A compactness theorem for Hamilton circles in infinite graphs.

by

Daryl Funk
B.Sc., Simon Fraser University, 1992
M.Ed., University of Calgary, 2002

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

A compactness theorem for Hamilton circles in infinite graphs.

by

Daryl Funk
B.Sc., Simon Fraser University, 1992
M.Ed., University of Calgary, 2002

Supervisory Committee

Dr. R. Brewster, Supervisor
(Department of Mathematics and Statistics)

Dr. G. MacGillivray, Departmental Member
(Department of Mathematics and Statistics)

Supervisory Committee

Dr. R. Brewster, Supervisor
(Department of Mathematics and Statistics)

Dr. G. MacGillivray, Departmental Member
(Department of Mathematics and Statistics)

ABSTRACT

The problem of defining cycles in infinite graphs has received much attention in the literature. Diestel and Kühn have proposed viewing a graph as 1-complex, and defining a topology on the point set of the graph together with its ends. In this setting, a *circle* in the graph is a homeomorph of the unit circle S^1 in this topological space. For locally finite graphs this setting appears to be natural, as many classical theorems on cycles in finite graphs extend to the infinite setting.

A *Hamilton circle* in a graph is a circle containing all the vertices of the graph. We exhibit a necessary and sufficient condition that a countable graph contain a Hamilton circle in terms of the existence of Hamilton cycles in an increasing sequence of finite graphs.

As corollaries, we obtain extensions to locally finite graphs of Zhan's theorem that all 7-connected line graphs are hamiltonian (confirming a conjecture of Georgakopoulos), and Ryjáček's theorem that all 7-connected claw-free graphs are hamiltonian. A third corollary of our main result is Georgakopoulos' theorem that the square of every two-connected locally finite graph contains a Hamilton circle (an extension of Fleischner's theorem that the square of every two-connected finite graph is Hamiltonian).

Contents

List of Figures

ACKNOWLEDGEMENTS

Thank-you to:

Richard Brewster,
for fearlessly shooting down crazy ideas, and gently encouraging crazy ideas.

The TRU Math & Stats Department,
for making me feel so much at home.

Gary MacGillivray,
for his openness, reasonableness, frankness, and friendliness.

The UVic Math & Stats Department,
for making me feel so much at home away from home.

Agelos Georgakopoulos,
for suggesting such an interesting problem.

I looked, and, behold, a new world! There stood before me,
visibly incorporate, all that I had before inferred, conjectured, dreamed,
of perfect Circular beauty.
A. Square

DEDICATION

to

Mom & Dad
thank-you for everything

Tierney Kristen
Natasha Aleah
the four most beautiful girls in the world!

&

my Michelle

$$\sum_{6\ 11\ 2005}^{\infty} \left\{ \mathcal{DJ} \cup \mathcal{MN} \right\}$$

I love you.

Chapter 1

Introduction

Cycles are foundational to the study of graphs. A *Hamilton* cycle in a graph is a cycle which contains all of its vertices; a graph containing such a cycle is said to be *hamiltonian*. The problem of finding a Hamilton cycle in a graph has been studied since at least 1857, when William Rowan Hamilton invented his Icosian Game, in which the goal is to find a Hamilton cycle in the graph of the dodecahedron.[1] The problem of finding a Hamilton cycle in a graph is in general a difficult one. Though Hamilton cycles have been much studied, there are not many natural sufficient conditions known which guarantee their existence. The following classical sufficiency results are perhaps some of the deepest known:

Theorem 1.1 (Tutte [12, 35]). *Every finite 4-connected planar graph is hamiltonian.*

Theorem 1.2 (Fleischner [12, 18]). *If G is finite and 2-connected, then G^2 is hamiltonian.*

Theorem 1.3 (Zhan [44]). *Every finite 7-connected line graph is hamiltonian.*

Theorem 1.4 (Ryjáček [31]). *Every finite 7-connected claw-free graph is hamiltonian.*

In this thesis we extend these last three results to locally finite infinite graphs.

1.1 Basic graph theoretic definitions

We follow [12] for graph theoretical definitions and notation. A graph $G = (V, E)$ is a set V of *vertices* and a set E of unordered pairs of distinct vertices, called *edges*. For

[1] Hamilton was able to sell his game to a London game dealer for 25 pounds, and the game was distributed throughout Europe [37].

a graph G, we write $V(G)$ for its vertex set, and $E(G)$ for its set of edges. A graph is *infinite* if its vertex set is, and likewise *countable* if its vertex set is. For two vertices x, y of G, if $e = \{x, y\} \in E$, we write $e = xy$, call x and y the *endvertices* of e, and say that x and y are *adjacent* or are *neighbours*. A vertex x is said to be *incident* with an edge e if $x \in e$. The *degree* of a vertex v is the cardinality of the set of edges incident with v. A graph is *locally finite* if each vertex has finite degree. The (graph theoretic) *neighbourhood* of a vertex x is the set $N_G(x) = \{y \in V : xy \in E\}$, denoted simply $N(x)$ if the graph G is clear from context.

A *walk* is a non-empty alternating sequence $v_0e_0v_1e_1 \cdots e_{k-1}v_k$ of vertices and edges such that $e_i = v_iv_{i+1}$ for all $i < k$. A walk with all edges distinct is a *trail*, and a walk in which all vertices are distinct is a *path*. We refer to a path by (one of the two) sequences of its vertices, and write $P = v_0v_1 \ldots v_k$; vertices v_0 and v_k are said to be *linked* by P, and are called the *terminal* vertices of P. If $P = v_0v_1 \ldots v_k$ is a path with $k \geq 2$, then the graph $C = (V(P), E(P) \cup \{v_kv_0\})$ is called a *cycle*. Note that paths and cycles are finite. An infinite graph of the form

$$V = \{x_0, x_1, x_2, \ldots\} \quad E = \{x_0x_1, x_1x_2, x_2x_3, \ldots\}$$

is called a *ray*, and a *double ray* is an infinite graph of the form

$$V = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\} \quad E = \{\ldots, x_{-2}x_{-1}, x_{-1}x_0, x_0x_1, x_1x_2, \ldots\},$$

where in both cases the x_i are distinct. For sets of vertices A and B, if $e = uv$ is an edge with $u \in A$ and $v \in B$, we say e is an *A-B edge*; if $P = x_0 \ldots x_k$ is a path with $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$, we call P an *A-B path.* If $A = B$ we call P an A-path. A graph G is *connected* if any two of its vertices are linked by a path in G; G is k-connected (for $k \in \mathbb{N}$) if $|V| > k$ and $G - S$ is connected for every set $S \subset V$ with $|S| < k$.

If $G = (V, E)$ and $H = (W, F)$ are graphs with $W \subseteq V$ and $F \subseteq E$, then we say H is a *subgraph* of G, that G *contains* H, and write $H \subseteq G$. For a subset $W \subseteq V$ of vertices G, the *induced subgraph* on W in G, denoted $G[W]$, is the graph on W whose edges are exactly the edges of G with both ends in W.

The *n-th power* G^n of a graph G is the graph on $V(G)$ in which two vertices are adjacent if and only if they have distance at most n in G. The *line graph* $L(G)$ of a graph G is the graph with $V(L(G)) = E(G)$ and in which $x, y \in E(G)$ are adjacent in

$L(G)$ if and only if x and y have a common endvertex in G. A graph is *claw-free* if it contains no copy of the complete bipartite graph $K_{1,3} = (\{w, x, y, z\}, \{wx, wy, wz\})$ as an induced subgraph.

1.2 How to define an infinite Hamilton cycle (informal discussion)

In the context of infinite graphs, two natural questions to ask are:

Question 1. Is there a reasonable infinite analogue of the concept of hamiltonicity? and if so,

Question 2. Which infinite graphs are hamiltonian?

Following the lead of Erdős, Grünwald, and Vazsonyiet, Nash-Williams proposed spanning rays and spanning double rays as an appropriate answer to Question 1 [30]. (Erdős et. al. characterized the infinite graphs containing a one-way infinite Euler trail, and those containing a two-way infinite Euler trail [30].)

Consider for a moment just how differently infinite graphs may behave than their finite counterparts. Properties of finite graphs often do not carry through in a straightforward way to infinite graphs. For example, perhaps as foundational a theorem about cycles as there is,

Theorem (Euler, 1736)**.** *A connected graph has an Euler tour if and only if every vertex has even degree.*

is false for infinite graphs. The graph in Figure 1.1 has all vertices of even degree, but has no Euler tour or infinite Euler trail. On the other hand, the graph in Figure 1.2 is both Eularian in this sense of Erdős et. al., and hamiltonian in the sense of Nash-Williams, containing the two-way infinite Euler trail

$$\ldots u_{-2}v_{-2}v_{-1}u_{-2}u_{-1}v_{-1}v_0u_{-1}u_0v_0v_1u_0u_1v_1v_2u_1u_2v_2v_3u_2u_3\ldots$$

and spanning double ray

$$\ldots v_{-1}u_{-1}v_0u_0v_1u_1\ldots.$$

From this perspective, the graph in Figure 1.1 is neither Eularian nor hamiltonian, as a double ray can head off in only two directions at once, and here there are four. The

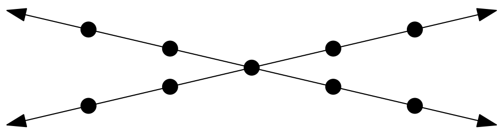

Figure 1.1: A graph of all even degree, but which has neither an Euler tour nor an infinite Euler trail.

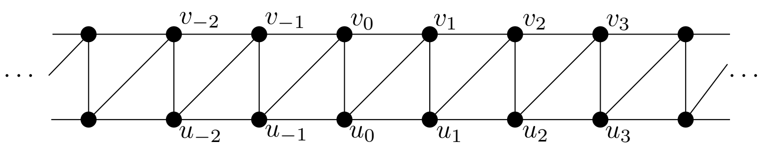

Figure 1.2: Deleting a finite set of vertices always leaves at most two infinite components.

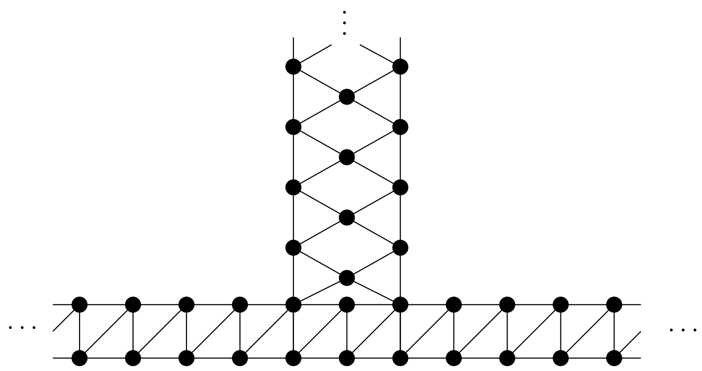

Figure 1.3: Deleting a finite set of vertices may result in three infinite components.

same problem occurs in the graph in Figure 1.3, which is exhibited by Nash-Williams in [30].

Nash-Williams observed that a necessary condition for a graph to have an infinite Euler trail, a spanning ray, or a spanning double ray, is that the deletion of any finite set of edges or vertices cannot result in more than two infinite components. Thus the graph in Figure 1.3 is not hamiltonian or Eularian in the sense of infinite spanning paths or trails, even though it is an edge-disjoint union of two Eularian subgraphs. For finite Eularian graphs, of course, this cannot happen.

Nash-Williams therefore conjectured that Theorem 1.1 holds for infinite graphs with the property that deleting a finite $S \subset V$ never leaves more than two infinite components, where we call an infinite graph hamiltonian if it contains a spanning ray (when no more than one infinite component may be left) or a spanning double ray (when no more then two infinite components may be left) [30]. Both these conjectures have now in fact been proved: the first by Dean, Thomas and Yu [8], and the second more recently by Yu [39, 40, 41, 42, 43].

We now turn our attention to developing the concept of an infinite Hamilton cycle. Consider again just the bottom subgraph of Nash-William's example, the graph in Figure 1.2. For any $n \in \mathbb{N}$, the finite induced subgraph on vertices $u_{-n}, \ldots, u_0, \ldots, u_n$ and $v_{-n}, \ldots, v_0, \ldots, v_n$ admits Hamilton cycle

$$u_{-n} \ldots u_0 \ldots u_n v_n \ldots v_0 \ldots v_{-n} u_{-n}.$$

Consider for a moment the Real Line $(-\infty, \infty)$. For any $n \in \mathbb{N}$, the removal of closed interval $[-n, n]$ results in two infinite open intervals, $(-\infty, n)$ and (n, ∞). Informally, we may compactify the Real Line by adding a point at infinity ∞, corresponding to the infinite sequence of nested intervals $\{(n,\infty)\}_{n\in\mathbb{N}}$, and a point $-\infty$, corresponding to the infinite sequence of nested intervals $\{(-\infty,-n)\}_{n\in\mathbb{N}}$. Similarly, removing the finite set of vertices $\{u_{-n},\ldots,u_0,\ldots,u_n\}\cup\{v_{-n},\ldots,v_0,\ldots,v_n\}$ for each $n\in\mathbb{N}$ from the graph in Figure 1.2, yields two infinite sequences of nested infinite components. Following such a topological approach allows a similar compactification for graphs.

In fact, such an approach allows a natural answer to Question 1. (Precise definitions are given in Chapter 2.) Informally, we add a point "at infinity", which we call an *end*, for each nested sequence of infinite components left behind by the deletion of finite sets of vertices. We indicate these points by isolated dots, ω,ω', as in Figure 1.4. Now the upper and lower rays heading to the right from v_0 and u_0 respectively

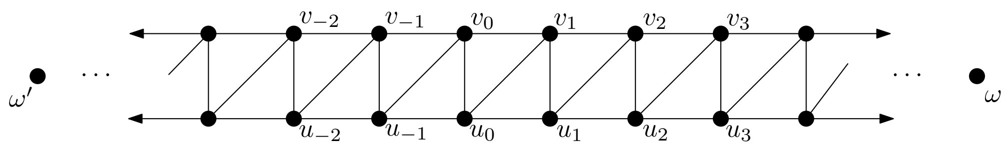

Figure 1.4: An infinite graph with its "points at infinity".

may be thought of as converging to ω, and similarly the upper and lower rays heading left, as converging to ω'. In a sense which will be made precise in Chapter 2, we now have a *Hamilton circle*

$$v_0v_{-1}v_{-2}\cdots\omega'\cdots u_{-2}u_{-1}u_0u_1u_2\cdots\omega\cdots v_2v_1v_0$$

consisting of the upper and lower double rays together with ends ω and ω'.

One advantage of such an approach is that we retain the sense of a Hamilton cycle, which returns to the point at which it began. (In Chapter 2 we will see that a circle in a graph, in a precise way, really *is* a circle). Another is that graphs which have more than two infinite components after the removal of some finite $S\subset V$, now have the possibility of being hamiltonian, in a natural way. Consider again Nash-William's example of Figure 1.3, but let us add the graph's ends as in Figure 1.5. Together with its ends, the graph contains a Hamilton circle, shown by the bold edges in Figure 1.5. This circle is a union of rays, pairs of which converge to a common end, one for each of the three infinite components left by the deletion of any large enough finite set of vertices. In fact, it is even possible that a graph with uncountably many ends may contain a Hamilton circle. We will demonstrate this in Section 2.4 for the graph shown in Figure 2.2 on page 25.

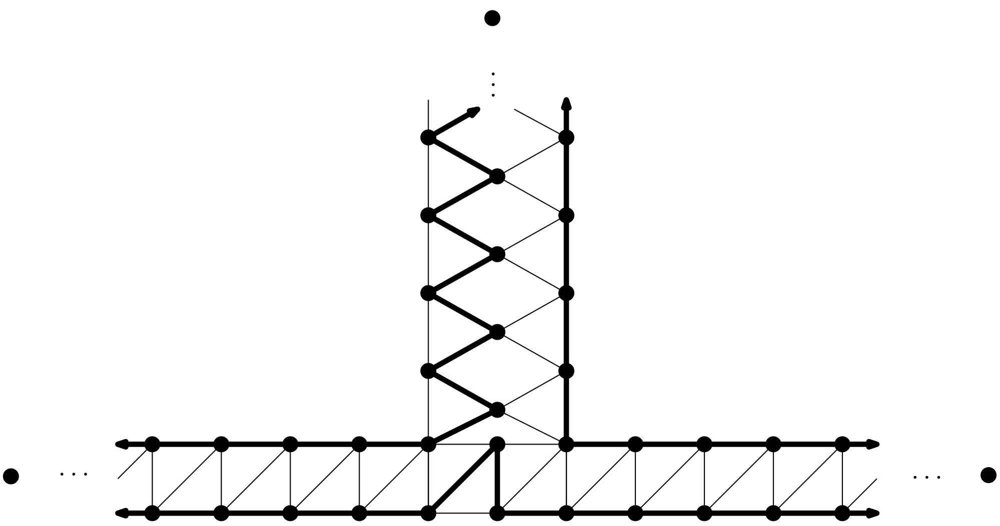

Figure 1.5: A Hamilton circle (bold) in a graph with three ends.

As will be made clear by our main result (Theorem 1.7 below) and its corollaries, this approach also goes some way to providing some answers to Question 2.

These concepts were introduced by Diestel and Kühn [15, 16, 17] as part of an ambitious project whose goal is a natural extension to infinite graphs of the foundational concepts and properties of the cycle space of a finite graph. This project has seen considerable success. We see our results as providing additional evidence that these concepts are, in some sense, the "right" ones for infinite graphs.

1.3 Previous hamiltonicity results for infinite graphs

In the spirit of Nash-Williams' concept of spanning rays and double rays as infinite Hamilton paths, Thomassen has generalized Fleischner's Theorem 1.2 on the hamiltonicity of the square of a graph:

Theorem (Thomassen [33])**.** *If G is a 2-connected locally finite 1-ended graph, then G^2 contains both a spanning ray and a spanning double ray.*

In a similar spirit, to deal with infinite graphs with more than one or two ends, Halin [26] defined a notion of a *Hamilton tree* — a spanning tree T which is either a spanning ray or a tree with no leaves, such that for any two disjoint rays P_1 and P_2 in T, there is a finite set $S \subset V(G)$ such that $G - S$ has no P_1-P_2 path.

Theorem (Halin [26])**.** *If G is a connected locally finite graph, then G^3 contains a Hamilton tree.*

There have also been some successful extensions of finite hamiltonicity results to infinite graphs using the Diestel-Kühn approach. Georgakopoulos has recently extended to locally finite graphs two finite results on powers of graphs [21]. These are Fleischner's Theorem 1.2, and the fact that the third power of any finite connected graph is hamiltonian ([28, 32]):

Theorem 1.5 ([21])**.** *The square of a locally finite 2-connected graph has a Hamilton circle.*

Theorem 1.6 ([21])**.** *The cube of a locally finite connected graph has a Hamilton circle.*

Secondly, as a partial extension of Tutte's Theorem 1.1, Bruhn and Yu [6] have shown

Theorem ([6]). *Every locally finite 6-connected planar graph with at most finitely many ends has a Hamilton circle.*

1.4 Main results

Our main result gives a necessary and sufficient condition for hamiltonicity of a countable graph G in terms of the hamiltonicity of a nested sequence of finite subgraphs. Informally (precise definitions will be given in Chapter 3), if $\{v_1, v_2, \ldots\}$ is an enumeration of $V(G)$, then for each positive integer n, we define a finite graph on $\{v_1, v_2, \ldots, v_n\}$, which we denote G_n^*. Each G_n^* contains the induced subgraph $G[\{v_1, v_2, \ldots, v_n\}]$, and as $n \to \infty$, $G_n^* \to G$. Our main result as stated for locally finite graphs is

Theorem 1.7. *Let G be a locally finite graph. Then G is hamiltonian if and only if there is a positive integer m such that for all $n \geq m$, G_n^* is hamiltonian.*

As a corollary to Theorem 1.7, we confirm a conjecture of Georgakopoulos (personal communication, 2008; also in the unpublished [23]) extending Zhan's Theorem 1.3 to locally finite graphs:

Corollary 1.8. *Every locally finite 7-connected line graph is hamiltonian.*

Furthermore, Ryjáček's Theorem 1.4 also extends to locally finite graphs as a consequence of Theorem 1.7:

Corollary 1.9. *Every locally finite 7-connected claw-free graph is hamiltonian.*

In addition, Georgakopoulos' Theorems 1.5 and 1.6 are obtained as corollaries to our Theorem 1.7. This provides a shorter proof for Theorem 1.5 than that given in [21].

Ryjáček [31] has shown that a well-know conjecture of Thomassen

Conjecture 1.10 ([34]). *Every finite 4-connected line graph is hamiltonian.*

and a conjecture of Matthews and Sumner (of which Thomassen's conjecture is a special case, since every line graph is claw-free)

Conjecture 1.11 ([29]). *Every finite 4-connected claw-free graph is hamiltonian.*

are equivalent. Our main result also has as a consequence that these conjectures are also equivalent for locally finite graphs.

We also prove a version of Theorem 1.7 (Theorem 4.3) for arbitrary countable graphs.

The remainder of this thesis is organized as follows. Chapter 2 develops the concepts required in order to state and prove Theorem 1.7, our main result for locally finite graphs. The proof of Theorem 1.7 is given in Chapter 3. In Chapter 4 we extend Theorem 1.7 to arbitrary countable graphs. In Chapter 5 we prove several corollaries of Theorem 1.7. These include Corollaries 1.8 and 1.9, as well as Theorems 1.5 and 1.6.

