The Cauchy problem for the 3D relativistic Vlasov-Maxwell system and its Darwin approximation

by

Reinel Sospedra-Alfonso
B.Sc., InsTEC, Havana, Cuba, 2002
M.Sc., The Abdus Salam ICTP, Trieste, Italy, 2005

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ABSTRACT

The relativistic Vlasov-Maxwell system (RVM for short) is a kinetic model that arises in plasma physics and describes the time evolution of an ensemble of charged particles that interact only through their self-induced electromagnetic field. Collisions among the particles are neglected and they are assumed to move at speeds comparable to the speed of light. If the particles are allowed to move in the three dimensional space, then the main open problem concerning this system is to prove (or disprove) that solutions with sufficiently smooth Cauchy data do not develop singularities in finite time. Since the RVM system is essential in the study of dilute hot plasmas, much effort has been directed to the solution of its Cauchy problem. The underlying hyperbolic nature of the Maxwell equations and their nonlinear coupling with the Vlasov equation amount for the challenges imposed by this system.

In this thesis, we show that solutions of the RVM system with smooth, compactly supported Cauchy data develop singularities only if the charge density blows-up in finite time. In particular, solutions can not break-down due to shock formations, since in this case scenario the solution would remain bounded while its derivative blows-up.

On the other hand, if the transversal component of the displacement current is neglected from the Maxwell equations, then the RVM system reduces to the so-called relativistic Vlasov-Darwin (RVD) system. The latter has useful applications in

numeric simulations of collisionless plasma, since the hyperbolic RVM is now reduced to a more tractable elliptic system while preserving a fully coupled magnetic field. As for the RVM system, the main open problem for the RVD system is to prove whether *classical* solutions with unrestricted Cauchy data exist globally in time.

In the second part of this thesis, we show that classical solutions of the RVD system exist provided the Cauchy datum satisfies some suitable smallness assumption. The proof presented here does not require estimates derived from the conservation of the total energy nor those on the transversal component of the electric field. These have been crucial in previous results concerning the RVD system. Instead, we exploit the potential formulation of the model equations. In particular, the Vlasov equation is rewritten in terms of the *generalized* variables and coupled with the equations satisfied by the scalar and vector Darwin potentials. This allows to use standard estimates for singular integrals and a recursive method to produce the existence of local in time classical solutions. Hence, by means of a bootstrap argument, we show that such solutions can be made global in time provided the Cauchy data is sufficiently small.

Contents

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DEDICATION

A mi familia.
A mis padres. A Sita.

Chapter 1

Introduction

A plasma is, simply said, an ionized gas. In contrast with a neutral gas, where the particles have no charge and they interact almost entirely through collisions, in a plasma the particle dynamics is also determined by long-range interactions. Specifically, charge separations produce electric forces, and charged particle flows produce currents and magnetic fields that exert forces on the moving charged particles themselves. If the ionized gas is sufficiently dense so that collisions among the particles are frequent, then the plasma can be considered as an electrically conducting fluid. In this regime, unique phenomena arise that can not be described by the equations of neutral fluid dynamics. The so-called *magnetohydrodynamic* equations should be used instead. Such plasmas are characterized by the high density of particles, high frequency of collisions and the low temperatures. Magnetohydrodynamic models find applications in power-up fusion reactors and the physics of stars.

On the other hand, for dilute or low density plasmas, the collisions among the particles are relatively rare or almost non-existing. In such a regime, the fluid approach mentioned above is no longer accurate. Moreover, the lack of collisions keeps the system far from the statistical equilibrium, and therefore we must work explicitly with the non-equilibrium distribution of the particles in the phase-space. This is precisely the realm of the *kinetic theory of plasmas*. In this framework, the plasmas studied are essentially characterized by the low density of particles, the absence of collisions and the high temperatures. They are usually called Vlasov plasmas, as we make clear below. Kinetic plasma models are used, for instance, in the study of solar winds, nebulae, Van Allen radiation belts and tails of comets [1]. They also find applications in the study of the Earth's ionosphere, which is considered a *partially* ionized gas [2]. For a detailed characterization of these types of plasmas see, for instance, [3] and [4].

In this work, we shall primarily be concerned with the Cauchy problem for the relativistic Vlasov-Maxwell system (RVM). This is a kinetic plasma model that describes the time evolution of an ensemble of relativistic charged particles that interact *only* through their self-induced electromagnetic field. In particular, collisions among the particles are neglected, thus the RVM system is suitable to describe dilute and hot plasmas as those mentioned above. The corresponding model equations for a single species particles of rest mass m and charge e consist of the Vlasov's equation[1]

$$\partial_t f + v \cdot \nabla_x f + e\,(E + v \times B) \cdot \nabla_p f = 0,$$

coupled with the Maxwell's equations

$$\begin{aligned} \nabla \times B - \frac{1}{c}\partial_t E &= \frac{4\pi}{c} j, & \nabla \cdot B &= 0, \\ \nabla \times E + \frac{1}{c}\partial_t B &= 0, & \nabla \cdot E &= 4\pi\rho, \end{aligned}$$

by means of the charge and current densities, defined respectively by

$$\rho := e \int_{\mathbb{R}^3} f dp, \qquad j := e \int_{\mathbb{R}^3} v f dp.$$

Here $v := m^{-1} p \left(1 + m^{-2} c^{-2} |p|^2\right)^{-1/2}$ denotes the relativistic velocity, c being the speed of light. The one-particle distribution function $f = f(t, x, p)$ depends on time $t \in]0, \infty[$, position $x \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$. The vector-valued functions $E = E(t, x)$ and $B = B(t, x)$ stand for the (self-induced) electric and magnetic fields respectively. The Cauchy problem for this system is defined once we impose appropriate values for (f, E, B) at time $t = 0$.

If we formally set $c = \infty$ in the relativistic velocity, then $v = m^{-1} p$ and the set of equations given above becomes the non-relativistic Vlasov-Maxwell (nRVM) system. In contrast with the RVM system, the former is not invariant under Lorentz transformation and it is considered a hybrid. Several results have first been obtained for the nRVM system and then been adapted to the relativistic case.

The global in time existence of solutions for both the RVM and nRVM systems with sufficiently smooth, unrestricted Cauchy data remain unsolved. Local existence of unique solutions of the nRVM system, with sufficiently smooth Cauchy data and $f|_{t=0}$ having compact support, was first proved by S. Wollman in [5] by generalizing

[1] Which explain the 'Vlasov plasma' nomenclature used earlier.

an abstract theorem due to T. Kato [6, Th. II, p. 195]. This result can be easily adapted to the relativistic counterpart. In [7], P. Degond slightly improved Wollman's result by avoiding the compact support of f_0 in the space variable. His proof relies on energy estimates and a bootstrap argument derived from a Sobolev embedding theorem. A similar result was given independently and about the same time by K. Asano in [8].

However, the break-through in the existence problem for the RVM system came with the work by R. Glassey and W. Strauss in [9]. They not only proved the local existence and uniqueness of solutions for smooth, compactly supported Cauchy data, but also gave conditions for which these solutions can be extended globally in time. Specifically, they showed that if the momenta of the particles are controlled, i.e. if the momentum support of the (local in time) distribution function remains bounded, then the solution exists for arbitrary times. Hence, the implication that a singularity could occur only if some particles travel at speeds arbitrary close to the speed of light. Later on, this continuation criterion was used to prove the global existence of solutions for small data [10], for close to neutral data [11], and for close to spherically symmetric data [12]. In lower dimensional scenarios, analogous results have been obtained for unrestricted Cauchy data in [13, 14, 15, 16]. A global existence and uniqueness result for a modified RVM system can be found in [17]. Also, the result given in [9] has been revisited in [18, 19] following two completely different approaches. For a more detailed account on the relevant literature for the RVM system, cf. [1].

Based on [9], additional continuation criteria for the RVM system have been given in [20] and [21]. These results will be briefly discussed in Chapter 3. In the same spirit, we shall introduce another continuation criterion that weakens the previously cited. Precisely, we shall prove that *a solution of the RVM system having smooth, compactly supported Cauchy data, can be continued globally in time provided that the charge density remains bounded*. The proof relies on the observation that, via characteristics, one can estimate the time integral of the electric field acting on the individual particles in terms of the kinetic energy of the (individual) particles themselves, provided the charge density remains bounded. Hence, the kinetic energy of the single particle, irrespective of the particle chosen, can be estimated uniformly in time via Gronwall's lemma, which in turn implies a uniform bound on the momentum support of the one-particle distribution function. The problem is then reduced to the one studied by Glassey and Strauss in [9], from which the existence of global in time classical solutions follows. Thus, we conclude that singularities could occur only if the charge density of the particles blows up in finite time, that is, only if some particles can reach positions sufficiently close to each other. Therefore, no break-down could occur due to shock formations, since in this case scenario the solution itself would remain bounded while its derivative blows up. This result is proved in Chapter 3.

On the other hand, the existence of global in time *weak* solutions corresponding to both the RVM and nRVM systems was first proved by R. DiPerna and P. Lions in [22]. This result can be also found in [1, Chapter 7], where the approach in [23] is used. In [24], the existence result for the RVM system is revisited. Uniqueness of this type of solutions, however, remains unsolved. It is also unknown whether weak solutions preserve the total energy at least almost everywhere in time. Notice that the latter is a desirable feature of a meaningful solution concept for a conservative system. As part of this work, we have shown that *weak solutions of the RVM system conserve the total energy in time, provided that they satisfy some additional regularity and integrability conditions*. We do not include this result in the present thesis, but we provide the corresponding reference at the end of this section.

It is well known that the Maxwell system of equations can be rewritten in terms of scalar and vector potentials as a system of two second order partial differential equations. An important fact is that, although the equation satisfied by these potentials depend on the *gauge condition* chosen, the electromagnetic field does not. On the other hand, if we study the trajectories of the charged particles in the *generalized* phase-space, a Vlasov-like equation can be derived whose structure is determined by an incompressible vector field independently of the imposed gauge condition. Therefore, the RVM system can be conveniently reformulated in terms of the scalar and vector potentials. In the second part of this work, we use this formulation to prove a small data result for the Darwin approximation of the RVM system. Specifically, we show that *the so-called relativistic Vlasov-Darwin (RVD) system, supplemented with a sufficiently smooth compactly supported Cauchy datum f_0, has a unique global in time solution provided f_0 satisfies additional smallness conditions*. We do so by first producing a local result, and then showing that under suitable conditions on the Cauchy datum, the local solutions are actually global in time. Both local and global results are obtained in Chapter 5. We present the state of the art as well as the relevant references for this system in Section 5.3 of Chapter 5. We emphasize that the formulation we shall present here is essentially different from those previously used.

This thesis is organized as follows. In Chapter 2, we present the system of equations and define the Cauchy problem for the RVM system. We discuss the main properties of the equations involved, deduce a representation of the field in terms of the charge and current densities -the so-called Jefimenko representation-, and finally discuss the main conservation laws. Chapter 3 is devoted to the study of the Cauchy problem previously defined. First, by following a somehow different approach to that of Glassey and Strauss in [9], we recall the local existence and uniqueness result for the RVM system. Then, in Section 3.3, we discuss the continuation criteria to extend local solutions globally in time. It is in this section that we introduce our main new result concerning the RVM system. In Chapter 4, we explore the potential representation of the RVM system in terms of the two most common *gauges*. Hence, in Chapter 5, we present the Darwin approximation of the Maxwell equations and study their solutions: the so-called Darwin potentials. Then, we define the RVD system in Section 5.2 and present our local and global existence results for small Cauchy data. A detailed review on this system will be given in Section 5.3. Finally, we provide the concluding remarks in Chapter 6.

The papers supporting this thesis are:

- *Classical solvability of the relativistic Vlasov-Maxwell system with bounded spatial density.* R. Sospedra-Alfonso and R. Illner. Mathematical Methods in the Applied Sciences, 33:751-757 (2010).

- *On the energy conservation by weak solutions of the relativistic Vlasov-Maxwell system.* R. Sospedra-Alfonso. Communications in Mathematical Sciences, in press. Available online at `archiv:math.AP/0910.3956`, 2009.

- *Global classical solutions of the relativistic Vlasov-Darwin system with small Cauchy data: the generalized variables approach.* R. Sospedra-Alfonso and R. Illner. In preparation.

1.1 Notation

Although the notation we use throughout this work is standard, it is opportune to provide some additional specifications. We denote by I -sometimes J- an open interval of $\mathbb{R}$ with $0 \in I$. The interior of I is denoted by I^o and its closure by $\bar{I}$. We shall deal with scalar and, in general, tensor valued functions whose arguments are time $t \in I$, position $x \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$, or some of these. We shall frequently use the shorthand notation $z := (x, p)$.

We denote the standard basis of $\mathbb{R}^n$ by $\{\hat{e}_i\}_1^n$. For points $x, y \in \mathbb{R}^n$, $x \cdot y$ denotes the Euclidean scalar product

$$x \cdot y := \sum_{i=1}^{n} x^i y^i,$$

and $|x| := \sqrt{x \cdot x}$ denotes the Euclidean norm. We shall use the (repeated index) summation convention, thus $x \cdot y \equiv x^i y^i$. We may use both upper or lower indexes without distinction. Occasionally, we set $r := |y - x|$, and we make extensive use of the unit vector $\omega := r^{-1}(y - x)$. The open ball of radius $R > 0$ about x is denoted by

$$\Omega_R(x) := \left\{ y \in \mathbb{R}^3 : |y - x| < R \right\},$$

and its boundary by $\partial\Omega_R$. The symbol $\otimes$ is reserved for dyadic products. Thus, the object $\omega \otimes \omega$ is a tensor of entries $[\omega \otimes \omega]_{ij} := \omega^i \omega^j$, with $i, j \in \mathbb{N}$. In particular, we denote the identity matrix by $\texttt{id} = \delta_{ij} \hat{e}_i \otimes \hat{e}_j$, being δ_{ij} the Kronecker delta function -recall summation convention!-. For any tensor T of entries $T_{i_1,\ldots,i_k}$, we define $|T|$ by the Frobenius norm, i.e.,

$$|T| := \left(\sum_{i_1} \cdots \sum_{i_k} |T_{i_1,\ldots,i_k}|^2 \right)^{1/2}.$$

For any differentiable vector function $G = G(t, x, p)$, we denote by $\partial_t G$ the vector function of components $\partial_t G^i$, $i = 1, \ldots, n$. We denote by $\partial_x G$ the tensor of entries $\partial_{x^k} G^i$, $i = 1, \ldots, n$, $k = 1, 2, 3$, and similarly for $\partial_p G$. In particular, for scalar functions $g = g(t, x, p)$, we have that $\partial_x g \equiv \nabla g$ is the gradient of g. For the sake of clarity, we may specify the variables for the gradient. For instance, $\nabla_p g$ is the gradient of g with respect to the momentum variable p, etc. If no confusion arises, we may simplify $\partial_k G^i \equiv \partial_{x^k} G^i$. We may also combine $\partial_{(t,x)} G$, etc, which is defined in the obvious way. Similarly, for higher order derivatives, we denote, for instance, $\partial_x^2 G := \partial_x \partial_x G$ as the tensor of entries $\partial_j \partial_k G^i$, etc, and thus we may write $\partial_{xp}^2 G$, etc. The cross product of $\mathbb{R}^3$ is denoted by '$\times$', thus the curl$_x$ of $G \in \mathbb{R}^3$ reads $\nabla \times G$.

For $t \in I$ fixed, we denote by $g(t)$ the map $g(t) : \mathbb{R}^6 \ni z \mapsto g(t, z)$. If $g(t)$ is continuous on $\mathbb{R}^6$, then we write $g(t) \in C(\mathbb{R}^6; \mathbb{R})$. If it is also bounded, then $g(t) \in C_b(\mathbb{R}^6; \mathbb{R})$. We denote by $\texttt{supp} g(t)$ the support of $g(t)$, that is, the closure of $\{z : g(z) \neq 0\}$. In particular, we may use p-$\texttt{supp} g(t)$ meaning the closure of $\{p : \exists x : g(x, p) \neq 0\}$, and similarly for x. Clearly, if $\texttt{supp} g(t) \subset \mathbb{R}^6$ is bounded, then $g(t)$ has compact support. In that case we write $g(t) \in C_0(\mathbb{R}^6; \mathbb{R})$. Trivially,

$C_0 \subset C_b$. On the other hand, if $g(t)$ is Hölder continuous of order $0 < \alpha < 1$, then $g(t) \in C^\alpha(\mathbb{R}^6;\mathbb{R})$. Also, if $g(t)$ has all derivatives up to the order k continuous on $\mathbb{R}^6$, then we write $g(t) \in C^k(\mathbb{R}^6;\mathbb{R})$. The definitions for $C_b^k(\mathbb{R}^6;\mathbb{R})$, $C_0^k(\mathbb{R}^6;\mathbb{R})$ and $C_0^{k,\alpha}(\mathbb{R}^6;\mathbb{R})$ are now obvious, with the understanding that $C^0 \equiv C$, etc. Lastly, if $g(t)$ has continuous derivatives up to order m with respect to $z \in \mathbb{R}^6$ and it is continuously differentiable up to order k with respect to $t \in I$, we write $g \in C^k(I, C^m(\mathbb{R}^6);\mathbb{R})$. The spaces C_b, C_0, C_0^k, etc are all normed with the (uniform) sup-norm $\|\cdot\|_{L^\infty}$.

As usual, $L^q(\mathbb{R}^6,\mathbb{R})$ with $1 \leq q \leq \infty$ is the Lebesgue space with norm $\|\cdot\|_{L^q}$. We may write $\|\cdot\|_{L^q_x}$, etc if we want to specify variables. We also write

$$\|g\|_{L^r_p;L^s_x} := \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |g(x,p)|^r \, dp \right)^{s/r} dx \right)^{1/s},$$

with the corresponding definition for either $s = \infty$ or $r = \infty$. In particular, we write $\|\cdot\|_{L^q_{x,p}} \equiv \|\cdot\|_{L^q_p;L^q_x}$. For tensor valued functions of entries $G_{i_1,\ldots,i_k}$, we generalize the definition of the L^q-norm by

$$\|G\|_{L^q} := \left(\sum_{i_1} \cdots \sum_{i_k} \|G_{i_1,\ldots,i_k}\|^q_{L^q} \right)^{1/q}.$$

Sobolev spaces are denoted by $W^{k,q}(\mathbb{R}^6;\mathbb{R})$ with the usual norm

$$\|g\|_{W^{k,q}_{x,p}} := \left(\|g\|^q_{L^q_{x,p}} + \left\|\partial_{(x,p)} g\right\|^q_{L^q_{x,p}} + \ldots + \left\|\partial^k_{(x,p)} g\right\|^q_{L^q_{x,p}} \right)^{1/q}.$$

Mostly, we will be concerned with the cases $k = 0, 1$. Finally, for $t \in I$ we define

$$\|g(t)\|_{D^{1,q}_{x,p}} := \left(\|g(t)\|^q_{W^{1,q}_{x,p}} + \|\partial_t g(t)\|^q_{L^q_{x,p}} \right)^{1/q}.$$

Little c will always denote the speed of light, which in occasions may be set to one. Capital C denotes a universal constant unless we specify otherwise. We allow ‘constants’ to depend on time, the Cauchy data, etc. In such cases we may write, for instance, $C(t)$ and C^0 or $C(f_0)$, or, for both dependences combined, $C^0(t)$ or $C(t; f_0)$. In general, all such constants may change values from line to line.

Chapter 2

Equations and the Cauchy Problem

We present the equations that define the relativistic Vlasov-Maxwell (RVM) system. On one hand, we have the Vlasov equation which is a scalar linear first-order partial differential equation. On the other hand, we have the full set of Maxwell equations which are linear as well. The two components are non-linearly coupled to form the so-called RVM system. First, we shall discuss the properties of the Vlasov and Maxwell set of equations separately. In doing so, we obtain the Jefimenko representation of the electromagnetic fields in terms of the charge and current densities. This will be the prelude to the representation of the fields in terms of the one-particle distribution function, which is given in Chapter 3. Then, we define the RVM system and state the Cauchy problem. Finally, we provide the main associated conservation laws.

2.1 The Maxwell equations

Definition 1. *Let $j \in C(I \times \mathbb{R}^3; \mathbb{R}^3)$ and $E_0, B_0 \in C^1(\mathbb{R}^3; \mathbb{R}^3)$ be given. The field $E, B \in C^1(I \times \mathbb{R}^3; \mathbb{R}^3)$ is said to be a classical solution of the Maxwell equations if*

$$\nabla \times B - \frac{1}{c}\partial_t E = \frac{4\pi}{c} j \tag{2.1.1}$$

$$\nabla \times E + \frac{1}{c}\partial_t B = 0 \tag{2.1.2}$$

holds on $I \times \mathbb{R}^3$. In addition, the field (E, B) is said to be a classical solution of the Cauchy problem if $(E, B)|_{t=0} = (E_0, B_0)$.

Lemma 1. *The classical solution (E, B) of the Cauchy problem to the Maxwell equations is well defined and unique. Moreover, it satisfies*

(a) *Let $\rho \in C(I \times \mathbb{R}^3; \mathbb{R})$ be given and satisfy the continuity equation*

$$\partial_t \rho + \nabla \cdot j = 0 \quad in \;\; \mathcal{D}'(I^o \times \mathbb{R}^3) \tag{2.1.3}$$

Denote $\rho_0 = \rho|_{t=0}$. Then, we have that

$$\nabla \cdot E_0 = 4\pi\rho_0 \;\; on\; \mathbb{R}^3 \quad \Leftrightarrow \quad \nabla \cdot E = 4\pi\rho \;\; on\; I \times \mathbb{R}^3.$$

(b) *We also have*

$$\nabla \cdot B_0 = 0 \;\; on\; \mathbb{R}^3 \quad \Leftrightarrow \quad \nabla \cdot B = 0 \;\; on\; I \times \mathbb{R}^3.$$

Proof. The equations (2.1.1)-(2.1.2) form a non-homogeneous, linear symmetric hyperbolic system. The existence and uniqueness of solutions of their corresponding Cauchy problem is proved, for instance, in [25, Corollary 12.1.b.2].

To prove (a), notice that $\nabla \cdot (\partial_t E + 4\pi j) = c\nabla \cdot (\nabla \times B) \equiv 0$ in $\mathcal{D}'(I^o \times \mathbb{R}^3)$. Therefore, from (2.1.3) we find that $\partial_t (\nabla \cdot E - 4\pi\rho) = 0$ in $\mathcal{D}'(I^o \times \mathbb{R}^3)$. The continuity of the function $\nabla \cdot E - 4\pi\rho$ provides the result. As for (b), the proof runs exactly the same way since $\partial_t (\nabla \cdot B) \equiv 0$ in $\mathcal{D}'(I^o \times \mathbb{R}^3)$ and $\nabla \cdot B$ is continuous. □

In view of the previous definition and lemma, we will refer to the following set of equations as the *Maxwell equations*:

$$\nabla \times B - \frac{1}{c}\partial_t E = \frac{4\pi}{c} j \tag{2.1.4}$$

$$\nabla \times E + \frac{1}{c}\partial_t B = 0 \tag{2.1.5}$$

$$\nabla \cdot E = 4\pi\rho \tag{2.1.6}$$

$$\nabla \cdot B = 0 \tag{2.1.7}$$

The functions ρ and j are the charge and current densities respectively. Starting from the top, the first three equations are known as the *Ampère-Maxwell* (2.1.4), *Faraday* (2.1.5) and *Coulomb* (2.1.6) laws. $\nabla \cdot B = 0$ (2.1.7) denotes the absence of free magnetic poles. This system is consistent as long as the *continuity equation* (2.1.3) holds, since by Lemma 1 the relations (2.1.6) and (2.1.5) can be regarded as mere initial data. We point out that the continuity equation implies the conservation of charge, which we discuss in the Section 2.5 as well as other conservation laws.

Among the phenomena predicted by the Maxwell equations, the propagation of the field as electromagnetic waves is perhaps the most remarkable of all. This is better understood if we rewrite the Maxwell equations in their second order form. To this end, consider a smooth solution of (2.1.4)-(2.1.7). After rewriting (2.1.4) conveniently, take its partial time derivative and combine the resultant equation with (2.1.5). We obtain

$$\begin{aligned} \partial_t^2 E + 4\pi \partial_t j &= c\nabla \times \partial_t B = -c^2 \nabla \times \nabla \times E \\ &= -c^2 \nabla (\nabla \cdot E) + c^2 \Delta E = -4\pi\, c^2 \nabla \rho + c^2 \Delta E, \end{aligned}$$

where in the second line we have used the elementary vector identity for the double rotational and the equation (2.1.6). Similarly, by taking the time derivative of (2.1.5) and combining with (2.1.4) we find

$$\begin{aligned} \partial_t^2 B &= -c\nabla \times \partial_t E = 4\pi c \nabla \times j - c^2 \nabla \times \nabla \times B \\ &= 4\pi c \nabla \times j - c^2\, \nabla (\nabla \cdot B) + c^2 \Delta B = 4\pi c \nabla \times j + c^2 \Delta B, \end{aligned}$$

where the same vector identity and (2.1.7) have been used. Gathering these two relations yield the *electromagnetic wave equations*:

$$\Delta E - \frac{1}{c^2} \partial_t^2 E = 4\pi \nabla \rho + \frac{4\pi}{c^2} \partial_t j, \tag{2.1.8}$$

$$\Delta B - \frac{1}{c^2} \partial_t^2 B = -\frac{4\pi}{c} \nabla \times j. \tag{2.1.9}$$

The Jefimenko representation

Based on (2.1.8)-(2.1.9), we now look for a representation of the electromagnetic field in terms of the charge and current densities. To that effect, we first recall some standard results useful to our purposes. Details can be found, for instance, in [25, 26].

Let $g \in C^2(I \times \mathbb{R}^3; \mathbb{R})$. Denote $r = |y - x|$ and the unit vector $\omega = r^{-1}(y - x)$. Consider the Cauchy problem of a generic wave equation:

$$\Delta u - \frac{1}{c^2} \partial_t^2 u = -4\pi g \tag{2.1.10}$$

$$u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \tag{2.1.11}$$

where $u_0 \in C^3(\mathbb{R}^3; \mathbb{R})$ and $u_1 \in C^2(\mathbb{R}^3; \mathbb{R})$. By virtue of the Duhamel principle, the general solution $u \in C^2(I \times \mathbb{R}^3;\mathbb{R})$ of (2.1.10) is

$$u(t,x) = u_H(t,x) + \int_{\Omega_{ct}(x)} g(t - \frac{1}{c}\left|y-x\right|, y)\frac{dy}{|y-x|}, \tag{2.1.12}$$

where the function u_H solves (2.1.10) for $g=0$ and is given by the Kirchhoff formula

$$u_H(t,x) = \frac{1}{4\pi t^2}\int_{\partial\Omega_{ct}(x)}\left[u_0(y) + c\,(y-x)\cdot\nabla_y u_0(y) + \frac{t}{c}u_1(y)\right]dS_y. \tag{2.1.13}$$

The second term in the right-hand side of (2.1.12) solves (2.1.10) when $u_0 \equiv 0 \equiv u_1$. It is obtained by the method of retarded potentials and it is often called the *retarded solution.* If we set $y = x + c\omega(t-s)$, it can be also written as

$$\int_{\Omega_{ct}(x)} g(t-\frac{1}{c}\left|y-x\right|, y)\frac{dy}{|y-x|} \equiv c\int_0^t (t-s)\int_{|\omega|=1} g(s, x + c\omega(t-s))d\omega ds.$$

In the future, we shall use either representation without further notice. We shall also use the notation

$$\left[g(t,y)\right]_{\mathtt{ret}}(x) := g(t - \frac{1}{c}\left|y-x\right|, y),$$

meaning that the function g is evaluated at a retarded time. If we write the space gradient as $\nabla_y = (\partial_1, \partial_2, \partial_3)$, it is straightforward to check that

$$\partial_t \left[g(t,y)\right]_{\mathtt{ret}} = \left[\partial_t g(t,y)\right]_{\mathtt{ret}} \tag{2.1.14}$$

$$\partial_k \left[g(t,y)\right]_{\mathtt{ret}} = \left[\partial_k g(t,y)\right]_{\mathtt{ret}} - \frac{\omega^k}{c}\left[\partial_t g(t,y)\right]_{\mathtt{ret}}, \quad k=1,2,3. \tag{2.1.15}$$

In particular, if $G : I \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ we have

$$\nabla_y \left[g(t,y)\right]_{\mathtt{ret}} = \left[\nabla_y g(t,y)\right]_{\mathtt{ret}} - \frac{\omega}{c}\left[\partial_t g(t,y)\right]_{\mathtt{ret}} \tag{2.1.16}$$

$$\nabla_y \cdot \left[G(t,y)\right]_{\mathtt{ret}} = \left[\nabla_y \cdot G(t,y)\right]_{\mathtt{ret}} - \frac{\omega}{c}\cdot\left[\partial_t G(t,y)\right]_{\mathtt{ret}} \tag{2.1.17}$$

$$\nabla_y \times \left[G(t,y)\right]_{\mathtt{ret}} = \left[\nabla_y \times G(t,y)\right]_{\mathtt{ret}} - \frac{\omega}{c}\times\left[\partial_t G(t,y)\right]_{\mathtt{ret}}. \tag{2.1.18}$$

Indeed, the identity (2.1.14) is trivial and (2.1.15) is a straightforward consequence of the chain rule. Identities (2.1.16) to (2.1.18) follow easily from linearity and (2.1.15). They will all be used several times throughout this work.

We now turn to the representation of the field.

Lemma 2. *Let $E_0, B_0 \in C^3(\mathbb{R}^3;\mathbb{R}^3)$ and $E_1, B_1 \in C^2(I \times \mathbb{R}^3;\mathbb{R}^3)$. Let also $j \in C^1(I \times \mathbb{R}^3;\mathbb{R}^3)$ and $\rho \in C^1(I \times \mathbb{R}^3;\mathbb{R})$. The solutions E, B of the electromagnetic wave equations (2.1.8)-(2.1.9) with Cauchy data given by $(E,B)|_{t=0} = (E_0, B_0)$ and $(\partial_t E, \partial_t B)|_{t=0} = (E_1, B_1)$ satisfy the representation*

$$E(t,x) = (E)_0\,(t,x) - \int_{\Omega_{ct}(x)} [\rho(t,y)]_{ret} \frac{\omega dy}{|y-x|^2} \\ -\frac{1}{c}\int_{\Omega_{ct}(x)} [\partial_t \rho(t,y)]_{ret} \frac{\omega dy}{|y-x|} - \frac{1}{c^2}\int_{\Omega_{ct}(x)} [\partial_t j(t,y)]_{ret} \frac{dy}{|y-x|} \quad (2.1.19)$$

and

$$B(t,x) = (B)_0\,(t,x) - \frac{1}{c}\int_{\Omega_{ct}(x)} [j(t,y)]_{ret} \times \frac{\omega dy}{|y-x|^2} \\ -\frac{1}{c^2}\int_{\Omega_{ct}(x)} [\partial_t j(t,y)]_{ret} \times \frac{\omega dy}{|y-x|}, \quad (2.1.20)$$

where

$$\begin{aligned} (E)_0\,(t,x) &= \frac{1}{4\pi t^2}\int_{\partial\Omega_{ct}(x)} \left[E_0(y) + ct\,(\omega\cdot\nabla_y)\,E_0(y) + t\nabla_y \times B_0(y)\right] dS_y \\ &\quad -\frac{1}{ct}\int_{\partial\Omega_{ct}(x)} \rho(0,y)\omega dS_y \qquad (2.1.21) \\ (B)_0\,(t,x) &= \frac{1}{4\pi t^2}\int_{\partial\Omega_{ct}(x)} \left[B_0(y) + ct\,(\omega\cdot\nabla_y)\,B_0(y) - t\nabla_y \times E_0(y)\right] dS_y \\ &\quad -\frac{1}{c^2 t}\int_{\partial\Omega_{ct}(x)} j(0,y)\times\omega dS_y. \qquad (2.1.22) \end{aligned}$$

We shall refer to the retarded solutions in (2.1.19) and (2.1.20) as the Jefimenko representation of the electromagnetic field [27, Sec. 15.7].

Proof. Consider the wave equation for the electric field in (2.1.8). Owing to (2.1.12), the solution is

$$\begin{aligned} 4\pi E(t,x) &= E_H(t,x) - \int_{\Omega_{ct}(x)} \left([\nabla_y \rho(t,y)]_{\mathtt{ret}} + \frac{1}{c^2}[\partial_t j(t,y)]_{\mathtt{ret}} \right) \frac{dy}{|y-x|} \\ &= E_H(t,x) - \int_{\Omega_{ct}(x)} \nabla_y [\rho(t,y)]_{\mathtt{ret}} \frac{dy}{|y-x|} \\ &\quad -\frac{1}{c}\int_{\Omega_{ct}(x)} [\partial_t \rho(t,y)]_{\mathtt{ret}} \frac{\omega dy}{|y-x|} - \frac{1}{c^2}\int_{\Omega_{ct}(x)} [\partial_t j(t,y)]_{\mathtt{ret}} \frac{dy}{|y-x|}, \end{aligned}$$

where E_H is given by (2.1.13) with $u_0 \equiv E_0$ and $u_1 \equiv E_1 = c\nabla \times B_0$. Notice in the last step that we have used the identity (2.1.16) introduced above. Now, as a consequence of the divergence theorem, the second term in the right-hand side of the last equality becomes

$$\int_{\Omega_{ct}(x)} \nabla_y \left[\rho(t,y)\right]_{\mathtt{ret}} \frac{dy}{|y-x|} = \int_{\partial\Omega_{ct}(x)} \rho(0,y) \frac{\omega dS_y}{ct} + \int_{\Omega_{ct}(x)} \left[\rho(t,y)\right]_{\mathtt{ret}} \frac{\omega dy}{|y-x|^2}.$$

Hence, by combining the last two equations, the expression (2.1.19) for the electric field easily follows. In particular, the boundary term added to E_H provides (2.1.21).

As for the magnetic field, we proceed in the same way, but now with the source term $c\nabla \times j$ in the corresponding wave equation, as given by (2.1.9). Thus, following (2.1.12) and recalling the identity (2.1.18), we obtain

$$\begin{aligned} 4\pi B(t,x) &= B_H(t,x) - \frac{1}{c}\int_{\Omega_{ct}(x)} \nabla_y \times \left[j(t,y)\right]_{\mathtt{ret}} \frac{dy}{|y-x|} \\ &\quad - \frac{1}{c^2}\int_{\Omega_{ct}(x)} \left[\partial_t j(t,y)\right]_{\mathtt{ret}} \times \frac{\omega dy}{|y-x|} \\ &= B_H(t,x) - \frac{1}{c}\int_{\partial\Omega_{ct}(x)} j(0,y) \times \frac{\omega dS_y}{ct} - \frac{1}{c}\int_{\Omega_{ct}(x)} \left[j(t,y)\right]_{\mathtt{ret}} \times \frac{\omega dy}{|y-x|^2} \\ &\quad - \frac{1}{c^2}\int_{\Omega_{ct}(x)} \left[\partial_t j(t,y)\right]_{\mathtt{ret}} \times \frac{\omega dy}{|y-x|}, \end{aligned}$$

where B_H is given by (2.1.13) with $u_0 \equiv B_0$ and $u_1 \equiv B_1 = -c\nabla \times E_0$. Again, the second equality is a consequence of the divergence theorem. Since the above expression is precisely (2.1.20), the proof of the theorem is complete. □

2.2 The Vlasov equation

Definition 2. *Let $f_0 \in C^1(\mathbb{R}^6;\mathbb{R})$. Let the vector fields $v \in C(I, C^1(\mathbb{R}^6);\mathbb{R}^3)$ and $K \in C(I, C^1(\mathbb{R}^6);\mathbb{R}^3)$ be bounded on $\bar{J} \times \mathbb{R}^6$ for every compact subinterval $\bar{J} \subset I$, and given such that $\nabla_x \cdot v + \nabla_p \cdot K \equiv 0$. The function $f \in C^1(I \times \mathbb{R}^6;\mathbb{R})$ is said to be a classical solution of the linear Vlasov equation if*

$$\partial_t f + v \cdot \nabla_x f + K \cdot \nabla_p f = 0, \quad \text{on } I \times \mathbb{R}^6. \tag{2.2.1}$$

Moreover, f is said to be a classical solution to the Cauchy problem if $f|_{t=0} = f_0$.

Definition 3. *Denote $z = (x, p) \in \mathbb{R}^3 \times \mathbb{R}^3$. For every $t \in I$ and $z \in \mathbb{R}^6$, the set of ordinary differential equations*

$$\begin{aligned} \dot{X}(s,t,z) &= v(s, X(s,t,z), P(s,t,z)) && (2.2.2) \\ \dot{P}(s,t,z) &= K(s, X(s,t,z), P(s,t,z)) && (2.2.3) \end{aligned}$$

is called the characteristic system of (2.2.1). The curves $Z := (X, P)(\cdot, t, z) : I \to \mathbb{R}^6$ satisfying (2.2.2)-(2.2.3) and $Z(t, t, z) \equiv z$ are called the characteristic flow of (2.2.1).

Lemma 3. *For any $t \in I$ and $z \in \mathbb{R}^6$ fixed, there exists a unique solution $Z(s, t, z)$ of the system (2.2.2)-(2.2.3) with $Z(t, t, z) = z$. Thus, the characteristic flow is well defined and we have*

(a) $Z \in C^1(I \times I \times \mathbb{R}^6; \mathbb{R}^6)$.

(b) *For any $s, t \in I$ fixed, the map $Z(s, t, \cdot) : \mathbb{R}^6 \to \mathbb{R}^6$ is a C^1-diffeomorphism with inverse $Z^{-1}(s, t, z) = Z(t, s, z)$ and Jacobian determinant*

$$\det \frac{\partial Z(s,t,z)}{\partial z} \equiv 1,$$

i.e., it satisfies the volume preserving property.

(c) *If f is a classical solution of the Cauchy problem corresponding to (2.2.1), then f is constant along the characteristic flow. Conversely, the function defined by $f(t, z) := f_0(Z(0, t, z))$ on $I \times \mathbb{R}^6$ is the unique classical solution of the Cauchy problem to (2.2.1). If $f_0 \geq 0$, then $f \geq 0$. Also, for $t \in I$*

$$\textbf{\textit{supp}} f(t) = Z(t, 0, \textbf{\textit{supp}} f_0)$$

and for each $1 \leq q \leq \infty$, $t \in I$,

$$\|f(t)\|_{L^q_{x,p}} = \|f_0\|_{L^q_{x,p}}.$$

Proof. The first part of the lemma, including (a) and (b), can be proved by standard theory of first order ordinary differential equations, cf. [28, ch. II and V]. In particular, part (b) follows from -cf. [28, Corollary V.3.1]-

$$\det \frac{\partial Z(s,t,z)}{\partial z} = \exp \int_t^s \left[\nabla_x \cdot v + \nabla_p \cdot K\right](\sigma, Z(\sigma, t, z)) d\sigma \neq 0, \qquad (2.2.4)$$

which implies that for every $s, t \in I$ fixed, the map $z \mapsto Z(s,t,z)$ is of class C^1 and so is its inverse. By uniqueness, $Z(s,t,Z(t,s,z)) = Z(s,s,z) = z$, thus $Z^{-1}(s,t,z) = Z(t,s,z)$. The volume preserving property is consequence of the vanishing divergence assumed in Definition 2, i.e., (2.2.4) equals 1.

As for the part (c), the proof follows the standard Cauchy's method of characteristics (cf. [28, ch. VI]). In particular, if $z(s)$ is a solution of (2.2.2)-(2.2.3), then

$$\frac{d}{ds} f(s, z(s)) = \left[\partial_t f + v \cdot \nabla_x f + K \cdot \nabla_p f\right](s, z(s)).$$

The properties of the solution f are straightforward consequence of part (b). $\square$

Remark 1. To ease notation, we shall write the characteristic flow emanating from $z \in \mathbb{R}^6$ at $t = 0$ simply as $Z(s, z_0)$, i.e., $Z(s, 0, z) \equiv Z(s, z_0)$, $s \in I$, $z_0 \in \mathbb{R}^6$, unless we specify otherwise.

Remark 2. The velocity field v in Definition 2 is given in a very general setting. Throughout this chapter, $C^\infty(\mathbb{R}^3; \mathbb{R}^3) \ni v : p \mapsto v(p)$ and thus trivially $\nabla_x \cdot v \equiv 0$. The generality of v will become apparent in Chapter 4.

Let us briefly discuss the nature of the so-called Vlasov equation. Consider a set of N interacting particles with identical rest mass m. Associate to the i-th particle the six phase-space coordinates $z_i \equiv (x_i, p_i) \in \mathbb{R}^3 \times \mathbb{R}^3$, being x and p position and momentum respectively. We assume that the interaction is by pairs, so we denote the force exerted on particle i by particle j as $F_{i,j}$ with $i \neq j$.

Any particular *state* of this system can be represented as a point in the $6N$-dimensional space $[\mathbb{R}^3 \times \mathbb{R}^3]^N$. The probability of finding that point in the volume $dz_1 dz_2 \cdots dz_N$ at the time t is given by the quantity $D(t, z_1, z_2 \ldots z_N) dz_1 dz_2 \cdots dz_N$, where $D(t, z_1, z_2 \ldots z_N)$ is the N-particle probability distribution function. In the absence of external forces, the Liouville's theorem for statistical mechanics asserts that D evolves in time according to

$$\partial_t D + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} D + \frac{1}{m} \sum_{i=1}^{N} \sum_{j \neq i}^{N} F_{i,j} \cdot \nabla_{p_i} D = 0, \tag{2.2.5}$$

where v_i stands for the velocity of the i-th particle. In the non-relativistic regime velocities and momenta satisfy the trivial relation $v_i = m^{-1} p_i$, but this not need to be the case in general. For the time being, no distinction is made.

The Vlasov equation is obtained via the BBGKY hierarchy [3, 29] as a limit case of the Liouville's equation (2.2.5). In such scenario, it can be written as

$$\partial_t f(t, z_1) + v_1 \cdot \nabla_{x_1} f(t, z_1) + \frac{1}{m} \left[\int_{\mathbb{R}^6} F_{1,2} f(t, z_2) dz_2 \right] \cdot \nabla_{p_1} f(t, z_1) = 0, \qquad (2.2.6)$$

where f denotes the one-particle probability distribution function, normalized so that $N^{-1} \int f(t, z) dz = 1$. The field in brackets denotes the mean force exerted on particle 1 by the remaining particles in the system. Thus, the Vlasov equation provides a mean field approach to the dynamics of the particle ensemble and it avoids having to deal with each pair of interactions separately[1]. Clearly, the latter is prohibitive when the system is made of a very large number of particles.

To actually study the evolution of f, the force $F_{1,2}$ must be specified. Typically, potential forces are considered, where the force is the gradient of some scalar function. That is the case, for instance, of the Coulomb force[2], where $F_{1,2} \equiv \nabla \Phi_{1,2}$ with $\Phi_{1,2} \propto 1/r^2$, being $r = |x_1 - x_2|$. But even with such apparently simple interactions, the equation (2.2.6) is far from simple, since the scalar function $\Phi_{1,2}$ is singular and the term involving the force is non-linear. If the mean force is *given*, then the equation (2.2.6) simplifies enormously. In such a case it reduces to a *linear* Vlasov equation as the one we introduced in Definition 2.

2.3 The Vlasov-Maxwell system

Now, consider a system of several species of collisionless charged particles moving at speeds comparable to the speed of light. In this regime, we can not simply assume the interactions by pairs as we did earlier, since the induced electromagnetic field propagates at finite speed and so the force at one particle due to others depends on their state of motion at *retarded* times. This is hinted by the Jefimenko representation in Section 2.1, cf. (2.1.19)-(2.1.20), where the electromagnetic field is given in terms of the retarded charge and current densities. Nevertheless, a Vlasov equation can still be deduced. Its derivation, however, must rely on heuristic arguments that we present next.

[1]Notice that in such a limit, collisions have been neglected. If collisions are to be taken into account, then a Boltzmann equation should be considered instead. If such collisions produce only small changes in the momenta of the particles, then the simpler Vlasov-Fokker-Planck equation is preferred. Details can be found, for instance, in [3, 29, 30].

[2]Coulomb interactions yields the widely studied Vlasov-Poisson system; cf. Chapter 5.

Indeed, consider a relativistic particle of species α, with charge e_α and rest mass m_α, under the influence of some *given* smooth electromagnetic field (E, B). Its equations of motion are

$$\dot{x} = v_\alpha, \quad \dot{p} = e_\alpha \left(E + \frac{v_\alpha}{c} \times B \right) =: K_\alpha, \tag{2.3.1}$$

where K_α denotes the Lorentz force acting on the particle and v_α stands for its relativistic velocity, i.e.,

$$v_\alpha = \frac{cp}{\sqrt{m_\alpha^2 c^2 + |p|^2}}. \tag{2.3.2}$$

The Lorentz force satisfies $\nabla_p \cdot K_\alpha \equiv 0$, which can be easily verified. At a time t, the probable number of particles of species α in the volume $dxdp$ of the phase-space is equal to $dN = f_\alpha(t, x, p)dxdp$. After the infinitesimal time interval dt, their location must change according to the transformation $(x, p) \mapsto (x + v_\alpha dt, p + K_\alpha dt)$, as suggested by (2.3.1). But this is essentially the map in Lemma 3 (b), which is volume preserving. Therefore, in the absence of collisions we have

$$\left[f_\alpha(t + dt, x + v_\alpha dt, p + K_\alpha dt) - f_\alpha(t, x, p) \right] dxdp = 0.$$

It follows that the total differential of f_α equals zero, thus -compare to Lemma 3 (c)-

$$\partial_t f_\alpha + v_\alpha \cdot \nabla_x f_\alpha + e_\alpha \left(E + \frac{v_\alpha}{c} \times B \right) \cdot \nabla_p f_\alpha = 0. \tag{2.3.3}$$

We recognize (2.3.3) as a linear Vlasov equation.

