

A THEOREM ON WIDDER'S POTENTIAL  
TRANSFORM AND ITS APPLICATIONS

by

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In the present paper the authors prove a Parseval–Goldstein type theorem involving the classical Laplace transform, the Fourier sine transform, and Widder's potential transform. The theorem is then shown to yield a simple algorithm for evaluating infinite integrals. Some illustrative examples are also given.

## 1. INTRODUCTION, DEFINITIONS, AND THE MAIN RESULT

Over two decades ago, Widder [10] presented a systematic account of the integral transform:

$$F_p(x) = \mathcal{P}\{f(t); x\} = \int_0^{\infty} \frac{t f(t)}{x^2 + t^2} dt, \quad (1)$$

which, by an exponential change of variable, becomes a convolution transform with a kernel belonging to a general class treated by Hirschman and Widder [4]. In fact, as Widder pointed out, the kernel of the integral transform (1) is the same as that appearing in the familiar Poisson integral representation of a function which is harmonic in a half-plane. With this interesting connection in view, Widder [10] gave several inversion formulas for the potential transform (1) and applied his results to harmonic functions. Subsequently, Srivastava and Singh [9] gave the following Parseval–Goldstein type theorem:

$$\int_0^{\infty} x F_p(x) g(x) dx = \int_0^{\infty} x G_p(x) f(x) dx, \quad (2)$$

where  $F_p(x)$  and  $G_p(x)$  are the potential transforms of the functions  $f(t)$  and

$g(t)$ , respectively. There are numerous analogous results in the literature on integral transforms. (See, for instance, Goldstein [3], Srivastava and Panda [8], and Yürekli [12].)

The objective of this paper is first to establish a Parseval–Goldstein type theorem involving the potential transform (1), the classical Laplace transform:

$$F_L(x) = \mathcal{L}\{f(t); x\} = \int_0^{\infty} e^{-xt} f(t) dt, \quad (3)$$

and the Fourier sine transform:

$$F_S(x) = \mathcal{F}_S\{f(t); x\} = \int_0^{\infty} \sin(xt) f(t) dt. \quad (4)$$

We also show how the theorem leads to a simple algorithm for evaluating infinite integrals.

The following result will be required in our investigation.

**LEMMA** (*cf.* Widder [11]). *The identities*

$$\mathcal{L}\left\{\mathcal{F}_S\{f(t); u\}; x\right\} = \mathcal{P}\{f(t); x\} \quad (5)$$

and

$$\mathcal{L}\{f(t); x\} = \frac{2}{\pi} \mathcal{P}\left\{\mathcal{F}_S\{f(t); u\}; x\right\} \quad (6)$$

*hold true, provided that the integrals involved converge absolutely.*

We now state our main result contained in the following

**THEOREM.** Let  $F_L(x)$  be the classical Laplace transform of  $f(t)$ ,  $G_P(x)$  be the Widder potential transform of  $g(t)$ , and  $G_S(x)$  be the Fourier sine transform of  $g(t)$ .

Then

$$\int_0^{\infty} F_L(x) G_S(x) dx = \int_0^{\infty} f(x) G_P(x) dx \quad (7)$$

provided that the integrals involved converge absolutely.

**Proof.** From the definition (3) of the Laplace transform, we obtain

$$\int_0^{\infty} F_L(x) G_S(x) dx = \int_0^{\infty} G_S(x) \left[ \int_0^{\infty} e^{-xt} f(t) dt \right] dx. \quad (8)$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, we find from (8) that

$$\begin{aligned} \int_0^{\infty} F_L(x) G_S(x) dx &= \int_0^{\infty} G_S(x) \left[ \int_0^{\infty} e^{-xt} f(t) dt \right] dx \\ &= \int_0^{\infty} f(t) \mathcal{L}\{G_S(x); t\} dt. \end{aligned} \quad (9)$$

Now the assertion (7) follows from (5) and (9). This evidently completes the proof of the theorem under the hypothesis stated already.

## 2. A USEFUL COROLLARY

An interesting consequence of the theorem may be stated as the following

**COROLLARY.** *Let  $G_L(x)$  denote the Laplace transform of  $g(t)$ . Suppose also that  $\text{Re}(\mu) > 0$ .*

*Then*

$$\int_0^{\infty} \frac{g(x)}{x^{\mu}} dx = \frac{1}{\Gamma(\mu)} \int_0^{\infty} x^{\mu-1} G_L(x) dx, \quad (10)$$

*provided that each of the integrals exists.*

**Proof.** In the theorem we set

$$f(t) = t^{\mu-1}, \quad (11)$$

so that

$$F_L(x) = \mathcal{L}\{t^{\mu-1}; x\} = \Gamma(\mu) x^{-\mu} \quad (\text{Re}(\mu) > 0). \quad (12)$$

Making use of the inversion formula [2, p. 63, Equation 2.1(1)] for the Fourier sine transform (4), we obtain

$$\Gamma(\mu) \int_0^{\infty} \frac{g(x)}{x^{\mu}} dx = \frac{2}{\pi} \int_0^{\infty} t^{\mu-1} \mathcal{P} \left\{ \mathcal{F}_s \{g(x); u\}; t \right\} dt. \quad (13)$$

Now the result (10) follows from (13) and (6), and the proof of the corollary is thus completed.