Chapter 2

Definitions and basic facts

In this chapter, we provide the basic concepts and tools required in order to state and prove Theorem 1.7.

2.1 Ends of a graph

The concept of an end of a graph $G = (V, E)$ was introduced by Halin in [25], as an equivalence class of rays. Subrays of a ray or double ray are called *tails*. Every ray has infinitely many tails; any two tails of the same ray differ only on a finite initial segment. An equivalence relation on the set of rays of G is defined in which two rays R and R' are *equivalent* if for every finite $S \subset V$, both R and R' have a tail in the same component of $G - S$. The *ends* of G are the equivalence classes under this relation. We denote the set of ends of G by $\Omega = \Omega(G)$, and write $G = (V, E, \Omega)$ for the graph with vertex set V, edge set E, and end set Ω.

It is useful to observe that two rays are in the same end if and only if they can be linked by infinitely many disjoint paths (some of which may be trivial; in particular, if one is a tail of the other all such paths are trivial). To see this, suppose R and Q have a tail in the same component of $G-S$ for some finite $S \subset V$. Then R and Q are linked by at least one path, say P_1 in $G-S$. Set $S_1 = S \cup V(P_1)$; since R and Q both have a tail in the same component of $G - S_1$, there is similarly a path P_2 linking R and Q in $G - S_1$, which is disjoint from P_1. Set $S_2 = S_1 \cup V(P_2)$. Continuing in this manner, we inductively construct an infinite set of disjoint paths $\{P_i : i \in \mathbb{N}\}$ linking R and Q. Conversely, if there is a collection of infinitely many disjoint paths linking R and Q, then no finite $S \subset V$ may separate R and Q in $G - S$, so R and Q are in the same end.

2.2 Topological tools

We are going to define a topology on a graph $G = (V, E, \Omega)$ in Section 2.2.3. Before doing so, however, we need some basic concepts from topology and geometry.

2.2.1 Basic topological definitions

A *topology* on a set X is a collection τ of subsets of X, such that

1. any union of elements of τ belongs to τ,
2. any finite intersection of elements of τ belongs to τ, and
3. $\emptyset$ and X belong to τ.

The subsets of X contained in τ are called the *open* sets of X. We call (X, τ), or simply X when there is no confusion about τ, a *topological space*.

If $x \in X$, a *neighbourhood* of x is a set U which contains an open set containing x. The collection $\mathscr{U}_x$ of all neighbourhoods of x is called the *neighbourhood system* at x. A *neighbourhood base* at x is a subcollection $\mathscr{B}_x \subset \mathscr{U}_x$ such that

$$\mathscr{U}_x = \{U \subset X : B \subset U \text{ for some } B \in \mathscr{B}_x\} .$$

The elements of a chosen neighbourhood base are called *basic neighbourhoods*. Since the open neighbourhoods of x form a neighbourhood base at x, there is no loss of generality if we refer only to basic open neighbourhoods containing x. If (X, τ) is a topological space, a *base* for τ (or a *base* for X when there is no confusion about τ) is a collection $\mathscr{B} \subset \tau$ such that

$$\tau = \left\{ \bigcup_{B \in \mathscr{D}} B : \mathscr{D} \subset \mathscr{B} \right\} .$$

A collection $\mathscr{B}$ of open sets in a set X is a base for X if and only if for each $x \in X$, $\mathscr{B}_x = \{B \in \mathscr{B} : x \in B\}$ is a neighbourhood base at x ([38], Theorem 5.4). Hence we may specify a topology on X by specifying a collection of open sets for each point $x \in X$. These sets are then called the *basic open sets around* a point $x \in X$.

A subset $A \subseteq X$ is *closed* if and only if its complement $X \setminus A$ is open. If X is a topological space and $W \subseteq X$, the *closure* of W in X is the set denoted $\overline{W}$ defined by

$$\overline{W} = \bigcap \{A \subseteq X : A \text{ is closed and } W \subseteq A\}\,.$$

Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* at $x \in X$ if and only if for each neighbourhood N of $f(x)$ in Y there is a neighbourhood U of x in X such that $f(U) \subset N$. The function f is *continuous on* X if and only if f is continuous at each $x \in X$. A bijection $f : X \to Y$ such that both f and f^{-1} are continuous is a *homeomorphism* of X to Y; we say X and Y are *homeomorphic*. In the case that $f : X \to Y$ is a continuous injection and $f^{-1} : f(X) \to X$ is also continuous, i.e. f is a homeomorphism of X to $f(X)$, f is called an *embedding* of X into Y; we say X is *embedded* in Y by f.

A *path* in X is a continuous image of the real unit interval $[0, 1]$ in X. The images of 0 and 1 are the *endpoints* of the path. An *arc* in X is an embedding of the unit interval in X. We say an arc *links* its endpoints, the images of 0 and 1 under the arc. We denote the set of all inner points of an arc A by $\mathring{A}$, i.e. if A links s and t, $\mathring{A} = A \setminus \{s, t\}$. A *loop* in a topological space X is a continuous image of the unit circle $S^1 \subset \mathbb{R}^2$ in X. A *circle* in X is an embedding of S^1 into X. For an arc A defined by $\alpha : [0, 1] \to X$, and a circle C defined by $\sigma : S^1 \to X$, we call both the embedding α and its image $A = \alpha([0, 1])$ an arc in X; similarly, we call both the embedding σ and its image $C = \sigma(S^1)$ a circle in X.

An *orientation* of an arc A is the linear order defined on its points induced by a homeomorphism $h : [0, 1] \to A$. This is the ordering given by, for $a, b \in A$, $a < b$ if $h^{-1}(a) < h^{-1}(b)$ in $[0, 1]$. Given an oriented arc A, we write aA for the oriented subarc of A consisting of all points $b \in A$ such that $a \leq b$, Aa for the oriented subarc of A consisting of all points $b \in A$ such that $b \leq a$, and aAb for the oriented subarc of A consisting of all points $c \in A$ such that $a \leq c \leq b$. An *orientation* of a circle σ is a choice of one of the two orientations of every subarc $A \subset \sigma$ such that all these orientations are compatible on their intersections. Given an oriented circle σ, and $a, b \in \sigma$, $a \neq b$, we write $a\sigma b$ for the oriented subarc of σ consisting of all points $c \in \sigma$ with $a \leq c \leq b$.

For all other topological definitions and notation we follow [38].

2.2.2 Complexes

We say that k points in Euclidean n-space $\mathbb{R}^n$ are in *general position* if any proper subset of them spans a strictly smaller hyperplane. A subset $C \subseteq \mathbb{R}^n$ is *convex* if for all $x, y \in C$ and all $t \in [0,1]$, the point $tx + (1-t)y$ is also in C (i.e. every point on the line segment connecting x and y is in C). Given a set of points X in $\mathbb{R}^n$, we call the minimal convex set containing X the *convex hull* of X. (The following definitions are from [1].)

Definition. *A* simplex of dimension k *(or a* k-simplex*) is the convex hull of a set of* $(k+1)$ *points in general position in* $\mathbb{R}^n$*, for some* $n \geq k$*. The convex hull of any nonempty subset of the* $k+1$ *defining points of an* k*-simplex is called a* face *of the simplex; if such a subset has size* $m+1$ *it is called an* m*-face.*

For example, a point on the real line is a 0-simplex, a line segment in $\mathbb{R}^n$ ($n \geq 1$) is a 1-simplex, a triangle in $\mathbb{R}^n$ (all interior points included, $n \geq 2$) is a 2-simplex, and a tetrahedron in $\mathbb{R}^n$ (all interior points included, $n \geq 3$) is a 3-simplex. The terminology here coincides with that of graph theory: the defining points of a k-simplex, as sets of size 1, are 0-faces, and are called the *vertices* of the simplex, and the 1-faces are called *edges* of the simplex.

Definition. *A* simplicial complex $\mathcal{K}$ *is a collection of simplices in* $\mathbb{R}^n$ *which satisfies the following conditions:*

1. *every face of a simplex in* $\mathcal{K}$ *is also in* $\mathcal{K}$*, and*
2. *the intersection of any two simplices* $K_1, K_2 \in \mathcal{K}$ *is a face of both* K_1 *and* K_2*.*

If the largest dimension of a simplex in $\mathcal{K}$ *is* k*, then we say* $\mathcal{K}$ *is a* simplicial k-complex*, or, simply a* k-complex*.*

Let each simplex in $\mathcal{K}$ carry its topology from the subspace topology it receives from $\mathbb{R}^n$. For points x contained in a face F which is the intersection of n simplices $K_1, K_2, \ldots, K_n$, take as basic open sets the unions of the basic open sets containing x in each of $K_1, K_2, \ldots, K_n$. We also allow the possibility that there may be a face F which is the intersection of infinitely many simplices, $K_1, K_2, \ldots$; a basic open set around a point $x \in F$ is then an infinite union of basic open sets containing x in each of $K_1, K_2, \ldots$. A complex $\mathcal{K}$, when regarded in this way as a topological space, is denoted $|\mathcal{K}|$.

2.2.3 Defining a topology on a graph together with its ends

We first of all consider a graph $G = (V, E)$ as a 1-complex. Hence each edge with its incident vertices becomes a 1-simplex, homeomorphic to the real unit interval $[0, 1]$, and the interiors of edges do not intersect; edges may meet only at their endvertices. We define a topology on the point set of the 1-complex G together with the set of ends $\Omega(G)$ by specifying a collection of basic open sets for each of these points.

The basic open sets around points $x \in V \cup E$ are those of G as a 1-complex. For edge $e = uv$ we write $\mathring{e} = (u, v)$ for the set of its *inner points* on the line segment between u and v, and $e = [u, v] = \{u\} \cup (u, v) \cup \{v\}$ for the 1-simplex e, its inner points together with its incident vertices. The basic open sets for inner points of an edge are just the open intervals containing it on the edge. Since for each edge $e = [u, v] \in G$ (the 1-complex), there is a homeomorphism $h_e : [0, 1] \to [u, v]$, we may think of the basic open sets around inner points of an edge as corresponding to open intervals in $(0, 1)$, from which the edge receives the usual metric and topology. A basic open set around a vertex v is a union of half-open intervals $[v, z_e)$, with z_e an inner point of e for each edge e at v.

We will freely switch between viewing our objects of study as graphs and subgraphs or as topological spaces and subspaces, depending on which is more convenient. If $F \subseteq E$ we denote by $\mathring{F}$ the set of all inner points of edges in F, that is, $\mathring{F} = \bigcup\{\mathring{e} : e \in F\}$. When referring to a subgraph $H \subseteq G$, we will also mean the corresponding point set $V(H) \cup \mathring{E}(H)$ of H.

This topology is extended to the set of ends Ω of a graph G by defining for each end $\omega \in \Omega$ a collection of basic open sets as follows. For every finite set $S \subset V$, let $C_G(S, \omega)$, or just $C(S, \omega)$ if G is clear from context, denote the unique component of $G - S$ which contains a ray in ω (so $C(S, \omega)$ contains a tail of every ray in ω). We say ω *belongs* to $C_G(S, \omega)$. Let $\Omega(S, \omega)$ denote the set of all ends of G with a ray in $C(S, \omega)$. Let $E'(S, \omega)$ denote a union of half-open intervals $[v, z)$, one for each S-$C(S, \omega)$ edge $e = vu$ of G, with $v \in C(S, \omega)$ and z an inner point of e. We take as our collection of basic open sets around ω all sets of the form

$$\widehat{C}_G(S, \omega) = C_G(S, \omega) \cup \Omega(S, \omega) \cup E'(S, \omega)$$

as S ranges over all finite subsets of V and E' over all inner points of all corresponding S-$C(S, \omega)$ edges. See Figure 2.1 for an example of some basic open sets around some ends in a graph.

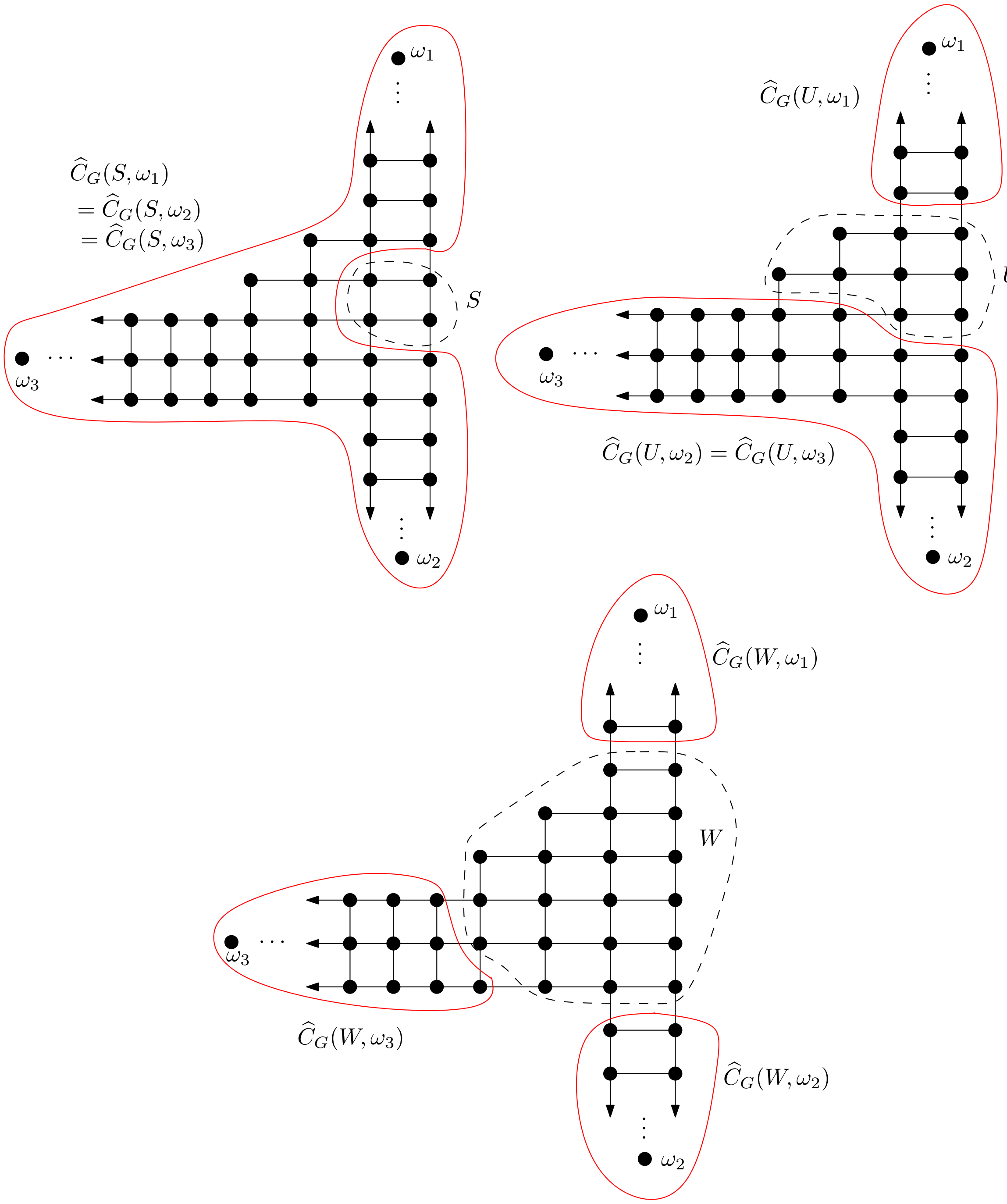

Figure 2.1: Finite subsets $S, U, W \subset V$ and resulting basic open neighbourhoods of ω_1, ω_2, and ω_3.

We denote by $|G|$ the topological space of the point set $V \cup \mathring{E} \cup \Omega$ endowed with this topology. When G is locally finite, this is the Freudenthal compactification of the 1-complex G [14]. Note that $|G|$ is Hausdorff (for any two distinct points $x, y \in |G|$ there are disjoint open sets $U, W \subset |G|$ with $x \in U$ and $y \in W$).

This is the topology used in [3, 4, 5, 9, 11, 15, 16, 17, 20, 21, 22, 24, 27], and denoted Top in [13]. It is often referred to as the "standard topology" in the literature.

Definition. *A* Hamilton circle *is a circle in* $|G|$ *which contains all the vertices of* $|G|$*. If* $|G|$ *contains a Hamilton circle we say* G *is* hamiltonian*.*

We will see that such a circle must contain all the ends of G as well.

2.3 Basic facts

The following simple lemma is a powerful tool we use throughout this thesis:

Lemma 2.1 (König's Infinity Lemma, [12])**.** *Let* $V_0, V_1, \ldots$ *be an infinite sequence of disjoint non-empty finite sets, and let* G *be a graph on their union. If every vertex* $v \in V_n$*,* $n \geq 1$ *has a neighbour* $f(v) \in V_{n-1}$*, then* G *has a ray* $v_0 v_1 \ldots$ *with* $v_n \in V_n$ *for all* n*.*

We will sometimes refer to König's Infinity Lemma as simply "the Infinity Lemma". Once a predecessor for each element of each V_n has been specified, a ray whose existence is then guaranteed by the Infinity Lemma, will be said to have been *returned by* the Infinity Lemma.

An *accumulation point* of a set X in a topological space Y is a point $y \in Y$ such that each basic neighbourhood of y contains some point of X other than y. A basic theorem of topology ([38], Theorem 4.10) tells us that the topological closure of a set is the set itself together with its accumulation points. Hence we note that for any graph $G = (V, E, \Omega)$, in $|G|$:

- $\overline{V} = V \cup \Omega$, since every neighbourhood of an end contains a vertex.
- $\mathring{E}$ contains no accumulation points of V.
- For any edge $e = [u, v]$, $\overline{\mathring{e}} = \overline{(u, v)} = e$; i.e. the closure of the inner points of an edge is the edge together with its endvertices.

- If $R \subseteq G$ is a ray belonging to end ω, then $\overline{R} = R \cup \omega$, since every neighbourhood of ω contains infinitely many vertices of R.
- For any finite $S \subseteq V$, $\overline{C(S,\omega)} = C(S,\omega) \cup \Omega(S,\omega)$, since $C(S,\omega)$ contains tails of every ray in every end in $\Omega(S,\omega)$.

It is a basic topological fact that if X is closed and $W \subseteq X$, then $\overline{W} \subseteq X$. Since the homeomorphic image of a closed set is closed ([38] Theorem 7.9), and S^1 is closed in $\mathbb{R}^2$, a circle is closed in $|G|$. Since a Hamilton circle σ in $|G|$ contains V, σ also contains $\overline{V} = V \cup \Omega$.

We also make use of the following important facts:

Lemma 2.2 ([1], Theorem 3.7)**.** *Every continuous injective map from a compact space to a Hausdorff space is a topological embedding.*

A family $\mathcal{A}$ of subsets of a set X has the *finite intersection property* if and only if the intersection of any finite subcollection from $\mathcal{A}$ is nonempty.

Proposition 2.3 ([38], Theorem 17.4)**.** *Let X be a topological space. Then X is compact if and only if every family of closed subsets of X with the finite intersection property has nonempty intersection.*

2.3.1 Relationships between G and $|G|$

The following lemmas are meant to exhibit the nature of the correspondence between a graph G and its space $|G|$, and to enable us to freely move between graph theoretical walks, paths, rays, and double rays in G and their corresponding topological paths, arcs, and circles in $|G|$. They also assure us that circles in $|G|$ uniquely correspond to cycles or unions of double rays in G.

Recall that a topological space X is *pathwise connected* if and only if for any two points $x, y \in X$, there is a path $f : [0,1] \to X$ with $f(0) = x$ and $f(1) = y$. A space is *locally pathwise connected* if and only if each point has a neighbourhood base consisting of pathwise connected sets. Since the basic open sets defining $|G|$ are locally pathwise connected, $|G|$ is locally pathwise connected. If G is (graph theoretically) connected, then $|G|$ is certainly (topologically) connected (there do not exist two disjoint non-empty open subsets whose union is $|G|$). Since a connected, locally pathwise connected space is pathwise connected ([38], Theorem 27.5), we have:

Proposition 2.4. *1. If G is connected, $|G|$ is pathwise connected.*

2. *Every open topologically connected subset of $|G|$ is pathwise connected.*

A graph theoretical walk in a graph G corresponds naturally to a topological path in $|G|$, while a graph theoretical path naturally defines an arc in $|G|$. We will often implicitly make use of the following fact.

Lemma 2.5 ([38], Corollary 31.6). *A topological path in a Hausdorff space with distinct endpoints x and y contains an arc linking x and y.*

So just as every walk in a graph contains a (graph theoretical) path, every (topological) path in $|G|$ contains a (topological) arc.

Finally, we have the following lemma which assures us that an arbitrary circle in $|G|$ does in fact correspond uniquely to either a finite cycle or a disjoint union of double rays in G. Similarly, an arc in $|G|$ uniquely determines either a finite path or a disjoint union of rays and double rays in G.

Proposition 2.6 ([15], [16]). *For any arc α with endpoints $x, y \in V \cup \Omega$, and any circle σ in $|G|$,*

1. *α (respectively σ) includes every edge of G of which it contains an inner point;*
2. *the (point) sets $\alpha \cap G$ and $\sigma \cap G$ are dense in α and σ, respectively;*
3. *if v is a vertex in $\alpha \setminus \{x, y\}$ (respectively in σ), then α (respectively σ) contains exactly two edges at v (which in G are incident with v).*

Hence every arc α (and every circle σ) has a well-defined set of edges $E(\alpha)$ (respectively, $E(\sigma)$), and $\overline{E(\alpha)} = \alpha$ (and likewise $\overline{E(\sigma)} = \sigma$). In other words, the topological closure of the edge set of a circle in $|G|$ is the circle. Thus a circle is uniquely determined by its edges, and a circle has a uniquely determined edge set; similarly for arcs. Moreover, every end ω contained in a circle σ contains exactly two internally disjoint arcs which meet at ω. In G, these arcs define a union of rays or double rays contained in $\sigma \cap G$ which converge to ω. Similarly this is the case for every end contained as an inner point of an arc.

