SOME CONVOLUTION SERIES IDENTITIES

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Abstract

A representation of a convolution series involving arbitrary sequences is obtained in terms of the derivatives of known generating functions. Another variation of the main result is given, and applications are shown to yield certain combinatorial identities and addition formulas.

1. Introduction and the Main Result

Several results giving convolution type series identities for certain sequences of numbers or various classes of functions are frequently found in the literature (see, for instance, [4] and [7]). The so-called addition theorems associated with the classical orthogonal polynomials (or their generalizations) are also essentially of the convolution types. The method of approach in the derivation of such results is either direct or uses certain techniques of series manipulations and operational calculi.

In this paper we derive a result which expresses a convolution series involving arbitrary sequences in terms of the derivatives of known generating functions. We consider some applications leading to certain combinatorial identities. Another variation of the main result is also given and examples are cited to illustrate the applications.

The main result of the paper is contained in

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Keywords. Convolution series, generating functions, combinatorial identities, addition formulas, orthogonal polynomials, series manipulations, operational calculi, Taylor's series, Cauchy product, fractional derivative, Vandermonde's convolution, binomial expansion, Gould's expansion formula, Rothe's identity, Laguerre polynomials.

Theorem 1. *For $\lambda, \mu \in \mathbb{C}$ and $m \in \mathbb{N}$ ($\mathbb{N} := \{1, 2, 3, \cdots\}$), define the power series expansions by*

$$\sum_{n=0}^{\infty} A_{\lambda,n}\, z^{mn} = f_m(\lambda, w) \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} B_{\mu,n}\, z^{n} = g(\mu, w), \tag{1.2}$$

where $\{A_{\lambda,n}\}$ and $\{B_{\mu,n}\}$ denote certain arbitrary sequences for fixed constants λ and μ, respectively, and w is a function of z defined implicitly by

$$w = z\; h(w) \quad \text{and} \quad w(0) = 0. \tag{1.3}$$

Then

$$\begin{aligned}&\sum_{k=0}^{[n/m]} A_{\lambda,k}\; B_{\mu,n-mk}\\ &\quad = \frac{1}{[n/m]!}\, D_z^{[n/m]} \left\{ f_m(\lambda, z)\, g(\mu, z) \left(1 - \frac{z\, h'(z)}{h(z)}\right) \left(h(z)\right)^{[n/m]} \right\}\Bigg|_{z=0},\end{aligned} \tag{1.4}$$

where $[\lambda]$ denotes the greatest integer in λ.

2. Proof of Theorem 1

In view of a familiar series rearrangement property, (1.1) and (1.2) readily yield

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} A_{\lambda,k}\; B_{\mu,n-mk}\; z^n = f_m(\lambda, w)\, g(\mu, w), \tag{2.1}$$

where w is a function of z defined by (1.3).

A comparison with Taylor's series expresses the Cauchy product coefficients in (2.1) in the form:

$$\begin{aligned}\sum_{k=0}^{[n/m]} A_{\lambda,k}\; B_{\mu,n-mk} &= \frac{1}{[n/m]!}\, D_z^{[n/m]} \{f_m(\lambda, w)\, g(\mu, w)\}\big|_{z=0}\\ &= \frac{1}{[n/m]!}\, D_{w/h(w)}^{[n/m]}\; \{f_m(\lambda, w)\, g(\mu, w)\|\big|_{w=0}.\end{aligned} \tag{2.2}$$

Now we appeal to the following particular form of a known result of Osler [11, p. 288, Equation (1.2)]:

$$D^{\rho}_{u(z)}\{F(z)\} = D^{\rho}_{z}\left\{F(z)\,u'(z)\left(\frac{z-\zeta}{u(z)-u(\zeta)}\right)^{\rho+1}\right\}\Big|_{\zeta=z}, \tag{2.3}$$

where $D^{\rho}_{u(z)}\{F(z)\}$ denotes the *fractional derivative* (of *order* ρ) of $F(z)$ with respect to $u(z)$. We thus arrive at the assertion (1.4), and the proof of Theorem 1 is evidently completed.

