ON THE EXISTENCE OF CELESTIAL BODIES WITH UNPREDICTABLE MOTION IN THE SOLAR SYSTEM

by

LENNARD F. BAKKER and FLORIN N. DIACU

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On the Existence of Celestial Bodies with Unpredictable Motion in the Solar System

Lennard F. Bakker * *and* Florin N. Diacu **
Department of Mathematics and Statistics
University of Victoria, Victoria, B.C., Canada

1. **Introduction.** In a recent paper [Xia, 1991] it was shown that a chaotic phenomenon, known as Arnold diffusion, exists in the elliptic restricted three body problem when the mass of the one primary is small relative to the mass of the other primary and when the eccentricity of their orbits is small. The existence of Arnold diffusion implies there are orbits on which the motion of the zero mass is "random." This led the authors to consider the possible existence of "small" celestial bodies in the solar system with unpredictable orbits in the sense that the motion is sensitive to initial conditions. Since the Arnold diffusion exists in a neighbourhood of nonresonant motion, we consider the implications this has on asteroids in the asteroid belt near nonresonance. This leads us to the idea of nonresonant *quasigaps* in the distribution of the mean motions of the asteroids. Note that the applications of the elliptic restricted three body problem need not be wholly confined to the solar system.

[Xia, 1992] has also shown that Arnold diffusion exists in the general planar three body problem when the mass of the third particle is small relative to a sufficiently small eccentricity of the orbits of the primaries. In this paper we will follow the approach of [Xia, 1991]. In doing so, we will neglect to go into all the mathematical details, preferring to foster an intuitive understanding in the reader. We begin by transforming the equations of motion for the zero mass into a form more suitable for analysis. We will consider the elliptic restricted three body problem as a perturbation of the circular restricted three body problem, the eccentricity of the orbits of the primaries being the perturbation parameter. The structure of the orbits of the circular problem are analyzed by using a so called Poincaré section and its associated Poincaré map. Perturbing the eccentricity of the orbits of the primaries, KAM Theory is applied, and in conjunction with a Theorem of [Xia, 1991] and the so called λ-lemma, the existence of Arnold diffusion in the elliptic restricted three body problem is shown. All relevant terms and concepts will be defined and illustrated by means of simple geometric examples.

2. **The Elliptic Restricted Three Body Problem.** Let P_1, P_2, and P_3 be three point masses in the plane, each with mass $1-\mu, \mu$, and zero respectively, where $0 < \mu \ll 1$. Suppose the center of mass is fixed at the origin. Under these conditions, the primaries P_1 and P_2 move in elliptic (in particular circular), hyperbolic, parabolic or collinear orbits. Suppose the primaries move in elliptic orbits with eccentricity e. Let $\vec{q} = (q_1, q_2)$ be the position of P_3 in the plane, and let $\vec{p} = (p_1, p_2)$ be its momentum. The distance between

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the primaries is the length of the vector (x_{12}, y_{12}) where

$$x_{12} = (1 - e\cos\psi)\cos\psi + o(e)$$

$$y_{12} = (1 - e\cos\psi)\sin\psi + o(e)$$

and

$$\psi = t + 2e\sin t + O(e^2).$$

Since the mass of P_3 is zero, the potential function is

$$\begin{aligned} U(q_1, q_2, \psi) = & \frac{1-\mu}{(q_1 - \mu x_{12})^2 + (q_2 - \mu y_{12})^2} \\ & + \frac{\mu}{\sqrt{(q_1 + (1-\mu)x_{12})^2 + (q_2 + (1-\mu)y_{12})^2}} \end{aligned} \tag{1}$$

where x_{12} and y_{12} are considered functions of ψ. So the equations of motion for P_3 are

$$\begin{aligned} q_1' &= p_1 \\ q_2' &= p_2 \\ p_1' &= \partial U/\partial q_1 \\ p_2' &= \partial U/\partial q_2 \end{aligned} \tag{2}$$

where the prime denotes differentiation with respect to the independent time variable t.

