The coupling of uniform spanning trees and quantitative Russo–Seymour–Welsh for random walk on random graphs

by

Tingzhou Yu
B.Sc., Hebei Normal University, 2019

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

The coupling of uniform spanning trees and quantitative Russo–Seymour–Welsh for random walk on random graphs

by

Tingzhou Yu
B.Sc., Hebei Normal University, 2019

Supervisory Committee

Dr. Gourab Ray, Supervisor
(Department of Mathematics and Statistics)

Dr. Anthony Quas, Departmental Member
(Department of Mathematics and Statistics)

ABSTRACT

The central concern of this thesis is the study of the Russo-Seymour-Welsh (RSW) theory. The first contribution of this thesis is a macroscopic decorrelation result for uniform spanning trees (USTs) on random planar graphs based on the RSW assumption. A similar result was established on a fixed graph in [BLR20, Theorem 4.21]. We extend this result to USTs on random graphs. In particular, we show that a similar coupling can be obtained for a collection of graphs, which has a high probability. This is the key missing step in the application of the proof strategy in [BLR20] for random graphs, which established the scaling limits of height function of dimer model to a Gaussian free field on a fairly general class of fixed graphs.

The second contribution of this thesis is the RSW type results for random walks on two concrete and natural examples: the unique infinite cluster of supercritical bond percolation in $\mathbb{Z}^2$ and the Poisson-Delaunay triangulation in $\mathbb{R}^2$. We show that random walks crossing a rectangle without exiting occurs with a stretched exponentially high in the scale. The main tool used in the proof is heat kernel estimates for random walks on the supercritical bond percolation. The proof of RSW for bond percolation is a quick application of a combination of Barlow's results. However, we cannot apply Barlow's results for the Delaunay triangulation directly since there is no uniform bound on degree. A key input is a quantitative isoperimetric inequality for the Delaunay triangulation, which we consider to be another novel contribution of this thesis.

Contents

List of Figures

DEDICATION

To my parents and girlfriend.

ACKNOWLEDGEMENTS

Firstly, I would like to thank my advisor, Gourab Ray, for his invaluable guidance and support during my time at the University of Victoria. His door was always open if I needed guidance. He was always glad to discuss my recent progress in research and found a way to point me in the right direction.

I would like to thank Benoit Laslier and Anthony Quas for serving as a committee member of my thesis defense.

At the University of Victoria, I want to thank Chris Bose, Ryan Budney, Yu-Ting Chen, Michelle Miranda, and Lei zhao for teaching me so much in the past two years and patiently answering my questions. I would also like to thank Amy Almeida, Patti Arts, Kelly Choo, and Carol Anne Sargent for their hard work.

Also, I want to thank Edwin Perkins and Mathav Murugan for marking my assignments and writing detailed comments in Math 610E at the University of British Columbia. I want to thank Brendan Pass for teaching me the optimal transport at the University of Alberta. I want to thank organizers of Online Open Probability School and CRM-PIMS Summer School in Probability. I would like to thank Nina Gantert, Frank den Hollander, Gady Kozma, Elena Pulvirenti and the rest of the professors in OOPS for responding to emails and questions that I had in the series of mini-courses. I would like to thank Jean-Christophe Mourrat for teaching me a fascinating connection among spin glasses, statistical inference, and PDE and for giving plenty of exercises in the CRM-PIMS summer school.

I want to thank friends during my studies: Chenlin Gu, Hannah Cairns, Changhao (Nancy) Guo, Yuying Huang, Jeremy Hume, Yangming (Derek) Li, Jingyi Wu, Zhuoyu Xiao.

Chapter 1

Introduction

The **Russo-Seymour-Welsh (RSW)** theory was first introduced in [Rus78] and [SW78] in the Bernoulli percolation model, which aim to prove uniform positivity of the probability of a crossing of a rectangle in critical Bernoulli percolation. These are a crucial input into establishing the more refined properties of percolation processes, such as the sharpness of the phase transition and scaling limits for the interfaces of percolation clusters. The RSW theory plays a central role in the study of two-dimensional statistical physics models, such as RSW for Voronoi percolation was established in [Tas16], RSW for FK percolation was proved in [DCHN11], and RSW for level sets of the planar Gaussian free field in [DCMT18]. Recently, the author in [KST20] extends the Russo-Seymour-Welsh theory to general percolation measures. In [BLR20], RSW was used to study decorrelation of uniform spanning trees in a fixed planar infinite graphs. Very roughly, this type of estimate leads to rough Harnack type inequality and also Beurling type hitting estimates. This led to a result that the scaling limits of height function of dimer model to a Gaussian free field on a fairly general class of graphs. This is later extended to graphs on multiply connected Riemann surfaces in paper [BLR19a] and [BLR19b]. There are two main assumptions of the graph in [BLR20]. The first assumption is that random walk on the graph converges weakly to Brownian motion. This assumption is robust under reasonable perturbations of the underlying graph, see [Rou15] for Delaunay triangulations. The second assumption is that the random walk crosses a rectangle larger than a fixed scale horizontally without exiting it with a probability uniform in the scale, which depends only on the aspect ratio. This assumption is called a RSW type assumption. It can be shown that this holds for isoradial graphs (i.e., planar graphs embedded in the plane in such a way that every face is inscribed in a circle of radius one). Let us remark that the RSW assumption is in some sense related to the uniformity in the rate of convergence of the random walk to a Brownian motion depending on the location of the graph. Indeed, for this reason, RSW for the square lattice for example is a simple consequence of the invariance principle. On the other hand in the presence of some local irregularities, it is not clear at all if such an estimate is even true.

This thesis extend the RSW type results to random planar graphs which are not necessarily 'uniformly elliptic' in the sense of the examples considered so far. For example, the key examples are the unique infinite cluster of a Bernoulli bond percolation on $\mathbb{Z}^2$, and Delaunay triangulation (the dual graph of Voronoi tessellation). Obviously, RSW assumption does not hold for rectangles larger than any fixed scale uniformly over the location of the graph in both two cases (for example, an arbitrarily large rectangle is empty at some location almost surely). However, we show that the RSW assumption holds with a exponentially high probability in the scale in this thesis. The main input for the random walk RSW results is from [Bar04], which states that a quadratic volume growth and Poincaré inequality ensures a good heat kernel bound for random walks on the infinite cluster of supercritical bond percolation. The informal form of our first result is as follows.

Theorem 1.0.1. *The RSW assumption holds for*

- *the unique infinite cluster of a Bernoulli bond percolation on* $\mathbb{Z}^2$
- *Delaunay triangulations*

with stretched exponentially high probability in the scale.

Remark 1.0.2. *See Theorem 4.2.1 and Theorem 4.3.1 for mathematical statement.*

Then we apply the RSW assumption to establish a decorrelation result for uniform spanning trees in random graphs, which extend one result in [BLR20, Theorem 4.21]. More precisely, Let $G = (V(G), E(G))$ be a random graph sample from a probability measure μ which is supported on infinite, proper, embedding planar graphs. Assume that μ is shift invariant, RSW assumption holds, and bounded density assumption holds which is the number of vertices inside a square $[-n, n]^2$ is less than Cn^2 for a constant $C > 0$ with exponentially high probability in n. Our second result is as follows.

Theorem 1.0.3. *We sample a uniform spanning tree* $\mathcal{T}$ *with wired boundary condition for a small mesh size graph on a simply connected domain* $D \subset \mathbb{R}^2$. *Fix two points* x, y *in* D. *One can couple two independent USTs* $\mathcal{T}_1$, $\mathcal{T}_2$ *with* $\mathcal{T}$ *so that* $\mathcal{T}$ *and* $\mathcal{T}_1$ *have same law on a small neighborhood of* x, *and* $\mathcal{T}$ *and* $\mathcal{T}_2$ *have same law on a small neighborhood of* y. *Moreover, the radius of these two neighborhood are random and one can obtain a polynomial bound on the lower tail of the radius. This result holds not just for two but for any finite number of points in* D.

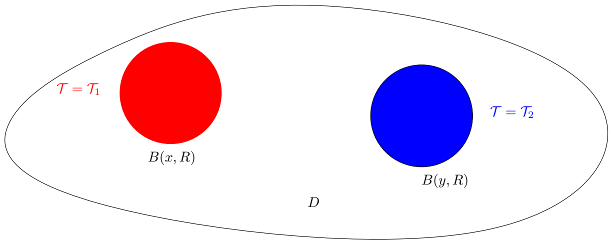

Figure 1.1: An illustration of the coupling of two USTs $\mathcal{T}$ and $\mathcal{T}_1$ around a ball $B(x, R)$ with radius R and center x (in red color) and two USTs $\mathcal{T}$ and $\mathcal{T}_2$ on the ball $B(y, R)$ with radius R and center y (in blue color).

Remark 1.0.4. *See Theorem 3.1.2 for mathematical statement.*

Another consequence of the quantitative RSW and decorrelation result for uniform spanning trees is a scaling limit result for dimer height function on such random planar graphs. This is the key missing step in the application of the proof strategy of [BLR20] for random planar graphs (see [RY21, Section 6.1] for a detailed discussion).

1.1 Organization of the thesis

In Chapter 2, we review a variety of classical mathematical results used including USTs and heat kernel estimate. The main contribution of the thesis begins in Chapter 3. We will introduce the decorrelation result as in Theorem 3.1.2 based on RSW type estimates assumption. In Chapter 4, we prove that RSW type results on unique infinite cluster of bond percolation in Theorem 4.2.1 and the Poisson-Delaunay triangulation in Theorem 4.3.1. The results in Chapters 3 and 4 are included in [RY21].

Chapter 2

Preliminaries

2.1 Uniform spanning trees and scaling limits

Most of the concepts and proofs featured in this section were covered in the course *Planar maps, random walks and the circle packing theorem*, which was taught by Prof. Asaf Nachmias on 48th Probability Summer School Saint-Flour (France). The other parts were covered in the course *Uniform spanning trees in high dimension* taught by Dr. Tom Hutchcroft on Online Open Probability School held by University of British Columbia during the Summer of 2020. Other notable sources include [Nac20], [LP17b], and [LP17a].

2.1.1 Uniform spanning trees

Let $G = (V(G), E(G))$ be a finite connected graph. A **spanning tree** $\mathcal{T}$ of G is a connected subgraph of G that contains no cycles and spans $V(G)$ (i.e., contains all the vertices of G). Obviously, the number of spanning trees of a given finite connected graph is finite. So we can choose one uniformly at random. This random tree is called **uniform spanning tree** (UST) of G. We define $\mathbf{UST}_G$ to be the law on spanning trees of G that assigns equal mass to each spanning tree of G.

The uniform spanning tree was first studies by Kirchhoff in [Kir47] who established a formula for the number of spanning trees of a fixed graph and presented a connection with the theory of electric networks. We refer the reader to [LP17b, Chapter 2] for more details about the electric networks. Uniform spanning trees have played a central role in the development of probability theory over the last twenty years. This model has surprising connections to lots of subjects in probability, such as loop-erased random walk (see e.g.,[Law79, BLPS01a]), the Gaussian free field (see e.g., [BLR20]), and domino tilings (see e.g., [Ken00]). One result was that the study of the scaling limit of the UST that led Oded Schramm to introduce the **Schramm–Loewner evolution** process in [Sch00], which has revolutionized the study of two dimensional models in statistical physics. Moreover, one result in [LSW11] shows that the scaling limit of Peano curve of USTs is SLE_8. Recently, the scaling limits of uniform spanning trees of higher dimension was studied (see e.g. [ACHTS20, HS20]).

After the definition of USTs of finite graph, one may ask the following question:

Question 2.1.1. *Is there a natural way to define a UST probability measure on an infinite connected graph ?*

Let $G = (V(G), E(G))$ be an infinite connected graph. One natural way is to take USTs on each of a increasing sequence of finite subgraphs $\{G_n\}$ so that $\{G_n\}$ exhaust the whole infinite graph G (i.e., $\cup_n G_n = G$). One result is that the UST probability measure on G_n converges weakly to some probability measure on subsets of $E(G)$, which was proved in paper [Pem91]. There are two ways to make it work as follows.

Let $\{V_n\}_{n\geq 1}$ be an exhaustion of the vertex set of G by finite sets (i.e., $V_n \subset V_{n+1} \subset \dots$) and $\cup_{n\geq 1} V_n = V$. For each $n \geq 1$, we define G_n to be the subgraph induced by V_n. One way is to define the **free uniform spanning forest** (FUSF), which is the weak limit of USTs of G_n, that is,

$$\mathbf{FUSF} = \lim_{n\to\infty} \mathbf{UST}_{G_n}.$$

This weak limits are well-define and do not depend on the choice of exhaustion (see e.g., [Pem91]). The proof is a consequence of Rayleigh's monotonicity in [LP17b, Chapter 2] and Kolmogorov's extension theorem.

Here is an example from [Pem91] to explain the reason for the change of term from 'tree' to 'forest'. One result in [Pem91] shows that a sample of **FUSF** on $\mathbb{Z}^d$ is almost surely connected when $d \leq 4$ and almost surely disconnected when $d \geq 5$. The term 'free' is from that we have not assumed any boundary conditions in **FUSF**.

We also define the **wired uniform spanning forest** (WUSF) by taking a limit of the uniform spanning tree measures over exhaustion with wired boundary. Let G be an infinite connected graph and let $\{G_n\}$ be a finite exhaustion of it as above. Denote G^* by identifying set of vertices $G \setminus G_n$ to a single vertex δ_n and erasing the loops at δ_n formed by this identification. We say that $\{G_n^*\}$ is a **wired finite exhaustion** of G. The WUSF of an infinite graph will be defined similarly as FUSF, that is

$$\mathbf{WUSF} = \lim_{n\to\infty} \mathbf{UST}_{G_n^*}.$$

To summarize,

Theorem 2.1.2 ([Pem91])**.** *Let G be an infinite connected graph and let $\{G_n\}_{n\geq 1}$ be an* *exhaustion of G as above. Then the weak limits*

$$\mathbf{FUSF} = \lim_{n\to\infty} \mathbf{UST}_{G_n}.$$

and

$$\mathbf{WUSF} = \lim_{n\to\infty} \mathbf{UST}_{G_n^*}.$$

exists and do not depend on the exhaustion $\{G_n\}_{n\geq 1}$.

2.1.2 Wilson's algorithm

A graph typically has an enormous number of spanning trees. Because of this, it is not obvious easy to choose one uniformly at random in a reasonable amount of time. We will introduce a beautiful method for sampling a uniform spanning tree, which is due to Wilson [Wil96]. The method we describe for generating random spanning trees is the fastest method known. To describe Wilson's method, we introduce the important idea of **loop-erased random walk** (LERW), due to Lawler [Law79]. It is obtain by performing a random walk on a graph and then erasing the loops of the random walk path in chronological order. More precisely, if γ is any finite path $\{x_0, x_1, \ldots, x_n\}$ in a directed or undirected graph G, we define the **loop erasure** of a path γ by deleting all cycles that the path traces out in the order they appear, denoted $\mathrm{LE}(\gamma) = \{y_0, y_1, \cdots, y_m\}$. More precisely, set $y_0 = x_0$. If $x_n = x_0$, we let $m = 0$ and do nothing. Otherwise, let $y_1 = x_{i+1}$ for $i = \min\{j : x_j = x_0\}$. If $x_n = y_1$, then we let $m = 1$ and do nothing. Otherwise, let y_2 be the first vertex in γ after the last visit to y_1, and so on.

Wilson's algorithm. Let G be a finite connected graph G. List a path $\gamma = \{v_0, v_1, \ldots, v_n\}$ of graph G. We define an increasing sequence of random subtrees of G recursively as follows:

1. Let $T_0 = \{v_0\}$ and no edges in this tree.

2. Given $1 \leq i \leq n$ and T_{i-1}, we pick an arbitrary vertex v_i not contained in T_{i-1} and started a random walk X^i at v_i. We stopped it when it first hits the vertex set of T_{i-1}. Let T_i be the union of T_{i-1} with the loop-erasure of X^i. If v_i was already contained in T_{i-1}, let $T_i = T_{i-1}$.

The final tree T_i is clearly a spanning tree of G (no cycles due to loop erasure). Moreover, it is distributed as a uniform spanning tree of G from Wilson's Theorem as follows.

Theorem 2.1.3 ([Wil96]). *The random tree T_n is distributed as a uniform spanning tree of G.*

In particular, the choice of enumeration of G does not affect the distribution of T_n.

Remark 2.1.4. *There are several algorithms for sampling USTs. For example, Kirchhoff presented one algorithm using matrix tree to sample USTs. David Aldous and Andrei Broder bring up a connection between uniform spanning trees and random walk in [Ald90] and [Bro89]. This algorithm is called* ***Aldous-Broder algorithm****.*

It is obvious how to extend Wilson's algorithm to infinite recurrent graphs. (For the square lattice $\mathbb{Z}^2$, we can choose $(0,0)$ as the root and follow the same procedure as above.) In [BLPS01b], the authors showed that how to extend Wilson's algorithm to sample the wired uniform spanning forest of any transient graph. Their extension is called Wilson's algorithm rooted at infinity. We refer the reader to [LP17b, Chapter 10] for more details.

2.2 Two random graph models

In this section, we will introduce two random graph models. One is the bond percolation model, and other one is Voronoi tesselation.

2.2.1 Bernoulli Percolation

Percolation is a typical model in statistical physics. It is one of the simplest models that displays a phase transition. There are a number of textbooks available with percolation as their major topic, most notably [Gri99] as a general reference on the topic. We refer the reader to [DC18] for relevant history and references of this very popular model.

Suppose that $G = (V(G), E(G))$ is a graph. Fix $p \in [0,1]$. **Bernoulli bond percolation** on G is a probability measure $\mathbb{P}_p$ on $\omega = (\omega(e) : e \in E) \in \{0,1\}^E$ for which each edge of E is open with probability p and closed with probability $1-p$, independently of the states of other edges. The $\sigma-$algebra of measurable events is the smallest $\sigma-$algebra containing events depending on finitely many edges. More precisely, if $E \subset E(G)$ is a finite subset of edges and $\eta \in \{0,1\}^{E(G)}$, then the cylinder set around E at η is the set

$$C_{\eta,E} := \{\omega \in \{0,1\}^{E(G)} : \omega(e) = \eta(e) \text{ for all } e \in E\}.$$

The $\sigma-$algebra $\mathcal{F}$ is generated by all cylinders, that is,

$$\mathcal{F} = \sigma(C_{\eta,E} : \eta \in \{0,1\}^{E(G)}, E \subset E(G), |E| < \infty).$$

A **cluster** is a connected component induced by the open edges. Let $\mathcal{C}(x)$ be the cluster containing x in ω with $|\mathcal{C}(x)|$ denoting the number of vertices in it. We say x is in an **infinite cluster** if $|\mathcal{C}(x)| = \infty$ which is denoted by $\mathcal{C}_\infty(x)$. This model was introduced by Broadbent and Hammersley in [BH57].

We will often define Bernoulli percolation on the square lattice. We therefore consider the probability space $(\{0,1\}^{E(\mathbb{Z}^2)}, \mathcal{F}, \mathbb{P}_p)$. For the bond percolation on $\mathbb{Z}^2$, we are interested in the percolation probability

$$\theta(p) := \mathbb{P}_p(0 \longleftrightarrow \infty).$$

If $\theta(p) > 0$, it is very likely that there will be an infinite cluster for percolation configuration. Define the parameter

$$p_c := \inf\{p \in [0,1] : \theta(p) > 0\}.$$

We will see the following phase transition of the bond percolation.

Theorem 2.2.1. *For Bernoulli bond percolation on $\mathbb{Z}^2$, there exists $p_c \in (0,1)$ such that*

$$\theta(p) = 0, \quad \text{if } p < p_c;$$

$$\theta(p) > 0, \quad \text{if } p > p_c.$$

Proof. See [Gri99]. □

Remark 2.2.2. *From Theorem 2.2.1, there is a phase transition between a regime without infinite cluster and a regime with infinite cluster. The $p < p_c$ regime is called **subcritical**, $p = p_c$ regime is called **critical**, and $p > p_c$ regime is called **supercritical**.*

Remark 2.2.3. *In fact, we knew that $p_c = 1/2$ and $\theta(p_c) = 0$ for Bernoulli bond percolation on $\mathbb{Z}^2$ by [Kes80].*

Remark 2.2.4. *We conjecture that $\theta(p_c) = 0$, but it is the major open problems for $\mathbb{Z}^d$ with $3 \leq d \leq 10$. Surprisingly , this is also known to be the case for $d \geq 19$, which is a highly nontrivial result by Hara and Slade in [HS94]. Recently, the authors in [FvdH17] sharpened the methods of Hara and Slade to prove the case for $d > 10$.*

One theorem about the number of infinite clusters on the supercritical regime by Aizenman, Kesten and Newman [AKN87] is as follows.