Hence an end $\omega \in \Omega$ of G is a point in $|G|$, which may be an endpoint or inner point of a topological arc or circle in $|G|$, just as vertices may. (While there is no reason that an inner point of an edge may not also be an endpoint of an arc in $|G|$, we avoid this, so that an arc in $|G|$ corresponds to either a finite path or a union of rays in G.)

If P is a path in G and vertices $x, y \in P$, we denote both the path in G from x to y on P by xPy and the topological (oriented) arc in $|G|$ linking x and y via P by xPy. Similarly, if C is a circle in $|G|$ for which an orientation has been chosen, and x, y are vertices contained in C, we write xCy for the oriented arc contained in C linking x and y. This also defines either a finite path or a union of rays in G.

2.3.2 Some important properties of $|G|$

We close this section with some important facts about when the space $|G|$, or its subspaces $V \cup \Omega$ and Ω, are compact.

Proposition 2.7 ([12], Proposition 8.5.1)**.** *If G is connected and locally finite, then $|G|$ is compact.*

Under certain conditions, $V \cup \Omega$ or Ω may be compact subsets of $|G|$, even if $|G|$ is not compact. First, we require some basic concepts about spanning trees.

A partial order can be defined on the vertices of a (finite or infinite) tree T by fixing a specified vertex r, called the *root* of T; T is then called a *rooted tree.* The *tree-order* associated with T and r is defined by $x \leq y$ if and only if $x \in rTy$ (*i.e.*, x is on the unique path in T from r to y). The set $\lceil y \rceil = \{x : x \leq y\}$ is called the *down-closure* of y. A rooted spanning tree $T \subseteq G$ is called *normal* in G if the endvertices of every edge in G are comparable in the tree-order of T. (For finite graphs, depth-first search trees are normal.)

Lemma 2.8 ([12], Lemma 1.5.5.)**.** *Let T be a normal spanning tree in a graph G. Any two vertices $x, y \in T$ are separated in G by the set $\lceil x \rceil \cap \lceil y \rceil$.*

A ray in a normal spanning tree which begins at the root of the tree is called a *normal ray.*

Lemma 2.9 ([12], Lemma 8.2.3)**.** *If T is a normal spanning tree of a graph G, then every end of G contains exactly one normal ray of T.*

Proposition 2.10 ([12], Theorem 8.2.4)**.** *Every countable connected graph has a normal spanning tree.*

Equipped with these tools, we may now characterize the graphs for which $V \cup \Omega$ and Ω are compact subsets of $|G|$.

Proposition 2.11. *Let $G = (V, E, \Omega)$ be a countable graph containing a vertex of infinite degree. Then*

1. *$|G|$ is not compact.*

2. *$V \cup \Omega$ is a compact subset of $|G|$ if and only if for every finite $S \subset V$, there are only finitely many components of $G - S$.*

3. *Ω is a compact subset of $|G|$ if and only if for every finite $S \subset V$, only finitely many components of $G - S$ contain a ray.*

(We collect these facts here and provide proofs for convenience: (1) is simply an easy observation; (2), though surely a well-known fact, is not explicitly stated in the literature in this form for spaces $|G|$; (3) is stated without proof in [12].)

Proof. (1) Consider an open cover in which each edge e_i incident with a vertex of infinite degree contains a point covered by exactly one open set $O_i \subset \mathring{e}_i$ of the cover. Such a cover has no finite subcover.

(2) ($\Longrightarrow$) Suppose G has a finite $S \subset V$ such that $G - S$ has infinitely many components. For every vertex $v \in G - S$, let $C(S, v)$ denote the component of $G - S$ containing v. Consider an open cover in which each component contains a vertex v_i covered by exactly one open set $O_i \subseteq C(S, v_i)$. Such a cover has no finite subcover.

($\Longleftarrow$) Without loss of generality we may assume G is connected, since otherwise we may apply the argument to each component of G (by assumption there can be only finitely many of them). Let $\mathcal{O}$ be an open cover of $V \cup \Omega$. We show that $\mathcal{O}$ has a finite subcover. Let T be a normal spanning tree of G (by Proposition 2.10, G has one). We observe that T is locally finite. For suppose to the contrary there is a vertex, z, of infinite degree in T. Then $\lceil z \rceil$ is finite, and $U = \{u \in N(z) : u > z\}$ is infinite. But then by Lemma 2.8, any two vertices $u_i, u_j \in U$ are separated in G by $\lceil u_i \rceil \cap \lceil u_j \rceil = \lceil z \rceil$. But then we have a finite set, $\lceil z \rceil$, with $G - \lceil z \rceil$ having infinitely many components, a contradiction.

Let S_n be the set of vertices at distance (in T) less than n from the root of T, and let D_n be the set of vertices at distance (in T) n from T's root. Since T is locally finite, S_n and D_n are both finite, and for each positive integer n, $G - S_n$ has only finitely many components. For every $v \in D_n$, let $C(v)$ denote the vertex set of the component of $G - S_n$ containing v, and let $\overline{C(v)}$ be the topological closure of $C(v)$. Then S_n and $\{\overline{C(v)} : v \in D_n\}$ together partition $V \cup \Omega$.

We show now that we may take n large enough so that the topological closure of every component of $G - S_n$ is contained in some open set $O \in \mathcal{O}$. Since $G - S_n$ has only finitely many components, we may therefore take these open sets O (which form a finite subcover of Ω), and combine them with a finite subcover from $\mathcal{O}$ of the vertices of S_n (which is compact since it is finite) to obtain our required finite subcover of $V \cup \Omega$.

Suppose to the contrary that for all n, there is a component of $G - S_n$ whose topological closure is not contained in O, for any $O \in \mathcal{O}$. For each n, let $V_n = \{v \in D_n : \text{no set from } \mathcal{O} \text{ contains } \overline{C(v)}\}$. By assumption, each V_n is non-empty, and since each D_n is finite, so is each V_n. Moreover, for the neighbour $u \in D_{n-1}$ of $v \in V_n$, since $S_{n-1} \subseteq S_n$, we have $C(v) \subseteq C(u)$. Therefore $u \in V_{n-1}$. For each vertex $v \in V_n$, let $f(v)$ be such a vertex u. By König's Infinity Lemma (Lemma 2.1) there is a ray $R = v_0v_1\ldots$ with $v_n \in V_n$ for all n. Let ω be the end containing R, and let O be an open set of $\mathcal{O}$ containing ω. Since O is open, O contains a basic open neighbourhood of ω. Hence there exists a finite $S \subset V$ such that $\widehat{C}(S,\omega) \cap (V \cup \Omega) \subseteq O$. Note that $\widehat{C}(S,\omega) \cap (V \cup \Omega) = V(C(S,\omega)) \cup \Omega(S,\omega) = \overline{V(C(S,\omega))}$.

Now take n large enough that $S_n \supseteq S$. Then $C(v_n)$ is contained in a component of $G-S$. Since $C(v_n)$ contains the tail of R from v_n, v_nR, and $R \in \omega$, this component must be $C(S,\omega)$. Hence $C(v_n) \subseteq V(C(S,\omega))$. Also then (by a basic topological fact, [38] Lemma 3.6) $\overline{C(v_n)} \subseteq \overline{V(C(S,\omega))}$. Therefore

$$\overline{C(v_n)} \subseteq \overline{V(C(S,\omega))} \subseteq O \in \mathcal{O}.$$

But this is a contradiction, as $v_n \in V_n$.

(3) ($\Longrightarrow$) Suppose G has a finite $S \subset V$ such that $G - S$ has infinitely many components containing a ray. Consider an open cover in which each end $\omega_i \in \Omega$ is covered by exactly one open set $O_i \subseteq C(S,\omega_i)$ of the cover. Such a cover has no finite subcover.

($\Longleftarrow$) Again without loss of generality we may assume G is connected, since otherwise we may apply the argument to each component of G containing a ray (by assumption there can be only finitely many of them). Let $\mathcal{O}$ be an open cover of Ω; we show that $\mathcal{O}$ has a finite subcover. Again we use a normal spanning tree of G, but ignore its leaves. Let T be a normal spanning tree of G. Every end $\omega \in \Omega$ contains exactly one ray R_ω in T beginning at r, the root of T (Lemma 2.9). Let $R = \{v \in V(R_\omega) : \omega \in \Omega\}$ be the set of vertices of all normal rays of T. Let D_n be the set of vertices in R at distance (in T) n from T's root, and let $S_n = D_0 \cup D_1 \cup \cdots \cup D_{n-1}$.

We claim T can have no vertex z adjacent to infinitely many vertices of R. For suppose z sends edges to infinitely many vertices v_i ($i = 1, 2, \ldots$), all in R. Then there infinitely many v_i with $v_i > z$, and each of these is a vertex in a distinct ray in T. But by Lemma 2.8, these infinitely many distinct rays of T are separated in G by the finite set $\lceil z \rceil$. In other words, $G - \lceil z \rceil$ has infinitely many components containing a ray, a contradiction.

Therefore each D_n, and each S_n, is finite. As in the proof of (2), for every $v \in D_n$, let $C(v)$ denote the vertex set of the component of $G - S_n$ containing v. Let $C(v)_\Omega = \overline{C(v)} \cap \Omega$. Then the sets $\{C(v)_\Omega : v \in D_n\}$ partition Ω. Since for each $n \in \mathbb{N}$, D_n is finite, $\{C(v)_\Omega : v \in D_n\}$ is finite.

We now show that we may take n large enough that each of the sets $C(v)_\Omega$ is contained in some open set $O \in \mathcal{O}$; the proof is similar to the proof of (2).

Suppose to the contrary that for all $n \geq N$, there is a set $C(v)_\Omega$ not contained in O, for any $O \in \mathcal{O}$. For each n, let $V_n = \{v \in D_n : \text{no set from } \mathcal{O} \text{ contains } C(v)_\Omega\}$. By assumption each V_n is non-empty, and since each D_n is finite, so is each V_n. Moreover, for the neighbour $u \in D_{n-1}$ of $v \in V_n$, since $S_{n-1} \subseteq S_n$, we have $C(v) \subseteq C(u)$. Therefore $u \in V_{n-1}$. For each vertex $v \in V_n$, let $f(v)$ be such a vertex u. By König's Infinity Lemma (Lemma 2.1) there is a ray $R = v_0 v_1 \ldots$ with $v_n \in V_n$ for all n. Let ω be the end containing R, and let O be an open set of $\mathcal{O}$ containing ω.

Since O is open, O contains a basic open neighbourhood of ω. Hence there exists a finite $S \subset V$ such that $\widehat{C}(S, \omega) \cap \Omega = \Omega(S, \omega) \subseteq O$ (the basic open neighbourhoods in Ω are those given by the subspace topology on $\Omega \subseteq |G|$, namely, the basic open neighbourhoods of $|G|$ intersected with Ω).

Now take n large enough that $S_n \supseteq S$. Then $C(v_n)$ is contained in a component of $G - S$. Since $C(v_n)$ contains the tail of R from v_n, $v_n R$, and $R \in \omega$, this component must be $C(S, \omega)$. Hence $C(v_n) \subseteq V(C(S, \omega))$, and so $\overline{C(v_n)} \subseteq \overline{V(C(S, \omega))}$. Therefore

$$C(v_n)_\Omega \subseteq \Omega(S, \omega) \subseteq O \in \mathcal{O}.$$

But this is a contradiction, as $v_n \in V_n$. $\square$

Definition. *A graph G is* t-tough *if for any finite separating set $S \subset V(G)$, $G - S$ has at most $|S|/t$ components.*

Being 1-tough is an elementary necessary condition for any graph, finite or infinite, to be hamiltonian.

Corollary 2.12. *Let $G = (V, E, \Omega)$ be a 1-tough countable graph. Then both $V \cup \Omega$ and Ω are compact subsets of $|G|$.*

Proof. If for every finite $S \subset V$, $G - S$ has at most $|S|$ components, G satisfies the conditions of Proposition 2.11 (2) and (3). □

Since in any embedding of S^1 defining a circle $\sigma \subseteq |G|$ (or any embedding of $[0, 1]$ defining an arc $\alpha \subseteq |G|$) every edge has a rational number contained in the subinterval of S^1 (respectively $[0, 1]$) mapped to it, $|E(\sigma)|$ (respectively $|E(\alpha)|$) is always countable. Since we may easily put the set of vertices $V(\sigma)$ contained in σ (respectively $V(\alpha)$ minus an endpoint) into bijective correspondence with the edges it contains, $V(\sigma)$ (respectively $V(\alpha)$) must also be countable. An uncountable graph therefore cannot contain a Hamilton circle.

Furthermore, any graph $G = (V, E, \Omega)$ with $V \cup \Omega$ not compact cannot contain a Hamilton circle. From a graph theoretical perspective, by Corollary 2.12 such a graph would not be 1-tough. From a topological perspective, this follows from the fact that a closed subset of a compact space is compact: Suppose $|G|$ contains a Hamilton circle σ. Since σ is the continuous image of a compact space, σ is compact. Since $V \cup \Omega$ is a closed subset of σ, $V \cup \Omega$ must be compact.

While a hamiltonian graph must be countable, its set of ends need not be. The next section gives an example of such a graph G. We exhibit a Hamilton circle h which, though it contains $V(G)$, which is countable, and though it traverses a subset of $E(G)$, which of course is also countable, also traverses the uncountable set of ends of G. By Proposition 2.6, each of these ends has, contained in $h \cap G$, a countable union of rays converging to it.

2.4 An example

We show the graph G in Figure 2.2 is hamiltonian, by exhibiting an embedding of S^1 in $|G|$ which includes all vertices of G. This graph is given in [12] as an example (without proof) of a hamiltonian graph with uncountably many ends. We give a proof of this fact.

This graph is constructed from the infinite binary tree T_2. First let us construct T_2: take as the vertex set of T_2 the set of all finite binary sequences, including the empty sequence as its root. Put, for each finite binary sequence l, an edge joining

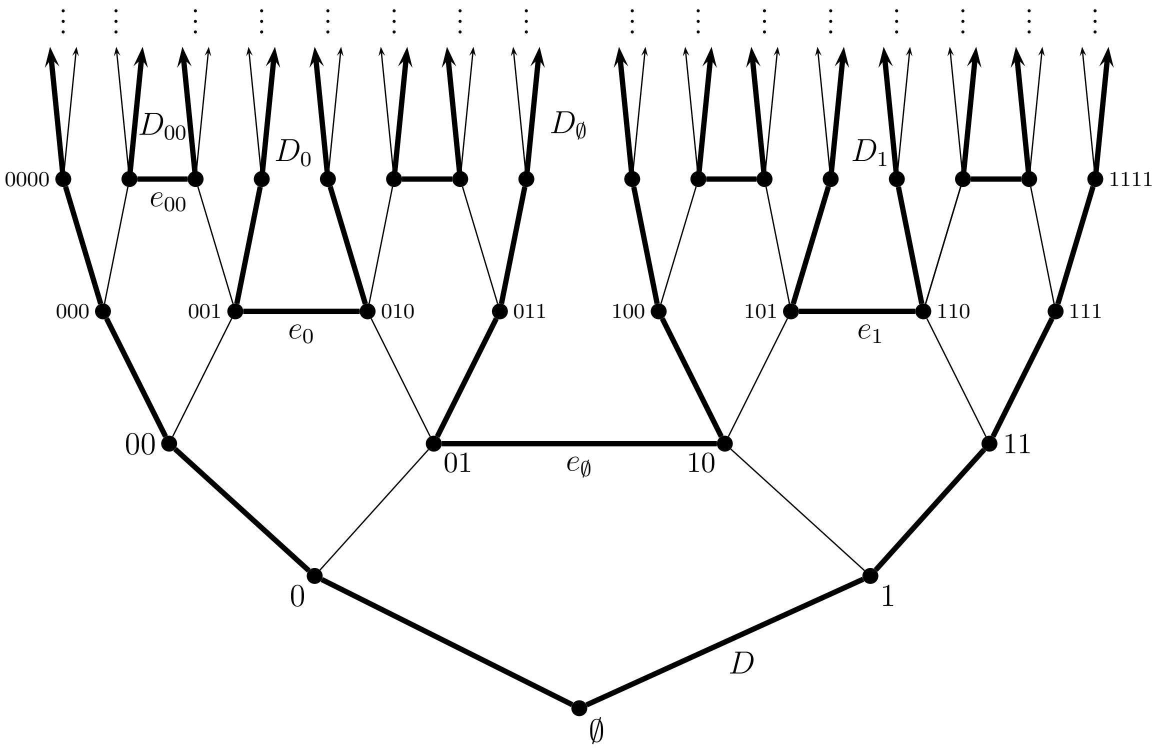

Figure 2.2: A hamiltonian graph with uncountably many ends (edges contained in the Hamilton circle are drawn bold).

l and its two one-digit extensions, $l0$ and $l1$. We now obtain G by adding, for each finite binary sequence l, another edge e_l between vertices $l01$ and $l10$.

Two rays in a tree are equivalent if and only if they share a tail, and each end of T_2 contains exactly one ray starting at $\emptyset$. In constructing G, any two rays of T_2 beginning at $\emptyset$ have at most one added edge joining them. Thus non-equilvalent rays in T_2 are still non-equivalent in G. Moreover, each end of G contains a ray from T_2. Given any ray R in G, we may find a unique ray R' in T_2 in the same end as R as follows: R either contains infinitely many vertices beginning with 0 or beginning with 1. If 0, then R either contains infinitely many vertices beginning 00 or 01; if 1, then R either contains infinitely many vertices beginning 10 or 11. Continuing in this manner, we construct a unique ray R' in T_2 starting at $\emptyset$. Since R and R' may not be finitely separated, they are contained in the same end of G. Hence the ends of G correspond bijectively to its rays starting at $\emptyset$ which are also contained in T_2, and so to the set of all infinite binary sequences. Just as we label the vertices of G with the finite binary sequences, let us label each end ω of G with the unique infinite binary sequence given by the unique ray in T_2 in ω beginning at $\emptyset$.

Let D denote the double ray $\ldots 000\ 00\ 0\ \emptyset\ 1\ 11\ 111 \ldots$. For each finite binary sequence l, let D_l denote the double ray containing e_l, $\ldots l0111\ l011\ l01\ l10\ l100\ l1000 \ldots$. Each D_l has exactly two ends, with subray $l01\ l011\ l0111 \ldots$ converging to end $l0111\ldots$ and subray $l10\ l100\ l1000 \ldots$ converging to end $l10000\ldots$.

Let us regard S^1 as a quotient of the interval $[0, 2]$ under the identification of 0 and 2. First let h map $(1, 2)$ to D, continuously and injectively, such that in the orientation induced by h on D, $1 < \emptyset < 0$. We next map $[0, 1]$ to $|G| \setminus D$, continuously and injectively; 0 will be mapped to end $000\ldots$ and 1 will be mapped to end $111\ldots$, so that patching these maps together will define an embedding of S^1 in $|G|$.

We construct our required homeomorphism $h : S^1 \to |G|$ by defining h on $[0, 1]$ using the usual construction of a ternary Cantor set $\mathbb{C} \subset [0, 1]$. We iteratively remove open intervals from $[0, 1]$, mapping them to double rays D_l as we go. We then map the endpoints of these intervals — the points of $\mathbb{C}$ — to the ends of G. We do so in such a way that the points of $\mathbb{C}$ are mapped bijectively to $\Omega(G)$, continuously with the D_l.

Define closed subsets $A_1 \supset A_2 \supset \cdots$ in $[0, 1]$ as follows. Let $A_1 = [0, 1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right)$. Let h map $\left(\frac{1}{3}, \frac{2}{3}\right)$ continuously and injectively to $D_\emptyset$, such that in the orientation induced by h on $D_\emptyset$, $01 < 10$. Let $A_2 = A_1 \setminus \left\{\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right\}$. Let h map $\left(\frac{1}{9}, \frac{2}{9}\right)$ continuously and injectively to D_0 and $\left(\frac{7}{9}, \frac{8}{9}\right)$ continuously and injectively to D_1, such that in the orientation induced on D_0, $001 < 010$, and in the orientation on D_1, $101 < 110$.