Now, if we regard (E, B) as the mean electromagnetic field induced by the charged particles themselves, then E and B can be computed by means of the Maxwell equations (2.1.4)-(2.1.7) once we have defined the charge and current densities as -summation runs over all particle species-

$$\rho(t, x) = \sum_\alpha \int_{\mathbb{R}^3} e_\alpha f_\alpha(t, x, p)dp, \quad j(t, x) = \sum_\alpha \int_{\mathbb{R}^3} v_\alpha e_\alpha f_\alpha(t, x, p)dp. \tag{2.3.4}$$

The equation (2.3.3), coupled via (2.3.4) to the Maxwell equations (2.1.4)-(2.1.7) and complemented with (2.3.2), describe what is known as the *relativistic Vlasov-Maxwell* system. They model a hot, low density plasma formed by charged particles interacting only through the self-induced electromagnetic field and moving at speeds comparable to the speed of light.

2.4 The Cauchy problem

We can now define the Cauchy problem for the relativistic Vlasov-Maxwell (RVM) system. For simplicity, we shall consider a single particle species only since all results can be trivially adapted to the multi-species case. Also, we shall set all physical constants to one, namely the rest mass and charge of the particles as well as the speed of light.

Definition 4. *Let $f_0 \in C^1(\mathbb{R}^6;\mathbb{R})$, $f_0 \geq 0$ and $E_0, B_0 \in C^1(\mathbb{R}^3;\mathbb{R}^3)$. The triplet (f, E, B) is said to be a classical solution of the RVM system if $f \in C^1(I \times \mathbb{R}^6;\mathbb{R}^6)$ and $E, B \in C^1(I \times \mathbb{R}^3;\mathbb{R}^3)$; for every compact subinterval $\bar{J} \subset I$ the electromagnetic field (E, B) is bounded on $\bar{J} \times \mathbb{R}^3$; and (f, E, B) satisfies*

$$\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_p f = 0, \tag{2.4.1}$$

$$\nabla \times B - \partial_t E = 4\pi j \tag{2.4.2}$$

$$\nabla \times E + \partial_t B = 0 \tag{2.4.3}$$

$$\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0 \tag{2.4.4}$$

where

$$j = \int_{\mathbb{R}^3} v f dp, \quad \rho = \int_{\mathbb{R}^3} f dp, \tag{2.4.5}$$

and

$$v = \frac{p}{\sqrt{1 + |p|^2}}. \tag{2.4.6}$$

Moreover, the triplet (f, E, B) is said to be a classical solution of the Cauchy problem if $(f, E, B)|_{t=0} = (f_0, E_0, B_0)$.

2.5 Conservation laws

In this section we introduce the main conservation laws that are formally satisfied by the solutions of (2.4.1)-(2.4.6). Indeed, let (f, E, B) be a classical solution of the RVM system. As pointed out earlier, the Lorentz force satisfies $\nabla_p \cdot (E + v \times B) \equiv 0$, thus the Vlasov equation (2.4.1) can be written as

$$\partial_t f + \nabla_x \cdot [vf] + \nabla_p \cdot [(E + v \times B)\, f] = 0.$$

Integrating with respect to p yields the continuity equation

$$\partial_t \rho + \nabla_x \cdot j = 0$$

which denotes the local law for the *conservation of the charge*. If we further integrate over all $x \in \mathbb{R}^3$, we obtain the global counterpart, i.e., $\|\rho(t)\|_{L^1_x} = \|\rho(0)\|_{L^1_x}$, $t \in I$. In view of (2.4.5), the latter is just the particular case of the Lemma 3 (c) when $q = 1$.

To derive the law for the *conservation of the total energy* we proceed as follows. Let W be the total energy of a single relativistic particle of rest mass m in the absence of interactions, defined by

$$W(p) = mc^2 \sqrt{1 + \frac{|p|^2}{m^2 c^2}}.$$

We shall call W the *kinetic* energy of the single relativistic particle[3]. Setting m and c equal to one, we define the *kinetic energy density* function by

$$h(t,x) = \int_{\mathbb{R}^3} \sqrt{1 + |p|^2} f(t,x,p) dp. \tag{2.5.1}$$

Now, take the partial time derivative in both sides of (2.5.1). A use of the Vlasov equation implies

$$\begin{aligned}
\partial_t h &= -\int_{\mathbb{R}^3} \sqrt{1 + |p|^2} \left(\nabla_x \cdot [vf] + \nabla_p \cdot [Kf] \right) dp \\
&= -\nabla_x \cdot \int_{\mathbb{R}^3} \sqrt{1 + |p|^2} v f dp + \int_{\mathbb{R}^3} \nabla_p \sqrt{1 + |p|^2} \cdot K f dp \\
&= -\nabla_x \cdot \int_{\mathbb{R}^3} p f dp + j \cdot E. \qquad (2.5.2)
\end{aligned}$$

Notice the integration by parts in the second equality. Also, the use of the identity $\nabla_p \sqrt{1 + |p|^2} \cdot K \equiv v \cdot E$ resulting from $v \cdot (v \times B) \equiv 0$, as well as the definition of the current density j in the third equality.

On the other hand, we define the *electromagnetic field energy density* as

$$u(t,x) = \frac{1}{8\pi} \left(|E(t,x)|^2 + |B(t,x)|^2 \right). \tag{2.5.3}$$

[3]Actually, the kinetic energy is $W - mc^2$, where mc^2 is the energy of the particle at rest.

By taking the partial time derivative in both sides of (2.5.3) and invoking the Maxwell equations (2.4.2) and (2.4.3), we find

$$\begin{aligned}\partial_t u &= \frac{1}{4\pi}\left(E\cdot\partial_t E + B\cdot\partial_t B\right)\\ &= \frac{1}{4\pi}E\cdot(\nabla\times B - 4\pi j) + \frac{1}{4\pi}B\cdot(-\nabla\times E)\\ &= -\frac{1}{4\pi}\nabla\cdot(E\times B) - j\cdot E, \end{aligned}\tag{2.5.4}$$

where in the last equality we have used the elementary vector identity for the divergence of the cross product. Incidentally, integration in (2.5.4) with respect to x yields the so-called Poynting theorem [31].

Hence, by combining (2.5.2) and (2.5.4), we obtain the local law for the *conservation of the total energy*

$$\partial_t e + \nabla\cdot\sigma = 0,$$

where

$$\begin{aligned}e(t,x) &:= \int_{\mathbb{R}^3}\sqrt{1+|p|^2}f(t,x,p)dp + \frac{1}{8\pi}\left(|E(t,x)|^2 + |B(t,x)|^2\right),\\ \sigma(t,x) &:= \int_{\mathbb{R}^3} pf(t,x,p)dp + \frac{1}{4\pi}\left(E(t,x)\times B(t,x)\right)\end{aligned}$$

denote the total energy and flux densities respectively. In addition, if we integrate over all $x\in\mathbb{R}^3$, the corresponding global conservation law follows, i.e., the *total energy*

$$\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\sqrt{1+|p|^2}f(t,x,p)dxdp + \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(|E(t,x)|^2 + |B(t,x)|^2\right)dx \tag{2.5.5}$$

is preserved in time.

Gronwall-Bellman-Bihari lemma

For the sake of reference, we conclude this chapter by stating the well-known and useful Gronwall-Bellman-Bihari lemma:

Lemma 4. *Let $u,h\in C([t_0,T[)$, $T\le\infty$ be non-negative functions. Let the constant*

$C > 0$ and assume that $\omega \in C(]0, \infty[)$ is non-negative and non-decreasing. If

$$u(t) \le C + \int_{t_0}^{t} h(s)\omega(u(s))ds, \qquad t \in [t_0, T), \tag{2.5.6}$$

then, for $t \in [t_0, t_1]$

$$u(t) \le \Omega^{-1}\left[\Omega(C) + \int_{t_0}^{t} h(s)ds\right], \tag{2.5.7}$$

where

$$\Omega(u) := \int_{u_0}^{u} \frac{dz}{\omega(z)}, \qquad u_0, u > 0.$$

and $t_1 \in [t_0, T[$ is the largest number such that

$$\Omega(k) + \int_{t_0}^{t_1} h(s)ds \le \Omega(\infty)$$

Proof. The proof was originally given by I. Bihari and can be found in [32]. □

We distinguish two particular cases used several times throughout this work.

Corollary 1. (i) *If $\omega(u) = u$, then (2.5.6) reduces to the Gronwall's inequality and (2.5.7) becomes*

$$u(t) \le C \exp \int_{t_0}^{t} h(s)ds. \tag{2.5.8}$$

In particular, if $C = 0$ then (2.5.8) holds for every $C > 0$ and by letting $C \to 0$, we have $u(t) \equiv 0$.

(ii) *For $u > 1$, if $\omega(u) = u \ln u$, then (2.5.7) becomes*

$$u(t) \le C \exp\exp \int_{t_0}^{t} h(s)ds.$$

Similarly, if $C = 0$, then $u(t) \equiv 0$.

Chapter 3

Classical Solutions

This chapter is devoted to the study of the Cauchy problem for the RVM system. The main open problem concerning this system is to prove whether solutions with sufficiently smooth Cauchy data develop singularities in finite time. In this direction, the new result of this chapter is that a classical solution of the RVM system launched by smooth, compactly supported Cauchy data, becomes singular only if the charge density blows up in finite time. In particular, this result weakens previous assumptions used to extend classical solutions globally in time.

For the sake of consistency, we shall recall the local existence result for the RVM system as given in the pioneering work by Glassey and Strauss in [9]. However, our approach differs from the latter in several aspects. Among others, we do not use the abstract operators introduced in [9] to represent the field and its derivatives. Instead, we deal directly with the more 'natural' Jefimenko representation of the electromagnetic fields and the identities for the derivatives of *retarded* solutions, previously given in Chapter 2. We obtain an explicit representation of the electromagnetic field in terms of the *velocity* and *acceleration fields* -cf. Remark 4-, which somewhat simplifies the computations needed in the reminder of the section.

The chapter is organized as follows. In Section 3.1 we provide the preliminaries and a-priori bounds used to produce the local existence result in Section 3.2. Then, we discuss the continuation criteria for the local solutions in Section 3.3. We review the known conditional results for the existence of global solutions and then we conclude with our result: if the charge density remains bounded, then the corresponding classical solution of the RVM system can be extended globally in time.

3.1 Preliminaries

Let $f_0 : \mathbb{R}^6 \to \mathbb{R}$ be smooth, non-negative and with compact support. Also, let the pair $(E_0, B_0) : \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ be smooth and satisfy the compatibility conditions

$$\nabla \cdot E_0 = 4\pi \int_{\mathbb{R}^3} f_0 dp, \qquad \nabla \cdot B_0 = 0.$$

Define the map $v : \mathbb{R}^3 \ni p \mapsto p\left(1 + |p|^2\right)^{-1/2}$, thus $v \in C_b^\infty(\mathbb{R}^3; \mathbb{R}^3)$ satisfies $|v| \leq 1$. Throughout this section, we shall assume that the vector field $(E, B) : \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ is *given* and smooth. We abbreviate the Lorentz force by $K := E + v \times B$ and let $Q := K - v\,(v \cdot E)$. For the given field, we denote by f^* the solution of the *linear* Vlasov equation (2.4.1) with initial datum $f^*|_{t=0} = f_0$. Then, for the characteristic system (2.2.2)-(2.2.3) associated to (2.4.1), the boundedness of (v, K) yields

$$\left|\dot{Z}(s, t, z)\right| \leq C(s, t), \qquad (s, t) \in I \times I.$$

As a result, and by virtue of Lemma 3(c), the compact support of f_0 implies that for each $t \in I$ the function $f^*(t)$ has compact support in $\mathbb{R}^6$ as well. Define ρ^* and j^* according to (2.4.5). Integrating (2.4.1) over all $p \in \mathbb{R}^3$ yields the continuity equation

$$0 = \partial_t \rho^* + \nabla_x \cdot j^* + \int_{\mathbb{R}^3} \nabla_p \cdot (K f^*) dp = \partial_t \rho^* + \nabla \cdot j^*.$$

Then, we denote by (E^*, B^*) the solution of the Maxwell equations (2.4.2)-(2.4.4) with charge and current densities ρ^* and j^* respectively. Notice that by virtue of Section 2.1, the pair (E^*, B^*) also solves the wave equations (2.1.8)-(2.1.9) with Cauchy data (E_0, E_1) and (B_0, B_1), where

$$E_1 := \nabla \times B_0 - 4\pi \int_{\mathbb{R}^3} v f_0 dp, \qquad B_1 := -\nabla \times E_0.$$

For $t \in I$, we further define

$$\bar{P}^*(t) := \sup\left\{|p| : \exists 0 \leq s \leq t, x \in \mathbb{R}^3 : f^*(s, x, p) \neq 0\right\}. \tag{3.1.1}$$

Remark 3. $\bar{P}^*(t)$ is non-decreasing in t. Thus, for all $p \in \mathtt{supp} f^*(s)$, $0 \leq s \leq t$, we have that $|p| \leq \bar{P}^*(t)$. Since for every $t \in I$ and $x \in \mathbb{R}^3$ the function f^* has compact support in p, then $\bar{P}^*(t) < \infty$. Therefore, $|v| \leq \bar{P}^*(t)\left(1 + \bar{P}^{*2}(t)\right)^{-1/2} < 1$ strictly on the p-$\mathtt{supp}f^*(s)$, $0 \le s \le t$. We heavily exploit this fact throughout the section.

For $x, y \in \mathbb{R}^3$, set $r := |y-x|$ and the unit vector $\omega := r^{-1}(y-x)$. In order to ease future calculations, we collect some identities which are easily verifiable:

$$\begin{aligned}
\nabla_y r &= \omega & (3.1.2)\\
\nabla_y \frac{1}{r} &= -\frac{\omega}{r^2} & (3.1.3)\\
\nabla_y \omega^i &= \frac{1}{r}\left(\hat{e}_i - \omega^i \omega\right) & (3.1.4)\\
\nabla_y (v\cdot\omega) &= \frac{1}{r}\left(v - \omega (v\cdot\omega)\right) & (3.1.5)\\
\nabla_y (v\times\omega)^i &= \frac{1}{r}\left(\hat{e}_k v^j - \hat{e}_j v^k - \omega (v\times\omega)^i\right) & (3.1.6)\\
\nabla_p v^i &= \left(1+|p|^2\right)^{-1/2}\left(\hat{e}_i - v^i v\right) & (3.1.7)\\
\nabla_p (v\cdot\omega) &= \left(1+|p|^2\right)^{-1/2}\left(\omega - v(v\cdot\omega)\right) & (3.1.8)\\
\nabla_p (v\times\omega)^i &= \left(1+|p|^2\right)^{-1/2}\left(\left(\hat{e}_j\omega^k - \hat{e}_k\omega^j\right) - v(v\times\omega)^i\right) & (3.1.9)
\end{aligned}$$

Here $i \neq j \neq k$ and $i, j, k = 1, 2, 3$. Finally, we recall the elementary vector identity for the triple cross product: for any vectors a, b and c

$$a \times (b \times c) = (a\cdot c)\, b - (a \cdot b)\, c. \tag{3.1.10}$$

In particular, if u is a unit vector and b is perpendicular to u, we have

$$(u \times b) \times u = b. \tag{3.1.11}$$

3.1.1 Representation of the electromagnetic field

The aim is to deduce a representation of the field (E^*, B^*) in terms of the one-particle probability distribution function f^*. We will do so by combining the Jefimenko representation (2.1.19)-(2.1.20) with the Vlasov equation (2.4.1). We start by setting

$$a(v,\omega) := \frac{v+\omega}{1+v\cdot\omega}, \qquad b(v,\omega) := \frac{v\times\omega}{1+v\cdot\omega} \tag{3.1.12}$$

on the support of f^*. By virtue of the Remark 3, $|v| < 1$ and so we have the strict inequality $1 + v\cdot\omega > 0$. Therefore, a and b are not singular on the p-$\mathtt{supp}f^*$.

Lemma 5. *For $r>0$, the following identities hold:*

$$\left(v\cdot\nabla_y\right)\frac{a}{r}=\frac{\left(v+\omega\right)\left(1-|v|^2\right)}{r^2\left(1+v\cdot\omega\right)^2}-\frac{\omega}{r^2} \tag{3.1.13}$$

and

$$\left(\frac{K}{r}\cdot\nabla_p\right)a=\frac{\omega\times\left(Q\times\left(v+\omega\right)\right)\sqrt{1-|v|^2}}{r\left(1+v\cdot\omega\right)^2}. \tag{3.1.14}$$

Proof. We prove (3.1.13) first. Helped by (3.1.2-3.1.5), a lengthy but elementary computation shows that

$$\nabla_y\frac{a^i}{r}=-\frac{\omega\left(v^i+\omega^i\right)}{r^2\left(1+v\cdot\omega\right)}+\frac{\hat{e}_i-\omega^i\omega}{r^2\left(1+v\cdot\omega\right)}-\frac{\left(v^i+\omega^i\right)\left(v-\omega\left(v\cdot\omega\right)\right)}{r^2\left(1+v\cdot\omega\right)^2}.$$

Hence, recalling that $v^i=v\cdot\hat{e}_i$, we have

$$\begin{aligned}\left(v\cdot\nabla_y\right)\frac{a}{r} &= \frac{-\left(v+\omega\right)\left(v\cdot\omega\right)+v-\omega\left(v\cdot\omega\right)}{r^2\left(1+v\cdot\omega\right)}-\frac{\left(v+\omega\right)\left(|v|^2-\left(v\cdot\omega\right)^2\right)}{r^2\left(1+v\cdot\omega\right)^2}\\ &= \frac{\left(v+\omega\right)\left(1-v\cdot\omega\right)-\omega\left(1+v\cdot\omega\right)}{r^2\left(1+v\cdot\omega\right)}-\frac{\left(v+\omega\right)\left(|v|^2-\left(v\cdot\omega\right)^2\right)}{r^2\left(1+v\cdot\omega\right)^2}\\ &= \frac{1}{r^2\left(1+v\cdot\omega\right)^2}\Big[\left(v+\omega\right)\left(1-\left(v\cdot\omega\right)^2\right)-\omega\left(1+v\cdot\omega\right)^2\\ &\qquad -\left(v+\omega\right)\left(|v|^2-\left(v\cdot\omega\right)^2\right)\Big],\end{aligned}$$

from where we easily get (3.1.13).

As for (3.1.14), by using (3.1.7) and (3.1.8) we have

$$\nabla_p a^i=\frac{\hat{e}_i-v^iv}{\sqrt{1+|p|^2}\left(1+v\cdot\omega\right)}-\frac{\left(v^i+\omega^i\right)\left(\omega-v\left(v\cdot\omega\right)\right)}{\sqrt{1+|p|^2}\left(1+v\cdot\omega\right)^2}.$$

Hence, noticing that $v\cdot(v\times B)=0$ so $v\cdot K=v\cdot E$, and recalling the identity (3.1.10) and that $|\omega| = 1$, we have

$$\begin{aligned}
\left(\frac{K}{r} \cdot \nabla_p\right) a &= \frac{K - v\,(v \cdot E)}{r\sqrt{1 + |p|^2}\,(1 + v \cdot \omega)} - \frac{(v + \omega)\,[\omega \cdot (K - v\,(v \cdot E))]}{r\sqrt{1 + |p|^2}\,(1 + v \cdot \omega)^2} \\
&= \frac{Q + Q\,(v \cdot \omega) - v(\omega \cdot Q) - \omega\,(\omega \cdot Q)}{r\sqrt{1 + |p|^2}\,(1 + v \cdot \omega)^2} \\
&= \frac{Q\,(\omega \cdot \omega) - \omega\,(\omega \cdot Q) + Q\,(v \cdot \omega) - v(\omega \cdot Q)}{r\sqrt{1 + |p|^2}\,(1 + v \cdot \omega)^2} \\
&= \frac{\omega \times (Q \times \omega) + \omega \times (Q \times v)}{r\sqrt{1 + |p|^2}\,(1 + v \cdot \omega)^2}.
\end{aligned}$$

Since $\left(1 + |p|^2\right)^{-1} = 1 - |v|^2$ as it follows from (2.4.6), it is straightforward to check that the identity (3.1.14) holds. This conclues the proof of the lemma. □

Lemma 6. *For $r > 0$, the following identities also hold:*

$$(v \cdot \nabla_y)\,\frac{b}{r} = \left(\frac{(v + \omega)\left(1 - |v|^2\right)}{r^2\,(1 + v \cdot \omega)^2} - \frac{v}{r^2}\right) \times \omega \tag{3.1.15}$$

and

$$\left(\frac{K}{r} \cdot \nabla_p\right) b = \left(\frac{\omega \times (Q \times (v + \omega))\,\sqrt{1 - |v|^2}}{r\,(1 + v \cdot \omega)^2}\right) \times \omega. \tag{3.1.16}$$

Proof. Similarly, a lengthy but elementary computation, and the use of (3.1.2-3.1.6), imply that

$$\nabla_y \frac{b^i}{r} = -\frac{\omega\,(v \times \omega)^i}{r^2\,(1 + v \cdot \omega)} + \frac{\hat{e}_k v^j - \hat{e}_j v^k - \omega\,(v \times \omega)^i}{r^2\,(1 + v \cdot \omega)} - \frac{(v \times \omega)^i\,(v - \omega\,(v \cdot \omega))}{r^2\,(1 + v \cdot \omega)^2}.$$

Thus, noting that $v \cdot \left(\hat{e}_k v^j - \hat{e}_j v^k\right) \equiv v^k v^j - v^j v^k = 0$, we obtain

$$\begin{aligned}
(v \cdot \nabla_y)\,\frac{b}{r} &= -\frac{2\,(v \cdot \omega)\,(v \times \omega)}{r^2(1 + v \cdot \omega)} - \frac{(v \times \omega)\left(|v|^2 - (v \cdot \omega)^2\right)}{r^2\,(1 + v \cdot \omega)^2} \\
&= \frac{v \times \omega}{r^2\,(1 + v \cdot \omega)^2}\left((v \cdot \omega)^2 - 2\,(v \cdot \omega)\,(1 + v \cdot \omega) - |v|^2\right) \\
&= \frac{v \times \omega}{r^2\,(1 + v \cdot \omega)^2}\left(1 - |v|^2 - (1 + v \cdot \omega)^2\right),
\end{aligned}$$

from where (3.1.15) readily follows.

As for (3.1.16), we use (3.1.7-3.1.9) to find

$$\nabla_p b^i = \frac{\hat{e}_j \omega^k - \hat{e}_k \omega^j - v\,(v\times\omega)^i}{\sqrt{1+|p|^2}\,(1+v\cdot\omega)} - \frac{(v\times\omega)^i\,(\omega - v\,(v\cdot\omega))}{\sqrt{1+|p|^2}\,(1+v\cdot\omega)^2}.$$

Hence, since $(v\cdot K) = (v\cdot E)$ and recalling that $Q = K - v\,(v\cdot E)$, we get

$$\begin{aligned} \left(\frac{K}{r}\cdot\nabla_p\right) b &= \frac{(K - v\,(v\cdot E))\times\omega}{r\sqrt{1+|p|^2}\,(1+v\cdot\omega)} - \frac{(v\times\omega)\,[\omega\cdot(K - v\,(v\cdot E))]}{r\sqrt{1+|p|^2}\,(1+v\cdot\omega)^2} \\ &= \frac{Q\times\omega + (Q\,(v\cdot\omega) - v\,(\omega\cdot Q))\times\omega}{r\sqrt{1+|p|^2}\,(1+v\cdot\omega)^2} \\ &= \frac{Q\times\omega + (\omega\times(Q\times v))\times\omega}{r\sqrt{1+|p|^2}\,(1+v\cdot\omega)^2}. \end{aligned} \tag{3.1.17}$$

But it is elementary that $(\omega\times(Q\times\omega))\times\omega = Q\times\omega$, as we recalled in (3.1.11). Thus, if we substitute this in (3.1.17) and use that $\left(1+|p|^2\right)^{-1} = 1 - |v|^2$, the identity (3.1.16) easily follows and the proof of the lemma is complete. □

Theorem 1. *Let* $Q := E + v\times B - v\,(v\cdot E)$*. Denote the kernel*

$$\mathcal{K}(v,\omega) := \frac{(v+\omega)\left(1-|v|^2\right)}{(1+v\cdot\omega)^2}. \tag{3.1.18}$$

The electromagnetic field (E^*, B^*) *satisfies the representation*

$$\begin{aligned} E^*(t,x) = (E^*)_0\,(t,x) &- \int\int_{\Omega_t(x)\times\mathbb{R}^3} \mathcal{K}(v,\omega)\,[f^*(t,y,p)]_{ret}\,\frac{dpdy}{|y-x|^2} \\ &+ \int\int_{\Omega_t(x)\times\mathbb{R}^3} \omega\times\left(\frac{\mathcal{K}(v,\omega)}{\sqrt{1-|v|^2}}\times[Qf^*(t,y,p)]_{ret}\right)\frac{dpdy}{|y-x|} \end{aligned} \tag{3.1.19}$$

and

$$\begin{aligned} B^*(t,x) = (B^*)_0\,(t,x) &- \int\int_{\Omega_t(x)\times\mathbb{R}^3} (\omega\times\mathcal{K}(v,\omega))\,[f^*(t,y,p)]_{ret}\,\frac{dpdy}{|y-x|^2} \\ &+ \int\int_{\Omega_t(x)\times\mathbb{R}^3} \omega\times\left(\omega\times\left(\frac{\mathcal{K}(v,\omega)}{\sqrt{1-|v|^2}}\times[Qf^*(t,y,p)]_{ret}\right)\right)\frac{dpdy}{|y-x|}, \end{aligned} \tag{3.1.20}$$

where $(E^*)_0\,(t,x)$ *and* $(B^*)_0\,(t,x)$ *are functionals of the Cauchy data only.*

Remark 4. The acceleration of a single relativistic charged particle is given by $\dot{v} = \dot{p}\cdot\nabla_p v \equiv \left(1+|p|^2\right)^{-1/2} Q$. Thus, if we exclude the first term in (3.1.19) and (3.1.20) respectively, which depend on the Cauchy data only, we have that the electric and magnetic fields are both 'naturally' decomposed into *velocity* and *acceleration fields*. The velocity fields do *not* depend on the acceleration of the particles and can be essentially regarded as the contribution of the static fields induced by the charges, with decay $O(|x|^{-2})$ as $|x| \to \infty$. On the other hand, the last term in (3.1.19) and (3.1.20) respectively depend linearly on the acceleration of the charged particles. They amount for the radiation fields of the system. Finally, notice the mutually transverse nature of the electric and magnetic fields induced by each particle. More on the physics of radiation by moving charged particles can be found in [31, Ch. 14].

Proof of Theorem 1. We show (3.1.19) first. To this end, we use the definition of the charge and current density functions (2.4.5) and rewrite the Jefimenko representation for the electric field (2.1.19) as

$$\begin{aligned} E^*(t,x) &= (E^*)_0\,(t,x) - \int\int_{\Omega_t(x)\times\mathbb{R}^3} [f^*(t,y,p)]_{\mathtt{ret}} \frac{\omega dpdy}{|y-x|^2} \\ &\quad - \int\int_{\Omega_t(x)\times\mathbb{R}^3} (v+\omega)\,[\partial_t f^*(t,y,p)]_{\mathtt{ret}} \frac{dpdy}{|y-x|}. \end{aligned} \tag{3.1.21}$$

Helped by the identity (2.1.17) in Section 2.1, the *linear* Vlasov equation and the chain rule imply

$$\begin{aligned} [\partial_t f^*(t,y,p)]_{\mathtt{ret}} &= -\left[\left(v\cdot\nabla_y f^* + K\cdot\nabla_p f^*\right)(t,y,p)\right]_{\mathtt{ret}} \\ &= -\left[\nabla_y\cdot v f^*(t,y,p)\right]_{\mathtt{ret}} - \left[\nabla_p\cdot K f^*(t,y,p)\right]_{\mathtt{ret}} \\ &= -\left(v\cdot\omega\right)[\partial_t f^*(t,y,p)]_{\mathtt{ret}} - \nabla_y\cdot[v f^*(t,y,p)]_{\mathtt{ret}} \\ &\quad -\nabla_p\cdot[K f^*(t,y,p)]_{\mathtt{ret}}\,. \end{aligned}$$

Hence, in view of the Remark 3, we have that $|v| < 1$ and so $1 + v\cdot\omega > 0$. Therefore,

$$[\partial_t f^*(t,y,p)]_{\mathtt{ret}} = -\left(1+v\cdot\omega\right)^{-1}\left(\nabla_y\cdot[v f^*(t,y,p)]_{\mathtt{ret}} + \nabla_p\cdot[K f^*(t,y,p)]_{\mathtt{ret}}\right). \tag{3.1.22}$$

Recall $r = |y-x|$. We may now write the right-hand side of the above equality into the expression (3.1.21), to find that

$$E^*(t,x) = (E^*)_0(t,x) - \int\int_{\Omega_t(x)\times\mathbb{R}^3} \frac{\omega}{r^2}\left[f^*(t,y,p)\right]_{\mathtt{ret}} dpdy$$
$$+ \int\int_{\Omega_t(x)\times\mathbb{R}^3} \frac{v+\omega}{r\left(1+v\cdot\omega\right)} \nabla_y \cdot \left[v f^*(t,y,p)\right]_{\mathtt{ret}} dpdy$$
$$+ \int\int_{\Omega_t(x)\times\mathbb{R}^3} \frac{v+\omega}{r\left(1+v\cdot\omega\right)} \nabla_p \cdot \left[K f^*(t,y,p)\right]_{\mathtt{ret}} dpdy. \quad (3.1.23)$$

Abbreviate $a = (1+v\cdot\omega)^{-1}(v+\omega)$. As a consequence of the divergence theorem, and since $[f^*(r,y,p)]_{\mathtt{ret}} = f_0^*(y,p)$, the second integral becomes

$$\int\int_{\Omega_t(x)\times\mathbb{R}^3} \frac{a}{r}\nabla_y \cdot \left[v f^*(t,y,p)\right]_{\mathtt{ret}} dpdy$$
$$= \frac{1}{t}\int\int_{\partial\Omega_t(x)\times\mathbb{R}^3} a\left(v\cdot\omega\right) f_0^*(y,p) dpdS_y$$
$$- \int\int_{\Omega_t(x)\times\mathbb{R}^3} \left(v\cdot\nabla_y \frac{a^i}{r}\right)\hat{e}_i \left[f^*(t,y,p)\right]_{\mathtt{ret}} dpdy. \quad (3.1.24)$$

Notice that the first integral in the right-hand side of the latter equation depends on the Cauchy data only, so we include it in $(E^*)_0(t,x)$. As for the third integral in (3.1.23), we apply the divergence theorem as well, noticing that the boundary term vanishes in view of the compact support of f^* in the p variable. Hence, we find that

$$E^*(t,x) = (E^*)_0(t,x) - \int\int_{\Omega_t(x)\times\mathbb{R}^3} \frac{\omega}{r^2}\left[f^*(t,y,p)\right]_{\mathtt{ret}} dpdy$$
$$- \int\int_{\Omega_t(x)\times\mathbb{R}^3} \left(v\cdot\nabla_y \frac{a^i}{r}\right)\hat{e}_i \left[f^*(t,y,p)\right]_{\mathtt{ret}} dpdy$$
$$- \int\int_{\Omega_t(x)\times\mathbb{R}^3} \left(\frac{\left[K(t,y,p)\right]_{\mathtt{ret}}}{r}\cdot\nabla_p a^i\right)\hat{e}_i \left[f^*(t,y,p)\right]_{\mathtt{ret}} dpdy.$$

If we now invoke Lemma 5, it is easy to check that (3.1.19) hold, which proves the representation for the electric field. As for the magnetic field, the proof runs similarly. The corresponding Jefimenko representation in terms of the charge and current density reads,

$$
\begin{aligned}
B^*(t,x) = (B^*)_0\,(t,x) - \int\int_{\Omega_t(x)\times\mathbb{R}^3} (v\times\omega)\,[f^*(t,y,p)]_{\mathtt{ret}}\,\frac{dpdy}{|y-x|^2} \\
-\int\int_{\Omega_t(x)\times\mathbb{R}^3} (v\times\omega)\,[\partial_t f^*(t,y,p)]_{\mathtt{ret}}\,\frac{dpdy}{|y-x|}.
\end{aligned}
$$

Denote $b = (1+v\cdot\omega)^{-1}(v\times\omega)$. If we now use the representation of the time derivative (3.1.22) in the above expression, and then apply the divergence theorem, we find that

$$
\begin{aligned}
B^*(t,x) &= (B^*)_0\,(t,x) - \int\int_{\Omega_t(x)\times\mathbb{R}^3} \frac{(v\times\omega)}{r}\,[f^*(t,y,p)]_{\mathtt{ret}}\,dpdy \\
&\quad -\int\int_{\Omega_t(x)\times\mathbb{R}^3} \left(v\cdot\nabla_y\frac{b^i}{r}\right)\hat{e}_i\,[f^*(t,y,p)]_{\mathtt{ret}}\,dpdy \\
&\quad -\int\int_{\Omega_t(x)\times\mathbb{R}^3} \left(\frac{[K(t,y,p)]_{\mathtt{ret}}}{r}\cdot\nabla_p b^i\right)\hat{e}_i\,[f^*(t,y,p)]_{\mathtt{ret}}\,dpdy,
\end{aligned}
$$

where the boundary term corresponding to the integration in y has been already included in $(B^*)_0\,(t,x)$, and the one corresponding to p vanishes due to the compact support of f^*. Invoke Lemma 6. It is easy to check that (3.1.20) holds and the proof of the theorem is complete. □

3.1.2 Representation of the derivatives of the field

Theorem 2. *Denote the space gradient $\nabla = (\partial_1,\partial_2,\partial_3)$. For each $k = 1,2,3$ we have*

$$
\begin{aligned}
\partial_k E^*(t,x) = (\partial_k E^*)_0\,(t,x) + \int_{\mathbb{R}^3} \hat{S}^k(v) f^*(t,x,p)dp \\
+\int\int_{\Omega_t(x)\times\mathbb{R}^3} \hat{A}^k(v,\omega)\,[f^*(t,y,p)]_{ret}\,\frac{dpdy}{|y-x|^3} \\
+\int\int_{\Omega_t(x)\times\mathbb{R}^3} \hat{B}(v,\omega)\,[Kf^*(t,y,p)]_{ret}\,\frac{\omega^k dpdy}{|y-x|^2} \\
+\int\int_{\Omega_t(x)\times\mathbb{R}^3} \hat{C}(v,\omega)\,[(f^*\partial_k Q + Q\partial_k f^*)(t,y,p)]_{ret}\,\frac{dpdy}{|y-x|},
\end{aligned}
\tag{3.1.25}
$$

where the kernels $\hat{S}^k:\mathbb{R}^3\to\mathbb{R}^3$, $\hat{A}^k:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ and $\hat{B},\hat{C}:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^9$ are smooth and bounded on the support of f^*. Moreover, for all $v\in\mathbb{R}^3$ with $|v|<1$, the *average of the kernels $\hat{A}^k$ on the unit sphere vanishes, i.e.,*

$$\int_{|\omega|=1} \hat{A}^k(v,\omega)d\omega = 0. \tag{3.1.26}$$

The magnetic field B^* satisfies an analogue representation with kernels $\tilde{S}^k$, $\tilde{A}^k, \tilde{B}$ and $\tilde{C}$ having the same properties as those for the electric field. In particular, the average of $\tilde{A}^k$ on the unit sphere vanishes as well.

Proof. For $\mathcal{K}$ defined as (3.1.18), denote by $\hat{C}$ the matrix whose columns are given by

$$\hat{C}_i(v,\omega) = \frac{\omega \times (\mathcal{K}(v,\omega) \times \hat{e}_i)}{\sqrt{1-|v|^2}}. \tag{3.1.27}$$

A direct estimation shows that $\left|\hat{C}_i(v,\omega)\right| \leq C\,(1+v\cdot\omega)^{-2}$. Thus, in view of the Remark 3, all elements of $\hat{C}$ are smooth and bounded on the support of f^*. Now, after changing variables $y = x+z$, take the partial space derivative ∂_k to both sides of the representation (3.1.19) of E^*. Then, after returning to our original variables, we obtain

$$\begin{aligned} \partial_k E^*(t,x) &= (\partial_k E^*)_0\,(t,x) - \int\int_{\Omega_t(x)\times\mathbb{R}^3} \mathcal{K}(v,\omega)\,[\partial_k f^*(t,y,p)]_{\mathtt{ret}}\,\frac{dpdy}{r^2} \\ &\quad + \int\int_{\Omega_t(x)\times\mathbb{R}^3} \hat{C}_i(v,\omega)\left[\partial_k\left(Q^i f^*\right)(t,y,p)\right]_{\mathtt{ret}}\frac{dpdy}{r}, \end{aligned} \tag{3.1.28}$$

where $(\partial_k E^*)_0\,(t,x) = \partial_k\,(E^*)_0\,(t,x)$ depends on the Cauchy data only. By the product rule, the last term in the right-hand side of (3.1.28) is just the last term in the right-hand side of (3.1.25). Therefore, we only have to find the terms in (3.1.25) involving the kernels $\hat{S}^k$, $\hat{A}^k$ and $\hat{B}$ respectively.

To this end, notice first that by combining the identity (2.1.15) with the representation (3.1.22) of $[\partial_t f^*(t,y,p)]_{\mathtt{ret}}$ we have

$$\begin{aligned} [\partial_k f^*(t,y,p)]_{\mathtt{ret}} &= \partial_k\,[f^*(t,y,p)]_{\mathtt{ret}} + \omega^k\,[\partial_t f^*(t,y,p)]_{\mathtt{ret}} \\ &= \partial_k\,[f^*(t,y,p)]_{\mathtt{ret}} - \frac{\omega^k}{1+v\cdot\omega}\nabla_y\cdot[v f^*(t,y,p)]_{\mathtt{ret}} \\ &\quad - \frac{\omega^k}{1+v\cdot\omega}\nabla_p\cdot[K f^*(t,y,p)]_{\mathtt{ret}}\,. \end{aligned} \tag{3.1.29}$$

Hence, we can rewrite the second term in the right-hand side of (3.1.28) as

$$\begin{aligned}\int\int_{\Omega_t(x)\times\mathbb{R}^3} &\mathcal{K}(v,\omega)\left[\partial_k f^*(t,y,p)\right]_{\mathtt{ret}}\frac{dpdy}{r^2}\\ &= \int\int_{\Omega_t(x)\times\mathbb{R}^3}\frac{\mathcal{K}(v,\omega)}{r^2}\left(\partial_k - \frac{\omega^k v\cdot\nabla_y}{1+v\cdot\omega}\right)\left[f^*(t,y,p)\right]_{\mathtt{ret}} dpdy\\ &\quad -\int\int_{\Omega_t(x)\times\mathbb{R}^3}\frac{\mathcal{K}(v,\omega)}{1+v\cdot\omega}\nabla_p\cdot\left[Kf^*(t,y,p)\right]_{\mathtt{ret}}\frac{\omega^k dpdy}{r^2}. \qquad (3.1.30)\end{aligned}$$

Now, consider the last integral in (3.1.30). Integration by parts yields

$$\begin{aligned}\int\int_{\Omega_t(x)\times\mathbb{R}^3}&\frac{\mathcal{K}(v,\omega)}{1+v\cdot\omega}\nabla_p\cdot\left[Kf^*(t,y,p)\right]_{\mathtt{ret}}\frac{\omega^k dpdy}{r^2}\\ &= -\int\int_{\Omega_t(x)\times\mathbb{R}^3}\partial_{p_i}\left(\frac{\mathcal{K}(v,\omega)}{1+v\cdot\omega}\right)\left[K^i f^*(t,y,p)\right]_{\mathtt{ret}}\frac{\omega^k dpdy}{r^2}\\ &\equiv -\int\int_{\Omega_t(x)\times\mathbb{R}^3}\hat{B}_i(v,\omega)\left[K^i f^*(t,y,p)\right]_{\mathtt{ret}}\frac{\omega^k dpdy}{r^2},\end{aligned}$$

which is the term in (3.1.25) involving the kernel $\hat{B}$. The explicit form of the $\hat{B}_i$'s are unimportant, although we can easily verify that they are smooth and satisfy $\left|\hat{B}_i(v,\omega)\right| \le c\,(1+v\cdot\omega)^{-3}$. Hence, they are bounded on the support of f^*.

The first integral in the right-hand side of (3.1.30), on the other hand, needs extra care due to the singularity r^{-2} at $r=0$. To overcome this difficulty, let $\epsilon > 0$ and define $\Omega_t^\epsilon(x) := \{y\in\mathbb{R}^3 : \epsilon < r \le t\}$. Integration by parts then implies

$$\begin{aligned}\int\int_{\Omega_t^\epsilon(x)\times\mathbb{R}^3}&\frac{\mathcal{K}(v,\omega)}{r^2}\left(\partial_k - \frac{\omega^k v\cdot\nabla_y}{1+v\cdot\omega}\right)\left[f^*(t,y,p)\right]_{\mathtt{ret}} dpdy\\ &= \frac{1}{t^2}\int\int_{\partial\Omega_t(x)\times\mathbb{R}^3}\frac{\omega^k\mathcal{K}(v,\omega)}{1+v\cdot\omega}f_0^*(y,p)dpdS_y\\ &\quad -\frac{1}{\epsilon^2}\int\int_{\partial\Omega_\epsilon(x)\times\mathbb{R}^3}\frac{\omega^k\mathcal{K}(v,\omega)}{1+v\cdot\omega}f^*(t-\epsilon,y,p)dpdS_y\\ &\quad -\int\int_{\Omega_t^\epsilon(x)\times\mathbb{R}^3}\left\{\left(\partial_k - v\cdot\nabla_y\frac{\omega^k}{1+v\cdot\omega}\right)\frac{\mathcal{K}(v,\omega)}{r^2}\right\}\left[f^*(t,y,p)\right]_{\mathtt{ret}} dpdy. \qquad (3.1.31)\end{aligned}$$

The first of the two boundary terms depends on the Cauchy data and so we include it in $\left(\partial_k E^*\right)_0(t,x)$. It is easy to check that it is bounded for all $t\in I$. As for the second one, notice that after changing variables $y = x + \omega\epsilon$ we have

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3}\int_{|\omega|=1}\frac{\omega^k\mathcal{K}(v,\omega)}{1+v\cdot\omega}f^*(t-\epsilon, x+\omega\epsilon, p)d\omega dp$$
$$= \int_{\mathbb{R}^3}\int_{|\omega|=1}\frac{\omega^k\mathcal{K}(v,\omega)}{1+v\cdot\omega}f^*(t,x,p)d\omega dp \equiv \int_{\mathbb{R}^3}\hat{S}^k(v)f^*(t,x,p)dp.$$

Clearly, $\hat{S}^k$ is smooth and bounded on the support of f^*. Therefore, the last integral in (3.1.31) also converges and we can pass to the limit as $\epsilon \to 0$.

Finally, we must show that

$$\left\{\left(\partial_k - v\cdot\nabla_y\frac{\omega^k}{1+v\cdot\omega}\right)\frac{\mathcal{K}(v,\omega)}{r^2}\right\} \equiv \frac{\hat{A}^k(v,\omega)}{r^3}. \tag{3.1.32}$$

We omit this elementary but lengthy computation which is done in [9, Appendix]. There, it is shown that $\hat{A}^k$ is smooth and satisfies $|\hat{A}^k(v,\omega)| \le c\,(1+v\cdot\omega)^{-4}$, so it is bounded on the support of f^*. Moreover, it is found that the average of $\hat{A}^k$ on the unit sphere vanishes, cf. [9, p.69], which provides (3.1.26) and completes the proof of the theorem as far as the electric field E^* is concerned.

The analogue results for the magnetic field can be verified with little extra effort, since the representation (3.1.20) of B^* is very similar to that of E^*. In particular, the analogue of (3.1.25) holds with kernels $\tilde{S}^k$, $\tilde{A}^k$, $\tilde{B}$ and $\tilde{C}$ that can be found in a similar fashion. They are smooth and satisfies the same estimates as $\hat{S}^k$, $\hat{A}^k$, $\hat{B}$ and $\hat{C}$ respectively. Hence, in order to conclude, it has to be shown that (3.1.26) also holds for the kernel $\tilde{A}^k$. This result can be found in [9, p.70] as well. □

3.1.3 Estimates on the field

The following lemma is almost trivial but useful:

Lemma 7. *(i)* $|v+\omega|^2 \le 2\,(1+v\cdot\omega)$ *and* *(ii)* $1-|v|^2 \le 2\,(1+v\cdot\omega)$.

Proof. It is straightforward that $|v+\omega|^2 = |v|^2 + |\omega|^2 + 2\,(v\cdot\omega) \le 2\,(1+v\cdot\omega)$. On the other hand, the elementary identity $(v\cdot\omega)^2 + |v\times\omega|^2 = |v|^2$ is used to find that

$$\begin{aligned}1-|v|^2 &= 1-(v\cdot\omega)^2 - |v\times\omega|^2\\ &\le 1-(v\cdot\omega)^2 \le (1-v\cdot\omega)\,(1+v\cdot\omega)\\ &\le 2\,(1+v\cdot\omega)\,,\end{aligned}$$

which completes the proof. □

Lemma 8. *The kernel $\mathcal{K}$ defined in (3.1.18) satisfies*

$$|\mathcal{K}(v,\omega)| \leq 4\sqrt{1+|p|^2}. \tag{3.1.33}$$

Also, for $Q = E + v \times B - v\,(v \cdot E)$, we have that

$$|Q\,(\omega \cdot \mathcal{K})| \leq 4\,(|E| + |B|) \quad \textit{and} \quad |\mathcal{K}\,(\omega \cdot Q)| \leq 8\,(|E| + |B|). \tag{3.1.34}$$

Proof. We use (i) first and then (ii) from Lemma 7 to estimate $\mathcal{K}$, namely

$$\begin{aligned} \left| \frac{(v+\omega)\left(1-|v|^2\right)}{(1+v\cdot\omega)^2} \right| &\leq \frac{\sqrt{2}\,\left(1-|v|^2\right)}{(1+v\cdot\omega)^{\frac{3}{2}}} = \frac{\sqrt{2}}{\sqrt{1-|v|^2}} \left(\frac{1-|v|^2}{1+v\cdot\omega} \right)^{3/2} \\ &\leq \frac{2^2}{\sqrt{1-|v|^2}} = 4\sqrt{1+|p|^2}, \end{aligned}$$

which proves (3.1.33).