Theorem 2.2.5 ([AKN87])**.** *If $p \in [0,1]$ is such that $\theta(p) > 0$, then*

$$\mathbb{P}_p(\textit{there exists exactly one infinite open cluster}) = 1$$

Remark 2.2.6. *Let $q \in (0,1)$. Similarly, we can define the **site percolation** on $\mathbb{Z}^2$, where it is a probability measure $\mathbb{P}_q$ on $\omega \in \{0,1\}^{V(\mathbb{Z}^2)}$ and makes the $\omega(x)$ i.i.d. Bernoulli random variable with $\mathbb{P}_q(\omega(x) = 1) = q$.*

2.2.2 Delaunay triangulation

In this section, we will introduce the Voronoi tessellation and its dual graph called Delaunay triangulation in $\mathbb{R}^2$. Next, we are going to focus on some results of the Delaunay triangulation, which we mainly concern about in thesis. Let's briefly introduce the history. The Voronoi tessellation was introduced by Voronoi in [Vor08]. This model has been applied to lots of different areas, for example, probability (see e.g. [Gol10, Cal03, MM82]), computational geometry (see e.g. [Yap87, ACV05]), and mathematical biology (see e.g. [BTKA10]).

The Delaunay triangulation was named after Boris Delaunay for his work in [DVLDG34]. Some results about the random walk on such graphs have been studied, for example, Addario-Berry and Sarkar proved the recurrence of simple random in $\mathbb{R}^2$ in [ABS05] and Rousselle got a quenched invariance principle result of the simple random walks on Delaunay triangulations in [Rou15]. Let's begin with the definition of **Poisson point processes** (PPP) with constant intensity. Let D be a domain of $\mathbb{R}^2$ with finite and positive area $\mathbf{Vol}(D)$. Let X be a random subsets of D consisting of finitely many points. We call X a point process on D and let $X(A)$ denote the number of points in X of A for $A \subset D$.

Definition 2.2.7. *A point process X is called a Poisson point process with intensity $\lambda \geq 0$ on D if*

- *$X(A_1)$ and $X(A_2)$ are independent for disjoint subsets $A_1, A_2 \subset D$;*
- *$X(A)$ is a Poisson distribution with expectation $\lambda \mathbf{Vol}(A)$ for $A \subset D$, that is*

$$\mathbb{P}(X(A) = k) = \frac{(\lambda \mathbf{Vol}(A))^k}{k!} \exp(-\lambda \mathbf{Vol}(A)),\ k \in \mathbb{N}.$$

Now we describe the Voronoi diagram associated to a Poisson point process in $\mathbb{R}^2$ with intensity 1. Given a homogeneous Poisson point process Π of intensity 1 in $\mathbb{R}^2$, the **Voronoi cell** (or called **Voronoi tessellation**) of a point $x \in \Pi$ is defined by

$$V(x) := \{y \in \mathbb{R}^2 : \|x - y\| = \min_{x' \in \xi} \|x' - y\|\}$$

where $\|\cdot\|$ is the ℓ^2 norm. The point x is called the **nucleus** of the cell. The **Voronoi diagram** (or called **Voronoi tessellation**) of Π is the collection of the Voronoi cells.

The **Delaunay triangulation** $\mathrm{DT}(\Pi) = (\Pi, E(\mathrm{DT}(\Pi)))$ is the dual graph of the Voronoi diagram. There is an edge between x and x' in $\mathrm{DT}(\Pi)$ if $V(x)$ and $V(x')$ share an entire edge. One useful property of $\mathrm{DT}(\Pi)$ is as following. A triangle is a cell of $\mathrm{DT}(\Pi)$ if and only if there is no point of Π in interior of its circumscribed sphere.

Lemma 2.2.8 ([ABS05])**.** *If e is an edge of* $\mathrm{DT}(\Pi)$*, then one of the half-circles with diameter e contains no points of* Π.

2.3 Heat kernel estimates

The main sources for the concepts and results covered in this section are [Bar17], [SC97], [Woe00], and [Gri18, Chapter 5].

We start with some basic definitions. For a graph $G = (V, E)$, we consider the each edge e of G as corresponding to a pair of **oriented** edges, that is, an oriented edge $e^{\rightarrow} \in E$ is oriented from its tail e^- to its head e^+, and has reversal denotes by $e^{\leftarrow}$. We write $E^{\rightarrow}$ for the set of oriented edges of G, $E^{\rightarrow}_v$ for the set of oriented edges of G emanating from the vertex v, and $E^{\rightarrow}_{uv}$ for the set of oriented edges of G starting in u and ending in v. We write $u \sim v$ to mean $\{u, v\} \in E$ and v is a **neighbor** of u. The **degree** of a vertex is defined to be $\deg(v) := |E^{\rightarrow}_v|$, and a graph is said to be **locally finite** if $\deg(v) < \infty$ for every $v \in V$.

A **path** γ in graph G is a sequence $u_0, u_1, \dots, u_n$ with $u_{i-1} \sim u_i$ for $1 \le i \le n$. The length of a path γ is the number of edges in γ. We define $d(x, y)$ to be the length n of the shortest path $x = u_0, u_1, \dots, u_n = y$. If there is no such path then we set $d(x, y) = \infty$. Write for $x \in V, A \subset V$

$$d(x, A) = \min\{d(x, y) : y \in A\}.$$

We say G is **connected** if $d(x, y) < \infty$ for all x, y.

Let $H \subset V$. Then the subgraph **induced by** H is the graph with vertex set H and edge set

$$E(H) := \{\{u, v\} \in E : u, v \in H\}.$$

A weighted graph will be a pair (G, μ) where G is a finite unoriented connected graph with vertex set V and edge set E, and $\mu : E \to (0, \infty)$ is a function assigning a positive **weight** to each edge of G. We often write $\mu_{uv} = \mu(u, v)$ for $\mu(\{u, v\})$ where $\{u, v\} \in E$. Clearly, $\mu_{uv} = \mu_{vu}$. Note that if G is locally finite, we have $\mu(A) = \sum_{x \in A} \mu(x) < \infty$.

We may define a transition matrix $P \in [0, 1]^{V^2}$ by the formula

$$P(u, v) := \frac{\mu(u, v)}{\mu(u)} \tag{2.1}$$

where $\mu(u) = \sum_{v : v \sim u} \mu(u, v)$.

Let X be the discrete time Markov chain $X = (X_n, n \ge 0, \mathbb{P}^x, x \in V)$ with transition matrix $(P(x, y))$. Here $\mathbb{P}^x$ is the law of the chain with $X_0 = x$ and the transition probabilities are given by

$$\mathbb{P}(X_{n+1} = y | X_n = x) = P(x, y).$$

We say that X is the (weighted) **random walk on graph** G[1]. This Markov chain is reversible with respect to the probability π defined by $\pi(u) := \mu(u)/\mu_G$, where $\mu_G := \sum_{u \in V} \mu(u)$. Since

$$\pi(u)P(u,v) = \frac{\mu(u)}{\mu_G}\frac{\mu(u,v)}{\mu(u)} = \frac{\mu(v)}{\mu_G}\frac{\mu(v,u)}{\mu(v)} = \pi(v)P(v,u), \tag{2.2}$$

then π is stationary by the detailed balance equation.

Set

$$P_n(x, y) = \mathbb{P}^x(X_n = y).$$

From now on, let $G = (V, E)$ be an infinite, locally finite, connected graph. Whenever we discuss a graph without explicit mention of weights, we will assume we are using the natural weights.

It is often convenient to consider the **heat kernel** (also called **transition density**) of random walk X with respect to the measure μ rather than working with the transition probabilities $P(x, y)$, which is defined as in (2.1). Here we define the **(discrete time) heat kernel** by

$$p_n(x, y) = \mu_y^{-1} P_n(x, y) = \mu_y^{-1} \mathbb{P}^x(X_n = y). \tag{2.3}$$

where $p_1(x, y) = p(x, y) = \frac{\mu(x,y)}{\mu(x)\mu(y)}$ and $p_0(x, y) = \frac{\mathbb{1}_x(y)}{\mu_x}$.

Define the function space $C(V) = \{f : V \to \mathbb{R}\}$. We write

$$\int f d\mu := \sum_{x \in V} f(x)\mu(x)$$

For $1 \le p \le \infty$ and $f \in C(V)$, let

$$\|f\|_p^p = \int |f|^p d\mu := \sum_{x \in V} |f(x)|^p \mu(x),$$

and

$$L^p = L^p(V, \mu) := \{f \in C(V) : \|f\|_p < \infty\}.$$

Set $\|f\|_\infty = \sup_x |f(x)|$ and $L^\infty(V, \mu) = \{f : \|f\|_\infty < \infty\}$.

[1] Here is a special case of a weighted random walk on graph. If μ is the natural weight, X is the simple random walk on G with transition probabilities $P(u, v) = \frac{\mathbb{1}_{\{v \sim u\}}}{deg(u)}$.

We define $|\nabla f| : E \to \mathbb{R}$ by $|\nabla f|(e) = |f(x) - f(y)|$ for $e = \{x, y\}$.

2.3.1 Isoperimetric inequalities

Isoperimetric problem is one of the oldest variational problem of mathematics. The solution to the isoperimetric problem in the plane is usually expressed in the form of an inequality. We refer reader to [Oss78] for the history of isoperimetric inequality. In graph theory, [HLW06] considered the application of isoperimetric inequalities in the study of expander graphs. A discrete isoperimetric inequality on lattices was established in [Ham14]. We refer reader to [Chu04] for classical resutls of isoperimetric inequalities and a number of applications in extremal graph theory and random walks.

The study of the close connections between random walks and isoperimetric inequalities was opened by Varopoulos [Var85]. The paper [Tho92] make a thorough study of the relation between isoperimetric inequality and transience of the graph. More related papers see e.g. [CF07, Tel03].

Assume that G is a locally finite connected graph with natural weight. For $A, B \subset V$, set

$$\partial_E(A, B) := \{\{x, y\} : x \in A, y \in B\}.$$

Let

$$i(A) = \frac{\mu(\partial_E(A, V - A))}{\mu(A)}.$$

Definition 2.3.1 (Isoperimetric inequality). *Generally, in [Bar17, Chapter 3] the author define more general isoperimetric inequality. Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing. We say that graph G satisfies the $\Psi-$isoperimetric inequality if there exists a constant $0 < C_0 < \infty$ such that*

$$\frac{\mu(\partial_E(A, V - A))}{\Psi(\mu(A))} \geq C_0^{-1}, \quad \text{for every finite non-empty } A \subset V.$$

Example 2.3.2. *(i) The Euclidean lattice $\mathbb{Z}^d$ satisfies $\Psi(t) = t^{1-1/d}$.*

(ii) The binary tree satisfies $\Psi(t) = t$ with constant $C_0 = 3$.

For a finite connected graph $H \subset G$ with natural weight, consider the induced subgraph $\mathcal{H} = (H, E(H))$ on H. Let $\mu_0(x, y)$ be the measure on $E(H)$ for $\{x, y\} \in E(H)$ and $\mu_0(x)$ be the measure on H, which are in the same way as $\mu(x, y)$ and $\mu(x)$ defined on G. More precisely, write

$$\mu_0 = \sum_{y \in H} \mu_{xy}, \quad \mu_0(A) = \sum_{x \in A} \mu_0(x).$$

We define the **isoperimetric constant** for $A \subset H$

$$I_H := \min_{0<\mu_0(A)\leq \frac{1}{2}\mu_0(H)} \frac{\mu(\partial_E(A, H-A))}{\mu_0(A)}.$$

This is closely related the **Cheeger constant** for a finite graph, which is defined by

$$\chi(A) := \frac{\mu_0(H)\mu(\partial_E(A, H-A))}{\mu_0(A)\mu_0(H_A)}.$$

Set

$$J_H := \min_{A\neq\emptyset, H} \chi(A).$$

A very important characterization of when J_H takes the minimum value was introduced in the paper [MR04]. We state it as follows.

Lemma 2.3.3. *The minimum in J_H is attained by a set A such that A and $H - A$ are connected.*

Proof. See [MR04, Section 3.1]. □

Proposition 2.3.4. *[Bar17, Proposition 3.27] Let $\mathcal{H} = (H, E(H))$ be a finite graph with weight μ_0. Then for any $f : H \to \mathbb{R}$*

$$\min_a \sum_{x\in H} |f(x) - a|^2\mu_0(x) \leq 2I_H^{-2}\mathcal{E}_H(f, f). \tag{2.4}$$

Next, we will introduce a second kind of isoperimetric inequality, which is the **weak Poincaré inequality**. From now on, we consider infinite graphs G. Let $B = B(x, R) := \{y : d(x, y) \leq R\}$.

Definition 2.3.5 (Weak Poincaré inequality)**.** *We say that an infinite graph $G = (V, E)$ with weight μ satisfies a **weak Poincaré inequality** (PI) if there exist constant $0 < C_P < \infty$ and $\lambda \geq 1$ such that for all $x \in V$, $R \geq 1$, and every function $f : B^* = B(x, \lambda R) \to \mathbb{R}$,*

$$\int_B (f(x) - \overline{f}_B)^2 d\mu(x) \leq C_p R^2 \int_{E(B^*)} |\nabla f(e)|^2 d\mu(e) \tag{2.5}$$

where

$$\overline{f}_B = \mu(B)^{-1} \int_B f(x)d\mu(x)$$

Note that the PI gives a family of inequalities which hold for all balls $B(x, \lambda R)$ for $\lambda \geq 1$ in G.

Remark 2.3.6. *We say that (G, μ) satisfies a **strong Poincaré inequality** if the weak PI holds with $\lambda = 1$.*

Corollary 2.3.7. *$\mathbb{Z}^d$ satisfies the (weak) PI.*

Proof. See [Bar17, Corollary 3.30]. □

2.3.2 General bounds of continuous time heat kernel

In this section, we will see some results about the Gaussian type upper and lower bound for the continuous time heat kernel.

Given a discrete time random walk $X_n, n \in \mathbb{N}$ and an independent Poisson process $\{N_t\}_{t\geq 0}$ with rate 1, we denote the **continuous time random walk** by $Y_t := X_{N_t}, t \in [0, \infty)$. The random walk Y_t waits an exp(1) time at each vertex x and then jumps to some neighbors y with the probability $\mathcal{P}(x, y) := \mu_{xy}/\mu_x$. We write $\mathbb{P}^x$ to represent the law of Y_t with starting point $Y_0 = x$ and write $q_t(x, y)$ for the **continuous time heat kernel**, that is,

$$q_t(x, y) = \mu_y^{-1}\mathbb{P}^x(Y_t = y). \tag{2.6}$$

Here are some general bounds on q_t in the following. This was also same as [Bar04, Lemma 1.1].

Lemma 2.3.8. *[Bar04, Lemma 1.1] Let $x, y \in V(G)$ and $D = d(x, y) \geq 1$. Then*

(i) If $D \leq et$, then

$$q_t(x, y) \leq 4(\mu_x\mu_y)^{-1/2} \exp\left(-\frac{D^2}{e^2 t}\right). \tag{2.7}$$

(ii) If $D \geq et > 0$, then

$$q_t(x, y) \leq (\mu_x \vee \mu_y)^{-1} \exp\left(-t - D\log\frac{D}{et}\right).$$

(iii) Assume that G has controlled weights. If $D \geq t > 0$, then there exist $c_1, c_2 > 0$ such that

$$q_t(x, y) \geq c_1(\mu_x \wedge \mu_y)^{-1} \exp\left(-c_2 D - D\log\frac{D}{t}\right).$$

Let $A \subset V$. Write

$$\tau_A := \min\{t \geq 0 : Y_t \notin A\}$$

for the **time of the exit from** A.

Define the **killed heat kernel** for the process Y killed on exiting from A by

$$q_t^A(x,y) = (\mu(y))^{-1}\mathbb{P}^x(Y_t = y, t < \tau_A).$$

Define

$$\tau(x,r) := \inf\{t : Y_t \notin B(x,r)\}$$

Clearly, the event $\{Y_t \notin B(x,r)\} \subset \{\tau(x,r) \le t\}$.

Chapter 3

The coupling of uniform spanning trees on random planar graphs

3.1 Main result

Let graph $G := (V(G), E(G))$ be an infinite and planar graph. The **embedding** of G into $\mathbb{R}^2$ is a drawing of the edges of G in $\mathbb{R}^2$ so that no two edges cross each other. When a planar graph is drawn in this way, it divides the plane into regions called **faces**. Assume every face is bounded by finitely many edges. We also assume that the union of vertices, edges and faces is $\mathbb{R}^2$. We call graph G along with the specification of the embedding, an **infinite, proper, embedded, planar** graph.

Let $\Omega := \{A \subset \mathbb{R}^2 : \ A \text{ is locally compact}\}$. Define the **Hausdorff distance**

$$d_H(X,Y) := \max\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\}$$

where X, Y are two non-empty subsets. This induces a metric space on subsets of $\mathbb{R}^2$. We say **locally Hausdorff topology** if $X_n \to X$ for every compact set K, then $d_H(X_n \cap K, X\cap K) \to 0$ as $n \to \infty$. Let τ be the locally Hausdorff topology and $\mathcal{F}$ is the σ-algebra generated by τ. We consider probability measure μ on the measurable space $(\Omega, \mathcal{F})$. For $z \in \mathbb{R}^2$, let $T_c : z \mapsto z + c$, $c \in \mathbb{R}^2$, then we say that μ is **translation invariant** if for all $D \subset \mathbb{R}^2$ and $c \in \mathbb{R}^2$, $\mu(T_c^{-1}D) = \mu(D)$.

Let $G = (V(G), E(G))$ be an infinite, locally finite, one ended, random and planar graphs embedded in a proper way in the plane as above. Let μ be a probability measure supported on the graph G. Recall that $d(x, y)$ is the graph distance on G as in Section 2.3. Recall that graph distance ball is defined by $B(x, R) := \{y : d(x, y) \leq R\}$. Let Λ_n denote the square $[-n, n]^2$ with $\Lambda_n(x) = x + \Lambda_n$. Let $\Lambda_{m,n} = [-m, m] \times [-n, n]$ be a rectangle and similarly $\Lambda_{m,n}(x) = x + \Lambda_{m,n}$.

Recall that $\{X_t\}_{t\geq 0}$ is a random walk on the graph G as in Section 2.3. The random walk is defined by that the walk jumps from u to v at rate $w(u,v)$ where $w(u,v)$ is the weight of the edge $\{u,v\}$. Let $\mathbb{P}_v^G$ be the law of random walks starting from $v \in G$. Let $\Lambda^{(1)} := z + \Lambda_{0.5m}((-2m,0))$ be a starting square and $\Lambda^{(2)} := z + \Lambda_{0.5m}((2m,0))$ be a target square inside the rectangle $\Lambda_{3m,m}(z)$. In this thesis, our main interest is the following.

Definition 3.1.1. *Given a constant $c_\mu > 0$, we say $\Lambda_{3m,m}(z)$ is $c_\mu-$**crossable** if $V(G) \cap \Lambda^{(1)} \neq \emptyset$ and for all $z \in \mathbb{C}$ and $v \in \Lambda^{(1)}$,*

$$\mathbb{P}_v^G(X_t \textit{ hits } \Lambda^{(2)} \textit{ before exiting } \Lambda_{3m,m}(z)) \geq c_\mu. \tag{3.1}$$

See Figure 3.1. All such events are defined by

$$\mathcal{G}_n(z, c_\mu) := \{G : \Lambda_{3n,n}(z) \text{ is } c_\mu - \text{crossable}\}.$$

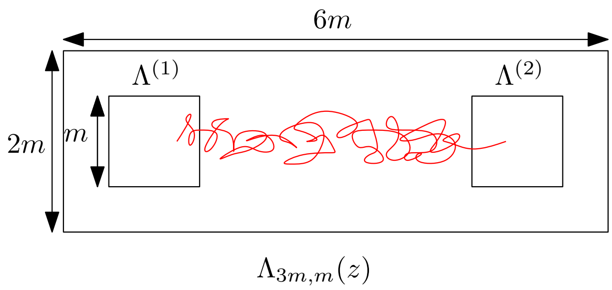

Figure 3.1: A rectangle $\Lambda_{3m,m}(z)$ is $c_\mu-$**crossable** if an event like above has probability at least c_μ.

Assume that G sampled from μ satisfies the following assumptions:

(i) The law μ is translation invariant and invariant under $\pi/2-$rotations of the plane.

(ii) **(Crossing estimate)** There exist constants $\alpha, \beta, c_\mu > 0$ such that for all $n \geq 1$, $z \in \mathbb{R}^2$,

$$\mu(\mathcal{G}_n(z, c_\mu)) \geq 1 - e^{-\alpha n^\beta}. \tag{3.2}$$

(iii) **(Bounded density)** There exist constants $C_\mu, \gamma > 0$ such that for $n \geq 1$

$$\mu(|\Lambda_n(z) \cap V(G)| \geq C_\mu n^2) \leq e^{-\gamma n^2}. \tag{3.3}$$

We also call the second assumption the **Russo-Seymour-Welsh (RSW) type estimate**. Note by Assumption (i) of μ, if (3.2) is satisfied for $z = (0,0)$, then it also holds for any other $z \in \mathbb{R}^2$.

