

K-THEORY RELATIVE TO A SEMIFINITE FACTOR

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Abstract. Let M be a II_1 factor, let $N = M \bar{\otimes} \mathcal{B}(H)$ be the corresponding II_∞ factor, and let \mathcal{K}_N be the closed two-sided ideal in N generated by the elements of finite trace. For an arbitrary C^* -algebra A , we define the abelian groups, $K_0^N(A)$ and $K_1^N(A)$. $K_0^N(A)$ is roughly speaking the "groupification" of the equivalence classes of projections in $A \otimes \mathcal{K}_N$, while $K_1^N(A)$ is the group of connected components of invertible operators in $(A \otimes \mathcal{K}_N)^1$. Our main result states that $K_*^N(A)$ and $K_*(A \otimes M)$ are naturally isomorphic and thus K_*^N is a stable, half-exact, homotopy function satisfying Bott periodicity. This generalizes (with completely different proofs) the results of M. Breuer who had defined the function K_*^N for commutative C^* -algebras using the idea of vector bundles relative to \mathcal{K}_N .

We also exhibit a B.D.F. pairing:

$$K_1^N(A) \otimes_{\mathbb{Z}} \text{Ext}^N(A)^{-1} \rightarrow \mathbb{R}$$

where $\text{Ext}^N(A)^{-1}$ is the group of invertible extensions of A by \mathcal{K}_N defined by P. Fillmore. For a large class of algebras, A , this pairing yields a "universal coefficient theorem", that is, an isomorphism of \mathbb{R} -vector spaces:

$$\text{Ext}^N(A)^{-1} \cong \text{Hom}_{\mathbb{R}}[K_1^N(A), \mathbb{R}].$$

This is, of course, closely related to the universal coefficient theorem of J. Rosenberg and C. Schochet.

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Introduction

In reading Georges Skandalis' paper [13] "On the group of extensions relative to a semifinite factor" I began to wonder if there might be some analogous results for K-theory. This paper is the end product of that wondering.

Before going further, let me state Skandalis' result and the attendant definitions. Let M be a II_1 factor and let $N = M \bar{\otimes} \mathcal{K}(H)$ be the corresponding II_∞ factor, where H is a separable Hilbert space. Let \mathcal{K}_N be the closed two-sided ideal in N generated by the elements of finite trace. Then, roughly speaking, $\text{Ext}^N(A)$ is the semigroup of equivalence classes of extensions of the form $0 \rightarrow \mathcal{K}_N \rightarrow B \rightarrow A \rightarrow 0$ for a fixed C^* -algebra A . In a similar vein, $\text{Ext}(A, M)$ is the group of equivalence classes of extensions of the form $0 \rightarrow M \otimes \mathcal{K} \rightarrow B \rightarrow A \rightarrow 0$. Now, it is known that \mathcal{K}_N is much larger than $M \otimes \mathcal{K}$ and, in fact, does not contain a countable approximate identity. Thus, a priori, $\text{Ext}^N(A)$ does not fit into Kasparov's extension theory. Nevertheless, Skandalis proved that $\text{Ext}^N(A)^{-1} \cong \text{Ext}(A, M)$ where $\text{Ext}^N(A)^{-1}$ denotes the invertible elements in $\text{Ext}^N(A)$.

In searching for a similar result for K-theory, I discovered that the K-theory relative to a II_∞ factor had only been defined (by Breuer [5]) for commutative C^* -algebras. His definition was in terms of vector bundles, naturally enough. Part of the problem then, is a suitable definition for the functors K_0^N and K_1^N . The obvious answer is to replace \mathcal{K} , the ideal of compact operators, wherever it appears in the definition of ordinary K-theory with the ideal \mathcal{K}_N . Thus, for example, $K_0^N(A)$ should be the "groupification" of the equivalence classes of projections in $A \otimes \mathcal{K}_N$. This is exactly what we do. The next question is: What is the analogue of $\text{Ext}(A, M)$ for K-theory? Here, the answer is: replace \mathcal{K} with $M \otimes \mathcal{K}$. Thus, for example, we should look at the "groupification" of the equivalence classes of projections in $A \otimes M \otimes \mathcal{K}$. However, this is just $K_0(A \otimes M)$. Thus, our result should say that $K_*^N(A)$ and $K_*(A \otimes M)$ are naturally isomorphic.

This is, in fact, our main result. One nice corollary of this theorem is that K_*^N is a stable, homotopy functor satisfying Bott periodicity.

In section 1 we begin by quickly reviewing ordinary K -theory with definitions most suited to our purposes. We then briefly examine the difference between the algebras $M \otimes \mathcal{K}$ and \mathcal{K}_N . Here we give completely new and more conceptual proofs of some results of Brown, Green, and Rieffel, and Akemann and Pedersen. We then carefully define our functors K_0^N and K_1^N and prove our main result that the embedding $M \otimes \mathcal{K} \rightarrow \mathcal{K}_N$ induces a natural isomorphism $K_*(A \otimes M) \cong K_*^N(A)$ for any C^* -algebra A .

In section 2 we observe that the long exact sequence relative to an ideal; the Pimsner-Voiculescu exact sequence relative to a crossed product by \mathbb{Z} ; and Connes' Thom Isomorphism relative to a crossed product by \mathbb{R} all hold to some degree in the K_*^N setting.

In section 3 we exhibit the analogue of the B.D.F. pairing:

$$K_1^N(A) \otimes_{\mathbb{Z}} \text{Ext}^N(A)^{-1} \rightarrow \mathbb{R}.$$

For a large class of algebras, this pairing yields an isomorphism of vector spaces:

$$\text{Ext}^N(A)^{-1} \rightarrow \text{Hom}_{\mathbb{R}}[K_1^N(A), \mathbb{R}]$$

which is naturally dubbed the "universal coefficient theorem". Of course we are forced to examine these vector space structures more closely, which we do. We explain here the close link of our universal coefficient theorem with that of Rosenberg and Schochet.

Finally, in section 4 we give the expected connection of the K^N -theory with the homotopy groups of $GL_1(A \otimes \mathcal{K}_N)$.