**REMARK.** For  $\mu = 1$ , the assertion (10) reduces at once to a known property of the classical Laplace transform. (See also Example 3 below.)

### 3. SOME ILLUSTRATIVE EXAMPLES

A simple illustration of the corollary is provided by

**EXAMPLE 1.** We shall show that (cf. [2, p. 310, Equation 6.2(19)])

$$\int_0^{\infty} \frac{x^{\mu-1}}{(x+a)^{\nu}} dx = \frac{\Gamma(\mu)\Gamma(\nu-\mu)}{\Gamma(\nu)} a^{\mu-\nu}, \quad (14)$$

where, for convergence,  $\operatorname{Re}(\nu) > \operatorname{Re}(\mu) > 0$ .

In the corollary we set

$$g(x) = x^{\nu-1} e^{-ax}, \quad (15)$$

so that

$$G_L(x) = \Gamma(\nu)(x+a)^{-\nu} \quad (\operatorname{Re}(\nu) > 0). \quad (16)$$

Upon substituting these values into (10), we obtain

$$\int_0^{\infty} x^{\nu-\mu-1} e^{-ax} dx = \frac{\Gamma(\nu)}{\Gamma(\mu)} \int_0^{\infty} \frac{x^{\mu-1}}{(x+a)^{\nu}} dx. \quad (17)$$

The integral on the left-hand side of (17) is the Laplace transform of  $x^{\nu-\mu-1}$ .

Thus the result (14) follows immediately from (17).

In the following example we show how Weber's integral formula (19) below can be deduced by making use of the theorem and Example 1. For an alternative method of evaluation of this integral, see Andrews [1, p. 210, Example 2]; see also Erdélyi *et al.* [2, p. 326, Equation 6.8(1)].

**EXAMPLE 2.** For the Bessel function  $J_\nu(z)$  defined by

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}z)^{\nu+2n}}{n! \Gamma(\nu+n+1)} \quad (|z| < \infty), \quad (18)$$

we derive *Weber's integral formula*:

$$\int_0^{\infty} x^{2m-p-1} J_p(ax) dx = \frac{2^{2m-p-1} \Gamma(m)}{\Gamma(p-m+1)} a^{p-2m}, \quad (19)$$

where, for convergence,  $a > 0$  and  $0 < \text{Re}(2m) < \text{Re}(p) + \frac{3}{2}$ .

We begin by setting

$$f(x) = x^{2p-2m}, \quad (20)$$

so that

$$F_L(x) = \Gamma(2p-2m+1) x^{2m-2p-1} \quad (\text{Re}(p-m) > -\frac{1}{2}; \text{Re}(x) > 0). \quad (21)$$

Furthermore, in terms of the Heaviside function  $H(t)$ , we let

$$g(x) = \frac{2^{p+1} a^p}{\sqrt{\pi} \Gamma(\frac{1}{2}-p)} H(x-a) (x^2-a^2)^{-p-\frac{1}{2}}, \quad (22)$$

so that (*cf.* [2, p. 100, Equation 2.12(8); p. 182, Equation 4.14(7)])

$$G_S(x) = x^p J_p(ax) \quad (-1 < \operatorname{Re}(p) < \tfrac{1}{2}; a > 0), \quad (23)$$

and

$$G_P(x) = \frac{2^p a^p}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}+p) (x^2+a^2)^{-p-\frac{1}{2}} \quad (\operatorname{Re}(p) > -\tfrac{1}{2}; \operatorname{Re}(x) > |\operatorname{Im}(a)|), \quad (24)$$

where we have employed the identity (5). Substituting these values into (7), we find from the theorem that

$$\int_0^\infty x^{2m-p-1} J_p(ax) dx = \frac{(2a)^p \Gamma(p+\frac{1}{2})}{\sqrt{\pi} \Gamma(2p-2m+1)} \int_0^\infty x^{2p-2m} (x^2+a^2)^{-p-\frac{1}{2}} dx. \quad (25)$$