For $z \in \mathbb{R}^2$, let $A(z, r, R) := \Lambda_R(z) \setminus \Lambda_r(z)$ be an annulus. We say that an annulus $A(z, n, 3n)$ is c_μ^4-crossable if we put four copies of $6n \times 2n$ rectangles by translation or rotation by 90 degrees in $A(z, n, 3n)$ (see Figure 3.2) and all the four rectangles are $c_\mu-$crossable. Indeed, by Markov property of random walk, the probability for a random walk to make a full turn.

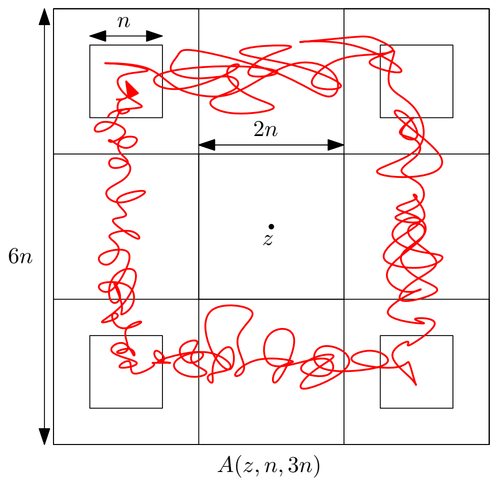

Figure 3.2: An annulus $A(v, n, 3n)$ is $c_\mu-$crossable where all four copies of rectangles are $c_\mu-$crossable.

Define $G^\delta = \delta G$ be a rescaling of the embedded graph G by mesh size δ. Take a finitely and simply connected open domain $D \subset \mathbb{R}^2$. Let $D^\delta = (V^\delta(D), E^\delta(D))$ where $V^\delta(D)$ is the set of vertices of G^δ in D and $E^\delta(D)$ is the set of edges of G^δ with both vertices in $V^\delta(D)$. Recall that the uniform spanning tree is defined as in Section 2.1. We say **wired uniform spanning trees** if the uniform spanning tree is sampled by a given set of 'boundary' vertices to be the root. Let $\mathcal{T}^\delta$ be a wired uniform spanning tree in D^δ. Define $\mathbf{P}^G_{D^\delta}$ be the law of wired UST on D^δ. Recall that random walks and USTs are intimately related to each other via the Wilson's algorithm, which was introduced in Section 2.1.2.

The following is a statement of our first main result.

Theorem 3.1.2. *Suppose G sampled from μ satisfies the above assumptions for some positive constants $c_\mu, \alpha, \beta, C_\mu, \gamma$. Fix a domain $\Lambda_1 \subset D \subset \Lambda_{10}$ and $v_i \in D, 1 \le i \le k$. Let $r = \min_{i \neq j} |z_i - z_j| \wedge dist(v_i, \partial D)$. Let $\mathcal{T}^\delta$, $\mathcal{T}^\delta_1, \mathcal{T}^\delta_2 \ldots, \mathcal{T}^\delta_k$ be copies of wired uniform spanning trees in D^δ. There exist constants $c = c(c_\mu, \alpha, \beta, C_\mu, \gamma), c' = c'(c_\mu, \alpha, \beta, C_\mu, \gamma) > 0$ such that for all $\varepsilon, \varepsilon' > 0$, the following holds for all $\delta \le \varepsilon \wedge \delta_0(\varepsilon')$ small enough. There exists a collection of graphs $\mathcal{G}$ with $\mu(\mathcal{G}) \ge 1 - \varepsilon$ such that for all graphs $G \in \mathcal{G}$, one can couple $(\mathcal{T}^\delta, \mathcal{T}^\delta_i), 1 \le i \le k$ with law $\mathbf{P}^G$ so that*

(i) $\{\mathcal{T}^\delta_i\}_{1 \le i \le k}$ are i.i.d. copies of $\mathcal{T}^\delta$.

(ii)

$$\mathcal{T}^\delta \cap \Lambda_R(v_i) = \mathcal{T}_i^\delta \cap \Lambda_R(v_i), \quad 1 \le i \le k.$$

where R is a random variable satisfying

$$\mathbf{P}^G(R \le r\varepsilon') \le c\varepsilon' + \left(\frac{\varepsilon'}{r}\right)^{c'}.$$

Remark 3.1.3. *If we fix $\varepsilon' > 0$ and apply Theorem 3.1.2 for a sequence $\delta_k = \varepsilon_k = 2^{-k}$ with $2^{-k} < \delta_0(\varepsilon')$, then by Borel-Cantelli Lemma, there exists a collection $\mathcal{G}$ with $\mu(\mathcal{G}) = 1$ such that for any $G \in \mathcal{G}$, the coupling $\mathbf{P}^G$ as in Theorem 3.1.2 holds for all k large enough depending on G.*

In [BLR20, Theorem 4.21], an analogous version was proved but for a fixed graph, where the RSW condition was valid above a certain fixed scale δ_0. The new input in Theorem 3.1.2 is that an analogous result holds with high probability with more general RSW condition.

3.2 Outline of the proofs

We first obtain some RSW type estimates in Section 3.3.1. The key estimate is the Beurling type hitting estimate as Lemma 3.3.11. Armed with this estimate, the rest of the proof of Theorem 3.1.2 follows the same line of argument as in [BLR20]. The proof is mainly divided into two stages. First we couple around a single point as in Section 3.3.3. Then if it fails, we iterate until we succeed as in Section 3.3.4. Second we apply the same way of the coupling around a single point to finitely many points in a fix domain in Section 3.3.5.

3.3 Coupling of spanning trees on random graphs

Let μ be a probability measure as specified in Section 3.1. Let G be a sample from measure μ. Let $\mathcal{T}^\delta$ be a wired uniform spanning tree in $D^\delta = \delta G \cap D$ as Section 3.1. We want to establish a coupling between k independent copies of full plane UST $\{\mathcal{T}_i^\delta\}_{1\le i\le k}$ and a wired UST $\mathcal{T}^\delta$ satisfying that $\mathcal{T}_i^\delta$ and $\mathcal{T}^\delta$ agree on a random neighborhood N_i of v_i. The diameter of the neighbourhoods N_i being very small with a high probability.

Fix $z \in \mathbb{R}^2$. Let $\mathcal{A}(z, n)$ be the event that there exists an open circuit of G surrounding z lying completely inside $A(z, n, 3n)$.

Lemma 3.3.1. *There exist constants $c, c' > 0$ such that ,*

$$\mu(\mathcal{A}(z,n)) \geq 1 - 4e^{-cn^{c'}}$$

Proof. Let $\Lambda_{3n,n}(v_i), i = 1,2,3,4$ be four rectangles arranged clockwise as Figure 3.2 with centers v_i and events $\mathcal{G}_n(v_i, c_\mu)$ be that the rectangle $\Lambda_{3n,n}(v_i)$ is c_μ-crossable. Note that if the random walk crosses the rectangle with positive probability, then there exists a path crossing $A(z, n, 3n)$ by joining paths in events $\mathcal{G}_n(v_i, c_\mu)$. So we have $\bigcap_{i=1}^4 \mathcal{G}_n(v_i, c_\mu) \subset \mathcal{A}(z,n)$. Then using (3.2) we have

$$\begin{aligned}
\mu(\mathcal{A}(z,n)) &\geq 1 - \mu\left(\bigcup_{i=1}^{4} \mathcal{G}_n(v_i, c_\mu)^c\right) \\
&\geq 1 - \sum_{i=1}^{4} \mu\left(\mathcal{G}_n(v_i, c_\mu)^c\right) \\
&\geq 1 - 4e^{-\alpha n^\beta}
\end{aligned}$$

where $\mathcal{G}_n(v_i, c_\mu)^c$ is the complement of event $\mathcal{G}_n(v_i, c_\mu)$.

□

Fix $z \in \mathbb{R}^2$, we try to estimate the size of the subgraph of G in which the rectangles are c_μ-crossable, and that should be a large subgraph given (3.2). More precisely, we estimate the maximal k such that all of the annuli $\{A_j(z)\}_{j\geq k} := \{A(z, 2^j, 2^{j+1})\}_{j\geq k}$ are $c_\mu-$crossable as $j \geq k$. Denote

$$R^\delta = R^\delta(z) := \max\{2^j\delta : A(z, 2^j\delta, 2^{j+1}\delta) \text{ is } \textbf{not } c_\mu\text{-crossable in } \delta G\} \tag{3.4}$$

and

$$R^\delta_{\max} := \max_{z\in\Lambda_{10}\cap\delta\mathbb{Z}^2} R^\delta(z)$$

.

Lemma 3.3.2. *There exists a constant $C > 0$ such that for all $\varepsilon, \delta > 0$,*

$$\mu(R^\delta_{\max} > R^\delta_0) \leq \varepsilon,$$

where

$$R^\delta_0 = R^\delta_0(\varepsilon) = \delta\left(\frac{1}{\alpha}\log\left(\frac{C}{\varepsilon\delta^2}\right)\right)^{1/\beta} \tag{3.5}$$

and α and β are as in (3.2).

Proof. Fix $k \geq 1$ and fix $z \in \Lambda_{10} \cap \delta\mathbb{Z}^2$. Let

$$\mathcal{B} = \bigcup_{j \geq k} \{A(z, 2^j\delta, 2^{j+1}\delta) \text{ is not } c_\mu\text{−crossable in } \delta G\}.$$

From Lemma 3.3.1, we get

$$\mu(\mathcal{B}) \leq \sum_{j \geq k} 4\exp(-\alpha 2^{\beta j}) = 4\exp(-\alpha 2^{\beta k}) + 4\sum_{j \geq 1} \left(\exp(-\alpha 2^{\beta k})\right)^{2^{\beta j}} \leq C' \exp(-\alpha 2^{\beta k})$$

for a constant $C' > 0$ independent of other constants. By translation invariance, the same bound is true for any other $z \in \delta\mathbb{Z}^2$.

Since there are at most $400/\delta^2$ many points in $\Lambda_{10} \cap \delta\mathbb{Z}^2$, we obtain an union bound for all $z \in \Lambda_{10} \cap \delta\mathbb{Z}^2$

$$\mu(R^\delta_{\max} > R^\delta_0) \leq \frac{400}{\delta^2} C' \exp(-\alpha 2^{\beta k}) \leq \varepsilon,$$

where $R^\delta_0 = 2^k\delta$ and $C = 400C'$. This gives us the desired result. □

Remark 3.3.3. *Note that for any ε which is at least δ^m for some $m > 0$, we have $R^\delta \to 0$. Moreover, for a choice of the sequence $\delta_k = \varepsilon_k = 2^{-k}$, by Borel-Cantelli Lemma, μ−a.s. $R^\delta_{\max} \leq R^\delta_0$ for all k large enough.*

3.3.1 Russo-Seymour-Welsh type estimates

We say a random walk starting from v **does a full turn** in $A(z, r, R)$ if the random walk trajectory intersects every curve in the plane starting from circle of radius r and ending at circle of radius R. In this section, we fix a domain D such that $\Lambda_1 \subset D \subset \Lambda_{10}$ and an annulus $A(z, r, R) \subset D$. The **Euclidean distance** between a point x and a set A is given by $\text{dist}(x, A) := \inf\{y \in A : |x - y|\}$.

An application of Lemma 3.3.2 is that for a large enough rectangle depending on $R^\delta_{\max}$, RSW is true.

Lemma 3.3.4. *Fix the graph G satisfying assumptions in Section 3.1. Recall that $R^\delta(v)$ is defined as (3.4). Let $R^\delta_{\max}(\Lambda_{11}) = \max_{z \in \Lambda^\delta_{11}} R^\delta(z)$. For $n \geq 100R^\delta_{\max}(\Lambda_{11})$, all of rectangles $\Lambda_{4n,n}(v)$ completely inside the domain D are c_μ−crossable.*

Proof. Recall that $R^\delta(z) := \max\{2^i\delta : A(z, 2^i\delta, 2^{i+1}\delta) \text{ is not } c_\mu\text{-crossable}\}$. Given a rectangle $\Lambda_{4n,n}(v) \subset D$, we choose a point z such that the straight line between z (see Figure 3.3) and center v is vertical to the long side of $\Lambda_{2n,0.5n}(v)$.

Let the distance between z and $\partial\Lambda_{2n,0.5n}(v)$ be $\text{dist}(z, \partial\Lambda_{2n,0.5n}(v))$. We want to choose a increasing sequence of annuli $\{A(z, 2^i R^\delta(z), 2^{i+1}R^\delta(z))\}_{i \geq 1}$ around z such that there ex-

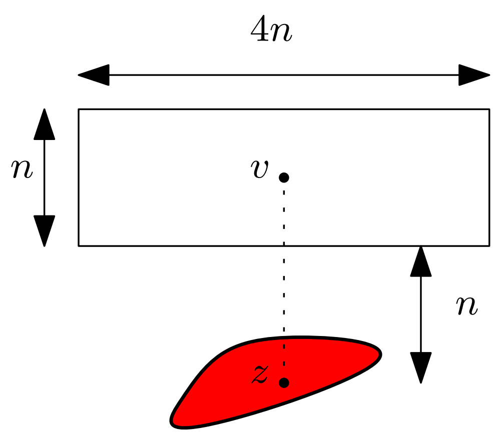

Figure 3.3: Illustration of the position of point z in a bad region (red color) which is below the rectangle $\Lambda_{2n,0.5n}(v)$ and the distance from point z to the boundary of rectangle $\Lambda_{2n,0.5n}(v)$ is n.

ists an annulus $A(z, 2^N R^\delta(z), 2^{N+1} R^\delta(z))$ consisting of four copies of $\Lambda_{2n,0.5n}(v)$ as Figure 3.2. We choose z such that $\text{dist}(z, \partial\Lambda_{2n,0.5n}(v)) = n$ and $N > 0$ such that $2^N R^\delta(z) = 2n$. Indeed, because $n \geq 100R^\delta_{\max}(\Lambda_{11}) \geq 100R^\delta(z)$, here we choose $N = \log_2(200) + 1$. This completes the proof. □

Remark 3.3.5. *Here we replace rectangle $\Lambda_{3n,n}$ with $\Lambda_{4n,n}$. Actually, we can show that the RSW estimate* (3.2) *is true for any general rectangle $\Lambda_{\rho n,n}$ for $\rho > 1$.*

Remark 3.3.6. *The same statement as in Theorem 3.3.4 is also true for every rectangle lying completely inside D, which is a translation and a $\pi/2-$rotation of $\Lambda_{4n,n}(v)$*

Lemma 3.3.7. *Recall that an annulus $A(z, r, R)$ and $R^\delta_{\max}$ are defined as before. For an annulus $A(z, r, R) \subset D$ on graph G where $0 < r < R$ and $R - r \geq 2000R^\delta_{\max}(\Lambda_{11})$, there exists a constant $c > 0$ depending on R/r and c_μ such that for all $x \in A(z, r + \frac{R-r}{3}, R - \frac{R-r}{3})$,*

$$\mathbb{P}^G_x(\textit{random walk does a full turn before exiting } A(z, r, R)) \geq c.$$

Proof. Since $R - r \geq 2000R_{\max}(\Lambda_{11})$, then we have $R/r \geq 1 + 2000R_{\max}(\Lambda_{11})/r$. Consider a sequence of rectangles $\{\Lambda^{(i)}_{4n,n}\}_{1\leq i\leq k} \subset A(z, r, R)$ with side length $n = (R - r)/10 \geq 200R_{\max}$ such that

- the target ball of $\Lambda^{(i)}_{4n,n}$ coincides with the starting ball of $\Lambda^{(i+1)}_{4n,n}$;
- x is in the first starting ball of $\Lambda^{(1)}_{4n,n}$.

By Lemma 3.3.4, these rectangles are $c_\mu-$crossable and if a walker crosses them in order, we get a order full turn in $A(v, r, R)$. The number of rectangles is about $k := cR/(R - r)$ depending on R/r where c is a universal constant independent of everything else. Applying the inequality (3.1), we get the probability of the event that random walk does a full turn before exiting $A(z, r, R)$ is bounded below by $c^k_\mu =: c$. □

Lemma 3.3.8. *Let $\Lambda_{3n,n}(w)$ be a rectangle with center w inside $A(v,r,R) \subset D$ where $0 < r < R$ and $R - r \geq 2000R^{\delta}_{\max}(\Lambda_{11})$. Fix $\varepsilon = (R-r)/6$. Let squares $\Lambda^{(1)} := w + \Lambda_{0.5n}((-2n,0))$ and $\Lambda^{(2)} := w + \Lambda_{0.5n}((2n,0))$. Let τ be the stopping time when the walker exits $A(z,r,R)$. There is a constant $\eta = \eta(R/r, \varepsilon/r) > 0$ such that following holds. For all $x \in \Lambda^{(1)}$ and $u \in \partial A(v,r,R)$ such that $\mathbb{P}^G_x(X_\tau = u) > 0$,*

$$\mathbb{P}^G_x(X \text{ hits } \Lambda^{(2)} \text{ before exiting } \Lambda_{3n,n}(w) | X_\tau = u) > \eta.$$

Proof. Our proof is based on [BLR20, Lemma 4.4] and combine with Lemma 3.3.4.

Let $h(x) = \mathbb{P}^G_x(X_\tau = u)$. We first show that $h(x) \asymp_c h(x')$ for all $x, x' \in A(x, r+\varepsilon, R-\varepsilon)$, i.e. there exists a constant $c > 0$ such that $c^{-1}h(x) \geq h(x') \geq ch(x)$. Since X is irreducible and h is harmonic, we can find a path $\gamma = \{x = x_0, x_1, \ldots, x_k\}$ from x to $\partial A(x,r,R)$ where $x_k \in \partial A(v,r,R)$. Denote τ_γ be a hitting time of $\gamma \cup \partial A(v,r,R)$ by a simple random walk. Since h is harmonic and bounded, we have

$$h(x') = \mathbb{E}_{x'}(h(X_{\tau_\gamma})) \geq h(x)\mathbb{P}^G_{x'}(X_{\tau_\gamma} \in \gamma)$$

Since the event that X does a full turn in $A(x,r,r+\varepsilon)$ and in $A(x,R,R-\varepsilon)$ before exiting $A(x,r,R)$ is contained by the event that the walker hits $\gamma \cup \partial A(v,r,R)$. Hence, from Lemma 3.3.7 there exists a constant $c = c(R/r, \varepsilon/r) > 0$ such that

$$\frac{1}{c}h(x) \geq h(x') \geq ch(x).$$

Next, combine with crossing estimate (3.1) and Markov property we have

$$\begin{aligned}\mathbb{P}^G_x(X \text{ hits } \Lambda^{(2)} &\text{ before exiting } \Lambda_{3n,n}(w),\ X_\tau = u) \\ &\geq \mathbb{P}^G_x(X \text{ hits } \Lambda^{(2)} \text{ before exiting } \Lambda_{3n,n}(w)) \inf_{x' \in \Lambda^{(2)}} h(x') \\ &> c_\mu c h(x).\end{aligned}$$

Dividing by $h(v)$ on the both side which gives the desired result. □

Corollary 3.3.9. *Suppose we are in the setup of Lemma 3.3.8. Assume that τ is defined as in Lemma 3.3.8. Then there exists a constant $\eta > 0$ such that*

$$\mathbb{P}^G_x(X \text{ does a full turn in } A(z,r,R) | X_\tau = u) \geq \eta$$

Proof. Like Lemma 3.3.7, let v be surrounded by some rectangles in $A(z,r,R)$. Then from Lemma 3.3.8 we get the desired result. □

Proposition 3.3.10. *Let $u, v \in D$. Let $r = |u-v| \wedge \ dist(v, \partial D) \wedge \ dist(u, \partial D)$. There exists a constant $\eta > 0$ independent of u, v , and r such that for a loop erased random walk γ starting from v^δ (near v) until it exits the domain D^δ,*

$$\mathbb{P}^G(\ dist(u, \gamma) < 6^{-n} r) < (1-\eta)^n$$

for all $n \le \log_6 \left(\frac{r}{2000 R^\delta_{max}(\Lambda_{11})} \right)$.

Proof. See [BLR20, Proposition 4.11] and combine with Corollary 3.3.9. □

3.3.2 Local coupling of uniform spanning trees

Recall that we fix a domain D as in Section 3.3.1. One standard application of Lemma 3.3.7 is a Beurling type hitting estimate for a random walk. The Beurling estimate is a classical result for the hitting probability of a two-dimensional Brownian motion. We refer the reader to [LL04] for a introduction of the Beurling estimate for random walks.

Lemma 3.3.11. *Let $K \subset D$ be a connected set. There exists $c_1, c_2 > 0$ such that for all $\varepsilon > 0$ and for all $\delta \le \varepsilon$ the following holds. Fix a graph G sampled from μ such that $R^\delta_{\max} \le R^\delta_0(\varepsilon)$. Let X be a simple random walk in G started from v. Then for $K \cap \Lambda_{dist(v,\partial D)}(v) \neq \emptyset$,*

$$\mathbb{P}^G_v(X \ \text{exits} \ \Lambda_{dist(v,\partial D)}(v) \ \text{before hitting} \ K) \le c_1 \left(\frac{dist(v, K) \vee R^\delta_0(\varepsilon)}{dist(v, \partial D)} \right)^{c_2}$$

Proof. Suppose that $2^j = \mathrm{dist}(v, K) \vee R^\delta_0$ and $2^{j'} = \mathrm{dist}(v, \partial D)$ for some $j, j' \ge 0$. Recall that a sequence of annuli is defined as $\{A_k(v)\}_{k \ge n} = \{A(v, 2^k, 2^{k+1})\}_{k \ge n}$.