§1 K-Theory

1.1 Grothendieck group.

If S is an abelian semigroup, then we let $G_p(S)$ denote the Grothendieck group of S . That is, $G_p(S)$ consists of equivalence classes of formal differences, $x-y$, of elements $x, y \in S$ where $x-y \sim x'-y'$ if and only if there exists $z \in S$ such that $x+y'+z = x'+y+z$. Then $G_p(S)$ is an abelian group with the obvious operations. Moreover, the map $x \mapsto [x-0]$ is a semigroup homomorphism of S into $G_p(S)$ with the property that any homomorphism $S \rightarrow H$ where H is an abelian group extends uniquely to a homomorphism $G_p(S) \rightarrow H$. See [14] for details.

1.2 K-theory of C^* algebras.

With this in mind, let A be a unital C^* -algebra and let \mathcal{K} denote the compact operators on a separable Hilbert space. Let $P(A \otimes \mathcal{K})$ denote the set of projections in $A \otimes \mathcal{K}$. We define equivalence in $P(A \otimes \mathcal{K})$ to be the usual Murray-von Neumann equivalence of projections. That is, $e \sim f$ in $P(A \otimes \mathcal{K})$ if there exists $v \in A \otimes \mathcal{K}$ with $v^*v = e$ and $vv^* = f$. Let $[e], [f] \in P(A \otimes \mathcal{K})/\sim$. We define $[e] + [f] = [e'+f']$ where $e' \sim e$ and $f' \sim f$ are chosen so that $e'f' = 0 = f'e'$. This addition is well-defined and makes $P(A \otimes \mathcal{K})/\sim$ into an abelian semigroup with zero element. We then define:

$$K_0(A) = G_p\left[P(A \otimes \mathcal{K})/\sim\right].$$

The map $A \mapsto K_0(A)$ is easily seen to be functorial from unital C^* -algebras with unit preserving homomorphisms to abelian groups with homomorphisms. If A is not necessarily unital then let A^1 denote the unitization of A and let $\phi: A^1 \rightarrow \mathbb{C}$ be the canonical homomorphism. We define K_0 to be the kernel of the map

$\phi_*: K_0(A^1) \rightarrow K_0(\mathbb{C})$. One easily checks that the two definitions of $K_0(A)$ agree if A already has a unit (see section 1.3.4).

Now, if A is a C^* -algebra we let $U_1(A)$ (respectively, $GL_1(A)$) denote the group of unitaries (respectively, invertibles) in A^1 of the form $1 + a$ where $a \in A$. We let $U_1^0(A)$ (respectively, $GL_1^0(A)$) denote the subgroup consisting of the connected component of the identity. It is well-known (and not hard to prove) that the inclusion of $U_1(A)$ into $GL_1(A)$ and the polar decomposition map $x \mapsto u(x) = x|x|^{-1}$ are homotopy inverses and so we have that $U_1(A)/U_1^0(A) = GL_1(A)/GL_1^0(A)$. We then define $K_1(A) = U_1(A \otimes \mathcal{K})/U_1^0(A \otimes \mathcal{K}) = GL_1(A \otimes \mathcal{K})/GL_1^0(A \otimes \mathcal{K})$.

As usual, one can prove that $K_1(A)$ is an abelian group and that the map $A \mapsto K_1(A)$ is functorial.

1.3 K-theory relative to a II_{∞} factor.

Now, let M be a II_1 factor and let $N = M \bar{\otimes} \mathcal{B}(H)$ be the corresponding II_{∞} factor, where $\mathcal{B}(H)$ denotes the algebra of all bounded operators in the (separable) Hilbert space, H . Let \mathcal{K}_N denote the closed two sided ideal in N generated by the positive elements of finite trace. We observe that $M \otimes \mathcal{K} \subseteq \mathcal{K}_N$. That these algebras are different is due independently to Brown, Green and Rieffel [8] who refer to Blackadar, and Akemann and Pedersen [1] who refer to Elliott. We give a completely different and much simpler proof of their result in the following proposition.

1.3.1 Proposition: If N is a II_{∞} factor, then \mathcal{K}_N does not have a countable approximate identity.

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be a countable approximate identity for \mathcal{K}_N with $0 \leq x_n \leq 1$ for all n . Then by spectral theory we can find projections e_n in \mathcal{K}_N with $\|e_n x_n - x_n\| < \frac{1}{n}$ for each n . By an $\frac{\epsilon}{3}$ argument, we see that $\{e_n\}_{n=1}^{\infty}$ is an approximate identity for \mathcal{K}_N . If we let $E_n = \sup\{e_1, \dots, e_n\}$ then $\{E_n\}_{n=1}^{\infty}$ is an increasing approximate identity of projections in \mathcal{K}_N , which we can assume is strictly increasing. For each $n \geq 1$ choose a nonzero projection $f_n \leq [E_{n+1} - E_n]$ with $\text{trace}(f_n) \leq \frac{1}{2^n}$. Let $a = \sum_{n=1}^{\infty} f_n$. Then $a \in \mathcal{K}_N$ and $\|E_n a - a\| = 1$ for all $n \geq 1$, which is an obvious contradiction. ■

1.3.2 Corollary: If $N = M \bar{\otimes} \mathcal{K}(H)$ where M is a II_1 factor, then \mathcal{K}_N is not stably isomorphic to M .

Proof: $M \otimes \mathcal{K}$ has a countable approximate identity. If $\mathcal{K}_N \otimes \mathcal{K}$ had a countable approximate identity, $\{x_n\}$ then $\{(1 \otimes e)x_n(1 \otimes e)\}$ would be a countable approximate identity for $\mathcal{K}_N \otimes e \cong \mathcal{K}_N$ where e is a rank one projection in \mathcal{K} . ■

We now give a new and very different proof of a theorem of Brown, Green and Rieffel [8] which shows that even if we weaken our notion of stable equivalence to allow for non-separable Hilbert spaces, \mathcal{K}_N is never equivalent to any II_1 factor.

1.3.3 Theorem: Let M be any unital C^* -algebra, let N be any II_{∞} factor and let H_i for $i = 1, 2$ be (possibly nonseparable) Hilbert spaces. Then, $M \otimes \mathcal{K}(H_1)$ is not isomorphic to $\mathcal{K}_N \otimes \mathcal{K}(H_2)$.