Continue in this manner, at each step, obtaining A_n from A_{n-1} by removing the open middle thirds from each of the 2^{n-1} closed intervals which make up A_{n-1}. For each binary sequence l of length $n-1$, let h map one of the 2^{n-1} middle thirds removed from A_{n-1} continuously and injectively to double ray D_l, always so that in the orientation h induces on D_l, $l01 < l10$, as follows:

At step n, 2^{n-1} intervals are removed from A_{n-1}. Each interval has two endpoints whose ternary expansions have length n, $\sum_{i=1}^{n} a_i/3^i$, and which agree on their first $n-1$ digits. Furthermore, each $a_i \in \{0,2\}$ for $i = 1, \ldots, n-1$, the left endpoint is $0.a_1a_2\ldots a_{n-1}1$, and the right endpoint is $0.a_1a_2\ldots a_{n-1}2$. There are 2^{n-1} sequences of length $n-1$ using only 0s and 2s, and each of these $n-1$ intervals removed has exactly one of these sequences $a_1a_2\cdots a_{n-1}$ as its first $n-1$ digits. Using infinite ternary expansions, the intervals removed from A_{n-1} can be written as

$$(0.a_1a_2\ldots a_{n-1}0222\ldots,\ 0.a_1a_2\ldots a_{n-1}2000\ldots)\,.$$

For each digit a_i ($i = 1, 2, \ldots$) let $b_i = a_i/2$. Since each $a_i \in \{0,2\}$, each $b_i \in \{0,1\}$. Let h map each of these open intervals continuously and injectively to $D_{b_1b_2\cdots b_{n-1}}$. Further, let h map each point in $[0,1]$ whose infinite ternary expansion can be written using only 0s and 2s, $\sum_{i=1}^{\infty} a_i/3^i$ with $a_i \in \{0,2\}$ for all i (that is, each point in $\mathbb{C}$), to the infinite binary sequence given by taking each $b_i = a_i/2$:

$$\sum_{i=1}^{\infty} \frac{a_i}{3^i} \ \mapsto\ b_1b_2\ldots.$$

This mapping h is continuous and injective on $V \cup E \subset |G|$, since each vertex and edge of $G - D$ is contained in exactly one D_l. It is also surjective on $V - V(D)$, since every vertex of $G - V(D)$ is contained in some D_l.

We now show that h is also continuous and bijective on $\Omega = \Omega(G)$. Let $\omega \in \Omega$, and suppose $\omega = b_1b_2\cdots$. Let

$$x = \sum_{i=1}^{\infty} \frac{2b_i}{3^i}.$$

Then $h(x) = \omega$, so h maps $\mathbb{C}$ onto $\Omega(G)$. Suppose $x \neq y$ are two points mapped to ends of G. Then the infinite ternary expansions of x and y using only 0s and 2s differ on some digit: if $x = \sum_i x_i/3^i$ and $y = \sum_i y_i/3^i$, then for some i, $x_i \neq y_i$. Then $h(x) \neq h(y)$, so h is injective on Ω.

It remains to show that h is continuous at each $\omega \in \Omega$. Suppose $h(x) = \omega$. Let N be any neighbourhood of ω. We find an open interval $J \subset S^1$ containing x such that $h(J) \subseteq N$. Suppose $\omega = b_1 b_2 b_3 \cdots$. Let R be the unique ray in T_2 contained in ω beginning at $\emptyset$. Then $R = \emptyset\ b_1\ b_1 b_2\ b_1 b_2 b_3 \ldots$. There is a basic open set $\widehat{C}(S, \omega) \subset N$, for some finite $S \subset V$. Since $R \in \omega$, R has a tail contained in $C(S, \omega)$. Let $v \in R$ be a vertex contained in $C(S, \omega)$ with binary sequence longer than that of any vertex $u \in S$. Suppose $v = b_1 b_2 \ldots b_n$.

Note $h^{-1}(v)$ agrees with x on the first n digits of x, and

$$h^{-1}(v0) < h^{-1}(v000\ldots) \leq h^{-1}(\omega) = x \leq h^{-1}(v111\ldots) < h^{-1}(v1).$$

Take $J = (h^{-1}(v0), h^{-1}(v1))$.

Since for any $y \in J$ which is mapped to a vertex or end of G, $h(y)$ agrees with v on its first n digits, $h(y) \in \overline{C(S, \omega)}$. Every point in J mapped to an edge by h is mapped to an edge which occurs between two vertices contained in $C(S, \omega)$, and so is contained in $C(S, \omega)$. Hence $h(J) \subseteq N$.

Similarly, given any neighbourhood N containing end $000\ldots$ or end $111\ldots$, open intervals in S^1 containing $0 = \{0, 2\}$ or 1 may be found to show continuity in the cases $h(0) = 000\ldots$ and $h(1) = 111\ldots$.

Chapter 3

A necessary and sufficient condition for hamiltonicity of locally finite graphs

In this chapter we prove our main result, Theorem 1.7.

Let $G = (V, E, \Omega)$ be a locally finite graph, and $\{v_1, v_2, \ldots\}$ an enumeration of V. We define an infinite sequence of finite graphs, $(G^*_n)_{n\in\mathbb{N}}$, each containing an induced subgraph of G. For any finite subset S of V, let G_S denote the graph obtained from G by replacing each component C of $G - S$ with a single vertex u_C, where u_C is adjacent to each vertex in S which has a neighbour in C. In other words, contract each component C of $G - S$ to a single vertex u_C, deleting loops and identifying multiple edges with a single edge; we therefore call u_C a *contracted vertex*. Denote by G^*_S the graph obtained from G_S by adding all edges among the neighbours of each contracted vertex u_C so that u_C together with its neighbours in G_S is a clique, which we denote K_{u_C}. For the graph G and subsets $S, U, W \subset V(G)$ shown in Figure 2.1 on page 16, Figure 3.1 shows G_S, G_U, G_W, and G^*_S, G^*_U, and G^*_W. For convenience, let $S_n = \{v_1, \ldots, v_n\}$ and denote by G_n the graph G_{S_n} and by G^*_n the graph $G^*_{S_n}$.

For any sequence of sets $(X_n)_{n\in\mathbb{N}}$, the set

$$\liminf(X_n) = \bigcup_{n\in\mathbb{N}} \bigcap_{i>n} X_i$$

is the set of elements eventually in all X_i for large enough i (i.e. the elements in all X_i with $i > n$ for some sufficiently large $n \in \mathbb{N}$). Note that our sequence of graphs

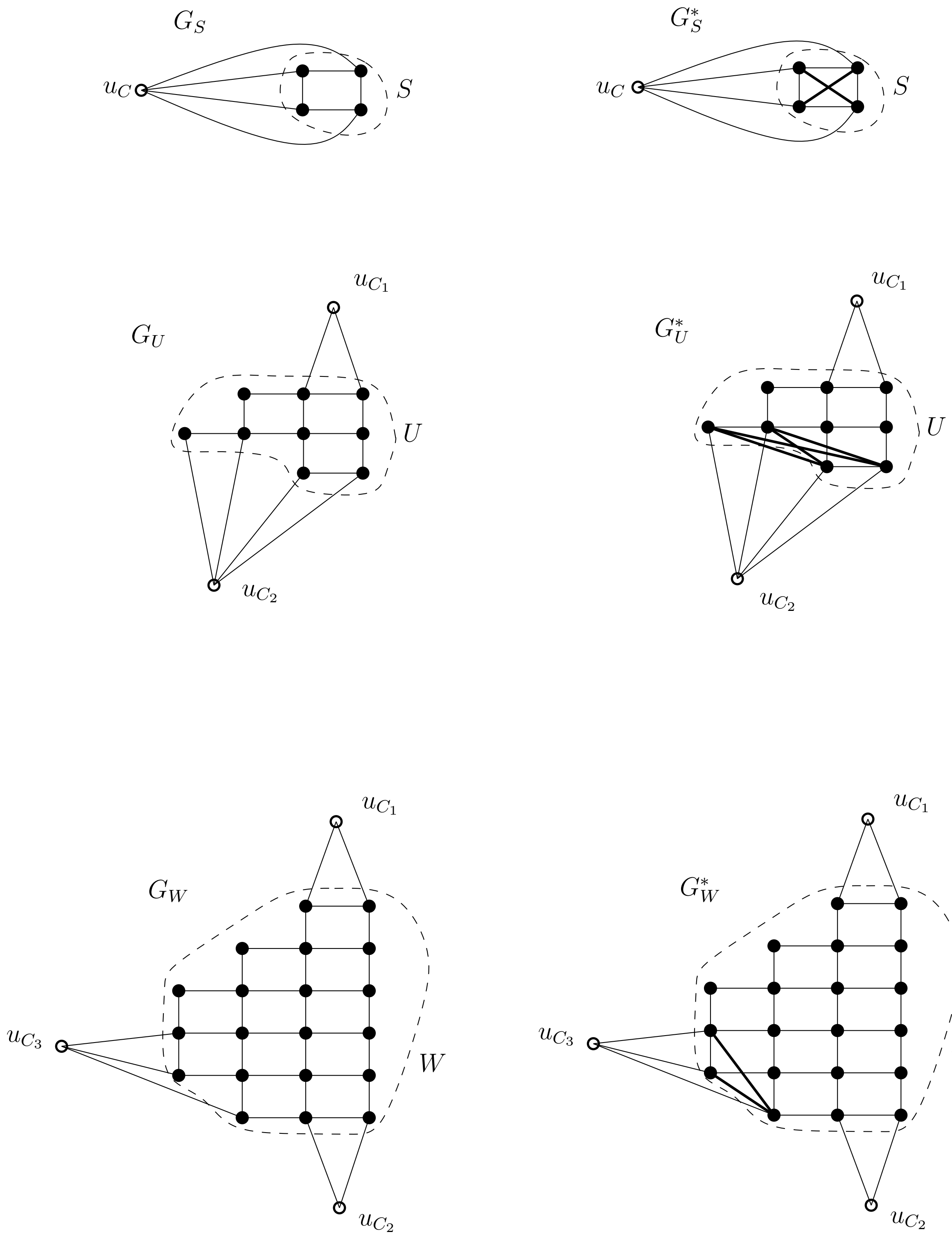

Figure 3.1: Some examples of our "contraction graphs", G_S, G_S^*, G_U, G_U^*, and G_W, G_W^*. Edges added between neighbours of contracted vertices are shown bold.

$(G_n^*)_{n\in\mathbb{N}}$ has $\liminf G_n^* = G$.

We may now state our main result:

Theorem 1.7 (Main result for locally finite graphs)**.** *Let G be a locally finite graph. Then G is hamiltonian if and only if there is a positive integer m such that for all $n \geq m$, G_n^* is hamiltonian.*

3.1 Proof of Theorem 1.7

3.1.1 Sufficiency

Suppose there is an $m \in \mathbb{N}$ such that for all $n \geq m$, G_n^* is hamiltonian. The strategy of the proof is as follows: We obtain a sequence of Hamilton cycles $(H_n)_{n\geq m}$, each cycle $H_n \subseteq G_n^*$. This sequence will be constructed in such a way that we may use it to define an embedding η of S^1 in $|G|$ which contains all the vertices of G.

This section of the proof is inspired by and adapts the general approach of Georgakopoulos in his study of topological Euler tours [24]. In particular, the idea of contracting components of $G - S$ for increasing finite subsets $S \subset V$ to a single vertex to obtain a sequence of finite graphs, and then using a sequence of mappings to define a "limit map", appear in [24]. However, complications arise in the application of these ideas to Hamilton circles. While a topological Euler tour (like its finite counterpart) may traverse a vertex or an end any number of times (in the case of an end, even infinitely many times), we need our limit map to be bijective on $V(G) \cup \Omega(G)$. This requires quite a bit more care than in the case of finding a topological Euler tour. In fact, our finite contracted graphs G_n^* differ from those used in [24], enabling the construction of a particular sequence of Hamilton cycles $(H_n)_{n\geq m}$ from which a limit map, injective on $\Omega(G)$, may be defined.

Defining a sequence of Hamilton cycles

For each positive integer $n \geq m$, let $\mathcal{V}_n$ be the set of all Hamilton cycles in G_n^*. We wish to apply König's Infinity Lemma (Lemma 2.1) to obtain a sequence $(H_n)_{n\geq m}$ of Hamilton cycles with each $H_n \in \mathcal{V}_n$, which we use to define a Hamilton circle in $|G|$. To satisfy the conditions of the lemma, observe that by assumption each $\mathcal{V}_n$ is non-empty, and that since G_n^* is finite, so is each $\mathcal{V}_n$. Now to each Hamilton cycle $H \in \mathcal{V}_{n+1}$ we associate an element in $\mathcal{V}_n$, which we denote $H|n$, as described below.

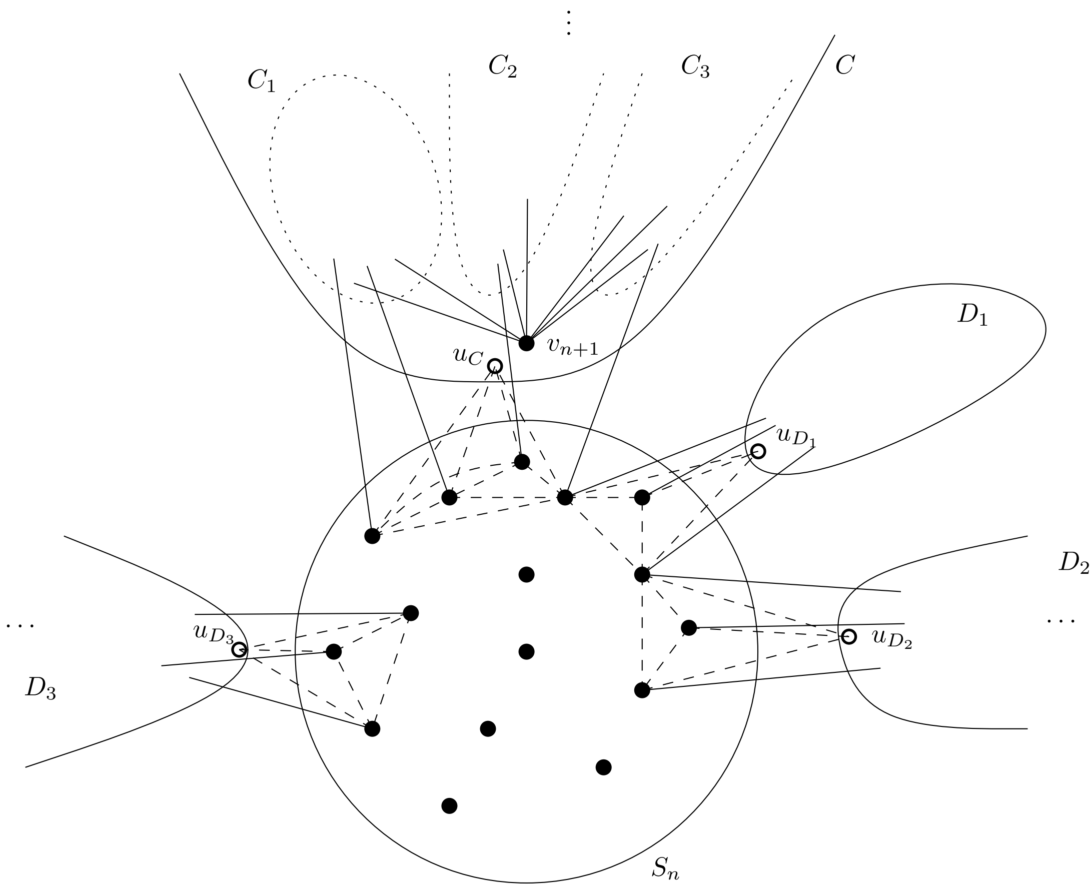

Figure 3.2: S_n, the components of $G - S_n$, and the components of $G - S_{n+1}$. Components $C_1, C_2, C_3 \subset C$ contain a neighbour of v_{n+1}. Edges of the cliques K_{u_C}, $K_{u_{D_1}}$, $K_{u_{D_2}}$, and $K_{u_{D_3}}$ are shown dashed; edges between vertices of S_n are not shown.

Let $u_{C_1}, u_{C_2}, \ldots, u_{C_k}$ be the contracted vertices in G^*_{n+1} which are adjacent to v_{n+1}. Then $C_1, C_2, \ldots, C_k$ are the components of $G - S_{n+1}$ containing a neighbour of v_{n+1} in G. Let C be the component of $G - S_n$ containing v_{n+1}. For any $x, y \in V(C_1 \cup C_2 \cup \cdots \cup C_k)$, there is an x-y path in $G - S_n$, so $C_1 \cup C_2 \cup \cdots \cup C_k \subseteq C$. (See Figure 3.2.) Moreover, each component of $C - v_{n+1}$ must be C_i for some i. Hence $C = C_1 \cup C_2 \cup \cdots \cup C_k \cup \{v_{n+1}\}$. Let u_{D_i} ($i \in \{1, 2, \ldots, l\}$) be the contracted vertices in G^*_{n+1} which are not adjacent to v_{n+1}. Then D_i ($i \in \{1, 2, \ldots, l\}$) are the components of $G - S_{n+1}$ not containing a neighbour of v_{n+1}, and so each of these components are also components of $G - S_n$. Hence their corresponding contracted vertices u_{D_i} are also contracted vertices in G^*_n, corresponding to exactly the same components. Furthermore, since v_{n+1} does not send any edges to any of these components, each of these u_{D_i} has the same vertices in its neighbourhood in both G_n and G_{n+1}, and so

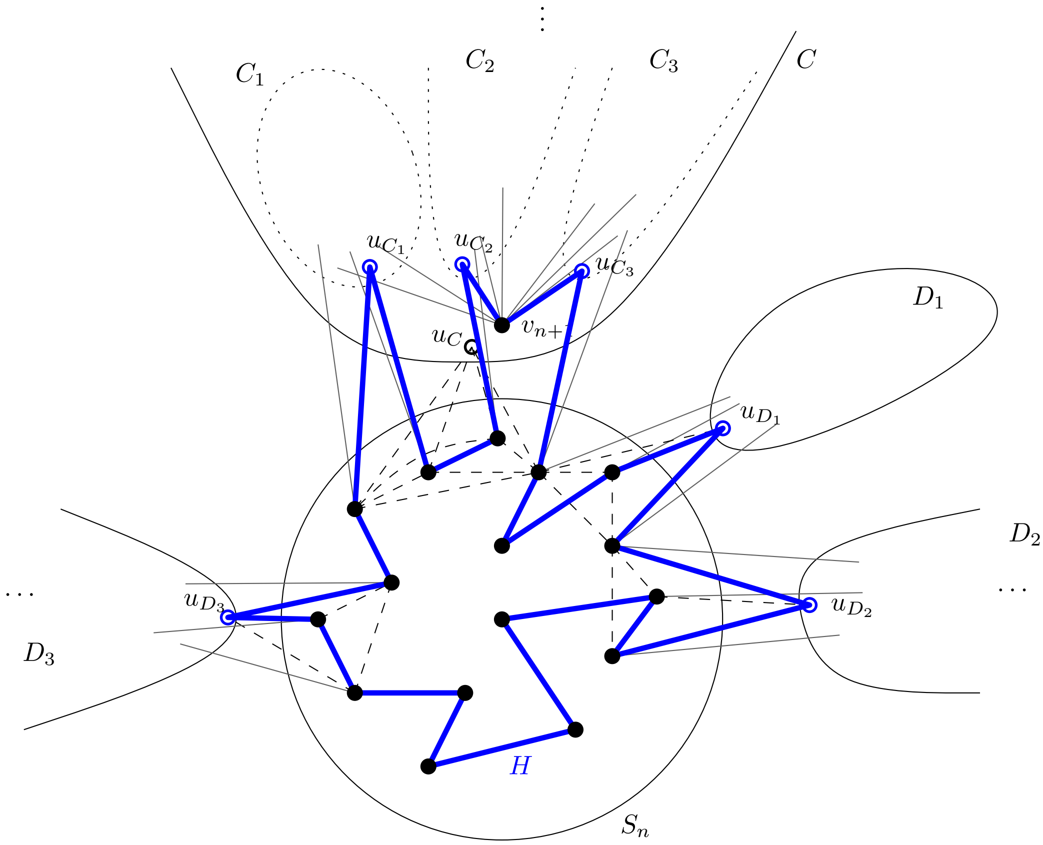

Figure 3.3: H is a Hamilton cycle in G^*_{n+1}.

each clique $K_{u_{D_i}}$ $(i \in \{1, 2, \dots, l\})$ is identical in both G^*_n and G^*_{n+1}.