Setting  $x^2 = t$  in the integral on the right-hand side of (25), and using Example 1 with  $\mu = p - m + \frac{1}{2}$  and  $\nu = p + \frac{1}{2}$ , we obtain

$$\begin{aligned} \int_0^\infty x^{2m-p-1} J_p(ax) dx &= \frac{2^{p-1} a^p \Gamma(p+\frac{1}{2})}{\sqrt{\pi} \Gamma(2p-2m+1)} \int_0^\infty t^{p-m-\frac{1}{2}} (t+a^2)^{-p-\frac{1}{2}} dt \\ &= \frac{2^{p-1} a^{p-2m} \Gamma(p-m+\frac{1}{2}) \Gamma(m)}{\sqrt{\pi} \Gamma(2p-2m+1)}, \end{aligned} \quad (26)$$

provided that the integral on the left-hand side of (26) exists, that is, that

$$a > 0 \quad \text{and} \quad 0 < \operatorname{Re}(2m) < \operatorname{Re}(p) + \frac{3}{2},$$

as already stated with Weber's integral formula (19).

Finally, since

$$\Gamma(2p-2m+1) = \frac{2^{2p-2m}}{\sqrt{\pi}} \Gamma(p-m+\frac{1}{2}) \Gamma(p-m+1), \quad (27)$$

by *Legendre's duplication formula* for the  $\Gamma$ -function (*cf.* [1, p. 58, Equation (2.24)]), the last member of (26) leads immediately to the right-hand side of (19).

**EXAMPLE 3.** If, in the assertion (7) of our theorem, we set

$$g(x) = \frac{2}{\pi} x^{-\lambda} \sin(\tfrac{1}{2}\lambda\pi), \quad (28)$$

so that (*cf.* [10, p. 360])

$$G_P(x) = x^{-\lambda} \quad (0 < \operatorname{Re}(\lambda) < 2), \quad (29)$$

and [2, p. 68, Equation 2.3(1)]

$$G_S(x) = \frac{x^{\lambda-1}}{\Gamma(\lambda)} \quad (0 < \operatorname{Re}(\lambda) < 2), \quad (30)$$

we shall immediately obtain [*cf.* Equation (10)]

$$\int_0^\infty x^{-\lambda} f(x) dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} F_L(x) dx, \quad (31)$$

provided that each of the integrals exists.

Thus we have given a simpler derivation of the assertion (10) of the above corollary. Furthermore, we may apply (31) by setting

$$f(x) = x^{\rho-1} \prod_{j=1}^n \left\{ J_{\mu_j} (z_j x) \right\}, \quad (32)$$

so that (*cf.* [6, p. 2, Equation (2.2)])

$$F_L(x) = \frac{(\frac{1}{2}z_1)^{\mu_1} \cdots (\frac{1}{2}z_n)^{\mu_n} \Gamma(M)}{x^M \Gamma(\mu_1+1) \cdots \Gamma(\mu_n+1)} \cdot F_C^{(n)} \left[ \frac{1}{2}M, \frac{1}{2}M+\frac{1}{2}; \mu_1+1, \dots, \mu_n+1; -\frac{z_1^2}{x^2}, \dots, -\frac{z_n^2}{x^2} \right], \quad (33)$$

where, for convenience,

$$M = \rho + \mu_1 + \cdots + \mu_n, \quad (34)$$

$F_C^{(n)}$  denotes the third type of Lauricella's hypergeometric functions of  $n$  variables, defined by (*cf.* [5]; see also [7, p. 33, Equation 1.4(3)])

$$F_C^{(n)} \left[ \alpha, \beta; \gamma_1, \dots, \gamma_n; z_1, \dots, z_n \right] = \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_n} (\beta)_{m_1+\dots+m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!} \quad (35)$$

$$(\sqrt{|z_1|} + \cdots + \sqrt{|z_n|}) < 1; \quad (\lambda)_m = \Gamma(\lambda+m)/\Gamma(\lambda),$$

and the Laplace transform (33) exists (by the principle of analytic continuation) when

$$\operatorname{Re}(M) > 0 \quad \text{and} \quad \operatorname{Re}(x) > \sum_{j=1}^n |\operatorname{Im}(z_j)|. \quad (36)$$

