcal{I}_l} \ltimes \Omega) \to \mathbb{B}(\mathcal{H})$ is faithful if and only if the restriction of π to the C*-subalgebra $C_r^*(\mathcal{C}^* \ltimes \Omega)$ is faithful.* $\square$

Corollary 4.20. *If the canonical map $C^*(\overline{\mathcal{I}_l} \ltimes \Omega) \to C_r^*(\overline{\mathcal{I}_l} \ltimes \Omega)$ is an isomorphism, e.g. if $\overline{\mathcal{I}_l} \ltimes \Omega$ is amenable, then any representation of $C^*(\overline{\mathcal{I}_l} \ltimes \Omega)$ that is faithful on the C*-subalgebra $C^*(\mathcal{C}^* \ltimes \Omega)$ is faithful.* $\square$

Remark 4.21. If the left cancellative small category is a monoid, i.e. it has one object, and the monoid embeds in a group, then the groupoid $\overline{\mathcal{I}_l} \ltimes \Omega$ is isomorphic to a partial transformation groupoid and is therefore Hausdorff. In this case Theorem 4.19 recovers [19, Theorem 5.1.]. On the other hand there are left cancellative monoids which do not embed in a group for which the groupoid $\overline{\mathcal{I}_l} \ltimes \Omega$ is Hausdorff so our result extends this theorem non trivially.

5 Future and Ongoing Work

In [2] Becky Armstrong extends the uniqueness theorem [3, Theorem 3.1.] for Hausdorff étale groupoids to twisted groupoid C*-algebras. It is ongoing work of the author to extend her theorem to more general Fell bundles. More precisely, for a C*-algebra A and an étale groupoid G if $\mathcal{E}$ is an action of $\mathrm{Bis}(G)$ on A by Hilbert bimodules then there is an action $\mathcal{E}_{\mathrm{Iso}}$ of $\mathrm{Bis}(\mathrm{Iso}(G)^{\circ})$ on A by Hilbert bimodules. When G is Hausdorff we expect that the reduced cross section C*-algebra of $\mathcal{E}_{\mathrm{Iso}}$ detects ideals in the reduced cross section C*-algebra of $\mathcal{E}$. More generally we expect to be able to detect ideals in the latter algebra using subalgebras associated to a family of subgroupoids $\mathcal{H}$ such that G is relatively topologically principal to $\mathcal{H}$. In this setting topological freeness conditions of the groupoid should have a formulation in terms of modules as in [15]. In [13] Kennedy, Kim, Li, Raum, and Ursu make extensive use of a Furstenburg boundary and its identification with the spectrum of the G-equivariant injective hull of $C(X)$ to prove a uniqueness theorem for the inclusion $C_0(X) \hookrightarrow C^*_{\mathrm{ess}}(G)$ where G is not assumed to be Hausdorff. In our setting of groupoid or inverse semigroup Fell bundles the Hamana boundary should be replaced with the G-equivariant injective hull of the unit fiber A, see [12] for the group case. Then one relates ideal detection properties of subalgebras in the essential section algebra with ideal detection properties of corresponding subalgebras in the boundary section algebra.

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