Proof: Let $\{E_n\}_{n=1}^\infty \subseteq M \otimes \mathcal{K}(H_1)$ be any increasing sequence of projections. Then we can find projections $F_n = 1_M \otimes Q_n$ in $M \otimes \mathcal{K}(H_1)$ which we can take to be increasing such that

1. $\|F_n E_n - E_n\| < \frac{1}{n}$ for each n , and
2. $F_n \rightarrow F \in M \left[M \otimes \mathcal{K}(H_1) \right]$ strictly (here $F = 1_M \otimes Q$ where $Q = \sup_n Q_n$), where clearly $F \geq E_n$ for all n .

If $\mathcal{K}_N \otimes \mathcal{K}(H_2)$ also had this property then we could choose E_n in the following way. Let $\{\tilde{E}_n\}$ be an increasing sequence of projections in \mathcal{K}_N so that $E = \sup_n \tilde{E}_n$ is infinite in N . Let e be a rank one projection in $\mathcal{K}(H_2)$ and let $E_n = \tilde{E}_n \otimes e$. Then $\{E_n\}$ is an increasing sequence of projections in $\mathcal{K}_N \otimes \mathcal{K}(H_2)$. Suppose we can choose projections $\{F_n\}$ in $\mathcal{K}_N \otimes \mathcal{K}(H_2)$ such that

1. $\|F_n E_n - E_n\| < \frac{1}{n}$ for each n , and
2. $F_n \rightarrow F \in M \left[\mathcal{K}_N \otimes \mathcal{K}(H_2) \right]$ strictly where $F \geq E_n$ for all n .

Let $E = \tilde{E} \otimes e = \sup_n E_n$ in $N \bar{\otimes} \mathcal{K}(H_2)$. Then $E \in N \bar{\otimes} e \subseteq M \left[\mathcal{K}_N \otimes \mathcal{K}(H_2) \right]$. Now

$$\mathcal{K}_{\tilde{E}N\tilde{E}} = \tilde{E} \mathcal{K}_N \tilde{E} \cong (\tilde{E} \otimes e) \left[\mathcal{K}_N \otimes \mathcal{K}(H_2) \right] (\tilde{E} \otimes e) = (FE) \left[\mathcal{K}_N \otimes \mathcal{K}(H_2) \right] (FE)$$

since $FE = E = EF$. But, $EF_n E$ is in this latter algebra and since $EF_n E \rightarrow EFE = E$ strictly, we see that $\{EF_n E\}$ is a countable approximate identity for this latter algebra. Thus, $\mathcal{K}_{\tilde{E}N\tilde{E}}$ has a countable approximate identity, contradicting

1.3.1. ■

1.3.4 \mathcal{K}_*^N -theory. Now, given a countably decomposable II_∞ factor N , one can show directly that for any C^* -algebra A , $P(A \otimes \mathcal{K}_N) / \sim$ is an abelian semigroup with 0. However, this will follow from theorem 1.3.7. We then define $K_0^N(A)$ for unital

A to be $G_p \left[P(A \otimes \mathcal{K}_N) / \sim \right]$. If A is not necessarily unital we define $K_0^N(A)$ to be the kernel of the induced map $K_0^N(A^1) \rightarrow K_0^N(\mathbb{C})$. To see that this does not conflict with our previous definition in case A is unital we observe that if A has a unit, e , then $A^1 = A \oplus \mathbb{C}(1-e)$ as algebras and so we get $K_0^N(A^1) = K_0^N(A) \oplus K_0^N[\mathbb{C}(1-e)]$. Since $A^1 \rightarrow \mathbb{C}$ has the obvious splitting $\mathbb{C} \rightarrow \mathbb{C}(1-e)$, the kernel of the map $K_0^N(A^1) \rightarrow K_0^N(\mathbb{C})$ is just $K_0^N(A)$ in the sense of unital algebras. For completeness, we observe that a choice of trace on N yields an isomorphism $K_0^N(\mathbb{C}) \cong \mathbb{R}$.

We now define $K_1^N(A) = U_1(A \otimes \mathcal{K}_N) / U_1^0(A \otimes \mathcal{K}_N) = GL_1(A \otimes \mathcal{K}_N) / GL_1^0(A \otimes \mathcal{K}_N)$. Clearly, K_1^N is a functor from C^* -algebras to groups which preserves direct sums. It follows from results below that $K_1^N(A)$ is an abelian group for any C^* -algebra, A . We observe that an easy application of the Spectral theorem shows that $K_1^N(\mathbb{C}) = \{0\}$. By considerations similar to those used in defining K_0^N we see that $K_1^N(A) \cong K_1^N(A^1)$ via the injection $A \rightarrow A^1$.

1.3.5 Lemma. If e is a projection in $A \otimes \mathcal{K}_N$ and $\epsilon > 0$, then there is a projection f in $A \otimes \mathcal{K}_N$ and a unitary u in N so that

- (1) $\|f-e\| < \epsilon$, and
- (2) $(1_A \otimes u)y(1_A \otimes u^*) \in A \otimes M \otimes \mathcal{K}$ for all y in $f(A \otimes \mathcal{K}_N)f$.

In particular, $1_A \otimes u$ is in $M(A \otimes \mathcal{K}_N)$ and so $(1_A \otimes u)f(1_A \otimes u^*)$ is equivalent to f in $A \otimes \mathcal{K}_N$ via the partial isometry $(1_A \otimes u)f$.