We would like to transform the equations (2) into a form more suitable for analysis. We begin by computing the first order approximation of the potential function U about $\mu = 0$:

$$\begin{aligned} \tilde{U}(q_1, q_2, \psi) = & \frac{1}{(q_1^2 + q_2^2)} + \mu \Bigg(\frac{-1}{(q_1^2 + q_2^2)} + \frac{q_1\cos\psi + q_2\sin\psi}{(q_1^2 + q_2^2)^{3/2}} \\ & + \frac{1}{((q_1 + \cos\psi)^2 + (q_2 + \sin\psi)^2)^{1/2}} \Bigg) + O(\mu^2). \end{aligned}$$

Next we define new variables x, y, θ, ρ by the transformation

$$\Gamma(x, \theta, y, \rho, \psi) = (q_1, q_2, p_1, p_2, \psi)$$

where

$$\begin{aligned} q_1 &= x^{-2}\cos\theta \\ q_2 &= x^{-2}\sin\theta \\ p_1 &= y\cos\theta - x^2\rho\sin\theta \\ p_2 &= y\sin\theta + x^2\rho\cos\theta \\ \psi &= \psi \end{aligned}$$

Figure 1 illustrates this transformation of variables. We can write this transformation of variables as

$$\Gamma(x, \theta, y, \rho, \psi) = (\Gamma_1(x, \theta, \psi), \Gamma_2(x, \theta, y, \rho)).$$

Under these new variables, the approximation of potential function (1) becomes

$$\hat{U}(x,\psi-\theta)=\tilde{U}\circ\Gamma_1(x,\theta,\psi)$$

where

$$\hat{U}(x,\psi-\theta)=x^2+\mu x^2\left(-1+x^2cos(\psi-\theta)+\frac{1}{(1+2x^2cos(\psi-\theta)+x^4)^{3/2}}\right)+O(\mu^2)$$

and the equations of motion for P_3 are

$$\begin{aligned}
x' &= -\frac{1}{2}x^3y\\
y' &= -x^4+x^6\rho^2+\mu g_1(x,\psi-\theta)+O(\mu^2)\\
\theta' &= x^4\rho\\
\rho' &= \mu g_2(x,\psi-\theta)+O(\mu^2)
\end{aligned}\tag{3}$$

where

$$\begin{aligned}
g_1(x,\psi-\theta) &= x^4\left(1-2x^2cos(\psi-\theta)-\frac{1}{(1+2x^2cos(\psi-\theta)+x^4)^{3/2}}\right)\\
g_2(x,\psi-\theta) &= x^4sin(\psi-\theta)\left(1-\frac{1}{(1+2x^2cos(\psi-\theta)+x^4)^{3/2}}\right)
\end{aligned}$$

and

$$\psi=t+2esint+O(e^2).$$

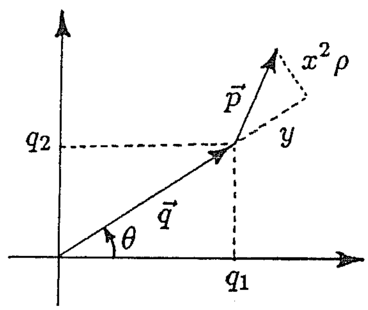

Figure 1
The transformation of variables Γ

3. **The circular Restricted Three Body Problem.** For $e=0$ the equations (3) are

$$\begin{aligned}
x' &= -\frac{1}{2}x^3y\\
y' &= -x^4+x^6\rho^2+\mu g_1(x,t-\theta)+O(\mu^2)\\
\theta' &= x^4\rho\\
\rho' &= \mu g_2(x,t-\theta)+O(\mu^2).
\end{aligned}\tag{4}$$

These equations (4) describe the motion of P_3 in the circular restricted three body problem. This system has the well known Jacobi integral

$$\frac{1}{2}y^2 + \frac{1}{2}x^4\rho^2 - \hat{U}(x, t-\theta) - \rho = C \tag{5}$$

where C is the Jacobi constant, and is a constant of motion.

Notice that the independent time variable t always appears in the form $t - \theta$ in the equations (4). We define a new independent time variable $s \in S^1$ by

$$s = t - \theta. \tag{6}$$

Here S^1 is the real line segment $[0, 2\pi]$ with the points 0 and 2π identified, i.e. a circle. Differentiating (6) with respect to s and using the fact that $d\theta/dt = x^4\rho$, we have

$$\frac{dt}{ds} = \frac{1}{1 - x^4\rho}. \tag{7}$$

Using (6) we can replace the equation for θ' in (4) by (7). By the Jacobi integral (5) we can find ρ when given x, y, C and s, that is $\rho = \rho(x, y, C, s)$. So we can replace the equation for ρ' in (4) by $dC/dt = 0$. (Recall that C is a constant of motion.) Finally, applying (7) to the equations for x' and y' in (4) we have the equations of motion for P_3 in the new independent time variable s:

$$\begin{aligned} \frac{dx}{ds} &= \frac{-\frac{1}{2}x^3 y}{1 - x^4\rho} \\ \frac{dy}{ds} &= \frac{-x^4 + x^6\rho^2 + \mu g_1(x, s)}{1 - x^4\rho} + O(\mu^2) \\ \frac{dC}{ds} &= 0 \\ \frac{dt}{ds} &= \frac{1}{1 - x^4\rho}. \end{aligned} \tag{8}$$

We can partially solve these equations, in that $dC/ds = 0$ yields C is a constant, and

$$t = \int \frac{ds}{1 - x^4\rho} + t_0 \tag{9}$$

where $x = x(s)$ and $\rho = \rho(x, y, C, s)$. Thus t is a function of x, y, C, s and t_0. Since the equations for dx/ds and dy/ds in (8) do not depend on t, then it suffices to solve for x and y. Once we have $x = x(s)$ and $y = y(s)$i, then we can explicitly solve for $t = t(s)$ by (9). This reduction will be used in the next section.

4. **Poincaré Sections and Maps.** The space with x, y, C, t and s as coordinates is called the *phase space* of (8). Note that this space is five dimensional. Consider the solution

$$\alpha(s; \xi_0) = (x(s), y(s), C, t(s), s)$$

of (8) for the initial condition $\xi_0 = (x_0, y_0, C, t_0, s_0)$; that is, $\alpha(0;\xi_0) = \xi_0$. Since C is a constant of motion, $C(s) = C$ for all s. As s varies, the solution $\alpha(s;\xi_0)$ traces out a curve in phase space. Each curve in phase space corresponds to a solution of (8). The collection of all these curves in phase space is called the flow and this flow is determined by the equation

$$\xi' = f(\xi)$$

where

$$f(\xi) = f(x,y,C,t,s) = \left(\frac{-\frac{1}{2}x^3y}{1-x^4\rho}, \frac{-x^4+x^6\rho^2+\mu g_1(x,s)}{1-x^4\rho}, 0, \frac{1}{1-x^4\rho}, 1\right). \tag{10}$$

The only place that s explicitly occurs in (10) is in the function g_1 in the equation for dy/ds. This function is 2π-periodic in s; that is

$$g_1(x, s+2\pi) = g_1(x,s).$$

So we have that the function f defined in (10) is 2π-periodic in s; that is

$$f(x,y,C,t,s+2\pi) = f(x,y,C,t,s). \tag{11}$$

This allows us to define a "global cross section" Σ which consists of the points in phase space with $s = s_0$ fixed and x, y, C, t arbitrary. This section satisfies the following two conditions. The dimension of Σ is one less than that of phase space; that is, the dimension of Σ is four. And the flow in phase space is *transversal* to Σ. Let us explain what transversal means. Let $\xi \in \Sigma$ be arbitrary. The vector at ξ tangent to the curve passing through ξ is given by $\xi' = f(\xi)$. A unit vector normal to $f(\xi)$ is $n_f = (0,0,1,0,0)$. A unit normal to Σ at ξ is given by $n_\Sigma = (0,0,0,0,1)$. So $n_f \cdot n_\Sigma = 0$ and so n_f is not a multiple of n_Σ. Thus the curve passing through ξ is not tangent to the section Σ at ξ. Since ξ is arbitrary, the flow in phase space is never tangent to the section Σ. When the flow in phase space is never tangent to a section Σ, we say that the section Σ is transversal to the flow. In Figure 2, drawing the section Σ as a line we illustrate the concept of transversality. A section which has dimension one less than that of phase space and is transversal to the flow in phase space is called a Poincaré section.