From Lemma 3.3.7, there exists $c > 0$ such that

$$\begin{aligned}
\mathbb{P}^G_v(X \text{ exits } \Lambda_{\mathrm{dist}(v,\partial D)}(v) \text{ before hitting } K) &\le \mathbb{P}^G_v(\text{no full turn in annuli } \{A_k(v)\}_{j \le k \le j'}) \\
&\le (1-c)^{j'-j} \\
&\le \exp\left(\log_2 \left(\frac{\mathrm{dist}(v, K) \vee R^\delta_0}{\mathrm{dist}(v, \partial D)} \right) \log \frac{1}{1-c} \right)
\end{aligned}$$

by the definition of R^δ_0. We complete the proof. □

We now describe the **good algorithm** from [BLR20, Lemma 4.18].

Sample a graph G from μ. For $\Lambda_1 \subset D \subset \Lambda_{10}$, we choose a point $z \in D$. Let r be small enough such that $\Lambda_{2r}(z) \subset D$. We will introduce a way of sampling the branches of the wired uniform spanning tree $\mathcal{T}^\delta$ from the vertices of $\Lambda_r(z)^\delta$. We denote by $\mathcal{Q}_j$ the collection of vertices in G^δ which are farthest from z in each cell of $\Lambda^\delta_{(1+2^{-j})r}(z) \cap r6^{-j}\mathbb{Z}^2$

at step j and are not sampled before. If there is no such vertex, we ignore that cell. At each step j, we sample the branches $\mathcal{T}_j^\delta$ from each vertex of $\mathcal{Q}_j$ in any order. This results in a proportion tree $\mathcal{T}_j^\delta$, which is the union of all the branches sampled in steps 1 up to j. We repeat this process until we exhaust all the vertices in $\Lambda_r^\delta(z)$.

Lemma 3.3.12 (Schramms's finiteness Lemma)**.** *Fix D, z, r as above. For all $\varepsilon, \varepsilon' > 0$, there exists a constant $j_0 = j_0(\varepsilon') > 0$ such that for all $\delta < \delta_0(\varepsilon', r) \wedge \varepsilon$ the following two events hold with probability at least $1 - \varepsilon'$. Fix a graph G sampled from μ such that $R^\delta_{\max}(\Lambda_{11r}) \le R_0^\delta(\varepsilon)$ where $R_0^\delta(\varepsilon)$ is as in (3.5) and $|G^\delta \cap \Lambda_{10}| \le 100 C_\mu \delta^{-2}$. Then*

(i) The random walks emanating from all vertices in $\mathcal{Q}_j$ for $j \ge j_0$ stay in the square $\Lambda_{2r}(z)$.

(ii) All the branches of $\mathcal{T}^\delta$ sampled from vertices in $\mathcal{Q}_j$ for $j \ge j_0$ until they hit $\mathcal{T}^\delta_{j_0} \cup \partial D^\delta$ have Euclidean diameter at most $\varepsilon' r$.

Proof. For each step $j \ge 1$, there are at most 6^{2j} cells. So the number of vertices chosen at each step is at most equal to 6^{2j}. Let $j_{\max} = \log_6(4r/R_0^\delta)$. Fix $j < j_{\max}$, the Euclidean distance between one vertex in $\mathcal{Q}_{j-1}$ and another one in $\mathcal{Q}_j$ is at most $4 \times 6^{-j} r$. Let X_t be a simple random walk started from a vertex in $\mathcal{Q}_j$. Then using Lemma 3.3.11, there exist constants $c, c' > 0$ such that

$$\begin{aligned} \mathbb{P}^G(X_t \text{ reach distance } C_0 6^{-j} r \text{ without hitting } \mathcal{T}^\delta_{j-1}) &\le c\Big(\frac{4 \times 6^{-j} r \vee R^\delta_{\max}}{C_0 6^{-j} r}\Big)^{c'} \\ &\le c\Big(\frac{4 \times 6^{-j} r \vee R^\delta_0}{C_0 6^{-j} r}\Big)^{c'} \\ &= c\Big(\frac{4 \cdot 6^{-j} r}{C_0 6^{-j} r}\Big)^{c'} \\ &\le \frac{1}{2} \end{aligned} \tag{3.6}$$

where the second inequality comes from the fact that $R^\delta_{\max}(\Lambda_{11r}) \le R_0^\delta$, the equality comes from the choice of $j < j_0$. The final inequality holds for a large enough choice of C_0, depending only on c, c'.

Denote $\mathcal{D}(w, j)$ to be the event that diameter of the random walks emanating from a vertex $w \in \mathcal{Q}_j$ is greater than $j^2 6^{-j} r$. Then applying the bound (3.6) j^2/C_0 times and using the Markov property of the random walk, we have

$$\mathbb{P}^G(\mathcal{D}(w, j)) \le \left(\frac{1}{2}\right)^{j^2 C_0^{-1}}. \tag{3.7}$$

If $j > j_{\max}$, we define the similar event $\widetilde{\mathcal{D}}(w, j)$ to be that the random walks emanating from a vertex $w \in \mathcal{Q}_j$ reaches distance greater than $j^2_{\max} 6^{-j_{\max}} r$ without hitting $\mathcal{T}^\delta_j$. The total number of vertices in $\cup_{j\geq j_{\max}}\mathcal{Q}_j$ is at most $100C_\mu\delta^{-2}$ by the choice of $\mathcal{G}$. Since $6^{-j} < R_0^\delta$, we cannot apply Lemma 3.3.11. However, since $j_{\max}^2 6^{-j_{\max}} > j^2 6^{-j}$ as $j_{\max} < j$, we still have

$$\mathbb{P}^G(\widetilde{\mathcal{D}}(w,j)) \leq \left(\frac{1}{2}\right)^{j_{\max}^2 C_0^{-1}}.$$

We will use this same upper bound for all $w \in \mathcal{Q}_j$ for $j \geq j_{\max}$. Denote $\mathcal{D} := \cup_{j\geq j_0}\cup_{w\in\mathcal{Q}_j}\mathcal{D}(w,j)$. Note that $\mathcal{D}^c$ contains the event that none of the random walks emanating from a vertex $w \in \bigcup_{j\geq j_0}\mathcal{Q}_j$ reaches distance $j^2 6^{-j} r$ from its starting point. Consequently, they all stay in $\Lambda_{2r}(z)$ which is item (i).

Note that

$$\mathbb{P}^G(\mathcal{D}) \leq \mathbb{P}^G\left(\bigcup_{j_0\leq j\leq j_{\max}}\bigcup_{w\in\mathcal{Q}_j}\mathcal{D}(w,j)\right) + \mathbb{P}^G\left(\bigcup_{j\geq j_{\max}}\bigcup_{w\in\mathcal{Q}_j}\widetilde{\mathcal{D}}(w,j)\right) \tag{3.8}$$

$$\leq \sum_{j_0\leq j\leq j_{\max}} 6^{j^2}\cdot\left(\frac{1}{2}\right)^{j^2 C_0^{-1}} + \frac{100C_\mu}{\delta^2}\left(\frac{1}{2}\right)^{j_{\max}^2 C_0^{-1}} \tag{3.9}$$

$$\leq \frac{\varepsilon'}{2} + \frac{100C_\mu}{\delta^2}\left(\frac{1}{2}\right)^{j_{\max}^2 C_0^{-1}}$$

where the first term in (3.9) is from

$$\sum_{j_0\leq j\leq j_{\max}} 6^{j^2}\cdot\left(\frac{1}{2}\right)^{j^2 C_0^{-1}} \leq \sum_{j\geq j_0} 6^{j^2}\cdot\left(\frac{1}{2}\right)^{j^2 C_0^{-1}} \leq \frac{\varepsilon'}{2}$$

for large enough choice of $j_0 = j_0(\varepsilon')$ by the ratio test.

Note that

$$\begin{aligned} j_{\max} = \log_6\frac{R_0^\delta}{4r} &= \log_6\left(\frac{\delta}{4r}\right) - \frac{1}{\beta}\log_6(4r\alpha) + \frac{1}{\beta}\log_6\left(\log\left(\frac{C}{\varepsilon\delta^2}\right)\right) \\ &\geq c'\log(\delta r^{-c''}) + C'\log\log(\varepsilon^{-1}\delta^{-2}) \end{aligned}$$

for some constants $c', c'', C' > 0$.

Hence, plug the lower bound of $j_{\max}$ in the second term of (3.9)

$$\begin{aligned} \frac{100C_\mu}{\delta^2}\left(\frac{c}{2}\right)^{j_{\max}^2 C_0^{-1}} &\leq \frac{100C_\mu}{\delta^2}(1/2)^{\left(c'\log(\delta r^{-c''})+C'\log\log(\varepsilon^{-1}\delta^{-2})\right)^2 C_0^{-1}} \\ &\leq \frac{100C_\mu}{\delta^2}\exp(-C''\log^2(\varepsilon^{-1}r^{-1}\delta)) \\ &\leq \varepsilon'/2. \end{aligned}$$

for some constants $C'' > 0$ where the last inequality holds for δ small enough.

Hence,

$$\mathbb{P}^G(\mathcal{D}) \le \varepsilon'$$

for large enough $j_0 = j_0(\varepsilon')$ and small enough choice of $\delta = \delta(\varepsilon', r) \wedge \varepsilon$.

Let $\mathcal{E}(w, j)$ be a event that the branch starting at w and ending at $\mathcal{T}^\delta_{j-1}$ has diameter at least $j^2 6^{-j} r$. Note that $\mathbb{P}^G(\mathcal{E}(w, j)|\mathcal{T}^\delta_{j-1})$ does not depend on the order of points in $\mathcal{Q}_j$ by Wilson's algorithm. Hence, we can assume that w is the first point when we compute this conditional probability. Hence,

$$\begin{aligned}\mathbb{P}^G(\mathcal{E}(w, j)|\mathcal{T}^\delta_{j-1}) &\le \mathbb{P}^G(\mathcal{D}(w, j)) \\ &\le (1/2)^{(j \wedge j_{\max})^2/C_0}\end{aligned}$$

The complement of event $\mathcal{E} := \cup_{j \ge j_0} \cup_{w \in \mathcal{Q}_j} \mathcal{E}(w, j)$ is that for each point $w \in \mathcal{Q}_j$ connected to a point in $\mathcal{T}^\delta_{j_0}$ by a path of diameter at most $\sum_{j \ge j_0} j^2 6^{-j} r \le \varepsilon' r$ as $j_0 = j_0(\varepsilon')$ large enough by the ratio test.

Furthermore, $\mathbb{P}^G(\mathcal{E}) \le \varepsilon'$ if $j_0 = j_0(\varepsilon')$ is large enough. The argument is similar as the inequality (3.8). This gives the desired result item (ii). □

3.3.3 Base Coupling

We fix a domain D such that $\Lambda_1 \subset D \subset \Lambda_{10}$ and an annulus $A(z, mn, (m+1)n) \subset D$ with $n \ge 2000 R^\delta_{\max}(\Lambda_{11})$ and $m \in \mathbb{N}$.

First, our goal is to describe the wired coupling between the uniform spanning tree in D^δ and a uniform spanning tree in $\widetilde{D}^\delta = \Lambda^\delta_{10}$ so that they match within a neighborhood of a fixed vertex $v \in D^\delta$. We call this coupling **base coupling**.

Base coupling. Recall that R^δ_0 is defined as in (3.5). Fix $\varepsilon' = 1/2, \varepsilon > 0$, we choose a collection

$$\mathcal{G} = \{G : R^\delta_{\max}(\Lambda_{11}) \le R^\delta_0, |G^\delta \cap \Lambda_1| \le C_\mu/\delta^2\}$$

with δ small enough so that $\mu(\mathcal{G}) \ge 1 - \varepsilon$ exactly in the same way as in Lemma 3.3.12. Given a graph $G \in \mathcal{G}$, we choose $j_0 = j_0(1/2)$ as in Lemma 3.3.12 so that the two events defined in Lemma 3.3.12 hold. A base coupling comes with a scale $0 < r < \mathrm{dist}(v, \partial_1)/2$. For two domains D^δ and $\tilde{D}^\delta$ in G, we try to couple the branches $\mathcal{T}^\delta$ emanating from the D^δ to the branches $\widetilde{\mathcal{T}}^\delta$ emanating from the $\tilde{D}^\delta$ within a square neighborhood $\Lambda^\delta_{0.9r}(v) \subset D^\delta$ of a vertex v and r. We call this **base coupling at scale** r. Given a vertex w, let $\gamma(w)$ (resp. $\tilde{\gamma}(w)$) be a branch of $\mathcal{T}^\delta$ (resp. $\widetilde{\mathcal{T}}^\delta$) sampled from a vertex w in D^δ (resp. $\tilde{D}^\delta$) via Wilson's algorithm (see Section 2.1.2 for the description of Wilson's algorithm).

(i) Fix a point $w_1 \in A(v, 0.8r, 0.9r)$, we sample $\gamma(w_1)$ and $\tilde{\gamma}(w_1)$ independently until hitting the boundary of either D^δ or $\tilde{D}^\delta$ respectively. Let E_1 be the event that both

$\gamma(w_1)$ and $\tilde{\gamma}(w_1)$ stay outside $\Lambda_{0.7r}(v)$.

(ii) Conditional on the event E_1 holding, we couple the loop-erased random walk emanating from a vertex $w_2 \in A(v, 0.3r, 0.4r)$. Here we sample a loop-erased random walk until hitting either $\gamma(w_1) \cup \partial D^\delta$ or $\tilde{\gamma}(w_1) \cup \partial \tilde{D}^\delta$. Without loss of generality, we assume that the path of the random walk corresponding to $\gamma(w_2)$ intersects $\gamma(w_1) \cup \partial D^\delta$ at time t_1. Then we continue the random walk from that point until it is in the $\tilde{\gamma}(w_1) \cup \partial \tilde{D}^\delta$ at time t_2 and its path is denoted by $\tilde{\gamma}(w_2)$. Let E_2 be the event that $\gamma(w_2)$ and $\tilde{\gamma}(w_2)$ agree in $\Lambda_{0.6r}(v)$. See Figure 3.4.

(iii) Suppose that events E_1 and E_2 hold. Fix a $j_0 = j_0(1/2)$ as defined in Lemma 3.3.12. As the description of good algorithm above, let $\mathcal{Q}_j$ be a set of vertices in $\{0.1r6^{-j}\mathbb{Z}^2\}_{j\geq 0} \cap \Lambda_{0.1r}(v)$ which are chosen that each one is furthest away from v within the small square. Define the event E_3 to be the branches emanating from all the vertices in $\cup_{j\leq j_0}\mathcal{Q}_j$ of $\mathcal{T}^\delta$ and $\widetilde{\mathcal{T}}^\delta$ agree in $\Lambda_{0.5r}(v)$.

(iv) Assume that events E_1, E_2 and E_3 hold. Let E_4 be the event that the remaining branches starting from vertices in $\cup_{j>j_0}\mathcal{Q}_j$ of $\mathcal{T}^\delta$ and $\widetilde{\mathcal{T}}^\delta$ agree in $\Lambda_{0.1r}(v)$.

Let $\mathbf{P}^G_{D,\tilde{D}} = \mathbf{P}^G$ be the probability measure defined by the above coupling.

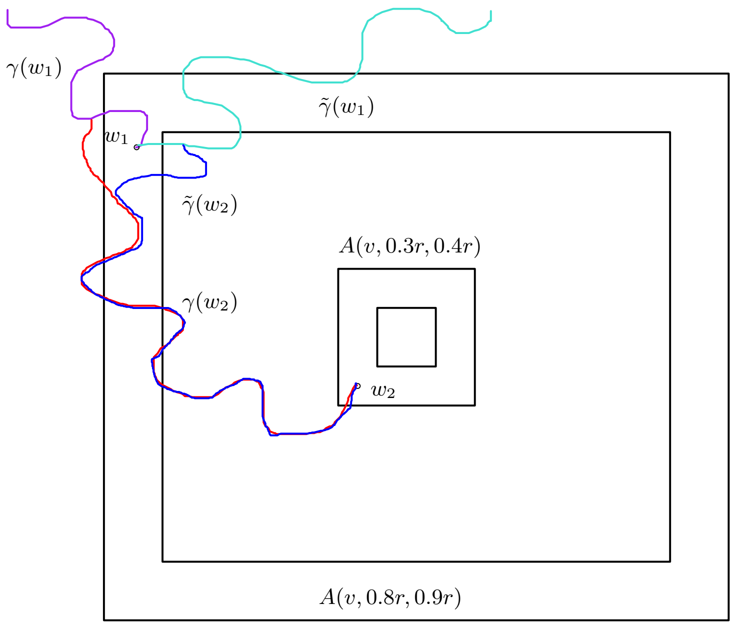

Figure 3.4: Base coupling: we sample $\gamma(w_1)$ (purple color) and $\tilde{\gamma}(w_1)$ (nattier blue color) from a point $w_1 \in A(v, 0.8r, 0.9r)$, and then the path of LERW is $\gamma(w_2)$ (red color) from a vertex $w_2 \in A(v, 0.3r, 0.4r)$ and we continue the random walk from that point form the blue path $\tilde{\gamma}(w_2)$.

We will see below that the base coupling succeeds with a uniformly positive probability. We say that the base coupling has **failed** if the intersection of all the events $\cap_{1\leq i\leq 4} E_i$

does not occur. Now, we prove lower and upper bound of the probability of $\cap_{1\le i\le 4} E_i$ as following.

Lemma 3.3.13. *Suppose that we are in the setup as above. There exists constants* $0 < p_1 < p_2 < 1$ *such that for all* $\delta > 0$ *and* $r > 2 \cdot 10^4 R_0^\delta$,

$$p_1 < \mathbf{P}^G(\textit{base coupling has failed}) < p_2$$

Proof. From Proposition 3.3.10, the loop-erased random walk does not get close to w_1 with positive probability. So $\mathbf{P}^G(E_1)$ is bounded below by $p_1' > 0$. Also, we can get the lower bound of the probability of the complement of the event E_1

$$\mathbf{P}^G(E_1^c) \ge c > 0.$$

by Lemma 3.3.4 for δ small enough. Indeed, the crossing probability of any rectangle of size large or equal than $1000R_{\max}^\delta$ is uniformly positive. So $\mathbf{P}^G(E_1) < p_2$.

By Lemma 3.3.7, the random walk started from w_2 has a positive probability to exit $\Lambda_{0.7r}(v)$ and then makes a full turn in $A(v, 0.9r, 0.95r)$. It is easy to see that this event is contained in event E_2. So the walk would stop in the $A(v, 0.8r, 0.9r)$ but first hit $\Lambda_{0.6r}(v)$ which implies $\mathbf{P}^G(E_2|E_1) \ge p_3'$.

Suppose that the event $E_1 \cap E_2$ holds. For $j \le j_0$, the probability of the event that the walk starting from $w_3 \in \mathcal{Q}_j$ makes a full turn before exiting $A(v, 0.4r, 0.5r)$ is at least p_4 by Lemma 3.3.7. Hence, the corresponding branches $\gamma(w_3)$ and $\widetilde{\gamma}(w_3)$ will agree in $\Lambda_{5n}(v)$ since the walk will hit the either $\gamma(w_2)$ or $\widetilde{\gamma}(w_2)$ on this event. Hence,

$$\mathbf{P}^G(E_3|E_1 \cap E_2) \ge 6^{j_0} p_4 = p_4'.$$

For the event E_4, we have

$$\mathbf{P}^G(E_4|E_1 \cap E_2 \cap E_3) \ge \frac{1}{2}$$

by Lemma 3.3.12.

Finally, we get

$$\mathbf{P}^G\left(\bigcap_i E_i\right) \ge \frac{p_1' p_3' p_4'}{2} =: p_1$$

This completes the proof. □

If the base coupling fails, we can retry the coupling in a new smaller neighborhood that was not intersected of the sampled paths. We say a vertex v has isolation radius

6^{-k} at scale r at any step in the above base coupling if $\Lambda_{6^{-k}r}(v)$ does not intersect any sampled branches. Let I_v be the minimal of such k at the time when the base coupling fails.