Proof: Since the net of increasing projections in \mathcal{K}_N form an approximate identity for \mathcal{K}_N we can find a projection E in \mathcal{K}_N so that $1_A \otimes E$ almost dominates e . Then, by the usual functional calculus arguments we can find a projection f in $(1_A \otimes E)(A \otimes \mathcal{K}_N)(1_A \otimes E) = A \otimes E \mathcal{K}_N E$ with $\|e-f\| < \epsilon$. Since $\text{Tr}(E) < +\infty$, we can find a projection $F = 1_M \otimes Q$ in $M \otimes \mathcal{K}$ with $\text{Tr}(F) \geq \text{Tr}(E)$. By the comparison theorem there is a unitary u in N with $uEu^* \leq F$ so that

$(uEu^*)N(uEu^*) \subseteq \text{FNF}$. Now,

$$\begin{aligned}
& (1_A \otimes u) [f(A \otimes \mathcal{K}_N) f] (1_A \otimes u^*) \\
\subseteq & (1_A \otimes u) [(1_A \otimes E)(A \otimes \mathcal{K}_N)(1_A \otimes E)] (1_A \otimes u^*) \\
= & A \otimes (uE) \mathcal{K}_N (Eu^*) \\
\subseteq & A \otimes (uE) N(Eu^*) \\
= & A \otimes (uEu^*) N(uEu^*) \\
\subseteq & A \otimes \text{FNF} \\
= & A \otimes [(1_M \otimes Q)(M \otimes \mathcal{S}(H))(1_M \otimes Q)] \\
= & A \otimes [M \otimes Q \mathcal{S}(H) Q] \\
= & A \otimes [M \otimes Q \mathcal{S}(H) Q] \text{ since } Q \text{ is finite-dimensional} \\
\subseteq & A \otimes (M \otimes \mathcal{K}) \text{ as required. } \blacksquare
\end{aligned}$$

1.3.6 Theorem: Let A be a unital C^* -algebra. Then the inclusion map $A \otimes M \otimes \mathcal{K} \hookrightarrow A \otimes \mathcal{K}_N$ induces an isomorphism $P(A \otimes M \otimes \mathcal{K}) / \sim \rightarrow P(A \otimes \mathcal{K}_N) / \sim$. This yields a natural isomorphism $K_0(A \otimes M) \cong K_0^N(A)$ and hence K_0^N defines a stable homotopy functor on unital C^* -algebras.

Proof: Surjectivity. Let e be a projection in $A \otimes \mathcal{K}_N$ and let $0 < \epsilon < 1$ then by the lemma choose f in $A \otimes \mathcal{K}_N$ with $\|e - f\| < \epsilon$ so that $f \sim e$, and also $u \in N$ satisfying (2). then $f' = (1_A \otimes u)f(1_A \otimes u^*)$ is in $A \otimes M \otimes \mathcal{K}$ and is equivalent to f and hence equivalent to e . Hence, the obvious map $P(A \otimes M \otimes \mathcal{K}) / \sim \rightarrow P(A \otimes \mathcal{K}_N) / \sim$ is surjective. It is clearly a well-defined map. The surjectivity also shows that addition makes sense in $P(A \otimes \mathcal{K}_N) / \sim$ and that the map is a homomorphism of semigroups.

Injectivity. Suppose e and f are in $A \otimes M \otimes \mathcal{K}$ and are equivalent in the larger algebra $A \otimes \mathcal{K}_N$. Then we can find a finite rank projection Q in \mathcal{K} so that $1_A \otimes 1_M \otimes Q$ almost dominates e and f . Therefore, there exist projections \tilde{e}, \tilde{f}

dominated by $1_A \otimes 1_M \otimes Q$ which are equivalent to e and f respectively in $A \otimes M \otimes \mathcal{K}$. Let v be a partial isometry in the larger algebra $A \otimes \mathcal{K}_N$ implementing the equivalence between \tilde{e} and \tilde{f} . In particular, $v = \tilde{e}v\tilde{f}$ and hence $v = (1_A \otimes 1_M \otimes Q)v(1_A \otimes 1_M \otimes Q)$ is in $A \otimes M \otimes Q \mathcal{B}(H)Q \subseteq A \otimes M \otimes \mathcal{K}$. Thus \tilde{e} and \tilde{f} (and hence e and f) are equivalent in $A \otimes M \otimes \mathcal{K}$. Thus the map is injective, and hence extends to an isomorphism $K_0(A \otimes M) \cong K_0^N(A)$. ■

1.3.7 Theorem: Let A be an arbitrary C^* -algebra. Then the inclusion map $A \otimes M \otimes \mathcal{K} \hookrightarrow A \otimes \mathcal{K}_N$ induces a natural isomorphism $K_0(A \otimes M) \cong K_0^N(A)$. Hence, K_0^N is a stable homotopy functor for C^* -algebras.

Proof: Consider the commutative square

$$\begin{array}{ccc} A^1 \otimes M \otimes \mathcal{K} & \longrightarrow & A^1 \otimes \mathcal{K}_N \\ \downarrow & & \downarrow \\ \mathbb{C} \otimes M \otimes \mathcal{K} & \longrightarrow & \mathbb{C} \otimes \mathcal{K}_N \end{array}$$

This induces the commutative square

$$\begin{array}{ccc} K_0(A^1 \otimes M) \cong K_0^N(A^1) & & \\ \downarrow & & \downarrow \\ K_0(\mathbb{C} \otimes M) \cong K_0^N(\mathbb{C}) & & \end{array}$$

which induces a natural isomorphism of the kernels of the two vertical maps. The kernel of the right hand map is $K_0^N(A)$ by definition. On the left hand side we apply the long exact sequence of K -theory to the short exact sequence:

$$0 \rightarrow A \otimes M \rightarrow A^1 \otimes M \rightarrow \mathbb{C} \otimes M \rightarrow 0$$

and observe that $K_1(\mathbb{C} \otimes M) = \{0\}$ to see that the kernel of the left hand side is just $K_0(A \otimes M)$ as required. ■

Now, in order to prove a similar result for K_1^N , one could attempt to work directly along the lines of the previous theorem. Some difficulties are encountered in this approach, however, While it is not hard to show directly that the map $K_1(A \otimes M) \rightarrow K_1^N(A)$ is surjective, injectivity seems to be much more difficult to prove.

We get around this difficulty by using the result for K_0^N together with the suspension isomorphism for ordinary K-theory. It is curious (and extremely fortunate) that this approach does not first require proving a suspension isomorphism in the K_*^N theory. First, we need a lemma.

1.3.8 Lemma. The map $B \otimes M \otimes \mathcal{K} \hookrightarrow B \otimes \mathcal{K}_N$ induces an isomorphism $K_0(B \otimes M \otimes \mathcal{K}) \rightarrow K_0(B \otimes \mathcal{K}_N)$.