To prove (3.1.34), we first notice that $\omega \cdot \mathcal{K} = (1+v\cdot\omega)^{-1}\left(1-|v|^2\right)$, and that $|Q| \leq 2\,(|E|+|B|)$. Hence, owing to Lemma 7 (ii), $|Q\,(\omega\cdot\mathcal{K})| \leq 4\,(|E|+|B|)$. On the other hand, since the triple product $v \cdot (v \times B) = 0$, we have that

$$\begin{aligned} \omega \cdot Q &= \omega \cdot (E + v \times B - v\,(v \cdot E)) \\ &= (\omega - v\,(v\cdot\omega)) \cdot (E + v \times B) \\ &= (v + \omega - v\,(1 + v\cdot\omega)) \cdot (E + v \times B)\,. \end{aligned}$$

Thus, another use of (i) and then (ii) from Lemma 7 yield

$$\begin{aligned} |\mathcal{K}\,(\omega \cdot Q)| &\leq \left(\frac{|v+\omega|^2\left(1-|v|^2\right)}{(1+v\cdot\omega)^2} + \frac{|v|\,|v+\omega|\left(1-|v|^2\right)}{1+v\cdot\omega} \right)(|E|+|B|) \\ &\leq \frac{4\left(1-|v|^2\right)}{1+v\cdot\omega}\,(|E|+|B|) \leq 8\,(|E|+|B|)\,, \end{aligned}$$

which concludes the proof of the lemma. □

With these results at hand, it is an easy matter to estimate the field.

Lemma 9. *Denote $\bar{K} := |E| + |B|$. The electric field E^* satisfies*

$$\begin{aligned} |E^*(t,x)| &\leq C^0(t) + 4\int\int_{\Omega_t(x)\times\mathbb{R}^3} \sqrt{1+|p|^2}\,[f^*(t,y,p)]_{ret}\,\frac{dpdy}{|y-x|^2} \\ &\quad +12\int\int_{\Omega_t(x)\times\mathbb{R}^3} \sqrt{1+|p|^2}\,\left[\bar{K}f^*(t,y,p)\right]_{ret}\,\frac{dpdy}{|y-x|}. \end{aligned} \tag{3.1.35}$$

where $C^0(t)$ is a non-negative, continuous function of t, otherwise depending on the Cauchy data only. The same estimate holds for the magnetic field B^*.

Proof. Consider the representation (3.1.19) of the electric field E^*, which we rewrite here for the sake of convenience

$$\begin{aligned} E^*(t,x) = (E^*)_0\,(t,x) - \int\int_{\Omega_t(x)\times\mathbb{R}^3} \mathcal{K}(v,\omega)\,[f^*(t,y,p)]_{\mathtt{ret}}\,\frac{dpdy}{|y-x|^2} \\ +\int\int_{\Omega_t(x)\times\mathbb{R}^3} \omega\times\left(\frac{\mathcal{K}(v,\omega)}{\sqrt{1-|v|^2}}\times[Qf^*(t,y,p)]_{\mathtt{ret}}\right)\frac{dpdy}{|y-x|}. \end{aligned}$$

Estimating the second term in the right-hand side it is a straightforward consequence of (3.1.33) in Lemma 8. To estimate the third one, the vector identity (3.1.10) produces

$$|\omega\times(\mathcal{K}\times Q)| = |\mathcal{K}\,(\omega\cdot Q) - Q\,(\omega\cdot\mathcal{K})| \leq |\mathcal{K}\,(\omega\cdot Q)| + |Q\,(\omega\cdot\mathcal{K})|\,.$$

Then, since $\left(1-|v|^2\right) = \left(1+|p|^2\right)^{-1}$, the expected estimate follows from the relations (3.1.34) in Lemma 8.

Finally, we show that there is a non-negative continuous function $C^0(t)$ of t, otherwise depending on the initial data only, such that $|(E^*)_0\,(t,x)| \leq C^0(t)$. To this end, we first recall that $(E^*)_0\,(t,x)$ is defined by

$$\begin{aligned} (E^*)_0(t,x) &= \frac{1}{t^2}\int_{\partial\Omega_t(x)} \left[E_0(y) + t\,(\omega\cdot\nabla_y)\,E_0(y) + t\nabla_y\times B_0(y)\right]dS_y \\ &\quad -\frac{1}{t}\int\int_{\partial\Omega_t(x)\times\mathbb{R}^3} \frac{\omega - v\,(v\cdot\omega)}{1+v\cdot\omega} f_0(y,p)dpdS_y, \end{aligned} \tag{3.1.36}$$

which is the result of the boundary term in (3.1.24) added to (2.1.21). In view of the assumptions made on (f_0, E_0, B_0) at the beginning of Section 3.1, a direct estimate shows that there is a constant $C = C(\bar{P}_0, \|f_0\|_{L^\infty_{x,p}}, \|E_0\|_{W^\infty_x}, \|B_0\|_{W^\infty_x})$ such that

$$|(E^*)_0\,(t,x)| \leq C(\bar{P}_0, \|f_0\|_{L^\infty_{x,p}}, \|E_0\|_{W^\infty_x}, \|B_0\|_{W^\infty_x})t \equiv C^0(t), \tag{3.1.37}$$

which proves the claim. Here $\bar{P}_0$ denotes the upper bound on the p-$\mathtt{supp} f_0$, which is defined according to (3.1.1).

As for the magnetic field, the representation (3.1.20) trivially implies that the same estimate holds for B^*, thus concluding the proof of the lemma. □

It will be convenient to rewrite (3.1.35) in terms of the kinetic energy density function (2.5.1), which we recall:

$$h^*(t,x) = \int_{\mathbb{R}^3} \sqrt{1+|p|^2} f^*(t,y,p)dp. \tag{3.1.38}$$

By doing the change of variables $y = x + \omega\,(t-s)$ in (3.1.35), we end up with the following alternative formulation of Lemma 9:

Corollary 2. *Denote $\bar{K} = |E| + |B|$. The electric field E^* satisfies*

$$\begin{aligned} |E^*(t,x)| &\leq C^0(t) + 4\int_0^t \int_{|\omega|=1} h^*(s, x+\omega\,(t-s))d\omega ds \\ &\quad 12\int_0^t (t-s)\int_{|\omega|=1} \bar{K} h^*\,(s, x+\omega\,(t-s))\,d\omega ds. \end{aligned} \tag{3.1.39}$$

The same estimates hold for the magnetic field B^*.

We conclude this paragraph with the following result.

Lemma 10. *Let*

$$u^*(t) := 1 + \left(\|E^*(t)\|_{L^\infty_x} + \|B^*(t)\|_{L^\infty_x}\right).$$

The field (E^*, B^*) satisfies the estimate

$$u^*(t) \leq C^0(t)\left(1 + \int_0^t u(s)\,\|h^*(s)\|_{L^\infty_x}\,ds\right), \tag{3.1.40}$$

where u is defined accordingly and $C^0(t)$ is a non-negative, continuous function of t, otherwise depending on the Cauchy data only.

Remark 5. In view of Remark 3 and the boundedness of f^*, it is straightforward from (3.1.40) that there is a positive $C^*(t) \equiv C(t; \bar{P}^*(t), \|f_0\|_{L^\infty_{x,p}}, \|E_0\|_{W^\infty_x}, \|B_0\|_{W^\infty_x})$, which for each t is continuous in its arguments, such that

$$u^*(t) \le C^*(t)\left(1 + \int_0^t u(s)ds\right).$$

Recall that $\bar{P}^*(t)$ is defined by (3.1.1).

Proof of Lemma 10. The estimate (3.1.39) from Corollary 2 produces

$$\begin{aligned} |E^*(t,x)| \;&\le\; C^0(t) + 4\int_0^t \|h^*(s)\|_{L^\infty_x} ds \\ &\quad +12t\int_0^t \|h^*(s)\|_{L^\infty_x}\left(\|E(s)\|_{L^\infty_x} + \|B(s)\|_{L^\infty_x}\right) ds. \end{aligned}$$

But the same estimate holds for the magnetic field, so we have that

$$\begin{aligned} &1 + |E^*(t,x)| + |B^*(t,x)| \\ &\le C^0(t) + C^0(t)\int_0^t \|h^*(s)\|_{L^\infty_x}\left[1 + \left(\|E(s)\|_{L^\infty_x} + \|B(s)\|_{L^\infty_x}\right)\right] ds, \end{aligned}$$

irrespective of $x \in \mathbb{R}^3$. Take the supremum over all $x \in \mathbb{R}^3$ and use the Gronwall's lemma. The estimate (3.1.40) then follows. □

3.1.4 Estimates on the derivatives of the field

Abbreviate the representation (3.1.25) of the space derivatives of E^* as

$$\begin{aligned} \partial_k E^*(t,x) = \left(\partial_k E^*\right)_0(t,x) + \int_{\mathbb{R}^3} \hat{S}^k(v) f^*(t,x,p)dp \\ +\partial_k E^*_A(t,x) + \partial_k E^*_B(t,x) + \partial_k E^*_C(t,x). \end{aligned} \tag{3.1.41}$$

The meanings of $\partial_k E^*_A$, $\partial_k E^*_B$ and $\partial_k E^*_C$ are obvious as they correspond to the terms in (3.1.25) with kernels $\hat{A}^k$, $\hat{B}$ and $\hat{C}$ respectively. We estimate one at a time.

We start with $\partial_k E^*_A$, whose singularity r^{-3} at $r = 0$ precludes us from taking a direct estimate. We overcome this difficulty by using the vanishing average of the kernel $\hat{A}^k$ on the unit sphere (3.1.26). After changing variables $y = x + \omega\,(t-s)$ and by virtue of (3.1.26), we may rewrite this term as

$$\begin{aligned} \partial_k E_A^*(t,x) &= \int_0^t \int_{|\omega|=1} \int_{\mathbb{R}^3} \hat{A}^k(v,\omega) f^*(s, x+\omega(t-s), p) dp d\omega \frac{ds}{t-s} \\ &= \int_0^{t-d} \int \int \hat{A}^k(v,\omega) f^*(s, x+\omega(t-s), p) dp d\omega \frac{ds}{t-s} \\ &\quad + \int_{t-d}^t \int \int \hat{A}^k(v,\omega) \left[f^*(s, x+\omega(t-s), p) - f^*(s,x,p) \right] dp d\omega \frac{ds}{t-s}, \end{aligned} \tag{3.1.42}$$

where the parameter $0 < d < t$ is to be determined. Since the time integral corresponding to the first term in the last equality is taken on $[0, t-d]$, no singularity occur there and the estimate is straightforward, namely

$$\sup_{v,\omega} \left| \hat{A}^k(v,\omega) \right| \frac{4\pi}{3} \bar{P}^*(t)^3 \|f_0\|_{L^\infty_{x,p}} \int_0^{t-d} \frac{ds}{t-s} \leq C^*(t) \|f_0\|_{L^\infty_{x,p}} \log \frac{t}{d}.$$

As for the second term, the mean value theorem provides the bound

$$\sup_{v,\omega} \left| \hat{A}^k(v,\omega) \right| \frac{4\pi}{3} \bar{P}^*(t)^3 \|\nabla_x f^*\|_{L^\infty_{t,x,p}} \int_{t-d}^t ds \leq C^*(t) \|\nabla_x f^*\|_{L^\infty_{t,x,p}} d.$$

Then, we chose $d = t \|\nabla_x f^*\|^{-1}_{L^\infty_{t,x,p}}$ if $\|\nabla_x f^*\|_{L^\infty_{t,x,p}} \geq 1$, otherwise $d = t \|\nabla_x f^*\|_{L^\infty_{t,x,p}}$, to find that (henceforth the constant $C^*(t)$ may change from line to line)

$$\begin{aligned} |\partial_k E_A^*(t,x)| &\leq C^*(t) \left(\log \frac{t}{d} + d \right) \\ &\leq C^*(t) \left(1 + \log^+ \sup_{0 \leq s \leq t} \|f^*(s)\|_{D^\infty_{x,p}} \right), \end{aligned} \tag{3.1.43}$$

where we have defined $\log^+ a = a$ for $0 \leq a \leq 1$ and $\log^+ a = 1 + \log a$ for $a \geq 1$.

On the other hand, the term $\partial_k E_B^*$ can be written as

$$\partial_k E_B^*(t,x) = \int_0^t \int_{|\omega|=1} \int_{|p| \leq \bar{P}^*_t} \hat{B}(v,\omega) K f^*(s, x+\omega(t-s), p) \omega^k dp d\omega ds.$$

Thus, by the boundedness of the kernel $\hat{B}$ on the compact support of f^*, we get

$$|\partial_k E_B^*(t,x)| \leq C^*(t) \int_0^t \left(\|E(s)\|_{L^\infty_x} + \|B(s)\|_{L^\infty_x} \right) \|f^*(s)\|_{L^\infty_{x,p}} ds, \tag{3.1.44}$$

which is the estimate for the term involving the kernel $\hat{B}$.

Lastly, we proceed to estimate $\partial_k E^*_C$, which we split in two integrals

$$\begin{aligned}\partial_k E^*_C(t,x) &= \int\int_{\Omega_t(x)\times\mathbb{R}^3} \hat{C}(v,\omega)\left[f^*\partial_k Q(t,y,p)\right]_{\mathtt{ret}} \frac{dpdy}{|y-x|} \\ &\quad + \int\int_{\Omega_t(x)\times\mathbb{R}^3} \hat{C}(v,\omega)\left[Q\partial_k f^*(t,y,p)\right]_{\mathtt{ret}} \frac{dpdy}{|y-x|}.\end{aligned}$$

Since $\|\partial_k Q(t)\|_{L^\infty_{x,p}} \leq 2\left(\|\partial_k E(t)\|_{L^\infty_x} + \|\partial_k B(t)\|_{L^\infty_x}\right)$, and the elements of $\hat{C}$ are bounded on the support of f^*, the first integral can be estimated by

$$C^*(t)\int_0^t \left(\|E(s)\|_{D^\infty_x} + \|B(s)\|_{D^\infty_x}\right)\|f^*(s)\|_{L^\infty_{x,p}}\, ds. \tag{3.1.45}$$

The second one, on the other hand, needs some extra care. Indeed, let us first recall the identity (3.1.29), which we rewrite here as

$$\left[\partial_k f^*(t,y,p)\right]_{\mathtt{ret}} = \left(\partial_k - \frac{\omega^k v\cdot\nabla_y}{1+v\cdot\omega}\right)\left[f^*(t,y,p)\right]_{\mathtt{ret}} - \frac{\omega^k}{1+v\cdot\omega}\nabla_p\cdot\left[Kf^*(t,y,p)\right]_{\mathtt{ret}}.$$

Hence, after using this identity and doing integration by parts, we obtain

$$\begin{aligned}&\int\int_{\Omega_t(x)\times\mathbb{R}^3} \hat{C}(v,\omega)\left[Q\partial_k f^*(t,y,p)\right]_{\mathtt{ret}} \frac{dpdy}{|y-x|} \\ &\quad = -\int\int_{\Omega_t(x)\times\mathbb{R}^3} \left\{\left(\partial_k - \frac{\omega^k v\cdot\nabla_y}{1+v\cdot\omega}\right)\frac{\hat{C}\left[Q\right]_{\mathtt{ret}}}{|y-x|}\right\}\left[f^*(t,y,p)\right]_{\mathtt{ret}} dpdy \\ &\quad\quad + \int\int_{\Omega_t(x)\times\mathbb{R}^3} \partial_p\left(\frac{\hat{C}\left[Q\right]_{\mathtt{ret}}}{1+v\cdot\omega}\right)\left[Kf^*(t,y,p)\right]_{\mathtt{ret}} \frac{\omega^k dpdy}{|y-x|}. \\ &\quad \equiv I + II.\end{aligned}$$

For the sake of simplicity we have not written the boundary terms as they depend on the derivatives of the Cauchy data and can be included in $(\partial_k E^*)_0$. Notice that the singularity r^{-1} does not produce a boundary term because of its small power.

Now, it is straightforward from (3.1.7) that $|\partial_p Q| \leq C(|E| + |B|)$. Also, it is not difficult to check that $\left|\partial_p(1+v\cdot\omega)^{-1}\hat{C}\right| \leq C(1+v\cdot\omega)^{-4}$, cf. definition of $\hat{C}$ in (3.1.27). Then, after applying the product rule and doing the usual change of variables we find that the second integral satisfies

$$|II| \leq C^*(t) \int_0^t \left(\|E(s)\|^2_{L^\infty_x} + \|B(s)\|^2_{L^\infty_x} \right) \|f^*(s)\|_{L^\infty_{x,p}} \, ds.$$

As for the first integral, recall that $\partial_k [Q]_{\mathtt{ret}} = [\partial_k Q]_{\mathtt{ret}} - \omega^k [\partial_t Q]_{\mathtt{ret}}$ in view of the identity (2.1.15). Hence, since the resulting singularity in I after differentiation is at most r^{-2}, we can directly estimate this integral to find that

$$|I| \leq C^*(t) \int_0^t \left(\|E(s)\|_{D^\infty_x} + \|B(s)\|_{D^\infty_x} \right) \|f^*(s)\|_{L^\infty_{x,p}} \, ds.$$

Combining the last two estimates with (3.1.45), we obtain

$$|\partial_k E^*_C(t,x)| \leq C^*(t) \int_0^t \Big(\|E(s)\|^2_{L^\infty_x} + \|B(s)\|^2_{L^\infty_x} + \|E(s)\|_{D^\infty_x} + \|B(s)\|_{D^\infty_x} \Big) \|f^*(s)\|_{L^\infty_{x,p}} \, ds. \quad (3.1.46)$$

Finally, estimating $(\partial_k E)_0$ and the term involving the kernel $\hat{S}^k$ in (3.1.41) are easy since the former depends on the derivatives of the Cauchy data, and the latter is dominated by $C^* \|f_0\|_{L^\infty_{x,p}}$ in view of $\|f^*(t)\|_{L^\infty_{x,p}} \leq \|f_0\|_{L^\infty_{x,p}}$. Hence, we gather (3.1.43), (3.1.44) and (3.1.46), and use the boundedness of f^* to find that

$$|\partial_k E^*(t,x)| \leq C^*(t) \Big[1 + \log^+ \sup_{0 \leq s \leq t} \|f^*(s)\|_{D^\infty_{x,p}} + \|E\|^2_{L^\infty_{t,x}} + \|B\|^2_{L^\infty_{t,x}} + \int_0^t \left(\|E(s)\|_{D^\infty_x} + \|B(s)\|_{D^\infty_x} \right) ds \Big]. \quad (3.1.47)$$

With the help of Theorem 2 and Theorem 1, it is not difficult to check that the same estimate holds for the space derivatives of the magnetic field B^*. Actually, it holds for both space and time derivatives of the electromagnetic field:

Lemma 11. *The field (E^*, B^*) satisfies the estimate*

$$\|E^*(t)\|_{D^\infty_x} + \|B^*(t)\|_{D^\infty_x} \leq C^*(t) \Big[1 + \log^+ \sup_{0 \leq s \leq t} \|f^*(s)\|_{D^\infty_{x,p}} + \|E\|^2_{L^\infty_{t,x}} + \|B\|^2_{L^\infty_{t,x}} + \int_0^t \left(\|E(s)\|_{D^\infty_x} + \|B(s)\|_{D^\infty_x} \right) ds \Big], \quad (3.1.48)$$

where for each t the positive $C^*(t) \equiv C(t; \bar{P}^*(t), \|f_0\|_{L^\infty_{x,p}}, \|E_0\|_{W^\infty_x}, \|B_0\|_{W^\infty_x})$ is con-

tinuous in its arguments.

Proof. We have already shown that the estimate (3.1.48) holds for all space derivatives of the field with $C^*(t)$ satisfying the stated properties. To prove that it also holds for the time derivatives of the field, we invoke the Maxwell equations (2.1.4)-(2.1.7). Since $|v| \leq 1$ so that $|j^*| \leq \rho^* = (4\pi)^{-1} \nabla \cdot E^*$, we have for each $t \in I$ and $x \in \mathbb{R}^3$

$$\begin{aligned} |\partial_t E^*(t,x)| &\leq |\nabla \times B^*(t,x)| + |\nabla \cdot E^*(t,x)| \\ |\partial_t B^*(t,x)| &\leq |\nabla \times E^*(t,x)| \,. \end{aligned}$$

Hence, since the time derivatives can be estimated by means of the space derivatives, the estimate (3.1.48) readily follows. □

3.2 Local existence

This section is devoted to prove the following result:

Theorem 3. *Let $f_0 \in C_0^1(\mathbb{R}^6)$, $f_0 \geq 0$, and let $E_0, B_0 \in C^2(\mathbb{R}^3)$ satisfy the constraints (2.4.4). Then, for some $T > 0$ there is a unique classical solution (f, E, B) of the RVM system on $[0, T[$ satisfying $(f, E, B)|_{t=0} = (f_0, E_0, B_0)$. Moreover, for each $0 \leq t < T$ the function $f(t)$ is non-negative and has compact support.*

In essence, the proof we present here follows the one by Glassey and Strauss in [9]. However, they did not explicitly show the existence of local solutions there. Their result basically states that, for any given time $T > 0$, if the momentum support of any sequence of approximative solutions (as defined below) is uniformly bounded, then a unique classical solution of the RVM system exists on $[0, T[$. Here, we show that there exists a finite time for which this condition holds, thus providing local existence and uniqueness of classical solutions.

The plan is to construct a sequence of approximations $\{f^n, E^n, B^n\}$ and to show that for some time $T > 0$ the momentum support of the f^n's are uniformly bounded. Then, all the estimates found in previous sections will hold on $\{f^n, E^n, B^n\}$ uniformly in n. As a result, we can show that this sequence is Cauchy in the C-uniform norm and in the C^1-uniform norm, on any subinterval of $[0, T[$. Hence, the limit exists which is the classical solution of the RVM system on $[0, T[$. For the sake of clarity, we shall present the proof in several steps or claims. Also, notice that henceforth all constants may change values from line to line.

Proof of Theorem 3. By density arguments, we can assume $f_0 \in C_0^\infty(\mathbb{R}^6)$, $f_0 \geq 0$ and $E_0, B_0 \in C^\infty \cap C_b(\mathbb{R}^3)$. Set $E_1 := \nabla \times B_0 - 4\pi \int_{\mathbb{R}^3} v f_0 dp$ and $B_1 := -\nabla \times E_0$. We construct the iterative scheme as follows. For $t \geq 0$ and $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$ define

$$f^0(t,x,p) = f_0(x,p), \quad E^0(t,x) = E_0(x), \quad B^0(t,x) = B_0(x).$$

For $n \in \mathbb{N}$, assume that $E^{n-1}, B^{n-1} \in C^2 \cap C_b(I \times \mathbb{R}^3; \mathbb{R}^3)$ are already given. Let $K^{n-1} := E^{n-1} + v \times B^{n-1}$ and denote by $Z_n := (X_n, P_n)(s,t,z)$ the solution of the characteristic system

$$\dot{X}_n(s,t,z) = v(P_n(s,t,z)) \tag{3.2.1}$$
$$\dot{P}_n(s,t,z) = K^{n-1}(t, X_n(s,t,z), P_n(s,t,z)), \tag{3.2.2}$$

with $Z_n(t,t,z) \equiv z$. We define the n-th iterate of the one-particle distribution function by

$$f^n(t,z) := f_0(Z_n(0,t,z)),$$

which in view of Lemma 3(a) and (c) is a C^2 solution of the (linear) initial value problem

$$\begin{cases} \partial_t f^n + v \cdot \nabla_x f^n + K^{n-1} \cdot \nabla_p f^n = 0 \\ f^n(0,z) = f_0(z). \end{cases} \tag{3.2.3}$$

Since E^{n-1} and B^{n-1} are both bounded functions, then f^n has compact support in the p variable as follows from (3.2.2). Therefore,

$$\rho^n(t,x) := \int_{\mathbb{R}^3} f^n(t,x,p)dp, \quad j^n(t,x) := \int_{\mathbb{R}^3} v f^n(t,x,p)dp$$

are well defined C^2 functions. Thus, integrating (3.2.3) over all $p \in \mathbb{R}^3$ yields the continuity equation

$$0 = \partial_t \rho^n + \nabla_x \cdot j^n + \int_{\mathbb{R}^3} \nabla_p \cdot \left(K^{n-1} f^n\right) = \partial_t \rho^n + \nabla_x \cdot j^n.$$

Hence, we define E^n and B^n as the solution of the Maxwell equations

$$\begin{aligned} \nabla \times B^n - \partial_t E^n &= 4\pi j^n \\ \nabla \times E^n + \partial_t B^n &= 0 \\ \nabla \cdot E^n &= 4\pi \rho^n \\ \nabla \cdot B^n &= 0, \end{aligned}$$

which as shown in Section 2.1 also solve

$$\Delta E^n - \partial_t^2 E^n = 4\pi \nabla \rho^n + 4\pi \partial_t j^n \tag{3.2.4}$$

$$\Delta B^n - \partial_t^2 B^n = -4\pi \nabla \times j^n \tag{3.2.5}$$

with the initial conditions (E_0, E_1) and (B_0, B_1) given above.

Step 1. The sequence $\{(f^n, E^n, B^n)\}$ is well defined. To see this, we prove that if f^n is a C^2 solution of (3.2.3) and (E^n, B^n) solves (3.2.4)-(3.2.5), then E^n and B^n are C^2 bounded functions. Notice that the solutions E^n and B^n are known to be C^1 in view of the Jefimenko representation (2.1.19) and (2.1.20) and the fact that ρ^n and j^n are C^2 functions. We prove that E^n and B^n are C^2 and bounded by induction.

Indeed, the field (E^0, B^0) is in $C^\infty \cap C_b$ as given by the Cauchy data. Now, assume that E^{n-1}, B^{n-1} are $C^2 \cap C_b$ functions and so is $Q^{n-1} = E^{n-1} + v \times B^{n-1} - v\,(v \cdot E^{n-1})$. The equation (3.2.2) implies that f^n has compact support in p. Hence, we can use the representation of the electric field in Theorem 1 with the iterative scheme abbreviated as $f^n = (f^{n-1})^*$, $E^n = (E^{n-1})^*$ and $B^n = (B^{n-1})^*$ to find that

$$\begin{aligned} E^n(t,x) = (E^n)_0\,(t,x) &- \int\int_{\Omega_t(x)\times\mathbb{R}^3} \mathcal{K}(v,\omega)\,[f^n(t,y,p)]_{\mathtt{ret}} \frac{dpdy}{|y-x|^2} \\ &+ \int\int_{\Omega_t(x)\times\mathbb{R}^3} \omega \times \left(\frac{\mathcal{K}(v,\omega)}{\sqrt{1-|v|^2}} \times \left[Q^{n-1} f^n(t,y,p)\right]_{\mathtt{ret}} \right) \frac{dpdy}{|y-x|}. \end{aligned}$$

For the given Cauchy data, the term $(E^n)_0\,(t,x)$ is $C^2 \cap C_b$, as follows from its definition (3.1.36). Also, the assumption made on E^{n-1} and B^{n-1} readily implies that the remaining terms in the above representation are $C^2 \cap C_b$ as well. Hence, the field E^n is $C^2 \cap C_b$ and since we can do similarly with B^n, the claim follows.

Step 2. Next, let $P_0(t,0,z) = P_n(0,0,z) \equiv p \in \mathtt{supp} f_0$ and for $n \in \mathbb{N}$ define

$$\begin{aligned}\bar{P}_n(t) &= \sup\left\{|p| : \exists 0 \le s \le t, x \in \mathbb{R}^3 : f^n(s,x,p) \neq 0\right\} \\ &\equiv \sup\left\{|P_n(s,0,z)| : 0 \le s \le t, z \in \mathtt{supp} f_0\right\},\end{aligned}$$

where the equivalence is a consequence of Lemma 3(c). Notice that $\bar{P}_n(t)$ is a non-decreasing function of t. We claim that for some $T > 0$ there is a non-negative function $\mathcal{P} \in C([0,T[;\mathbb{R})$ depending on the Cauchy data only such that

$$\bar{P}_n(t) \le \mathcal{P}(t) \quad \text{for all } n \in \mathbb{N}_0,\ 0 \le t < T. \tag{3.2.6}$$

In order to prove the claim, we first notice that for $n \in \mathbb{N}$, the equation (3.2.2) of the characteristics implies that

$$|P_{n+1}(t,x_0,p_0)| \le |p_0| + \int_0^t \left(\|E^n(s)\|_{L^\infty_x} + \|B^n(s)\|_{L^\infty_x}\right) ds. \tag{3.2.7}$$

Then, helped by the results presented in the previous section, we aim for a suitable estimate on the approximate field (E^n, B^n). To this end, define the n-th iterate of the kinetic energy density function as, cf. (3.1.38)

$$h^n(t,x) = \int_{\mathbb{R}^3} \sqrt{1+|p|^2} f^n(t,x,p)dp$$

In view of the support of f^n, this is a well defined C^2 function which satisfies

$$\begin{aligned}h^n(t,x) &\le \int_{|p|\le \bar{P}_n(t)} \sqrt{1+|p|^2} f^n(t,x,p)dp \\ &\le \frac{4\pi}{3} \|f^n\|_{L^\infty_{t,x,p}} \bar{P}_n^3(t)\sqrt{1+\bar{P}_n^2(t)} \\ &\le C\|f^n\|_{L^\infty_{t,x,p}} \left(1+\bar{P}_n^2(t)\right)^2.\end{aligned}$$

Hence, since by Lemma 3(c) we have $\|f^n\|_{L^\infty_{t,x,p}} = \|f_0\|_{L^\infty_{x,p}}$, $n \in \mathbb{N}$, then Lemma 10 implies

$$\begin{aligned} u^n(t) &\equiv 1 + \|E^n(t)\|_{L^\infty_x} + \|B^n(t)\|_{L^\infty_x} \\ &\leq C^0(t) + C^0(t) \int_0^t u^{n-1}(s) \, \|h^n(s)\|_{L^\infty_x} \, ds \\ &\leq C^0(t) + C^0(t) \int_0^t u^{n-1}(s) \left(1 + \bar{P}_n^2(s)\right)^2 ds, \end{aligned} \tag{3.2.8}$$

where $C^0(t)$ is a non-negative, continuous function of t, otherwise depending on the Cauchy data only. We now define the sequences

$$\tilde{u}^n(t) = \sup_{k \leq n} u^k(t), \quad \tilde{P}_n(t) = \sup_{k \leq n} \bar{P}_k(t).$$

Fix n and consider (3.2.8) for every $k \leq n$. If we take the supremum over $k \leq n$, we obtain the Gronwall's inequality

$$\tilde{u}^n(t) \leq C^0(t) + C^0(t) \int_0^t \tilde{u}^n(s)(1 + \tilde{P}_n^2(s))^2 ds.$$

It follows that

$$\tilde{u}^n(t) \leq C^0(t) \exp\left\{ C^0(t) \int_0^t (1 + \tilde{P}_n^2(s))^2 ds \right\}. \tag{3.2.9}$$

But (3.2.9) dominates the integrand of (3.2.7), irrespective of the characteristic curve chosen. Then, we may take the supremum over all characteristics to find that

$$\begin{aligned} \tilde{P}_{n+1}(t) &\leq \bar{P}_0 + \int_0^t \tilde{u}^n(s) ds \\ &\leq \bar{P}_0 + \int_0^t C^0(s) \exp\left\{ C^0(s)(1 + \tilde{P}_n^2(s))^2 \right\} ds. \end{aligned} \tag{3.2.10}$$

Here we have used that $\tilde{P}_n(t)$ is a non-decreasing function of t and the value of $C^0(t)$ for each t have changed from the previous line. Now, let $T > 0$ be the life span of the solution of the integral equation

$$\mathcal{P}(t) = \bar{P}_0 + \int_0^t C^0(s) \exp\left\{ C^0(s)(1 + \mathcal{P}^2(s))^2 \right\} ds. \tag{3.2.11}$$

Clearly, $\tilde{P}_0(t) \equiv \bar{P}_0(t) \leq \mathcal{P}(t)$. Suppose that $\tilde{P}_n(t) \leq \mathcal{P}(t)$ for some $n \in \mathbb{N}$. Then, if we write $\tilde{P}_n(t)$ instead of $\bar{P}_n(t)$ in (3.2.6), the claim holds by induction due to

(3.2.10). Therefore, we have $\bar{P}_n(t) \leq \tilde{P}_n(t) \leq \mathcal{P}(t)$, $0 \leq t < T$, uniformly in n. The function $\mathcal{P}(t)$ depends via $C^0(t) = C(t; \bar{P}_0, \|f_0\|_{L^\infty_{x,p}}, \|E_0\|_{W^\infty_x}, \|B_0\|_{W^\infty_x})$ on the initial data only. We emphasize that $C^0(t)$ is a non-negative continuous function of its arguments, essentially given by (3.1.37).

Step 3. Fix $0 < \bar{T} < T$. In view of (3.2.9), for all $0 \leq t \leq \bar{T}$ we define

$$u(t) = C^0(t) \exp\left\{ C^0(t) \int_0^t (1 + \mathcal{P}^2(s))^2 ds \right\},$$

which is non-negative and continuous on $[0, \bar{T}]$. We further define

$$\bar{P}_{\bar{T}} = \sup_{0 \leq t \leq \bar{T}} \mathcal{P}(t), \qquad u_{\bar{T}} = \sup_{0 \leq t \leq \bar{T}} u(t). \tag{3.2.12}$$

Notice that $\bar{P}_{\bar{T}} < \infty$ and $u_{\bar{T}} < \infty$ depend on $\bar{T}$ and the Cauchy data only. In particular, $u_{\bar{T}}$ does *not* depend on $t \in [0, \bar{T}]$. This fact will hold true along the proof although the actual value of $u_{\bar{T}}$ may change from line to line.

Now, by the definition of the sequence $u^n(t)$ provided in (3.2.8), the uniform bound $u^n(t) \leq u_{\bar{T}}$ readily follows. Thus, since $\|f^n(t)\|_{L^\infty_{x,p}} = \|f_0\|_{L^\infty_{x,p}}$, we conclude that

$$\|f^n(t)\|_{L^\infty_{x,p}} + \|E^n(t)\|_{L^\infty_x} + \|B^n(t)\|_{L^\infty_x} \leq u_{\bar{T}}, \quad n \in \mathbb{N}_0,\ 0 \leq t \leq \bar{T}. \tag{3.2.13}$$

Furthermore, we claim that there is a constant $u_{1\bar{T}}$ also depending on $\bar{T}$ and the Cauchy data only, such that

$$\|f^n(t)\|_{D^\infty_{x,p}} + \|E^n(t)\|_{D^\infty_x} + \|B^n(t)\|_{D^\infty_x} \leq u_{1\bar{T}}, \quad n \in \mathbb{N}_0,\ 0 \leq t \leq \bar{T}. \tag{3.2.14}$$

To proceed, we first take the time and space partial derivatives $\partial_{(t,x)}$ to the Vlasov equation (3.2.3), namely

$$\begin{aligned}\left[\partial_t + v \cdot \nabla_x + (E^{n-1} + v \times B^{n-1}) \cdot \nabla_p\right] \partial_{(t,x)} f^n \\ = -\left[\partial_{(t,x)} E^{n-1} + v \times \partial_{(t,x)} B^{n-1}\right] \cdot \nabla_p f^n.\end{aligned} \tag{3.2.15}$$

Thus, on each characteristic curve solution of (3.2.1)-(3.2.2), we have

$$\begin{aligned}\left|\partial_{(t,x)} f^n(t)\right| \;\leq\; & \left\|\partial_{(t,x)} f^n(0)\right\|_{L^\infty_{x,p}} \\ & + \int_0^t \left(\left\|E^{n-1}(s)\right\|_{D^\infty_x} + \left\|B^{n-1}(s)\right\|_{D^\infty_x} \right) \|f^n(s)\|_{D^\infty_{x,p}}\, ds.\end{aligned} \tag{3.2.16}$$

Similarly, if we apply the momentum partial derivative ∂_p to (3.2.3) we get

$$\left[\partial_t + v\cdot\nabla_x + (E^{n-1} + v\times B^{n-1})\cdot\nabla_p\right]\partial_p f^n \\ = -\partial_p v\cdot\nabla_x f^n - \partial_p v\times B^{n-1}\cdot\nabla_p f^n. \tag{3.2.17}$$

Hence, since $v\in C_b^\infty(\mathbb{R}^3;\mathbb{R}^3)$ so that $|\partial_p v|\le C$, we have on each characteristic curve that solves (3.2.1)-(3.2.2)

$$|\partial_p f^n(t)| \le \|\partial_p f^n(0)\|_{L^\infty_{x,p}} + 2\int_0^t \left(1+\left\|B^{n-1}(s)\right\|_{L^\infty_x}\right)\|f^n(s)\|_{D^\infty_{x,p}}\,ds. \tag{3.2.18}$$

Now we gather the estimates (3.2.16) and (3.2.18) for the time, space and momentum partial derivatives of f^n. Since the right-hand side of the resultant inequality does not depend on the characteristic curve chosen, we find that

$$\|f^n(t)\|_{D^\infty_{x,p}} \le C^0 + C\int_0^t \left(\left\|E^{n-1}(s)\right\|_{D^\infty_x} + \left\|B^{n-1}(s)\right\|_{D^\infty_x}\right)\|f^n(s)\|_{D^\infty_{x,p}}\,ds.$$

Therefore, the Gronwall's lemma applies and we obtain

$$\|f^n(t)\|_{D^\infty_{x,p}} \le C^0\exp\left\{C\int_0^t\left(\left\|E^{n-1}(s)\right\|_{D^\infty_x} + \left\|B^{n-1}(s)\right\|_{D^\infty_x}\right)ds\right\}. \tag{3.2.19}$$

At this point, we invoke Lemma 11 in the previous section. In terms of the iteration scheme, and by virtue of (3.2.13), this lemma states that

$$\|E^n(t)\|_{D^\infty_x} + \|B^n(t)\|_{D^\infty_x} \le C^n(t)\left[1+\log^+\sup_{0\le s\le t}\|f^n(s)\|_{D^\infty_{x,p}} + u^2_{\bar{T}} \right.\\ \left. + \int_0^t\left(\left\|E^{n-1}(s)\right\|_{D^\infty_x} + \left\|B^{n-1}(s)\right\|_{D^\infty_x}\right)ds\right],$$

where $C^n(t)$ is a non-negative continuous function of t, which for each $0\le t\le\bar{T}$ is continuous with respect to $\bar{P}_n(t)$ and depends otherwise on the Cauchy data only.

But as it follows from (3.2.6) in Step 2 and the definition of $\bar{P}_{\bar{T}}$ in (3.2.12), for all $p\in\mathtt{supp}f^n(t)$ we have $|p|\le\bar{P}_n(t)\le\bar{P}_{\bar{T}}$ uniformly in $n\in\mathbb{N}_0$, $0\le t\le\bar{T}$. Thus, there is a non-negative constant $C^0_{\bar{T}}$ depending on $\bar{T}$ and the Cauchy data only, such that $C^n(t)\le C^0_{\bar{T}}$ for all $n\in\mathbb{N}_0$, $0\le t\le\bar{T}$. Hence, since $\log^+ s\le\max\{1, 1+\log s\}$, we can combine the last two inequalities to find that

$$\|E^n(t)\|_{D_x^\infty} + \|B^n(t)\|_{D_x^\infty} \le C_{\bar{T}}^0 + C_{\bar{T}}^0 \int_0^t \left(\left\|E^{n-1}(s)\right\|_{D_x^\infty} + \left\|B^{n-1}(s)\right\|_{D_x^\infty} \right) ds.$$

If we now iterate this inequality we have (recall that $C_{\bar{T}}^0$ may change from line to line)

$$\|E^n(t)\|_{D_x^\infty} + \|B^n(t)\|_{D_x^\infty} \le C_{\bar{T}}^0 \left(1 + C_{\bar{T}}^0 t + \cdots + \frac{\left(C_{\bar{T}}^0 t\right)^n}{n!} \right) \le C_{\bar{T}}^0 e^{C_{\bar{T}}^0 \bar{T}},$$

which in turn provides a bound on (3.2.19). It is now clear that there is a constant $u_{1\bar{T}}$ such that the uniform bound (3.2.14) indeed holds.

Step 4. We claim that $\{(f^n, E^n, B^n)\}$ is Cauchy in the C-uniform norm. Indeed, for $m, n \in \mathbb{N}_0$ abbreviate

$$f^{m,n} = f^m - f^n, \quad E^{m,n} = E^m - E^n, \quad B^{m,n} = B^m - B^n, \tag{3.2.20}$$

with $K^{m,n} = K^m - K^n \equiv E^{m,n} + v \times B^{m,n}$ and $Q^{m,n} = K^{m,n} - v\,(v \cdot E^{m,n})$. Also, let

$$\bar{f}^{m,n}(t) = \|f^{m,n}(t)\|_{L_{x,p}^\infty}, \; \bar{E}^{m,n}(t) = \|E^{m,n}(t)\|_{L_x^\infty}, \; \bar{B}^{m,n}(t) = \|B^{m,n}(t)\|_{L_x^\infty},$$

and $\bar{K}^{m,n}(t) = \|E^{m,n}(t)\|_{L_x^\infty} + \|B^{m,n}(t)\|_{L_x^\infty}$. Clearly, $\bar{f}^{m,n}(0) = \bar{f}^{00}(t) \equiv 0$ and the same holds for the iterates of the field.

Since $Q^{m-1} f^m - Q^{n-1} f^n = Q^{m-1,n-1} f^m + Q^{n-1} f^{m,n}$, the representation (3.1.19) of the electric field implies

$$\begin{aligned} E^{m,n}(t,x) = &- \int\int_{\Omega_t(x) \times \mathbb{R}^3} \mathcal{K}(v,\omega) \left[f^{m,n}(t,y,p)\right]_{\mathtt{ret}} \frac{dpdy}{r^2} \\ &+ \int\int_{\Omega_t(x) \times \mathbb{R}^3} \omega \times \left(\frac{\mathcal{K}(v,\omega)}{\sqrt{1-|v|^2}} \times \left[Q^{m-1,n-1} f^m(t,y,p)\right]_{\mathtt{ret}} \right) \frac{dpdy}{r} \\ &+ \int\int_{\Omega_t(x) \times \mathbb{R}^3} \omega \times \left(\frac{\mathcal{K}(v,\omega)}{\sqrt{1-|v|^2}} \times \left[Q^{n-1} f^{m,n}(t,y,p)\right]_{\mathtt{ret}} \right) \frac{dpdy}{r}. \end{aligned}$$

We now apply the Corollary 2 of Lemma 9 to the above representation to find that

$$\begin{aligned}|E^{m,n}(t,x)| \leq &\int_0^t \int_{|\omega|=1} \int_{|p|\leq \bar{P}_{\bar{T}}} \sqrt{1+|p|^2} f^{m,n} dpd\omega ds \\ &+C\int_0^t (t-s) \int_{|\omega|=1} \int_{|p|\leq \bar{P}_{\bar{T}}} \sqrt{1+|p|^2} Q^{m-1,n-1} f^m dpd\omega ds \\ &+C\int_0^t (t-s) \int_{|\omega|=1} \int_{|p|\leq \bar{P}_{\bar{T}}} \sqrt{1+|p|^2} Q^{n-1} f^{m,n} dpd\omega ds,\end{aligned}$$

where the integrands are evaluated at $(s, x+\omega(t-s), p)$. The same estimate holds for each $B^{m,n}$. Hence, in view of the uniform estimate (3.2.13), and by noticing that $Q^{m,n} \leq C\bar{K}^{m,n}$, it is easy to check that

$$\bar{K}^{m,n}(t,x) \leq C_{\bar{T}} \int_0^t \left(\bar{K}^{m-1,n-1}(s) + \bar{f}^{m,n}(s)\right) ds. \tag{3.2.21}$$

On the other hand, by subtracting the Vlasov equations (3.2.3) corresponding to the index m and n respectively, we obtain that

$$\partial_t f^{m,n} + v\cdot\nabla_x f^{m,n} + K^{n-1}\cdot\nabla_p f^{m,n} = -K^{m-1,n-1}\cdot\nabla_p f^m.$$

The structure of this equation implies that along the solutions of the characteristic system (3.2.1)-(3.2.2) we have

$$\dot{f}^{m,n} = -\left(K^{m-1,n-1}\cdot\nabla_p f^m\right)(s, Z_n(s,t,z)).$$

We then estimate the right-hand side of this equation along the characteristic curves. Hence, since the resultant estimate does not depend on the characteristic curve chosen and since $f^{m,n}(0, Z_n(0,t,z)) \equiv 0$, we find that

$$\bar{f}^{m,n}(t) \leq C_{\bar{T}} \int_0^t \bar{K}^{m-1,n-1}(s) ds. \tag{3.2.22}$$

Here we have used that $\|\nabla_p f^n(t)\|_{L^\infty_{x,p}} \leq u_{1\bar{T}}$ for all $n \in \mathbb{N}_0$ and $0 \leq t \leq \bar{T}$ as it follows from (3.2.14). With (3.2.22), we can now estimate (3.2.21) by noticing that

$$\begin{aligned}\int_0^t \int_0^s \bar{K}^{m-1,n-1}(\sigma)d\sigma ds &= \int_0^t \int_\sigma^t \bar{K}^{m-1,n-1}(\sigma)dsd\sigma \\ &= \int_0^t (t-\sigma)\bar{K}^{m-1,n-1}(\sigma)d\sigma \\ &\leq C_{\bar{T}} \int_0^t \bar{K}^{m-1,n-1}(\sigma)d\sigma.\end{aligned} \tag{3.2.23}$$

Therefore,

$$\bar{K}^{m,n}(t) \leq C_{\bar{T}} \int_0^t \bar{K}^{m-1,n-1}(s)ds. \tag{3.2.24}$$

Bearing the equality (3.2.23) in mind, we then iterate the above relation to find

$$\begin{aligned}\bar{K}^{m,n}(t) &\leq C_{\bar{T}}^2 \int_0^t (t-\sigma)\bar{K}^{m-2,n-2}(\sigma)d\sigma \\ &\leq \cdots \\ &\leq C_{\bar{T}}^k \int_0^t \frac{(t-\sigma)^{k-1}}{(k-1)!}\bar{K}^{m-k,n-k}(\sigma)d\sigma \\ &\leq u_{\bar{T}} C_{\bar{T}}^k \frac{\bar{T}^k}{k!}\end{aligned} \tag{3.2.25}$$

for $m \geq k$ and $n \geq k$. Notice that in the last inequality we have used the uniform bound $\bar{K}^{m,n} \leq u_{\bar{T}}$ which is consequence of the triangle inequality and (3.2.14). It follows that $\bar{K}^{m,n} \to 0$ as $m, n \to \infty$, uniformly on $[0, \bar{T}]$. Hence, the iterates $\{(E^n, B^n)\}$ are Cauchy sequences in the uniform norm, and so is $\{f^n\}$ in view of (3.2.22).