Let $H \in \mathcal{V}_{n+1}$. (Figure 3.3 shows such a Hamilton cycle H in G^*_{n+1}.) Let $H|n$ be the cycle in G^*_n obtained by modifying H as follows: The vertices of S_n partition H into n S_n-paths. Each of these paths has exactly one of the following forms:

1. An edge $e \in E(G)$ between two vertices $v_p, v_q \in S_n$;
2. an edge in a clique $E(K_{u_{D_i}}) \setminus E(G)$ for some contracted vertex u_{D_i};
3. an edge e in clique $E(K_{u_{C_i}}) \setminus E(G)$ for some contracted vertex u_{C_i} (note that since $C_i \subset C$, e is an edge in clique K_{u_C} in G^*_n);
4. a path of length two between two vertices v_p, v_q of S_n which includes a contracted vertex u_{D_i} $(i \in \{1, 2, \dots, l\})$;
5. a path of the form $v_p u_{C_i} v_q$ for some contracted vertex u_{C_i} $(i \in \{1, 2, \dots, k\})$ (note that since $C_i \subset C$, $u_{C_i} \notin V(G^*_n)$, and $v_p v_q$ is an edge in clique K_{u_C} in G^*_n);

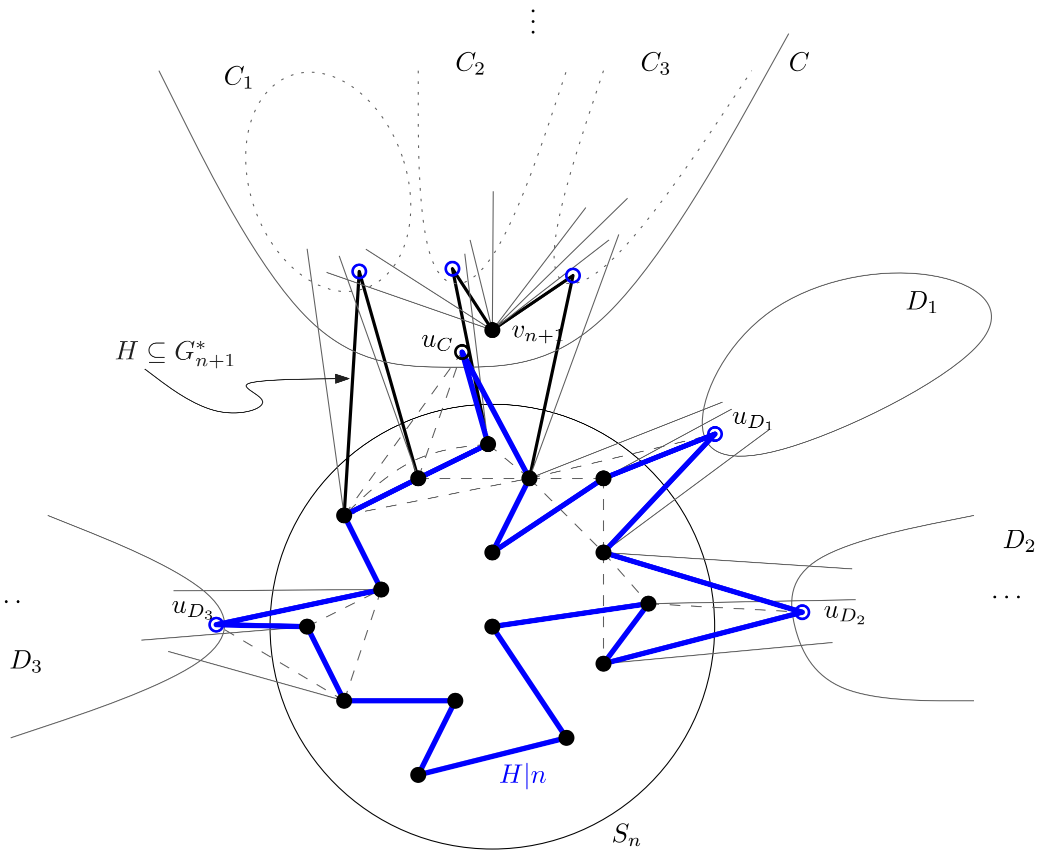

Figure 3.4: $H \in \mathcal{V}_{n+1}$ and $H|n \in \mathcal{V}_n$.

6. a path which includes v_{n+1}. Note that if v_p and v_q are the vertices in S_n which are the terminal vertices of this path, then this path could be of the form $v_p v_{n+1} v_q$, $v_p v_{n+1} u_{C_i} v_q$, $v_p u_{C_i} v_{n+1} v_q$, or $v_p u_{C_i} v_{n+1} u_{C_j} v_q$, for some $i, j \in \{1, 2, \ldots, k\}$. Any contracted vertices u_{C_i} or u_{C_j} occurring on this path correspond to components $C_i, C_j \subset C$, and so neither u_{C_i}, u_{C_j} is in the vertex set of G^*_n.

Each S_n-path in H of form 1, 2, 3, or 4 above, is also an edge or path in G^*_n; simply leave each of these S_n-paths in $H|n$. Modify H by replacing each S_n-path of form 5 with the edge $v_p v_q \in G^*_n$, and replacing each S_n-path of form 6 with path $v_p u_C v_q$ in G^*_n. (Figure 3.4 shows this process for the Hamilton cycle H in G^*_{n+1} of the graph in Figure 3.3.)

Since every vertex in S_n, and every contracted vertex of G^*_n, is included in $H|n$ exactly once, $H|n$ is a Hamilton cycle in G^*_n. So $H|n \in \mathcal{V}_n$.

Hence by König's Infinity Lemma, there is a sequence of Hamilton cycles $(H_n)_{n \geq m}$ such that each $H_n \in \mathcal{V}_n$ and $H_{n+1}|n = H_n$ for all $n \geq m$.

Note on future use of the Infinity Lemma

From this point on in this thesis, given a sequence of graphs of the form $(G_n^*)_{n\geq m}$, a sequence of Hamilton cycles $(H_n)_{n\geq m}$ returned by the Infinity Lemma will always be such that in the application of the Lemma, the predecessor element $H|n$ of each Hamilton cycle $H \subseteq G_{n+1}^*$ has been chosen as in the construction above, unless explicitly stated otherwise. When applying the Infinity Lemma in this manner, we will also always assume that m is a positive integer such that G_n^* is hamiltonian for all $n \geq m$.

Defining $\eta : S^1 \to |G|$

We now use this sequence $(H_n)_{n\geq m}$ to inductively define a mapping $\eta : S^1 \to |G|$ as the limit of a sequence of homeomorphisms $\eta_n : S^1 \to H_n \subseteq |G_n^*|$ for each $n \geq m$. The function η will be a continuous injection whose image contains $V \cup \Omega$. Since a continuous injection from a compact space to a Hausdorff space is a homeomorphism onto its image (Lemma 2.2), η will be our required Hamilton circle.

Let η_m be a homeomorphism of S^1 onto $H_m \subseteq |G_m^*|$. Now suppose that for some $n \geq m$ we have defined a mapping $\eta_n : S^1 \to |G_n^*|$ so that η_n is continuous, injective, and $\eta_n(S^1) = H_n$. We use η_n to define a mapping $\eta_{n+1} : S^1 \to H_{n+1} \subseteq |G_{n+1}^*|$ as follows.

Recall that any graph-theoretic path $P = x_1 \dots x_k$ in H_n corresponds to a topological arc A in $|G_n^*|$. Arc A is a closed connected subset of $|G_n^*|$. Since η_n is a homeomorphism, $\eta_n^{-1}(A)$ is closed and connected, i.e., a closed subinterval of S^1. Suppose $P_1 = x_0 \dots x_k$ and $P_2 = y_0 \dots y_l$ are two internally disjoint paths in H_n, and A_1 and A_2 are their corresponding arcs in $|G_n^*|$. Let $\mathring{A}_1 = A_1 \setminus \{x_1, x_k\}$ and $\mathring{A}_2 = A_2 \setminus \{y_1, y_l\}$. Since η_n is bijective, $\eta_n^{-1}(\mathring{A}_1) \cap \eta_n^{-1}(\mathring{A}_2) = \emptyset$. If P_1 and P_2 share just one terminal vertex, say $v = x_k = y_0$, then $\eta_n^{-1}(A_1) \cap \eta_n^{-1}(A_2) = \eta_n^{-1}(v)$, a single point in S^1.

As above, S_n partitions H_{n+1} into n S_n-paths. Our application of the Infinity Lemma has given us $H_{n+1}|n = H_n$ for all $n \geq m$. By the predecessor construction, each S_n-subpath Q of H_{n+1} corresponds to a subpath P of H_n between the same terminal vertices. By the above paragraph, $\eta_n^{-1}(P)$ is a closed interval of S^1, which we denote I_Q. We now define η_{n+1} as a homeomorphism which maps each of these intervals I_Q to Q, as follows. (Note that by the above paragraph, for distinct S_n-subpaths P_1, P_2 of H_n, $\eta_n^{-1}(P_1)$ and $\eta_n^{-1}(P_2)$ intersect in at most their endpoints, in which case such endpoints are mapped to the terminal vertex shared by P_1 and P_2 in S_n by η_n.)

Each of these S_n-subpaths Q of H_{n+1} in G^*_{n+1} has exactly one of the following forms:

1. An edge v_iv_j where $v_i, v_j \in S_n$ and $v_iv_j \in E(G)$. Then $v_iv_j \in E(H_n)$. We have nothing to do here: define $\eta_{n+1}(s) = \eta_n(s)$ for all $s \in I_Q = \eta_n^{-1}([v_i, v_j])$.

2. A path of the form of a path $v_iu_Dv_j$ where $v_i, v_j \in S_n$ and u_D is a contracted vertex of G^*_{n+1}, where component $D \notin \{C_1, \ldots, C_k\}$ (as above; i.e. D contains no neighbours of v_{n+1}). Therefore D is also a component of $G - S_n$, $u_D \in V(G^*_n)$, and subpath $v_iu_Dv_j$ of H_{n+1} is also a subpath of H_n. Again we have nothing to do here: for all $s \in \eta_n^{-1}(v_iu_Dv_j) = I_Q$, define $\eta_{n+1}(s) = \eta_n(s)$.

3. A path of the form $v_iu_{C_l}v_j$ where C_l, $1 \leq l \leq k$, is a component of $G - S_{n+1}$ containing a neighbour of v_{n+1}; the subpath $v_iu_{C_l}v_j$ in H_{n+1} corresponds to edge $v_iv_j \in E(H_n)$ in the clique K_{u_C} in G^*_n.

   Define η_{n+1} so that it continuously and bijectively maps $\eta_n^{-1}([v_i, v_j]) = I_Q$ to $Q = v_iu_{C_l}v_j$, with $\eta_{n+1}(\eta_n^{-1}(v_i)) = v_i$, $\eta_{n+1}(\eta_n^{-1}(v_j)) = v_j$, and such that the midpoint of I_Q is mapped to u_{C_l}.

4. A path of the form v_iTv_j where $v_i, v_j \in S_n$ and T is a subpath of H_{n+1} containing v_{n+1}. The path T is one of the following: v_{n+1}, $v_{n+1}u_{C_l}$, $u_{C_l}v_{n+1}$, or $u_{C_l}v_{n+1}u_{C_p}$ $(1 \leq l < p \leq k)$, where C_l and C_p are components of $G - S_{n+1}$ contained in the component C of $G - S_n$ which contains v_{n+1}. The subpath $v_iu_Cv_j$ is the corresponding subpath of H_n.

   Define η_{n+1} so that η_{n+1} continuously and bijectively maps $\eta_n^{-1}(v_iu_Cv_j) = I_Q$ to $Q = v_iTv_j$, with $\eta_{n+1}(\eta_n^{-1}(v_i)) = v_i$ and $\eta_{n+1}(\eta_n^{-1}(v_j)) = v_j$. Furthermore, if v_iTv_j has length r ($r = 2, 3$, or 4), define η_{n+1} such that the preimage of each edge in v_iTv_j has length $|I_Q|/r$.

For distinct subpaths P_1, P_2 of H_n, $\eta_n^{-1}(P_1) = I_{Q_1}$ and $\eta_n^{-1}(P_2) = I_{Q_2}$ intersect in at most their endpoints. Suppose $I_{Q_1} \cap I_{Q_2} = \{s\}$. In this case, when η_{n+1} is defined on I_{Q_1}, η_{n+1} is defined so that it maps s to vertex $\eta_n(s) \in S_n$, and on I_{Q_2}, η_{n+1} is defined also to map s to vertex $\eta_n(s) \in S_n$. Hence η_{n+1} has been defined to be a continuous bijection from S^1 onto H_{n+1}, and so by Lemma 2.2, is a homeomorphism.

We next use these maps η_n to define our Hamilton circle η in $|G|$. Observe that η_{n+1} agrees with η_n on all points $s \in S^1$ which are mapped by η_n to a vertex $v_i \in S_n$ or to an inner point of an edge $e \in E(G[S_n])$. Hence if $\eta_n(s)$ is a vertex or inner point of an edge in G, then $\eta_k(s) = \eta_n(s)$ for all $k \geq n$.

So we may now define a "limit map" $\eta : S^1 \to |G|$ as follows. Let $s \in S^1$. If there is an $n \in \mathbb{N}$ such that $\eta_k(s) = \eta_n(s) = y$ for all $k \geq n$, define $\eta(s) = y$. In this case, y must be a vertex or inner point of an edge in G, so that the maps η_n agree on s for all n large enough that $y \in |G[S_n]|$. Otherwise, for each n there exists a contracted vertex u_{C_n} such that $\eta_n(s)$ is either an inner point of an edge in the clique $K_{u_{C_n}}$ or the contracted vertex u_{C_n} itself. Since for all n, $S_n \subset S_{n+1}$, the corresponding components C_n are nested, i.e. $C_n \supseteq C_{n+1}$. Let $\overline{C_n}$ denote the topological closure of C_n in $|G|$. Since these components are nested and non-empty, any finite collection of $\overline{C_n}$ has non-empty intersection. Since $|G|$ is compact, by the finite intersection property, $U = \bigcap_{n \geq m} \overline{C_n}$ is non-empty.

Claim. *$U \subseteq \Omega$, and U is a singleton.*

Proof of claim. For any vertex $v \in V$, there is a positive integer k such that $v \in S_k$. Then $v \notin C_k$. Since $\overline{C_k} = C_k \cup \Omega(C_k)$ and $v \notin \Omega(C_k)$, $v \notin \overline{C_k}$. Hence $v \notin \bigcap_{n \geq m} \overline{C_n}$.

Similarly, for any edge $e = [u, w]$, we may take k large enough that $u, w \in S_k$. Then $e = uw \notin C_k$, and so no inner points contained in $\mathring{e}$ are in C_k. As clearly $e \notin \Omega(C_k)$, $e \notin \overline{C_k}$. Hence no inner point $x \in \mathring{e}$ is in $\bigcap_{n \geq m} \overline{C_n}$. Therefore $U \subseteq \Omega$.

For any two distinct ends $\omega, \xi \in \Omega$, there is a finite vertex set S such that the tails of rays in ω and the tails of rays in ξ are in different components of $G - S$. Hence as soon as $S \subseteq S_n$, $\overline{C_n}$ cannot contain both ω and ξ. Therefore U is just a single end ω. □

Define $\eta(s) = \omega$. Note that

$$\eta(s) \in \overline{C_n} \text{ for every } n \geq m. \tag{3.1}$$

η is an embedding of S^1 in $|G|$

Since every injective continuous map from a compact space into a Hausdorff space is a homeomorphism onto its range (Lemma 2.2), we just have to show that η is continuous and injective. From this it follows that $\eta(S^1)$ is an embedding of S^1 in $|G|$.

First we consider points $s \in S^1$ which are mapped to a vertex or an inner point of an edge of $|G|$. Let $s \in S^1$ be such a point. Then for some sufficiently large n, $\eta(s) = \eta_k(s)$ for all $k \geq n$. Recall that all these η_k are continuous and injective at s. We will establish that therefore η is continuous and injective at s as well. We first show continuity, then injectivity.

Given any neighbourhood O of the point $\eta(s)$, consider a basic open neighbourhood N of $\eta(s)$ contained in O. We will find an open neighbourhood J of s such that $\eta(J) \subseteq N$.

Claim. *We may take n large enough that η_n and η agree on $\eta_n^{-1}(N)$.*

Proof of Claim. First suppose $\eta(s) = y \in \mathring{e}$ is an inner point of an edge of $|G|$. Then $e = [v_i, v_j]$ for some $v_i, v_j \in V(G)$. Take n large enough that both $v_i, v_j \in S_n$. Then $\eta_n^{-1}(\mathring{e})$ is an open interval of S^1. By construction, η and η_n agree on this interval. Since any basic open neighbourhood N of $\eta(s)$ is contained in $\mathring{e}$, η_n agrees with η on N.

Now suppose $\eta(s) = y \in V(G)$. Since y has finite degree, simply take n large enough that y and all its neighbours are in S_n. In particular, the path $v_i y v_j$ in H_n is also a path in G. Hence η_n and η agree on $\eta_n^{-1}(v_i y v_j)$. Since any basic open neighbourhood N of y consists of y together with a union of half-edges $[y, z)$, one for each edge incident with y (z an inner point of the edge), $\eta_n^{-1}(N) \subseteq \eta_n^{-1}(v_i y v_j)$. □

So now take n large enough that η_n and η agree on $\eta_n^{-1}(N)$. Take as J an open interval around s contained in $\eta_n^{-1}(N) \subseteq S^1$. Then $\eta(J) = \eta_n(J) \subseteq N$. So η is continuous on points of S^1 mapped to vertices or inner points of edges of $|G|$.

To see that η is injective on such points, suppose $\eta(x) = \eta(y)$ for some $x, y \in S^1$ mapped to a vertex or inner point of an edge. Then taking n large enough that η_n agrees with η on both x and y implies $x = y$, since η_n is injective.

To complete our analysis, let s be a point of S^1 mapped by η to an end $\omega \in |G|$. To show continuity, for every neighbourhood O of ω we have to find a neighbourhood $J \subset S^1$ of s such that $\eta(J) \subseteq O$. Since O contains a basic open neighbourhood $\widehat{C}(S, \omega)$ for some finite $S \subset V$, taking n large enough that $S \subseteq S_n$ yields a component C of $G - S_n$ such that $\overline{C} \subseteq O$. This component C corresponds to a vertex $u_C \in G_n^*$, and $\eta_n(s)$ is an inner point of an edge in K_{u_C} or the contracted vertex u_C itself. Let P_n be the S_n-path in G_n^* containing $\eta_n(s)$. Let v_i, v_j be the terminal vertices of $P_n \subseteq |G_n^*|$, and let $J_n = \eta_n^{-1}(P_n) \subset S^1$. Since G is locally finite, we can take k large enough that all neighbours of v_i and v_j are contained in S_k. Hence both edges incident with v_i and both edges incident with v_j in H_k are edges of G. Thus, there are vertices $v_p, v_q \in C$ with $v_iv_p, v_jv_q \in E(H_k)$ and $v_iv_p, v_jv_q \in E(G)$. Then η_k maps J_n to a path in $|G_k^*|$ linking v_i and v_j, and this path contains edges $v_iv_p, v_jv_q \in E(G)$. This path is of the form $v_iv_pP_kv_qv_j$, i.e. $\eta_k(J_n) = v_iv_pP_kv_qv_j$, where P_k is a v_p-v_q path in $|G_k^*|$. Now $\eta_k(s)$ is an inner point (an inner point of an edge in K_{u_D} or the contracted vertex u_D itself) of a subpath of P_k between two neighbours of a contracted vertex u_D, for some component D of $G - S_k$ contained in C. Take as J an open interval around s contained in $\eta_k^{-1}(P_k)$. Since v_p and v_q are both contained in C, we claim:

Claim. $\eta(J) \subseteq \overline{C}$.

Proof of Claim. To see that this is true, first suppose $x \in J$ is mapped by η to a vertex or inner point of an edge of $|G|$. Then there is some $l \geq k$ such that $\eta_l(x) = \eta(x)$. Since $\eta_k(J) \subseteq P_k$, by construction $\eta_l(x) \in C \subseteq \overline{C}$.

Now suppose $x \in J$ mapped by η to an end. Then $\eta_l(x)$ is an inner point of a path in $K_{u_{C_l}}$ for all $l \geq k$, with all $C_l \subseteq C$. By (3.1), for all l, $\eta(x) \in \overline{C_l}$. Since $C_l \subseteq C$ implies $\overline{C_l} \subseteq \overline{C}$, $\eta(x) \in \overline{C}$. □

Since $\overline{C} \subseteq O$, $\eta(J) \subseteq O$.

We now show that η is injective on Ω. Suppose $\eta(x) = \omega_1$ and $\eta(y) = \omega_2$ for some $\omega_1, \omega_2 \in \Omega$, and $x \neq y$. Then there exist two sequences of nested components $(C_n)_{n \geq m}$ and $(D_n)_{n \geq m}$, such that for all $n \geq m$, x is mapped by η_n to either an inner point of an edge in clique $K_{u_{C_n}}$ or to contracted vertex u_{C_n}, and y is mapped by η_n to an inner point of an edge in a clique $K_{u_{D_n}}$ or to contracted vertex u_{D_n}. Our goal is to show that for some positive integer n, $C_n \neq D_n$.

Since $x \neq y$, we may choose an open neighbourhood N containing x and an open neighbourhood M containing y such that $N \cap M = \emptyset$. Let $v_n, u_n \in S_n$ be the terminal vertices of the path $P_n \subseteq K_{u_{C_n}}$ containing $\eta_n(x)$. Let $w_n, z_n \in S_n$ be the terminal vertices of the path $Q_n \subseteq K_{u_{D_n}}$ containing $\eta_n(y)$. Since η is bijective on V, $\eta^{-1}(v_n)$, $\eta^{-1}(u_n)$, $\eta^{-1}(w_n)$, and $\eta^{-1}(z_n)$ are distinct points in S^1; let us denote these points x_n^1, x_n^2, y_n^1, and y_n^2 respectively.

We now observe that

$$\left|\eta_n^{-1}(P_n)\right| \to 0 \text{ as } n \to \infty.$$

This follows from the definition of η. Indeed, for each n, our path P_n is of one of the forms (2), (3), or (4) in Section 3.1.1. Since G is locally finite, v_n and u_n together have only finitely many neighbours in C_n, and so one of these neighbours must become adjacent to either v_n or u_n in G_k^* for some $k > n$. Hence for each n, there is a minimal vertex in our enumeration of V, contained in component C_n, which is adjacent to one of v_n or u_n in $H_k \subseteq G_k^*$ (i.e. P_n can only be of form (2) for some finite number of steps). At step $k-1$, $P_n = P_{k-1}$ is of the form (3) or (4) in Section 3.1.1. If P_n is of form (3), $|\eta_k^{-1}(P_k)| = \frac{1}{2}|\eta_n^{-1}(P_n)|$, and if P_n is of form (4), $|\eta_k^{-1}(P_k)| \leq \frac{2}{3}|\eta_n^{-1}(P_n)|$. Therefore for all $n \in \mathbb{N}$, there is a positive integer $k > n$ such that

$$\left|\eta_k^{-1}(P_k)\right| \leq \frac{2}{3}\left|\eta_n^{-1}(P_n)\right|.$$

Hence the lengths of our intervals $\eta_n^{-1}(P_n)$ can be made arbitrarily small by taking n sufficiently large. Note also that for all n, $x \in \eta_n^{-1}(P_n)$.