Proof: We calculate:

$$\begin{aligned} K_0(B \otimes M \otimes \mathcal{K}) &= G_P(P((B \otimes M \otimes \mathcal{K}) \otimes \mathcal{K}) / \sim) \\ &\cong G_P(P(\mathcal{K} \otimes (B \otimes M \otimes \mathcal{K})) / \sim) \text{ via switch isomorphism} \\ &\cong G_P(P(\mathcal{K} \otimes (B \otimes \mathcal{K}_N)) / \sim) \text{ via theorem 1.3.7} \\ &\cong G_P(P((B \otimes \mathcal{K}_N) \otimes \mathcal{K}) / \sim) \text{ via switch undoing previous switch} \\ &= K_0(B \otimes \mathcal{K}_N) \quad \blacksquare \end{aligned}$$

1.3.9 Theorem. The embedding $A \otimes M \otimes \mathcal{K} \hookrightarrow A \otimes \mathcal{K}_N$ induces a natural isomorphism $K_1(A \otimes M) \cong K_1^N(A)$.

Proof: We first apply the suspension isomorphism of ordinary K-theory to the previous lemma with $B = SA$ and see that the embedding $A \otimes M \otimes \mathcal{K} \hookrightarrow A \otimes \mathcal{K}_N$ induces an isomorphism $K_1(A \otimes M \otimes \mathcal{K}) \cong K_1(A \otimes \mathcal{K}_N)$.

Now, we have the commutative diagram:

$$\begin{array}{ccc}
 K_1(A \otimes M \otimes \mathcal{K}) & \xrightarrow{\cong} & K_1(A \otimes \mathcal{K}_N) \\
 \parallel & & \parallel \\
 \frac{GL_1((A \otimes M \otimes \mathcal{K}) \otimes \mathcal{K})}{GL_1^0((A \otimes M \otimes \mathcal{K}) \otimes \mathcal{K})} & \xrightarrow{\cong} & \frac{GL_1((A \otimes \mathcal{K}_N) \otimes \mathcal{K})}{GL_1^0((A \otimes \mathcal{K}_N) \otimes \mathcal{K})} \\
 \uparrow \cong & & \uparrow \cong \\
 \frac{GL_1(A \otimes M \otimes \mathcal{K})}{GL_1^0(A \otimes M \otimes \mathcal{K})} & \xrightarrow{\quad} & \frac{GL_1(A \otimes \mathcal{K}_N)}{GL_1^0(A \otimes \mathcal{K}_N)} \\
 \parallel & & \parallel \\
 K_1(A \otimes M) & \xrightarrow{\quad} & K_1^N(A)
 \end{array}$$

$x \mapsto x \otimes e$

This clearly implies that the map $K_1(A \otimes M) \rightarrow K_1^N(A)$ is injective. To see that it is surjective let a be an element in $GL_1(A \otimes \mathcal{K}_N)$ so that $a = 1 + x$ with x in $A \otimes \mathcal{K}_N$. Since the injection $A \hookrightarrow A^1$ induces isomorphisms in both K_1 and K_1^N we can without loss of generality assume that A has an identity. Thus, given $\epsilon > 0$ we can find a projection e in $A \otimes \mathcal{K}_N$ with $\|exe - x\| < \epsilon$. Now, choose f in $A \otimes \mathcal{K}_N$ and $u \in N$ according to lemma 1.3.5 so that $\|fxf - x\| < 2\epsilon$ and $(1_A \otimes u)(fxf)(1_A \otimes u^*) \in A \otimes M \otimes \mathcal{K}$. Then $1 + fxf$ is in $GL_1(A \otimes \mathcal{K}_N)$ and is equivalent to $1 + x$ there provided $\epsilon > 0$ is sufficiently small. Let $u_t, t \in [0, 1]$, be a norm-continuous path of unitaries in N connecting u to 1_N . Then, $(1_A \otimes u_t) \in M(A \otimes \mathcal{K}_N)$ for all t

and

$$(1_A \otimes u_t)(1 + fxf)(1_A \otimes u_t^*) = 1 + (1_A \otimes u_t)fxf(1_A \otimes u_t^*)$$

is a path of invertibles in $GL_1(A \otimes \mathcal{K}_N)$ connecting $[1 + fxf]$ to $[1 + (1_A \otimes u)fxf(1_A \otimes u^*)] \in GL_1(A \otimes M \otimes \mathcal{K})$. Thus the map $K_1(A \otimes M) \rightarrow K_1^N(A)$ is surjective and hence an isomorphism. ■

1.3.10 Corollary. There are natural isomorphisms:

1. $K_1^N(A) \cong K_0^N(SA)$
2. $K_0^N(A) \cong K_1^N(SA)$
3. $K_1^N(A) \cong K_1^N(S^2A)$, $K_0^N(A) \cong K_0^N(S^2A)$
4. $K_*^N(A) \cong K_*^N(\mathcal{K} \otimes A)$ induced by $a \mapsto e \otimes a$ where e is a minimal projection in \mathcal{K} .

Proof: This follows immediately from the corresponding properties for ordinary K-theory and the natural isomorphism $K_*^N(A) \cong K_*(A \otimes M)$. ■

2. Exact sequences in the K_*^N theory

2.1 Ideals

If J is an ideal in the C^* -algebra A , then the exact sequence $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$ induces a sequence $0 \rightarrow J \otimes M \rightarrow A \otimes M \rightarrow A/J \otimes M \rightarrow 0$ which may fail to be exact in the centre. However, if A/J is nuclear then one can easily show that we have a short exact sequence, $0 \rightarrow J \otimes M \rightarrow A \otimes M \rightarrow A/J \otimes M \rightarrow 0$. Then the long exact sequence of ordinary K-theory coupled with theorems 1.3.9 and 1.3.7 show that we

get the long exact sequence for K_*^N :

$$\begin{array}{ccccc} K_0^N(J) & \longrightarrow & K_0^N(A) & \longrightarrow & K_0^N(A/J) \\ \uparrow & & & & \downarrow \\ K_1^N(A/J) & \longleftarrow & K_1^N(A) & \longleftarrow & K_1^N(J). \end{array}$$

Now, even though a II_1 factor M is not nuclear, it is possible but not likely that it is C^* -exact so that one would have $0 \rightarrow J \otimes M \rightarrow A \otimes M \rightarrow A/J \otimes M \rightarrow 0$ is exact, in general. This, of course, would imply that the long exact sequence for K_*^N holds in general. More hopeful, perhaps, is the possibility that the map $A \otimes M/J \otimes M \rightarrow (A/J) \otimes M$ is some sort of KK -equivalence: we have no idea how this might be proved.