Step 5. Moreover, we claim that $\{(f^n, E^n, B^n)\}$ is Cauchy in the C^1-uniform norm. To proceed, we further abbreviate

$$\bar{f}_1^{m,n}(t) = \|f^{m,n}(t)\|_{D_{x,p}^\infty},\ \bar{E}_1^{m,n}(t) = \|E^{m,n}(t)\|_{D_x^\infty},\ \bar{B}_1^{m,n}(t) = \|B^{m,n}(t)\|_{D_x^\infty}$$

and let $\bar{K}_1^{m,n}(t) = \|K^{m,n}(t)\|_{D_{x,p}^\infty}$.

Now, for the indexes m and n, consider the representation (3.1.25) of $\partial_x E^m$ and $\partial_x E^n$ respectively. We write their difference $\partial_x E^{m,n}$ as, cf. (3.1.41)

$$\begin{aligned}\partial_x E^{m,n}(t,x) = \int_{\mathbb{R}^3} \hat{S}(v) f^{m,n}(t,x,p)dp \\ +\partial_x E_A^{m,n}(t,x) + \partial_x E_B^{m,n}(t,x) + \partial_x E_C^{m,n}(t,x),\end{aligned}$$

where the last three terms in the right-hand side correspond to those with kernels $\hat{A}$, $\hat{B}$ and $\hat{C}$ respectively. We estimate one at a time. Notice that estimating the integral with kernel $\hat{S}$ is straightforward from (3.2.22).

To bound the term $\partial_x E_A^{m,n}$, write it as in (3.1.42) and use the property (3.1.26) of kernel $\hat{A}$. It is then immediate that

$$|\partial_x E_A^{m,n}(t,x)| \leq C_{\bar{T}} \int_0^t \bar{f}_1^{m,n}(s) ds.$$

To bound $\partial_x E_B^{m,n}$, recall that $K^{m-1} f^m - K^{n-1} f^n = K^{m-1,n-1} f^m + K^{n-1} f^{m,n}$. Hence, exactly as we did for (3.1.44), and by using the uniform estimate (3.2.14) on the sequence of solutions, we find that

$$|\partial_x E_B^{m,n}(t,x)| \leq C_{\bar{T}} \int_0^t \left(\bar{K}^{m-1,n-1}(s) + \bar{f}^{m,n}(s)\right) ds.$$

Finally, the same steps used to obtain (3.1.46) combined with the uniform bound (3.2.14) on the iterates yield

$$|\partial_x E_C^{m,n}(t,x)| \leq C_{\bar{T}} \int_0^t \left(\bar{K}_1^{m-1,n-1}(s) + \bar{f}^{m,n}(s)\right) ds.$$

Collecting terms, we have the bound on $\partial_x E^{m,n}$. As usual, the bound for $\partial_x B^{m,n}$ follows suit. Then, we use the linearity of the Maxwell equations and we reason as in the proof of Lemma 11 to find that

$$\begin{aligned}
&|\partial_t E^{m,n}(t,x)| + |\partial_t B^{m,n}(t,x)| \\
&\qquad \leq C_{\bar{T}} \left(|f^{m,n}(t,x)| + |\nabla \times E^{m,n}(t,x)| + |\nabla \times B^{m,n}(t,x)|\right) \\
&\qquad \leq C_{\bar{T}} \int_0^t \left(\bar{K}_1^{m-1,n-1}(s) + \bar{f}_1^{m,n}(s)\right) ds.
\end{aligned}$$

Notice in the last inequality that we have used (3.2.22) as well. Hence, we gather these estimates on the time and space partial derivatives of the field and take the supremum over all $x \in \mathbb{R}^3$ to find that

$$\bar{K}_1^{m,n}(t) \leq C_{\bar{T}} \int_0^t \left(\bar{K}_1^{m-1,n-1}(s) + \bar{f}_1^{m,n}(s)\right) ds. \tag{3.2.26}$$

On the other hand, to estimate $\bar{f}_1^{m,n}$, we first invoke (3.2.15) which we rewrite here for the sake of convenience,

$$\left[\partial_t + v \cdot \nabla_x + K^{n-1} \cdot \nabla_p\right] \partial_{(t,x)} f^n = -\partial_{(t,x)} K^{n-1} \cdot \nabla_p f^n.$$

The associated characteristic system reads, cf. (3.2.1)-(3.2.2)

$$\dot{Z}_n = \left(v, K^{n-1}\right)(s, Z_n(s,t,z)) \tag{3.2.27}$$

$$\partial_{(t,x)} \dot{f}^n = -(\partial_{(t,x)} K^{n-1} \cdot \nabla_p f^n)(s, Z_n(s,t,z)), \tag{3.2.28}$$

with $Z_n(t,t,z) \equiv z$. To simplify notation, we subsequently drop the dependence on the initial positions and momenta unless we specify otherwise. Now, consider (3.2.27) with the index m instead. The difference of the two implies that

$$\begin{aligned}
&|Z_m(s) - Z_n(s)| \\
&\quad \leq \int_s^t \left|\left(v, K^{m-1}\right)(\sigma, Z_m(\sigma)) - \left(v, K^{n-1}\right)(\sigma, Z_n(\sigma))\right| d\sigma \\
&\quad \leq \int_s^t \Big(\left|\left(v, K^{m-1}\right)(\sigma, Z_m(\sigma)) - \left(v, K^{m-1}\right)(\sigma, Z_n(\sigma))\right| \\
&\qquad\qquad + \left|\left(v, K^{m-1}\right)(\sigma, Z_n(\sigma)) - \left(v, K^{n-1}\right)(\sigma, Z_n(\sigma))\right|\Big) d\sigma \\
&\quad \leq C_{\bar{T}} \int_s^t \left(|Z_m(\sigma) - Z_n(\sigma)| + \left|K^{m-1}(\sigma, Z_n(\sigma)) - K^{n-1}(\sigma, Z_n(\sigma))\right|\right) d\sigma
\end{aligned}$$

where the first term in the last inequality follows since the relativistic velocity v is by definition a C^1 function with bounded first derivative, and the field K^{m-1} has uniformly bounded C^1-norm in view of (3.2.14). The second term is obvious. Now, as it was shown in Step 4, the sequence $\{K^n\}$ is Cauchy in the uniform norm, thus

$$|Z_m(s) - Z_n(s)| \leq \delta_{\bar{T}}^{m,n} + C_{\bar{T}} \int_0^t |Z_m(\sigma) - Z_n(\sigma)| \, d\sigma$$

for some $\delta_{\bar{T}}^{m,n} \to 0$ uniformly on $[0, \bar{T}]$ as $m, n \to \infty$. By taking the supremum over the initial positions and momenta, and applying the Gronwall's lemma, we find that

$$|Z_m(s,t,z) - Z_n(s,t,z)| \to 0, \quad \text{uniformly on } [0,\bar{T}] \times [0,\bar{T}] \times \mathbb{R}^6 \tag{3.2.29}$$

as $m, n \to \infty$. Now, the differential equation (3.2.28) in integral form reads

$$\partial_{(t,x)} f^n(t,z) = \partial_{(t,x)} f_0(Z_n(0)) - \int_0^t (\partial_{(t,x)} K^{n-1} \cdot \nabla_p f^n)(s, Z_n(s)) ds.$$

Hence, by taking the difference between this equation and the same one but with index m, we find that

$$\begin{aligned} \left|\partial_{(t,x)} f^m(t,z) - \partial_{(t,x)} f^n(t,z)\right| &\leq \left|\partial_{(t,x)} f_0(Z_m(0)) - \partial_{(t,x)} f_0(Z_n(0))\right| \\ &+ \int_0^t \left|(\partial_{(t,x)} K^{m-1} \cdot \nabla_p f^m)(s, Z_m(s)) - (\partial_{(t,x)} K^{n-1} \cdot \nabla_p f^n)(s, Z_n(s))\right| ds. \end{aligned}$$

In view of the assumptions on the Cauchy data, the first term in the right-hand side converges to zero as $m, n \to 0$. As for the second term, it can be estimated by

$$\begin{aligned} &\int_0^t \left|\partial_{(t,x)} K^{m-1}(s, Z_m(s))\right| \left|\nabla_p f^m(s, Z_m(s)) - \nabla_p f^m(s, Z_n(s))\right| ds \\ &\quad + \int_0^t \left|\nabla_p f^m(s, Z_n(s))\right| \left|\partial_{(t,x)} K^{m-1}(s, Z_m(s)) - \partial_{(t,x)} K^{n-1}(s, Z_m(s))\right| ds \\ &\quad + \int_0^t \left|\partial_{(t,x)} K^{n-1}(s, Z_m(s))\right| \left|\nabla_p f^m(s, Z_n(s)) - \nabla_p f^n(s, Z_n(s))\right| ds \\ &\quad + \int_0^t \left|\nabla_p f^n(s, Z_n(s))\right| \left|\partial_{(t,x)} K^{n-1}(s, Z_m(s)) - \partial_{(t,x)} K^{n-1}(s, Z_n(s))\right| ds \\ &\leq \epsilon_{\bar{T}}^{m,n} + C_{\bar{T}} \int_0^t \left(\left|\partial_{(t,x)} K^{m-1,n-1}(s, Z_m(s))\right| + \left|\nabla_p f^{m,n}(s, Z_n(s))\right|\right) ds \\ &\leq \epsilon_{\bar{T}}^{m,n} + C_{\bar{T}} \int_0^t \left(\bar{K}_1^{m-1,n-1}(s) + \bar{f}_1^{m,n}(s)\right) ds \end{aligned} \tag{3.2.30}$$

with $\epsilon_{\bar{T}}^{m,n} \to 0$ as $m, n \to \infty$. We justify the first inequality as follows. Lines 1 and 4 in the left-hand side are bounded by $\epsilon_{\bar{T}}^{m,n}$ in view of the uniform bound on the derivative of the functions given in (3.2.14), and the continuity of the derivatives of the functions and (3.2.29). The estimate for lines 2 and 3, on the other hand, follows from the uniform bound (3.2.14) and the definitions (3.2.20). Hence, it is clear that

$$\left|\partial_{(t,x)} f^m(t,z) - \partial_{(t,x)} f^n(t,z)\right| \leq \epsilon_{\bar{T}}^{m,n} + C_{\bar{T}} \int_0^t \left(\bar{K}_1^{m-1,n-1}(s) + \bar{f}_1^{m,n}(s)\right) ds$$

for some other $\epsilon_{\bar{T}}^{m,n} \to 0$ as $m, n \to \infty$. We can now take the supremum of the left-hand side over all $z \in \mathbb{R}^6$. Hence, since as a consequence of (3.2.17) we essentially have the same estimate for derivatives of the type ∂_p, we find that

$$\bar{f}_1^{m,n}(t) \leq \epsilon_{\bar{T}}^{m,n} + C_{\bar{T}} \int_0^t \left(\bar{K}_1^{m-1,n-1}(s) + \bar{f}_1^{m,n}(s)\right) ds$$

with yet another $\epsilon_{\bar{T}}^{m,n} \to 0$ uniformly on $[0, \bar{T}]$ as $m, n \to \infty$. Gronwall's lemma then implies (below, both $C_{\bar{T}}$ and $\epsilon_{\bar{T}}^{m,n}$ keep changing from line to line)

$$\bar{f}_1^{m,n}(t) \leq \epsilon_{\bar{T}}^{m,n} + C_{\bar{T}} \int_0^t \bar{K}_1^{m-1,n-1}(s) ds, \tag{3.2.31}$$

which combined with (3.2.26) gives

$$\begin{aligned} \bar{K}_1^{m,n}(t) &\leq \epsilon_{\bar{T}}^{m,n} + C_{\bar{T}} \int_0^t \bar{K}_1^{m-1,n-1}(s) ds + C_{\bar{T}} \int_0^t \int_0^s \bar{K}_1^{m-1,n-1}(\sigma) d\sigma ds \\ &\leq \epsilon_{\bar{T}}^{m,n} + C_{\bar{T}} \int_0^t \bar{K}_1^{m-1,n-1}(s) ds. \end{aligned}$$

Notice the use of (3.2.23) to obtain the last inequality. The triangle inequality and the uniform bound (3.2.14) imply that $\bar{K}_1^{m,n}(t) \leq u_{1\bar{T}}$ uniformly in $m, n \in \mathbb{N}_0$ and $0 \leq t \leq \bar{T}$. Then, iteration similar to that in (3.2.25) implies that

$$\begin{aligned} \bar{K}_1^{m,n}(t) &\leq \epsilon_{\bar{T}}^{m,n} (1 + C_{\bar{T}} t) + C_{\bar{T}}^2 \int_0^t (t-\sigma) \bar{K}_1^{m-2,n-2}(\sigma) d\sigma \\ &\leq \cdots \\ &\leq \epsilon_{\bar{T}}^{m,n} \sum_{k=1}^{l} \frac{C_{\bar{T}}^{k-1} t^{k-1}}{(k-1)!} + C_{\bar{T}}^l \int_0^t \frac{(t-\sigma)^{l-1}}{(l-1)!} \bar{K}_1^{m-l,n-l}(\sigma) d\sigma \\ &\leq \epsilon_{\bar{T}}^{m,n} e^{C_{\bar{T}} \bar{T}} + u_{1\bar{T}} C_{\bar{T}}^l \frac{\bar{T}^l}{l!} \end{aligned}$$

for $m \geq l$ and $n \geq l$ and $\epsilon_{\bar{T}}^{m,n} \to 0$ uniformly on $[0, \bar{T}]$ as $m, n \to \infty$. It follows that the sequence $\{(E^n, B^n)\}$ is Cauchy in the C^1 uniform norm, and so is $\{f^n\}$ in view of the estimate (3.2.31).

Step 6. Hence, there exist $f \in C^1([0, \bar{T}] \times \mathbb{R}^6; \mathbb{R})$ and $E, B \in C^1([0, \bar{T}] \times \mathbb{R}^3; \mathbb{R}^3)$ such that $f^n \to f$ uniformly on $[0, \bar{T}] \times \mathbb{R}^6$ and $E^n, B^n \to E, B$ uniformly on $[0, \bar{T}] \times \mathbb{R}^3$ as well as their derivatives. It is an easy matter to check that the limit (f, E, B) satisfies the Vlasov equation and the second order form of the Maxwell equations in the sense of distribution. Since this is true on any subinterval $[0, \bar{T}]$ of $[0, T[$, then

(f, E, B) exists on $[0, T[$. Clearly, $f \geq 0$ on $[0, T[$ and for each $0 \leq t < T$

$$\mathtt{supp} f(t) \subseteq \left\{ (x, p) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| \leq \mathcal{X}(t), |p| \leq \mathcal{P}(t) \right\},$$

where $\mathcal{P}(t) < \infty$ was already found in Step 2 and, as a result of (3.2.1) and $|v| \leq 1$, we have defined $\mathcal{X}(t) := t + \sup\{|x| : \exists p \in \mathbb{R}^3 : f_0(x, p) \neq 0\} < \infty$. Moreover, (E, B) satisfies the representation (3.1.19)-(3.1.20) on $[0, T[$. In particular, Remark 5 adapted to (f, E, B) implies that the electromagnetic field is bounded on $\bar{J} \times \mathbb{R}^3$ for every compact subinterval $\bar{J} \subset I$. Thus, we have established the existence of local classical solutions of the RVM system.

Step 7. Regarding uniqueness, we let (f_1, E_1, B_1) and (f_2, E_2, B_2) be two such solutions on $[0, T[$ with the same Cauchy data. These solutions satisfy the representations and estimates of Section 3.1. Also, the estimates (3.2.13) and (3.2.14) hold for these solutions as well. We define

$$f = f_1 - f_2, \quad E = E_1 - E_2, \quad B = B_1 - B_2.$$

Exactly as we obtained for indexes m and n the estimates (3.2.21) and (3.2.22) in Step 4, we get here for both solutions that on any subinterval $[0, \bar{T}]$ of $[0, T[$

$$\|E(t)\|_{L^\infty_x} + \|B(t)\|_{L^\infty_x} \leq c_{\bar{T}} \int_0^t \left(\|E(s)\|_{L^\infty_x} + \|B(s)\|_{L^\infty_x} + \|f(s)\|_{L^\infty_{x,p}} \right) ds$$

and

$$\|f(t)\|_{L^\infty_x} \leq c_{\bar{T}} \int_0^t \left(\|E(s)\|_{L^\infty_x} + \|B(s)\|_{L^\infty_x} \right) ds.$$

Hence, we combine these two inequalities and use the Gronwall's lemma to find that for any $0 \leq \bar{T} < T$, $f_1 = f_2$ on $[0, \bar{T}] \times \mathbb{R}^6$, and $E_1 = E_2$ and $B_1 = B_2$ on $[0, \bar{T}] \times \mathbb{R}^3$, which proves uniqueness. This concludes the proof of the Theorem 3. □

3.3 Criteria for continuation of solutions

In this section, we show that a classical solution (f, E, B) with smooth, compactly supported initial data becomes singular only if the charge density ρ blows-up in finite time. That is, a local solution can be extended globally in time provided the charge density remains bounded. Hence, we conclude that *no* break-down could occur due to shock formations since, in this case scenario, the solution itself would remain bounded while its derivative blows-up. In order to prove this result, we show that the size of the momentum support of f is controlled as long as the charge density remains bounded. Hence, we invoke the celebrated result by Glassey and Strauss in [9]: if the solution were to develop a singularity, then the size of the momentum support of f would become infinite in the same time.

Recently, C. Pallard [21] has shown that a break-down of the solution implies a blow-up of a range of non-zero moments of f in the momentum variable. Here, what we prove is that the 0-moment limit case also holds true, which corresponds to the boundedness of the charge density ρ. In order to be systematic, we state his result in Theorem 6 below and we recall the simpler 1-moment case (in Theorem 5), given earlier by Glassey and Strauss in [20].

We start by introducing some notation. Let (f, E, B) be a C^1 solution to the RVM system with life span T. We recall that the characteristic flow associated to the Vlasov equation (2.4.1) are the solutions $X = X(t, x_0, p_0)$, $P = P(t, x_0, p_0)$ of the ordinary differential system

$$\dot{X}(t, x_0, p_0) = v(P(t, x_0, p_0)) \tag{3.3.1}$$

$$\dot{P}(t, x_0, p_0) = K(t, X(t, x_0, p_0), P(t, x_0, p_0)), \tag{3.3.2}$$

satisfying the initial condition $(X(0, x_0, p_0), P(0, x_0, p_0)) \equiv (x_0, p_0) \in \mathrm{supp} f_0$. To ease notation, we subsequently drop the explicit dependence on the initial positions and momenta, unless we specify otherwise. Thus, we denote the characteristic flow simply as $X = X(t)$ and $P = P(t)$. We recognize (3.3.1) and (3.3.2) as the equations of movement of the particles, while the characteristic flow denotes just the trajectories of the particles in the phase space. For all $t < T$, we further define the quantity

$$\begin{aligned} \bar{P}(t) &:= \sup\left\{|p| : \exists 0 \le s \le t, x \in \mathbb{R}^3 : f(s, x, p) \neq 0\right\} \\ &\equiv \sup\left\{|P(s, x_0, p_0)| : 0 \le s \le t, (x_0, p_0) \in \mathtt{supp} f_0\right\}, \end{aligned} \tag{3.3.3}$$

where in the second line the supremum is taken over the time interval $[0, t]$ and the volume in the phase space accessible to the particles at time $t = 0$. In view of Lemma 3(c) this is equivalent to take the supremum over the support of f spanned on the time interval $[0, t]$, as given in the first line. It follows that $\bar{P}(t)$ is a non-decreasing function of t. By definition of the relativistic velocity (2.4.6), we denote $\bar{v}(t) := \bar{P}(t)\left(1 + \bar{P}(t)\right)^{-1/2}$. As stated in the Theorem 3, for any fixed $0 \le t < T$ the solution $f(t)$ has compact support in p and so

$$|\dot{X}(t)| \leq \bar{v}(t) < 1. \tag{3.3.4}$$

The map $t \mapsto X(t, x_0, p_0)$ satisfying (3.3.4) is called a *time-like* curve, to emphasize that it lies within the *light cone* with vertex $(t = 0, x = x_0)$. The condition (3.3.4) will be exploited in the Theorem 7 below. We now establish the first continuation criterion shown in this section, which is essentially the achievement of [9]:

Theorem 4. *Let $f_0 \in C^1_0(\mathbb{R}^6; \mathbb{R})$, $f_0 \geq 0$, and let $E_0, B_0 \in C^2(\mathbb{R}^3; \mathbb{R}^3)$ satisfy the constraints (2.4.4). For the Cauchy data (f_0, E_0, B_0), let (f, E, B) be a classical solution of the RVM system with life span $T > 0$ (whose existence is guaranteed by Theorem 3). Then*

$$\bar{P}_T = \sup_{0 \leq t < T} \bar{P}(t) < \infty \quad \Rightarrow \quad T = \infty.$$

That is, if $\bar{P}_T < \infty$, then the solution is global in time.

Proof. Assume that $\bar{P}_T < \infty$ but $T < \infty$. We show that this is a contradiction.

Set $\bar{P}_0 = \bar{P}(0)$. As shown in Step 6 of the proof of Theorem 3, for each $0 \leq t < T$ and $p \in \mathtt{supp} f(t)$, we have $|p| \leq \mathcal{P}(t)$ where $\mathcal{P} \in C[0, T[$ solves, cf. (3.2.11)

$$\mathcal{P}(t) = \bar{P}_0 + \int_0^t C^0(s) \exp\left\{C^0(s)\left(1 + \mathcal{P}^2(s)\right)^2\right\} ds. \tag{3.3.5}$$

The actual function $C^0(t) = C(t; \bar{P}_0, \|f_0\|_{L^\infty_{x,p}}, \|E_0\|_{W^\infty_x}, \|B_0\|_{W^\infty_x})$ is unimportant although we know it is non-negative and continuous in its arguments. On the other hand, in view of (3.2.13) and $\bar{P}_T < \infty$, for all $0 \leq t < T$ the solution (f, E, B) satisfies the uniform estimate

$$\|f(t)\|_{D^\infty_{x,p}} + \|E(t)\|_{D^\infty_x} + \|B(t)\|_{D^\infty_x} \leq u_T < \infty, \tag{3.3.6}$$

where u_T depends on T and the Cauchy data (f_0, E_0, B_0) only.

Now, fix $0 \leq t_0 < T$ and consider

$$f^0(x,p) := f(t_0, x, p), \quad E^0(x) := E(t_0, x), \quad B^0(x) := B(t_0, x)$$

as Cauchy data of the RVM system. By invoking a density argument if necessary, Theorem 3 implies that (f^0, E^0, B^0) launches a unique classical solution on $[t_0, t_0+\epsilon[$, for some $\epsilon > 0$. We claim that ϵ does *not* depend on t_0.

Indeed, the analogue of (3.3.5) for the new Cauchy data (f^0, E^0, B^0) involves a non-negative $C(t; t_0) = C(t; \bar{P}(t_0), \|f(t_0)\|_{L^\infty_{x,p}}, \|E(t_0)\|_{W^\infty_x}, \|B(t_0)\|_{W^\infty_x})$ which is continuous in its arguments. Hence, since $\bar{P}_T < \infty$, in view of (3.3.6) we have that

$$\bar{C}_T := \sup_{0\le t<T} \sup_{0\le t_0<T} C(t; t_0) < \infty$$

does *not* depend on (f^0, E^0, B^0). Now, let $\epsilon > 0$ be the life span of the solution to the equation

$$\mathcal{P}(t) = \bar{P}_T + \bar{C}_T \int_{t_0}^{t} \exp\left\{\bar{C}_T(1+\mathcal{P}^2(s))^2\right\} ds.$$

It follows from Steps 2 and 3 in the proof of Theorem 3 that all necessary estimates on the corresponding sequence of approximate solutions, which guarantee the existence of a classical solution on $[t_0, t_0+\epsilon[$, hold there. Since neither $\bar{C}_T$ nor $\bar{P}_T$ depend on t_0, then neither does ϵ and the claim follows.

Now, we could have fixed t_0 arbitrary close to $T < \infty$, and so extend the unique solution (f, E, B) beyond its life span T. But this is a contradiction. Therefore $\bar{P}_T < \infty$ implies $T = \infty$ and the solution is global in time. □

Next, we show that the boundedness of the kinetic energy density h on $[0, T[$ suffices to control the momenta of the particles, and therefore, to extend local solutions globally in time. For convenience sake, we recall that h is defined as

$$h(t, x) = \int_{\mathbb{R}^3} \sqrt{1+|p|^2} f(t, y, p) dp. \tag{3.3.7}$$

Essentially, we just have to estimate the characteristic equation (3.3.2) to obtain

$$|P(t)| \le |p_0| + \int_0^t \left(|E(s, X(s))| + |B(s, X(s))|\right) ds \tag{3.3.8}$$

and invoke Lemma 10, which guarantees the boundedness of the field (E, B) provided that h is itself bounded. This combined with the previous inequality provides the result. We make the statement precise:

Theorem 5. *Let $f_0 \in C_0^1(\mathbb{R}^6; \mathbb{R})$, $f_0 \ge 0$, and let $E_0, B_0 \in C^2(\mathbb{R}^3; \mathbb{R}^3)$ satisfy the constraints (2.4.4). For the Cauchy data (f_0, E_0, B_0), let (f, E, B) be a classical* *solution of the RVM system with life span $T > 0$. Then*

$$h_T = \sup_{0\le t<T}\sup_{x\in\mathbb{R}^3} h(s,x) < \infty \quad \Rightarrow \quad \sup_{0\le t<T} \bar{P}(t) < \infty. \tag{3.3.9}$$

In particular, the solution is global in time, i.e., $T = \infty$.

Proof. As we pointed out in the Step 6 of the proof of Theorem 3, the field (E, B) satisfies the representation (3.1.19)-(3.1.20). Therefore, (E, B) also satisfies the estimate (3.1.40) in Lemma 10. But our assumption is that $h_T < \infty$, thus there is a positive constant C_T which may depend on T such that

$$\sup_{x\in\mathbb{R}^3} (|E(t,x)| + |B(t,x)|) \le C_T \left\{ 1 + \int_0^t \sup_{x\in\mathbb{R}^3} (|E(s,x)| + |B(s,x)|)\, ds \right\}$$

for all $0 \le t < T$. The Gronwall's lemma applies and the field is uniformly bounded on $[0, T[$. This, together with the compact support of f_0 in the momentum variable, provides the uniform bound on $\bar{P}(t)$ via (3.3.8). The implication (3.3.9) then follows. But the situation is now reduced to the one in Theorem 4, thus the solution can be continued for all times. □

In [21], Pallard observed that to bound the momenta of the particles via (3.3.8), we actually do not need a sharp estimate on the L^∞-norm of the field. Instead, we could look for an estimate on its *time integral taken along characteristics*. By doing so, he was able to improve the previous result, at least for initial fields having compact support. Incidentally, the latter is used to control the L^2-norm of the field via conservation of the total energy (2.5.5), a required ingredient of the proof.

Theorem 6. *Let $f_0 \in C_0^1(\mathbb{R}^6;\mathbb{R})$, $f_0 \ge 0$, and let $E_0, B_0 \in C_0^2(\mathbb{R}^3;\mathbb{R}^3)$ satisfy the constraints (2.4.4). For the Cauchy data (f_0, E_0, B_0), let (f, E, B) be a classical solution of the RVM system with life span $T > 0$. Define the set*

$$\mathcal{F} = \{(\theta, q) : \theta > 4/q\,, 6 \le q \le \infty\}\,.$$

Then for any pair $(\theta, q) \in \mathcal{F}$, we have that

$$\sup_{0\le t<T} \left\| \int_{\mathbb{R}^3} \left(1 + |p|^2\right)^{\theta/2} f(t,\cdot,p)dp \right\|_{L^q_x} < \infty \quad \Rightarrow \quad \sup_{0\le t<T} \bar{P}(t) < \infty. \tag{3.3.10}$$

In particular, the solution is global in time, i.e., $T = \infty$.

Remark 6. The inequality $\theta > 4/q$ in $\mathcal{F}$ is strict!

Proof. Reference [21] is devoted to the proof of this theorem. □

Notice that (3.3.9) is a particular case of (3.3.10) with $\theta = 1$ and $q = \infty$. Hence, as far as a compactly supported initial data is concerned, Theorem 6 weakens the assumption made in Theorem 5, since $0 < \theta < 1$ instead of $\theta = 1$ suffices to extend local solutions for all times. Moreover, for $\theta = 1$ we could replace the sup-norm in (3.3.9) by any L^q-norm with $6 \leq q < \infty$. Notice that for a such choice of θ, the value $q = 1$ is desired since the global existence would follow as a consequence of the conservation of the total energy[1]. On the other hand, Theorem 6 says nothing about the limit case $\theta = 0$, which corresponds to the boundedness of the charge density function ρ. As commented earlier, this would rule out singularities of the solution caused by shock formations and it would weaken even further the previous continuation criteria. The remainder of this section is devoted to prove that this limit case also holds true.

Theorem 7. *Let $f_0 \in C_0^1(\mathbb{R}^6;\mathbb{R})$, $f_0 \geq 0$, and let $E_0, B_0 \in C_0^2(\mathbb{R}^3;\mathbb{R}^3)$ satisfy the constraints (2.4.4). For the Cauchy data (f_0, E_0, B_0), let (f, E, B) be a classical solution of the RVM system with life span $T > 0$. Then*

$$\sup_{0\leq t<T}\sup_{x\in\mathbb{R}^3}\rho(t,x) < \infty \quad \Rightarrow \quad \sup_{0\leq t<T}\bar{P}(t) < \infty. \tag{3.3.11}$$

In particular, the solution is global in time, i.e., $T = \infty$.

We first introduce some preliminary results and postpone the actual proof of Theorem 7 to the end of this section. We start by recalling that the kinetic energy of the relativistic particle is

$$W(p) = \sqrt{1 + |p|^2}.$$

Clearly, $\dot{W}(p) = v \cdot \dot{p}$. Then, noticing that the scalar triple product $v \cdot (v \times B) = 0$, we have that along the characteristic flow solving (3.3.1)-(3.3.2)

$$\dot{W}(P(t)) = v(P(t)) \cdot E(t, X(t)). \tag{3.3.12}$$

It follows that

$$W(t) \leq W(0) + \int_0^t |E(s, X(s))|\, ds, \tag{3.3.13}$$

[1]In fact, it can be showed by interpolation methods that $q = 4/3$ would suffice, cf. [33].

where we have abbreviated $W(t) = W(P(t))$ and will continue to do so unless we specify otherwise. We further define $\bar{W}(t) := \sqrt{1+\bar{P}^2(t)}$, with $\bar{P}(t)$ given by (3.3.3). Clearly, $\bar{W}(t)$ dominates $\bar{P}(t)$. Therefore, in contrast with [21] where $\bar{P}(t)$ was estimated directly, our strategy will be to reduce (3.3.13) to a Gronwall's type inequality by estimating the time-integral of the electric field along characteristics in terms of $\bar{W}(t)$. In turn, this will provide the desired uniform bound on the p-support of f.

Notice that the equation (3.3.12) is the one-particle analogue to the equation (2.5.4) in Section 2.5. As commented there, $W(p)$ is just the kinetic energy of a single relativistic particle in the absence of interactions, while the equations (3.3.2) and (3.3.12) describe its general motion under the influence of an external electromagnetic field. For the present case, that field is induced by the remaining charges of the system and is computed by means of the Maxwell equations (2.4.2)-(2.4.4). Both (3.3.2) and (3.3.12) arise naturally in the covariant formulation of the electrodynamics for the relativistic particle, cf. [31, Sec. 12.5]. On the other hand, notice the elementary fact that the magnetic force does no work on charged particles, and so it does not contribute to their change of kinetic energy per unit time. Indeed, as we have already noticed $v \cdot (v \times B) \equiv 0$. Consequently, a representation of the magnetic field is not needed in any of our next calculations.

Bounding $W(p)$

For an arbitrary non-negative function g, define the integrals

$$\begin{aligned} I_k(g;t) &= \int_0^t \int_0^s (s-\sigma)^k \int_{|\omega|=1} g(\sigma, X(s)+\omega(s-\sigma))\,d\omega d\sigma ds \\ &= \int_0^t \int_\sigma^t (s-\sigma)^k \int_{|\omega|=1} g(\sigma, X(s)+\omega(s-\sigma))\,d\omega ds d\sigma, \end{aligned} \tag{3.3.14}$$

with $k = 0, 1$. For $\sigma \le \sigma_1 \le \sigma_2 \le t$, we further define

$$\mathcal{I}_k(g;\sigma_1,\sigma_2) = \int_{\sigma_1}^{\sigma_2} (s-\sigma)^k \int_{|\omega|=1} g(\sigma, X(s)+\omega(s-\sigma))\,d\omega ds. \tag{3.3.15}$$

It follows that

$$I_k(g;t) = \int_0^t \mathcal{I}_k(g;\sigma,t)\,d\sigma. \tag{3.3.16}$$

Now, in view of the Corollary 2, the electric field E can be estimated as

$$\begin{aligned} |E(t,x)| &\leq C_T + 4\int_0^t \int_{|\omega|=1} h(s, x+\omega\,(t-s))d\omega ds \\ & 12\int_0^t (t-s)\int_{|\omega|=1} \bar{K}h\,(s, x+\omega\,(t-s))\,d\omega ds, \end{aligned}$$

where $\bar{K} = |E| + |B|$ and h is the kinetic energy density function. Hence, if we combine this estimate with (3.3.14), we find that

$$\begin{aligned} W(t) &\leq W(0) + \int_0^t |E(s, X(s))|\,ds \\ &\leq \bar{W}(0) + C_T + cI_0(h;t) + CI_1(h|\bar{K}|;t). \end{aligned} \tag{3.3.17}$$

Bounding $W(t)$ is now reduced to estimate the integrals $I_0(h;t)$ and $I_1(|\bar{K}|h;t)$. To that effect, we first focus on the generic integrals $\mathcal{I}_k(g;\sigma,t)$, $k=0,1$, as defined in (3.3.15). The following lemma is due to C. Pallard [21] and provides a change of variables that is crucial to our purposes. For the sake of reference, we give its proof in the Appendix A.

Lemma 12. *Let $\Omega_{\sigma_1,\sigma_2}$ be the set $(\sigma_1,\sigma_2)\times(0,2\pi)\times(0,\pi)$. The map defined by*

$$\pi_\sigma : \begin{array}{c} \Omega_{\sigma_1,\sigma_2} \to \pi_\sigma\left(\Omega_{\sigma_1,\sigma_2}\right) \subset \mathbb{R}^3 \\ (s,\theta,\phi) \mapsto X(s) + \omega(s-\sigma) \end{array}$$

and satisfying (3.3.4) is a C^1-diffeormorphism with Jacobian determinant

$$\det J_{\pi_\sigma}(s,\theta,\phi) = -\left(1 + \dot{X}(s)\cdot\omega\right)(s-\sigma)^2 \sin\phi.$$

Proof. cf. Appendix A. □

With this result at hand, we can now prove the following lemma:

Lemma 13. *For $0 \leq \sigma < t$, the integrals $\mathcal{I}_k(g;\sigma,t)$, $k=0,1$, satisfy the estimates*

$$\mathcal{I}_0(g;\sigma,t) \leq C\,\|g(\sigma)\|_{L^\infty_x}\,t \tag{3.3.18}$$

$$\mathcal{I}_1(g;\sigma,t) \leq C\frac{\|g(\sigma)\|_{L^2_x}}{\sqrt{t-\sigma}}\int_\sigma^t \left[1 + \ln \bar{W}(s)\right]ds. \tag{3.3.19}$$

Proof. Let $k = 0$ in (3.3.15). It is straightforward that

$$\begin{aligned} \mathcal{I}_0(g;\sigma,t) &\leq \int_\sigma^t \int_{|\omega|=1} g(\sigma, X(s)+\omega(s-\sigma))d\omega ds \\ &\leq 4\pi \left\| g(\sigma) \right\|_{L^\infty_x} t, \end{aligned}$$

which is (3.3.18). As for (3.3.19), let us rewrite $\mathcal{I}_1(g;\sigma,t)$ in spherical coordinates

$$\mathcal{I}_1(g;\sigma,t) = \int_\sigma^t (s-\sigma) \int_0^\pi \int_0^{2\pi} g(\sigma, X(s)+\omega(s-\sigma)) \sin\phi d\theta d\phi ds,$$

where $\omega = \omega(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. Lemma 12 shows that the map $\pi_\sigma : (s,\theta,\phi) \mapsto X(s)+\omega(s-\sigma)$ is a C^1 diffeomorphism whose Jacobian determinant has the form

$$J_{\pi_\sigma}(s,\theta,\phi) = -\left(1+\dot{X}(s)\cdot\omega\right)(s-\sigma)^2 \sin\phi.$$

Therefore, the Cauchy-Schwarz inequality implies that

$$\begin{aligned} \mathcal{I}_1(g;\sigma,t) &\leq \left(\int_\sigma^t \int_0^\pi \int_0^{2\pi} g^2(\sigma, X(s)+\omega(s-\sigma)) \left| J\pi_\sigma(s,\theta,\phi) \right| d\theta d\phi ds \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_\sigma^t \int_0^\pi \int_0^{2\pi} \frac{\sin\phi d\theta d\phi ds}{1+\dot{X}(s)\cdot\omega} \right)^{\frac{1}{2}} \\ &\leq \left\| g(\sigma) \right\|_{L^2_x} \left(\int_\sigma^t \int_0^\pi \int_0^{2\pi} \frac{\sin\phi d\theta d\phi ds}{1+\dot{X}(s)\cdot\omega} \right)^{\frac{1}{2}}. \end{aligned}$$

To estimate the angular integral, we notice that

$$\left|\ln\left(1-\bar{V}(s)\right)\right| = 2\ln\bar{W}(s) + \ln(1+\bar{V}(s)) \leq 2\left(1+\ln\bar{W}(s)\right).$$

Hence, since without lose of generality we can assume that $|\dot{X}(s)| \neq 0$, we have

$$\begin{aligned} \int_0^\pi \int_0^{2\pi} \frac{\sin\phi d\theta d\phi}{1+\dot{X}(s)\cdot\omega} &= \int_{-1}^1 \frac{2\pi du}{1+|\dot{X}(s)|u} \\ &\leq C\left(1+|\ln(1-\bar{V}(s))|\right) \\ &\leq C\left(1+\ln\bar{W}(s)\right). \end{aligned}$$

Finally, combining these estimates yield

$$
\begin{aligned}
\mathcal{I}_1(g;\sigma,t) &\leq C\,\|g(\sigma)\|_{L^2_x}\left(\int_\sigma^t \left[1+\ln \bar{W}(s)\right]ds\right)^{\frac{1}{2}} \\
&\leq C\frac{\|g(\sigma)\|_{L^2_x}}{\sqrt{t-\sigma}}\int_\sigma^t \left[1+\ln \bar{W}(s)\right]ds
\end{aligned}
$$

which is (3.3.19). Notice that we have used $0 < t-\sigma \leq \int_\sigma^t \left[1+\ln \bar{W}(s)\right]ds$ in the final step. The proof of the lemma is complete. □

We now turn to the proof of Theorem 7.

Proof of Theorem 7. The assumption made in the theorem is that for some constant C_T which may depend on T

$$
\sup_{0\leq t<T}\sup_{x\in\mathbb{R}^3}\rho(t,x) \leq C_T. \tag{3.3.20}
$$

Then, in view of (3.3.3), and recalling the definitions of ρ and h given by (2.4.5) and (3.3.7) respectively, we have that

$$
h(t,x) \leq \int_{|p|\leq \bar{P}(t)}\sqrt{1+|p|^2}f(t,x,p)dp \leq C_T\bar{W}(t).
$$

It follows that $\|h(t)\|_{L^\infty_x} \leq C_T\bar{W}(t)$. Since classical solutions preserve the total energy in time (2.5.5), then $\left\||\bar{K}|(t)\right\|_{L^2_x} \leq C$ uniformly on $[0,T[$ and

$$
\left\|\left(h|\bar{K}|\right)(t)\right\|_{L^2_x} \leq \|h(t)\|_{L^\infty_x}\left\||\bar{K}|(t)\right\|_{L^2_x} \leq C_T\bar{W}(t).
$$

Now, combining these inequalities with Lemma 13, we find that the integrals $I_0(h;t)$ and $I_1(|\bar{K}|h;t)$ satisfy the estimates -recall (3.3.16)-

$$
I_0(h;t) \leq C_T t\int_0^t \bar{W}(s)ds, \tag{3.3.21}
$$

$$
\begin{aligned}
I_1(h|\bar{K}|;t) &\leq C_T\int_0^t\int_\sigma^t \frac{\bar{W}(\sigma)}{\sqrt{t-\sigma}}\left[1+\ln\bar{W}(s)\right]ds\,d\sigma \\
&= C_T\int_0^t\int_0^s \frac{\bar{W}(\sigma)}{\sqrt{t-\sigma}}\left[1+\ln\bar{W}(s)\right]d\sigma\, ds \\
&\leq C_T\sqrt{t}\int_0^t \bar{W}(s)\left[1+\ln\bar{W}(s)\right]ds,
\end{aligned} \tag{3.3.22}
$$

where in the last inequality we have used that $\bar{W}(t)$ is non-decreasing in t. If we now go back to (3.3.17), we see that

$$\begin{aligned} W(t) &\le \bar{W}(0) + C_T + CI_0(h;t) + CI_1(|\bar{K}|h;t) \\ &\le \bar{W}(0) + C_T + C_T \int_0^t \bar{W}(s) \left[1 + \ln \bar{W}(s)\right] ds \end{aligned}$$

irrespective of the characteristic curves. Therefore,

$$\bar{W}(t) \le \bar{W}(0) + C_T + C_T \int_0^t \bar{W}(s) \left[1 + \ln \bar{W}(s)\right] ds.$$

But f_0 is assumed to have compact support, thus $\bar{W}(0) \le C$. Hence, we can invoke Corollary 1 to conclude that $\bar{P}(t) < \bar{W}(t) \le C_T$ for all $0 \le t < T$, where C_T depends on T and the Cauchy data only. This proves the implication (3.3.11). Therefore, Theorem 4 applies and the solution can be continued globally in time. This concludes the proof of the theorem. □

In view of Theorems 3, 4 and 7, we can summarize this section with the following theorem, concerning the well-posedness of the relativistic Vlasov-Maxwell system:

Theorem 8. *Let $f_0 \in C_0^1(\mathbb{R}^6;\mathbb{R})$, $f_0 \ge 0$, and let $E_0, B_0 \in C^2(\mathbb{R}^3;\mathbb{R}^3)$ satisfy the constraints (2.4.4). Then, for some $T > 0$ there is a unique classical solution (f, E, B) of the RVM system on $[0, T[$ satisfying $(f, E, B)|_{t=0} = (f_0, E_0, B_0)$. For each $0 \le t < T$ the function $f(t)$ is non-negative and has compact support. Moreover, if $T > 0$ is the life span of (f, E, B), then*

$$\sup\left\{|p| : \exists 0 \le t < T, x \in \mathbb{R}^3 : f(t, x, p) \ne 0\right\} < \infty$$

implies that the solution is global in time. If the initial field (E_0, B_0) is assumed to have compact support, then also

$$\sup\left\{\rho(t, x) : 0 \le t < T, x \in \mathbb{R}^3\right\} < \infty$$

implies that the solution is global in time, i.e., $T = \infty$.

Chapter 4

The Potential Representation

In this chapter, we reformulate the RVM system by introducing the *generalized* Vlasov transport equation -to be specified in Section 4.2- and the potential representation of the Maxwell equations in terms of the two most common *gauges*, namely, the *Lorentz* and *Coulomb* gauges. As we know from classical electrodynamics, the electric and magnetic fields can be expressed in terms of a scalar and vector potential so that the full set of Maxwell equations can be replaced by two second-order partial differential equations satisfied by these potentials. Those equations -and so the potentials themselves- depend upon the imposed gauge condition, while the electromagnetic field remains invariant to the gauge under consideration. In Section 4.1, we briefly recall the concepts of *gauge invariance* and *gauge transformation*.