Lemma 3.3.14. *Fix $r > 2 \cdot 10^4 R_0^\delta$. Suppose we perform base coupling at scale r. If we define I_v as above, then there exist constants $c_1, c_2, c_3 > 0$ such that for all $i > 0$ and $\delta > 0$ small enough,*

$$\mathbf{P}^G\left(I_v \geq i \middle| \textit{ coupling fails }\right) \leq c_1 e^{-c_2 i} + \frac{100 C_\mu}{\delta^2} \exp(-c_3 \log^2(r^{-1}\delta)).$$

Proof. We first consider the case that one of the $\mathcal{E}_k$ fails for $k = 1, 2, 3$. In fact, the event that loop erased random walk comes within distance $r6^{-i}$ of v contains the event that the isolation radius is at least i if base coupling fails at a step. Let $H_{k,i}$ be the event that the isolation radius is at least i on the step $\mathcal{E}_k$, $k = 1, 2, 3$. By Proposition 3.3.10, we get

$$\mathbf{P}^G(H_k) \leq (1-\eta)^i.$$

Thus we have

$$\begin{aligned}\mathbf{P}^G(I_v \geq i; \text{coupling fails}) &\leq \mathbf{P}^G\left(\bigcup_{k=1}^{3} H_k\right) + \mathbf{P}^G(H_4; \text{coupling fails})\\ &\leq \sum_{k=1}^{3} \mathbf{P}^G(H_k) + \mathbf{P}^G(H_4; \text{coupling fails})\\ &\leq 3(1-\eta)^i + \mathbf{P}^G(H_4; \text{coupling fails}) \qquad (3.10)\end{aligned}$$

where the first inequality is by inclusion.

For the rest of the proof, it is enough to get the upper bound of $\mathbf{P}^G(H_4)$ and we exclusively deal with $j \geq j_0$ in this step. Note that the number of vertices in $\mathcal{Q}_j$ is at most 6^{2j}. Let $\mathcal{A}(i,j)$ be the event that coupling fails in step $j \geq j_0$ and $I_v \geq i$ at this step.

Next, we split i into two cases: $i \in \left(j^2, \log_6\left(\frac{r}{2000 R_0^\delta}\right)\right)$ and $i < j^2$.

For $i \in \left(j^2, \log_6\left(\frac{r}{2000 R_0^\delta}\right)\right)$, $\mathbf{P}^G(\mathcal{A}(i,j))$ is bounded by the probability of the event that a branch $\gamma(\omega)$ from one of the vertices $\omega \in \mathcal{Q}_j$ comes within distance $r6^{-i}$ of v, that is,

$$\mathbf{P}^G(\mathcal{A}(i,j)) \leq \mathbf{P}^G\left(\bigcup_{\omega \in \mathcal{Q}_j} \operatorname{dist}(\gamma(\omega), v) < r6^{-i}\right).$$

Note that

$$\mathbf{P}^G\left(\operatorname{dist}(\gamma(\omega), v) < r6^{-i}\right) \leq (1-\eta)^{i-j}$$

where $n = i - j$ and r is substituted by $r6^{-j}$ in Proposition 3.3.10.

Then we obtain

$$\mathbf{P}^G\left(\bigcup_{j\le\sqrt{i}}\bigcup_{\omega\in\mathcal{Q}_j}\mathcal{A}(i,j)\right) \le \sum_{j\le\sqrt{i}} 6^{2j}(1-\eta)^{i-j}$$
$$\le \sum_{j\le\sqrt{i}} 6^{2\sqrt{i}}(1-\eta)^{i-\sqrt{i}}$$
$$\le \sqrt{i}6^{2\sqrt{i}}(1-\eta)^{i-\sqrt{i}}. \tag{3.11}$$

It is obvious that the upper bound (3.11) of the probability is exponentially small in i.

For $i < j^2$, the probability of $\mathcal{A}(i,j)$ is at most the probability of the event that diameter of one branches sampled in step j is greater than $j^2 6^{-j} r$ around ω. Then we bound the probability of $\mathcal{A}(i,j)$ by the inequality (3.7) which we got in the proof of Lemma 3.3.12. Note that the later one is also a bound on the probability that one of these branches leaving the square $\Lambda_{0.1r}(v)$. So

$$\mathbf{P}^G(\mathcal{A}(i,j)) \le 6^{2j}\left(\frac{1}{2}\right)^{C_0^{-1}j^2} \le C(1-\eta')^{j^2}$$

for $j < j_{\max}$ and some constants $\eta' > 0$ and $C > 0$.

Then we get

$$\mathbf{P}^G\left(\bigcup_{j\ge\sqrt{i}}\mathcal{A}(i,j)\right) \le \mathbf{P}^G\left(\bigcup_{\sqrt{i}\le j<j_{\max}}\mathcal{A}(i,j)\right) + \underbrace{\mathbf{P}^G\left(\bigcup_{j\ge j_{\max}}\mathcal{A}(i,j)\right)}_{(*)} \tag{3.12}$$

$$\le \sum_{j\ge\sqrt{i}} C(1-\eta')^{j^2} + \frac{100C_\mu}{\delta^2}\left(\frac{1}{2}\right)^{j_{\max}^2 C_0^{-1}} \tag{3.13}$$
$$\le C(1-\eta')^i$$

for $\delta > 0$ small enough where the term $(*)$ in (3.12) is bounded by the second term in (3.9) and the second term in (3.13) is much smaller than first term by the same argument as (3.9) in Lemma 3.3.12.

Hence, combine (3.12) and (3.11) estimates

$$\mathbf{P}^G(H_4;\text{coupling fails}) \le \mathbf{P}^G\left(\bigcup_{j\ge\sqrt{i}}\mathcal{A}(i,j)\right) + \mathbf{P}^G\left(\bigcup_{j\le\sqrt{i}}\bigcup_{\omega\in\mathcal{Q}_j}\mathcal{A}(i,j)\right) \le c_4 e^{-c_5\eta'} \tag{3.14}$$

for some constants $c_4, c_5 > 0$.

To sum up, from (3.10) and (3.14) we have

$$\mathbf{P}^G(I_v \geq i;\ \text{coupling fails }) \leq \mathbf{P}^G\left(\bigcup_{j\geq j_0}\mathcal{A}(i,j)\right) \leq c_6 e^{-c_7\eta'} \tag{3.15}$$

for some constants $c_6, c_7 > 0$ and $\delta > 0$ small enough.

Hence, using Lemma 3.3.13 and (3.15)

$$\mathbf{P}^G(I_v \geq i|\ \text{coupling fails }) \leq p_1^{-1}\mathbf{P}^G(I_v \geq i;\ \text{coupling fails }) \leq p_1^{-1}c_6 e^{-c_7\eta'}. \tag{3.16}$$

Now we record the case of $i > \log_6\left(\frac{r}{2000R_0^\delta}\right)$. For $i > \log_6\left(\frac{r}{2000R_0^\delta}\right)$, we still bound the probability of $\mathcal{A}(i,j)$ by the second term in (3.9). So for some constants $C' > 0$

$$\mathbf{P}^G\left(I_v \geq i\right) \leq \frac{100C_\mu}{\delta^2}\exp(-C'\log^2(r^{-1}\delta)). \tag{3.17}$$

Combine (3.16) and (3.17), we complete the proof. □

3.3.4 Iteration of base coupling around a single point

In this section, we describe a way to iterate the base coupling which decreases the scale at each step, and in the end we want to conclude that the coupling succeeds after geometric many tries. Also, we want to conclude that after geometric many tries, there is enough space around a single point that we perform the base coupling so that the USTs are coupled.

Fix $r > 2\cdot 10^4 R_{\max}^\delta$ and domain $\Lambda_1 \subset D \subset \Lambda_{10}$. The definition of base coupling at scale r from Section 2.3. We now provide a strategy to iterate the base coupling around a vertex v until we either succeed or abort in the following procedure.

(i) If the base coupling succeeds, we are done.

(ii) If not, set $6^{-I_{v,1}}r$ to be the isolation radius around v at scale r (i.e., none of the branches sampled in the base coupling intersects $B(v, 6^{-I_{v,1}}r)$). Define $\mathcal{T}_0^\delta$ (resp. $\widetilde{\mathcal{T}}_0^\delta$) to be the part of the UST in D^δ (resp. $\widetilde{D}^\delta$) sampled as in the base coupling in this step.

case 1: If $I_{v,1} < \log_6\left(\frac{r}{2000R_0^\delta}\right)$, we perform a new base coupling in $\Lambda_{6^{-I_{v,1}}r}(v)$ with domains $D^\delta \setminus \mathcal{T}_0^\delta$ (resp. $\widetilde{D}^\delta \setminus \widetilde{\mathcal{T}}_0^\delta$).

case 2: If $I_{v,1} \geq \log_6\left(\frac{r}{2000R_0^\delta}\right)$, then we abort the whole process and the full coupling fails.

(iii) If the coupling fails in case 1 of (ii), let $6^{-I_{v,1}-I_{v,2}}r$ be the isolation radius of v. Let branches $\mathcal{T}_1^\delta$ (resp. $\widetilde{\mathcal{T}}_1^\delta$) be the portion of UST up to step (ii) where $\mathcal{T}_0 \subset \mathcal{T}_1$ and $\widetilde{\mathcal{T}}_0 \subset \widetilde{\mathcal{T}}_1$.

case 1: If $I_{v,1} + I_{v,2} < \log_6\left(\frac{r}{2000R_0^\delta}\right)$, we perform the base coupling with $D^\delta \setminus \mathcal{T}_1^\delta$ (resp. $\widetilde{D}^\delta \setminus \widetilde{\mathcal{T}}_1^\delta$) in $\Lambda_{6^{-I_{v,1}-I_{v,2}}r}(v)$.

case 2: If $I_{v,1} + I_{v,2} \geq \log_6\left(\frac{r}{2000R_0^\delta}\right)$, we abort the whole process.

(iv) Then we repeat the above steps until we either succeed or abort this process.

case 1: If we succeed at step m, then we get a coupling between $\mathcal{T}_m^\delta$ and $\widetilde{\mathcal{T}}_m^\delta$ which agree in $\Lambda_{6^{-\sum_{l=1}^m I_{v,l}}r}(v)$.

case 2: If we abort this process at step m or the coupling succeeds, let $m = N$. If we abort, we say whole coupling has failed.

We denote $I := \sum_{l=1}^{N-1} I_{v,l}$ by convention. Actually, we can show that I would not be too large.

Theorem 3.3.15. *Fix $r > 2 \cdot 10^4 R_{\max}^\delta$. There exist constants $c_1, c_2, c_3 > 0$ such that for all $i > 0$,*

$$\mathbf{P}^G(I \geq i) \leq c_1 e^{-c_2 i} + \left(\frac{\delta}{r}\right)^{c_3}.$$

Proof. Let event $\mathcal{A}$ be that we abort the coupling. From Lemma 3.3.13, the base coupling has probability uniformly bounded below by $1 - p_2$ to succeed at every step, that means N is stochastic dominated by a geometric distribution. Also, $I_{v,l}$ is independent and has uniformly exponential tail conditionally on all previous steps $I_{v,k}, 1 \leq k \leq l-1$ by Lemma 3.3.14. Let $\mathcal{A}_n$ be the event that the coupling is not aborted by step n. Then there exists constants C, λ such that

$$\mathbb{E}(e^{\lambda I_{v,l}} | I_{v,k}, 1 \leq k \leq l-1; \mathcal{A}_l^c) < C \tag{3.18}$$

by Lemma 3.3.14.

Hence,

$$\mathbb{E}(e^{\lambda I_{v,1} + \lambda I_{v,2}}) = \mathbb{E}(\mathbb{E}(e^{\lambda I_{v,1} + \lambda I_{v,2}} | I_1)) = \mathbb{E}(e^{\lambda I_{v,1}} \mathbb{E}(e^{\lambda I_{v,2}} | I_1)) \leq \mathbb{E}(C e^{\lambda I_1}) \leq C^2.$$

By induction, we obtain

$$\mathbb{E}(e^{\sum_{l=1}^n \lambda I_{v,l}}) \leq C^n. \tag{3.19}$$

By the Markov's inequality and above estimations, we have

$$
\begin{aligned}
\mathbf{P}^G(I \geq i; \mathcal{A}^c) = \mathbf{P}^G\left(\sum_{l=1}^{N-1} I_{v,l} \geq i; \mathcal{A}^c\right) &\leq \mathbf{P}^G\left(N \geq \eta i\right) + \mathbf{P}^G\left(\sum_{l=1}^{N-1} I_{v,l} \geq i, N < \eta i\right) \\
&\leq \mathbf{P}^G(N \geq \eta i) + \mathbf{P}^G\left(\exp\left(\sum_{l=1}^{\eta i-1} I_{v,l}\right) \geq e^i\right) \\
&\leq (1-p_2)^{\eta i} + \mathbb{E}\left(\exp\left(\sum_{l=1}^{\eta i-1} I_{v,l}\right)\right) e^{-i} \\
&\leq e^{-c\eta i} + C^{\eta i} e^{-i} \\
&\leq c_1 e^{-c_2 i} \qquad (3.20)
\end{aligned}
$$

for some constants $c_1, c_2 > 0$, where choose η sufficiently small such that $0 < \eta < 1/\log C$.

Also, note that

$$
\begin{aligned}
\mathbf{P}^G(\mathcal{A}) &\leq \mathbf{P}^G\left(\sum_{l=1}^{N-1} I_{v,l} \geq \log_6\left(\frac{r}{2000 R_0^\delta}\right)\right) \\
&\leq \mathbf{P}^G\left(N > \eta \log(\delta^{-2})\right) + \mathbf{P}^G\left(\sum_{l=1}^{N-1} I_{v,l} \geq \log_6\left(\frac{r}{2000R_0^\delta}\right), N \leq \eta\log(\delta^{-2})\right) \\
&\leq \mathbf{P}^G\left(N \geq \eta\log(\delta^{-2})\right) + \mathbb{E}\left(\exp\left(\sum_{l=1}^{\eta\log(\delta^{-2})} I_{v,l}\right)\right)\left(\exp\left(\log_6\left(\frac{r}{2000R_0^\delta}\right)\right)\right)^{-1} \\
&\leq \delta^{2c\eta} + \delta^{-2\eta\log C}\left(\frac{\delta\log(\delta^{-2})}{r}\right)^{c'} \leq \left(\frac{\delta}{r}\right)^{c_3}
\end{aligned}
$$

for some constants $c_3 > 0$, where we take $\eta < 1/(10\log C)$.

□

3.3.5 Full coupling

After describing the iteration of base coupling around a single point, it is natural to ask how to perform the full coupling around some fixed points. Indeed, we need to make the iteration of base coupling on each point. However, these processes are not independent. The key point is that we first sample all branches emanating from a cutset separating each vertex. Conditional on these sampled branches, the neighborhood of these vertices are independent now. Then we can show that "unexplored" neighborhoods around each point are still large with high probability and apply the iteration of base coupling for each "unexplored" neighborhood.

Let $v_1, \ldots, v_k$ be k distinct points and $0 < r < \frac{1}{2}\min\{\mathrm{dist}(v_i, \partial\Lambda_1), |v_i - v_j|, i \neq j\}$.

Let $\Lambda^\delta_{0.9r}(v_i)$ be disjoint squares and $\Lambda^\delta_{0.9r}(v_i) \subset D^\delta$ for $1 \le i \le k$. We repeat the previous setting in Section 3.3.3. As we assumed in base coupling, we fix $\varepsilon > 0, \varepsilon' = 1/2$ and choose a collection

$$\mathcal{G} = \{G : R^\delta_{\max}(\Lambda_{11}) \le R^\delta_0, |G^\delta \cap \Lambda_1(v_i)| \le C_\mu \delta^{-2}, i = 1, 2 \dots, k\} \tag{3.21}$$

with R^δ_0 defined as in (3.5) chosen so that $\mu^\delta(\mathcal{G}) \ge 1 - \varepsilon$. Fix a graph $G \in \mathcal{G}$ and choose $j_0 = j_0(1/2)$ so that the two events defined in Lemma 3.3.12 hold. We also assume $r > 2 \cdot 10^4 R^\delta_0$ to use results in Section 3.3.4.

Let K_i be a set of vertices in $A(v_i, 0.4r, 0.5r)$ and let $K = \cup_i K_i$. Then we sample all branches emanating from K_i by Wilson's algorithm, inducing a portion of UST defined by $\mathcal{T}^\delta_K$. This step is called by **cutset exploration**. We denote J_{v_i} by the minimal j so that $\Lambda_{6^{-j}r}(v_i) \cap \mathcal{T}^\delta_K = \emptyset$. Let $J := \max_i J_{v_i}$. We say that we **abort the cutset exploration** if $J_{v_i} \ge \log_6 \left(\frac{r}{2000 R^\delta_0}\right), i = 1, 2, \dots, k$.

We hope that there exists a loop of open edges around the neighborhood of v_i because we need a cutset so that v_i disconnect from ∂D^δ and other branches. Let $\mathcal{A}(v_i)$ be the event that there is a loop in $A(v_i, 0.4r, 0.5r)$. Indeed, using Lemma 3.3.1, we have

$$\mu(\mathcal{A}(v_i)) \ge 1 - 4 \exp\left(-\frac{c r^{c'}}{\delta^{c'}}\right).$$

So for all $\varepsilon > 0$, $v_i, i = 1, 2, \dots, k$, we can assume that $\mathcal{G}$ is contained in $\cap_{i=1}^k \mathcal{A}(v_i)$ with $\mu(\mathcal{G}) \ge 1 - 2\varepsilon$ for δ small enough.

Next, we will show that perform the iteration of base coupling around $v_i, 1 \le i \le k$ on the remaining unexplored domain $D^\delta \setminus \mathcal{T}^\delta_K$. Conditional on the previous branches $\mathcal{T}^\delta_K$, we assume that we have not aborted this process so far. The remaining tree of $\mathcal{T}^\delta$ is distributed as a wired UST on the component containing v_i of $D^\delta \setminus \mathcal{T}^\delta_K$. Then we perform the iterated base coupling of this wired UST and a UST $\mathcal{T}^\delta_i$ of $\tilde{D}^\delta$ on $\Lambda_{6^{-J}r}(v_i)$.

Finally, we obtain conditionally independent subtrees $\{\mathcal{T}^\delta_i\}_{1 \le i \le k}$ given $\mathcal{T}^\delta_K$. Also, we abort the full coupling if we either aborted at the cutset exploration or in any of steps of the iterated base coupling on the event we have not aborted. Let $I_i := \sum_{l=1}^{N_i - 1} I_{v_i, l} + J$ be the final isolation radius around v_i.

3.3.6 Proof of theorem 3.1.2

Proof of theorem 3.1.2. We choose $\delta(\varepsilon')$ sufficiently small so that for all $\delta < \delta_0(\varepsilon')$ and $(\delta \log(\delta^{-1}))^{c_1} \le \varepsilon'$. If needed, we even let $\delta \le \varepsilon$. Hence, we have a collection $\mathcal{G}$ as in (3.21) so that $\mu(\mathcal{G}) \ge 1 - \varepsilon$. Fix such a graph $G \in \mathcal{G}$ and the choice of δ. Then we perform the full coupling described as in Section 3.3.5.

Recall that $I_i = \sum_{l=1}^{N_i - 1} I_{v_i,l} + J$. Note that Lemma 3.3.14 allows us to upper bound the tail probability $\mathbf{P}^G(J \geq n)$. We follow the exact same lines as Theorem 3.3.15. Let $I := \max_{1 \leq i \leq k} I_i$ and $R = r6^{-I}$. Then from Theorem 3.3.15 we obtain

$$\mathbf{P}^G(R \leq r\varepsilon') = \mathbf{P}^G(I \geq \log_6((\varepsilon')^{-1})) \leq c_2\varepsilon' + \left(\frac{\varepsilon'}{r}\right)^{c_3}$$

for some constants $c_2, c_3 > 0$. This completes the proof. □

Chapter 4

Quantitative Russo-Seymour-Welsh for random walk on random graphs

In this chapter, we consider the application of the main theorem 3.1.2 in concrete examples: the unique infinite cluster of bond percolation and Delaunay triangulation (or called **Voronoi triangulation**) [1]. To apply our main theorem, we need to verify two assumptions as in Section 3.1: crossing estimate (RSW estimate) and bounded density. Obviously, the bounded density assumption is true for bond percolation and Poisson point processes. The rest thing is to prove the RSW type estimates. In fact, the main input is a result by [Bar04], which in two dimensions states that a quadratic volume growth and Poincaré inequality ensures a good heat kernel bound for random walks on graphs. One can establish the Poincaré inequality with a good control on the volume growth and isoperimetric constant on the graph. We collect these geometric criteria in Lemma 4.1.5. We need to be a bit more careful than the treatment in [Bar04] because there is no uniform bound on degree in our assumption. Actually, the uniform bound on degree was assumed in [Bar04] as the main motivation to study heat kernel bounds for bond percolation in $\mathbb{Z}^d$. Next, we show that criteria as in Lemma 4.1.5 hold for the main two applications in this thesis about the unique infinite cluster of bond percolation in $\mathbb{Z}^2$ and Delaunay triangulation. A key input is a quantitative isoperimetric inequality for Delaunay triangulation.