Therefore, the points $(x_n^1)_{n\geq m}$ and $(x_n^2)_{n\geq m}$ define sequences both converging to x. Similarly, the points $(y_n^1)_{n\geq m}$ and $(y_n^2)_{n\geq m}$ define sequences both converging to y. Hence we may now fix n large enough that for all $k \geq n$ both $x_k^1, x_k^2 \in N$ and $y_k^1, y_k^2 \in M$. Since η_n is continuous and injective, and $x_n^i \neq y_n^i$ ($i \in \{1,2\}$) (the x_n^i are in N, the y_n^i in M), $P_n \neq Q_n$.

If for this fixed n, $u_{C_n} \neq u_{D_n}$, then we are done. In this case, $C_n \neq D_n$ and so $\overline{C_n} \cap \overline{D_n} = \emptyset$. Since $\omega_1 \in \overline{C_n}$ and $\omega_2 \in \overline{D_n}$, $\omega_1 \neq \omega_2$. So suppose now that $u_{C_n} = u_{D_n}$. Since H_n is a Hamilton cycle, not both P_n and Q_n may contain u_{C_n}. So without loss of generality, suppose $P_n = u_n v_n$ is an edge of $K_{u_{C_n}}$. Consider the successor H_k, for some $k > n$, of H_n in which edge $u_n v_n$ is replaced by a path $P_k = u_n u_{C_k} v_n$ (we have case (3) in Section 3.1.1). At this step, C_k is a component of $G - S_k$ containing a neighbour of v_k, $v_k \in C_n$, and $C_k \subset C_n$. In $|G_k^*|$, η_k maps $\eta_n^{-1}(P_n)$ to path $P_k = u_n u_{C_k} v_n$. In fact, we can choose k large enough that η_k also maps $\eta_n^{-1}(Q_n)$ to a path containing a contracted vertex u_{D_k} with $u_{D_k} \neq u_{C_k}$ (though also $D_k \subset C_n$). Hence $C_k \neq D_k$, and $\overline{C_k} \cap \overline{D_k} = \emptyset$. Since $\omega_1 \in \overline{C_k}$ and $\omega_2 \in \overline{D_k}$, $\omega_1 \neq \omega_2$.

η is a Hamilton circle in $|G|$

Let $v \in V$. Then for some sufficiently large n, $v \in S_n$, and so $v \in H_k$ for all $k \geq n$. Hence $v \in \eta(S^1)$.

This proves sufficiency.

We commented earlier that a Hamilton circle in $|G|$ must in fact contain all the ends of G as well. This is straightforward. The continuous image of a compact space is compact ([38], Theorem 17.7), and a compact subset of a Hausdorff space is closed ([38], Theorem 17.5). Hence, as $S^1 \subset \mathbb{R}^2$ is compact, $\eta(S^1)$ is a closed subspace of $|G|$.

A closed set contains the closure of any subset it contains, and so, since $V \subset \eta(S^1)$, $\Omega \subset \overline{V} \subset \eta(S^1)$.

3.1.2 Necessity

Let $\{v_1, v_2, \ldots\}$ be an enumeration of V, and let φ be a Hamilton circle in $|G|$. For each G_n^*, $n \geq 2$, we define a mapping $\varphi_n : S^1 \to |G_n^*|$ induced by φ in $|G_n^*|$. We then use φ_n to construct a Hamilton cycle in G_n^*.

Let $\theta_n : V \to |G_n^*|$ be the vertex projection map

$$\theta_n(v) = \begin{cases} v & \text{if } v \in S_n, \\ u_C & \text{if } C \text{ is the component of } G - S_n \text{ containing } v. \end{cases}$$

Recall that every topological edge $e = [u, v]$ is the image of the unit interval under some homeomorphism $h_e : [0, 1] \to [u, v]$. Let $\rho_n : \mathring{E} \to |G_n^*|$ be the edge projection map

$$\rho_n(x) = \begin{cases} x & \text{if } x \in \mathring{e} = (u, v) \text{ where } u, v \in S_n \text{ and } uv \in E(G), \\ u_C & \text{if } x \in \mathring{e} \text{ and } e \text{ is contained in component } C \text{ of } G - S_n, \\ y & \text{if } x \in \mathring{e} = (v_i, v_j), \text{ where } v_i \in S_n, v_j \in C, \text{ a component of } G - S_n, \text{ and } y = h_{v_i u_C} \circ h_e^{-1}(x). \end{cases}$$

Let $\vartheta_n : \Omega(G) \to |G_n^*|$ be the end projection map

$$\vartheta_n(\omega) = u_C \text{ if } \omega \in \overline{C}, \text{ where } C \text{ is a component of } G - S_n.$$

And finally now let $\Theta_n : |G| \to |G_n^*|$ be the projection map

$$\Theta_n(x) = \begin{cases} \theta_n(x) & \text{if } x \in V, \\ \rho_n(x) & \text{if } x \in \mathring{E}, \\ \vartheta_n(x) & \text{if } x \in \Omega. \end{cases}$$

Let $\varphi_n = \Theta_n \circ \varphi$. Then $\varphi_n \subseteq |G_n^*|$ is a topological loop which defines a closed walk, which we also call φ_n, in G_n^* which contains every vertex of G_n^*. (See Figure 3.5 for an example.)

If φ_n is a cycle, we are done. If not, φ_n visits some vertex more than once. Suppose

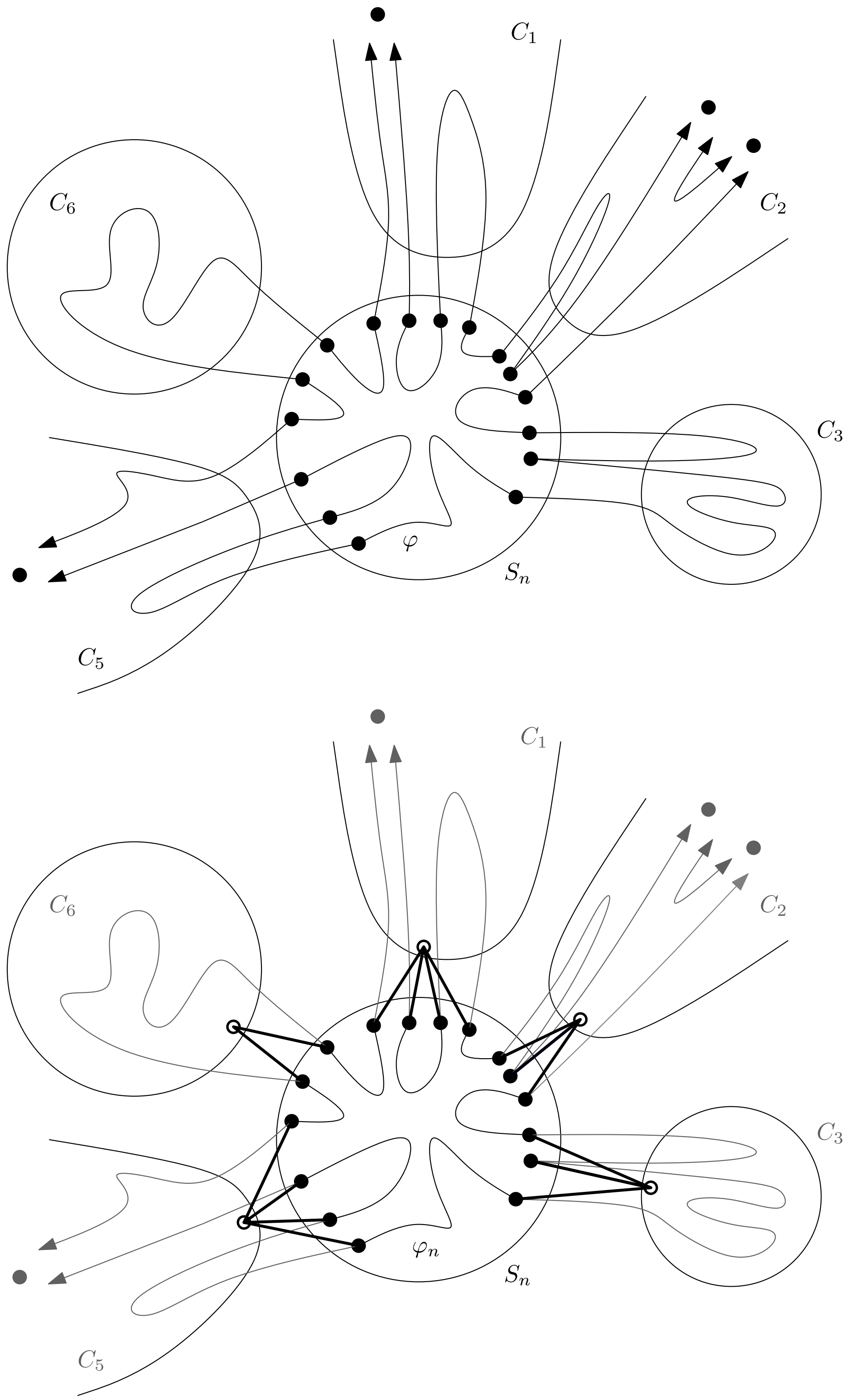

Figure 3.5: Circle φ and loop φ_n.

that a vertex $v \in S_n$ is visited twice by φ_n. Then v is the image of two distinct points $s, t \in S^1$ under φ, a contradiction, since φ is injective. Hence any such repeated vertex in φ_n must be a contracted vertex u_C for some component C of $G - S_n$. By construction of G_n^*, contracted vertex u_C is contained in a clique $K_{u_C} = u_C \cup N_{G_n^*}(u_C)$. Therefore we may modify φ_n as follows to obtain a Hamilton cycle $H_n \subseteq G_n^*$.

Choose an orientation for φ. This orientation induces an orientation on all subarcs of φ, and on φ_n, and thus on all subarcs of φ_n. For each path of the form $u_i u_C u_j \subseteq \varphi_n$ $(u_i, u_j \in S_n)$ there is a unique subarc $u_i \varphi u_j \subseteq \varphi$ such that $\Theta_n(u_i \varphi u_j) = u_i u_C u_j$.

Let v_l be the minimal vertex in our enumeration of V which is in component C of $G - S_n$. Since φ is a Hamilton circle in $|G|$, v_l must be on exactly one subarc $u_i \varphi u_j$. If $v_l \in u_i \varphi u_j$, leave path $u_i u_C u_j$ in H_n. Otherwise, replace path $u_i u_C u_j$ in φ_n with edge $u_i u_j$ in G_n^*. Since there are only finitely many contracted vertices in G_n^*, repeating this process for each of them yields a Hamilton cycle $H_n \subseteq G_n^*$. (Figure 3.6 shows H_n obtained by modifying the loop φ_n shown in Figure 3.5 in this way.)

This completes the proof of Theorem 1.7.

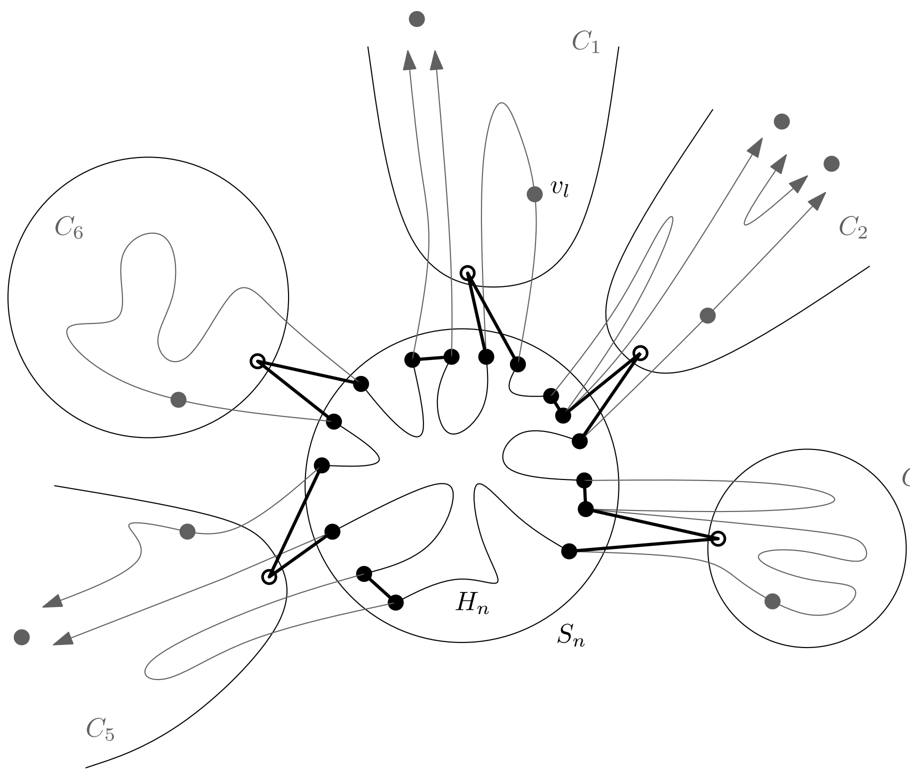

Figure 3.6: The cycle H_n constructed from φ_n. Marked vertices in each component C_i represent the vertex which is minimal in the enumeration of V in C_i.

Chapter 4

Extending Theorem 1.7 to countable graphs

Theorem 1.7 fails in general for non-locally finite graphs. The graph shown in Figure 4.1 has G_n^* hamiltonian for all $n \geq 1$ and yet fails to be hamiltonian itself.

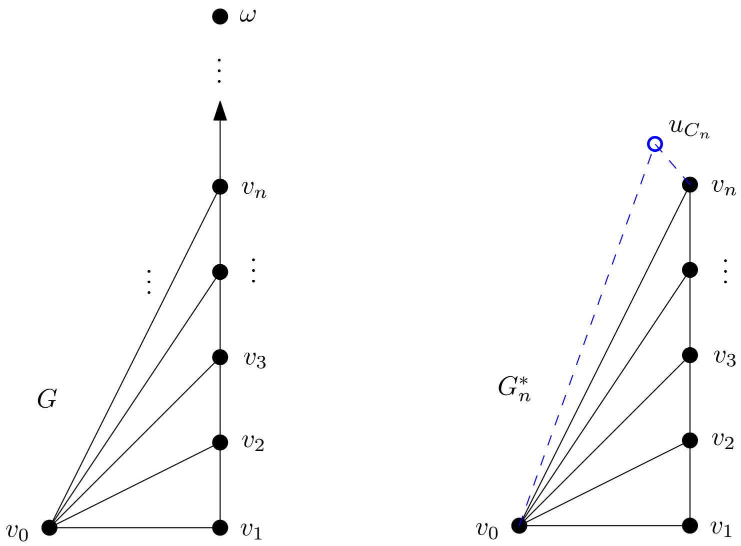

Figure 4.1: For all $n \geq 1$, G_n^* is hamiltonian, but $|G|$ does not contain a Hamilton circle.

However, with an additional hypothesis, and without a great deal of effort, we may extend Theorem 1.7 to non-locally finite graphs endowed with the standard topology. We need a definition in order to state the extra condition we require.

Given a sequence of Hamilton cycles $(H_n)_{n\geq m}$ as returned by the Infinity Lemma in Section 3.1.1, we say a vertex $v \in V$ *fits* in $(H_n)_{n\geq m}$ if there is a positive integer k

such that both edges incident with v in H_k are edges of G. We say G *fits* in $(H_n)_{n\geq m}$ if all its vertices do. If there exists such a sequence of Hamilton cycles, we say G is *fitting*.

In Figure 4.1, v_0 does not fit in the unique sequence of Hamilton cycles with $H_n \subseteq G_n^*$. Consider now the graph G in Figure 4.2. Take as S_n vertex u_0 together with the first n vertices on rays R and Q, and denote by C_n the single component of $G - S_n$ (added edges of G_n^* are shown dashed). Then u_0 fits the sequence $(H_n)_{n\geq 1}$ where, for $n \geq 2$, $H_n = u_0 r_1 q_1 \cdots q_n u_{C_n} r_n \cdots r_2 u_0$ (bold edges).

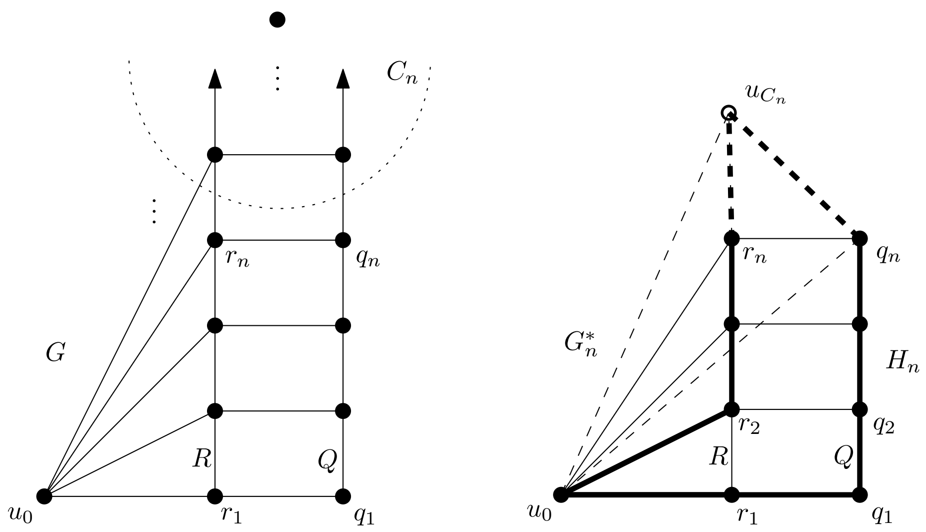

Figure 4.2: G is fitting.

Lemma 4.1. *Let $(H_n)_{n\geq m}$ be a sequence of Hamilton cycles, with each $H_n \subseteq G_n^*$, as returned by the Infinity Lemma. If $v \in V$ is of finite degree, then v fits in $(H_n)_{n\geq m}$.*

Proof. Take n large enough that v and all its neighbours in G are in S_n. Since v has no neighbours in any components of $G - S_n$, the two edges of H_n incident with v are edges of G. □

Corollary 4.2. *Let $(H_n)_{n\geq m}$ be a sequence of Hamilton cycles, with each $H_n \subseteq G_n^*$, as returned by the Infinity Lemma. If G is locally finite, then G is fitting.*

Proof. All vertices of G are of finite degree, and so by Lemma 4.1 fit any sequence $(H_n)_n$ returned by the Infinity Lemma. □

Thus requiring that G fit a sequence of Hamilton cycles in its finite contracted graphs G_n^*, is a way of saying that in some sense, each vertex of infinite degree must "act like" a vertex of finite degree in any Hamilton circle in $|G|$.

We may now state our main result for countable graphs.

Theorem 4.3 (Main Result for countable graphs)**.** *Let G be a countable graph. Then $|G|$ contains a Hamilton circle if and only if G is fitting.*

4.1 Proof of Theorem 4.3

4.1.1 Sufficiency

Let $G = (V, E, \Omega)$ be a countable graph, with enumeration $V = \{v_1, v_2, \ldots\}$. Suppose first that G is fitting. We have our sequence of Hamilton cycles $(H_n)_{n \geq m}$, each $H_n \subseteq G_n^*$, with $H_{n+1}|n = H_n$, for some $m \in \mathbb{N}$, by assumption.

The first issue we deal with is that of compactness. While for G locally finite, $|G|$ is compact, this is not so when G is not locally finite. In our proof of Theorem 1.7, we require $|G|$ compact. In particular, in the case that for all n, $\eta_n(s)$ is mapped to an edge in a clique $K_{u_{C_n}}$ or to contracted vertex u_{C_n} itself, we used the fact that $|G|$ is compact to define $\eta(s)$. However, if each G_n^* is hamiltonian, then we can show that $V \cup \Omega \subset |G|$ is compact.

We will again use the vertex projection map $\theta_n : V \to V(G_n^*)$ defined by

$$\theta_n(v) = \begin{cases} v & \text{if } v \in S_n, \\ u_C & \text{if } C \text{ is the component of } G - S_n \text{ containing } v. \end{cases}$$

Note that for any fixed n, for $v \in S_n$, $\theta_n^{-1}(v) = \{v\}$. For a contracted vertex $u_C \in V(G_n^*)$, we have $V(C) = \theta_n^{-1}(u_C) = \{v \in V : v \in C\}$.

Lemma 4.4. *Let G be a countable graph. If there is a positive integer m such that for all $n \geq m$, G_n^* is hamiltonian, then G is 1-tough.*

Proof. Suppose to the contrary that there is some finite $S \subset V$ such that the number of components of $G - S$ is greater than $|S|$. Take n large enough that $S \subseteq S_n$. We show that the number of components of $G_n^* - S$ is greater than $|S|$, a contradiction since G_n^* is hamiltonian. (In other words, if G is not 1-tough, then G_n^* is not 1-tough and so cannot be hamiltonian, a contradiction.) We do this by showing that if $x, y \in V(G_n^*)$ are both in the same component of $G_n^* - S$, then $\theta_n^{-1}(x)$ and $\theta_n^{-1}(y)$ are both in the same component of $G - S$, so the number of components of $G_n^* - S$ is at least as many as the number of components of $G - S$.