2.2 Crossed Products

If $\alpha: G \rightarrow \text{Aut } A$ is an action of the locally compact group G on the C^* -algebra A , then $\alpha \otimes \text{id}$ defines an action of G on $A \otimes M$ and there is a natural isomorphism $(A \otimes M) \rtimes_{\alpha} G \cong (A \rtimes_{\alpha} G) \otimes M$. Theorems 1.3.7 and 1.3.9 then allow us to obtain a Pimsner-Voiculescu exact sequence [2] in the K_*^N theory when α is an action of \mathbb{Z} on A :

$$\begin{array}{ccccc} K_0^N(A) & \xrightarrow{1-\alpha_*} & K_0^N(A) & \xrightarrow{i_*} & K_0^N(A \rtimes_{\alpha} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1^N(A \rtimes_{\alpha} \mathbb{Z}) & \xleftarrow{i_*} & K_1^N(A) & \xleftarrow{1-\alpha_*} & K_1^N(A). \end{array}$$

Similarly, if α is an action of \mathbb{R} on A , we get Connes' Thom Isomorphism [2]: $K_i^N(A \rtimes_{\alpha} \mathbb{R}) \cong K_{1-i}^N(A)$ for $i = 0, 1$.

3. Pairing with Ext_*^N .

Let $[u] \in K_1^N(A)$ and $[\tau] \in \text{Ext}^N(A)^{-1}$. Then u is a unitary in $U_1(A \otimes \mathcal{K}_N)$ and $\tau: A \rightarrow N/\mathcal{K}_N$ is a $*$ -monomorphism. We want $\tau \otimes \text{id}: A \otimes \mathcal{K}_N \rightarrow N/\mathcal{K}_N \otimes \mathcal{K}_N$ to define a map into $N \bar{\otimes} N/\mathcal{K}_{N \bar{\otimes} N}$. However, tensor product difficulties make this a little tricky. We proceed as follows. Let $\phi: A \rightarrow N$ be a completely positive lifting for τ , so that ϕ is a multiplicative modulo \mathcal{K}_N . Then, $\phi \otimes \text{id}: A \otimes \mathcal{K}_N \rightarrow N \otimes \mathcal{K}_N \subseteq N \bar{\otimes} N$ is completely positive. Moreover, an easy calculation on $A \otimes \mathcal{K}_N$ shows that $\phi \otimes \text{id}$ is multiplicative modulo $\mathcal{K}_N \otimes \mathcal{K}_N \subseteq \mathcal{K}_{N \bar{\otimes} N}$, and, that $\phi \otimes \text{id}$ is independent of ϕ modulo $\mathcal{K}_N \otimes \mathcal{K}_N \subseteq \mathcal{K}_{N \bar{\otimes} N}$. Thus, we obtain a $*$ -homomorphism $A \otimes \mathcal{K}_N \rightarrow N \bar{\otimes} N/\mathcal{K}_{N \bar{\otimes} N}$ which we denote by $\tau \otimes \text{id}$. Then $(\tau \otimes \text{id})^1: U_1(A \otimes \mathcal{K}_N) \rightarrow U_1(N \bar{\otimes} N/\mathcal{K}_{N \bar{\otimes} N}) \cong U(N \bar{\otimes} N/\mathcal{K}_{N \bar{\otimes} N})$ since the algebra $N \bar{\otimes} N/\mathcal{K}_{N \bar{\otimes} N}$ has an identity. If we fix a trace, Tr , on N we get a trace $\tilde{\text{Tr}}$ on $N \bar{\otimes} N$ and so we define $\langle [u], [\tau] \rangle = \text{ind}_{\tilde{\text{Tr}}}((\tau \otimes \text{id})^1(u)) \in \mathbb{R}$. Where $\text{ind}_{\tilde{\text{Tr}}}$ is the Breuer index [3, 4] for unitaries in $N \bar{\otimes} N/\mathcal{K}_{N \bar{\otimes} N}$ relative to the trace, $\tilde{\text{Tr}}$. This is easily seen to define a group homomorphism $K_1^N(A) \otimes_{\mathbb{Z}} \text{Ext}^N(A)^{-1} \rightarrow \mathbb{R}$ which we also think of as a homomorphism: $\text{Ext}^N(A)^{-1} \rightarrow \text{Hom}_{\mathbb{Z}}(K_1^N(A), \mathbb{R})$. This construction of the homomorphism, using the extension τ to define a homomorphism from $A \otimes \mathcal{K}_N \rightarrow N \bar{\otimes} N/\mathcal{K}_{N \bar{\otimes} N}$ is a (rough) analogue of the construction used in [7, section 4] for $\mathcal{B}(\mathcal{H})$ in place of N . The other way of constructing this homomorphism is to consider the short exact sequence

$$0 \rightarrow \mathcal{K}_N \rightarrow D \xrightarrow{\tau} A \rightarrow 0$$

(where $D = \pi^{-1}(\tau(A)) \subseteq N$ and $\pi: N \rightarrow N/\mathcal{K}_N$ is the canonical map). One then applies