Let's recall some notations and definitions in Section 2.3. Recall that given a graph G, we denote by $V(G)$ its vertex set and $E(G)$ its edge set. Let Λ_n denote the square $[-n, n]^2$ with $\Lambda_n(x) = x+\Lambda_n$. Sometimes we will also deal with rectangles $\Lambda_{m,n} = [-m, m]\times[-n, n]$ and similarly $\Lambda_{m,n}(z) = z+\Lambda_{m,n}$. For $S \subset V(G)$, denote by $|S|_E$ the sum of the weights of the edges incident to vertices of G in S, while $|S|$ simply denotes the number of vertices in S. With a slight abuse of notation, for any $S \subset \mathbb{R}^2$, we use $|S|_E$ to denote the sum of the

[1] See Section 2.2 for definition of these two models.

weights of the edges which is incident to a vertex in S, and $|S|$ to denote the cardinality of the set of vertices in S. Let $\deg(v)$ denotes the degree of the vertex v. Recall that $d(x, y)$ is the graph distance in G from x to y and $B(x, r)$ denote the graph distance ball of radius r centered around the vertex x.

4.1 A general criterion for RSW

In this section, we work with a fixed graph $G = (G(E), V(E))$ embedded in $\mathbb{R}^2$. The goal is to summarize certain geometric properties of the graph which ensures that a simple random walk on it behaves in a nice manner. The main quantity of interest in this section is the criteria of a c_μ−crossable rectangle, which is defined as in Definition 3.1.1.

Let $G = (V, E)$ be a finite planar graph, properly embedded in $\mathbb{R}^2$. For a function $f : V \mapsto \mathbb{R}$, recall that ∇f is a function from the oriented edges of the graph to $\mathbb{R}$ as in Section 2.3, satisfying

$$\nabla f((e^-, e^+)) = f(e^+) - f(e^-).$$

We will denote by $|\nabla f|$ the function which takes absolute value of ∇f for each unoriented edge in E. We now borrow the notions of 'good' and 'very good' from [Bar04]. Let o denote the vertex in G which is closest to the origin in $\mathbb{R}^2$.

Definition 4.1.1. *([Bar04, Definition 1.7]) Let $C_P, C_V > 0$ and $C_W \geq 1$ be fixed. We say that $B(o, n)$ is (C_P, C_V, C_W)-**good** if it satisfies*

$$\text{(Vol)} \qquad |B(o, n)|_E \geq C_V n^2.$$

*and every $f : B(o, C_W n) \mapsto \mathbb{R}$ satisfies the **weak Poincaré inequality**, i.e.,*

$$\text{(P)} \qquad \sum_{v \in B(o,n)} (f(v) - \bar{f})^2 \deg(v) \leq C_P n^2 \sum_{e \in E(B(o, C_W n))} |\nabla f(e)|^2$$

*where $\bar{f} = |V(B(o, n))|_E^{-1} \sum_{v \in B(o,n)} f(v) \deg(v)$. We say $B(o, n)$ is (C_P, C_V, C_W)-**very good** if there exists an integer $N_{B(o,n)} \leq n^{1/4}$ such that every $B(y, r) \subset B(o, n)$ is good for every $N_B \leq r \leq n$.*

We will sometimes drop the constants in the definition of good and very good when they are clear from the context.

Remark 4.1.2. *Note that the weak Poincaré inequality has been discussed in Definition 2.3.5.*

We now extend the notion of good and very good to Euclidean rectangles and squares. We say Λ_n is $(C_{\mathsf{Euc}}, C_P, C_V, C_W)$-**very good** if and only if there exists a constant $C_{\mathsf{Euc}} > 0$

such that

$$B(o, C_{\mathsf{Euc}}^{-1} n) \subset \Lambda_n \subset B(o, C_{\mathsf{Euc}} n)$$

and $B(o, C_{\mathsf{Euc}} n)$ is (C_P, C_V, C_W)-very good.

Lemma 4.1.3. *Suppose there exist constants $C_{\mathsf{Euc}}, C_P, C_V, C_W, c_0, d > 0$ such that for all $n \geq 1$ the following is true.*

(i) $\Lambda_{n \log n}$ *is* $(C_{\mathsf{Euc}}, C_P, C_V, C_W)$*-very good with* $N_{\Lambda_{n \log n}} \leq n^{1/8}$.

(ii) $|\Lambda^{(2)}| \geq dn^2$ *where* $\Lambda^{(2)}$ *is as in Definition 3.1.1.*

(iii) The graph distance between any vertex in $\Lambda_{2.5m,m/2}$ *and any vertex outside* $\Lambda_{3m,m}$ *is at least* $c_0 n$ *with* $m = n/4$.

Then there exists a constant $c = c(C_P, C_V, C_W, C_{\mathsf{Euc}}, c_0, d)$ *such that* $\Lambda_{3m,m}$ *is c-crossable.*

Proof. Let Y be the continuous time random walk with q_t^0 denoting its density killed upon exiting $\Lambda_{3m,m}$. More precisely, denoting τ to be the infimum over times when Y is not in $\Lambda_{3m,m}$,

$$q_t^0(x, y) = \mathbb{P}(Y_t = y, Y_0 = x, \tau > t).$$

Using the same argument as in [Bar04, Lemma 5.8], we can show that there exists a constant $c = c(C_P, C_V, C_W, C_{\mathsf{Euc}}, c_0)$ such that for any $x \in B_1$ and $y \in B_2$,

$$q_t^0(x, y) \geq \frac{c}{t}, \quad cn^2 \leq t \leq c^{-1} n^2$$

Let us provide some details of this fact. We can write for any $z \in \Lambda_{2.5m,m/2}$ and any $t > 0$,

$$\begin{aligned} q_t^0(x, z) &\geq q_t(x, z) - \mathbb{E}_x(\mathbb{1}_{\tau < t} q_{t-\tau}(Y_{\tau,z})) \\ &\geq q_t(x, z) - \sup_{0 \leq s \leq t} \sup_{w \in \partial'} q_s(w, z). \end{aligned}$$

where ∂' denote the set of vertices outside $\Lambda_{3m,m}$ with at least one neighbour in $\Lambda_{3m,m}$. Now fix $\varepsilon \in (0, 1/8)$, $z \in B(x, \delta n)$ and $t = \delta^2 n^2$. This allows us to apply the heat kernel bound of [Bar04, Theorem 5.3] to lower bound the first term $q_t(x, z)$ by $t^{-1} c e^{-c'}$ (for ease of reference, we point out that we choose $x_0 = x_1 = o$, $R \log R = C_{\mathsf{Euc}} n \log n$ in the notations of that theorem). On the other hand since the graph distance between any $w \in \partial'$ and $z \in \Lambda_{2.5m,m/2}$ is at least $c_0 n$ by the third item above, we can use [Bar04, Theorem 3.8] to upper bound the second term. Namely, writing $s = \theta t$,

$$\sup_{0 \leq s \leq t} \sup_{w \in \partial'} q_s(w, z) \leq \sup_{0 \leq \theta \leq 1} \frac{1}{\theta t} e^{-\frac{c' c_0^2 n^2}{\theta \delta^2 n^2}} = \sup_{0 \leq \theta \leq 1} \frac{1}{\theta} e^{-\frac{c' c_0^2}{\theta \delta^2}}.$$

(Again for ease of reference, we point out that we choose $x_0 = o$, $R = C_{\mathsf{Euc}} n \log n$ in the notations of [Bar04, Theorem 3.8].) Actually to be more precise for very small values of θ we go outside the range of times when [Bar04, Theorem 3.8] is applicable, in which case we use the upper bound [Bar04, Lemma 1.1] instead. Choosing δ small enough we obtain

$$q_t^0(x,z) \geq \frac{c(\delta)}{t} = \frac{c(\delta)}{\delta^2 n^2}.$$

Now we use the standard chaining argument. Namely, we choose a sequence of balls inside $\Lambda_{2.5m,m}$ of Volume $O(\delta^2 n^2)$ such that on the event that the walk iteratively lands on these sequence of balls, without leaving $\Lambda_{2.5m,m}$ the walk enters B_2. Using the Markov property of the walk, this event has probability at least $c(\delta)^{O(\delta^{-1})}$. The proof is complete as the probability of crossing is at least this constant. $\square$

In light of Lemma 4.1.3, it is clear that in the setting of random graphs, we require a box to be very good with high probability. To that end, it is useful to find a geometric condition for a box being very good. While (Vol) is a very simple geometric condition, (P) is analytic. We present below a lemma which essentially states that a relevant isoperimetric inequality implies (P).

We now recall some relevant definitions regarding isoperimetry of general graphs as in Section 2.3.1. Take a finite and connected graph H. For any $A \subset V(H)$, let

$$i_H(A) := \frac{\partial_E(A, H \setminus A)}{|A|_E}.$$

where $\partial_E(A, H \setminus A)$ denotes the collection of edges with one endpoint in A and another in $H \setminus A$. We say A is connected if the subgraph induced by A is connected. Define the isoperimetric constant I_H as

$$I_H := \inf\{i_H(A) : 0 < |A|_E \leq \frac{1}{2}|V(H)|_E : A \text{ and } H \setminus A \text{ are connected}\}.$$

A subgraph H' of H is a graph induced by a subset of vertices of H. Notice that the definition of $I_{H'}$ ignores the edges not present in $E(H')$. Observe the assertion of connectedness in A and $H \setminus A$ being connected is slightly non-standard, however [Bar04, Lemma 1.3] ensures that removing this connectedness assertion only changes the constant by a factor of 2. This leads us to the following rephrasing of [Bar04, Proposition 1.4(a)]:

Lemma 4.1.4. *Then there exists $c > 0$ such that for any subgraph H of G and any function $f : V(H) \mapsto \mathbb{R}$,*

$$\sum_{v \in V(H)} (f(v) - \bar{f}_H)^2 \deg_H(v) \leq \frac{c}{I_H^2} \sum_{e \in E(H)} |\nabla f(e)|^2$$

where $\bar{f}_H = |V(H)|^{-1}_{E(H)} \sum_{v \in V(H)} f(v)\deg(v)$ *and* $\deg_H(v)$ *is the degree of* v *and* $|V(H)|_{E(H)}$ *is the sum over degrees of vertices in* $V(H)$ *counting edges only in* H.

This allows us to describe an equivalent geometric criterion which will ensure a crossing estimate.

Lemma 4.1.5. *Suppose there exist constants* $C_{\mathsf{Euc}}, C_V, C_V, C_W, C_I, d, c_0$ *such that for all* $n \geq 1$,

(i) $B(o, C_{\mathsf{Euc}}^{-1} n \log n) \subset \Lambda_{n \log n} \subset B(o, C_{\mathsf{Euc}} n \log n)$.

(ii) For all $n^{1/9} \leq r \leq C_{\mathsf{Euc}} n \log n$,

$$C_V r^2 \leq |B(y,r)|_E \leq C'_V r^2$$

for all $B(y,r) \subset B(o, C_{\mathsf{Euc}} n \log n)$.

(iii) For all set of vertices $A \subseteq V(B(y,r)) \subseteq V(B(o, C_{\mathsf{Euc}} n \log n + 1))$ *inducing a connected subgraph such that* $n^{1/9} \leq |A|_E \leq \frac{|B(y,r+1)|_E}{2}$,

$$i_{B(y,r)}(A) \geq \frac{C_I}{\sqrt{|A|_E}} \tag{4.1}$$

(iv) $|\Lambda^{(2)}|_E \geq dn^2$ *where* $\Lambda^{(2)}$ *is as in* (3.1).

(v) Let $m = \frac{n}{4}$. *The graph distance between any vertex in* $\Lambda_{2.5m,m/2}$ *and any vertex outside* $\Lambda_{3m,m}$ *is at least* $c_0 n$.

Then there exists a constant $c > 0$ *(depending only on the constants above) such that for all* $n \geq 1$, $\Lambda_{3m,m}$ *is c-crossable.*

Proof. This is a straightforward application of Lemma 4.1.3 and Lemma 4.1.4. Indeed, if $B(o, C_{\mathsf{Euc}} n \log n)$ is very good then items (i), (iv) and (v) imply c-crossability by Lemma 4.1.3. The lower bound of item (ii) establishes (Vol), so we only need to establish (P) for every $B(y,r) \subset B(o, C_{\mathsf{Euc}} n \log n)$ for $n^{1/9} \leq r \leq C_{\mathsf{Euc}} n \log n$ (i.e., we choose $N_{B(o, C_{\mathsf{Euc}} n \log n)} = n^{1/9}$).

Fix y with $B(y,r) \subseteq B(o, C_{\mathsf{Euc}} n \log n)$, and write B_r for $B(y,r)$ to minimize notation. Notice that for any $B_r \subseteq B(o, C_{\mathsf{Euc}} n \log n)$ and any connected set $A \subseteq B_{r+1}$ with $n^{1/9} < |A|_E \leq \frac{1}{2}|B_{r+1}|_E$, we have

$$i_{B_{r+1}}(A) \geq C_I |A|_E^{-1/2} \geq \frac{C_I \sqrt{2}}{\sqrt{C'_V} r}$$

by the upper bound of item (ii) and the isoperimetric inequality (iii). On the other hand, if $|A|_E \le n^{1/9}$, then trivially $i_{B_{r+1}}(A) \ge |A|_E^{-1} \ge n^{-1/9} \ge r^{-1}$. Thus, by possibly decreasing C_I and increasing C_V if needed, we obtain that

$$I_{B_{r+1}} \ge \tilde{C} r^{-1}$$

with $\tilde{C} = C_I\sqrt{2/C_V}$.

Now choose $C_W = 2$, and for any function $f : B_{2r} \mapsto \mathbb{R}$. Restrict f to the graph induced by B_r union all the vertices in $\partial_E(B_r, G \setminus B_r)$.

$$\begin{aligned}\sum_{v\in V(B_r)} (f(v) - \bar{f}_{B_r})^2 \deg_G(v) &\le \sum_{v\in V(B_r)} (f(v) - \bar{f}_{B_{r+1}})^2 \deg_G(v) \\ &\le \sum_{v\in V(B_{r+1})} (f(v) - \bar{f}_{B_{r+1}})^2 \deg_{B_{r+1}}(v) \\ &\le \frac{cr^2}{\tilde{C}^2} \sum_{e\in E(B_{r+1})} |\nabla f(e)|^2.\end{aligned}$$

where c is as in Lemma 4.1.4. The first inequality follows from the fact that $\bar{f}_{B_r}$ is the minimum over a of $\sum_{v\in V(B_r)}(f(v)-a)^2 \deg_G(v)$. The second inequality is a trivial addition of non-negative terms along with the fact that $\deg_G(v) = \deg_{B_{r+1}(v)}$ if $v \in B_r$. The final inequality follows from Lemma 4.1.4 applied to B_{r+1}.

Notice that we cannot directly use Lemma 4.1.4 to bound the above inequalities, since the degrees are counted in G which could potentially by large as we have no assumption on the degree bound. Thus, we have established (P) with $C_P = c\tilde{C}^{-2}$ since adding $|\nabla f(e)|^2$ over the rest of the edges of B_{2r} only increases the right hand side. □

4.2 RSW for Bernoulli percolation

In this section, we focus on **Bernoulli bond percolation** on $\mathbb{Z}^2$. For $p \in [0, 1]$ let $\mathbb{P}_p$ denote the Bernoulli bond percolation probability measure induced by i.i.d. coin flips, one for each edge of $\mathbb{Z}^2$. We call an edge **open** if the edge is present, and **closed** otherwise. A cluster denotes a connected component of open edges. It is well known that for $p > p_c := 1/2$ (i.e. the percolation is supercritical) there exists a unique infinite cluster almost surely (see e.g. [Gri99]), call it $\mathcal{C}_\infty$. We refer to [DC18] for relevant history and references of this very popular model.

Our main result in this section is an RSW type estimate for random walk on $\mathcal{C}_\infty$. Recall the definition of c-crossable as in Definition 3.1.1. For each percolation configuration ω, we define a continuous time simple random walk $X = (\{X_t\}_{t\ge 0}, \mathbb{P}^\omega_x, x \in \mathcal{C}_\infty)$ on $\mathcal{C}_\infty$ which the walk starts in the infinite open cluster. The random walk X is the process that waits an exponential mean 1 at each vertex x, and then jumps along one of the open edges e containing x with each edge chosen uniformly. Then X is the Markov process with generator

$$\mathcal{L}_\omega f(x) = \frac{1}{\deg(x)} \sum_y \mathbb{1}_{xy}(f(y) - f(x)), \quad x \in \mathcal{C}_\infty.$$

From now on, we fix $p > p_c$. We also denote by $B(x, r)$ the graph distance balls in $\mathcal{C}_\infty$. We will prove the following theorem, which is a quick application of a combination of results in [Bar04] and Lemma 4.1.3.

Theorem 4.2.1. *Fix $p > p_c$. Let $\mathcal{C}_\infty$ be the unique infinite cluster of supercritical bond percolation in $\mathbb{Z}^2$. There exist positive constants $\alpha, \beta, c_p \in (0, 1]$ such that for all $n \geq 1$,*

$$\mathbb{P}_p(\Lambda_{3n,n} \text{ is } c_p\text{−crossable }) \geq 1 - e^{-\alpha n^\beta}. \tag{4.2}$$

Let us begin with a standard lemma:

Lemma 4.2.2. *There exist constants $C_{\mathsf{euc}} := C_{\mathsf{euc}}(p), c > 0$ such that for all $n \geq 1$,*

$$\mathbb{P}_p(B(o, C_{\mathsf{euc}}^{-1} n) \subseteq \Lambda_n \subset B(o, C_{\mathsf{euc}} n)) \geq 1 - e^{-cn}.$$

where recall that o is the vertex of $\mathcal{C}_\infty$ nearest to the origin.

Proof. It is easy to see by triangle inequality and the fact that graph distance in $\mathcal{C}_\infty$ is bigger than that in $\mathbb{Z}^2$, that for any point $x \in B(o, C_{\mathsf{euc}}^{-1} n)$, $|x| \leq |o| + C_{\mathsf{euc}}^{-1} n$ where $|\cdot|$ is the ℓ^1-norm. On the other hand, by [GM07, eq. (4) and references therein], $|o| \leq n/2$ with exponentially high probability in n. Thus the first inclusion is satisfied for a large enough choice of C_{euc} with exponentially high probability in n. The other inclusion is a similar standard application of [AP96, Theorem 1.1]. Indeed, there exists $\mathcal{C}_{\mathsf{euc}} > 0$ such that the chemical distance diameter of $\mathcal{C}_\infty \cap \Lambda_n$ is at most $\mathcal{C}_{\mathsf{euc}} n$ with probability exponentially high in n. So $\Lambda_n \subset B(o, \mathcal{C}_{\mathsf{euc}} n)$ with high probability. □

Lemma 4.2.3. *There exist positive constants $C_P, C_V, C_W, c, \alpha, \beta, d$ such that for all $n \geq 1$, the following events hold with probability at least $1 - ce^{-\alpha n^\beta}$.*

- *$|\Lambda^{(2)}| \geq dn^2$ where $\Lambda^{(2)}$ is a square of size n defined as in Lemma 4.1.3.*
- *$\Lambda_{n \log n}$ is $C_{\mathsf{euc}}, C_P, C_V, C_W$-very good with $N_B \leq n^{1/4}$ with C_{euc} as in Lemma 4.2.2.*

Proof. We apply Lemma 4.2.2 to first obtain a constant C_{euc} as required. By translation invariance, we can also assume C_{euc} is large enough so that $B(b, C_{\mathsf{Euc}}^{-1} n) \subset B_2$ where b is the closest point to $(n/2, 0)$ (i.e. the center of B_2 with $z = 0$). Thus we assume

$B(o, C_{\mathsf{Euc}}^{-1} n \log n) \subset \Lambda_{n \log n}$ and $B(b, C_{\mathsf{Euc}}^{-1} n) \subset B_2$ for the rest of the proof admitting a cost exponentially small in n.

We now show that $B(o, C_{\mathsf{euc}} n \log n)$ is C_P, C_V, C_W very good with stretched exponentially high probability and $B(b, C_{\mathsf{Euc}}^{-1} n)$. This is essentially a combination of [Bar04, Theorem 2.18 and Lemma 2.19], let us provide a brief explanation of the results there. In [Bar04, Theorem 2.18], it is proved for a box Q of any size, if certain events $H(Q, \alpha)$ and $D(Q, \alpha)$ hold, then the items in this lemma are satisfied. (The events $H(Q, \alpha)$ and $D(Q, \alpha)$ are certain geometric conditions whose exact definitions will not be important for us.) Later in [Bar04, Lemma 2.19], it is shown that $H(Q, \alpha)$ and $D(Q, \alpha)$ hold on the box Q with stretched exponentially high probability in the size of Q.