The potential framework in the Lorentz gauge has already been used in the study of the RVM system. In particular, applications of the smoothing effect resulting from the coupling of a transport and wave equations have been discussed by Bouchut *et al.* in [34]. Also, they used this formulation in [18] to provide an alternative proof of the local existence result by Glassey and Strauss [9]. In their proofs, however, the Vlasov equation was kept unchanged and so the electric and magnetic fields were still part of the model equations. In the present chapter, we not only reformulate the Maxwell equations but we also deduce a Vlasov-like equation whose structure derives naturally from the *Hamiltonian* of the relativistic charged particle; cf. Remark 9 below. In Section 4.2, we introduce the *generalized* momentum-space variables to obtain the Vlasov-like equation which is determined by an incompressible vector field irrespective of the gauge chosen. To our knowledge, this representation has not yet been used in the study of the RVM system. We shall use it in Chapter 5 to investigate the Cauchy problem for the corresponding Darwin approximation.

4.1 The scalar and vector potentials

In classical electrodynamics, it is well known [27, 31, 35] that a given electromagnetic field $(E, B) : I \times \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathbb{R}^3$ that is a smooth solution of the Maxwell equations (2.1.4)-(2.1.7) can be represented by a set of potentials $(\Phi, A) : I \times \mathbb{R}^3 \mapsto \mathbb{R} \times \mathbb{R}^3$ according to the expressions

$$E(t,x) = -\nabla\Phi(t,x) - \frac{1}{c}\partial_t A(t,x) \tag{4.1.1}$$

$$B(t,x) = \nabla \times A(t,x). \tag{4.1.2}$$

Essentially, the relation (4.1.2) is a consequence of the vanishing divergence of the magnetic field, i.e., $\nabla \cdot B = 0$; and the relation (4.1.1) follows from inserting (4.1.2) into the Faraday's law (2.1.5), that is, from

$$\nabla \times \left(E + \frac{1}{c}\partial_t A\right) = 0.$$

Since for any smooth scalar function Λ we have $\nabla \times \nabla\Lambda \equiv 0$, it is clear that such potentials are not uniquely determined. We may find another pair (Φ', A') given by

$$A'(t,x) = A(t,x) + \nabla\Lambda(t,x) \tag{4.1.3}$$

$$\Phi'(t,x) = \Phi(t,x) - \frac{1}{c}\partial_t\Lambda(t,x) \tag{4.1.4}$$

which also satisfies (4.1.1)-(4.1.2). The relations (4.1.3)-(4.1.4) are called a *gauge transformation* while Λ is called the *gauge function*. We see that, even though both sets of potentials (Φ, A) and (Φ', A') may satisfy different dynamical equations, they are both fully equivalent, since *they yield the same electric and magnetic fields*. Therefore, we say that the electromagnetic field is invariant under gauge transformations.

Usually, the explicit form of the gauge function is overlooked and the potentials are found by solving the equations that result from the gauge condition imposed. In Section 4.3, we shall discuss the two most common gauges and the corresponding dynamical equations. They are: the *Lorentz guage*, where the scalar potential propagates with finite speed $\nu = c$; and the *Coulomb gauge*, where the scalar potential propagates with infinite speed $\nu = \infty$. In a sense, they can both be thought as limit cases of a more general class known as the ν-gauges, for which the scalar potential propagates with an arbitrary speed $c < \nu < \infty$[1]. For a detailed discussion on the ν-gauges and the equivalence of potentials under gauge transformations cf. [35] and the references therein. In the following section, on the other hand, the results we present are irrespective of the gauge condition imposed to the potentials.

4.2 The Vlasov equation in terms of the potentials

Throughout this section, we assume that $\Phi \in C^2(I \times \mathbb{R}^3; \mathbb{R})$ and $A \in C^2(I \times \mathbb{R}^3; \mathbb{R}^3)$ are given in *some* gauge, and by virtue of (4.1.1)-(4.1.2) the electromagnetic field $E, B \in C^1(I \times \mathbb{R}^3; \mathbb{R}^3)$ is given as well. The aim is to introduce the generalized or *canonical* variables [31, Sec. 12.6] $(x, \mathtt{P}) \in \mathbb{R}^3 \times \mathbb{R}^3$ and to deduce the Vlasov-like equation satisfied by the one-particle distribution function $\tilde{f} = \tilde{f}(t, x, \mathtt{P})$, to be defined later on. As we shall prove, the resulting *linear* transport equation is determined by an *incompressible* vector field *irrespective of the gauge under consideration*. Therefore, all the 'nice' properties introduced in Lemma 3 will also apply in this case.

To start with, we recall that the Lorentz force K acting on a single particle moving at the relativistic velocity v is given by, cf. (2.3.1)-(2.3.2)

$$K = E + \frac{v}{c} \times B, \qquad v = \frac{cp}{\sqrt{c^2 + |p|^2}}. \tag{4.2.1}$$

For simplicity, we have set the mass and charge of the particle equal to one. Consider the characteristic system (2.2.2)-(2.2.3) in Definition 3 associated to the (linear) Vlasov equation (2.2.1) for K given by (4.2.1). Denoting $(X, P)(s)$ instead of $(X, P)(s, t, z)$ and using the relations (4.1.1)-(4.1.2) for the electromagnetic field, the characteristic system (2.2.2)-(2.2.3) can be written as

$$\dot{X}(s) = v(P(s)) \tag{4.2.2}$$

$$\dot{P}(s) = \left[-\nabla\Phi - \frac{1}{c}\partial_s A + \frac{v}{c} \times (\nabla \times A) \right](s, X(s), P(s)). \tag{4.2.3}$$

Hence, since by total differentiation

$$\dot{A}(s, X(s)) \equiv \frac{d}{ds} A(s, X(s)) = \left[\partial_s A + (v \cdot \nabla) A\right](s, X(s)),$$

[1] As pointed out in [35], the restriction $\nu > c$ is artificial since smaller values of ν may be considered as well. Regardless, we emphasize that the causality of the *electromagnetic fields* with propagation speed c remains unaltered by the arbitrary propagation of the ν-gauge potentials.

we have that (4.2.3) can be rewritten as

$$\dot{P}(s) = \left[-\frac{1}{c}\dot{A} - \nabla\Phi + \frac{v}{c} \times (\nabla \times A) + \left(\frac{v}{c} \cdot \nabla\right) A \right](s, X(s), P(s)). \tag{4.2.4}$$

The structure of this equation suggests that we could define a generalized or *canonical* momentum

$$\mathtt{P} = p + \frac{1}{c}A \tag{4.2.5}$$

such that (4.2.4) becomes

$$\dot{\Pi}(s) = \left[-\nabla\Phi + \frac{v}{c} \times (\nabla \times A) + \left(\frac{v}{c} \cdot \nabla\right) A \right](s, X(s), P(s)). \tag{4.2.6}$$

Here we have denoted $\Pi(s) := P(s) + c^{-1}A(s, X(s))$. On the other hand, the relativistic velocity can be redefined in terms of the generalized momentum $\mathtt{P}$ by inserting (4.2.5) into the expression for v in (4.2.1), namely

$$v_A := \frac{c^2\mathtt{P} - cA}{\sqrt{c^4 + |c\mathtt{P} - A|^2}}. \tag{4.2.7}$$

We denote the velocity by v_A, and will continue to do so, to emphasize the dependence on the vector potential A. Thus, we can reformulate the characteristic system (2.2.2)-(2.2.3) in terms of the generalized variables $\xi := (x, \mathtt{P})$ as

$$\dot{X}(s,t,\xi) \;=\; v_A(s, X(s,t,\xi), \Pi(s,t,\xi)) \tag{4.2.8}$$

$$\dot{\Pi}(s,t,\xi) \;=\; -\left[\nabla\Phi - \frac{v_A^i}{c}\nabla A^i\right](s, X(s,t,\xi), \Pi(s,t,\xi)), \tag{4.2.9}$$

where v_A is given by (4.2.7) and (4.2.9) follows from (4.2.6) by virtue of the identity

$$\frac{v}{c} \times (\nabla \times A) + \left(\frac{v}{c} \cdot \nabla\right) A = \frac{v_A^i}{c}\nabla A^i. \tag{4.2.10}$$

The computation of (4.2.10) is elementary and we postpone it to the Appendix B. As usual, repeated index means summation.

If we now go back to the proof of Lemma 3(a), we see that it can be easily adapted to show that for every $t \in I$ and $\xi \in \mathbb{R}^6$ fixed, there exist a unique local solution $\Xi := (X, \Pi)(s,t,\xi)$ of (4.2.8)-(4.2.9) satisfying $\Xi(t,t,\xi) = \xi$. Moreover,

$\Xi \in C^1(I \times I \times \mathbb{R}^6; \mathbb{R}^6)$. In turn, uniqueness implies that

$$Z := (X, \Pi - \frac{1}{c}A)(s, t, x, \mathtt{P} - \frac{1}{c}A) \tag{4.2.11}$$

is the unique solution of (2.2.2)-(2.2.3) with initial data $Z(t, t, z) = (x, \mathtt{P} - c^{-1}A)$.

Lemma 14. *For v_A given by (4.2.7), we have*

$$\nabla_x \cdot v_A + \nabla_P \cdot \left(-\nabla\Phi + \frac{v_A^i}{c} \nabla A^i \right) \equiv 0$$

Proof. Lengthy but elementary computations show that

$$\nabla_{\mathtt{P}} \cdot \left(-\nabla\Phi + \frac{v_A^i}{c} \nabla A^i \right) \equiv \frac{1}{c} \nabla_{\mathtt{P}} \cdot \left(v_A^i \nabla A^i \right) = \frac{\nabla \cdot A - v_A^i \left(v_A \cdot \nabla \right) A^i}{\sqrt{c^2 + |c\mathtt{P} - A|^2}} = -\nabla_x \cdot v_A.$$

□

As a consequence of Lemma 14, the proof of Lemma 3(b) can be easily adapted as well to the solutions of the characteristic system (4.2.8)-(4.2.9). Specifically, for any $s, t \in I$ fixed, the map $\Xi(s, t, \cdot) : \mathbb{R}^6 \mapsto \mathbb{R}^6$ is a C^1-diffeomorphism with inverse $\Xi^{-1}(s, t, \xi) = \Xi(t, s, \xi)$ and Jacobian determinant

$$\det J_{\Xi_{s,t}}(\xi) = \frac{\partial \Xi(s, t, \xi)}{\partial \xi} \equiv 1.$$

Therefore, the solutions of (4.2.8)-(4.2.9) satisfy the volume preserving property.

Lemma 15. *Let $\Phi \in C(I, C^2(\mathbb{R}^3); \mathbb{R})$ and $A \in C(I, C^2(\mathbb{R}^3); \mathbb{R}^3)$ be given in some gauge and let v_A be given by (4.2.7). Let $\tilde{f}_0 \in C^1(\mathbb{R}^6; \mathbb{R})$ and denote by $\Xi := (X, \Pi)$ the characteristic flow solving (4.2.8)-(4.2.9). Then, the function $\tilde{f}(t, \xi) := \tilde{f}_0(\Xi(0, t, \xi))$ defined on $I \times \mathbb{R}^6$ is the unique classical solution of the Cauchy problem for*

$$\partial_t \tilde{f} + v_A \cdot \nabla_x \tilde{f} - \left[\nabla\Phi - \frac{v_A^i}{c} \nabla A^i \right] \cdot \nabla_P \tilde{f} = 0. \tag{4.2.12}$$

Moreover, if $\tilde{f}_0 \geq 0$ then $\tilde{f} \geq 0$. Also, for $t \in I$

$$\boldsymbol{supp} \tilde{f}(t) = \Xi(t, 0, \boldsymbol{supp} \tilde{f}_0)$$

and for each $1 \le q \le \infty$, $t \in I$,

$$\left\| \tilde{f}(t) \right\|_{L^q_{x,\mathtt{P}}} = \left\| \tilde{f}_0 \right\|_{L^q_{x,\mathtt{P}}}.$$

Conversely, if $\tilde{f}$ *is a classical solution of the Cauchy problem for (4.2.12), then* $\tilde{f}$ *is constant along each solution of the characteristic system (4.2.8)-(4.2.9).*

Proof. By virtue of Lemma 14, this result is analogous to part (c) of Lemma 3. □

As we may expect, the solution of the Cauchy problem for (4.2.12) yields the solution of the Cauchy problem for (2.2.1).

Corollary 3. *For* (Φ, A) *as in Lemma 15, define* (E, B) *by (4.1.1)-(4.1.2). Also, for* $f_0 \in C(\mathbb{R}^6)$ *given, let* $\tilde{f}_0(x, \mathtt{P}) := f_0(x, p + c^{-1}A(0, x))$. *Then, the function defined on* $I \times \mathbb{R}^6$ *by* $f(t, x, p) := \tilde{f}(t, x, \mathtt{P} - c^{-1}A(t, x))$ *is the unique classical solution of the Cauchy problem for the linear Vlasov equation (2.2.1), with* v *and* K *given by (4.2.1).*

Proof. Combine Lemma 15, relation (4.2.11) and Lemma 3 to find that

$$\begin{aligned} \tilde{f}(t, x, \mathtt{P} - \frac{1}{c}A(t, x)) &= \tilde{f}_0\left((X, \Pi - \frac{1}{c}A)(0, t, x, \mathtt{P} - \frac{1}{c}A) \right) \\ &= f_0((X, P)(0, t, x, p)) \\ &= f(t, x, p), \end{aligned}$$

which proves the statement. □

Remark 7. Obviously, the transformation $p \mapsto p + c^{-1}A$ has Jacobian determinant equal to one. Therefore $\| f(t) \|_{L^q_{x,p}} = \left\| \tilde{f}(t) \right\|_{L^q_{x,\mathtt{P}}}$ for all $1 \le q \le \infty$, $t \in I$.

Remark 8. Clearly, $|v_A| \le c$ for all $t \in I$ and $(x, \mathtt{P}) \in \mathbb{R}^3 \times \mathbb{R}^3$; the constant c being the speed of light.

Remark 9. Perhaps, it would have been more elegant to introduce these results by using the Hamiltonian formulation of the relativistic charged particle[2]. Under the action of an external electromagnetic field of potentials (Φ, A), the Hamiltonian associated to the charged particle reads [31, Sec. 12.5]

$$\mathcal{H} = \sqrt{m^2 c^4 \left(\mathtt{P} - \frac{e}{c}A \right)^2} + e\Phi,$$

[2]Referring $\mathtt{P}$ as the *canonical* momentum becomes apparent in this framework.

where m and e denote its mass and charge respectively. Then, the 'characteristic system' (4.2.8)-(4.2.9) can be obtained by means of the Hamilton equations

$$\dot{x} = \nabla_{\mathtt{P}} \mathcal{H}, \quad \dot{\mathtt{P}} = -\nabla_x \mathcal{H},$$

and Lemma 14 becomes a consequence of $\partial^2_{x\mathtt{P}} \mathcal{H} \equiv \partial^2_{\mathtt{P}x} \mathcal{H}$. However, in order not to introduce additional nontrivial concepts and definitions, we have preferred to work in a more elementary setting.

4.3 Equations satisfied by the potentials

Formally, if we substitute the electric and magnetic fields given by (4.1.1)-(4.1.2) into the Ampère-Maxwell and Coulomb equations, cf. (2.1.4) and (2.1.6) respectively, we find that Φ and A satisfy

$$\Delta \Phi = -4\pi\rho - \frac{1}{c}\partial_t (\nabla \cdot A) \tag{4.3.1}$$

$$\Delta A - \frac{1}{c^2}\partial_t^2 A = -\frac{4\pi}{c} j + \nabla \left(\nabla \cdot A + \frac{1}{c}\partial_t \Phi \right). \tag{4.3.2}$$

These expressions can be further simplified once we impose a gauge condition on the potentials. For instance, if we set $\nabla \cdot A = 0$, these equations reduce to

$$\begin{aligned} \Delta \Phi &= -4\pi\rho \\ \Delta A - \frac{1}{c^2}\partial_t^2 A &= -\frac{4\pi}{c} j + \frac{1}{c}\nabla \partial_t \Phi. \end{aligned}$$

Then, we say that the potentials satisfy the *Coulomb gauge* condition, for which Φ satisfies a Poisson equation while A is determined by a wave equation whose source term depends on $\partial_t \Phi$. On the other hand, if we let $\nabla \cdot A + c^{-1}\partial_t \Phi = 0$, we obtain

$$\begin{aligned} \Delta \Phi - \frac{1}{c^2}\partial_t^2 \Phi &= -4\pi\rho \\ \Delta A - \frac{1}{c^2}\partial_t^2 A &= -\frac{4\pi}{c} j, \end{aligned}$$

in which case the potentials are related by the so-called *Lorentz gauge* condition and they are both determined by wave equations, coupled only through the continuity equation which is assumed to be satisfied by the charge and current densities.

Conversely, if the pair (Φ, A) solves (4.3.1)-(4.3.2) in *some* gauge, then the field (E, B) given by (4.1.1)-(4.1.2) is a solution of the Maxwell system of equations. We make this statement precise in terms of the two most common gauges:

4.3.1 The Lorentz gauge

Lemma 16. *Let $\rho \in C^1(I, C^2(\mathbb{R}^3); \mathbb{R})$ and $j \in C(I, C^2(\mathbb{R}^3); \mathbb{R}^3)$ satisfy the continuity equation (2.1.3). Let $A_0 \in C^3(\mathbb{R}^3; \mathbb{R}^3)$ and $A_1 \in C^2(\mathbb{R}^3; \mathbb{R}^3)$ such that $\nabla \cdot A_0 = 0$. Then, the following holds:*

(a) *There exists a unique $\Phi_L \in C^2(I \times \mathbb{R}^3; \mathbb{R})$ that solves*

$$\begin{aligned} &\Delta\Phi - \frac{1}{c^2}\partial_t^2\Phi = -4\pi\rho, \\ &\Phi|_{t=0} = 0, \quad \partial_t\Phi|_{t=0} = 0. \end{aligned} \tag{4.3.3}$$

(b) *There exists a unique $A_L \in C^2(I \times \mathbb{R}^3; \mathbb{R}^3)$ that solves*

$$\begin{aligned} &\Delta A - \frac{1}{c^2}\partial_t^2 A = -\frac{4\pi}{c}j, \\ &A|_{t=0} = A_0, \ \ \partial_t A|_{t=0} = A_1. \end{aligned} \tag{4.3.4}$$

(c) *The pair (Φ_L, A_L) satisfies the Lorentz gauge condition, i.e.,*

$$\nabla \cdot A_L + \frac{1}{c}\partial_t\Phi_L = 0, \quad \text{on } I \times \mathbb{R}^3.$$

Moreover, if we chose A_0 and A_1 such that $B_0 = \nabla \times A_0$ and $E_0 = -c^{-1}A_1$, then the electromagnetic field (E, B) defined by

$$\begin{aligned} E &:= -\nabla\Phi_L - \frac{1}{c}\partial_t A_L \\ B &:= \nabla \times A_L \end{aligned}$$

on $I \times \mathbb{R}^3$ is the unique classical solution of the corresponding Cauchy problem to the Maxwell equations (2.1.4)-(2.1.7).

Proof. (a) and (b) are standard results for wave equations. For instance, cf. [26, ch. 2]. To prove (c) set $g_L = \nabla \cdot A_L + c^{-1}\partial_t\Phi_L$. Owing to (4.3.3) and (4.3.4), the conditions on A_0 and A_1, and the assumptions on ρ and j, we have that g_L satisfies

$$\Delta g_L - \frac{1}{c^2}\partial_t^2 g_L = -\frac{4\pi}{c}\left(\partial_t \rho + \nabla \cdot j\right) = 0,$$
$$g_L\big|_{t=0} = 0, \quad \partial_t g_L\big|_{t=0} = 0.$$

in $\mathcal{D}'(I^0 \times \mathbb{R}^3)$. Thus, $g_L = 0$ in $\mathcal{D}'(I^0 \times \mathbb{R}^3)$, and since it is also continuous, then $g_L \equiv 0$ on $I \times \mathbb{R}^3$. On the other hand, if the pair (Φ_L, A_L) satisfies the Lorentz gauge condition, then the wave equations (4.3.3)-(4.3.4) can be rewritten as (4.3.1)-(4.3.2). Therefore, (E, B) as defined satisfies the equations (2.1.4) and (2.1.6). It is easy to check that the other two Maxwell equations are satisfied as well. Uniqueness completes the proof. □

4.3.2 The Coulomb gauge

Lemma 17. *Let $\rho \in C^1(I, C^2(\mathbb{R}^3); \mathbb{R})$ and $j \in C(I, C^2(\mathbb{R}^3); \mathbb{R}^3)$ satisfy the continuity equation (2.1.3) and have compact support on $\mathbb{R}^3$. Let $A_0 \in C^3(\mathbb{R}^3; \mathbb{R}^3)$ such that $\nabla \cdot A_0 = 0$. Then, the following holds:*

(a) *There exists a unique $\Phi_C \in C^1(I, C^3(\mathbb{R}^3); \mathbb{R})$ that solves*

$$\Delta \Phi = -4\pi\rho, \qquad \lim_{|x|\to\infty} \Phi(t, x) = 0 \tag{4.3.5}$$

for each $t \in I$.

(b) *There exists a unique $A_C \in C^2(I \times \mathbb{R}^3; \mathbb{R}^3)$ that solves*

$$\Delta A - \frac{1}{c^2}\partial_t^2 A = -\frac{4\pi}{c} j + \frac{1}{c}\nabla \partial_t \Phi,$$
$$A\big|_{t=0} = A_0, \quad \partial_t A\big|_{t=0} = 0. \tag{4.3.6}$$

(c) *The vector potential A_C satisfies the Coluomb gauge condition, i.e.,*

$$\nabla \cdot A_C = 0, \quad \text{on } I \times \mathbb{R}^3.$$

Moreover, if we chose A_0 such that $B_0 = \nabla \times A_0$ and let $\nabla \cdot E_0 = \rho\big|_{t=0}$, then *the electromagnetic field (E, B) defined by*

$$\begin{aligned} E &:= -\nabla\Phi_C - \frac{1}{c}\partial_t A_C \\ B &:= \nabla \times A_C \end{aligned}$$

on $I \times \mathbb{R}^3$ is the unique classical solution of the corresponding Cauchy problem to the Maxwell equations (2.1.4)-(2.1.7).

Proof. (a) is a standard result for Poisson equations. cf. [26, ch. 2] for both (a) and (b). As for (c), the proof goes exactly as in the proof of Lemma 16(c) after replacing g_L by $g_C \equiv \nabla \cdot A$ and (4.3.3)-(4.3.4) by (4.3.5)-(4.3.6). Indeed, notice that g_C satisfies

$$\begin{aligned} &\Delta g_L - \frac{1}{c^2}\partial_t^2 g_L = -\frac{4\pi}{c}\left(\nabla \cdot j - \partial_t \Delta\Phi\right) = 0, \\ &g_L|_{t=0} = 0, \quad \partial_t g_L|_{t=0} = 0. \end{aligned}$$

in $\mathcal{D}'(I^0 \times \mathbb{R}^3)$, thus the rest of the proof follows suit. □

4.4 The RVM system in terms of the potentials

In view of Lemma 16 and Lemma 17, the Cauchy problem for the Maxwell system of equations (2.1.4)-(2.1.7) can be replaced by either (4.3.3)-(4.3.4) or (4.3.5)-(4.3.6). On the other hand, by Corollary 3, we can replace the Vlasov equation (2.4.1) by (4.2.12). Therefore, we can combine these results to introduce two equivalent formulations of the Cauchy problem for the RVM system.

Lemma 18. *Let $f_0 \in C_0^1(\mathbb{R}^6; \mathbb{R})$, $f_0 \geq 0$ and $E_0, B_0 \in C^2(\mathbb{R}^3; \mathbb{R}^3)$ satisfy*

$$\nabla \cdot E_0 = 4\pi \int_{\mathbb{R}^3} f_0 dP, \qquad \nabla \cdot B_0 = 0.$$

Let $A_0 \in C^3(\mathbb{R}^3; \mathbb{R}^3)$ such that $\nabla \cdot A_0 = 0$ and $\nabla \times A_0 = B_0$. Then, solving the Cauchy problem for the RVM system can be reduced to solving the Cauchy problem for the following system of equations:

$$\partial_t f + v_A \cdot \nabla_x f - \left[\nabla\Phi - \frac{v_A^i}{c}\nabla A^i\right] \cdot \nabla_P f = 0, \tag{4.4.1}$$

$$v_A = \frac{c^2 \mathtt{P} - cA}{\sqrt{c^4 + |c\mathtt{P} - A|^2}}$$

on $I \times \mathbb{R}^3 \times \mathbb{R}^3$*, coupled with either*

$$\Delta\Phi - \frac{1}{c^2}\partial_t^2\Phi = -4\pi\rho, \tag{4.4.2}$$

$$\Delta A - \frac{1}{c^2}\partial_t^2 A = -\frac{4\pi}{c} j, \tag{4.4.3}$$

or

$$\Delta\Phi = -4\pi\rho, \qquad \lim_{|x|\to\infty} \Phi(t,x) = 0 \tag{4.4.4}$$

$$\Delta A - \frac{1}{c^2}\partial_t^2 A = -\frac{4\pi}{c} j + \frac{1}{c}\nabla\partial_t\Phi \tag{4.4.5}$$

on $I \times \mathbb{R}^3$*, via*

$$j = \int_{\mathbb{R}^3} v_A f d\mathtt{P}, \qquad \rho = \int_{\mathbb{R}^3} f d\mathtt{P}.$$

The Cauchy data is chosen as follows. For (4.4.2), $\Phi|_{t=0} = \partial_t\Phi|_{t=0} = 0$*. For (4.4.3),* $A|_{t=0} = A_0$ *and* $\partial_t A|_{t=0} = -cE_0$*. As for (4.4.5),* $A|_{t=0} = A_0$ *and* $\partial_t A|_{t=0} = 0$*.*

Chapter 5

The Darwin Approximation

Although the potential representation discussed in the previous chapter provides a promising framework to study the RVM system, the coupling between the time and space derivatives -due to the retarded solution of the electromagnetic wave equations- still brings significant difficulties to the problem. This coupling disappears if we simplify the system in the right way. For instance, if we formally let $c \to \infty$ in the model equations given in Lemma 18, then the generalized variables become the usual momentum-space variables, i.e.,

$$(x, p + \frac{1}{c}A) \to (x, p),$$

and the RVM system reduces to the so-called Vlasov-Poisson system

$$\partial_t f + p \cdot \nabla_x f + \nabla\Phi \cdot \nabla_p f = 0 \tag{5.0.1}$$

$$\Delta\Phi = -4\pi \int_{\mathbb{R}^3} f dp, \qquad \lim_{|x|\to\infty} \Phi(t,x) = 0. \tag{5.0.2}$$

This limit was rigorously studied in [36] and similar results were obtained in [7, 37]. Incidentally, none of them used the potential representation presented in the previous chapter. The system (5.0.1)-(5.0.2) was actually introduced as a model for collisionless plasma with negligible magnetic fields, and it has been extensively studied since the pioneering work by A. Vlasov in [38]. If the source term in the Poisson equation is taken positive, then we are dealing with the attractive version of the Vlasov-Poisson system. The latter was used as early as in 1915 to study the evolution of stellar cluster and galaxies, cf. [39]. In both the repulsive and attractive cases, the existence of global classical solutions for unrestricted smooth Cauchy data was obtained independently, and following completely different approaches, by Pfaffelmoser [40] and Lions-Perthame [41]. An extensive review on the Cauchy problem for the Vlasov-Poisson system as well as on the stability of its steady states[1] is given in [42].

On the other hand, if we replace the Vlasov equation (5.0.1) by

$$\partial_t f + \frac{cp}{\sqrt{c^2 + |p|^2}} \cdot \nabla_x f + \nabla\Phi \cdot \nabla_p f = 0 \tag{5.0.3}$$

and couple it with (5.0.2), we obtain the relativistic Vlasov-Poisson (RVP) system. Contrary to the non-relativistic case, the existence of global classical solution for unrestricted Cauchy data of (5.0.2)-(5.0.3) remains an open problem. It has been suggested in [43], that the difficulties are a consequence of the lack of Lorentz invariance in the system, since the non-relativistic Galilei invariant field equation is coupled with a relativistic transport equation.

The underlying elliptic structure of both the non-relativistic and the relativistic Vlasov-Poisson systems avoid the coupling of time and space derivatives mentioned earlier. However, these models provide a 'poor' approximation to the full RVM system when the intensity of the magnetic field is significant[2]. In this chapter, we study the relativistic Vlasov-Darwin (RVD) system, which has elliptic features yet preserves a fully coupled magnetic field. As discussed in Section 5.3, these are desirable properties for numerical simulations of collisionless plasma.

The structure of this Chapter is as follows. In Section 5.1 we define the Darwin approximation of the Maxwell equations. Then, we introduce the Darwin potentials as the unique solutions of the resulting equations, and we present their most relevant properties. We define the Cauchy problem for the RVD system within the potential framework in Section 5.2, and provide our main results of this chapter. First, we show that under suitable conditions on the Cauchy datum, there exists a time until which the RVD system has a unique classical solution, cf. Theorem 9. Then, we show that if the Cauchy datum is 'small' enough, this solution is actually global in time, cf. Theorem 10. In doing so, we obtain the decay estimate satisfied by the corresponding charge and current densities as well as the *space* derivatives, up to a second order, of the potentials induced by this solution. Finally, in Section 5.3 we

[1]The stability question is mainly considered in the attractive case.

[2]Except perhaps, if we assume spherically symmetric Cauchy data. In this case, the magnetic field can be shown to be constant in the whole space and thus, without loss of generality, can be set equal to zero. Then the model equations reduces naturally to the RVP system [43].

explore the advantages of the potential representation of the RVD system with respect to the usual representation. There, we discuss the state of the art for this system as well as open problems and the main difficulties to solve them.

We emphasize that the approach followed here is reminiscent to the one used for the RVP system, as given, for instance, in [42]. We have taken advantage of the potential representation of the RVD system -especially for the Vlasov equation- to adapt the techniques used previously to solve the Cauchy problem for the Vlasov-Poisson system. Clearly, we still have to deal with the difficulties introduced by the vector potential, which is not present in the Poisson case.

5.1 The Darwin potentials

Consider the generic wave equation $\Delta u - c^{-2}\partial_t^2 u = -4\pi g$ with vanishing initial data. As shown in Chapter 2, the solution of this equation is given by the method of retarded potentials according to the expression (2.1.12). We shall define the quasi-static limit of the solution (2.1.12) by absence of retardation in the source term, i.e.,

$$\int_{\Omega_{ct}(x)} g(t - \frac{1}{c}\left|y-x\right|, y)\frac{dy}{|y-x|} \quad \longrightarrow \quad \int_{\mathbb{R}^3} g(t,y)\frac{dy}{|y-x|},$$

where c denotes the speed of wave propagation. That is, we have formally let $c \to \infty$ and have replaced the retarded solution of the wave equation by the solution of a Poisson equation with source term $-4\pi g$. Therefore, this definition is formally equivalent to neglecting the term $c^{-2}\partial_t^2 u$ in the wave equation given above.

We define the Darwin approximation of the Maxwell equations as the quasi-static limit of the Maxwell equations given *in the Coulomb gauge*, cf. (4.3.5)-(4.3.6).

Definition 5. *Let $\rho \in C^1(I \times \mathbb{R}^3; \mathbb{R})$ and $j \in C(I, C^1(\mathbb{R}^3); \mathbb{R}^3)$ satisfy the continuity equation (2.1.3). The set of potentials (Φ, A) is said to be a classical solution of the Darwin equations if $\Phi \in C^1(I, C^2(\mathbb{R}^3); \mathbb{R})$, $A \in C(I, C^2(\mathbb{R}^3); \mathbb{R}^3)$ and on $I \times \mathbb{R}^3$*

$$\Delta\Phi = -4\pi\rho \tag{5.1.1}$$

$$\Delta A = -\frac{4\pi}{c}j + \frac{1}{c}\nabla\partial_t\Phi. \tag{5.1.2}$$

In this section, we shall introduce and investigate the Darwin potentials (Φ_D, A_D) as the unique classical solution of the system (5.1.1)-(5.1.2). We also explore their most relevant properties and deduce the a-priori estimates to be used in Section 5.2 below. Clearly, Φ_D must be the solution of the standard Poisson equation, but finding A_D turns out to be a bit more involved. For the sake of simplicity, we shall omit the dependence in time, to be recovered later on in Remark 10.

Definition 6. *For the charge density $\rho : \mathbb{R}^3 \mapsto \mathbb{R}$ and current density $j : \mathbb{R}^3 \mapsto \mathbb{R}^3$ we formally define the set of Darwin potentials $(\Phi_D, A_D) : \mathbb{R}^3 \mapsto \mathbb{R} \times \mathbb{R}^3$ by*

$$\begin{aligned} \Phi_D(x) &= \int_{\mathbb{R}^3} \rho(y) \frac{dy}{|y-x|} \\ A_D(x) &= \frac{1}{2c} \int_{\mathbb{R}^3} [\boldsymbol{id} + \omega \otimes \omega]\, j(y) \frac{dy}{|y-x|}. \end{aligned}$$

It will be convenient to introduce a different but equivalent representation for A_D. We do so in the form of a lemma.

Lemma 19. *Let $j \in C_0^1(\mathbb{R}^3; \mathbb{R}^3)$ be given. Then, the Darwin vector potential A_D in Definition 6 has the equivalent representation*

$$A_D(x) \equiv \frac{1}{c} \int_{\mathbb{R}^3} j(y) \frac{dy}{|y-x|} + \frac{1}{2c} \int_{\mathbb{R}^3} \nabla \cdot j(y) \frac{y-x}{|y-x|} dy. \tag{5.1.3}$$

Proof. In view of the compact support of the current density j, standard arguments imply that the integrals in the right-hand side (`RHS`) of (5.1.3) are well defined. Hence, the divergence theorem and the identity (3.1.4) applied to the second integral yield

$$\begin{aligned} \texttt{RHS} &= \frac{1}{c} \int_{\mathbb{R}^3} j(y) \frac{dy}{|y-x|} - \frac{1}{2c} \int_{\mathbb{R}^3} (j(y) \cdot \nabla)\, \omega dy \\ &= \frac{1}{c} \int_{\mathbb{R}^3} \left\{ j(y) - \frac{1}{2} [j(y) - \omega\, (j(y) \cdot \omega)] \right\} \frac{dy}{|y-x|} \\ &= \frac{1}{2c} \int_{\mathbb{R}^3} [j(y) + \omega\, (j(y) \cdot \omega)] \frac{dy}{|y-x|}, \end{aligned}$$

which is precisely the Darwin potential A_D in Definition 6. The use of the divergence theorem is justified by the following standard argument: we remove a small ball about $x \in \mathbb{R}^3$ in the domain of integration of the second integral so we can avoid the singularity at $y = x$, we shift variables and then use the divergence theorem, noticing that the boundary term corresponding to the small ball vanishes as its radii tends to 0. This completes the proof of the lemma. □

A useful tool will be the following estimate:

Lemma 20. *For $1 \le m < 3$ set $r_0 := 3/(3-m)$ and let $r < r_0 < s$. Then there exists a positive constant $C = C(m,r,s)$ such that for any $\Psi \in L^r \cap L^s(\mathbb{R}^n;\mathbb{R})$*

$$\left\| \int_{\mathbb{R}^n} \Psi(y) \frac{dy}{|y - \cdot|^m} \right\|_{L^\infty_x} \le C(m,r,s) \, \|\Psi\|_{L^r_x}^{1-\lambda} \, \|\Psi\|_{L^s_x}^{\lambda} ,$$

where $\lambda = (1 - r/r_0)/(1 - r/s)$. In particular, $C(m,1,\infty) = 3\,(4\pi/m)^{m/3} / (3-m)$.

Proof. cf. [44, Lemma 2.7]. ☐

Corollary 4. *Let $\rho \in L^1 \cap L^\infty(\mathbb{R}^3;\mathbb{R})$ and $j \in L^1 \cap L^\infty(\mathbb{R}^3;\mathbb{R}^3)$ be given. Then*

$$\|\Phi_D\|_{L^\infty_x} \le 3\,(\pi/2)^{1/3} \, \|\rho\|_{L^1_x}^{2/3} \, \|\rho\|_{L^\infty_x}^{1/3} , \qquad \|A_D\|_{L^\infty_x} \le 3^2\,(\pi/2)^{1/3} \, c^{-1} \, \|j\|_{L^1_x}^{2/3} \, \|j\|_{L^\infty_x}^{1/3} .$$

Lemma 21. *Let $\rho \in C^1_0(\mathbb{R}^3;\mathbb{R})$ and $j \in C^2_0(\mathbb{R}^3;\mathbb{R}^3)$. Then, the following holds*

(a) *The scalar potential Φ_D is the unique $C^2(\mathbb{R}^3;\mathbb{R})$ solution of*

$$\Delta\Phi(x) = -4\pi\rho(x), \qquad \lim_{|x|\to\infty} \Phi(x) = 0.$$

It satisfies

$$\nabla\Phi_D(x) = \int_{\mathbb{R}^3} \rho(y) \frac{\omega dy}{|y-x|^2}. \tag{5.1.4}$$

(b) *The following estimate holds,*

$$\|\nabla\Phi_D\|_{L^\infty_x} \le 3\,(2\pi)^{2/3} \, \|\rho\|_{L^1_x}^{1/3} \, \|\rho\|_{L^\infty_x}^{2/3} .$$

In addition, for any $0 < h \le R$, we have

$$\left\|\partial_x^2 \Phi_D\right\|_{L^\infty_x} \le C \left[R^{-3} \, \|\rho\|_{L^1_x} + h \, \|\nabla\rho\|_{L^\infty_x} + (1 + \ln(R/h)) \, \|\rho\|_{L^\infty_x} \right] ,$$

where $C > 0$ is independent h, R and ρ. In particular,

$$\left\|\partial_x^2 \Phi_D\right\|_{L^\infty_x} \le C \left[\|\rho\|_{L^1_x} + \left(1 + \|\rho\|_{L^\infty_x}\right) \left(1 + \ln^+ \|\nabla\rho\|_{L^\infty_x}\right) \right] .$$

(c) *The vector potential A_D is the unique $C^2(\mathbb{R}^3;\mathbb{R}^3)$ solution of*

$$\Delta A(x) = -\frac{4\pi}{c} \mathbb{P} j(x), \qquad \lim_{|x|\to\infty} |A(x)| = 0. \tag{5.1.5}$$

where

$$\mathbb{P}j(x) := j(x) + \frac{1}{4\pi}\nabla \int_{\mathbb{R}^3} \nabla \cdot j(y) \frac{dy}{|y-x|}.$$

For $i, k = 1, 2, 3$, *it satisfies*

$$\partial_k A_D^i(x) = \frac{1}{2c} \int_{\mathbb{R}^3} \left[\delta_{im}\omega^k - \delta_{km}\omega^i + \left(3\omega^i\omega^k - \delta_{ik}\right)\omega^m\right] j^m(y) \frac{dy}{|y-x|^2}.$$

In particular

$$\nabla \cdot A_D(x) = 0, \qquad \nabla \times A_D(x) = \frac{1}{c} \int_{\mathbb{R}^3} \omega \times j(y) \frac{dy}{|y-x|^2}. \tag{5.1.6}$$

(d) *We have the estimate,*

$$\|\partial_x A_D\|_{L^\infty_x} \leq 3^3 (2\pi)^{2/3} c^{-1} \|j\|_{L^1_x}^{1/3} \|j\|_{L^\infty_x}^{2/3}.$$

Moreover, for any $0 < h \leq R$

$$\left\|\partial_x^2 A_D\right\|_{L^\infty_x} \leq C\left[R^{-3}\|j\|_{L^1_x} + h\|\partial_x j\|_{L^\infty_x} + (1 + \ln(R/h))\|j\|_{L^\infty_x}\right],$$

where $C > 0$ *is independent of* h, R *and* j. *In particular,*

$$\left\|\partial_x^2 A_D\right\|_{L^\infty_x} \leq C\left[\|j\|_{L^1_x} + \left(1 + \|j\|_{L^\infty_x}\right)\left(1 + \ln^+\|\partial_x j\|_{L^\infty_x}\right)\right].$$

Remark 10. If ρ and j satisfy the continuity equation (2.1.3), then as a consequence of items (a) and (c), we have

$$\begin{aligned}
\Delta A_D(t,x) &= -\frac{4\pi}{c} j(t,x) - \frac{1}{c}\nabla \int_{\mathbb{R}^3} \nabla \cdot j(t,y) \frac{dy}{|y-x|} \\
&= -\frac{4\pi}{c} j(t,x) + \frac{1}{c}\nabla \partial_t \int_{\mathbb{R}^3} \rho(t,y) \frac{dy}{|y-x|} \\
&= -\frac{4\pi}{c} j(t,x) + \frac{1}{c}\nabla \partial_t \Phi_D.
\end{aligned}$$

Thus, (Φ_D, A_D) is the unique classical solution of the Darwin equations (5.1.1)-(5.1.2).

Remark 11. As expected, the vector potential A_D satisfies the Coulomb gauge condition $\nabla \cdot A_D = 0$. On the other hand, in general $\nabla \times A_D \neq 0$, and so despite of the approximation made to the Maxwell equations, we still have a system with a fully coupled magnetic field; cf. (4.1.2). However, if the current density j has spherical symmetry, then it is clear that $\nabla \times A_D = 0$. Therefore, the magnetic field vanishes and the *div-curl* representation of A_D [26, p. 139] implies that $A_D \equiv 0$ on $\mathbb{R}^3$. In this case, the Darwin equations (5.1.1)-(5.1.2) are simply reduced to the Poisson equation for the scalar potential Φ.

Proof of Lemma 21. Parts (a) and (b) are standard results for the Poisson equation, cf. [42, Lemma P1], [45, Propositions 1 and 2] and the references therein. In particular, the existence of the solution (in a much weaker sense) can be found in [46, Theorem 6.21] while its regularity is given in [46, Theorem 10.3]. Uniqueness is usually called Liouville's theorem [25, Theorem 7 Sec.4.2]. Also, the L^∞ estimate on $\nabla\Phi_D$ is a straightforward consequence of Lemma 20 given above. Estimating $\partial_x^2\Phi_D$, on the other hand, is the result of the following consideration, cf. [46, Eq.(10) p.227]: for any $h > 0$ and $x \in \mathbb{R}^3$

$$\begin{aligned} \partial_l\partial_k\Phi_D(x) &= \int_{|y-x|>h} \left(3\omega^l\omega^k - \delta_{lk}\right)\rho(y)\frac{dy}{|y-x|^3} \\ &\quad + \int_{|y-x|\le h} \left(3\omega^l\omega^k - \delta_{lk}\right)\left(\rho(y)-\rho(x)\right)\frac{dy}{|y-x|^3} - \frac{4\pi}{3}\delta_{lk}\rho(x). \end{aligned} \tag{5.1.7}$$

Notice that the singularity at $|y-x| = 0$ in the second integral is avoided by the difference $\rho(y) - \rho(x)$. Thus, by letting $0 < h \le R$, we obtain

$$\begin{aligned} \left|\partial_x^2\Phi_D(x)\right| &\le 4\|\rho\|_{L_x^\infty}\int_{h<|y-x|\le R}\frac{dy}{|y-x|^3} + 4\int_{|y-x|>R}\rho(y)\frac{dy}{|y-x|^3} \\ &\quad + 4\|\nabla\rho\|_{L_x^\infty}\int_{|y-x|\le h}\frac{dy}{|y-x|^2} + \frac{4\pi}{3}|\rho(x)| \\ &\le C\left[\ln(R/h)\|\rho\|_{L_x^\infty} + R^{-3}\|\rho\|_{L_x^1} + h\|\nabla\rho\|_{L_x^\infty} + \|\rho\|_{L_x^\infty}\right], \end{aligned}$$

which yields the first estimate. The second estimate on $\partial_x^2\Phi_D$ then follows by setting $R = 1$ and letting $h = \|\nabla\rho\|_{L_x^\infty}^{-1}$ if $\|\nabla\rho\|_{L_x^\infty} \ge 1$, otherwise $h = 1$.

To prove (c), we first rewrite the Poisson equation for the vector potential as

$$\begin{aligned} \Delta A(x) &= -\frac{4\pi}{c}\mathbb{P}j(x) \\ &= -\frac{4\pi}{c}j(x) - \frac{1}{c}\int_{\mathbb{R}^3}\nabla\cdot j(y)\frac{y-x}{|y-x|^3}dy, \end{aligned} \tag{5.1.8}$$

which follows by the regularity and compact support of j, and the same standard arguments needed to represent $\nabla\Phi_D$ in (a); i.e., we remove a small ball about $x \in \mathbb{R}^3$ in the domain of integration of the second term in the right-hand side of (5.1.5) so we can avoid the singularity at $y = x$; we then shift variables and take derivatives of $\nabla \cdot j$ under the integral; and finally integrate by parts noticing that the boundary term corresponding to the small ball vanishes as its radii tends to 0. This argument will be used several times in the remainder of the proof without further notice.