In terms of the first type of Laurcella's hypergeometric functions of  $n$  variables, defined by (*cf.* [5]; see also [7, p. 33, Equation 1.4(1)])

$$\begin{aligned} & F_A^{(n)} \left[ \alpha, \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; z_1, \dots, z_n \right] \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_n} (\beta_1)_{m_1} \dots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \dots (\gamma_n)_{m_n}} \frac{z_1^{m_1}}{m_1!} \dots \frac{z_n^{m_n}}{m_n!} \quad (37) \\ & \quad (|z_1| + \dots + |z_n| < 1), \end{aligned}$$

the choice (32) leads us also to the Laplace transform (*cf.* [2, p. 184, Equation 4.14(24)])

$$\begin{aligned} F_L(x) &= \frac{(\frac{1}{2}z_1)^{\mu_1} \dots (\frac{1}{2}z_n)^{\mu_n} \Gamma(M)}{(x+i\zeta)^M \Gamma(\mu_1+1) \dots \Gamma(\mu_n+1)} \\ & \cdot F_A^{(n)} \left[ M, \mu_1+\frac{1}{2}, \dots, \mu_n+\frac{1}{2}; 2\mu_1+1, \dots, 2\mu_n+1; \frac{2iz_1}{x+i\zeta}, \dots, \frac{2iz_n}{x+i\zeta} \right], \quad (38) \end{aligned}$$

which holds true under the constraints listed already in (36),  $\zeta$  being given by

$$\zeta = z_1 + \dots + z_n, \quad (39)$$

and  $M$  by (34). However, since (*cf.*, *e.g.*, [7, p. 331, Equation 9.4(225)])

$$\begin{aligned} F_G^{(n)} \left[ \frac{1}{2}\alpha, \frac{1}{2}\alpha + \frac{1}{2}; \gamma_1, \dots, \gamma_n; z_1^2, \dots, z_n^2 \right] \\ = (1+\zeta)^{-\alpha} F_A^{(n)} \left[ \alpha, \gamma_1 - \frac{1}{2}, \dots, \gamma_n - \frac{1}{2}; 2\gamma_1 - 1, \dots, 2\gamma_n - 1; \frac{2z_1}{1+\zeta}, \dots, \frac{2z_n}{1+\zeta} \right], \end{aligned} \quad (40)$$

where  $\zeta$  is given by (39), the Laplace transform formulas (33) and (38) are essentially the same.

Substituting for  $f(x)$  and  $F_L(x)$  in (31) from (32) and (33), respectively, and evaluating the resulting integral on the left-hand side by means of (*cf.* [6, p. 4, Equation (2.8)]); see also [7, p. 50, Equation 1.7(12)])

$$\begin{aligned} \int_0^\infty t^{\rho-1} \prod_{j=1}^n \left\{ J_{\mu_j} (z_j x) \right\} dx \\ = \frac{2^{\rho-1} z_1^{\mu_1} \dots z_{n-1}^{\mu_{n-1}} z_n^{\mu_n - M} \Gamma(\frac{1}{2}M)}{\Gamma(\mu_1+1) \dots \Gamma(\mu_{n-1}+1) \Gamma(\mu_n - \frac{1}{2}M+1)} \\ \cdot F_G^{(n-1)} \left[ \frac{1}{2}M, \frac{1}{2}M - \mu_n; \mu_1+1, \dots, \mu_{n-1}+1; \frac{z_1^2}{z_n^2}, \dots, \frac{z_{n-1}^2}{z_n^2} \right] \end{aligned} \quad (41)$$

$$(z_n > z_1 + \dots + z_{n-1}; \operatorname{Re}(1+\mu_1+\dots+\mu_n) > \operatorname{Re}(1-\rho) > -\frac{1}{2}n),$$

we finally have

$$\begin{aligned}
& \int_0^\infty x^{\lambda-1} F_C^{(n)} \left[ \begin{matrix} \frac{1}{2}\alpha, \frac{1}{2}\alpha + \frac{1}{2}; \gamma_1, \dots, \gamma_n; -\frac{z_1^2}{x^2}, \dots, -\frac{z_n^2}{x^2} \end{matrix} \right] dx \\
&= \frac{z_n^\lambda \Gamma(\lambda+\alpha) \Gamma(-\frac{1}{2}\lambda) \Gamma(\gamma_n)}{2^{\lambda+1} \Gamma(\frac{1}{2}\lambda+\gamma_n)} \\
&\quad \cdot F_C^{(n-1)} \left[ \begin{matrix} -\frac{1}{2}\lambda, 1-\frac{1}{2}\lambda-\gamma_n; \gamma_1, \dots, \gamma_{n-1}; \frac{z_1^2}{z_n^2}, \dots, \frac{z_{n-1}^2}{z_n^2} \end{matrix} \right], \tag{42}
\end{aligned}$$

provided that the integral exists, and

$$|z_n| > |z_1| + \dots + |z_{n-1}|.$$

In its special case when  $n = 1$ , this last result (42) would yield an infinite integral which is also derivable from a well-known Mellin transform formula [2, p. 336, Equation 6.9(3)] involving the Gaussian hypergeometric function.

We conclude by remarking that many other infinite integrals can be evaluated in this manner by applying the theorem and its corollary considered here.

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