Suppose $x, y \in V(G_n^*)$ are both contained in C_n, a component of $G_n^* - S$. Then there is an x-y path $P = xx_1x_2 \cdots x_{k-1}x_ky$ in $G_n^* - S$. If x is a contracted vertex, then $x_1 \in S_n$ (since two contracted vertices may not be adjacent); similarly if y is a contracted vertex, then $x_k \in S_n$. Hence if x is a contracted vertex, say $x = u_D$ for component D of $G - S_n$, there is a vertex $w \in \theta_n^{-1}(x) = V(D)$ adjacent to x_1. There is a path connecting w and every vertex of D in $G - S_n$, and so also connecting x_1 and every vertex of D in $G - S$. In other words, $\theta_n^{-1}(x)$ is contained in the same component of $G - S$ containing x_1. Similarly if y is a contracted vertex, say $y = u_{D'}$ for component D' of $G - S_n$, there is a vertex $z \in \theta_n^{-1}(y) = V(D')$ adjacent to x_k in $G - S_n$, and there is a path connecting z and every vertex of D' in $G - S_n$, and so also there is a path connecting x_k and every vertex of D' in $G - S$. So $\theta_n^{-1}(y)$ must be contained in the same component of $G - S$ containing x_k. Hence without loss of generality we may now assume $x, y \in S_n$.

Using P, we now construct an x-y walk Q in $G - S$. Consider each edge x_ix_{i+1} of P ($i = 0, 1, \ldots, k$, where $x_0 = x, x_{k+1} = y$). If $x_i, x_{i+1} \in S_n$ and $x_ix_{i+1} \in E$, leave x_ix_{i+1} as an edge in Q. If $x_i, x_{i+1} \in S_n$ and $x_ix_{i+1} \notin E$, then both x_i and x_{i+1} are neighbours of a contracted vertex $u_C \in V(G_n^*)$ for some component C of $G - S_n$. Hence there is a neighbour $w \in C$ of x_i and a neighbour $z \in C$ of x_{i+1}. Since both $w, z \in C$, there is a w-z path in C, and so a path of the form $x_iw \cdots zx_{i+1}$ in $G - S$; replace edge x_ix_{i+1} in P with this path $x_iw \cdots zx_{i+1}$ to form Q. Now suppose $x_i \in S_n$ and $x_{i+1} \notin S_n$, say $x_{i+1} = u_D$ for some component D of $G - S_n$. Then $x_{i+2} \in S_n$ (since two contracted vertices may not be adjacent). Then there is a vertex $w \in D$ adjacent to x_i, and a vertex $z \in D$ adjacent to x_{i+2}, and so a path of the form $x_iw \cdots zx_{i+2}$ in $G - S$. Replace subpath $x_ix_{i+1}x_{i+2}$ of P with this path $x_iw \cdots zx_{i+2}$ to form Q.

Once this process has been completed for $i = 0, 1, \ldots, k$, our walk $Q \subseteq G - S$. Hence $\theta_n^{-1}(V(C_n)) \subseteq C$, for some component C of $G - S$. □

Since G is 1-tough, by Corollary 2.12, $V \cup \Omega$ is a compact subset of $|G|$.

Defining $\eta : S^1 \to |G|$

We inductively define $\eta : S^1 \to |G|$ just as in the locally finite case (as in Section 3.1.1). However, in Section 3.1.1, in the case that $\eta_n(s)$ is mapped to an edge in a clique $K_{u_{C_n}}$ or to a contracted vertex u_{C_n} for all $n \geq m$, we used the fact that $|G|$ is compact when G is locally finite to show that $\eta(s)$ is well-defined. Since we no longer have compactness of $|G|$, in this section we adjust our compactness argument so that we only require compactness of $V \cup \Omega$.

Let $s \in S^1$. If there is an $n \in \mathbb{N}$ such that $\eta_k(s) = \eta_n(s) = y$ for all $k \geq n$, define $\eta(s) = y$. In this case, y must be a vertex or inner point of an edge, so that the maps η_n agree on s for all n large enough that $y \in |G[S_n]|$. If there is no such n, then $\eta_n(s)$ is an inner point (either an inner point of an edge or a contracted vertex) on a path between two neighbours of contracted vertices u_{C_n} for every n. Since for all n, $S_n \subset S_{n+1}$, the corresponding components C_n are nested, with $C_n \supseteq C_{n+1}$ for all n. Let $\overline{V(C_n)}$ denote the topological closure of $V(C_n)$ in $|G|$. As a subspace of $|G|$, $V \cup \Omega$ is compact and each $\overline{V(C_n)}$ is a closed subset of $V \cup \Omega$. Since these components are nested, any finite collection of $\overline{V(C_n)}$ has non-empty intersection. Hence by the finite intersection property, $U = \bigcap_{n \geq m} \overline{V(C_n)}$ is non-empty.

Claim. *$U \subseteq \Omega$, and U is a singleton.*

Proof of claim. Of course $U \subseteq V \cup \Omega$. Suppose there is a vertex $v \in U$. There is a $k \in \mathbb{N}$ such that $v \in S_k$. Then for all $n \geq k$, $v \notin V(C_n)$. Since for each $n \geq k$, $\overline{V(C_n)} = V(C_n) \cup \Omega(C_n)$ and $v \notin \Omega(C_n)$, $v \notin \overline{V(C_n)}$. Hence $v \notin \bigcap_{n \geq m} \overline{V(C_n)} = U$, a contradiction. Therefore $U \subseteq \Omega$.

For any two distinct ends $\omega, \xi \in \Omega$, there is a finite vertex set S such that the tails of rays in ω and the tails of rays in ξ are in different components of $G - S$. Hence as soon as $S \subseteq S_n$, $\overline{V(C_n)}$ cannot contain both ω and ξ. Therefore U is just a single end ω. □

Define $\eta(s) = \omega$. Similar to the locally finite case, we have

$$\eta(s) \in \overline{V(C_n)} \text{ for every } n. \tag{4.1}$$

η is an embedding of S^1 in $|G|$

Just as in the locally finite case, since every injective continuous map from a compact space into a Hausdorff space is a homeomorphism onto its range, we just have to show that η is continuous and injective.

The only differences now from the locally finite case are that in showing continuity, instead of using the fact that every vertex has finite degree to find our required open neighbourhoods in S^1, we use the fact that G fits $(H_n)_{n \geq m}$; and again we can only rely on the compactness of $V \cup \Omega$ rather than that of $|G|$.

First we consider points $s \in S^1$ which are mapped to a vertex or an inner point of an edge of $|G|$. Let $s \in S^1$ be such a point. Then for some sufficiently large n,

$\eta(s) = \eta_k(s)$ for all $k \geq n$. Recall that all these η_k are continuous and injective at s. We will establish that therefore η is continuous and injective at s as well. We first show continuity, then injectivity.

Given any neighbourhood O of the point $\eta(s)$, consider a basic open neighbourhood N of $\eta(s)$ contained in O. We will find an open neighbourhood J of s such that $\eta(J) \subseteq N$.

Claim. *We may take n large enough that η_n and η agree on $\eta_n^{-1}(N)$.*

Proof of Claim. First suppose $\eta(s) = y \in \mathring{e}$ is an inner point of an edge of $|G|$. Then $e = [v_i, v_j]$ for some $v_i, v_j \in V(G)$. Take n large enough that both $v_i, v_j \in S_n$. Then $\eta_n^{-1}(\mathring{e})$ is an open interval of S^1. By construction, η and η_n agree on this interval. Since any basic open neighbourhood N of $\eta(s)$ is contained in $\mathring{e}$, η_n agrees with η on $\eta_n^{-1}(N)$.

Now suppose $\eta(s) = y \in V(G)$. Since y fits in $(H_n)_{n \geq m}$, there is an $n \in \mathbb{N}$ such that the two edges incident with y in H_n are both edges of G. In particular, there is a path $v_i y v_j$ in H_n also a path in G, with $v_i, v_j \in S_n$. Hence η_n and η agree on $\eta_n^{-1}(v_i y v_j)$. Since any basic open neighbourhood N of y consists of y together with a union of half-edges $[y, z)$, one for each edge incident with y (z an inner point of the edge), $\eta_n^{-1}(N) \subseteq \eta_n^{-1}(v_i y v_j)$. □

As before, take as J an open interval around s contained in $\eta_n^{-1}(N)$. Then $\eta(J) = \eta_n(J) \subseteq N$. So η is continuous on points of S^1 mapped to vertices or inner points of edges of $|G|$.

We see that η is injective on such points just as in the locally finite case: If $\eta(x) = \eta(y)$ for some $x, y \in S^1$ mapped to a vertex or inner point of an edge, then taking n large enough that η_n agrees with η on both x and y implies $x = y$, since η_n is injective.

Now to show continuity and injectivity on Ω, we proceed as in the locally finite case, making use of the fact the G fits wherever previously we had relied on G being locally finite. To show continuity, for every neighbourhood O of ω we have to find a neighbourhood $J \subset S^1$ of s such that $\eta(J) \subseteq O$. Since O contains a basic open neighbourhood $\widehat{C}(S, \omega)$, for some finite $S \subset V$, taking n large enough that $S \subseteq S_n$ yields a component C of $G - S_n$ such that $\overline{C} \subseteq O$. This component C corresponds to a vertex $u_C \in G_n^*$, and $\eta_n(s)$ is an inner point of an edge in K_{u_C} or contracted vertex u_C. Let P_n be the S_n-path in G_n^* containing $\eta_n(s)$. Let v_i, v_j be the terminal vertices of $P_n \subseteq |G_n^*|$, and let $J_n = \eta_n^{-1}(P_n) \subset S^1$.

Since G fits in $(H_n)_{n\geq m}$, we can take k large enough that both edges incident with v_i and both edges incident with v_j in H_k are edges of G. Thus (now just as in the locally finite case), there are vertices $v_p, v_q \in C$ with $v_iv_p, v_jv_q \in E(H_k)$ and $v_iv_p, v_jv_q \in E(G)$. The rest of the proof of continuity is just as in the locally finite case, as is the proof of injectivity. Again, we simply use the fact that G fits to obtain whatever we had previously required finite degree to obtain (these arguments are given in Section 3.1.1, starting on page 37).

η is a Hamilton circle in $|G|$

Just as in the locally finite case, for any $v \in V$, $v \in S_n$ for some sufficiently large n. Hence $v \in H_k$ for all $k \geq n$, and so v is contained in $\eta(S^1)$.

This proves sufficiency.

4.1.2 Necessity

Let $\varphi \subseteq |G|$ be a Hamilton circle. Let φ_n be the topological loop $S^1 \to |G_n^*|$ defined by $\varphi_n = \Theta_n \circ \varphi$, and obtain $H_n \subseteq G_n^*$ just as in Section 3.1.2 (the argument there begins on page 41). This defines a sequence $(H_n)_{n\geq 2}$. To show that G is fitting, we first show that $H_{n+1}|n = H_n$, then that G fits this sequence.

In this construction (Section 3.1.2), we let v_l be the minimal vertex in our enumeration of V which is in component C of $G - S_n$, and observe that since φ is a Hamilton circle in $|G|$, v_l must be on exactly one subarc $u_i\varphi u_j$. We obtain H_n from φ_n by leaving path $u_iu_Cu_j$ in φ_n if $v_l \in u_i\varphi u_j$, and otherwise replacing path $u_iu_Cu_j$ in φ_n with edge u_iu_j in G_n^*.

We simply observe at this point that vertex v_{n+1} must be on exactly one of these subarcs of φ in exactly one component C of $G - S_n$, with corresponding contracted vertex u_C in G_n^*, and that v_{n+1} is minimal in C. Hence, as we show below, the construction of $H_n \subseteq G_n^*$ from loop φ_n, and the construction of $H_{n+1}|n$, from $H_{n+1} \subseteq G_{n+1}^*$ (which in turn was obtained from loop φ_{n+1}), agree.

Cycle $H_n \subseteq G_n^*$ is constructed from loop φ_n according the rule that the edge u_iu_j replaces $u_iu_Cu_j$ if $v_l \notin u_i\varphi u_j$, and path $u_iu_Cu_j$ remains if $v_l \in u_i\varphi u_j$. But in the construction of $H_{n+1}|n$, from $H_{n+1} \subseteq G_{n+1}^*$, we follow the same rule: $H_{n+1}|n$ remains the same as H_{n+1} except on paths between vertices of K_{u_C}, where C is the component of $G - S_n$ containing v_{n+1}. We replace each path $u_iu_{C_i}u_j$, where $C_i \subset C$ is a component of $G - S_{n+1}$, with edge u_iu_j, and replace the S_n-path containing v_{n+1} with path $u_i u_C u_j$. Observe that H_{n+1} is constructed from loop φ_{n+1}, and H_n is constructed from loop φ_n, and φ_{n+1} and φ_n agree on all their S_n-subarcs not traversing a vertex of C, where C is the component of $G - S_n$ containing v_{n+1}. Hence H_{n+1} and H_n agree on all S_n-subpaths of H_{n+1} not containing a vertex u_{C_i} with $C_i \subset C$. Therefore the construction of H_n from loop φ_n and the construction of $H_{n+1}|n$ agree. I.e., $H_{n+1}|n$ is precisely H_n.

To see that G fits $(H_n)_{n\geq 2}$, let $v \in V$, and consider the two neighbours of v in $\varphi(S^1)$, say v_i, v_j. Then $vv_i, vv_j \in E(G)$. Take n large enough that $v, v_i, v_j \in S_n$. Then $vv_i, vv_j \in E(G[S_n])$. Hence for all $k \geq n$, vv_i and vv_j are edges in $\varphi_k = \Theta_k \circ \varphi$, and so $vv_i, vv_j \in E(H_k)$. Therefore v fits in $(H_n)_{n\geq 2}$.

This completes the proof of Theorem 4.3.

Chapter 5

Corollaries

In this Chapter we obtain several corollaries of Theorem 1.7.

5.1 Locally finite claw-free and line graphs

As our first corollaries of Theorem 1.7, we obtain extensions of Zhan's Theorem 1.3 and Ryjáček's Theorem 1.4 to locally finite graphs.

Corollary 1.8. *Every locally finite 7-connected line graph is hamiltonian.*

Corollary 1.9. *Every locally finite 7-connected claw-free graph is hamiltonian.*

5.1.1 Locally finite 7-connected claw-free graphs are hamiltonian.

If a subset $\{a, b, c, d\} \subseteq V$ induces a claw in G, in which $\deg_G(a) = 3$ and $\deg_G(b) = \deg_G(c) = \deg_G(d) = 1$, we call a the *centre* of the claw. When listing the vertices of a claw, its centre will be listed first. We write $G[a, b, c, d]$ for $G[\{a, b, c, d\}]$.

Lemma 5.1. *Let G be a countable k-connected claw-free graph. Then G^*_n is k-connected and claw-free, for all $n \geq k$.*

Proof. Suppose first to the contrary that G^*_n contains a claw $G^*_n[w, x, y, z]$. Clearly w cannot be a contracted vertex, since a contracted vertex together with its neighbours is a clique in G^*_n. Suppose $x = u_C$ for some component C of $G - S_n$ and $y, z \in S_n$. Since $w \in N_{G_n}(u_C)$, there is a neighbour u of w, such that $u \in C$. Then $yu \notin E(G)$ and

$zu \notin E(G)$, for otherwise yu_C and zu_C would be edges in G_n^*. But then $G[w, u, y, z]$ is a claw in G, a contradiction.

A similar argument shows that none of x, y, z may be contracted vertices. Therefore each of $w, x, y, z \in S_n$. Since $G[S_n]$ is claw-free, $G_n^*[w, x, y, z]$ must contain an edge $e \in K_{u_C}$ added among the neighbourhood of a contracted vertex u_C for some component C of $G - S_n$. Suppose without loss of generality that $wx \in K_{u_C}$. Since $G_n^*[w, x, y, z]$ is a claw, neither y nor z is adjacent to u_C: if $yu_C \in E(G_n^*)$, then $xy \in E(G_n^*)$; similarly if $zu_C \in E(G_n^*)$, then $xz \in E(G_n^*)$. But then $G_n^*[w, u_C, y, z]$ is a claw in G_n^*, a contradiction since we have just shown that a claw in G_n^* cannot contain a contracted vertex.

We now show that G_n^* is k-connected for all $n \geq k$. Suppose that for some $n \geq k$, there is a minimal cut set $W \subseteq V(G_n^*)$ with $|W| < k$. We claim that W does not contain any contracted vertex u_C of G_n^*. For suppose so. Then for every path P in G_n^* traversing u_C there is an otherwise identical path avoiding u_C via the edge between the two neighbours of u_C in P. Hence removing u_C from W still leaves a cut set, contradicting the minimality of W.

Therefore $W \subseteq S_n$. Consider any two distinct vertices $u, v \in V(G_n^*)$ in different components of $G_n^* - W$, and consider $u' \in \theta_n^{-1}(u)$ and $v' \in \theta_n^{-1}(v)$ (note that $u' = u$ if $u \in S_n$; similarly for v'). Since G is k-connected, there is a u'-v' path P in G which avoids W. Since θ_n is injective on $W \subseteq S_n$, $\theta_n(P)$ is a u-v walk in G_n^* which avoids W, a contradiction. □

Proof of Corollary 1.9. Let G be a locally finite 7-connected claw-free graph. By Lemma 5.1, and Corollary 6 of [31] (which states that every finite 7-connected claw-free graph is hamiltonian), G_n^* is hamiltonian for all $n \geq 7$. Hence by Theorem 1.7, $|G|$ contains a Hamilton circle. □

Since line graphs are claw-free, Corollary 1.8 follows.

5.1.2 Locally finite 4-connected claw-free and line graphs

As discussed in Section 1.4, Thomassen has conjectured that all finite 4-connected line graphs are hamiltonian (Conjecture 1.10), and Matthews and Sumner that every finite 4-connected claw-free graph is hamiltonian (Conjecture 1.11). Ryjáček [31] has shown that these two conjectures are equivalent:

Lemma 5.2 ([31], Corollary 5)**.** *Every finite 4-connected line graph is hamiltonian if and only if every finite 4-connected claw-free graph is hamiltonian.*

This suggests the following:

Conjecture 5.3 (Georgakopoulos, [23])**.** *Every locally finite 4-connected line graph is hamiltonian.*

Conjecture 5.4. *Every locally finite 4-connected claw-free graph is hamiltonian.*

Theorem 1.7 implies that, just as in the finite case, these two conjectures are equivalent when applied to graphs endowed with the standard topology:

Corollary 5.5. *The following statements are equivalent.*

1. *Every locally finite 4-connected claw-free graph is hamiltonian.*

2. *Every locally finite 4-connected line graph is hamiltonian.*

Proof. ($\Longrightarrow$) Every line graph is claw-free.

($\Longleftarrow$) Suppose (2) holds and G is a locally finite 4-connected claw-free graph. Then by Lemma 5.1, for all $n \geq 4$, G_n^* is a finite 4-connected claw-free graph. Hence by assumption and Lemma 5.2, G_n^* is hamiltonian for all $n \geq 4$. Hence by Theorem 1.7, G is hamiltonian. □

Furthermore, if G is a locally finite 4-connected line graph, then G_n^* is 4-connected and claw-free (by Lemma 5.1). Hence if Thomasson's Conjecture 1.10 is true, then by Lemma 5.2, each G_n^* is hamiltonian, and so by Theorem 1.7, $|G|$ contains a Hamilton circle. Conversely, if Conjecture 5.3 holds, then since finite graphs are locally finite, so does Thomassen's Conjecture 1.10. Hence we have:

Corollary 5.6. *The following statements are equivalent:*

1. *Thomassen's Conjecture 1.10 is true.*

2. *Matthews and Sumner's Conjecture 1.11 is true.*

3. *Conjecture 5.3 is true.*

4. *Conjecture 5.4 is true.*

5.2 Powers of graphs

We now show that Georgakopoulos' Theorems 1.5 and 1.6 are also corollaries of Theorem 1.7.

Corollary 5.7. *If G is a locally finite 2-connected graph, then $|G^2|$ contains a Hamilton circle.*

Let $G = (V, E, \Omega)$ be a locally finite 2-connected graph. Let us label all edges in $E(G)$ *blue* and all edges in $E(G^2) \setminus E(G)$ *red*. In $(G^2)_n$, let all edges keep their labels in G^2: if $e \in E((G^2)_n)$ is blue in G^2, let e be blue in $(G^2)_n$, and if e is red in G^2, let e be red in $(G^2)_n$. If $e \in E((G^2)_n)$ is an edge between a vertex in S_n and a contracted vertex u_C for some component C of $G^2 - S_n$ which has had both a blue and a red edge identified with it in $(G^2)_n$, label e blue in $(G^2)_n$. A *blue path* (respectively *red path*) is a path all of whose edges are blue (respectively, red). We say two vertices are *red-adjacent* (respectively *blue-adjacent*) if they are endvertices of a red edge (respectively, blue), and we say these vertices are *red-neighbours* (respectively, blue-neighbours). Note that those edges added to produce a clique on a contracted vertex to form $(G^2)_n^*$ are not labeled blue or red. For each contracted vertex u_C in $(G^2)_n^*$, let us label *green* the subset of edges in K_{u_C} which are between blue-neighbours of u_C but which are not otherwise labeled blue or red. (Note that endvertices of green edges must belong to S_n.)

Consider the spanning subgraph H_n of $(G^2)_n^*$ with edge set precisely the blue and green edges of $(G^2)_n^*$.

Lemma 5.8. *H_n is 2-connected for all $n \geq 2$.*

Proof. Any cut-vertex of H_n would be a cut-vertex of G, but G is 2-connected. To show this in detail, we argue as follows.

Suppose there is a cut vertex x of H_n. Suppose first that x is a contracted vertex. We first show that there cannot be a component A of $H_n - x$ containing only red-neighbours of x: If $u \in A$ and ux is a red edge in H_n, then there is a vertex $u' \in \theta_n^{-1}(x)$ red-adjacent to u in G^2, so there must a vertex w and blue edges $uw, u'w$ in $E(G)$. If $w \in A$, then xw is a blue edge in H_n, and if $w \in \theta_n^{-1}(x)$, then ux is a blue edge in H_n. So every component of $H_n - x$ has a vertex blue-adjacent to x. By construction of H_n, these vertices are all green-adjacent, a contradiction.