We want to show that A_D in Definition 6 is a C^2 solution of (5.1.8), where $\mathbb{P}j$ has a decay $O(|x|^{-2})$ as $|x| \to \infty$ but does not have compact support. To this end, we rely on the representation for A_D given in Lemma 19, namely

$$A_D(x) \quad = \quad \frac{1}{c}\int_{\mathbb{R}^3} j(y)\frac{dy}{|y-x|} + \frac{1}{2c}\int_{\mathbb{R}^3} \nabla \cdot j(y)\frac{y-x}{|y-x|}dy. \tag{5.1.9}$$

We shall deduce the Poisson equation (5.1.8) from (5.1.9) by direct computation, in view of the regularity and compact support of j. To do so, we first notice that

$$\Delta\left\{\frac{1}{c}\int_{\mathbb{R}^3} j(y)\frac{dy}{|y-x|}\right\} = -\frac{4\pi}{c}j(x) \tag{5.1.10}$$

for all $x \in \mathbb{R}^3$, exactly as it does for the scalar potential Φ_D in part (a). Hence, we only have to deal with the second integral in the right-hand side of (5.1.9). Indeed, let $\epsilon > 0$. Since $j \in C_0^2(\mathbb{R}^3;\mathbb{R}^3)$, integration by parts yields

$$\begin{aligned}
\int_{|y-x|\geq\epsilon} \nabla_y \left\{\nabla \cdot j(y)\right\}\omega^i dy \quad &= \quad -\int_{|y-x|\geq\epsilon} \nabla \cdot j(y)\nabla_y \omega^i dy + \int_{|y-x|=\epsilon} \nabla \cdot j(y)\omega^i \omega dS_y \\
&= \quad -\int_{|y-x|\geq\epsilon} \nabla \cdot j(y)\left[\hat{e}_i - \omega^i\omega\right]\frac{dy}{|y-x|} \\
&\qquad +\epsilon^2\int_{|\omega|=1} \nabla \cdot j(x+\omega\epsilon)\omega^i\omega d\omega,
\end{aligned}$$

where the identity (3.1.4) has been used in the first term of the last equality. Clearly, all the above integrals converge as $\epsilon \to 0$. In particular, the last integral converges to zero. Thus, after shifting variables once, then taking the derivative under the integral, and shifting variables again, we obtain that

$$\begin{aligned}
\nabla_x\left\{\int_{\mathbb{R}^3} \nabla \cdot j(y)\omega^i dy\right\} \quad &= \quad \lim_{\epsilon\to 0}\int_{|y-x|\geq\epsilon} \nabla_y\left[\nabla \cdot j(y)\right]\omega^i dy \\
&= \quad -\int_{\mathbb{R}^3} \nabla \cdot j(y)\left[\hat{e}_i - \omega^i\omega\right]\frac{dy}{|y-x|}.
\end{aligned} \tag{5.1.11}$$

Set $r := |y - x|$. By using the identities (3.1.2)-(3.1.4), it is straightforward to check that $\omega \cdot \nabla_y \omega^i \equiv 0$ and so for $r > 0$, we obtain

$$\nabla_y \cdot \left\{ r^{-1} \left[\hat{e}_i - \omega^i \omega \right] \right\} \equiv -r^{-1} \omega^i \nabla_y \cdot \omega \equiv -2r^{-2} \omega^i. \tag{5.1.12}$$

Therefore, similar arguments to those used to find (5.1.11) yield

$$\nabla_x \cdot \left\{ - \int_{\mathbb{R}^3} \nabla \cdot j(y) \left[\hat{e}_i - \omega^i \omega \right] \frac{dy}{|y - x|} \right\} = -2 \int_{\mathbb{R}^3} \nabla \cdot j(y) \frac{\omega^i dy}{|y - x|^2}. \tag{5.1.13}$$

Hence, since for any vector field A we have that $\Delta A \equiv \hat{e}_i \nabla \cdot \nabla A^i$, we can combine (5.1.11) with (5.1.13) to find that, for each $x \in \mathbb{R}^3$,

$$\Delta \left\{ \frac{1}{2c} \int_{\mathbb{R}^3} \nabla \cdot j(y) \frac{y - x}{|y - x|} dy \right\} = -\frac{1}{c} \int_{\mathbb{R}^3} \nabla \cdot j(y) \frac{y - x}{|y - x|^3} dy. \tag{5.1.14}$$

Thus, by adding (5.1.10) to (5.1.14), we conclude that $\Delta A_D = -4\pi c^{-1} \mathbb{P} j$ holds on $\mathbb{R}^3$, and so A_D is a C^2 solution of the Poisson equation (5.1.5). Moreover, this solution is unique by Liouville's theorem; cf. [25, Theorem 7 Sec. 4.2].

We conclude by obtaining the explicit representation of $\partial_x A_D$. Indeed, since A_D is a C^2 solution of (5.1.5), we have for every $x \in \mathbb{R}^3$ that

$$\partial_x A_D(x) = \int_{\mathbb{R}^3} \partial_x \mathcal{K}(x, y) j(y) dy, \tag{5.1.15}$$

where $\mathcal{K}(x, y) := (2c)^{-1} |y - x|^{-1} [\mathtt{id} + \omega \otimes \omega]$. Then, similarly to (5.1.12), we find for $r > 0$ and $i, k, m = 1, 2, 3$ that

$$\partial_k \left\{ r^{-1} \left[\delta_{im} + \omega^i \omega^m \right] \right\} = r^{-2} \left[-\delta_{im} \omega^k + \delta_{km} \omega^i - 3\omega^i \omega^k \omega^m + \delta_{ik} \omega^m \right]. \tag{5.1.16}$$

This relation combined with (5.1.15) provide the result. The vanishing divergence and the relation for the rotational of A_D readily follow.

To prove (d), we start by noticing that the estimate on $\partial_x A_D$ is a straightforward consequence of Lemma 20. Then, we are only left to prove the estimates on $\partial_x^2 A_D$.

In order to do so, consider

$$\begin{aligned}\partial_l\partial_k A^i &\equiv \frac{1}{2c}\left\{\partial_l\int_{\mathbb{R}^3}\left[\delta_{im}\omega^k-\delta_{km}\omega^i-\delta_{ik}\omega^m\right]j^m(y)\frac{dy}{|y-x|^2}\right.\\ &\quad\left.+3\partial_l\int_{\mathbb{R}^3}j^m(y)\frac{\omega^i\omega^k\omega^m dy}{|y-x|^2}\right\}\\ &=: \frac{1}{2c}\left(I+3II\right).\end{aligned} \tag{5.1.17}$$

Estimating I is exactly as estimating the function $\partial_l\partial_k\Phi_D$. We just split I into five integrals: one corresponding to j^i, another to j^k and the remaining integrals corresponding to the dot product in the third term of I. These integrals are essentially the same as $\partial_l\left(\partial_k\Phi\right)$; we just have to replace the charge density ρ by the components of the current density j; cf. (5.1.4). Thus, for each $0<h\leq R$,

$$I\leq C\left[\ln\left(R/h\right)\|j\|_{L_x^\infty}+R^{-3}\|j\|_{L_x^1}+h\|\partial_x j\|_{L_x^\infty}+\|j\|_{L_x^\infty}\right]. \tag{5.1.18}$$

Estimating II, on the other hand, can be done as follows. We first compute for $r>0$

$$\Gamma(y-x):=\partial_l\left[\frac{\omega^i\omega^k\omega^m}{r^2}\right]=\frac{1}{r^3}\left[\delta_{ml}\omega^i\omega^k+\delta_{il}\omega^k\omega^m+\delta_{kl}\omega^i\omega^m-5\omega^i\omega^k\omega^m\omega^l\right],$$

with $i,k,l,m=1,2,3$. Hence, since $y^iy^ky^m\,|y|^{-5}$ is homogeneous of degree -2,

$$\int_{R_1<|y|<R_2}\Gamma(y)dy=\int_{|y|=R_2}\frac{y^l}{R_2}\frac{y^iy^ky^m}{|y|^5}dS_y-\int_{|y|=R_1}\frac{y^l}{R_1}\frac{y^iy^ky^m}{|y|^5}dS_y=0,$$

and thus

$$\int_{\mathbb{R}^3}\Gamma(y)dy=0. \tag{5.1.19}$$

Therefore, we can proceed as it is done for $\partial_l\partial_k\Phi_D$ in part (a) to obtain

$$\begin{aligned}II &= \int_{|y-x|>h}\left(5\omega^i\omega^k\omega^l\omega^m-\delta_{ml}\omega^i\omega^k-\delta_{il}\omega^k\omega^m-\delta_{kl}\omega^i\omega^m\right)j^m(y)\frac{dy}{|y-x|^3}\\ &\quad+\int_{|y-x|\leq h}\left(\cdots\right)\left(j^m(y)-j^m(x)\right)\frac{dy}{|y-x|^3}+j^m(x)\int_{|\omega|=1}\omega^i\omega^k\omega^m\omega^l d\omega.\end{aligned}$$

As a result, for $0 < h \leq R$

$$\begin{aligned} II &\leq C\left\{ \|j\|_{L^\infty_x} \int_{h<|y-x|\leq R} \frac{dy}{|y-x|^3} + \sup_{1\leq i\leq 3} \int_{|y-x|>R} \left|j^i(y)\right| \frac{dy}{|y-x|^3} \right. \\ &\quad \left. \|\partial_x j\|_{L^\infty_x} \int_{|y-x|\leq h} \frac{dy}{|y-x|^2} + \sup_{1\leq i\leq 3} \left|j^i(x)\right| \right\} \\ &\leq C\left[\ln(R/h) \|j\|_{L^\infty_x} + R^{-3} \|j\|_{L^1_x} + h \|\partial_x j\|_{L^\infty_x} + \|j\|_{L^\infty_x} \right]. \qquad (5.1.20) \end{aligned}$$

By combining (5.1.17), (5.1.18) and (5.1.20), the first estimate on $\partial_x^2 A_D$ readily follows. Finally, by setting $R = 1$ and letting $h = \|\partial_x j\|^{-1}_{L^\infty_x}$ if $\|\partial_x j\|_{L^\infty_x} \geq 1$, otherwise $h = 1$, the second estimate follows as well. This concludes the proof of the lemma. □

Remark 12. In the proof of part (c), where we showed that A_D is a C^2 solution of the Poisson equation (5.1.8), we encountered some difficulty due to the lack of compact support in the source term[3] $\mathbb{P}j$. Otherwise, we could just have reasoned as follows. On one hand, we have that the vector field $\bar{A} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$\begin{aligned} \bar{A}(x) &:= \frac{1}{c} \int_{\mathbb{R}^3} \mathbb{P}j(y) \frac{dy}{|y-x|} \\ &\equiv \frac{1}{c} \int_{\mathbb{R}^3} j(y) \frac{dy}{|y-x|} + \frac{1}{4\pi c} \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \nabla \cdot j(z) \frac{z-y}{|z-y|^3} dz \right) \frac{dy}{|y-x|} \qquad (5.1.21) \end{aligned}$$

is a solution of the Poisson equation (5.1.8), at least in the sense of distributions. Notice that $\bar{A}$ is well defined, since we know that $\mathbb{P}j$ satisfies $\|\mathbb{P}j\|_{L^q_x} \leq C(q) \|j\|_{L^q_x}$ for all $1 < q < \infty$ -e.g., see [44, Lemma 2.3]-. Thus, in view of Lemma 20 and the compact support of j, the integrals in the right-hand side exist almost everywhere.

On the other hand, in view of Lemma 19, the vector field $\tilde{A} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{aligned} \tilde{A}(x) &:= \frac{1}{c} \int_{\mathbb{R}^3} j(y) \frac{dy}{|y-x|} + \frac{1}{4\pi c} \int_{\mathbb{R}^3} \nabla \cdot j(z) \left(\int_{\mathbb{R}^3} \frac{z-y}{|z-y|^3} \frac{dy}{|y-x|} \right) dz \\ &= \frac{1}{c} \int_{\mathbb{R}^3} j(y) \frac{dy}{|y-x|} + \frac{1}{2c} \int_{\mathbb{R}^3} \nabla \cdot j(z) \frac{z-x}{|z-x|} dz \qquad (5.1.22) \end{aligned}$$

is the Darwin vector potential A_D. Here we have used the integral

$$\int_{\mathbb{R}^3} \frac{z-y}{|z-y|^3} \frac{dy}{|y-x|} = 2\pi \frac{z-x}{|z-x|}, \qquad (5.1.23)$$

[3]Even though j has compact support, the vector field $\mathbb{P}j$ does not. However, we do know that it satisfies $\mathbb{P}j(x) = O(|x|^{-2})$ as $|x| \to \infty$, which can be inferred from (5.1.8).

whose computation is relegated to the Appendix C, cf. (C.0.1).

Now, the right-hand side of (5.1.21) is formally the right-hand side of (5.1.22), with the corresponding exchange of integrals. Hence, *if* $\mathbb{P}j$ had compact support, we could have used Tonelli's and then Fubini's theorem to show that $\bar{A} = \tilde{A} \equiv A_D$ almost everywhere. This, and then showing that A_D has the required regularity, would have proved the claim. However, the vector field $\mathbb{P}j$ is not compactly supported, and the issue of whether the exchange of integrals holds for the whole space is not as simple as before. Therefore, in the proof given above, we have opted for using direct computation to show that $\tilde{A} \equiv A_D$ is a C^2 solution of the Poisson equation (5.1.8). Incidentally, uniqueness then implies that, indeed, $\bar{A} = \tilde{A}$ on $\mathbb{R}^3$.

Remark 13. For simplicity, we have assumed $j \in C_0^2(\mathbb{R}^3;\mathbb{R}^3)$ in Lemma 21(c)-(d). Nevertheless, *it seems* plausible to relax the regularity of the current density by $j \in C_0^{1,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$ with $0 < \alpha < 1$, and still be able to show that the Darwin vector potential A_D is the unique $C^2(\mathbb{R}^3;\mathbb{R}^3)$ solution of

$$\Delta A = -\frac{4\pi}{c}\mathbb{P}j(x), \qquad \lim_{|x|\to\infty} |A(x)| = 0.$$

Indeed, if $j \in C_0^{1,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$ with $0 < \alpha < 1$, then by standard arguments we can show that both $\tilde{A}$ and $\bar{A}$, as defined in (5.1.22) and (5.1.21) respectively, are C^2 vector-valued functions. Hence, the result would follow *if* we could justify the exchange of integrals mentioned in Remark 12.

5.2 The Vlasov-Darwin system

We defined the relativistic Vlasov-Darwin system by the coupling of the Vlasov equation (4.2.12) with the Darwin equations (5.1.1)-(5.1.2) via the charge and current densities, i.e.,

$$\partial_t f + v_A \cdot \nabla_x f - \left[\nabla\Phi - \frac{v_A^i}{c}\nabla A^i\right] \cdot \nabla_p f = 0, \tag{5.2.1}$$

$$v_A = \frac{c^2 p - cA}{\sqrt{c^4 + |cp - A|^2}}, \tag{5.2.2}$$

on $I \times \mathbb{R}^3 \times \mathbb{R}^3$, coupled with

$$\Delta\Phi = -4\pi\rho, \quad \lim_{|x|\to 0} \Phi(t,x) = 0 \tag{5.2.3}$$

$$\Delta A = -\frac{4\pi}{c} j + \frac{1}{c}\nabla\partial_t\Phi, \quad \lim_{|x|\to 0} A(t,x) = 0 \tag{5.2.4}$$

on $I \times \mathbb{R}^3$, via

$$j = \int_{\mathbb{R}^3} v_A f dp, \quad \rho = \int_{\mathbb{R}^3} f dp. \tag{5.2.5}$$

Notice that in contrast with the previous chapter, here we have denoted the generalized momentum variable by p instead of $\mathtt{P}$. Also, we have dropped the tilde from the one-particle distribution function in (4.2.12) to simplify notation. Moreover, in order to emphasize the dependence of j on the vector potential A, we shall henceforth denote the current density by j_A.

As shown in the Section 5.3 below, this definition of the RVD system is formally equivalent to the usual one, where the Darwin approximation is the result of neglecting the *transverse* part of the displacement current $\partial_t E$ in Ampère's law [44, 47, 48]. The system (5.2.1)-(5.2.5), on the other hand, deals directly with the potential representation of the corresponding equations. As usual, the electric and magnetic fields can be recovered by means of the relations (4.1.1)-(4.1.2). Notice, however, that Definition 5 is gauge dependent and thus the resulting field is *not* invariant under gauge transformations. We emphasize that the relativistic velocity satisfies $|v_A| \leq c$ for all $t \in I$ and $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$; c being the speed of light. This fact will be used extensively and without further notice in the upcoming computations.

Now, let the triplet (f, Φ, A) be smooth and satisfy the system (5.2.1)-(5.2.5). Assume that f has compact support in the (generalized) momentum variable. Then, as a result of Lemma 14, integrating (5.2.1) over all $p \in \mathbb{R}^3$ produces the continuity equation

$$0 = \partial_t\rho + \nabla_x \cdot j_A - \int_{\mathbb{R}^3} \nabla_p \cdot \left[\left(\nabla\Phi - \frac{v_A^i}{c}\nabla A^i \right) f \right] dp = \partial_t\rho + \nabla \cdot j_A.$$

Hence, by Lemma 21 and Remark 10, after combining Definition 6 with (5.2.5) we find that the potentials (Φ, A) can be represented in terms of the one-particle distribution function f according to

$$\Phi(t,x) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,y,p)\frac{dpdy}{|y-x|} \tag{5.2.6}$$

$$A(t,x) = \frac{1}{2c}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} [\mathtt{id}+\omega\otimes\omega]\, v_A f(t,y,p)\frac{dpdy}{|y-x|}, \tag{5.2.7}$$

where v_A itself depends on A via (5.2.2). Now, assume that f is *given* and consider (5.2.7) as an integral equation of unknown A. The following lemma shows that there exists a unique C^2 bounded solution of this equation provided that f satisfies some suitable conditions. Precisely, we have

Lemma 22. *Fix $t \in I$ and let $f(t) \in C_0^1(\mathbb{R}^6;\mathbb{R})$ be given. Define ρ by (5.2.5) and assume that f is such that*

$$C_*(t;f) := 3^3(4\pi)^{1/3}c^{-2}\,\|\rho(t)\|_{L_x^1}^{2/3}\,\|\rho(t)\|_{L_x^\infty}^{1/3} < 1, \quad t \in I. \tag{5.2.8}$$

Then, there exists a unique $A(t) \in C_b \cap C^2(\mathbb{R}^3;\mathbb{R}^3)$ satisfying (5.2.7).

Proof. Without loss of generality, we omit the time dependence. First, we show that there exists a unique $A_\infty \in C_b(\mathbb{R}^3;\mathbb{R}^3)$ solving (5.2.7). To this end, consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T[A](x) = \frac{1}{2c}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[\mathtt{id}+\omega\otimes\omega]\, v_A f(y,p)\frac{dpdy}{|y-x|}, \qquad v_A = \frac{c^2p - cA}{\sqrt{c^4 + |cp-A|^2}}.$$

For every fixed $p \in \mathbb{R}^3$, the map $A \mapsto v_A$ is C_b^∞ and so for all A_1, A_2 in $C_b(\mathbb{R}^3)$ an estimate $|v_{A_2} - v_{A_1}| \le C\,|A_2 - A_1|$ holds. Actually, $C = 6c^{-1}$ will do. Therefore

$$\begin{aligned} |T[A_2](x) - T[A_1](x)| &\le 2(3)^2c^{-2}\int_{\mathbb{R}^3}|A_2(y)-A_1(y)|\,\rho(y)\frac{dy}{|y-x|} \\ &\le 3^3(4\pi)^{1/3}c^{-2}\,\|\rho\|_{L_x^1}^{2/3}\,\|\rho\|_{L_x^\infty}^{1/3}\,\|A_2 - A_1\|_{L_x^\infty}, \end{aligned} \tag{5.2.9}$$

where Lemma 20 has been used. Hence, after taking the supremum over all $x \in \mathbb{R}^3$, the assertion follows in view of (5.2.8) and the Banach fixed point theorem.

Now, if we define j_{A_∞} according to (5.2.5), then the vector potential A_∞ satisfies

$$A_\infty(x) = \int_{\mathbb{R}^3}\mathcal{K}(x,y)j_{A_\infty}(y)dy, \tag{5.2.10}$$

where $\mathcal{K}(x,y) := (2c)^{-1}|y-x|^{-1}[\mathtt{id}+\omega\otimes\omega]$. Clearly, A_∞ has the form of a Darwin potential A_D with current density $j_{A_\infty}\in C_0(\mathbb{R}^3;\mathbb{R}^3)$. We want to show that A_∞ has the required regularity, but in contrast with Lemma 21(c) here j_{A_∞} does not have enough regularity to proceed by direct computation. Therefore, we must rely on the theory of distributions. We shall only sketch the proof as it is similar to the analogous result for the Poisson equation, cf. [46, Theorem 10.2].

Indeed, we do know that $j_{A_\infty}\in C(\mathbb{R}^3;\mathbb{R}^3)$ has compact support, and so we can prove that the distributional derivative of A_∞ is a function given by

$$\partial_x A_\infty(x) = \int_{\mathbb{R}^3} \partial_x\mathcal{K}(x,y) j_{A_\infty}(y)dy \tag{5.2.11}$$

for almost all $x\in\mathbb{R}^3$ -cf. Lemma 21(c) for the explicit representation of $\partial_x\mathcal{K}(x,y)$-. To prove so, we must show that the right-hand side of (5.2.11) is well defined for almost all $x\in\mathbb{R}^3$, and that for any test function $\phi\in C_0^\infty(\mathbb{R}^3;\mathbb{R})$

$$\int_{\mathbb{R}^3}\partial_x\phi(x)A_\infty(x)dx = -\int_{\mathbb{R}^3}\phi(x)\left\{\int_{\mathbb{R}^3}\partial_x\mathcal{K}(x,y)j_{A_\infty}(y)dy\right\}dx. \tag{5.2.12}$$

The former is consequence of $|\partial_x\mathcal{K}(x,y)|\le C|y-x|^{-2}$ and Lemma 20. To prove the latter, we notice that the integrability of the function $\partial_x\phi(x)\mathcal{K}(x,y)j_{A_\infty}(y)$ on $\mathbb{R}^3\times\mathbb{R}^3$ allows to use Fubini's theorem so that

$$\begin{aligned}\int_{\mathbb{R}^3}\partial_x\phi(x)A_\infty(x)dx &= \int_{\mathbb{R}^3}\left\{\int_{\mathbb{R}^3}\partial_x\phi(x)\mathcal{K}(x,y)dx\right\}j_{A_\infty}(y)dy\\ &= -\int_{\mathbb{R}^3}\left\{\int_{\mathbb{R}^3}\phi(x)\partial_x\mathcal{K}(x,y)dx\right\}j_{A_\infty}(y)dy,\end{aligned}$$

where the second equality is justified by a limiting process and integration by parts, similar to the argument following (5.1.8) in the proof of Lemma 21(c). Then, another use of Fubini's theorem yields (5.2.12), which is (5.2.11) in the sense of distributions.

Next, we have to show that the distributional derivative $\partial_x A_\infty$ is a continuous function. If so, then $A_\infty\in C^1(\mathbb{R}^3;\mathbb{R}^3)$ in view of the theorem for the equivalence of classical and distributional derivatives, cf. [46, Theorem 6.10]. We start by noticing that for any $0<\alpha<1$ and $x,y,z\in\mathbb{R}^3$ not equal, we have

$$|(\partial_x\mathcal{K})(x,z)-(\partial_x\mathcal{K})(y,z)|\le C|x-y|^\alpha\left(|z-x|^{-2-\alpha}+|z-y|^{-2-\alpha}\right), \tag{5.2.13}$$

whose proof is sketched in Appendix D. Hence, we obtain

$$\begin{aligned} \left|\left(\partial_x A_\infty\right)(x)-\left(\partial_x A_\infty\right)(y)\right| &\leq \int_{\mathbb{R}^3}\left|\left(\partial_x \mathcal{K}\right)(x,z)-\left(\partial_x \mathcal{K}\right)(y,z)\right|\left|j_{A_\infty}(z)\right| dz \\ &\leq C\left|x-y\right|^\alpha \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\left|z-x\right|^{-2-\alpha}\left|j_{A_\infty}(z)\right| dz \\ &\leq C(\alpha)\left|x-y\right|^\alpha \left\|j_{A_\infty}\right\|_{L^1_x}^{(1-\alpha)/3}\left\|j_{A_\infty}\right\|_{L^\infty_x}^{(\alpha+2)/3}. \end{aligned} \quad (5.2.14)$$

Notice the use of Lemma 20 in the last estimate, provided $0<\alpha<1$. Therefore, $A_\infty \in C^1(\mathbb{R}^3;\mathbb{R}^3)$ and so $j_{A_\infty} \in C^1_0(\mathbb{R}^3;\mathbb{R}^3)$. But such a regularity in the current density allows to proceed by direct computation, just as in the proof of Lemma 21(c). Thus, we conclude that $A_\infty \in C^2(\mathbb{R}^3;\mathbb{R}^3)$ and the proof of the lemma is complete. □

Remark 14. From (5.2.14), it follows that $\partial_x A_\infty$ is actually Hölder continuous of order α, i.e., $A_\infty \in C^{1,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$ with $0<\alpha<1$. We claim that $A_\infty \in C^{2,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$. Indeed, we still have that $j_{A_\infty} \in C^1_0(\mathbb{R}^3;\mathbb{R}^3)$ since we have assumed $f \in C^1_0(\mathbb{R}^3;\mathbb{R}^3)$ -we omit the time dependence-. Thus, after shifting variables once, taking derivatives under the integral, and shifting variables again, we have

$$\partial_x A_\infty(x) = \int_{\mathbb{R}^3}\mathcal{K}(x,z)\left(\partial_x j_{A_\infty}\right)(z)dz. \quad (5.2.15)$$

But (5.2.15) has the form of (5.2.10) with j_{A_∞} replaced by $\partial_x j_{A_\infty} \in C_0(\mathbb{R}^3;\mathbb{R}^9)$. Therefore, we can reason as above to conclude that $A_\infty \in C^{2,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$.

Remark 15. We could obtain such a regularity for the vector potential even with less regularity for the current density. More generally, for $k\geq 0$ and $0<\alpha<1$, if $j_{A_\infty} \in C^{k,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$ with compact support, then $A_\infty \in C^{k+2,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$. The analogous result for solutions of the Poisson equation is standard, and can be found in [46, Theroem 10.3]. It can be written mutatis mutandis for the Darwin vector potential

$$A_\infty(x) = \int_{R^3}\mathcal{K}(x,y)j_{A_\infty}(y)dy,$$

provided $\left|\partial_x^2\mathcal{K}(x,y)\right| \leq C\left|y-x\right|^{-3}$ and

$$\int_{\mathbb{R}^3}\partial_x^2\mathcal{K}(y)dy = 0.$$

These properties of the kernel can be easily verified along the lines of the proof of

Lemma 21(d). In particular, we use (5.1.19) to obtain the vanishing integral.

We shall define the Cauchy problem for the RVD system as follows:

Definition 7. *Let $f_0 \in C^1(\mathbb{R}^6;\mathbb{R})$, $f_0 \geq 0$. The function f is said to be a classical solution of the RVD system if $f \in C^1(I \times \mathbb{R}^6;\mathbb{R})$; it induces the scalar and vector potentials $\Phi \in C^1(I, C^2(\mathbb{R}^3);\mathbb{R})$ and $A \in C(I, C^2(\mathbb{R}^3);\mathbb{R}^3)$ via (5.2.6)-(5.2.7); for every compact subinterval $\bar{J} \subset I$ the fields $\nabla\Phi$ and $v_A^i \nabla A^i$ are bounded on $\bar{J} \times \mathbb{R}^3$; and the triplet (f, Φ, A) satisfies the RVD system (5.2.1)-(5.2.5) on $I \times \mathbb{R}^3 \times \mathbb{R}^3$. Moreover, f is said to be a classical solution of the Cauchy problem if $f|_{t=0} = f_0$.*

We conclude this section with a technical lemma that we shall use later on.

Lemma 23. *Let f_1 and f_2 be two classical solutions of the RVD having the same Cauchy datum f_0, and let A_1 and A_2 be the corresponding induced vector potentials. Define $C_*(t, f_2)$ according to (5.2.8) in Lemma 22. If $C_*(t, f_2) < 1$, then for all $t \in I$ there exist positive constants $C^0 = C(f_0)$ and $C^0(t) = C(t; f_0)$ such that*

$$\|A_2(t) - A_1(t)\|_{L^\infty_x} \;\; \leq \;\; C^0 \, \|f_2(t) - f_1(t)\|^{1/3}_{L^1_p;L^\infty_x} \tag{5.2.16}$$

and

$$\|\partial_x A_2(t) - \partial_x A_1(t)\|_{L^\infty_x} \;\; \leq \;\; C^0(t) \, \|f_2(t) - f_1(t)\|^{2/3}_{L^1_p;L^\infty_x} \, . \tag{5.2.17}$$

The function $C(t; f_0)$ is continuous in t.

Proof. The estimate (5.2.16) is proved first. Indeed, since the pairs (f_k, A_k), $k = 1, 2$ satisfy the RVD system, we can represent the vector potentials according to (5.2.7). Hence, since $A \mapsto v_A$ is C^∞_b and so $|v_{A_2} - v_{A_1}| \leq C\,|A_2 - A_1|$, we have that

$$\begin{aligned} |v_{A_2} f_2 - v_{A_1} f_1| \;\; &\leq \;\; |v_{A_2} - v_{A_1}|\, f_2 + |f_2 - f_1|\, |v_{A_1}| \\ &\leq \;\; C\,|A_2 - A_1|\, f_2 + |f_2 - f_1| \, . \end{aligned} \tag{5.2.18}$$

Therefore, the use of Lemma 20 yields

$$\begin{aligned}|A_2(t,x) - A_1(t,x)| &\leq C\int_{\mathbb{R}^3} |A_2(t,y) - A_1(t,y)|\,\rho_2(y)\frac{dy}{|y-x|} \\ &\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f_2(t,y,p) - f_1(t,y,p)|\,\frac{dpdy}{|y-x|} \\ &\leq C_*(t,f_2)\,\|A_2(t) - A_1(t)\|_{L^\infty_x} \\ &\quad + C\,\|f_2(t) - f_1(t)\|^{2/3}_{L^1_{x,p}}\,\|f_2(t) - f_1(t)\|^{1/3}_{L^1_p;L^\infty_x}. \qquad (5.2.19)\end{aligned}$$

Notice that estimating the second term in the right-hand side of the first inequality follows by letting $\Psi(t,y) := \|f_2(t,y) - f_1(t,y)\|_{L^1_p}$ in Lemma 20. Now, take the supremum over all $x \in \mathbb{R}^3$. The structure of the Vlasov equation implies that $\|f_1(t)\|_{L^1_{x,p}} = \|f_0\|_{L^1_{x,p}} = \|f_2(t)\|_{L^1_{x,p}}$; cf. Lemma 15. Hence, since we have assumed $C_*(t,f_2) < 1$, it is easy to check that (5.2.16) holds indeed.

To prove (5.2.17) we proceed similarly, except that we start with the representation for the space derivatives of the induced vector potentials. First, we denote the kernel $\mathcal{K}(x,y) := (2c)^{-1}\,|y-x|^{-1}\,[\mathtt{id} + \omega\otimes\omega]$. Then, it is clear that

$$|\partial_x A_2(t,x) - \partial_x A_1(t,x)| \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\partial_x \mathcal{K}(x,y)|\,|v_{A_2}f_2(t,y,p) - v_{A_1}f_1(t,y,p)|\,dpdy.$$

Thus, since $|\partial_x\mathcal{K}(x,y)| \leq C\,|y-x|^{-2}$; cf. (5.1.16), we have by virtue of (5.2.18) and Lemma 20 that

$$\begin{aligned}|\partial_x A_2(t,x) - \partial_x A_1(t,x)| &\leq C\,\|\rho_2(t)\|^{1/3}_{L^1_x}\,\|\rho_2(t)\|^{2/3}_{L^\infty_x}\,\|A_2(t) - A_1(t)\|_{L^\infty_x} \\ &\quad + C\,\|f_2(t) - f_1(t)\|^{1/3}_{L^1_{x,p}}\,\|f_2(t) - f_1(t)\|^{2/3}_{L^1_p;L^\infty_x}. \qquad (5.2.20)\end{aligned}$$

It is now an easy matter to check that (5.2.17) holds, we just have to use (5.2.16) to estimate the first term in the right-hand side and the triangle inequality to estimate the second one. Then, the regularity of solutions for the RVD system easily implies that the resulting $C^0(t) = C(t; f_0)$ is continuous in t. □

5.2.1 Local Solutions

Theorem 9. *Let $f_0 \in C^2_0(\mathbb{R}^6;\mathbb{R})$, $f_0 \geq 0$ and set*

$$\bar{P}_0 := \sup\left\{|p| : \exists x \in \mathbb{R}^3 : f_0(x,p) \neq 0\right\}.$$

If the Cauchy datum satisfies

$$\mathcal{A}_0 := 3^2(12\pi)^{2/3}c^{-2}\,\|f_0\|^{2/3}_{L^1_{x,p}}\,\|f_0\|^{1/3}_{L^\infty_{x,p}}\,\bar{P}_0 < 1, \tag{5.2.21}$$

then for some $T > 0$ there is a unique classical solution f of the RVD system on $[0, T[$ satisfying $f|_{t=0} = f_0$. For each $0 \le t < T$ the function $f(t)$ is non-negative and has compact support. Moreover, if $T > 0$ is the life span of f, then

$$\mathcal{A}_0\bar{P}_0^{-1}\sup\left\{|p| : \exists 0 \le t < T, x \in \mathbb{R}^3 : f(t,x,p) \neq 0\right\} < 1 \tag{5.2.22}$$

implies that the solution is global in time, i.e., $T = \infty$

One of the advantages of the formulation (5.2.1)-(5.2.5) is the similarity it has with the Vlasov-Poisson system. As the latter is well understood, it seems reasonable to transfer some of the known techniques to the Darwin model case. Our proof leans on the existence result for local solutions of the Vlasov-Poisson system as presented in [42]. In contrast, here we must deal with additional non-linear terms in the Vlasov equation that result from the inclusion of the vector potential A. Moreover, we have a non-linear equation satisfied by the vector potential itself.

The idea of the proof is the following. First, we construct a sequence of smooth approximations $\{f^n\}$ which induce the approximated potentials $\{\Phi^n, A_n\}$. Then, we show that there is a time interval $[0, \bar{T}]$ on which the momentum supports of the f^n's are uniformly bounded and the condition (5.2.8) holds, uniformly in n. In view of the known estimates and the smallness assumption (5.2.21), we are able to show that $\{f^n\}$ is uniformly Cauchy on $[0, \bar{T}] \times \mathbb{R}^6$ and so is $\{\Phi^n, A_n\}$ on $[0, \bar{T}] \times \mathbb{R}^3$. Hence, a limit (f, Φ, A) exists which, as it turns out, has the required regularity and satisfies the remaining properties of Definition 7.

As we did for the RVM system in Chapter 3, we shall present the proof in several steps. For simplicity, we shall omit details already discussed in the RVM's proof. Finally, notice that henceforth the speed of light is set to be one, i.e., $c = 1$ and that as usual all constants may change values from line to line.

Proof of Theorem 9. Let $f_0 \in C^2_0(\mathbb{R}^6; \mathbb{R})$, $f_0 \ge 0$ be fixed satisfying (5.2.21) and let $A_0 : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be the solution of the integral equation

$$A_0(x) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left[\mathtt{id} + \omega\otimes\omega\right]v_{A_0} f_0(y,p)\frac{dpdy}{|y-x|}, \quad v_{A_0} := \frac{p - A_0}{\sqrt{1 + |p - A_0|^2}}.$$

In view of Lemma 22, $A_0 \in C_b \cap C^2(\mathbb{R}^3;\mathbb{R}^3)$. We construct the iterative scheme as follows. For $t \in I$ and $z = (x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$, define

$$f^0(t,z) = f_0(z), \quad A_0(t,x) = A_0(x).$$

For $n \in \mathbb{N}$, assume that $f^n : I \times \mathbb{R}^6 \mapsto \mathbb{R}$ and $A_n : I \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ are given. We set

$$v_{A_n} := \frac{p - A_n}{\sqrt{1 + |p - A_n|^2}},$$

$$j^n(t,x) := \int_{\mathbb{R}^3} v_{A_n} f^n(t,x,p)dp, \quad \rho^n(t,x) := \int_{\mathbb{R}^3} f^n(t,x,p)dp,$$

and

$$\Phi^n(t,x) := \int_{\mathbb{R}^3} \rho^n(t,y)\frac{dy}{|y-x|}.$$

Then, we define the $(n+1)$-th iterate of the vector potential by

$$A_{n+1}(t,x) := \frac{1}{2}\int_{\mathbb{R}^3} [\mathtt{id} + \omega \otimes \omega]\, j^n(t,y)\frac{dy}{|y-x|},$$

and, if we denote by $Z_n := (X_n, P_n)(s,t,z)$ the solution of the characteristic system

$$\begin{aligned} \dot{X}_n(s,t,z) &= v_{A_{n+1}}(s, X_n(s,t,z), P_n(s,t,z)) \\ \dot{P}_n(s,t,z) &= -\left[\nabla\Phi^n - v^i_{A_{n+1}}\nabla A^i_{n+1}\right](s, X_n(s,t,z), P_n(s,t,z)) \end{aligned}$$

with $Z_n(t,t,z) = z$, we define the $(n+1)$-th iterate of the one-particle distribution function by

$$f^{n+1}(t,z) := f_0(Z_n(0,t,z)).$$

Step 1. By virtue of Lemmas 15, 21 and 22, it is not difficult to check that the iterates are well defined. In particular, we have that $f^n \in C^1(I \times \mathbb{R}^6;\mathbb{R})$, $f^n \geq 0$ and $A_n \in C^1(I, C^2(\mathbb{R}^3);\mathbb{R}^3)$; thus $j^n \in C^1(I \times \mathbb{R}^3;\mathbb{R}^3)$. Notice that by construction, $A_1(t,x) \equiv A_0(x)$ for all $t \in I$ and $x \in \mathbb{R}^3$, so A_n has the regularity of f_{n-1} with respect to t, for all $n \geq 2$. Finally, $\rho^n \in C^1(I \times \mathbb{R}^3;\mathbb{R})$, which produces $\Phi^n \in C^1(I, C^2(\mathbb{R}^3);\mathbb{R})$.

On the other hand, for each $t \in I$ we have $\mathtt{supp} f^n(t) = Z_{n-1}(0,t,\mathtt{supp} f_0)$. Set $\bar{X}_0(t) := \sup\{|x| : \exists p \in \mathbb{R}^3 : f_0(x,p) \neq 0\}$, $\bar{P}_0(t) := \sup\{|p| : \exists x \in \mathbb{R}^3 : f_0(x,p) \neq 0\}$

and let $P_0(t,0,z) = P_n(0,0,z) \equiv p \in \mathtt{supp} f_0$. For $n \in \mathbb{N}$ define

$$\begin{aligned} \bar{P}_n(t) &= \sup\left\{|p| : \exists 0 \le s \le t, x \in \mathbb{R}^3 : f^n(s,x,p) \neq 0\right\} \\ &\equiv \sup\left\{|P_{n-1}(s,0,z)| : 0 \le s \le t, z \in \mathtt{supp} f_0\right\}. \end{aligned}$$

It is clear that $\mathtt{supp} f^n(t) \subseteq \left\{(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| \le \bar{X}_0 + t, |p| \le \bar{P}_n(t)\right\}$, $t \in I$. Also, by Lemma 15, the iterates satisfy the estimates

$$\|f^n(t)\|_{L^q_{x,p}} = \|f_0\|_{L^q_{x,p}}, \quad 1 \le q \le \infty,\ t \in I,\ n \in \mathbb{N}$$

and

$$\|\rho^n(t)\|_{L^\infty_x} \le \frac{4}{3}\pi \|f_0\|_{L^\infty_{x,p}} \bar{P}_n^3(t).$$

In view of $|j^n| \le \rho^n$ and Corollary 4, there exists a positive $C^0 > 0$ depending on the Cauchy datum only such that

$$\|\Phi^n(t)\|_{L^\infty_x} + \|A_{n+1}(t)\|_{L^\infty_x} \le C^0 \bar{P}_n(t).$$

Finally, another use of Lemma 21 produces

$$\|\partial_x\Phi^n(t)\|_{L^\infty_x} + \|\partial_x A_{n+1}(t)\|_{L^\infty_x} \le C(f_0)\bar{P}_n^2(t), \quad C(f_0) := 30(2\pi)^{2/3} \|f_0\|_{L^1_x}^{1/3} \|f_0\|_{L^\infty_x}^{2/3}.$$

Step 2. For some $T > 0$ there is a non-negative, non-decreasing $\mathcal{P} \in C([0,T[;\mathbb{R})$ which depends on the Cauchy datum only, such that

$$\bar{P}_n(t) \le \mathcal{P}(t) \quad \text{for all } n \in \mathbb{N}_0,\ 0 \le t < T.$$

Indeed, for $n \in \mathbb{N}$ the characteristic equations and the previous estimate imply that

$$\begin{aligned} |P_n(s,0,z)| &\le |p| + \int_0^s \left(\|\partial_x\Phi^n(\tau)\|_{L^\infty_x} + \|\partial_x A_{n+1}(\tau)\|_{L^\infty_x}\right) d\tau, \\ &\le \bar{P}_0 + C(f_0)\int_0^t \bar{P}_n^2(\tau)d\tau. \end{aligned} \tag{5.2.23}$$

Now let $T > 0$ be the life span of the solution of the integral equation

$$\mathcal{P}(t) = \bar{P}_0 + C(f_0)\int_0^t \mathcal{P}^2(\tau)d\tau. \tag{5.2.24}$$

Trivially, $\bar{P}_0(t) \leq \mathcal{P}(t)$. Suppose that $\bar{P}_n(t) \leq \mathcal{P}(t)$ for some $n \in \mathbb{N}$, $0 \leq t < T$. Using (5.2.23), it is straightforward to check that it also holds for $n+1$. Thus, the claim follows by induction. As a result, all estimates in Step 1 are uniform in n on any subinterval $[0, \bar{T}]$ of $[0, T[$. In particular, for all $n \in \mathbb{N}_0$, $0 \leq t \leq \bar{T}$

$$\|\rho^n(t)\|_{L^\infty_x} + \|j^n(t)\|_{L^\infty_x} + \|\partial_x \Phi^n(t)\|_{L^\infty_x} + \|\partial_x A_n(t)\|_{L^\infty_x} \leq C^0_{\bar{T}} \equiv C(\bar{T}, f_0). \quad (5.2.25)$$

For future use, in view of the assumption (5.2.21) and the continuity of $\mathcal{P}(t)$, we can and will choose a smaller time, again denoted by T, such that it guarantees

$$C_*(t; f_0) := 3^2 (12\pi)^{2/3} \|f_0\|^{2/3}_{L^1_{x,p}} \|f_0\|^{1/3}_{L^\infty_{x,p}} \mathcal{P}(t) < 1, \quad 0 \leq t < T. \quad (5.2.26)$$

We can actually find such a time explicitly. Indeed, the maximal solution of (5.2.24) is given by

$$\mathcal{P}(t) = \frac{\bar{P}_0}{1 - C(f_0)\bar{P}_0 t}, \quad 0 \leq t < \frac{1}{C(f_0)\bar{P}_0}.$$

Therefore, in view of (5.2.26), we can set

$$T := \frac{1 - C_*(f_0)\bar{P}_0}{C(f_0)\bar{P}_0}, \quad C_*(f_0) := 3^2(12\pi)^{2/3} c^{-2} \|f_0\|^{2/3}_{L^1_{x,p}} \|f_0\|^{1/3}_{L^\infty_{x,p}}. \quad (5.2.27)$$

Notice that by the smallness assumption (5.2.21), $C_*(f_0)\bar{P}_0 \equiv \mathcal{A}_0 < 1$, thus $T > 0$.

Step 3. We claim that for every fixed $0 \leq \bar{T} < T$

$$\|\partial_x \rho^n(t)\|_{L^\infty_x} + \|\partial_x j^n(t)\|_{L^\infty_x} + \left\|\partial_x^2 \Phi^n(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A_n(t)\right\|_{L^\infty_x} \leq C^0_{\bar{T}}, \quad (5.2.28)$$

for all $n \in \mathbb{N}_0$, $0 \leq t \leq \bar{T}$. To show that this holds, we first compute the partial space derivatives of the characteristic curves $(X_n, P_n)(s) \equiv (X_n, P_n)(s, t, x, p)$. Recall that

$$v_{A_{n+1}}(s, X_n(s), P_n(s)) \equiv v(P_n(s), A_{n+1}(s, X_n(s))),$$

where v is C_b^∞ in its arguments. Hence, we have

$$\begin{aligned}
|\partial_x X_n(s)| &\leq |\partial_x X_n(t)| + \int_s^t |\partial_x [v(P_n(\tau), A_{n+1}(\tau, X_n(\tau)))]| \, d\tau \\
&\leq 1 + C \int_s^t \Big(|\partial_x P_n(\tau)| + \|\partial_x A_{n+1}(\tau)\|_{L_x^\infty} |\partial_x X_n(\tau)| \Big) \, d\tau \\
&\leq 1 + C_{\bar{T}}^0 \int_s^t (|\partial_x X_n(\tau)| + |\partial_x P_n(\tau))|) \, d\tau.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|\partial_x P_n(s)| &\leq |\partial_x P_n(t)| + \int_s^t \Big(\; |\partial_x [\partial_x \Phi^n(\tau, X_n(\tau))]| + |\partial_x [\partial_x A_{n+1}(\tau, X_n(\tau))]| \\
&\qquad + \|\partial_x A_{n+1}(\tau)\|_{L_x^\infty} |\partial_x [v(P_n(\tau), A_{n+1}(\tau, X_n(\tau)))]| \; \Big) \, d\tau \\
&\leq C_{\bar{T}}^0 \int_s^t \Big(1 + \left\|\partial_x^2 \Phi^n(\tau)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_{n+1}(\tau)\right\|_{L_x^\infty} \Big) \\
&\qquad \times \Big(\; |\partial_x X_n(\tau)| + |\partial_x P_n(\tau)| \; \Big) \, d\tau
\end{aligned}$$

After combining these two estimates and using Gronwall's lemma we find

$$|\partial_x X_n(s)| + |\partial_x P_n(s)| \leq \exp\left\{ C_{\bar{T}}^0 \int_s^t \Big(1 + \left\|\partial_x^2 \Phi^n(\tau)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_{n+1}(\tau)\right\|_{L_x^\infty} \Big) \, d\tau \right\}. \tag{5.2.29}$$

Take the supremum over all characteristic curves. It follows that

$$\begin{aligned}
\left|\partial_x \rho^{n+1}(t,x)\right| &\leq \int_{|p| \leq \mathcal{P}(t)} |\partial_x [f_0(Z_n(0,t,x,p))]| \, dp \\
&\leq C_{\bar{T}}^0 \Big(\|\partial_x X_n(0,t)\|_{L_{x,p}^\infty} + \|\partial_x P_n(0,t)\|_{L_{x,p}^\infty} \Big) \\
&\leq C_{\bar{T}}^0 \exp\left\{ C_{\bar{T}}^0 \int_0^t \Big(1 + \left\|\partial_x^2 \Phi^n(\tau)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_{n+1}(\tau)\right\|_{L_x^\infty} \Big) \, d\tau \right\}.
\end{aligned}$$

Similarly, after using the product rule and the above estimates,

$$\begin{aligned}
\left|\partial_x j^{n+1}(t,x)\right| &\leq \int_{|p| \leq \mathcal{P}(t)} |\partial_x [v(p, A_{n+1}(t,x)) f_0(Z_n(0,t,x,p))]| \, dp \\
&\leq C_{\bar{T}}^0 + C_{\bar{T}}^0 \Big(\|\partial_x X_n(0,t)\|_{L_{x,p}^\infty} + \|\partial_x P_n(0,t)\|_{L_{x,p}^\infty} \Big) \\
&\leq C_{\bar{T}}^0 \exp\left\{ C_{\bar{T}}^0 \int_0^t \Big(1 + \left\|\partial_x^2 \Phi^n(\tau)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_{n+1}(\tau)\right\|_{L_x^\infty} \Big) \, d\tau \right\}.
\end{aligned}$$

Thus, we invoke the estimates in Lemma 21(b) and (d) to obtain on $[0,\bar{T}[$ that

$$\begin{aligned}
&\left\|\partial_x^2\Phi^{n+1}(t)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_{n+2}(t)\right\|_{L_x^\infty} \\
&\qquad \leq C_{\bar{T}}^0\left(1+\ln^+\left\|\partial_x\rho^{n+1}(t)\right\|_{L_x^\infty} + \ln^+\left\|\partial_x j^{n+1}(t)\right\|_{L_x^\infty}\right) \\
&\qquad \leq C_{\bar{T}}^0 + C_{\bar{T}}^0\int_0^t \left(\left\|\partial_x^2\Phi^n(\tau)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_{n+1}(\tau)\right\|_{L_x^\infty}\right)d\tau
\end{aligned}$$

Bounding $\|\partial_x^2\Phi^0(\tau)\|_{L_x^\infty} + \|\partial_x^2 A_0(\tau)\|_{L_x^\infty} \leq C_{\bar{T}}^0$, the iteration in n implies that

$$\left\|\partial_x^2\Phi^n(t)\right\|_{L_x^\infty} + \left\|\partial_x^2 A_{n+1}(t)\right\|_{L_x^\infty} \leq C_{\bar{T}}^0 \exp\left\{C_{\bar{T}}^0\bar{T}\right\},$$

for all $n\in\mathbb{N}_0$, $0\leq t\leq\bar{T}$. In turn, this provides the uniform bound on the iterates of the derivatives of the current and density functions, which proves the claim.