the long exact sequence of K_*^N -theory and uses the connecting map $K_1^N(A) \rightarrow K_0^N(\mathcal{K}_N) \cong \mathbb{R}$ to obtain the image of $[\tau]$ in $\text{Hom}_{\mathbb{Z}}(K_1^N(A), \mathbb{R})$. We note that the fact that $[\tau]$ is invertible implies that we can assume as above, the existence of a completely positive splitting $\phi: A \rightarrow D$ for this short exact sequence and this implies that $0 \rightarrow \mathcal{K}_N \otimes B \rightarrow D \otimes B \rightarrow A \otimes B \rightarrow 0$ is exact for any C^* -algebra B . In particular, the sequence is exact for $B = M$ and so by 2.1 we have the long exact sequence for K_*^N -theory even though A may not be nuclear. A slight variation on this would be to look at the exact sequence $0 \rightarrow \mathcal{K}_N \otimes \mathcal{K}_N \rightarrow D \otimes \mathcal{K}_N \rightarrow A \otimes \mathcal{K}_N \rightarrow 0$ and to construct the connecting map directly in the K_*^N -theory: this would yield directly a map from $U_1(A \otimes \mathcal{K}_N) = K_1^N(A)$ to $G_p(P(\mathcal{K}_N \otimes \mathcal{K}_N)/\sim) = K_0^N(\mathcal{K}_N)$. By the arguments used in theorem 1.3.6 one can show that the injection $\mathcal{K}_N \otimes \mathcal{K}_N \rightarrow \mathcal{K}_{N \otimes N}$ induces an isomorphism $[P(\mathcal{K}_N \otimes \mathcal{K}_N)/\sim] \cong [P(\mathcal{K}_{N \otimes N})/\sim]$. Using this isomorphism and the construction of the connecting map, one easily checks that the two definitions of the homomorphism $\text{Ext}^N(A)^{-1} \rightarrow \text{Hom}_{\mathbb{Z}}(K_1^N(A), \mathbb{R})$ agree.

3.1 Vector space structure on K_*^N and $\text{Ext}_*^N(\)^{-1}$

Suppose $\alpha: \mathbb{R}^+ \rightarrow \text{Aut } N$ is a representation of the multiplicative group \mathbb{R}^+ which satisfies $\text{Tr} \circ \alpha_t = t \cdot \text{Tr}$. Such actions often exist: for example, if N is the hyperfinite II_{∞} factor. For a fixed action α of this kind, we can define actions of \mathbb{R}^+ via

$$t \cdot [u] = [(\text{id}_A \otimes \alpha_t)^1(u)] \text{ for } t \in \mathbb{R}^+, [u] \in K_1^N(A)$$

$$t \cdot [e] = [(\text{id}_A \otimes \alpha_t)(e)] \text{ for } t \in \mathbb{R}^+, [e] \in K_0^N(A)$$

$$t \cdot [\tau] = [\hat{\alpha}_t \circ \tau] \text{ for } t \in \mathbb{R}^+, [\tau] \in \text{Ext}^N(A)^{-1},$$

where $\hat{\alpha}_t$ is the induced automorphism on N/\mathcal{K}_N . Clearly, in each case we have

$$1 \cdot x = x \text{ and } (ts) \cdot x = t \cdot (s \cdot x).$$

One can also show that $t \cdot (x+y) = t \cdot x + t \cdot y$: this requires a little thought in the case of $\text{Ext}^N(A)^{-1}$, for one must first modify

α_t by an inner automorphism of N to insure that $\alpha_t(E) = E$ for some infinite, co-infinite projection E in N .

However, it is far from clear (to us) that we have $(t+s) \cdot x = t \cdot x + s \cdot x$ in any of the above cases. However, if we assume that A is in the category \mathcal{N} (i.e., the smallest subcategory of the category of separable nuclear C^* -algebras which contains the separable Type I algebras and is closed under the operations of taking extensions, inductive limits, stable isomorphism and crossed products by \mathbb{Z} and \mathbb{R}) then by Schochet's Künneth formula for tensor products [12] we have that

$$K_0^{\mathbb{N}}(A) \cong K_0(A) \otimes_{\mathbb{Z}} K_0(\mathbb{M}) \cong K_0(A) \otimes_{\mathbb{Z}} \mathbb{R}$$

and one easily checks that in this case our action of \mathbb{R}^+ on $K_0^{\mathbb{N}}(A)$ corresponds to the obvious one on $K_0(A) \otimes_{\mathbb{Z}} \mathbb{R}$ and so gives $K_0^{\mathbb{N}}(A)$ a vector space structure independent of α . A similar statement holds for $K_1^{\mathbb{N}}(A)$ via the suspension isomorphism.

Now, under the assumption that $A \in \mathcal{N}$, if we fix a $[\tau] \in \text{Ext}^{\mathbb{N}}(A)^{-1}$, then $\langle \cdot, [\tau] \rangle$ regarded as an element of $\text{Hom}_{\mathbb{Z}}(K_1^{\mathbb{N}}(A), \mathbb{R})$ is easily seen to be \mathbb{R} -linear and so we have a natural homomorphism

$$\text{Ext}^{\mathbb{N}}(A)^{-1} \rightarrow \text{Hom}_{\mathbb{R}}(K_1^{\mathbb{N}}(A), \mathbb{R}).$$

3.2 Theorem (Universal Coefficient Theorem): If A is in the category \mathcal{N} , then the natural map $\text{Ext}^{\mathbb{N}}(A) \rightarrow \text{Hom}_{\mathbb{R}}(K_1^{\mathbb{N}}(A), \mathbb{R})$ is an isomorphism which carries the \mathbb{R}^+ action on $\text{Ext}^{\mathbb{N}}(A)$ (when it exists) to the (vector space) action of \mathbb{R}^+ on $\text{Hom}_{\mathbb{R}}(K_1^{\mathbb{N}}(A), \mathbb{R})$. Thus, the action of \mathbb{R}^+ on $\text{Ext}^{\mathbb{N}}(A)$ extends to a scalar multiplication by \mathbb{R} and $\text{Ext}^{\mathbb{N}}(A) \cong \text{Hom}_{\mathbb{R}}(K_1^{\mathbb{N}}(A), \mathbb{R})$ as vector spaces.