To be more precise, we apply [Bar04, Theorem 2.18] for $Q = \Lambda_{C_{\mathsf{euc}} C n \log n}$ for $C = 3C_{\mathsf{euc}}/2$ and $\alpha = 1/8$. With this choice, the first item in this lemma holds with stretched exponentially high probability in n as per item (a) of [Bar04, Theorem 2.18] (with $r = C_{\mathsf{Euc}}^{-1} n$, $y = b$) since $|B_2| \geq |B_{C_{\mathsf{Euc}}^{-1} n}|$. Also, the second item holds with stretched exponentially high probability in n as per item (c) of [Bar04, Theorem 2.18] (with $R = C_{\mathsf{euc}} n \log n$ and $y = o$). This finishes the proof. □

Proof of Theorem 4.2.1. In order to apply Lemma 4.1.3, we need to check the conditions. Lemma 4.2.2 justifies the existence of $C_{\mathsf{euc}} \geq 1$ with $B(o, C_{Euc}^{-1} n) \subseteq B(o, n) \subseteq \Lambda_n \subset B(o, C_{\mathsf{euc}} n)$ losing a probability exponentially small in n. Furthermore, Lemma 4.2.3 justifies the requirement the very good condition and the lower bound on the volume of $\Lambda^{(2)}$ only losing a probability which is stretched exponentially small in n. The lower bound on the distance between the inner rectangle and the outer one is trivial since distances only increase in $\mathcal{C}_\infty$ (choosing $c_0 = 0.1$ suffices). An application of union bound on the above estimates finish the proof. □

4.3 RSW for Delaunay triangulation

Let Π be a Poisson point process in $\mathbb{R}^2$ with intensity 1. Recall that a Voronoi cell of $x \in \Pi$ is the set of points in $\mathbb{R}^2$ whose closest point (in Euclidean distance) in Π is x. Let $\mathbb{T}$ denote the Voronoi triangulation which is formed by joining to points in Π by a straight line if their cells share a common edge (it is a standard fact that this graph is a.s. a triangulation). We will denote by $d_{\mathbb{T}}$ the graph distance in $\mathbb{T}$ and the graph distance ball of radius r around x in $\mathbb{T}$ is denoted by $B_{\mathbb{T}}(x, r)$ (for a point x, $B(x, r)$ denotes the ball of radius r around a vertex in $\mathbb{T}$ closest to x). We will usually drop the subscript for notational convenience when the graph in question is unambiguous.

Recall the definition of c-crossable as in Definition 3.1.1. In this section we will prove the following theorem.

Theorem 4.3.1. *Let $\mathbb{T}$ be the Voronoi triangulation formed by a Poisson process of intensity 1. There exist constants $c_{\mathsf{RSW}}, c, \alpha \in (0,1]$ such that for all $n \geq 1$,*

$$\mathbb{P}(\Lambda_{3n,n} \text{ is } c_{\mathsf{RSW}}\text{-crossable}) \geq 1 - e^{-cn^{\alpha}}$$

We refer to [Rou15] for some results in this direction, but we failed find a reference to the quantitative nature of the estimates we need, hence we prove it in details.

Fix $s > 0$ (think of s as large but constant). Take the lattice $s\mathbb{Z}^2$. For $x \in s\mathbb{Z}^2$, divide the box $\Lambda_s(x)$ into 400 equal sized boxes of size $s/10$, and call them the *smaller boxes*. Call a box $\Lambda_s(x)$ A**-red** if each of the smaller boxes contain at least one and at most As^2 many points in Π. Clearly, for every $\varepsilon > 0$, one can choose a large s and A so that x is red with probability at least $1 - \varepsilon$ (since the number of points in the box $\sim$ Poisson $(4s^2)$). We call a box simply red if we let $A = \infty$ (i.e., we do not specify any upper bound of the number of vertices). We can think of the collection of red boxes of the form $\{\Lambda_s(x)\}_{x \in s\mathbb{Z}^2}$ as a site percolation configuration in $\{0,1\}^{s\mathbb{Z}^2}$ with a vertex being 1 if and only if it is red. Call this percolation configuration $\xi = \{\xi_x\}_{x \in s\mathbb{Z}^2} \in \{0,1\}^{s\mathbb{Z}^2}$. Observe that ξ is not necessarily an i.i.d. Bernoulli as the boxes corresponding to two adjacent vertices in $s\mathbb{Z}^2$ overlap. However, ξ is a 2-dependent percolation: as soon as graph distance between x and y is strictly greater than 2, $\Lambda_s(x) \cap \Lambda_s(y) = \emptyset$ and consequently ξ_x is independent of ξ_y.

Recall the notion of stochastic domination: ξ stochastically dominates ξ' if one can couple them in the same probability space with $\xi_x \geq \xi'_x$ for all $x \in s\mathbb{Z}^2$.

Lemma 4.3.2. *Fix $\varepsilon > 0$ and let s, ξ be chosen as above. Then ξ stochastically dominates a Bernoulli site percolation in $s\mathbb{Z}^2$ with parameter $1 - \varepsilon'(\varepsilon)$ with $\varepsilon' \to 0$ as $\varepsilon \to 0$.*

Proof. As observed above, ξ is a 2-dependent site percolation. So by a result of Liggett, Schonmann and Stacey [LSS97, Theorem 0.0], ξ dominates a Bernoulli $(1 - \varepsilon')$ site percolation ξ' with $\varepsilon' \to 0$ as $\varepsilon \to 0$. □

A path in a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that v_i is adjacent to v_{i+1} for all $0 \leq i \leq k$, and the edges (v_i, v_{i+1}) for all $0 \leq i \leq k$.

Lemma 4.3.3. *Let x, y be adjacent vertices in $s\mathbb{Z}^2$ and assume $\Lambda_s(x)$ and $\Lambda_s(y)$ are both A-red. Take any two smaller boxes of side length $s/10$ in the rectangle $\mathcal{R} := \Lambda_s(x) \cup \Lambda_s(y)$ which are at least Euclidean distance $s/4$ from the boundary of the rectangle. Then for any two points of Π in these boxes (which one can always find by definition of red), there exists a path in $\mathbb{T}$ with at most $L := 800As^2$ many vertices joining them, which lies completely inside $\mathcal{R}$.*

Proof. Join any two points in the first box and the second box by a straight line $\mathcal{L}$. We claim that the Voronoi cells $\mathcal{L}$ intersects can only belong to points of $\Pi \cap \mathcal{R}$. Indeed, if a cell corresponding to a point outside $\mathcal{R}$ intersects $\mathcal{L}$, then there is a point on $\mathcal{L}$ which is closer to a point outside $\mathcal{R}$ than any point inside $\mathcal{R}$. Thus the disk of radius at least $s/4$ from this point is empty. This means that one of the smaller boxes in $\mathcal{R}$ is empty, which is impossible since both the boxes $\Lambda_s(x)$ and $\Lambda_s(y)$ are red. This completes the proof of the claim. It is easy to construct a path in the Voronoi triangulation using only the vertices of the cells $\mathcal{L}$ intersects. Also since $\mathcal{R}$ is convex, all the edges of this path also lie inside $\mathcal{R}$. This path can have at most $800As^2$ many vertices as $\Lambda_s(x)$ and $\Lambda_s(y)$ are A-red and hence contains at most $800As^2$ points in total. $\square$

We now state a quick result for Bernoulli site percolation in a graph. An open vertex cluster is a connected component in the graph induced by the open vertices. Recall that in a percolation configuration, the *chemical distance* between two vertices in the same open cluster is the length shortest path in the cluster connecting those two points. It is well-known (see e.g. [Gri99]) that there exists a unique open infinite cluster in $\mathbb{Z}^2$ for $p > p_c \approx 0.59$.

Lemma 4.3.4. *There exist constants $c, c', C > 0$ such that for all $n, k \geq 1$ the following holds. Take a Bernoulli p-site percolation in $\mathbb{Z}^2$ with $p > 0.9$ and let $\mathcal{C}_\infty$ be the unique infinite open vertex cluster. Let $\mathcal{C}_n = \mathcal{C}_\infty \cap \Lambda_n$. Let $D_{n,k}$ be the following event*

- *The chemical distance diameter of $\mathcal{C}_n$ is at least $n/2$ and at most Cn.*
- *For any vertex in $x \in \Lambda_n$, $\Lambda_k(x) \cap \mathcal{C}_\infty \neq \emptyset$.*
- $\max\{|\mathcal{H}(x)| : x \in \Lambda_n \cap \mathbb{Z}^2\} \leq k^2$ *where $\mathcal{H}(x)$ denote the maximal $*$-connected cluster in containing x containing no vertex from $\mathcal{C}_n$ (call $\mathcal{H}(x)$ the **hole** containing x)*[2].

Then the probability of $D_{n,k}$ is at least $1 - cn^4e^{-c'\sqrt{n}} - cn^2e^{-c'k}$.

Proof. This lemma follows from some known results which we first gather. Let $\mathbb{P}_p$ denote the probability measure induced by the percolation and let D denote the chemical distance. Let $x \leftrightarrow y$ denote the event that x is connected to y, with $y = \infty$ meaning that x is in an infinite cluster. Let $|x|$ denote the graph distance in $\mathbb{Z}^2$. [AP96, Theorem 1.1] states that there is a constant $C = C(p)$ and c such that for all $x \in \mathbb{Z}^2$,

$$\mathbb{P}_p(0 \leftrightarrow x, D(0,x) > C|x|) \leq e^{-c|x|}. \tag{4.3}$$

[2] x, y are $*$-connected by closed vertices, if there is a path of closed vertices with two consecutive vertices at distance either 1 or 2 in $\mathbb{Z}^2$ (i.e. diagonally adjacent or regular adjacent) connecting x and y. It is well-known that a regular cluster is blocked by a $*$-connected circuit.

Although the result in [AP96] is about bond percolation, it can be easily extended to an analogous result for site percolation since we took p large enough. (e.g. by using [LSS97, Theorem 0.0] again). It also follows from [GM07, eq. (4) and references therein] and translation invariance that for any $x \in \mathbb{Z}^2$,

$$\mathbb{P}_p(\mathcal{C}_\infty \cap \Lambda_k(x) = \emptyset) \le e^{-ck} \tag{4.4}$$

Note again, that the results cited hold for bond percolation, but they can be easily translated to site percolation as we took p large enough.

Using (4.4), we see that $\mathcal{C}_\infty$ intersects $\Lambda_{n/2}$ with a probability which is exponentially high in n. This immediately implies that the chemical distance diameter of $\mathcal{C}_n$ is at least $n/2$ on this event (since $\mathcal{C}_n$ must intersect the complement of Λ_n). Also, (4.4) and a union bound over all $x \in \Lambda_n$ ensures the second item is valid with probability at least $1 - 4n^2 e^{-c'k}$.

We now show that the third item holds with probability $1 - cn^2 e^{-ck}$. Indeed by isoperimetry of $\mathbb{Z}^2$, if the volume of the $*$-connected hole $\mathcal{H}(x)$ is bigger than k^2 then there is a $*$-connected closed circuit separating $\mathcal{C}_\infty$ from x of diameter at least ck. It is known that this event has probability exponentially small in k. An union bound over all the vertices in $\Lambda_n \cap \mathbb{Z}^2$ upper bounds the probability of $\max\{|\mathcal{H}(x)| : x \in \Lambda_n \cap \mathbb{Z}^2\} > k^2$ by $4n^2 e^{-ck}$.

Furthermore, if the chemical distance diameter of $\mathcal{C}_n$ is bigger than Cn where C is chosen according to (4.3), then there must be vertices $x, y \in \Lambda_n \cap \mathbb{Z}^2$ with $D(x, y) > Cn$ and $x \leftrightarrow y$. For any pair with $\|x - y\| \ge \sqrt{n}/10$ this probability is exponentially small in $\sqrt{n}$ by (4.3). On the other hand if $\|x - y\| \le \sqrt{n}/10$ and the chemical distance is larger than Cn, then the cluster containing x and y must exit $\Lambda_x(\sqrt{n})$ but are not connected within $\Lambda_x(\sqrt{n})$. This must mean there is a $*$-connected closed cluster of diameter at least $c\sqrt{n}$ which separates these clusters. But this event also has probability exponentially small in $\sqrt{n}$. An union bound over the pairs x, y shows that the probability of the diameter of $\mathcal{C}_n$ being at least Cn is at most $Cn^4 e^{-c\sqrt{n}}$.[3]

The result follows by observing that $D_{n,k}$ contains the intersection of these events. □

Recall the notation $B(x, r)$ which denotes the graph distance ball of radius r in the Voronoi triangulation from a point in Π which is closest to x.

Lemma 4.3.5. *There exist constants $C_{\mathsf{euc}}, c, c' > 0$ such that for all $n \ge 1$, $\Lambda_n \subseteq B(o, C_{\mathsf{euc}} n)$ with probability at least $1 - ce^{-c'\sqrt{n}}$*

[3]This part of the bound is probably not optimal, but since we will be content with a stretched exponential bound anyway in the end, we do not pursue to make this optimal.

Proof. Choose A, s such that the 2-dependent percolation ξ as described in Lemma 4.3.2 dominates a Bernoulli $(1-\varepsilon')$ site percolation ξ' with $\varepsilon' < 0.01$. Couple ξ, ξ' so that ξ dominates ξ'. Let $\mathcal{C}_\infty$ be the unique infinite cluster of ξ' and let $\mathcal{C}_{2n} = \Lambda_{2n} \cap \mathcal{C}_\infty$. Choose C as in Lemma 4.3.4.

We will pick a box of the form $S := \Lambda_s(x) \subset \Lambda_n$ with $x \in s\mathbb{Z}^2$ and show that any point in $\Pi \cap S$ can be connected to o by a path of length $O(n)$ in $\mathbb{T}$ with stretched exponentially high probability. Also assume that if S is empty, this event is vacuously satisfied. Since there are at most n^2/s^2 many such boxes, a further union bound does the job.

Assume $|S \cap \Pi| \neq \emptyset$ and assume $\mathcal{D}'_n := \mathcal{D}_{2n,\sqrt{n}/10}$ occurs for ξ' where the event is as described in Lemma 4.3.4 and fix a sample of the Point process in $\mathcal{D}'_n$. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ denote the holes of ξ containing the box $S' := \Lambda_s(0)$ and S respectively, where holes are as defined in third item of Lemma 4.3.4. Note that $|\mathcal{H}_0| \leq n/100$ and $|\mathcal{H}_1| \leq n/100$ on $\mathcal{D}'_n$ since ξ dominates ξ'. These clusters are surrounded by open circuits in ξ each lying completely in $\mathcal{C}_{2n}$. Now using Lemma 4.3.3 by concatenating paths in $\mathbb{T}$ in the boxes corresponding to these circuits, we can find circuits C, C' in $\mathbb{T}$ completely surrounding S and S' respectively. This allows us to find a path from any point inside C to any point inside C' as follows. Find the shortest path in $\mathbb{T}$ until C, C' is hit and suppose they hit the circuits at vertices $u, v \in \mathbb{T}$, in boxes B_u, B_v respectively. Clearly, the length of these paths can be at most the volume of the holes, so at most $n/100$ each. Then find the shortest path in $s\mathbb{Z}^2$ using $\mathcal{C}_{2n}$ joining B_u and B_v. This path has length at most $2Cn$ on $\mathcal{D}'_n$. Furthermore since all the boxes corresponding to vertices in this path are A-red, using Lemma 4.3.3 we can find a path in $\mathbb{T}$ joining u and v of length at most $1600As^2Cn$. Thus the length of the path is at most $C'n$ with $C' = 1600As^2C + 1/50$ on $\mathcal{D}'_n$. We finish by applying Lemma 4.3.4 to lower bound the probability of $\mathcal{D}'_n$ by $1 - ce^{-c'\sqrt{n}}$ for appropriate choices of c, c'. $\square$

Now recall the following standard fact about Binomial random variables.

Lemma 4.3.6. *Let $X \sim$ Binomial $(n, 1-\varepsilon)$ and fix $c \in (0,1)$. Then for all $n \geq 1$*

$$\mathbb{P}(X < cn) \leq e^{-C(\varepsilon,c)n}$$

where $C(\varepsilon, c) \to \infty$ as $\varepsilon \to 0$.

Now we prove a lemma which states a quantitative bound on the probability that a path in the Voronoi triangulation has many long edges. For connected subgraph S in the Voronoi triangulation, its Euclidean diameter is denoted by $\mathrm{Diam}_{\mathsf{euc}}(S) := \sup\{|x - y|, x, y \in S\}$.

Lemma 4.3.7. *There exist constants $C_0, c_0 > 0$ such that for all $n \geq 1$ and for all $C > C_0$, the probability that there exists a connected set intersecting Λ_1 with at most n vertices and Euclidean diameter* [4] *at least Cn is at most $2^{-Cc_0 n}$.*

Proof. Fix $\varepsilon > 0$ and choose s large enough such that the site percolation ξ defined in Lemma 4.3.2 dominates a Bernoulli percolation with high probability, as asserted there. Take a connected subgraph D with at most n vertices intersecting Λ_1. Let $S(D)$ be the collection of squares of the form $\Lambda_s(x)$ with $x \in s\mathbb{Z}^2$ which intersect (some vertex, edge or triangle of) D. Let k be the number of squares in $S(D)$. Because of the large Euclidean diameter of D, $k > \frac{Cn}{2s}$. We say a square is *good* if it is red and all the squares intersected by it is red, otherwise we say it is bad. This corresponds to a 4-dependent site percolation in $s\mathbb{Z}^2$. Thus again using [LSS97, Theorem 0.0], and increasing s if necessary, we can ensure that the collection of good sites dominates a $(1-\varepsilon')$-Bernoulli site percolation in $s\mathbb{Z}^2$ with $\varepsilon' < \varepsilon$.

We claim that if a square A in $S(D)$ is good, the collection of squares A' of the form $\{\Lambda_s(x) : x \in s\mathbb{Z}^2\}$ which intersect A contain at least 1 vertex from D. To see this observe that if no edge intersects A then A is empty, which is not possible as A is red. On the other hand if an edge e intersects A, and one of the endpoints of this edge does not lie in one of A', then e must have length at least $2s$. From Lemma 2.2.8, we know that one of the semi-discs of the disc with diameter e is empty in a Voronoi triangulation. This must mean that one of the squares of side $s/10$ in one of the squares A' must be empty, which is a contradiction to the fact that all squares intersecting A are red. Thus one of the endpoints of e must be in some A'. One consequence of this is that the number of good squares is at most $9n$ (since we can overcount a vertex at most 9 times).

It is well known that the number of connected sets in $\mathbb{Z}^2$ with k vertices containing the origin is at most Δ^k for some $\Delta > 0$ (see e.g. [Ede61]). Choose ε small enough so that $e^{-C(\varepsilon,1/2)} < (2\Delta)^{-1}$ where $C(\varepsilon, 1/2)$ is as in Lemma 4.3.6. Now choose $C > C_0 := 36s = 36s(\varepsilon)$ and $c_0 = (2s)^{-1}$. For a fixed connected set in $s\mathbb{Z}^2$ containing the origin and containing k vertices, the number of good vertices X dominates a Binomial $(k, 1-\varepsilon)$. Recall that $k > \frac{Cn}{2s} > 18n$. Thus

$$\mathbb{P}(X < 9n) \leq \mathbb{P}(X < k/2) \leq e^{-C(\varepsilon,1/2)k} < (2\Delta)^{-k}$$

again using Lemma 4.3.6. Since the number of connected sets is at most Δ^k, an union bound gives that the required probability is at most

$$(1/2)^k \leq (1/2)^{Cn/s} = (1/2)^{Cc_0 n}$$

[4] Euclidean diameter of a set $A \subset \mathbb{R}^2$ is $\sup\{|x-y| : x, y \in A\}$.

as desired. □

An immediate corollary is the following:

Corollary 4.3.8. *There exists a constant $C, c > 0$ such that for all $n \geq 1$, $B(o,n) \subseteq \Lambda_{Cn}$ with probability at least $1 - e^{-cn}$.*

Proof. Note that the event $o \in \Lambda_{n/10}$ has probability at least e^{-cn^2} since Π is a Poisson process. Apply Lemma 4.3.7 to any path with n vertices starting from $O(n^2)$ many translates of Λ_1 inside $\Lambda_{n/10}$. □

We now use Lemma 4.3.7 to establish a quantitative isoperimetric inequality. We recall some topological notions first. For a finite connected set of vertices $A \subset \mathbb{T}$, we can consider the subgraph induced by A, which we also call A admitting an abuse of notation. This allows us to consider the faces which has all its incident edges in A. Overall, we can think of A as a subset of $\mathbb{R}^2$, by taking the union of all the vertices, edges and faces described as such. We define the *complement* of A to be the complement of the union of the faces and edges of A. We say A is *simply connected* if the complement has a unique component (which necessarily is the unbounded component a.s.). If A is not simply connected, it's complement may contain a certain number of finite components and one unique infinite component. Let $\partial_V(A)$ denote the vertex boundary of A, which is the collection of vertices of A which has some neighbour outside A.

Lemma 4.3.9. *There exist constants $\varepsilon_0, c, c', C > 0$ such that for all $n \geq 1, \alpha \in (0,2)$ the following holds. The probability that there exists a connected set $A \subset B(o,n)$ with $k \in [n^\alpha, \frac{1}{2}n^2]$ vertices but*

$$\mathcal{B}(A) := \{|\partial_V A| \leq \varepsilon_o \sqrt{k}\} \cup \{\text{Diam}(A) \leq C\varepsilon_0\sqrt{k}\}$$

holds is at most $ce^{-c'\sqrt{k}}$. Here Diam *denotes the graph distance diameter.*

Proof. Roughly, the idea is as follows: if $|\partial_V A|$ is small, then the Euclidean diameter must also be small up to a constant factor by Lemma 4.3.7, and consequently a square of small diameter containing at least as many points of A is unlikely. We now make this idea rigorous by carefully tracking the quantifiers.