3. Combinatorial Series Identities

We apply Theorem 1 to derive certain combinatorial series identities (for the case $m = 1$).

Let us put $h(z) = 1$, and choose the sequences $\{A_{\lambda,n}\}$ and $\{B_{\mu,n}\}$ as follows:

$$A_{\lambda,n} = \binom{\lambda}{n} \quad \text{and} \quad B_{\mu,n} = \binom{\mu}{n} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{3.1}$$

Then we at once find from (1.1) and (1.2) that

$$f_1(\lambda, z) = (1+z)^{\lambda} \quad \text{and} \quad g(\mu, z) = (1+z)^{\mu}, \tag{3.2}$$

since $w = z$ [*cf.* Equation (1.3)]. Theorem 1 with the above substitutions gives us the familiar formula (known as Vandermonde's convolution):

$$\sum_{k=0}^{n} \binom{\lambda}{k}\binom{\mu}{n-k} = \binom{\lambda+\mu}{n} \qquad (n \in \mathbb{N}_0), \tag{3.3}$$

involving arbitrary parameters λ and μ.

Next we consider the generating function [8, p. 3, Equation (2.1)]:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\lambda+\mu n}{\gamma+\beta n}\binom{\alpha+\beta n}{n} z^n \\ = (1+w)^{\alpha}\left[\frac{\gamma\mu(1+w)}{\beta(1+w-\beta w)} + \left(\lambda - \frac{\gamma\mu}{\beta}\right)\sum_{n=0}^{\infty} \mathcal{L}_n(\alpha,\beta,\gamma)\left(\frac{w}{1+w}\right)^n\right],\end{aligned} \tag{3.4}$$

where α, β, γ, λ, and μ are arbitrary parameters (independent of n), w is a function of z defined implicitly by

$$w = z(1+w)^{\beta} \quad \text{and} \quad w(0) = 0, \tag{3.5}$$

and

$$\mathcal{L}_n(\alpha,\beta,\gamma) = (-1)^n \binom{\alpha-\gamma}{n} \binom{n+\gamma/\beta}{n}^{-1}. \tag{3.6}$$

The formula (3.4) follows readily on using the binomial expansion [12, p. 355, Equation (5)] and Gould's expansion formula [3, p. 196] (see also [12, p. 356, Equation (13)]).

Now we set the parametric sequences as

$$A_{\lambda,n} = \Omega_n(\lambda,r,a,b,p) = \frac{p(\lambda+rn)}{p+bn}\binom{a+bn}{n} \qquad (n \in \mathbb{N}_0) \tag{3.7}$$

and

$$B_{\mu,n} = \Delta_n(\mu,s,c,b,q) = \frac{q(\mu+sn)}{q+bn}\binom{c+bn}{n} \qquad (n \in \mathbb{N}_0). \tag{3.8}$$

Then, in view of (3.4), on finding the unknown functions $f_1(\lambda,z)$, $g(\mu,z)$, and $h(z)$, and carrying out elementary simplification after the computation of derivatives, Theorem 1 is finally seen to lead to the following result (which is believed to be new):

$$\begin{aligned}
&\sum_{k=0}^{n} \Omega_k(\lambda,r,a,b,p)\, \Delta_{n-k}(\mu,s,c,b,q) \\
&\quad = M_1 \binom{P+1}{n} \sum_{j=0}^{n} \binom{n}{j} \frac{j!}{(2+P-n)_j} (b-1)^j + M_2 \binom{P}{n} \sum_{j=0}^{n} \binom{n}{j} \\
&\qquad \cdot \binom{P}{j}^{-1} \mathcal{L}_j(a,b,p) + M_3 \binom{P}{n} \sum_{j=0}^{n} \binom{P}{j}^{-1} L_j(c,b,q) \\
&\qquad + M_4 \binom{P-1}{n} \sum_{i,j=0}^{i+j=n} \mathcal{L}_i(a,b,p)\, \mathcal{L}_j(c,b,q) \binom{n}{i+j} \left\{ 1 + \frac{(b-1)(i+j-n)}{P-n} \right\},
\end{aligned} \tag{3.9}$$