Proof: We have only to check that it is an isomorphism. By Skandalis' result [13] we have that $\text{Ext}^N(A) \cong \text{Ext}(A, M)$ and so $\text{Ext}^N(\cdot)$ is a stable, periodic, homotopy-invariant functor satisfying the usual long exact sequence. As pointed out by Rosenberg [15], this implies (by an argument of Brown [6]) that $\text{Ext}^N(\cdot)$ satisfies a "Milnor \lim^1 sequence". Now the usual arguments as found in Rosenberg and Schochet [11] (see also [9], [10], and [12]) show that the map is an isomorphism for all $A \in \mathcal{N}$. ■

3.3 Remarks

1. Since $K_1^N(A) \cong K_1(A \otimes M) \cong K_1(A) \otimes K_0(M) = K_1(A) \otimes \mathbb{R}$ by Schochet's Künneth Theorem for tensor products (when $A \in \mathcal{N}$), we have that $\text{Hom}_{\mathbb{R}}(K_1^N(A), \mathbb{R}) \cong \text{Hom}_{\mathbb{R}}(K_1(A) \otimes \mathbb{R}, \mathbb{R}) \cong \text{Hom}_{\mathbb{Z}}(K_1(A), \mathbb{R})$. As observed by Skandalis the resulting isomorphism $\text{Ext}^N(A) \cong \text{Hom}_{\mathbb{Z}}(K_1(A), \mathbb{R})$ also follows directly from his result and Rosenberg and Schochet's universal coefficient formula in KK-theory.
2. On the other hand, if we could show that $\text{Ext}^N(\cdot)$ is a stable, periodic, homotopy functor satisfying the long exact sequence (without resorting to Skandalis' theorem) then we could use theorem 3.2 and Rosenberg and Schochet's universal coefficient formula to deduce Skandalis' theorem at least for $A \in \mathcal{N}$.

4. Homotopy groups.

Here we give the expected connection of the K^N -theory with the homotopy groups of $GL_1(A \otimes \mathcal{K}_N)$. The following lemma is certainly known, but we give the proof for the sake of completeness.

4.1 Lemma. If B is a Banach algebra then there is a natural isomorphism:

$$\pi_n(GL_1(B)) \cong \pi_{n-1}(GL_1(SB)) \text{ for } n \geq 1.$$

Proof: We observe that $S^n = (\mathbb{R}^n)^+$ so that elements of $\pi_n(GL_1(B))$ are (homotopy classes of) continuous functions $f: \mathbb{R}^n \rightarrow \{1 + b \mid b \in B \text{ and } b \text{ is quasi-invertible}\}$ satisfying $f(\omega) = 1$. Equivalently, we can look at functions $g: \mathbb{R}^n \rightarrow B' = \{b \in B \mid b \text{ is quasi-invertible}\}$ and $g(\omega) = 0$. This is exactly the set of quasi-invertible elements in the Banach algebra $C_0(\mathbb{R}^n, B)$. Similarly, $\pi_{n-1}(GL_1(SB))$ consists of (homotopy classes of) quasi-invertible elements in the Banach algebra $C_0(\mathbb{R}^{n-1}, C_0(\mathbb{R}, B))$. Since these two Banach algebras are naturally isomorphic as Banach algebras, the result follows. ■

4.2 Theorem: If A is a C^* -algebra and $n \geq 0$ then there are natural isomorphisms

$$\pi_n(GL_1(A \otimes \mathcal{K}_N)) \cong \begin{cases} K_1^N(A) & \text{if } n \text{ is even} \\ K_0^N(A) & \text{if } n \text{ is odd.} \end{cases}$$

So, in particular,

$$\pi_n(GL_1(\mathcal{K}_N)) \cong \begin{cases} \{0\} & \text{if } n \text{ is even} \\ \mathbb{R} & \text{if } n \text{ is odd.} \end{cases}$$

Proof: By repeated applications of the lemma

$$\begin{aligned} \pi_n(\mathrm{GL}_1(\mathbb{A} \otimes \mathcal{K}_N)) &\cong \pi_0(\mathrm{GL}_1(S^n \mathbb{A} \otimes \mathcal{K}_N)) \\ &= \mathrm{K}_1^N(S^n \mathbb{A}) \cong \begin{cases} \mathrm{K}_1^N(\mathbb{A}) & \text{if } n \text{ is even} \\ \mathrm{K}_0^N(\mathbb{A}) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Where the last isomorphisms follow from 1.3.10. ■

4.3 Corollary. The injection $\mathbb{A} \otimes \mathbb{M} \otimes \mathcal{K} \rightarrow \mathbb{A} \otimes \mathcal{K}_N$ induces a homotopy equivalence between $\mathrm{GL}_1(\mathbb{A} \otimes \mathbb{M} \otimes \mathcal{K})$ and $\mathrm{GL}_1(\mathbb{A} \otimes \mathcal{K}_N)$.

Proof: $\pi_n(\mathrm{GL}_1(\mathbb{A} \otimes \mathbb{M} \otimes \mathcal{K})) \cong \mathrm{K}_{[n+1]}(\mathbb{A} \otimes \mathbb{M}) \cong \mathrm{K}_{[n+1]}^N(\mathbb{A}) \cong \pi_n(\mathrm{GL}_1(\mathbb{A} \otimes \mathcal{K}_N))$ where $[n+1]$ denotes $n+1$ reduced mod 2. Thus the two groups are weakly homotopy equivalent, and hence homotopy equivalent by theorem is of [16]. ■

4.4 Remark. If we knew that $\mathrm{UM}(\mathbb{A} \otimes \mathcal{K}_N)$ is contractible (as it probably is), then applying the homotopy long exact sequence to

$$0 \rightarrow \mathrm{U}_1(\mathbb{A} \otimes \mathcal{K}_N) \rightarrow \mathrm{UM}(\mathbb{A} \otimes \mathcal{K}_N) \rightarrow \mathrm{U}^0(\mathrm{M}(\mathbb{A} \otimes \mathcal{K}_N)/(\mathbb{A} \otimes \mathcal{K}_N)) \rightarrow 0$$

would yield

$$\pi_n(\mathrm{U}^0(\mathrm{M}(\mathbb{A} \otimes \mathcal{K}_N)/(\mathbb{A} \otimes \mathcal{K}_N))) \cong \pi_{n-1}(\mathrm{U}_1(\mathbb{A} \otimes \mathcal{K}_N)) = \pi_{n-1}(\mathrm{GL}_1(\mathbb{A} \otimes \mathcal{K}_N)) \cong \mathrm{K}_{[n]}^N(\mathbb{A}) \text{ for } n \geq 1.$$

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