Step 4. We show that $\{f^n\}$ is Cauchy in the C-uniform norm on $[0,\bar{T}]\times\mathbb{R}^6$. To start with, notice that for $n\in\mathbb{N}$, $t\in[0,\bar{T}]$ and $z\in\mathbb{R}^6$

$$\begin{aligned}
\left|f^{n+1}(t,z)-f^n(t,z)\right| &= \left|f_0(Z_n(0,t,z))-f_0(Z_{n-1}(0,t,z))\right| \\
&\leq C^0\left|Z_n(0,t,z)-Z_{n-1}(0,t,z)\right|.
\end{aligned} \tag{5.2.30}$$

On the other hand, denoting $(X_n,P_n)(s)\equiv(X_n,P_n)(s,t,z)$ and bearing (5.2.25) in mind, it is easy to check that

$$\begin{aligned}
&|X_n(s)-X_{n-1}(s)| \\
&\qquad \leq \int_s^t \left|v(P_n(\tau),A_{n+1}(\tau,X_n(\tau))) - v(P_{n-1}(\tau),A_n(\tau,X_{n-1}(\tau)))\right| d\tau \\
&\qquad \leq C\int_s^t \left(|P_n(\tau)-P_{n-1}(\tau)| + |A_n(\tau,X_n(\tau))-A_n(\tau,X_{n-1}(\tau))|\right. \\
&\qquad\qquad \left. + |A_{n+1}(\tau,X_n(\tau))-A_n(\tau,X_n(\tau))|\right) d\tau \\
&\qquad \leq C_{\bar{T}}^0\int_s^t \left(|X_n(\tau)-X_{n-1}(\tau)| + |P_n(\tau)-P_{n-1}(\tau)| + \|A_{n+1}(\tau)-A_n(\tau)\|_{L_x^\infty}\right)d\tau
\end{aligned}$$

and, by also using (5.2.28)

$$
\begin{aligned}
&|P_n(s) - P_{n-1}(s)| \\
&\quad \le \int_s^t \big(\left|\nabla \Phi^n(\tau, X_n(\tau)) - \nabla \Phi^{n-1}(\tau, X_{n-1}(\tau))\right| \\
&\qquad + \left| \left(v^i_{A_{n+1}} \nabla A^i_{n+1}\right)(s, X_n(s), P_n(s)) - \left(v^i_{A_n} \nabla A^i_n\right)(s, X_{n-1}(s), P_{n-1}(s)) \right| \big)\, d\tau \\
&\quad \le C^0_{\bar T} \int_s^t \Big(\left\|\partial_x \Phi^n(\tau) - \partial_x \Phi^{n-1}(\tau)\right\|_{L^\infty_x} + \left|\partial_x \Phi^{n-1}(\tau, X_n(\tau)) - \partial_x \Phi^{n-1}(\tau, X_{n-1}(\tau))\right| \\
&\qquad + \Big| v(P_n(\tau), A_{n+1}(\tau, X_n(\tau))) - v(P_{n-1}(\tau), A_n(\tau, X_{n-1}(\tau))) \Big| \\
&\qquad + \Big[\|\partial_x A_{n+1}(s) - \partial_x A_n(s)\|_{L^\infty_x} + |\partial_x A_n(s, X_n(s)) - \partial_x A_n(s, X_{n-1}(s))| \Big] \Big)\, d\tau \\
&\quad \le C^0_{\bar T} \int_s^t \Big(|X_n(\tau) - X_{n-1}(\tau)| + |P_n(\tau) - P_{n-1}(\tau)| + \left\|\partial_x \Phi^n(\tau) - \partial_x \Phi^{n-1}(\tau)\right\|_{L^\infty_x} \\
&\qquad + \|A_{n+1}(\tau) - A_n(\tau)\|_{L^\infty_x} + \|\partial_x A_{n+1}(\tau) - \partial_x A_n(\tau)\|_{L^\infty_x} \Big)\, d\tau.
\end{aligned}
$$

Hence, these estimates and Gronwall's lemma yield

$$
\begin{aligned}
&|Z_n(0, t, z) - Z_{n-1}(0, t, z)| \\
&\quad \le C^0_{\bar T} \int_0^t \Big(\left\|\partial_x \Phi^n(\tau) - \partial_x \Phi^{n-1}(\tau)\right\|_{L^\infty_x} + \|A_{n+1}(\tau) - A_n(\tau)\|_{W^\infty_x} \Big)\, d\tau \\
&\quad \le C^0_{\bar T} \int_0^t \Big(\left\|\rho^n(\tau) - \rho^{n-1}(\tau)\right\|^{1/3}_{L^1_x} \left\|\rho^n(\tau) - \rho^{n-1}(\tau)\right\|^{2/3}_{L^\infty_x} \\
&\qquad + \|A_{n+1}(\tau) - A_n(\tau)\|_{W^\infty_x} \Big)\, d\tau \\
&\quad \le C^0_{\bar T} \int_0^t \Big(\left\|\rho^n(\tau) - \rho^{n-1}(\tau)\right\|_{L^\infty_x} + \|A_{n+1}(\tau) - A_n(\tau)\|_{W^\infty_x} \Big)\, d\tau \\
&\quad \le C^0_{\bar T} \int_0^t \Big(\left\|f^n(\tau) - f^{n-1}(\tau)\right\|_{L^\infty_{x,p}} + \|A_n(\tau) - A_{n-1}(\tau)\|_{L^\infty_x} \Big)\, d\tau,
\end{aligned}
$$

where we have used Lemma 21(b) in the second inequality, and the uniform bound of $\mathtt{supp} f^n(t)$ on $[0, \bar T]$ in the third and last inequalities. Also, in the last inequality we have used, adapted to the iterates, (5.2.16) and (5.2.17) from Lemma 23; cf. (5.2.19) and (5.2.20) respectively. Thus, if we combine the above estimate with (5.2.30) and take the supremum over all $z \in \mathbb{R}^6$, we find that

$$
\left\|f^{n+1}(t) - f^n(t)\right\|_{L^\infty_{x,p}} \le C^0_{\bar T} \int_0^t \Big(\left\|f^n(\tau) - f^{n-1}(\tau)\right\|_{L^\infty_{x,p}} + \|A_n(\tau) - A_{n-1}(\tau)\|_{L^\infty_x} \Big)\, d\tau. \tag{5.2.31}
$$

At this point, we wish to have a suitable estimate on the second term in the integrand of the latter inequality in order to close the 'Gronwall's loop' for the term in the left-hand side. Such estimate is provided by (5.2.16) in Lemma 23, which for the iterates can be rewritten as

$$\begin{aligned}
\|A_n(t) - A_{n-1}(t)\|_{L^\infty_x} &\leq C_* \left[\|A_{n-1}(t) - A_{n-2}(t)\|_{L^\infty_x} + C^0_{\bar{T}} \|f_{n-1}(t) - f_{n-2}(t)\|_{L^\infty_{x,p}} \right] \\
&\leq C_*^2 \|A_{n-2}(t) - A_{n-3}(t)\|_{L^\infty_x} + C^0_{\bar{T}} \Big[C_*^2 \|f_{n-2}(t) - f_{n-3}(t)\|_{L^\infty_{x,p}} \\
&\quad + C_* \|f_{n-1}(t) - f_{n-2}(t)\|_{L^\infty_{x,p}} \Big] \\
&\leq \cdots \\
&\leq C_*^{n-1} \|A_1(t) - A_0\|_{L^\infty_x} + C^0_T \sum_{k=1}^{n-1} C_*^{n-k} \left\| f^k(\tau) - f^{k-1}(\tau) \right\|_{L^\infty_{x,p}},
\end{aligned}$$

where $C_* \equiv C_*(\bar{T}; f_0) < 1$ is given by (5.2.26). But by the definition of the A^n's, we actually have that $A_1(t,x) \equiv A_0(x)$ for all $0 \leq t \leq \bar{T}$ and $x \in \mathbb{R}^3$. Therefore,

$$\|A_n(t) - A_{n-1}(t)\|_{L^\infty_x} \leq C^0_T \sum_{k=1}^{n-1} C_*^{n-k} \left\| f^k(\tau) - f^{k-1}(\tau) \right\|_{L^\infty_{x,p}}. \tag{5.2.32}$$

It is now an easy matter to estimate (5.2.31), namely

$$\left\| f^{n+1}(t) - f^n(t) \right\|_{L^\infty_{x,p}} \leq C^0_{\bar{T}} \int_0^t \sum_{k=1}^{n} C_*^{n-k} \left\| f^k(\tau) - f^{k-1}(\tau) \right\|_{L^\infty_{x,p}} d\tau. \tag{5.2.33}$$

Now, consider (5.2.33) for $n = 1, 2, 3 \ldots$. If we add the resultant inequalities, it is not difficult to check that for each $n \in \mathbb{N}$

$$\begin{aligned}
\sum_{k=2}^{n+1} \left\| f^k(t) - f^{k-1}(t) \right\|_{L^\infty_{x,p}} &\leq C^0_{\bar{T}} \int_0^t \sum_{k=1}^{n} \sum_{m=0}^{n-k} C_*^m \left\| f^k(\tau) - f^{k-1}(\tau) \right\|_{L^\infty_{x,p}} d\tau \\
&\leq C^0_{\bar{T}} \sum_{m=0}^{n} C_*^m \int_0^t \sum_{k=1}^{n} \left\| f^k(\tau) - f^{k-1}(\tau) \right\|_{L^\infty_{x,p}} d\tau \\
&\leq \frac{C^0_{\bar{T}}}{1 - C_*} \int_0^t \sum_{k=1}^{n} \left\| f^k(\tau) - f^{k-1}(\tau) \right\|_{L^\infty_{x,p}} d\tau.
\end{aligned}$$

Notice in the last step the use of the smallness assumption, i.e., $C_* < 1$. To prove the claim, it is enough to show that the latter sum converges as $n \to \infty$, uniformly on $[0, \bar{T}]$. Then, in view of the uniform bound on the L^∞-norm of the f^n's, we can add the missing term to the sum in the left-hand side to find that

$$S_{n+1}(t) := \sum_{k=1}^{n+1} \left\| f^k(t) - f^{k-1}(t) \right\|_{L^\infty_{x,p}} \leq C^0_{\bar{T}} + C^0_{\bar{T}} \int_0^t S_n(\tau) d\tau.$$

Finally, iteration in n produces $S_n(t) \leq C^0_{\bar{T}} \exp\left\{C^0_{\bar{T}}\bar{T}\right\}$, for all $n \in \mathbb{N}_0$, $0 \leq t \leq \bar{T}$, which implies the desired convergence.

Step 5. Hence, there is a function $f \in C([0,\bar{T}] \times \mathbb{R}^6; \mathbb{R})$ such that $f^n \to f$ uniformly on $[0,\bar{T}] \times \mathbb{R}^6$. The limiting function satisfies $f \geq 0$ and

$$\mathtt{supp} f(t) \subseteq \left\{(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| \leq \bar{X}_0 + t, |p| \leq \mathcal{P}(t)\right\}, \quad 0 \leq t \leq \bar{T}. \qquad (5.2.34)$$

Also, if we define

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,p) dp, \quad \Phi(t,x) = \int_{\mathbb{R}^3} \rho(t,y) \frac{dy}{|y-x|}, \quad x \in \mathbb{R}^3,\ 0 \leq t \leq \bar{T},$$

it is straightforward that $\rho^n \to \rho$ and $\Phi^n \to \Phi$ uniformly on $[0,\bar{T}] \times \mathbb{R}^3$.

As for the vector potential, the situation is a bit more involved. We have to explicitly show that $\{A_n\}$ is Cauchy in the C-uniform norm, since the current density depends on the vector potential itself. To this end, and in view of the estimate (5.2.32), similar arguments to those in Step 4 yield (recall $A_1 \equiv A_0$)

$$\begin{aligned}
\sum_{k=1}^{n} \|A_k(t) - A_{k-1}(t)\|_{L^\infty_x} &\leq C^0_T \sum_{k=1}^{n} \sum_{m=0}^{n-k} C^m_* \left\| f^k(\tau) - f^{k-1}(\tau) \right\|_{L^\infty_{x,p}} \\
&\leq C^0_{\bar{T}} \sum_{m=0}^{n} C^m_* \sum_{k=1}^{n} \left\| f^k(\tau) - f^{k-1}(\tau) \right\|_{L^\infty_{x,p}} \\
&\leq \frac{C^0_{\bar{T}}}{1 - C^*} \exp\left\{C^0_{\bar{T}}\bar{T}\right\}, \quad n \in \mathbb{N},\ 0 \leq t \leq \bar{T}.
\end{aligned}$$

This implies that the sum in the left-hand side converges as $n \to \infty$, uniformly on $[0,\bar{T}]$. As a consequence, the sequence $\{A_n\}$ is uniformly Cauchy and there is a vector field $A \in C([0,\bar{T}] \times \mathbb{R}^3; \mathbb{R}^3)$ such that $A^n \to A$ uniformly on $[0,\bar{T}] \times \mathbb{R}^3$. Also, if we define

$$v_A = \frac{p - A}{\sqrt{1 + |p - A|^2}}, \quad j_A(t,x) = \int_{\mathbb{R}^3} v_A f(t,x,p) dp, \quad x \in \mathbb{R}^3,\ 0 \leq t \leq \bar{T},$$

it not difficult to check that $v_{A_n} \to v_A$ and $v_{A_n} f^n \to v_A f$ uniformly on $[0, \bar{T}] \times \mathbb{R}^6$, and thus $j^n \to j_A$ uniformly on $[0, \bar{T}] \times \mathbb{R}^3$. Moreover,

$$A(t,x) \equiv \frac{1}{2} \int_{\mathbb{R}^3} [\mathtt{id} + \omega \otimes \omega]\, j_A(t,y) \frac{dy}{|y-x|}, \qquad x \in \mathbb{R}^3,\ 0 \le t \le \bar{T}.$$

Step 6. Actually, $(\Phi, A) \in C([0,\bar{T}], C^2(\mathbb{R}^3); \mathbb{R} \times \mathbb{R}^3)$ and $f \in C^1([0,\bar{T}] \times \mathbb{R}^6; \mathbb{R})$. Indeed, let us first show the regularity of the vector potential A. By Lemma 21(d), for every $i = 1, 2, 3$ we have

$$\begin{aligned} \|\partial_x A_n(t) - \partial_x A_m(t)\|_{L^\infty_x} &\le C \|j^n(t) - j^m(t)\|^{2/3}_{L^\infty_x} \|j^n(t) - j^m(t)\|^{1/3}_{L^1_x} \\ &\le C^0_{\bar{T}} \|j^n(t) - j^m(t)\|_{L^\infty_x}, \end{aligned}$$

and by letting $R = h$

$$\begin{aligned} \left\|\partial_x^2 A^n(t) - \partial_x^2 A^m(t)\right\|_{L^\infty_x} &\le C \Big(h^{-3} \|j^n(t) - j^m(t)\|_{L^1_x} + h \|\partial_x j^n(t) - \partial_x j^m(t)\|_{L^\infty_x} \\ &\qquad + \|j^n(t) - j^m(t)\|_{L^\infty_x} \Big) \\ &\le C^0_{\bar{T}} \Big[\left(1 + h^{-3}\right) \|j^n(t) - j^m(t)\|_{L^\infty_x} + h \Big], \end{aligned}$$

where in the last inequality we have used that $\|\partial_x j^n(t)\|_{L^\infty_x}$ is uniformly bounded on $[0, \bar{T}]$, a fact already proved in Step 3; cf. (5.2.28). It is now an easy matter to show that A is C^1, since j^n is uniformly Cauchy on $[0,\bar{T}] \times \mathbb{R}^3$. On the other hand, since $h > 0$ is arbitrary, for any given $\epsilon > 0$ we can let $0 < h < \epsilon$ and n, m large enough such that the left-hand side in the last estimate is dominated by ϵ, uniformly on $[0, \bar{T}]$. Again, it is now easy to show that A satisfies the expected regularity.

As for the scalar potential Φ, in view of the estimates in Lemma 21(b) we can run the proof in a similar fashion; it is a bit easier considering that we are now dealing with a scalar function. Then, due to the regularity of the potentials (Φ, A), we have that the curves $Z(s,t,z) \equiv (X,P)(s,t,x,p)$ defined by the uniform limit

$$Z := \lim_{n\to\infty} Z_n, \quad \texttt{on}\ [0,\bar{T}] \times [0,\bar{T}] \times \mathbb{R}^6$$

are the C^1 solutions of the characteristic system

$$\dot{X}(s,t,z) = v_A(s, X(s,t,z), P(s,t,z)) \tag{5.2.35}$$

$$\dot{P}(s,t,z) = -\left[\nabla\Phi - v_A^i \nabla A^i\right](s, X(s,t,z), P(s,t,z)) \tag{5.2.36}$$

with $Z(t,t,z)=z$. Therefore,

$$f(t,z) := \lim_{n\to\infty} f_0(Z_n(0,t,z)) = f_0(Z(0,t,z)) \tag{5.2.37}$$

and the claimed regularity follows.

Step 7. To conclude the existence proof, we have to show that the triplet (f,Φ,A) fulfills all the properties demanded by Definition 7. There is only one of those properties that is not trivially satisfied. Specifically, we must show that A solves the Darwin equation (5.2.4). In view of Lemma 21(c) and Remark 10, this would readily follow if for each $0 \leq t \leq \bar{T}$, we have $j_A(t) \in C^2(\mathbb{R}^3;\mathbb{R}^3)$ with compact support. We prove next that j_A has such a regularity.

Fix $0 < t \leq \bar{T}$. By the regularities of $f(t)$ and $A(t)$, and the compactness of $\mathtt{supp}f(\bar{T}) \supseteq \mathtt{supp}f(t)$, we have that $j_A(t) \in C^1(\mathbb{R}^3;\mathbb{R}^3)$ with compact support. But by virtue of Remark 14, this implies that $A(t) \in C^{2,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$ with $0<\alpha<1$. Similarly, we have $\rho(t) \in C^1(\mathbb{R}^3;\mathbb{R})$ and so $\Phi(t) \in C^{2,\alpha}(\mathbb{R}^3;\mathbb{R})$. Hence, by computations as those in Step 3 and 4, which we postpone to the Appendix E, we obtain via characteristics (5.2.35)-(5.2.36) that

$$|\partial_z Z(s,t,z_1) - \partial_z Z(s,t,z_2)| \leq C^0_{\bar{T}} |z_1 - z_2|^\alpha , \tag{5.2.38}$$

for all $z_1, z_2 \in \mathtt{supp}f(t)$ and $0 \leq s \leq t$, with $0<\alpha<1$. Therefore, by (5.2.37) and the assumption on f_0, we can easily verify that $f(t) \in C^{1,\alpha}(\mathbb{R}^6;\mathbb{R})$. This, together with the definitions of j_A and ρ, readily imply that $j_A(t) \in C^{1,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$ and $\rho(t) \in C^{1,\alpha}(\mathbb{R}^3;\mathbb{R})$, both with compact supports[4]. We are now half way to have proved the claim.

To prove the other half, we can reason as follows. In view of the regularities of the charge and current densities, we can invoke Remark 15 to find that $A(t) \in C^{3,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$ and similarly $\Phi(t) \in C^{3,\alpha}(\mathbb{R}^3;\mathbb{R})$. Moreover, since $\mathtt{supp}f(\bar{T}) \supseteq \mathtt{supp}f(t)$ is compact, then for each $0 \leq s \leq t$

$$\left\|\partial_z^2 Z(s,t)\right\|_{L^\infty_{x,p}} + \left\|\partial_x^3 \Phi(s)\right\|_{L^\infty_x} + \left\|\partial_x^3 A(s)\right\|_{L^\infty_x} \leq C^0_{\bar{T}}.$$

The estimates for the derivatives of the potentials can be found as in Lemma 21(b) and (d), once we replace the charge and current densities by their derivatives in (5.1.7) and (5.1.17), respectively. Notice that estimating $((\partial_x\rho)(y) - (\partial_x\rho)(x))$ uses the $C^{1,\alpha}$

[4]If the claim in Remark 13 was proved, then we could have relaxed the regularity of the Cauchy datum to $f_0 \in C^{1,\alpha}(\mathbb{R}^6)$ with $0<\alpha<1$, and stop here; cf. Remark 17.

regularity obtained above, and similarly for j_A. Also, the needed estimates for $\partial_x \rho$ and $\partial_x j_A$ follow from (5.2.28) in Step 3, written in terms of the solution. Arguments similar to those in Step 3 are also used to estimate $\partial_z^2 Z$, in particular the analogous to (5.2.29), obtained after taken a further derivative to the characteristic curves.

Hence, we can proceed as in the Appendix E to show that $Z(s,t) \in C^{2,\alpha}(\mathbb{R}^6;\mathbb{R}^6)$ for all $0 \leq s \leq t$. This, and the regularity of the Cauchy datum f_0, imply via (5.2.37) that $f(t) \in C^2(\mathbb{R}^6;\mathbb{R})$. As a result, and by the definition of j_A, the claim follows.

Having proved that, and since $0 \leq \bar{T} < T$ was arbitrary, we conclude that the function $f \in C^1([0,T[\times\mathbb{R}^6;\mathbb{R})$ is a classical solution of the Cauchy problem for the relativistic Vlasov-Darwin system.

Step 8. To prove uniqueness, we consider two classical solutions f_1 and f_2 that exist on $[0,\bar{T}]$ and have the same Cauchy datum f_0 satisfying (5.2.21). The size of the momentum support of each of these solutions can be estimated by

$$\bar{P}_k(t) := \sup\left\{|p| : \exists 0 \leq s \leq t, x \in \mathbb{R}^3 : f_k(s,x,p) \neq 0\right\}, \quad k = 1,2;$$

and in view of Definition 7, in particular the boundedness of the fields induced by the potentials, there is a positive time, again denoted by $\bar{T}$, such that $C_*(t;f_0) < 1$, $0 \leq t \leq \bar{T}$, where $C_*(t;f_0)$ is given by (5.2.26) after replacing $\mathcal{P}(t)$ by $\sup\left\{\bar{P}_1(t), \bar{P}_2(t)\right\}$. Therefore, by (5.2.16) we obtain

$$\|A_2(t) - A_1(t)\|_{L_x^\infty} \leq C_{\bar{T}}^0 \|f_2(t) - f_1(t)\|_{L_{x,p}^\infty},$$

and thus (5.2.31) written in terms of the solutions f_1 and f_2 yields

$$\|f_2(t) - f_1(t)\|_{L_x^\infty} \leq C_{\bar{T}}^0 \int_0^t \|f_2(\tau) - f_1(\tau)\|_{L_x^\infty} d\tau, \quad 0 \leq t \leq \bar{T}.$$

This combined with Gronwall's lemma provide the uniqueness result.

Step 9. To prove the continuation criterion we proceed similarly as we did for the Vlasov-Maxwell system in Theorem 4 above. Let f be the previously obtained classical solution of the RVD system which satisfies $f|_{t=0} = f_0$ and whose life span is $T := \left(C(f_0)\bar{P}_0\right)^{-1}\left(1 - C_*(f_0)\bar{P}_0\right)$, cf. (5.2.27). We recall that

$$C_*(f_0) := 3^2(12\pi)^{2/3} \|f_0\|_{L_{x,p}^1}^{2/3} \|f_0\|_{L_{x,p}^\infty}^{1/3}, \quad C(f_0) := 30(2\pi)^{2/3} \|f_0\|_{L_x^1}^{1/3} \|f_0\|_{L_x^\infty}^{2/3}.$$

Define the quantity $\bar{P}_T = \sup\{|p| : \exists 0 \leq t < T, x \in \mathbb{R}^3 : f(t,x,p) \neq 0\}$ and assume that $\mathcal{A}_0^* := C_*(f_0)\bar{P}_T < 1$ but $T < \infty$. We claim that this is a contradiction.

Indeed, fix $0 < t_0 < T$ and consider $f(t_0)$ as Cauchy datum of the RVD system. In view of Lemma (15), we have that

$$\|f(t_0)\|_{L^1_{x,p}} = \|f_0\|_{L^1_{x,p}}, \quad \|f(t_0)\|_{L^\infty_{x,p}} = \|f_0\|_{L^\infty_{x,p}}.$$

Thus, $C_*(f(t_0)) \equiv C_*(f_0)$ and $C(f(t_0)) \equiv C(f_0)$. Define $\epsilon := \left(C(f_0)\bar{P}_T\right)^{-1}(1 - \mathcal{A}_0^*)$, which is clearly *not* dependent on t_0. Steps 1, 2 and 3 imply that all uniform estimates on the sequence of approximate solutions induced by $f(t_0)$ hold on $[t_0, t_0 + \epsilon[$. Then, $f(t_0)$ launches a unique classical solution of the RVD system on that interval.

Now, we could have fixed t_0 arbitrary close to the life span $T < \infty$ of f and so extend this solution beyond the time T, which is a contradiction. Hence, the assumption $\mathcal{A}_0^* := C_*(f_0)\bar{P}_T < 1$ implies $T = \infty$ and so the solution f is global in time. This concludes the proof of Theorem 9. □

Remark 16. As long as the classical solution f exits, the relativistic velocity of the particles satisfy $|v_A| < 1$ strictly. To check this, notice that (5.2.34) implies that f having life span T satisfies for each $0 \le t < T$

$$\bar{P}(t) := \sup\left\{|p| : \exists 0 \le s \le t, x \in \mathbb{R}^3 : f(s,x,p) \neq 0\right\} < \infty. \tag{5.2.39}$$

In addition, Corollary 4 implies that the induced vector potential can be estimated by

$$\|A(t)\|_{L^\infty_x} \le C\bar{X}_0^2\bar{P}_0^2 \|f_0\|_{L^\infty_{x,p}} \bar{P}(t) < \infty, \quad 0 \le t < T.$$

Therefore, $|p - A(t,x)| \le C(t; f_0)$ for all $(x,p) \in \mathtt{supp} f(t)$, $0 \le t < T$. Hence, the strict inequality follows from the definition of v_A.

Remark 17. For simplicity, we have assumed that f_0 is a C^2 function. However, by virtue of Remark 13 and Step 7 in the above proof, *it seems* reasonable to relax the regularity of the Cauchy datum to f_0 in $\in C^{1,\alpha}$, with $0 < \alpha < 1$.

5.2.2 Global Solutions

It turns out that if some additional conditions are imposed to the Cauchy datum of the RVD system, then the local classical solution found in the previous section can be made global in time. The aim of this section is to prove this result. To this end, and by taking advantage of the formulation (5.2.1)-(5.2.5) of the RVD system, we adapt the analogous result for the Vlasov-Poisson system given in [42]. As commented earlier, in the Darwin case we must overcome some non-trivial difficulties arising from the inclusion of the vector potential in the model equations.

We start by defining the set where the Cauchy datum will be taken from. Let $\bar{X}_0 > 0$ and $\bar{P}_0 > 0$ be given such that

$$\mathcal{B}_0 := 3^2(4\pi)^{4/3}\bar{X}_0^2\bar{P}_0^3 < 1.$$

We define

$$\mathcal{D} = \Big\{ f_0 \in C_0^2(\mathbb{R}^6) : f_0 \geq 0, \|f_0\|_{L^\infty_{x,p}} \leq 1, \left\|\partial_{(x,p)} f_0\right\|_{L^\infty_{x,p}} \leq 1, \\ \mathcal{B}_0 < 1, \mathtt{supp} f_0 \subset \Omega_{\bar{X}_0}(0) \times \Omega_{\bar{P}_0}(0) \Big\}.$$

Clearly, the condition imposed on $\mathcal{B}_0$ guarantees that for any $f_0 \in \mathcal{D}$ the smallness assumption (5.2.21) in Theorem 9 holds, i.e.,

$$\mathcal{A}_0 := 3^2(12\pi)^{2/3} \|f_0\|_{L^1_{x,p}}^{2/3} \|f_0\|_{L^\infty_{x,p}}^{1/3} \bar{P}_0 \leq \mathcal{B}_0 \|f_0\|_{L^\infty_{x,p}} < 1. \tag{5.2.40}$$

Hence, any $f_0 \in \mathcal{D}$ launches a classical solution of the RVD system at least locally in time. The goal is to prove that such solution can be made global if the Cauchy datum is “small” enough.

Theorem 10. *There exists $\delta > 0$ such that, if $f_0 \in \mathcal{D}$ with $\|f_0\|_{L^\infty_{x,p}} \leq \delta$, then the classical solution of the RVD system with Cauchy datum f_0 is global in time. Moreover, for $t > 0$ this solution satisfies the decay estimates*

$$\begin{aligned} \|j_A(t)\|_{L^\infty_x} + \|\rho(t)\|_{L^\infty_x} &\leq Ct^{-3} \\ \|\partial_x\Phi(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} &\leq Ct^{-2} \\ \left\|\partial_x^2\Phi(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} &\leq Ct^{-3}\ln(1+t). \end{aligned}$$

The proof relies on a so-called bootstrap argument. That is, we show that a solution f of the RVD system with small enough Cauchy datum induces potentials whose space derivatives satisfy some suitable decay estimates. In turn, this estimates imply an even stronger decay of those derivatives on the whole interval of existence of f, and so we can continue the solution globally in time.

We first introduce some technical results and postpone the actual proof of the

Theorem 10 to the end of this section. To begin with, we show that a sufficiently small Cauchy datum yields a classical solution of the RVD system which exists on any given time interval and induces potentials whose space derivatives up to a second order can be made as small as desired. In particular, the self-induced electromagnetic field remains small as well[5].

Lemma 24. *Fix $\epsilon > 0$ and $T > 0$. There exists $\delta > 0$ such that, if $f_0 \in \mathcal{D}$ with $\|f_0\|_{L^\infty_{x,p}} \leq \delta$, then the classical solution of the RVD system with Cauchy datum f_0 exists on the time interval $[0, T]$ and induces potentials satisfying the estimate*

$$\|\partial_x \Phi(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{\infty_x} + \left\|\partial_x^2 \Phi(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} < \epsilon, \quad 0 \leq t \leq T. \quad (5.2.41)$$

Proof. As a consequence of (5.2.27) in Theorem 9 and the estimate (5.2.40), any $f_0 \in \mathcal{D}$ produces a classical solution f of the RVD system on the time interval $[0, \left(C(f_0)\bar{P}_0\right)^{-1}(1 - \mathcal{B}_0)\,[$, where

$$C(f_0) := 30(2\pi)^{2/3} \|f_0\|_{L^1_x}^{1/3} \|f_0\|_{L^\infty_x}^{2/3} \leq C_0 \|f_0\|_{L^\infty_{x,p}},$$

with a positive constant $C_0 = C(\bar{X}_0, \bar{P}_0)$. Now, for $T > 0$, assume that the Cauchy datum satisfies $\|f_0\|_{L^\infty_x} \leq \delta := \left(2C_0\bar{P}_0 T\right)^{-1}(1 - \mathcal{B}_0)$. Then, $C(f_0) \leq \left(2\bar{P}_0 T\right)^{-1}(1 - \mathcal{B}_0)$ and so the solution f exists on the time interval $[0, T] \subset [0, \left(C(f_0)\bar{P}_0\right)^{-1}(1 - \mathcal{B}_0)[$. This proves the first part of the lemma.

In order to prove (5.2.41), we notice that by (5.2.34) in Step 5 of Theorem 9 and uniqueness, the size of the momentum support of f defined in (5.2.39) satisfies

$$\bar{P}(t) \leq \mathcal{P}(t) \equiv \frac{\bar{P}_0}{1 - C(f_0)\bar{P}_0 t} \leq \frac{\bar{P}_0}{1 - (2T)^{-1}(1 - \mathcal{B}_0)\,t} =: \bar{\mathcal{P}}(t), \quad 0 \leq t \leq T,$$

where $\mathcal{P}(t)$ is the maximal solution of (5.2.24) and the second inequality follows from the above estimate on $C(f_0)$. Then, it is straightforward that

$$\|j_A(t)\|_{L^\infty_x} \leq \|\rho(t)\|_{L^\infty_x} \leq \frac{4\pi}{3}\bar{\mathcal{P}}^3(T)\,\|f_0\|_{L^\infty_{x,p}}$$

[5] Actually, here we get control only on the *longitudinal* part of the electric field $\nabla\Phi$ and on the magnetic field $\nabla \times A$; cf. (4.1.1)-(4.1.2). As shall be clear in Section 5.3, one of the advantages of our approach is that we do not have to deal neither with the *transversal* component of the electric field $E_T \equiv -\partial_t A$ nor with its derivatives.

and also

$$\|j_A(t)|_{L^1_x} \leq \|\rho(t)\|_{L^1_x} \leq \left(\frac{4\pi}{3}\right)^2 \bar{X}_0^3 \bar{P}_0^3 \|f_0\|_{L^\infty_{x,p}}.$$

In addition, we have that $\|\partial_x \rho(t)\|_{L^\infty_x} + \|\partial_x j_A(t)\|_{L^\infty_x} \leq C^0_T$, for all $0 \leq t \leq T$. This can be proven exactly as for the iterates in the Step 3 of the proof of the Theorem 9. Hence, we invoke Lemma 21(b) and (d) to estimate the first and second order space derivatives of the potentials, in terms of the charge and current densities. Thus, by making δ smaller if needed, it is easy to check that (5.2.41) indeed hold, which concludes the proof of the lemma. □

To proceed, we now define the so-called *free streaming condition* for classical solutions of the RVD system. More precisely,

Definition 8. *Fix $\alpha > 0$ and $a > 0$. A classical solution of the RVD system is said to satisfy the free streaming condition of parameter α (FSα) on the time interval $[0, a]$, if it exists on $[0, a]$ and induces potentials satisfying the estimates*

$$\begin{aligned} \|\partial_x \Phi(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} &\leq \alpha(1+t)^{-3/2}, \\ \left\|\partial_x^2 \Phi(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} &\leq \alpha(1+t)^{-5/2}, \end{aligned}$$

for all $0 \leq t \leq a$.

Lemma 25. *There exist $\delta > 0$, $\alpha > 0$ and a positive $C = C(\bar{X}_0, \bar{P}_0)$ such that any classical solution f of the RVD system having Cauchy datum $f_0 \in \mathcal{D}$ with $\|f_0\|_{L^\infty_{x,p}} \leq \delta$ and satisfying (FSα) on some interval $[0, a]$, also satisfies the estimates*

$$\|\partial_x \Phi(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} \leq Ct^{-2}, \tag{5.2.42}$$

$$\left\|\partial_x^2 \Phi(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} \leq Ct^{-3} \ln(1+t), \tag{5.2.43}$$

for all $0 < t \leq a$.

Proof. From Lemma 21(b) and (d), it is straightforward that the induced potentials satisfy the estimates

$$\|\partial_x \Phi(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} \leq C \|f_0\|^{1/3}_{L^1_{x,p}} \|\rho(t)\|^{2/3}_{L^\infty_x} \tag{5.2.44}$$

and, for $t > 1$,

$$\left\|\partial_x^2 \Phi(t)\right\|_{L_x^\infty} + \left\|\partial_x^2 A(t)\right\|_{L_x^\infty} \leq C\left[t^{-3}\left\|f_0\right\|_{L^1_{x,p}} + t^{-3}\left(\left\|\partial_x \rho(t)\right\|_{L_x^\infty} + \left\|\partial_x j_A(t)\right\|_{L_x^\infty}\right) + \left(1+\ln t^4\right)\left\|\rho(t)\right\|_{L_x^\infty}\right] \tag{5.2.45}$$

where the latter is consequence of setting $R = t$ and $h = t^{-3} \leq R$ in the cited lemma. In addition, *if* the charge and current densities satisfy

$$\left\|j_A(t)\right\|_{L_x^\infty} + \left\|\rho(t)\right\|_{L_x^\infty} \leq Ct^{-3} \tag{5.2.46}$$

$$\left\|\partial_x \rho(t)\right\|_{L_x^\infty} + \left\|\partial_x j_A(t)\right\|_{L_x^\infty} \leq C, \tag{5.2.47}$$

then it is immediate that (5.2.44) and (5.2.45) yield (5.2.42) and (5.2.43) respectively. Thus, we are led to show that for some $C = C(\bar{X}_0, \bar{P}_0) > 0$, both (5.2.46) and (5.2.47) indeed hold. To this end, we shall prove several technical results that we present as a series of steps:

Step 1. For the relativistic velocity v_A, consider the matrix Dv_A of entries

$$[Dv_A]_{ik} = \frac{\delta_{ik} - v_A^i v_A^k}{\sqrt{1 + |p-A|^2}}, \tag{5.2.48}$$

with δ_{ik} being the Kronecker delta function. Clearly, we have $|Dv_A| \leq C$. Denote $Dv_A(s) := Dv_A(P(s), A(s, X(s)))$ where the curves $(X,P)(s) = (X,P)(s,t,x,p)$ are the solution of the characteristic system (5.2.35)-(5.2.36) satisfying $|P(0)| \leq \bar{P}_0$. We claim that if the Cauchy datum f_0 satisfies the assumptions of the lemma, then for all $0 \leq s \leq t \leq a$ there exists a positive $C = C(\bar{X}_0, \bar{P}_0)$ such that

$$|Dv_A(t) - Dv_A(s)| \leq C\left[\alpha + (1+\alpha)\,\delta\right]. \tag{5.2.49}$$

This will enable the left-hand side to be arbitrary small as we shall need later on.

We start by noticing that in view of (FSα), the characteristic curves satisfy

$$\begin{aligned} |P(t)| &\leq \bar{P}_0 + \int_0^t \left(\left\|\partial_x \Phi(s)\right\|_{L_x^\infty} + \left\|\partial_x A(s)\right\|_{L_x^\infty}\right) ds \\ &\leq \bar{P}_0 + \int_0^t \alpha(1+s)^{-3/2} ds \;\leq\; \bar{P}_0 + 2\alpha, \end{aligned} \tag{5.2.50}$$

and so $\bar{P}(t) \leq \bar{P}_0 + 2\alpha$, where $\bar{P}(t)$ denotes the size of the momentum support of the solution f up to the time t; cf. (5.2.39). We emphasize that $\bar{P}(t)$ is a non-decreasing function of t. Hence, since $v_A = v(p, A)$ is C_b^∞ in its arguments, we have that

$$\begin{aligned} |Dv_A(t) - Dv_A(s)| &\leq C\Big(\, |P(t) - P(s)| + |A(t, X(t)) - A(s, X(s))| \, \Big) \\ &\leq C\left(\int_0^t |\dot{P}(\tau)| d\tau + \|A(t)\|_{L_x^\infty} + \|A(s)\|_{L_x^\infty} \right) \\ &\leq C\left[\alpha + \left(\bar{X}_0 \bar{P}_0 \right)^2 \|f_0\|_{L_{x,p}^\infty} \bar{P}(t) \right] \end{aligned} \tag{5.2.51}$$

where the first term in the last inequality is consequence of (FSα) via (5.2.50) and the second term follows from the usual estimates on the vector potential A; cf. Corollary 4. Thus, by the assumption on the Cauchy datum, we have $\|f_0\|_{L_{x,p}^\infty} \bar{P}(t) \leq \delta \left(\bar{P}_0 + 2\alpha \right)$, which combined with (5.2.51) imply (5.2.49).

Step 2. Let $0 < t \leq a$ and $x \in \mathbb{R}^3$ be fixed. Denote $(X, P)(s) = (X, P)(s, t, x, p)$ and consider the system

$$\begin{aligned} \xi(s) &:= \partial_p X(s) - (s - t) Dv_A(t) \\ \eta(s) &:= \partial_p P(s) - \mathtt{id}. \end{aligned}$$

Notice that $\xi(t) = \eta(t) = 0$. We claim that for some $C = C(\bar{X}_0, \bar{P}_0) > 0$

$$|\xi(s)| \leq C e^{\alpha C} \left[\alpha + (1 + \alpha) \delta \right] (t - s) \, . \tag{5.2.52}$$

Indeed, on the characteristics curves solution of (5.2.35)-(5.2.35), we have that

$$\begin{aligned} \dot{\xi}(s) &= \partial_p \dot{X}(s) - Dv_A(t) \\ &= Dv_A(s) \Big[\, \partial_p P(s) - \partial_x A(s, X(s)) \partial_p X(s) \, \Big] - Dv_A(t) \\ &= Dv_A(s) \Big\{ \eta(s) - \partial_x A(s, X(s)) \left[\xi(s) + (s - t) Dv_A(s) \right] \, \Big\} + Dv_A(s) - Dv_A(t). \end{aligned}$$

Therefore,

$$\begin{aligned}
|\xi(s)| &\leq C\int_s^t |\eta(\tau)|\, d\tau + C\int_s^t \|\partial_x A(\tau)\|_{L^\infty_x} |\xi(\tau)|\, d\tau \\
&\quad + C\int_s^t \Big[\|\partial_x A(\tau)\|_{L^\infty_x}(t-\tau) + |Dv_A(t) - Dv_A(\tau)|\Big]\, d\tau \\
&\leq C\int_s^t |\eta(\tau)|\, d\tau + C\int_s^t \alpha\,(1+\tau)^{-3/2}\, |\xi(\tau)|\, d\tau \\
&\quad + C\,(t-s)\int_s^t \alpha\,(1+\tau)^{-3/2}\, d\tau + C\int_s^t [\alpha + (1+\alpha)\,\delta]\, d\tau,
\end{aligned}$$

where in the last inequality we have used (FSα) and the estimate in Step 1. Hence, after combining the last two terms in the right-hand side of the above inequality, we deduce by Gronwall's lemma that

$$\begin{aligned}
|\xi(s)| &\leq C\left([\alpha + (1+\alpha)\,\delta]\,(t-s) + \int_s^t |\eta(\tau)|\, d\tau\right)\exp\left\{\alpha C\int_s^t (1+\tau)^{-3/2}\, d\tau\right\} \\
&\leq Ce^{\alpha C}\left([\alpha + (1+\alpha)\,\delta]\,(t-s) + \int_s^t |\eta(\tau)|\, d\tau\right). \qquad (5.2.53)
\end{aligned}$$

On the other hand, since $\dot\eta(s) = \partial_p \dot P(s)$, we also have

$$\dot\eta(s) = -\left[\partial_x^2\Phi(s, X(s)) - v_A^i\partial_x^2 A^i(s, X(s))\right]\partial_p X(s) + \partial_p\dot X^i(s)\nabla A^i(s, X(s)).$$

Then, by using as above $\partial_p \dot X(s) \equiv Dv_A(s)\left[\,\partial_p P(s) - \partial_x A(s, X(s))\partial_p X(s)\,\right]$ and the free streaming condition (FSα), we estimate

$$\begin{aligned}
|\eta(s)| &\leq C\int_s^t \Big(\left\|\partial_x^2\Phi(\tau)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(\tau)\right\|_{L^\infty_x} + \|\partial_x A(\tau)\|^2_{L^\infty_x}\Big)\, |\partial_p X(\tau)|\, d\tau \\
&\quad + C\int_s^t \|\partial_x A(\tau)\|_{L^\infty_x} |\partial_p P(\tau)|\, d\tau \\
&\leq C\int_s^t \alpha\left[(1+\tau)^{-5/2} + (1+\tau)^{-3}\right]\Big(\,|\xi(\tau)| + (t-\tau)\,\Big)\, d\tau \\
&\quad + C\int_s^t \alpha\,(1+\tau)^{-3/2}\Big(\,|\eta(\tau)| + 1\Big)\, d\tau \\
&\leq \alpha C\int_s^t (1+\tau)^{-5/2}\, |\xi(\tau)| d\tau + \alpha C\int_s^t (1+\tau)^{-3/2}\, |\eta(\tau)| d\tau \\
&\quad + \alpha C\int_s^t \left[(1+\tau)^{-5/2}\,(t-\tau) + (1+\tau)^{-3/2}\right] d\tau.
\end{aligned}$$

As in (5.2.53), a use of Gronwall's lemma yields

$$\begin{aligned}|\eta(s)| &\leq \alpha C e^{\alpha C}\int_s^t (1+\tau)^{-5/2}|\xi(\tau)|d\tau \\ &+\alpha C e^{\alpha C}\int_s^t \left[(1+\tau)^{-5/2}(t-\tau)+(1+\tau)^{-3/2}\right]d\tau. \qquad (5.2.54)\end{aligned}$$

Thus, if we combine (5.2.53) and (5.2.54) we find that

$$\begin{aligned}|\xi(s)| &\leq Ce^{\alpha C}\Big([\alpha+(1+\alpha)\delta](t-s)+\alpha\int_s^t\int_\tau^t (1+\sigma)^{-5/2}|\xi(\sigma)|d\sigma d\tau \\ &+\alpha\int_s^t\int_\tau^t\left[(1+\sigma)^{-5/2}(t-\sigma)+(1+\sigma)^{-3/2}\right]d\sigma d\tau\Big) \\ &\leq Ce^{\alpha C}\Big([\alpha+(1+\alpha)\delta](t-s)+\alpha\int_s^t\int_s^\sigma (1+\sigma)^{-5/2}|\xi(\sigma)|d\tau d\sigma \\ &+\alpha\int_s^t\int_s^\sigma\left[(1+\sigma)^{-5/2}(t-\sigma)+(1+\sigma)^{-3/2}\right]d\tau d\sigma\Big) \\ &\leq Ce^{\alpha C}\Big([\alpha+(1+\alpha)\delta](t-s)+\alpha\int_s^t (1+\sigma)^{-3/2}|\xi(\sigma)|d\sigma \\ &+\alpha\int_s^t\left[(1+\sigma)^{-3/2}(t-\sigma)+(1+\sigma)^{-1/2}\right]d\sigma\Big).\end{aligned}$$

Finally, since the last integral in the last inequality can be estimated by $3\alpha(t-s)$, another use of Gronwall's lemma readily implies (5.2.52).