First observe that it is enough to prove the bound for sets A which are simply connected, for otherwise we can simply fill in the finite holes, and this operation decreases boundary size but increases the size of A and also decreases $\text{Diam}(A)$. Thus $\partial_V A$ has a single connected component. It is a standard fact that the Euclidean diameter of A is the same as the diameter of its boundary. Thus for any point x in A, $\Lambda_{2m}(x) \supseteq A$ where $m = \text{Diam}_{\mathsf{euc}}(\partial_V A)$.

Fix $k \in [n^\alpha, \frac{1}{2}n^2]$. First using Corollary 4.3.8 find a C_1 such that $B(o,n) \subset \Lambda_{C_1 n}$ with probability at least $1-e^{-c_1 n}$. Suppose $\partial_V A$ intersects Λ_1 for some A with $|A| = k$ and $|\partial_V(A)| \le \varepsilon_0\sqrt{k}$ where ε_0 is to be fine tuned later. Now pick c_0, C_0 as in Lemma 4.3.7 and $C > C_0$. The probability that any connected set of size at most $\varepsilon_0\sqrt{k}$ intersecting Λ_1 has Euclidean diameter at most $C\varepsilon_0\sqrt{k}$ is at least $1-e^{-Cc_0\varepsilon_0\sqrt{k}}$. Thus on this event $A \subset \Lambda_{m'}$ with $m' = 2C\varepsilon_0\sqrt{k}$; and in particular, on this event, Λ_{2m} contains at least k vertices. Now pick an $\varepsilon_0 = \varepsilon_0(C)$ small enough so that the the probability that the number of vertices in Λ_{2m} is at least k is at most $e^{-c'k}$ for some c'. Thus overall for this choice of ε_0, the probability that there exists a set A intersecting Λ_1 such that $\mathcal{B}(A)$ holds but $|A| \ge k$ is at most $e^{-Cc_0\varepsilon_0\sqrt{k}} + e^{-c'k}$.

Finally, on the event that $B(o,n) \subset \Lambda_{C_1 n}$, we can take a union bound of the above replacing Λ_1 by at most $4C_1^2n^2$ many translates of Λ_1. This yields that the probability of the event in the lemma is at most

$$4C_1^2n^2(e^{-Cc_0\varepsilon_0\sqrt{k}} + e^{-c'k}) + e^{-c_1 n}.$$

Since $k \ge n^\alpha$ with $\alpha \in (0,1)$, we can find $c, c' > 0$ such that the above quantity is bounded above by $ce^{-c'\sqrt{k}}$. Taking a further union bound over all integers $k \in [n^\alpha, \frac{1}{2}n^2]$, we conclude by modifying the choice of c, c' appropriately. □

Notice that the boundary in Lemma 4.3.9 considers the vertex boundary in the whole Voronoi triangulation. However, the bound in (4.1) only counts the boundary edges of $A \subset B(y,r)$ inside $B(y,r)$. In the next lemma, we strengthen Lemma 4.3.9 to show that even discarding the edges going out of $B(y,r)$, there are many edges left over in the boundary with high probability. To do this, we require the following elementary geometric lemma.

Lemma 4.3.10. *Let $A \subset B(o,n)$ and suppose $\partial_V(A)$ has a single connected component. Assume that $\partial_V(A) \cap \partial_V(B(o,n)) \neq \emptyset$. Let $\partial_{\mathsf{int}}(A)$ denote the set of vertices in $\partial_V(A)$ with at least one neighbour in $B(o,n) \setminus A$ and assume $|\partial_{\mathsf{int}}(A)| = k$. Then for any $v \in \partial_{\mathsf{int}}(A)$, $A \subset B(v, 2k)$.*

Proof. Recall d denotes the graph distance in $\mathbb{T}$. Notice that

$$d(v, \partial_V(B(o,n))) \le k, \quad \text{for any } v \in \partial_{\mathsf{int}}$$

since $\partial_V(A) \cap \partial_V(B(o,n)) \neq \emptyset$ and hence ∂_{int} and $\partial_V(B(o,n))$ must be at distance 1. Thus triangle inequality yields $d(o,v) \ge n-k$. Observe that for any vertex $w \in A$, a geodesic from o to w must intersect $\partial_{\mathsf{int}}(A)$ since $\mathbb{T}$ is planar (since this geodesic must enter A through some vertex, and this vertex is necessarily in $\partial_{\mathsf{int}}(A)$). Let w' be such a vertex which is closest to o. By triangle inequality, and since $d(o,w') \geq n-k$, $d(w',w) = d(o,w) - d(o,w') \leq n-(n-k) = k$. Then $d(v,w) \leq d(v,w') + d(w',w) \leq 2k$. This completes the proof. □

Lemma 4.3.11. *There exist constants $\varepsilon_1, c, c' > 0$ such that for all $n \geq 1, \alpha \in (0,2)$ and $k \in [n^\alpha, \frac{1}{2}n^2]$ the following holds. The probability that there exists a connected set $A \subset B(o,n)$ with k vertices but with $|\partial_{\mathsf{int}}(A)| \leq \varepsilon_1\sqrt{k}$ is at most $ce^{-c'\sqrt{k}}$ where $\partial_{\mathsf{int}}(A)$ is defined as in Lemma 4.3.10.*

Proof. If all the vertices of A are in $B(o,n-1)$, then $\partial_V(A) = \partial_{\mathsf{int}}(A)$, and we simply choose $\varepsilon_1 = \varepsilon_0$ where ε_0 is as in Lemma 4.3.9. On the other hand, if A intersects $\partial B(o,n)$, then by Lemma 4.3.10, the graph distance diameter of A is at most $4|\partial_{\mathsf{int}}(A)|$. Thus again by Lemma 4.3.9, we can choose $\varepsilon_1 = \min\{\varepsilon_0, C\varepsilon_0/4\}$. □

Lemma 4.3.12. *For any $A \subseteq B(o,n)$,*

$$i_{B(o,n)}(A) = \frac{|\partial_E(A, B(o,n)\setminus A)|}{|A|_E} \geq \frac{|\partial_{\mathsf{int}}(A)|}{7|A|}$$

where $|A|_E$ is the sum of the degrees of vertices in A counting only edges in $B(o,n)$ and $\partial_{\mathsf{int}}(A)$ is as in Lemma 4.3.10.

Proof. Let E be the edge set of the subgraph induced by A. Notice that this subgraph is a subgraph of a triangulation, hence the faces form a collection of triangles and (potentially non-simple) polygons, call the latter outer faces. Without loss of generality, we can assume that there is only one outer face, as otherwise we can fill in the bounded faces, thereby decreasing $i_{B(o,n)}(A)$. Let $|P|$ denote the perimeter of the outer face which counts the number of edges in it, with the edges having both sides adjacent to P counted twice. Let F denote the set of triangles in this graph. By Euler's formula, $|A| - |E| + |F| + 1 = 2$. Also note, $2|E| = 3|F| + |P|$. Combining, we get $|E| = 3|A| - |P| - 3 \leq 3|A|$. Also note $|A|_E = 2|E| + |\partial_E(A, B(o,n)\setminus A))|$. Thus

$$i_{B(o,n)}(A) \geq \frac{|\partial_E(A, B(o,n)\setminus A))|}{6|A| + |\partial_E(A, B(o,n)\setminus A))|} \geq \frac{|\partial_{\mathsf{int}}(A)|}{7|A|}$$

The last inequality follows from $|\partial_E(A, B(o,n)\setminus A))| \geq |\partial_{\mathsf{int}}(A)|$, the fact that $x \mapsto x/(6+x)$ is increasing in x and $x/(6+x) \geq x/7$ for all $x \in (0,1)$. □

Proof of Theorem 4.3.1. We will simply prove that the items in Lemma 4.1.5 holds with stretched exponentially high probability for appropriate choice of constants (we use the notations there). Firstly, (iv) holds with exponentially high probability in n^2 for a small enough choice of d using standard estimate of a Poisson variable. It follows from Lemma

4.3.5 and Corollary 4.3.8 that (i) holds with stretched exponentially high probability in $n \log n$ for an appropriately large choice of C_{euc}. Now fix $r \in [n^{1/9}, C_{\mathsf{euc}} n \log n]$. It is easy to see that the volume of $B(o, r)$ is upper and lower bounded by some constant in r^2 with stretched exponentially high probability in r, again using Lemma 4.3.5, Corollary 4.3.8, and standard properties of a Poisson process. Applying Corollary 4.3.8 to $O(n^2)$ many translates of Λ_1, we can also ensure $(ii), (v)$ holds with exponentially high probability in r. Finally, choosing ε_1 as in Lemma 4.3.11, we can ensure that (iii) holds with $C_I = \varepsilon_1/7$ and with probability at least stretched exponentially high in r (and consequently stretched exponentially high in n.). Now we take an union bound over integers $r \in [n^{1/9}, C_{\mathsf{euc}} n \log n]$ to complete the proof. $\square$

Appendix A

Probability theory

A.1 Coupling theory

Coupling is a useful probabilistic technique in probability theory so that random variables can be compared with each other (see e.g., See [DH12, Roc15]). There are lots of applications of the coupling in discrete probability, for example, the coupling of Markov chains (see e.g. [Joh96, CM05]), percolation, and interacting particle systems. Here I will give some basic definitions.

Definition A.1.1. *A **coupling** of two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on the same measurable space $(\Omega, \mathcal{F})$ is a probability measure $\tilde{\mathbb{P}}$ on the product measurable space $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F})$ whose marginals are $\mathbb{P}_1$ and $\mathbb{P}_2$.*

Definition A.1.2. *A coupling of two random variables X and Y is any pair of random variables $(\tilde{X}, \tilde{Y})$ taking values in $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F})$ whose marginals have the same distribution as X and Y.*

Definition A.1.3. *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be probability measures on $\mathbb{R}$. The measure $\mathbb{P}_1$ is said to stochastically dominate $\mathbb{P}_2$ denoted by $\mathbb{P}_1 \geq \mathbb{P}_2$, if for all $x \in \mathbb{R}$*

$$\mathbb{P}_1(X > x) \geq \mathbb{P}_2(X' > x).$$

Bibliography

[ABS05] L Addario-Berry and A Sarkar. The simple random walk on a random voronoi tiling. *Preprint*, 2005.

[ACHTS20] Omer Angel, David A Croydon, Sarai Hernandez-Torres, and Daisuke Shiraishi. Scaling limits of the three-dimensional uniform spanning tree and associated random walk. *arXiv preprint arXiv:2003.09055*, 2020.

[ACV05] Helmut Alt, Otfried Cheong, and Antoine Vigneron. The voronoi diagram of curved objects. *Discrete & Computational Geometry*, 34(3):439–453, 2005.

[AKN87] Michael Aizenman, Harry Kesten, and Charles M Newman. Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Communications in Mathematical Physics*, 111(4):505–531, 1987.

[Ald90] David J Aldous. The random walk construction of uniform spanning trees and uniform labelled trees. *SIAM Journal on Discrete Mathematics*, 3(4):450–465, 1990.

[AP96] Peter Antal and Agoston Pisztora. On the chemical distance for supercritical bernoulli percolation. *The Annals of Probability*, pages 1036–1048, 1996.

[Bar04] Martin T Barlow. Random walks on supercritical percolation clusters. *The Annals of Probability*, 32(4):3024–3084, 2004.

[Bar17] Martin T Barlow. *Random walks and heat kernels on graphs*, volume 438. Cambridge University Press, 2017.

[BH57] Simon R Broadbent and John M Hammersley. Percolation processes: I. crystals and mazes. *Mathematical proceedings of the Cambridge philosophical society*, 53(3):629–641, 1957.

[BLPS01a] Itai Benjamini, Russell Lyons, Yuval Peres, and Oded Schramm. Special invited paper: uniform spanning forests. *Annals of probability*, pages 1–65, 2001.

[BLPS01b] Itai Benjamini, Russell Lyons, Yuval Peres, and Oded Schramm. Uniform spanning forests. *The Annals of Probability*, 29(1):1 – 65, 2001.

[BLR19a] Nathanaël Berestycki, Benoit Laslier, and Gourab Ray. The dimer model on Riemann surfaces, I. *arXiv preprint arXiv:1908.00832*, 2019.

[BLR19b] Nathanaël Berestycki, Benoit Laslier, and Gourab Ray. The dimer model on Riemann surfaces, II. *preparation*, 2019.

[BLR20] Nathanael Berestycki, Benoit Laslier, and Gourab Ray. Dimers and imaginary geometry. *The Annals of Probability*, 48(1):1–52, 2020.

[Bro89] Andrei Z Broder. Generating random spanning trees. In *FOCS*, volume 89, pages 442–447. Citeseer, 1989.

[BTKA10] Martin Bock, Amit Kumar Tyagi, Jan-Ulrich Kreft, and Wolfgang Alt. Generalized voronoi tessellation as a model of two-dimensional cell tissue dynamics. *Bulletin of mathematical biology*, 72(7):1696–1731, 2010.

[Cal03] Pierre Calka. Precise formulae for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional poisson-voronoi tessellation and a poisson line process. *Advances in Applied Probability*, pages 551–562, 2003.

[CF07] Pietro Caputo and Alessandra Faggionato. Isoperimetric inequalities and mixing time for a random walk on a random point process. *The Annals of Applied Probability*, 17(5-6):1707–1744, 2007.

[Chu04] Fan Chung. Discrete isoperimetric inequalities. *Surveys in differential geometry*, 9(1):53–82, 2004.

[CM05] Radu V Craiu and Xiao-Li Meng. Multiprocess parallel antithetic coupling for backward and forward markov chain monte carlo. *Annals of Statistics*, 33(2):661–697, 2005.

[DC18] Hugo Duminil-Copin. Sixty years of percolation. *In Proceeding of the International Congress. World Scientific*, 2018.

[DCHN11] Hugo Duminil-Copin, Clément Hongler, and Pierre Nolin. Connection probabilities and rsw-type bounds for the two-dimensional fk ising model. *Communications on pure and applied mathematics*, 64(9):1165–1198, 2011.

[DCMT18] Hugo Duminil-Copin, Ioan Manolescu, and Vincent Tassion. An RSW theorem for gaussian free field. *preparation. Zentralblatt MATH*, 6869097, 2018.

[DH12] Frank Den Hollander. Probability theory: The coupling method. *Lecture notes available online (http://websites. math. leidenuniv. nl/probability/lecturenotes/CouplingLectures. pdf)*, 2012.

[DVLDG34] Boris Delaunay, S Vide, A Lamémoire, and V De Georges. Bulletin de l'académie des sciences de l'urss. *Classe des sciences mathématiques et naturelles*, 6:793–800, 1934.

[Ede61] Murray Eden. A two-dimensional growth process. In *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability*, volume 4, pages 223–239. Univ of California Press Berkeley, 1961.

[FvdH17] Robert Fitzner and Remco van der Hofstad. Mean-field behavior for nearest-neighbor percolation in $d > 10$. *Electronic Journal of Probability*, 22(none):1 – 65, 2017.

[GM07] Olivier Garet and Régine Marchand. Large deviations for the chemical distance in supercritical bernoulli percolation. *The Annals of Probability*, 35(3):833–866, 2007.

[Gol10] André Goldman. The palm measure and the voronoi tessellation for the ginibre process. *The Annals of Applied Probability*, 20(1):90–128, 2010.

[Gri99] G Grimmett. Percolation springer-verlag. *Berlin (Second edition)*, 1999.

[Gri18] Alexander Grigor'yan. *Introduction to analysis on graphs*, volume 71. American Mathematical Soc., 2018.

[Ham14] Nao Hamamuki. A discrete isoperimetric inequality on lattices. *Discrete & Computational Geometry*, 52(2):221–239, 2014.

[HLW06] Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bulletin of the American Mathematical Society*, 43(4):439–561, 2006.

[HS94] Takashi Hara and Gordon Slade. Mean-field behaviour and the lace expansion. In *Probability and phase transition*, pages 87–122. Springer, 1994.

[HS20] Tom Hutchcroft and Perla Sousi. Logarithmic corrections to scaling in the four-dimensional uniform spanning tree. *arXiv preprint arXiv:2010.15830*, 2020.

[Joh96] Valen E Johnson. Studying convergence of markov chain monte carlo algorithms using coupled sample paths. *Journal of the American Statistical Association*, 91(433):154–166, 1996.

[Ken00] Richard Kenyon. The asymptotic determinant of the discrete laplacian. *Acta Mathematica*, 185(2):239–286, 2000.

[Kes80] Harry Kesten. The critical probability of bond percolation on the square lattice equals 1/2. *Communications in Mathematical Physics*, 74:41–59, 1980.

[Kir47] Gustav Kirchhoff. Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird. *Annalen der Physik*, 148(12):497–508, 1847.

[KST20] Laurin Köhler-Schindler and Vincent Tassion. Crossing probabilities for planar percolation. *arXiv preprint arXiv:2011.04618*, 2020.

[Law79] Gregory Francis Lawler. *A SELF-AVOIDING RANDOM WALK.* PhD thesis, Princeton University, 1979.

[LL04] Gregory Lawler and Vlada Limic. The Beurling Estimate for a Class of Random Walks. *Electronic Journal of Probability*, 9(none):846 – 861, 2004.

[LP17a] David A Levin and Yuval Peres. *Markov chains and mixing times*, volume 107. American Mathematical Soc., 2017.

[LP17b] Russell Lyons and Yuval Peres. *Probability on trees and networks*, volume 42. Cambridge University Press, 2017.

[LSS97] Thomas M Liggett, Roberto H Schonmann, and Alan M Stacey. Domination by product measures. *The Annals of Probability*, 25(1):71–95, 1997.

[LSW11] Gregory F Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. In *Selected Works of Oded Schramm*, pages 931–987. Springer, 2011.

[MM82] RE Miles and RJ Maillardet. The basic structures of voronoi and generalized voronoi polygons. *Journal of Applied Probability*, pages 97–111, 1982.

[MR04] Pierre Mathieu and Elisabeth Remy. Isoperimetry and heat kernel decay on percolation clusters. *The Annals of Probability*, 32(1A):100–128, 2004.

[Nac20] Asaf Nachmias. *Planar Maps, Random Walks and Circle Packing: École D'Été de Probabilités de Saint-Flour XLVIII-2018*. Springer Nature, 2020.

[Oss78] Robert Osserman. The isoperimetric inequality. *Bulletin of the American Mathematical Society*, 84(6):1182–1238, 1978.

[Pem91] Robin Pemantle. Choosing a spanning tree for the integer lattice uniformly. *The Annals of Probability*, pages 1559–1574, 1991.

[Roc15] Sebastien Roch. Modern discrete probability: An essential toolkit. *University Lecture*, 2015.

[Rou15] Arnaud Rousselle. Quenched invariance principle for random walks on delaunay triangulations. *Electronic Journal of Probability*, 20, 2015.

[Rus78] Lucio Russo. A note on percolation. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 43(1):39–48, 1978.

[RY21] Gourab Ray and Tingzhou Yu. Quantitative russo-seymour-welsh for random walk on random graphs and decorrelation of ust. *arXiv:2106.10323*, 2021.

[SC97] Laurent Saloff-Coste. *Lectures on finite Markov chains*, pages 301–413. Springer Berlin Heidelberg, Berlin, Heidelberg, 1997.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel Journal of Mathematics*, 118(1):221–288, 2000.

[SW78] Paul D Seymour and Dominic JA Welsh. Percolation probabilities on the square lattice. In *Annals of Discrete Mathematics*, volume 3, pages 227–245. Elsevier, 1978.

[Tas16] Vincent Tassion. Crossing probabilities for voronoi percolation. *The Annals of Probability*, 44(5):3385–3398, 2016.

[Tel03] András Telcs. Isoperimetric inequalities for random walks. *Potential Analysis*, 19(3):237–249, 2003.

[Tho92] Carsten Thomassen. Isoperimetric inequalities and transient random walks on graphs. *The Annals of Probability*, pages 1592–1600, 1992.

[Var85] Nicholas Varopoulos. Long range estimates for markov chains. *Bulletin des sciences mathématiques (Paris. 1885)*, 109(3):225–252, 1985.

[Vor08] Georges Voronoi. Nouvelles applications des paramètres continus à la théorie des formes quadratiques. deuxième mémoire. recherches sur les parallélloèdres primitifs. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1908(134):198–287, 1908.

[Wil96] David Bruce Wilson. Generating random spanning trees more quickly than the cover time. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 296–303, 1996.

[Woe00] Wolfgang Woess. *Random walks on infinite graphs and groups*, volume 138. Cambridge university press, 2000.

[Yap87] Chee K Yap. Ano (n logn) algorithm for the voronoi diagram of a set of simple curve segments. *Discrete & Computational Geometry*, 2(4):365–393, 1987.