where $\mathcal{L}_i(a,b,p)$ and $\mathcal{L}_j(c,b,q)$ are given by (3.6), and

$$\begin{cases} P = a + c + bn \\ M_1 = pqrs/b^2 \\ M_2 = \frac{qs}{b}\left(\lambda - \frac{pr}{b}\right) \\ M_3 = \frac{pr}{b}\left(\mu - \frac{qs}{b}\right) \\ M_4 = \left(\lambda - \frac{pr}{b}\right)\left(\mu - \frac{qs}{b}\right). \end{cases} \tag{3.10}$$

Two special cases of (3.9) are worth mentioning here. For $\lambda = a$ and $\mu = c$, and setting $r = s = 0$ and $\mu = b = 1$, Rothe's identity for binomial coefficients emerges from (3.9) which was rederived independently, by a markedly different line of approach, by Lee [5, p. 335] and also by Nasir and Koh [6]. On the other hand, when $\mu = 1$, $s = 0$, $p = a$, and $q = c$, (3.9) gives us the series identity:

$$\begin{aligned} \sum_{k=0}^{n} \frac{a(\lambda + rk)}{a + bk}\binom{a+bk}{k}\frac{c}{c + b(n-k)}\binom{c + b(n-k)}{n-k} \\ = \frac{\lambda(a+c) + arn}{a + c + bn}\binom{a+c+bn}{n}, \end{aligned} \tag{3.11}$$

which is precisely the same result as stated in [7, p. 169]. It may be observed that the summation identity considered in [5, p. 338] is contained in (3.11), and follows from it by merely putting $\lambda = b = 1$ and $r = 1/a$. It may further be observed that several other useful identities due to Nanjundiah, Krall, Andrews, Euyang, and Hsu (see [2] and [4]) are deducible from (3.9) by suitable specialization of parameters.

4. A Variation of Theorem 1

We now establish another result analogous to (1.4) involving certain sequences of functions. If $f(z)$ and $z^{-1}g(z)$ are analytic in the neighbourhood of the origin such that $f(0) \neq 0$ and $g'(0) \neq 0$, then, for the sequence of functions $\{F_n^{(\alpha)}(x)\}$ defined by

$$\sum_{n=0}^{\infty} F_n^{(\alpha)}(x)\frac{t^n}{n!} = \{f(z)\}^{\alpha} \exp(x\, g(z)), \tag{4.1}$$

it is known that [10, p. 472]

$$\sum_{n=0}^{\infty} F_n^{(\alpha+\lambda n)}(x+ny)\frac{t^n}{n!} = \frac{\{f(w)\}^{\alpha}\exp\bigl(x\,g(w)\bigr)}{1-w\{\lambda[f'(w)/f(w)]+y\,g'(w)\}}, \tag{4.2}$$

where w is given by

$$w = t\{f(w)\}^{\lambda}\exp\bigl(y\,g(w)\bigr). \tag{4.3}$$

Proceeding on lines similar to those detailed in the derivation of Theorem 1, we can establish the following result:

Theorem 2. *Corresponding to the sequence of functions* $\{F_n^{\alpha}(x)\}$ *defined by* (4.1),

$$\begin{aligned}\sum_{k=0}^{n} F_k^{(\alpha+\lambda k)}(x+ky)\,F_{n-k}^{(\beta+\lambda(n-k))}\bigl(u+(n-k)\,y\bigr)\\ = \frac{1}{n!}\,D_z^n\left\{\frac{\{f(z)\}^{\alpha+\beta+\lambda n}\exp((x+u+ny)\,g(z))}{1-z\{\lambda[f'(z)/f(z)]+y\,g'(z)\}}\right\}\Bigg|_{z=0},\end{aligned} \tag{4.4}$$

where α, β, *and* λ *are arbitrary parameters, and the functions involved satisfy the usual differentiability conditions.*