So $x \in S_n$. Consider any two distinct vertices $u, v \in V(H_n)$ in different components of $H_n - x$. Consider $u' \in \theta_n^{-1}(u)$ and $v' \in \theta_n^{-1}(v)$ (note that $u' = u$ if $u \in S_n$;

similarly for v'). Since G is 2-connected, there is a u'-v' path P in G not traversing x. Since θ_n is injective on S_n, $\theta_n(P)$ is a u-v path in H_n not traversing x, a contradiction. □

Lemma 5.9. *There are no contracted vertices at distance two in H_n.*

Proof. Suppose there are two contracted vertices u_C and u_D in H_n at distance two in H_n, with path $u_C v u_D$ between them. Edges vu_C and vu_D must be blue, since green edges of H_n only occur between blue-neighbours of a contracted vertex. Then there is an edge $vw \in E(G)$ for some vertex $w \in C$, and an edge $vx \in E(G)$ for some vertex $x \in D$. Hence wvx is a path in G, and so $wx \in E(G^2)$. But x and w are in different components of $G^2 - S_n$, so this is a contradiction. □

Lemma 5.9 immediately implies that there can be no edges in $(H_n)^2$ between contracted vertices.

Lemma 5.10. $E((H_n)^2) \subseteq E((G^2)^*_n)$.

Proof. We show that any two vertices of H_n which are adjacent in $(H_n)^2$ are also adjacent in $(G^2)^*_n$. Suppose $uv \in E((H_n)^2)$. If $uv \in E(H_n)$, then uv is a blue or a green edge in $(G^2)^*_n$.

So assume $uv \notin E(H_n)$. Hence there is a path uxv in H_n. Note that x cannot be a contracted vertex: for if so, then uv is a green edge in $(G^2)^*_n$. We can select a vertex $u' \in \theta_n^{-1}(u)$ and $v' \in \theta_n^{-1}(v)$ such that $u'xv'$ is a path in G.

If both ux, xv are blue edges, then there exist $u'x, xv' \in E(G)$, so $uv \in E((G^2)^*_n)$ (a red edge). If both ux, xv are green edges, then $u, x, v \in S_n$, and u, x are both blue-adjacent to a contracted vertex u_C, and x, v are both blue-adjacent to a contracted vertex u_D. If $u_C \neq u_D$, then $u_C x u_D$ is a path of length two in H_n, contradicting Lemma 5.9. Hence $u_C = u_D$, and u and v are both blue-adjacent to u_C. Therefore uv is a green edge in $(G^2)^*_n$.

Suppose now ux is green and xv is blue. Then u and x are both in S_n and both blue-neighbours of the same contracted vertex u_C. Therefore by Lemma 5.9, v cannot be a contracted vertex; i.e. $v \in S_n$. There is a vertex $y \in C$ such that $xy \in E(G)$. Hence v and y are at distance two in G, and so $vy \in E(G^2)$. Then $vu_C \in E((G^2)^*_n)$. Since both u and v are neighbours of u_C, $uv \in E((G^2)^*_n)$. □

We now obtain Theorem 1.5 (our Corollary 5.7):

Proof of Corollary 5.7. By Lemma 5.8 and Theorem 1.2, $(H_n)^2$ is hamiltonian for all $n \geq 2$. By Lemma 5.10 then, $(G^2)^*_n$ is hamiltonian for all $n \geq 2$. Therefore by Theorem 1.7, $|G^2|$ contains a Hamilton circle. □

Similarly, we obtain Theorem 1.6:

Corollary 5.11. *If G is a locally finite connected graph, then $|G^3|$ contains a Hamilton circle.*

Proof. Similarly to above, label all edges in $E(G)$ *blue*, all edges in $E(G^3) \setminus E(G)$ *red*, and label *green* any edges of $(G^3)^*_n$ which are blue-neighbours of the same contracted vertex which are not otherwise labeled blue or red. Define the graph H_n to be the graph on $V(G^3)^*_n$ with edge set precisely the blue and green edges of $(G^3)^*_n$.

Since G is connected, each H_n is connected. Analogously to Lemma 5.9, we claim

Claim. *There are no distinct contracted vertices in H_n at distance three or less.*

Proof of Claim. Suppose $u_C \neq u_D$ are contracted vertices in H_n at distance three or less. Then there is a u_C-u_D path $P = u_C xy u_D$ or $Q = u_C z u_D$ in H_n. If Q is such a path in H_n, then there is an edge $wz \in E(G)$ for some vertex $w \in C$ and an edge $zt \in E(G)$ for some vertex $t \in D$. Then wzt is a path in G, and so $wt \in E(G^3)$, a contradiction. If P is such a path in H_n, then there is an edge $wx \in E(G)$ for some vertex $w \in C$ and an edge $yt \in E(G)$ for some vertex $t \in D$. If xy is a blue edge, then $wxyt$ is a path in G, and so $wt \in E(G^3)$, a contradiction.

If xy is a green edge, then x and y are both blue-adjacent to some contracted vertex u_B. Hence there are edges xp and yq in G for some vertices $p, q \in B$. Then $wp \in E(G^3)$ and $tq \in E(G^3)$. Hence $C = B = D$, a contradiction. □

Analogously to Lemma 5.10, we now also claim

Claim. $E((H_n)^3) \subseteq E((G^3)^*_n)$.

Proof of Claim. Suppose $e = uv \in E((H_n)^3)$. If $uv \in E(H_n)$, we are done. So suppose $uv \notin E(H_n)$. Then there is a u-v path of length two or three in H_n.

As in the proof of Lemma 5.10, if u is a contracted vertex, then for every blue-neighbour x of u in $(G^3)^*_n$ there is a blue-neighbour $u' \in \theta_n^{-1}(u)$ of x in G; similarly for v we denote such a vertex v'. If $u \in S_n$, then $\theta_n^{-1}(u) = \{u\}$ and $u' = u$; similarly for v.

If u, v are at distance two in H_n, then a similar argument as in the proof of Lemma 5.10 shows $uv \in E((G^3)^*_n)$. If u, v are at distance three in H_n, let $P = uxyv$ be a path in H_n. If P is a blue-path, then we may choose vertices $u' \in \theta_n^{-1}(u)$, $v' \in \theta_n^{-1}(v)$, $y' \in \theta_n^{-1}(y)$, and $z' \in \theta_n^{-1}(z)$, such that $u'x'y'v'$ is a blue-path in G. Then $u'v' \in E(G^3)$, so $uv \in E((H_n)^3)$ (a red edge).

Suppose now P contains a green edge, say edge ux is green. Then both x and u are blue-neighbours of the same contracted vertex u_C. If $y = u_C$ then v is a blue-neighbour of u_C and $uv \in E((G^3)^*_n)$. If y is a different contracted vertex, then u_Cxy would be a path of length two in H_n, a contradiction. So $y \in S_n$. Now if v is a contracted vertex, then u_Cxyv is a path of length three in H_n, a contradiction. So $v \in S_n$. Since x is a blue-neighbour of u_C, there is a vertex $z \in C$ such that $xz \in E(G)$. Hence yz and vz are edges in G^3, and so $yu_C, vu_C \in E((G^3)^*_n)$. Hence $uv \in E((G^3)^*_n)$. The case that yv is a green edge is identical after relabeling.

If edge xy is green, the argument is similar: For some component C of $G^3 - S_n$, x and y are blue-adjacent to contracted vertex u_C. Neither u nor v can be contracted vertices, since then H_n would have a path of length two between contracted vertices, a contradiction. For some vertices $z, w \in C$ there are edges $xz, yw \in E(G)$. Hence $uz, vw \in E(G^3)$, and so $uu_C, vu_C \in E((G^3)^*_n)$. Since both u and v are neighbours of u_C in $(G^3)^*_n$, $uv \in E((G^3)^*_n$. □

Since the third power of any finite connected graph is hamiltonian ([28, 32]), for all $n \geq 2$, $(H_n)^3$ is hamiltonian. Since each $(H_n)^3$ is a spanning subgraph of $(G^3)^*_n$, for all $n \geq 2$, $(G^3)^*_n$ is hamiltonian. Therefore by Theorem 1.7, $|G^3|$ contains a Hamilton circle. □

Chapter 6

Directions for further research

6.1 Non-locally finite graphs

Having confirmed Georgakopoulos' conjecture that all locally finite 7-connected line graphs are hamiltonian (Corollary 1.8), it is natural to ask whether this result holds for arbitrary countable graphs. It may be that Theorem 4.3 can be used to prove extensions to countable graphs of Zhan's Theorem 1.3 and Ryjáček's Theorem 1.4.

While in the locally finite case it is enough to show, for a given graph G, that its finite contracted graphs G_n^* are hamiltonian, for arbitrary countable graphs we also have to show that G is fitting. In a line graph, the neighbourhood of any vertex is the disjoint union of at most two cliques, so a vertex of infinite degree must be contained in an infinite clique. Constructing a fitting sequence therefore should be possible. It can be shown that a vertex of infinite degree in a claw-free graph must also be contained in an infinite clique. However, in general, less structure is forced on a claw-free graph by the existence of such a vertex than in a line graph. For each of these families of graphs, it is the possible interactions of other structures in a graph G with an infinite clique contained in G which makes the problem of showing G is fitting non-trivial. Since line graphs possess more structure, it is here we may first expect to make progress.

Conjecture 6.1. *If G is a countable 7-connected line graph, then $|G|$ contains a Hamilton circle.*

The Ryjáček closure operation defined in [31] for finite graphs may also be defined for countable graphs. Applied to a countable claw-free graph G, this operation yields a line graph $cl(G)$. Analogously to the finite case in [31], we conjecture

Conjecture 6.2. *For any circle σ in $|cl(G)|$, there exists a circle φ in $|G|$ such that all vertices contained in σ are also contained in φ.*

Thus analogous to the finite case, as shown in [31], we would have that a countable graph G is hamiltonian if and only if $cl(G)$ is. If Conjecture 6.1 could be proved, then the following conjecture would follow.

Conjecture 6.3. *If G is a countable 7-connected claw-free graph, then $|G|$ contains a Hamilton circle.*

Related are the following two conjectures of Georgakopoulos, which would extend his work on powers of graphs from locally finite to countable graphs.

Conjecture 6.4 ([21])**.** *If G is a countable 2-connected graph, then G^2 has a Hamilton circle.*

Conjecture 6.5 ([21])**.** *If G is a countable connected graph, then G^3 has a Hamilton circle.*

Just as vertices of infinite degree in line and claw-free graphs are contained in infinite cliques, a similar structure occurs in squares of graphs: if v is a vertex of infinite degree in G, then $N_G(v) \cup \{v\}$ is an infinite clique in G^2. As we used Theorem 1.7 to obtain Theorems 1.5 and 1.6, perhaps Theorem 4.3 could be used to prove Conjectures 6.4 and 6.5.

Dealing with planar graphs using either of Theorems 1.7 or 4.3 would likely be more difficult still, since large cliques are formed in the contracted graphs G_n^*. However, an approach similar to that used in the proofs of Corollaries 5.7 and 5.11, in which we found a hamiltonian spanning subgraph of G_n^*, may be possible. It may be that judicious removal of edges from these cliques (judicious enough that connectivity is maintained) could allow us to find a planar 4-connected spanning subgraph of G_n^*. It is conceivable therefore that such an approach could yield progress toward Bruhn's conjecture,

Conjecture 6.6 ([10], [6])**.** *Let G be a locally finite 4-connected planar graph. Then $|G|$ contains a Hamilton circle.*

6.2 Other topologies

While the standard topology defined in Section 2.2.3 is legitimately applied to non-locally finite graphs, there may be good reasons for considering other topologies. One of the most natural is the following variation on $|G|$, introduced by Diestel and Kühn in [17] and referred to there as the *identification topology*, denoted $\widetilde{G}$.

Just as the rays in an end of G may not be finitely separated, a vertex of infinite degree v may not be finitely separated from any ray to which it is linked by infinitely many paths which are disjoint except at v. We say such a vertex *dominates* the end to which such a ray belongs (in Figure 4.1 v_0 dominates ω; in Figure 4.2 u_0 dominates the single end of G (pages 44-45)). A topology which respects the "infinite connectedness" of such structures may be obtained by simply identifying each such vertex with all the ends it dominates. The topological space $\widetilde{G}$ is the resulting quotient space obtained from $|G|$ in this way. (It turns out that the set of undominated ends of G are precisely the set of ends of G which correspond to the topological ends of G as a 1-complex, as originally defined by Freudenthal [19]; this is proved in [14].)

Extensions of the theorems and conjectures considered here to arbitrary countable graphs G with the identification topology $\widetilde{G}$ are desirable. However, the behaviour of circles in non-locally finite graphs may be strange. For instance, there are graphs G whose associated space $\widetilde{G}$ may contain circles with empty edge sets, or distinct circles having the same edge set. Furthermore, in the context of attempts to extend the theory of the cycle space of a finite graph to infinite graphs, unavoidable obstructions may occur in graphs containing two vertices linked by infinitely many internally disjoint paths. The Diestel-Kühn cycle space theory for $\widetilde{G}$, developed in [17], is therefore restricted to graphs in which no two vertices are linked by infinitely many internally disjoint paths. We have begun translating the work in this thesis to this setting, and it appears there is a natural analogue of Theorem 1.7 for the space $\widetilde{G}$. However, no countable line graph, claw-free graph, or power of a graph, satisfies the condition that no two vertices be linked by infinitely many internally disjoint paths as soon as it contains a vertex of infinite degree.

To deal with this situation, a more general approach, such as that begun by Vella, Richter, and Casteels [7, 36] may be required. This approach was begun by Vella and Richter in [36] in order to unify the notions of the cycle space of an infinite graph of [2] and that of Diestel and Kühn, and also to answer a call by Diestel and Kühn in [16] to characterize the cycle space of an infinite graph in purely combinatorial terms. In this approach, edges are open singletons in a topological space associated with a graph. Generalized "edge spaces" are considered; these contain *minimally connected subsets* analogous to spanning trees. Here, a *cycle* is a non-empty connected edge space (X, E) such that for every $e \in E$, $X \setminus e$ is (topologically) connected, but for every distinct $e, f \in E$, $X \setminus \{e, f\}$ is not (topologically) connected. Thus, the call of Diestel and Kühn is answered, after a fashion, in that a reliance on topological embeddings of S^1 is not required.

Vella and Richter also show that circles in $\widetilde{G}$ are also cycles under their definition. As long as no two vertices in G are linked by infinitely many edge-disjoint paths, Vella-Richter cycles also always correspond to circles in $\widetilde{G}$. Further, it is shown that (when G has no two vertices linked by infinitely many internally disjoint paths and is 2-connected) the Vella-Richter cycle space and the Diestel-Kühn cycle space in $\widetilde{G}$ are the same ([36], Theorem 30).

When G is locally finite, $|G| = \widetilde{G}$. Hence our results of Chapters 3 and 5 hold in Vella-Richter generalized edge spaces. It may be that such a general approach will also allow for statements about hamiltonicity of non-locally finite line graphs, claw-free graphs, or powers of graphs, or other families of graphs.

Bibliography

[1] Mark Anthony Armstrong, *Basic topology*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1983.

[2] C. Paul Bonnington and R. Bruce Richter, *Graphs embedded in the plane with a bounded number of accumulation points*, J. Graph Theory **44** (2003), no. 2, 132–147.

[3] Henning Bruhn, *The cycle space of a 3-connected locally finite graph is generated by its finite and infinite peripheral circuits*, J. Combin. Theory Ser. B **92** (2004), no. 2, 235–256.

[4] Henning Bruhn, Reinhard Diestel, and Maya Stein, *Menger's theorem for infinite graphs with ends*, J. Graph Theory **50** (2005), no. 3, 199–211.

[5] Henning Bruhn and Maya Stein, *MacLane's planarity criterion for locally finite graphs*, J. Combin. Theory Ser. B **96** (2006), no. 2, 225–239.

[6] Henning Bruhn and Xingxing Yu, *Hamilton cycles in planar locally finite graphs*, SIAM J. Discrete Math. **22** (2008), no. 4, 1381–1392.

[7] Karel Casteels and R. Bruce Richter, *The bond and cycle spaces of an infinite graph*, J. Graph Theory **59** (2008), no. 2, 162–176.

[8] Nathaniel Dean, Robin Thomas, and Xingxing Yu, *Spanning paths in infinite planar graphs*, J. Graph Theory **23** (1996), no. 2, 163–174.

[9] Reinhard Diestel, *The countable Erdős-Menger conjecture with ends*, J. Combin. Theory Ser. B **87** (2003), no. 1, 145–161.

[10] ______, *The cycle space of an infinite graph*, Combin. Probab. Comput. **14** (2005), no. 1-2, 59–79.

[11] ———, *The Erdős-Menger conjecture for source/sink sets with disjoint closures*, J. Combin. Theory Ser. B **93** (2005), no. 1, 107–114.

[12] ———, *Graph theory*, third ed., Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 2005.

[13] ———, *End spaces and spanning trees*, J. Combin. Theory Ser. B **96** (2006), no. 6, 846–854.

[14] Reinhard Diestel and Daniela Kühn, *Graph-theoretical versus topological ends of graphs*, J. Combin. Theory Ser. B **87** (2003), no. 1, 197–206.

[15] ———, *On infinite cycles. I*, Combinatorica **24** (2004), no. 1, 69–89.

[16] ———, *On infinite cycles. II*, Combinatorica **24** (2004), no. 1, 91–116.

[17] ———, *Topological paths, cycles and spanning trees in infinite graphs*, European J. Combin. **25** (2004), no. 6, 835–862.

[18] Herbert Fleischner, *The square of every two-connected graph is Hamiltonian*, J. Combinatorial Theory Ser. B **16** (1974), 29–34.

[19] Hans Freudenthal, *Über die Enden topologischer Räume und Gruppen*, Math. Z. **33** (1931), no. 1, 692–713.

[20] A. Georgakopoulos and P. Sprüssel, *Geodesic topological cycles in locally finite graphs*, Preprint, 2006.

[21] Agelos Georgakopoulos, *Infinite hamilton cycles in squares of locally finite graphs*, Advances in Mathematics **220** (2009), no. 3, 670–705.

[22] Angelos Georgakopoulos, *Connected but not path-connected subspaces of infinite graphs*, Combinatorica **27** (2007), no. 6, 683–698.

[23] ———, *Fleischners theorem for infinite graphs*, Oberwolfach Reports, no. 16, 2007.

[24] ———, *Topological circles and euler tours in locally finite graphs*, Preprint, 2008.

[25] R. Halin, *Über unendliche Wege in Graphen*, Math. Ann. **157** (1964), 125–137.

[26] ______, *On the notion of infinite Hamiltonian graph*, J. Graph Theory **7** (1983), no. 4, 437–440.

[27] Maya Jakobine Stein, *Arboricity and tree-packing in locally finite graphs*, J. Combin. Theory Ser. B **96** (2006), no. 2, 302–312.

[28] Jerome J. Karaganis, *On the cube of a graph*, Canad. Math. Bull. **11** (1968), 295–296.

[29] M. M. Matthews and D. P. Sumner, *Hamiltonian results in $K_{1,3}$-free graphs*, J. Graph Theory **8** (1984), no. 1, 139–146.

[30] C. St. J. A. Nash-Williams, *Hamiltonian lines in infinite graphs with few vertices of small valency*, Aequationes Math. **7** (1971), 59–81.

[31] Zdeněk Ryjáček, *On a closure concept in claw-free graphs*, J. Combin. Theory Ser. B **70** (1997), no. 2, 217–224.

[32] Milan Sekanina, *On an ordering of the set of vertices of a connected graph*, Spisy Přírod. Fak. Univ. Brno **1960** (1960), 137–141.

[33] Carsten Thomassen, *Hamiltonian paths in squares of infinite locally finite blocks*, Ann. Discrete Math. **3** (1978), 269–277, Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).

[34] ______, *Reflections on graph theory*, J. Graph Theory **10** (1986), no. 3, 309–324.

[35] W. T. Tutte, *A theorem on planar graphs*, Trans. Amer. Math. Soc. **82** (1956), 99–116.

[36] Antoine Vella and R. Bruce Richter, *Cycle spaces in topological spaces*, J. Graph Theory **59** (2008), no. 2, 115–144.

[37] Eric W. Weisstein, *Icosian game*, `http://mathworld.wolfram.com/IcosianGame.html`.

[38] Stephen Willard, *General topology*, Dover Publications Inc., Mineola, NY, 2004.

[39] Xingxing Yu, *Infinite paths in planar graphs. I. Graphs with radial nets*, J. Graph Theory **47** (2004), no. 2, 147–162.

[40] ______, *Infinite paths in planar graphs. II. Structures and ladder nets*, J. Graph Theory **48** (2005), no. 4, 247–266.

[41] ______, *Infinite paths in planar graphs. III. 1-way infinite paths*, J. Graph Theory **51** (2006), no. 3, 175–198.

[42] ______, *Infinite paths in planar graphs. IV. Dividing cycles*, J. Graph Theory **53** (2006), no. 3, 173–195.

[43] ______, *Infinite paths in planar graphs. V. 3-indivisible graphs*, J. Graph Theory **57** (2008), no. 4, 275–312.

[44] Siming Zhan, *On Hamiltonian line graphs and connectivity*, Discrete Math. **89** (1991), no. 1, 89–95.