Step 3. Therefore, $|\xi(0)| \leq Ce^{\alpha C}[\alpha+(1+\alpha)\delta]\,t$ where $\xi(0)=\partial_p X(0)+tDv_A(t)$. We claim that for some $\alpha>0$ and $\delta>0$ small enough and for some positive constant $C=C(\bar{X}_0,\bar{P}_0)$, the map $X(0,t,x,\cdot):\mathbb{R}^3\to\mathbb{R}^3$ has Jacobian determinant satisfying

$$|\det \partial_p X(0,t,x,p)| \geq Ct^3, \quad 0\leq t\leq a,\ x\in\mathbb{R}^3,\ p\in\mathbb{R}^3.$$

For $t=0$ this is obvious. Let $0<t\leq a$. Without loss of generality, we shall assume that $0<\alpha\leq 1/2$. Then, as shown in the Step 1 above, the size of the momentum support of f on the time interval $[0,a]$ satisfies $\bar{P}(t)\leq\bar{P}_0+1$. Also, as a consequence of the known estimate on the vector potential, cf. Corollary 4, and recalling that $f_0\in\mathcal{D}$ so $\|f_0\|_{L^\infty_{x,p}}\leq 1$, there exists a universal constant $C>0$ such that,

$$\|A(t)\|_{L^\infty_x}\leq C\bar{X}_0^2\bar{P}_0^2\left(\bar{P}_0+1\right).$$

Hence, by Remark 16 in the previous section, the relativistic velocity satisfies the estimate $|v_A| \le \beta < 1$, where β depends only on $\bar{X}_0$ and $\bar{P}_0$. Now, denote $g = |p - A|$ so that $Dv_A = (1+g^2)^{-1/2}\,[\mathtt{id} - v_A \otimes v_A]$, whose components are given by (5.2.48). Clearly, $g \le C = C(\bar{X}_0, \bar{P}_0)$. Then, if we write $\partial_p X(0)$ instead of $\partial_p X(0,t,x,p)$, we find that for small enough $\alpha > 0$ and $\delta > 0$

$$\begin{aligned}\left|\frac{\sqrt{1+g^2}}{t}\partial_p X(0) + \mathtt{id}\right| &\equiv \left|\frac{\sqrt{1+g^2}}{t}\xi(0) + v_A \otimes v_A\right| \\ &\le Ce^{\alpha C}\left[\alpha + (1+\alpha)\,\delta\right] + \beta \\ &=: \gamma \; < \; 1. \end{aligned} \tag{5.2.55}$$

Therefore, a positive constant $C = C(\bar{X}_0, \bar{P}_0)$ exists such that

$$|\mathtt{det}\partial_p X(0)| \equiv \frac{t^3}{(1+g^2)^{3/2}}\left|\mathtt{det}\left[\frac{\sqrt{1+g^2}}{t}\partial_p X(0) + \mathtt{id} - \mathtt{id}\right]\right| \ge Ct^3.$$

Step 4. Moreover, for every $0 < t \le a$ and $x \in \mathbb{R}^3$, the map $X(0,t,x,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is injective. Indeed, for $p, q \in \mathbb{R}^3$, let

$$p_\lambda := \lambda p + (1-\lambda)\,q, \quad g_\lambda = g(t,x,p_\lambda) := |p_\lambda - A(t,x)|\,, \quad 0 \le \lambda \le 1.$$

Then, in view of (5.2.55), we have

$$\begin{aligned}|X(0,t,x,p) - X(0,t,x,q)| &= \left|\int_0^1 \partial_p X(0,t,x,p_\lambda)(p-q)d\lambda\right| \\ &= \left|\int_0^1 \left[-\mathtt{id} + \mathtt{id} + \frac{\sqrt{1+g_\lambda^2}}{t}\partial_p X(0,t,x,p_\lambda)\right]\frac{t(p-q)}{\sqrt{1+g_\lambda^2}}d\lambda\right| \\ &\ge t\,|p-q|\int_0^1 \frac{d\lambda}{\sqrt{1+g_\lambda^2}} - \gamma t\,|p-q|\int_0^1 \frac{d\lambda}{\sqrt{1+g_\lambda^2}} \\ &\ge (1-\gamma)\,|p-q|\,t, \end{aligned} \tag{5.2.56}$$

and the assertion follows. In addition, we claim that the open range $X(0,t,x,\mathbb{R}^3)$ is all $\mathbb{R}^3$. If not, a boundary point x_0 exists so that $X(0,t,x,p_n) \to x_0 \notin X(0,t,x,\mathbb{R}^3)$ as $n \to \infty$, for some sequence $\{p_n\} \subset \mathbb{R}^3$. By (5.2.56), we have $p_n \to p_0$ for some $p_0 \in \mathbb{R}^3$ and so continuity implies that $X(0,t,x,p_0) = x_0$. But this is a contradiction and so we conclude that $X(0,t,x,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is actually a bijection.

Step 5. Therefore, Steps 3 and 4 imply that for every $0 < t \leq a$, $x \in \mathbb{R}^3$ the map $X(0,t,x,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-diffeomorphism. In particular, Step 3 implies that for some $C = C(\bar{X}_0, \bar{P}_0) > 0$, the inverse map $X^{-1}(0,t,x,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $X \mapsto p(X)$ has Jacobian determinant satisfying

$$\left|\det \partial_p X^{-1}(0,t,x,p(X))\right| \leq Ct^{-3}, \quad 0 < t \leq a,\ x \in \mathbb{R}^3.$$

Hence, we can now deduce the decay estimates (5.2.46)-(5.2.47) on both the current and charge densities as well as their space derivatives. Indeed, bearing in mind that $f_0 \in \mathcal{D}$ so $\|f_0\|_{L^\infty_{x,p}} \leq 1$, we have for the charge density

$$\begin{aligned} \rho(t,x) &= \int_{\mathbb{R}^3} f_0(X(0,t,x,p), P(0,t,x,p))dp \\ &= \int_{\mathbb{R}^3} f_0(X, P(0,t,x,p(X))) \left|\det \partial_p X^{-1}(0,t,x,p(X))\right| dX \\ &\leq Ct^{-3}, \end{aligned}$$

where $C = C(\bar{X}_0, \bar{P}_0) > 0$. Since $|j_A| \leq \rho$, it is clear that the (5.2.46) indeed holds.

Finally, we are done if we show that for some $C = C(\bar{X}_0, \bar{P}_0) > 0$

$$\|\partial_x \rho(t)\|_{L^\infty_x} + \|\partial_x j_A(t)\|_{L^\infty_x} \leq C, \quad 0 \leq t \leq a. \tag{5.2.57}$$

To that end, recalling that $|\partial_x v_A| \leq C\,|\partial_x A|$ and in view of (FSα), we notice that

$$\begin{aligned} \|\partial_x \rho(t)\|_{L^\infty_x} &\leq C\left(\bar{P}_0 + 1\right)^3 \|\partial_x f(t)\|_{L^\infty_{x,p}} \\ \|\partial_x j_A(t)\|_{L^\infty_x} &\leq C\left(\bar{P}_0 + 1\right)^3 \left(\|\partial_x A(t)\|_{L^\infty_x} \|f_0\|_{L^\infty_{x,p}} + \|\partial_x f(t)\|_{L^\infty_{x,p}}\right) \\ &\leq C\left(\bar{P}_0 + 1\right)^3 \left(\|f_0\|_{L^\infty_{x,p}} + \|\partial_x f(t)\|_{L^\infty_{x,p}}\right) \end{aligned}$$

and

$$\|\partial_x f(t)\|_{L^\infty_{x,p}} \leq \left\|\partial_{(x,p)} f_0\right\| \Big(\ |\partial_x X(0,t,x,p)| + |\partial_x P(0,t,x,p)|\ \Big).$$

Hence, since $f_0 \in \mathcal{D}$ so $\|f_0\|_{L^\infty_{x,p}} \leq 1$ and $\left\|\partial_{(x,p)} f_0\right\|_{L^\infty_{x,p}} \leq 1$, we obtain that

$$\|\partial_x \rho(t)\|_{L^\infty_x} + \|\partial_x j_A(t)\|_{L^\infty_x} \leq C\left(\bar{P}_0 + 1\right)^3 \Big(\ 1 + |\partial_x X(0,t,x,p)| + |\partial_x P(0,t,x,p)|\ \Big). \tag{5.2.58}$$

Thus, we are led to estimate the space partial derivatives of the characteristic curves.

To do so, we proceed as in Step 2 but with ∂_x instead of ∂_p. We find that for all $0 \leq s \leq t \leq a$ and some universal constant $C > 0$

$$|\partial_x X(s)| \leq 1 + C \int_s^t \left(|\partial_x P(\tau)| + \|\partial_x A(\tau)\|_{L^\infty_x} |\partial_x X(\tau)| \right) d\tau,$$

and in view of (FSα) and Gronwall's lemma

$$|\partial_x X(s)| \leq C \left(1 + \int_s^t |\partial_x P(\tau)| \, d\tau \right). \tag{5.2.59}$$

Similarly,

$$\begin{aligned} |\partial_x P(s)| &\leq C \int_s^t \left(\left\|\partial_x^2 \Phi(s)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(s)\right\|_{L^\infty_x} + \|\partial_x A(s)\|^2_{L^\infty_x} \right) |\partial_x X(\tau)| \, d\tau \\ &\quad + C \int_s^t \|\partial_x A(\tau)\|_{L^\infty_x} |\partial_x P(\tau)| \, d\tau \\ &\leq C \int_s^t (1+\tau)^{-5/2} |\partial_x X(\tau)| \, d\tau. \end{aligned} \tag{5.2.60}$$

Therefore, both (5.2.59) and (5.2.60) yield

$$\begin{aligned} |\partial_x X(s)| &\leq C + C \int_s^t \int_\tau^t (1+\sigma)^{-5/2} |\partial_x X(\sigma)| \, d\sigma d\tau \\ &\leq C + C \int_s^t \int_s^\sigma (1+\sigma)^{-5/2} |\partial_x X(\sigma)| \, d\tau d\sigma \\ &\leq C + C \int_s^t (1+\sigma)^{-3/2} |\partial_x X(\sigma)| \, d\sigma. \end{aligned}$$

Gronwall's lemma applied to the above inequality provides a uniform bound on $|\partial_x X(s)|$, which in turn produces a uniform bound on $|\partial_x P(s)|$ via (5.2.60). As a consequence

$$|\partial_x X(0,t,x,p)| + |\partial_x P(0,t,x,p)| \leq C, \quad 0 \leq t \leq a, \; x \in \mathbb{R}^3, \; p \in \mathbb{R}^3,$$

with combined with (5.2.58) imply (5.2.57) and concludes the proof of the lemma. □

With this results at hand, we can now turn to the proof of the theorem.

Proof of Theorem 10. Let $0 < \alpha < \bar{P}_0 \mathcal{B}_0^{-1} (1 - \mathcal{B}_0)$, $\delta > 0$ and $C = C(\bar{X}_0, \bar{P}_0) > 0$

be suitable for Lemma 25 to hold. Fix $T_0 > 1$ such that for all $t \geq T_0$

$$Ct^{-2} \leq \frac{\alpha}{2}(1+t)^{-3/2}, \quad C(1+\ln t)\,t^{-3} \leq \frac{\alpha}{2}(1+t)^{-5/2}. \tag{5.2.61}$$

It is not difficult to see that (5.2.61) still holds even if $\delta > 0$ is made smaller while keeping $\alpha > 0$, $C = C(\bar{X}_0, \bar{P}_0) > 0$ and $T_0 > 1$ as chosen.

Now, by letting $\delta > 0$ be smaller if necessary, Lemma 24 implies that the Cauchy datum $f_0 \in \mathcal{D}$ with $\|f_0\|_{L^\infty_{x,p}} \leq \delta$ produces a classical solution f of the RVD system on the maximal interval of existence $[0, T[$ with $T_0 < T$, and

$$\|\partial_x \Phi(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} + \left\|\partial_x^2 \Phi(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} < \frac{\alpha}{2}(1+T_0)^{-5/2},$$

for all $0 \leq t \leq T_0$. Hence, f satisfies the free streaming condition (FSα) on $[0, T_0]$. In fact, the continuity of the left-hand side of the above inequality implies that there exists a maximal $T_0 < T_1 \leq T$ such that f satisfies (FSα) on $[0, T_1[$. Therefore, Lemma 25 and (5.2.61) imply that for all $T_0 \leq t < T_1$

$$\begin{aligned} \|\partial_x \Phi(t)\|_{L^\infty_x} + \|\partial_x A(t)\|_{L^\infty_x} &\leq Ct^{-2} \leq \frac{\alpha}{2}(1+t)^{-3/2}, \\ \left\|\partial_x^2 \Phi(t)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(t)\right\|_{L^\infty_x} &\leq C(1+\ln t)\,t^{-3} \leq \frac{\alpha}{2}(1+t)^{-5/2}. \end{aligned}$$

Then, a continuation argument yields $T_1 = T$. Also, by means of the characteristic equation (5.2.36), the first of the last two inequalities and the upper bound on the chosen $0 < \alpha < \mathcal{B}_0^{-1}\bar{P}_0(1-\mathcal{B}_0)$ imply that for all $0 \leq t < T$

$$\mathcal{A}_0 \bar{P}_0^{-1} \bar{P}(t) \leq \mathcal{A}_0 \bar{P}_0^{-1}\left(\bar{P}_0 + \alpha\right) < \mathcal{A}_0 \mathcal{B}_0^{-1} \leq 1.$$

The strict inequality and the continuation criterion (5.2.22) implies that $T = \infty$. Thus, the solution f is global in time and the proof of Theorem 10 is complete. □

5.3 Remarks on the Vlasov-Darwin system

In classical electrodynamics, when describing phenomena in which the electric and magnetic fields are time-dependent, the static models derived from the Coulomb and Biot-Savart laws together with the static continuity equation are no longer accurate[6].

[6] We recall that the Coulomb law for Electrostatic is (5.3.3) and (5.3.5) while the Biot-Savart law for Magnetostatic is (5.3.4) and (5.3.6). The static continuity equation is $\nabla \cdot j = 0$.

Instead, we must deal with electromagnetic fields that should, in principle, be determined by the full set of Maxwell equations

$$\nabla \times B - \frac{1}{c}\partial_t E = \frac{4\pi}{c} j \tag{5.3.1}$$

$$\nabla \times E + \frac{1}{c}\partial_t B = 0 \tag{5.3.2}$$

$$\nabla \cdot E = 4\pi\rho \tag{5.3.3}$$

$$\nabla \cdot B = 0. \tag{5.3.4}$$

However, if the variations of the fields are much slower than the speed of light, then the so-called quasi-static limits provide an easier modeling alternative to the system (5.3.1)-(5.3.4). The quasi-static limits of (5.3.1)-(5.3.4) are: the electro-quasi-static (EQS), the magneto-quasi-static (MQS) and the Darwin approximation.

The EQS limit is used when capacitive but not inductive effects are taken into account. In this case, the term $\partial_t B$ is neglected in the Faraday's law (5.3.2), and so

$$\nabla \times E = 0. \tag{5.3.5}$$

Conversely, if inductive but not capacitive effects are taken into consideration, then the MQS limit is preferred. This limit is defined by neglecting the displacement current $\partial_t E$ in the Ampère-Maxwell law (5.3.1), which yields

$$\nabla \times B = \frac{4\pi}{c} j. \tag{5.3.6}$$

Lastly, if both the capacitive and inductive effects are to be considered, then the Darwin approximation is our best choice. In this case, we do not neglect the whole displacement current in the Ampère-Maxwell law, but only the transversal part $\partial_t E_T$. Clearly, the Darwin approximation is more inclusive than the EQS and MQS limits, and it provides a better approximation to the full set of Maxwell equations. For a detailed discussion on quasi-static limits, cf. [49] and the references therein.

To better understand the Darwin approximation, we decompose the electric field

$$E = E_T + E_L$$

where

$$\nabla \cdot E_T = 0, \quad \nabla \times E_L = 0. \tag{5.3.7}$$

The component E_T is called the *transversal* or solenoidal part of the electric field while E_L denotes its *longitudinal* or irrotational part. In [47], sufficient conditions to uniquely determine this decomposition are given. The Darwin approximation is then defined by

$$\nabla \times B - \frac{1}{c}\partial_t E_L = \frac{4\pi}{c} j \tag{5.3.8}$$

$$\nabla \times E_T + \frac{1}{c}\partial_t B = 0 \tag{5.3.9}$$

$$\nabla \cdot E_L = 4\pi\rho \tag{5.3.10}$$

$$\nabla \cdot B = 0 \tag{5.3.11}$$

together with (5.3.7). Notice that (5.3.8)-(5.3.11) differs from the Maxwell equations (5.3.1)-(5.3.4) only by the absence of $\partial_t E_T$ in the Ampère-Maxwell law, i.e., the transversal part of the displacement current. As a result, and as we shall make clear below, the Darwin system features an elliptic behavior with infinite speed propagation, in contrast to the hyperbolic nature of the full Maxwell system.

The well-posedness of (5.3.7)-(5.3.11) for perfectly conducting boundary conditions was studied in [47], where the tools used are likely to be adaptable to more general settings. From the physical point of view, the Darwin model is valid when there are no high frequency nor rapid current changes within the system of charged particles. This translates into slow variations of the electromagnetic field when compared to the speed of light. Actually, it is proved in [47] that the Darwin model gives first and second order relativistic corrections to the magnetic and electric fields, respectively. In this regime, the Darwin approximation is preferred for numerical purposes to the full Maxwell system, since its elliptic structure avoids error propagations inherent to hyperbolic equations [48, 50, 51]. Additional references on the numerical implementation of the Darwin model are [52, 53, 54], just to mention a few.

Now, if the system (5.3.7)-(5.3.11) is coupled via the charge and current densities to the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + \left(E_L + E_T + \frac{v}{c} \times B\right) \cdot \nabla_p f = 0, \tag{5.3.12}$$

with

$$v = \frac{cp}{\sqrt{c^2 + |p|^2}}, \tag{5.3.13}$$

then we obtain the relativistic Vlasov-Darwin system as usually defined. In contrast with our definition in Section 5.2, here the model equations are given in terms of the electromagnetic field and the ordinary momentum-space variables. Although formally equivalent -shown below-, these formulations have a crucial difference already exploited in our proofs. In particular, the component E_T does not appear in the potential representation of the RVD system. We shall return to this point later on.

The first existence result concerning (5.3.7)-(5.3.12) seems to be that in [48], where global *weak* solutions are shown to exist for sufficiently small Cauchy data. The smallness assumption was later removed in [44], where also the existence of local classical solutions was proven. We have learnt while preparing this work, that the existence of global classical solutions for small Cauchy data has recently been proven, cf. [55]. On the other hand, the global result for unrestricted Cauchy data remains unsolved. In Chapter 6, we discuss the possibility of using the potential framework to solve this problem at least in the non-relativistic setting.

We stress that *all* of the previous results on the RVD system rely on a-priori estimates derived from the conservation of the total energy as well as non-trivial estimates obtained for the transversal component E_T of the electric field. Deducing the latter has been the main difficulty when dealing with the RVD system. *In our proofs, neither the energy estimates nor those on the transversal component of the electric field are required.* Actually, as far as our results are concerned, no control on E_T is needed at all since E_T is given by the partial *time* derivative of the vector potential, which is absent from the Vlasov-Darwin model equations once they are written in terms of the generalized momentum-space variables; cf. (5.2.1)-(5.2.5).

Indeed, if the electromagnetic field is defined by

$$E_L = -\nabla \Phi_D, \quad E_T = -\frac{1}{c}\partial_t A_D, \quad B = \nabla \times A_D, \tag{5.3.14}$$

where (Φ_D, A_D) denotes the Darwin potentials given by Definition 6, then (E_L, E_T, B) formally solves the system (5.3.7)-(5.3.11) provided the charge and current densities satisfy the continuity equation (2.1.3). To see this, we first recall the equations satisfied by the potentials themselves, which we rewrite here for convenience's sake, cf. Definition 5. In view of Lemma 21 and Remark 10, they are

$$\Delta \Phi_D = -4\pi\rho, \quad \Delta A_D = -\frac{4\pi}{c} j + \frac{1}{c}\nabla \partial_t \Phi_D. \tag{5.3.15}$$

Also, we know that A_D satisfies the Coulomb gauge condition $\nabla \cdot A_D = 0$, which is proven in Lemma 21(c). Hence, it is straightforward that

$$\begin{aligned} \nabla \times B &= \nabla \times (\nabla \times A_D) \equiv \nabla (\nabla \cdot A_D) - \Delta A_D \\ &= \frac{4\pi}{c} j - \frac{1}{c} \partial_t \nabla \Phi_D = \frac{4\pi}{c} j + \frac{1}{c} \partial_t E_L, \end{aligned}$$

which yields (5.3.8). Similarly, (5.3.9) and (5.3.10) are the result of

$$\nabla \times E_T = -\frac{1}{c} \partial_t \nabla \times A_D = -\frac{1}{c} \partial_t B,$$

and

$$\nabla \cdot E_L = -\nabla \cdot \nabla \Phi = -\Delta \Phi = 4\pi \rho.$$

It is immediate to check that the equations in (5.3.7) and (5.3.11) also hold. Conversely, since the electromagnetic field ($E = E_L + E_T, B$) satisfies

$$\begin{aligned} E(t,x) &= -\nabla \Phi_D(t,x) - \frac{1}{c} \partial_t A_D(t,x) \\ B(t,x) &= \nabla \times A_D(t,x), \end{aligned}$$

then the equations for the potentials in (5.3.15) can be obtained as in Section 4.3. That is, we impose the Coulomb gauge condition $\nabla \cdot A = 0$ to (4.3.1)-(4.3.2) and neglect the term $c^{-2} \partial_t^2 A \equiv -c^{-1} \partial_t E_T$, which defines the Darwin approximation.

On the other hand, we know from Corollary 3 that the (linear) Vlasov equation written in terms of both the potentials and the electromagnetic field are equivalent. Hence, the formulations (5.2.1)-(5.2.5) and (5.3.7)-(5.3.12) for the RVD system are also equivalent, at least formally. We emphasize that the equivalence is formal since our classical solution for (5.2.1)-(5.2.5) does not require C^1-regularity of the vector potential A_D with respect to time, cf. Theorems 9 and 10. Therefore, the relation $-c^{-1} \partial_t A_D \equiv E_T$ may be satisfied only in the sense of distributions.

As commented earlier, previous results on the Cauchy problem for the RVD system rely on a-priori estimates provided by the conservation of the total energy, defined by

$$\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \sqrt{1 + |p|^2} f(t,x,p) dx dp + \frac{1}{8\pi} \int_{\mathbb{R}^3} \left(|E_L(t,x)|^2 + |B(t,x)|^2 \right) dx.$$

Since bounds on this functional provide no control on the component E_T of the electric field, then the latter has to be estimated differently. To this end, the elliptic structure of the Darwin system plays a crucial role. If we take the curl in (5.3.9) and combine the resulting equation with (5.3.8), then we find eventually that $\Delta E_T = -\mathbb{P}\partial_t j$; cf. Lemma (21)(c) for the definition of $\mathbb{P}j$. A major issue is the lack of regularity in $\partial_t j$. Then, the Vlasov equation is used to replace $\partial_t j$ and the resulting equation is treated by means of duality-type arguments. This is basically the approach followed in both [44, Sec. 2.3] and [48, Sec. 3.1].

Actually, a representation of E_T in terms of the electromagnetic field can be found by substitution of the Darwin vector potential A_D, as given in Definition 6, in the relation $E_T = -c^{-1}\partial_t A_D$. This, combined with the Vlasov equation, provides the result. Indeed, if $(f, E = E_L + E_T, B)$ solves the system (5.3.7)-(5.3.12) and we let $Q := E + v \times B - v\,(v \cdot E)$, then by virtue of (5.3.14) and after setting $c = 1$, we find that the electromagnetic field satisfies the representation

$$\begin{aligned}
E_L &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,y,p)\frac{\omega dpdy}{|y-x|^2} \\
E_T &= -\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[|v|^2 - 3\,(v\cdot\omega)^2\right] f(t,y,p)\frac{\omega dpdy}{|y-x|^2} \\
&\quad -\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sqrt{1-|v|^2}\,[\mathtt{id} + \omega\otimes\omega]\,Qf(t,y,p)\frac{dpdy}{|y-x|} \\
B &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,y,p)\frac{\omega\times v dpdy}{|y-x|^2}.
\end{aligned}$$

As expected, E_L and B are given by (5.1.4) and (5.1.6), respectively. On the other hand, the representation for E_T consists of two terms, the second one depending on the electromagnetic field itself. This representation hints at the difficulties to be faced when the component E_T, as well as its derivatives, are to be estimated.

We conclude by noticing that, as presented in (5.3.7)-(5.3.12), the RVD is an hybrid system whose field equations provide relativistic corrections to the electric and magnetic fields -of order one and two as mentioned above, cf. [47]- while the transport equation is itself assumed relativistic. As follows from [31, Sec. 12.6], the Darwin approximation is actually an order two inclusive relativistic correction to the interaction of charged particles and the self-induced electromagnetic field. Hence, if we Taylor expand the velocity (5.3.13) in powers of (p/c) and collect the terms up to the order two, the 'proper' Vlasov-Darwin system should read

$$\partial_t f + p\left(1 - \frac{|p|^2}{2c^2}\right)\cdot\nabla_x f + \left(E_L + E_T + \frac{p}{c}\times B\right)\cdot\nabla_p f = 0,$$

coupled with (5.3.7)-(5.3.11) via

$$\rho = \int_{\mathbb{R}^3} f dp, \quad j = \int_{\mathbb{R}^3} p \left(1 - \frac{|p|^2}{2c^2}\right) f dp.$$

This was noticed in [56], where the Vlasov-Darwin system was introduced as the next order approximation after the Vlasov-Poisson (5.0.1)-(5.0.2) to the RVM system, -cf. [31, 57, 58] for a physical perspective of this approximation-. Equivalently, the above model equations in the potential framework read

$$\partial_t f + v_A \cdot \nabla_x f - \nabla \left[\Phi - \frac{1}{c}\mathtt{P} \cdot A + \frac{1}{c^2}\left|A\right|^2\right] \cdot \nabla_{\mathtt{P}} f = 0$$

coupled with

$$\Delta\Phi = -4\pi \int_{\mathbb{R}^3} f d\mathtt{P}, \quad \Delta A = -\frac{4\pi}{c}\mathbb{P} \int_{\mathbb{R}^3} v_A f d\mathtt{P},$$

where

$$v_A := \left(\mathtt{P} - \frac{1}{c}A\right)\left(1 - \frac{1}{2c^2}|\mathtt{P} - \frac{1}{c}A|^2\right).$$

For simplicity, we have dealt with the relativistic hybrid version of the Vlasov-Darwin system, although the results obtained are likely to be adapted to the 'proper' version presented above.

Chapter 6

Concluding Remarks

In this work, we have proved that the relativistic Vlasov-Maxwell system complemented with sufficiently smooth, compactly supported Cauchy data, has global in time solutions provided that the charge density is controlled for all times. As a result, we conclude that a classical solution of the RVM system could develop a singularity only because of a blow-up of the charge density, i.e., only due to a concentration effect. In particular, shock formations are not allowed. This result weakens previous continuation criteria used to extend local solutions of the RVM system globally in time, provided that the Cauchy data are compactly supported.

Indeed, Glassey and Strauss [9] had showed that if a classical solution of the RVM system breaks down in finite time, then the size of the momentum support of the one-particle distribution function f would become infinite in the same time. This assumption was later relaxed in [20]. Afterward, Pallard [21] showed that the singularity of a classical solution would lead to the blow-up of a range of moments of f in the momentum variable. If the initial electromagnetic field is assumed compactly supported, then the result in [20] is a particular case of that in [21]. Here, we have weakened those assumptions by showing that the zero-moment limit also holds true.

In essence, the approach we followed was to control the kinetic energy of each particle separately. To do so, we singled out a particle and estimated its kinetic energy in terms of the time-integral of the electric field acting on the particle itself, which is induced by the remaining charges of the system. In turn, the latter was estimated by the (time-integral of the) suprema among the kinetic energies of the particles in the system. This was done after using a clever change of variables introduced by Pallard in [21], and by assuming that the charge density was uniformly bounded. Hence, since the resulting estimate was irrespective of the particle chosen, we used the Gronwall's lemma to control the single-particle kinetic energies, which in turn provided a control on their momenta. The result then followed by virtue of [9].

We have also proved in this work the global existence and uniqueness of classical solutions of the relativistic Vlasov-Darwin system with small Cauchy data. The approach we have followed does not require the estimates derived from the conservation of the total energy nor those on the transversal component of the electric field. These have been crucial in previous studies of the Cauchy problem for the RVD system. We have sorted this out by reformulating the Vlasov equations in terms of the generalized space-momenta variables. The resulting Vlasov equation has coefficients that does not depend on the transversal component of the electric field -i.e., the partial time derivative of the vector potential-, and therefore it is determined by a vector field that only involves the potentials and their space derivatives. This formulation allowed to adapt techniques previously used to study the Vlasov-Poisson system.

For a suitable small datum, we first produced a local result for the RVD system. Then, we showed that the local solution obtained induces potentials whose space derivatives, up to a second order, satisfy some suitable decay estimates. As it turns out, these estimates imply an even faster decay of those derivatives on the whole interval of existence, which allows to continue the solutions globally in time -a bootstrap argument-. It should be noticed, however, that the proof we provide requires a smallness assumption for the local as well as the global result. We used it to find a fix point to the integral equation deduced for the vector potential, cf. Lemma 22, and also to prove that the approximate sequence of vector potentials is uniformly Cauchy. It would be desirable to get rid of this assumption. To this end, a closer look at the integral equation (5.2.7) will be needed.

The general existence and uniqueness of global in time classical solutions of both the RVM and the RVD systems with un-restricted Cauchy data remain open. We expect that the potential formulation used here, in particular the generalized Vlasov equation, will shed some light on the solution of these problems. It may be possible, for instance, to adapt the Lions-Perthame proof [41] for global classical solutions of the Vlasov-Poisson system to the (elliptic) Darwin case, at least in the non-relativistic setting. This seems possible due to the generality of the Lions-Perthame's approach, which may be adapted to similar scenarios. We remark that the relativistic setting may be harder, since it has not been solved for the less demanding Poisson case. On the other hand, the existence result for the RVM system seems to require more than *just* adapting the proof in [41], due to the hyperbolic nature of the Maxwell equations.

Appendix A

Proof of Lemma 12. The map π_σ is injective. Indeed, suppose that $(s_1, \theta_1, \phi_1) \in \Omega_{\sigma_1,\sigma_2}$ and $(s_2, \theta_2, \phi_2) \in \Omega_{\sigma_1,\sigma_2}$ are such that $\pi_\sigma(s_1, \theta_1, \phi_1) = \pi_\sigma(s_2, \theta_2, \phi_2)$. Thus,

$$X(s_1) - X(s_2) = \omega_1 (s_1 - \sigma) - \omega_2 (s_2 - \sigma) \tag{A.0.1}$$

with $\omega_1 = \omega(\theta_1, \phi_1)$ and $\omega_2 = \omega(\theta_2, \phi_2)$. Hence

$$|X(s_1) - X(s_2)| = |\omega_1(s_1 - \sigma) - \omega_2(s_2 - \sigma)| \geq ||s_1 - \sigma| - |s_2 - \sigma|| = |s_1 - s_2| \, .$$

But $s \mapsto X(s)$ is a C^1 curve satisfying (3.3.4) and

$$X(s_1) - X(s_2) = \int_{s_1}^{s_2} \dot{X}(s) ds.$$

Therefore, if $s_1 \neq s_2$,

$$|s_1 - s_2| \leq |X(s_1) - X(s_2)| \leq \int_{s_1}^{s_2} \left| \dot{X}(s) \right| ds < |s_1 - s_2| \, ,$$

which is a contradiction. Thus $s_1 = s_2$ and (A.0.1) becomes $0 = (\omega_1 - \omega_2)(s_1 - \sigma)$. Since $\sigma \leq \sigma_1 < s_1$, it follows that $\omega_1 = \omega_2$ and so $(\theta_1, \phi_1) = (\theta_2, \phi_2)$, which concludes the proof of the claim. But π_σ is clearly surjective, so $\Omega_{\sigma_1,\sigma_2} \to \pi_\sigma (\Omega_{\sigma_1,\sigma_2})$ is actually a bijection.

Now π_σ is a C^1 map, thus, it remains to show that its Jacobian determinant is not singular on $\Omega_{\sigma_1,\sigma_2}$. Recalling that $\omega(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$, we have

$$J_{\pi_\sigma}(s, \theta, \phi) = \begin{vmatrix} \dot{X}_1(s) + \cos\theta \sin\phi & -(s - \sigma) \sin\theta \sin\phi & (s - \sigma) \cos\theta \cos\phi \\ \dot{X}_2(s) + \sin\theta \sin\phi & (s - \sigma) \cos\theta \sin\phi & (s - \sigma) \sin\theta \cos\phi \\ \dot{X}_3(s) + \cos\phi & 0 & -(s - \sigma) \sin\phi \end{vmatrix}$$

and so we get that

$$J_{\pi_\sigma}(s, \theta, \phi) = -(1 + \dot{X}(s) \cdot \omega)(s - \sigma)^2 \sin \phi.$$

But again, $s \mapsto X(s)$ is a C^1 curve satisfying (3.3.4), therefore $1 + \dot{X}(s) \cdot \omega > 0$. Thus, since $\phi \in (0, \pi)$ and $(s - \sigma) > 0$, we conclude that J_{π_σ} does not vanish on $\Omega_{\sigma_1, \sigma_2}$ and the proof of the lemma is complete. □

Appendix B

cf. (4.2.10). We show that for any smooth vector fields $F, G : \mathbb{R}^3 \to \mathbb{R}^3$, we have

$$F \times (\nabla \times G) + (F \cdot \nabla)\, G = F^i \nabla G^i, \tag{B.0.1}$$

where as usual, repeated index means summation. If we denote the space gradient as $\nabla = (\partial_1, \partial_2, \partial_3)$, a direct computation yields

$$\begin{aligned} G \times (\nabla \times F) &= \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ G^1 & G^2 & G^3 \\ \partial_2 F^3 - \partial_3 F^2 & \partial_3 F^1 - \partial_1 F^3 & \partial_1 F^2 - \partial_2 F^1 \end{vmatrix} \\ &\equiv G^i \left(\partial_k F^i - \partial_i F^k\right) \hat{e}_k. \end{aligned} \tag{B.0.2}$$

Hence, we use the elementary vector identity

$$\begin{aligned} F \times (\nabla \times G) + (F \cdot \nabla)\, G &= \nabla (F \cdot G) - (G \cdot \nabla)\, F - G \times (\nabla \times F) \\ &\equiv F^i \nabla G^i + \left[G^i \nabla F^i - (G \cdot \nabla)\, F\right] - G \times (\nabla \times F)\,. \end{aligned}$$

Since the term in brackets is clearly (B.0.2), then (B.0.1) readily follows.

Appendix C

cf. (5.1.23). Let $u \in \mathbb{R}^3 / \{0\}$. We show that

$$I := \int_{\mathbb{R}^3} \frac{v dv}{|v|^3 |u - v|} = 2\pi \frac{u}{|u|}. \tag{C.0.1}$$

Indeed, let $0 < \theta < 2\pi$ and $0 < \phi < \pi$ and set

$$\hat{e}_3 = |u|^{-1} u, \quad |v|^{-1} v = \cos\theta \sin\phi \hat{e}_1 + \sin\theta \sin\phi \hat{e}_2 + \cos\phi \hat{e}_3.$$

Therefore,

$$\begin{aligned} I &= 2\pi \hat{e}_3 \int_0^\infty \int_0^\pi \frac{\sin\phi \cos\phi d|v| d\phi}{\sqrt{|u|^2 + |v|^2 - 2|u||v|\cos\phi}} \\ &= 2\pi \hat{e}_3 \left(\int_0^{|u|} + \int_{|u|}^\infty \right) \int_0^\pi \frac{\sin\phi \cos\phi d|v| d\phi}{\sqrt{|u|^2 + |v|^2 - 2|u||v|\cos\phi}} \\ &=: 2\pi \hat{e}_3 (I_1 + I_2). \end{aligned}$$

To solve I_1, we set $r := |v||u|^{-1} \leq 1$. Hence

$$I_1 = \int_0^1 \int_0^\pi \frac{\sin\phi \cos\phi dr d\phi}{\sqrt{1 + r^2 - 2r\cos\phi}} = \frac{1}{3}.$$

To solve I_2, we set $r := |u||v|^{-1} \leq 1$. Hence

$$I_2 = \int_0^1 \int_0^\pi \frac{\sin\phi \cos\phi dr d\phi}{r\sqrt{1 + r^2 - 2r\cos\phi}} = \frac{2}{3}.$$

Appendix D

cf. (5.2.13). A similar proof is given in [46, Theorem 10.2 (iii)]. For $a > 0$, $(b/a) > 1$, $m \geq 1$ and $0 < \alpha < 1$, we have by Hölder's inequality that

$$\begin{aligned}\frac{1}{m}\left(1-(b/a)^{-m}\right) &= \int_1^{b/a} t^{-m-1}dt \\ &\leq \left(\int_1^{b/a} dt\right)^{\alpha}\left(\int_1^{\infty} t^{-(m+1)/(1-\alpha)}\right)^{1-\alpha} \leq ((b/a)-1)^{\alpha}.\end{aligned}$$

Therefore, we have the estimate

$$\left|b^{-m}-a^{-m}\right| \leq m\,|b-a|^{\alpha}\max\left\{a^{-m-\alpha}, b^{-m-\alpha}\right\}.$$

Let $x, y, z \in \mathbb{R}^3$ not equal. Since by triangle inequality $||z-x|-|z-y|| \leq |y-x|$, it follows that

$$\left||z-x|^{-m}-|z-y|^{-m}\right| \leq m\,|y-x|^{\alpha}\left(|z-x|^{-m-\alpha}+|z-y|^{-m-\alpha}\right).$$

Now, consider $\partial_x\mathcal{K}(x,z) \equiv \Omega(x,z)\,|x-z|^{-2}$, where the entries of $\Omega(x,z)$ are

$$\Omega^i_{km}(x,z) := \frac{1}{2c}\left[\delta_{im}\omega^k-\delta_{km}\omega^i+\left(3\omega^i\omega^k-\delta_{ik}\right)\omega^m\right].$$

Clearly, $|\Omega(x,z)| \leq C$ and so $|\partial_x\mathcal{K}(x,z)| \leq C\,|x-z|^{-2}$. Hence,

$$\left|\left(\partial_x\mathcal{K}\right)(x,z)-\left(\partial_x\mathcal{K}\right)(y,z)\right| \leq C\,|x-y|^{\alpha}\left(|z-x|^{-2-\alpha}+|z-y|^{-2-\alpha}\right).$$

Appendix E

cf. (5.2.38). Fix $0 < t \leq \bar{T}$. Since $\mathtt{supp} f(\bar{T}) \supseteq \mathtt{supp} f(t)$ is compact, then for each $0 \leq s \leq t$

$$\|\partial_z Z(s,t)\|_{L^\infty_{x,p}} + \|\partial_x \Phi(s)\|_{L^\infty_x} + \|\partial_x A(s)\|_{L^\infty_x} + \left\|\partial_x^2 \Phi(s)\right\|_{L^\infty_x} + \left\|\partial_x^2 A(s)\right\|_{L^\infty_x} \leq C^0_{\bar{T}}. \tag{E.0.1}$$

Also, since $\Phi(t) \in C^{2,\alpha}(\mathbb{R}^3;\mathbb{R})$ with $0 < \alpha < 1$, for any $z_1, z_2 \in \mathbb{R}^6$ we have

$$\left|\partial_x^2 \Phi(s, X(s,z_1)) - \partial_x^2 \Phi(s, X(s,z_2))\right| \leq C^0_T \left|X(s,z_1) - X(s,z_2)\right|^\alpha, \tag{E.0.2}$$

and similarly for $A(t) \in C^{2,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$.

Now, consider the characteristic system (5.2.35)-(5.2.36). In integral form, it reads

$$\begin{aligned} X(s,t,z) &= 1 - \int_s^t v(P(\tau,t,z), A(\tau, X(\tau,t,z)))d\tau & \text{(E.0.3)} \\ P(s,t,z) &= \int_s^t \left[\nabla\Phi - v_A^i \nabla A^i\right](\tau, X(\tau,t,z), P(\tau,t,z))d\tau. & \text{(E.0.4)} \end{aligned}$$

Take partial derivatives ∂_z in both sides of (E.0.3). If we write $Z := (X,P)(z)$ instead of $Z(s,t,z)$, then

$$\begin{aligned} \partial_z X(z) &= -\int_s^t \partial_z \left[v(P(z), A(\tau, X(z)))\right] d\tau \\ &= -\int_s^t (Dv_A) \left[\partial_z P(z) + \partial_x A(\tau, X(z)) \partial_z X(z)\right] d\tau, \end{aligned}$$

where Dv_A is evaluated accordingly and satisfies $|Dv_A| \leq C$ -recall that $v_A = v(p-A)$ is C_b^∞; cf. (5.2.48) for an explicit representation of Dv_A-. Hence, in view of (E.0.1), a lengthy computation shows that for any $z_1, z_2 \in \mathtt{supp} f(t)$

$$|\partial_z X(z_1) - \partial_z X(z_2)| \leq C_T^0 \int_0^t \Big(|X(z_1) - X(z_2)| + |P(z_1) - P(z_2)| \\ |\partial_z X(z_1) - \partial_z X(z_2)| + |\partial_z P(z_1) - \partial_z P(z_2)| \Big) d\tau. \quad \text{(E.0.5)}$$

Similarly, by taking the partial derivatives ∂_z in both sides of (E.0.4), we obtain

$$\partial_z P(z) = \int_s^t \Big\{ \left[\partial_x^2 \Phi(\cdot) + v_A^i(\cdot)\partial_x^2 A^i(\cdot)\right] \partial_z X(\cdot) + \partial_x A^i(\cdot)\partial_z \left[v^i(\cdot)\right] \Big\} d\tau.$$

Thus, bearing (E.0.1) and (E.0.2) in mind, an even lengthier but elementary computation yields, for any $z_1, z_2 \in \mathtt{supp} f(t)$

$$|\partial_z P(z_1) - \partial_z P(z_2)| \leq C_T^0 \int_0^t \Big(|X(z_1) - X(z_2)| + |P(z_1) - P(z_2)| \\ + |X(z_1) - X(z_2)|^\alpha + |\partial_z X(z_1) - \partial_z X(z_2)| + |\partial_z P(z_1) - \partial_z P(z_2)| \Big) d\tau. \quad \text{(E.0.6)}$$

Hence, by combining (E.0.5) with (E.0.6), using Gronwall's lemma, and recalling that for every $0 \leq s \leq t$, we have $Z(s,t) \in C^1(\mathbb{R}^6; \mathbb{R}^6)$, i.e., $Z(s,t) \in C^\alpha(\mathtt{supp} f(t); \mathbb{R}^6)$ with $0 < \alpha < 1$, we obtain

$$|\partial_z Z(s, z_1) - \partial_z Z(s, z_2)| \leq C_T^0 |z_1 - z_2|^\alpha, \quad z_1, z_2 \in \mathtt{supp} f(t).$$

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