Alternatively, the assertion (4.4) would also follow directly from (1.4) if we let

$$h(z) = \{f_1(p,z)\}^{\lambda}\exp\bigl(y\,g(q,z)\bigr),$$

and select the sequences $A_{p,n}$ and $B_{q,n}$ in accordance (and conformity) with the left-hand side of (4.4), and then apply (4.2) suitably.

The result (4.4) can find many applications which would yield various classes of addition formulas involving the classical orthogonal polynomials (or their generalizations). We illustrate it by considering the following example:

Recall Carlitz's generating-function relationship for the Laguerre polynomials [1, p. 525, Equation (5.5)]:

$$\sum_{n=0}^{\infty} L_n^{(a+bn)}(x+ny)\,t^n = \frac{(1+w)^{a+1}\exp(-xw)}{1-w[b-y(1+w)]}, \tag{4.5}$$

where w is given by

$$w = t(1+w)^{b+1}\exp(-yw). \tag{4.6}$$

With a similar generating function as given by (4.5) with a replaced by c, and x by u, the application of Theorem 2 in conjunction with (4.6) then yields

$$\begin{aligned}\sum_{k=0}^{n} L_k^{(a+bk)}(x+ky)\, L_{n-k}^{(c+b(n-k))}(u+(n-k)\,y) \\ = \frac{1}{n!}\, D_z^n \Big\{ (1+z)^{a+c+n(b+1)+1} \exp\big(-z(u+x+ny)\big) \\ \cdot \big(1+z(y-b)+yz^2\big)^{-1} \Big\}\Big|_{z=0}.\end{aligned} \tag{4.7}$$

Now, expressing the three functions on the right-hand side by their series expansions, the second member of (4.7) is seen to equal

$$\begin{aligned}\frac{1}{n!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \binom{\lambda}{i} (-1)^{j+r} \\ \cdot \frac{\mu^j (y-b)^r}{j!}\, D_z^n \left\{ z^{i+j+r} \left(1+\frac{yz}{y-b}\right)^r \right\}\Big|_{z=0},\end{aligned} \tag{4.8}$$

where

$$\begin{cases} \lambda = a+c+n(b+1)+1, \\ \mu = x+u+ny. \end{cases} \tag{4.9}$$

Upon simplifying the expression (4.8), we find from (4.7) that

$$\begin{aligned}\sum_{k=0}^{n} L_k^{(a+bk)}(x+ky)\, L_{n-k}^{(c+b(n-k))}(u+(n-k)\,y) \\ = \sum_{k=0}^{n} L_{n-k}^{(\lambda-n+k)}(\mu)\,(b-y)^k,\end{aligned} \tag{4.10}$$

where λ and μ are given by (4.9).

If we put $b=-1$ and $y=0$, replace c by $c+n$, and make use of the known identity:

$$\sum_{k=0}^{n} \binom{\lambda}{k} L_{n-k}^{(\alpha+k)}(x) = L_n^{(\alpha-\lambda)}(x), \tag{4.11}$$

we obtain the following addition formula [11, p. 444, Equation (5.12)]:

$$\sum_{k=0}^{n} L_k^{(a-k)}(x)\, L_{n-k}^{(c+k)}(u) = L_n^{(a+c)}(x+u), \tag{4.12}$$

which, as remarked by Srivastava *et al.* [11, p. 444], can be derived *directly* from a familiar generating function for the *modified* Laguerre polynomials $L_n^{(\alpha-n)}(x)$.

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