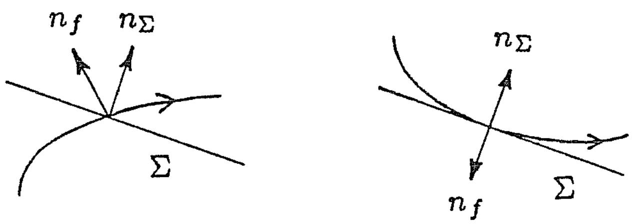

Figure 2
An example of a transversal section to a flow

There is a map naturally associated with a Poincaré section, and its construction is as follows. Let $\xi \in \Sigma$. Follow the curve through ξ as time moves forward until the curve intersects Σ. Denote this point of intersection by $\phi(\xi)$ (see Figure 3.) We call ϕ the first return or Poincaré map. From (11) it follows that the time elapsed between ξ and $\phi(\xi)$ is 2π; that is the curve intersects Σ whenever $s = s_0$ which occurs with every passing of 2π time units. In essence the Poincaré map is like a stroboscopic picture of the zero mass, with the strobe flashing every 2π time units.

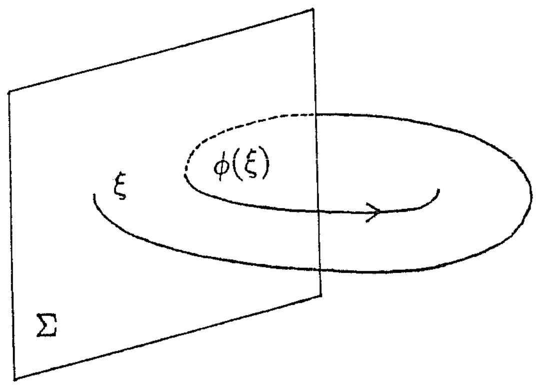

Figure 3
The Poincaré map

The Poincaré section Σ and its associated map ϕ contain information on the structure of the flow in phase space. For example, a periodic solution corresponds to a point $\xi \in \Sigma$ such that $\phi(\xi) = \xi$. By studying the Poincaré section and its associated map, we have "reduced the dimension" of the problem from an analysis of the flow in a five dimensional space to that of the analysis of a map defined on a four dimensional space.

Recall that the Poincaré section Σ consists of the points in phase space where $s = s_0$ is fixed and x, y, C, t are arbitrary. We are interested in the particular section when $s = \pi$. We denote this section by $\Sigma^{s=\pi}$. (The choice of $s_0 = \pi$ implies a symmetry which is called upon in the mathematical analysis to get the results.) Let ϕ be the Poincaré map associated with the section $\Sigma^{s=\pi}$ and write ϕ in component form:

$$\phi(x, y, C, t) = (\phi_1, \phi_2, \phi_3, \phi_4)$$

where, in general, $\phi_i = \phi_i(x, y, C, t), i = 1, 2, 3, 4$. Using the reduction described at the end of Section 4, ϕ_1, and ϕ_2 are shown to depend just on the variables x, y, C. ϕ_3 depends just on C, and ϕ_4 depends just on x, y, C and t (see (9).) Let $\tilde{\phi}$ be the map defined by $\tilde{\phi}(x, y) = (\phi_1(x, y, C), \phi_2(x, y, C))$. For this map [Xia, 1991] showed there exist periodic points of arbitrary period; that is, for each positive number there exists a point (x_n, y_n) such that

$$\begin{aligned} \tilde{\phi}^n(x_n, y_n) &= \tilde{\phi}(\tilde{\phi}(\cdots(\tilde{\phi}(x_n, y_n))\cdots)) \quad \text{n times} \\ &= (x_n, y_n). \end{aligned}$$

Figure 4 gives an illustration of a periodic point of period three.

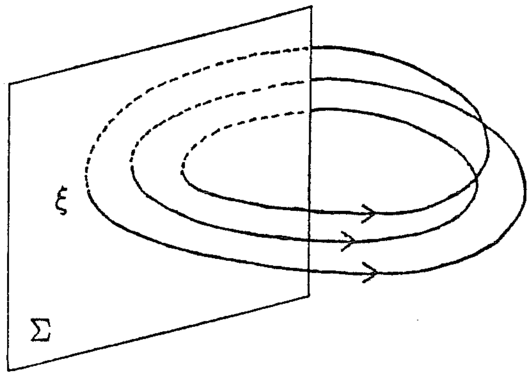

Figure 4
A point of period three

Let Ω_n be the set of points in $\Sigma^{s=\pi}$ such that $x = x_n, y = y_n, C \in J$ and $t \in S^1$, where J is the open interval $(\sqrt{2}+\delta_1, \sqrt{2}+\delta_2)$ with $0 < \delta_1 < \delta_2$. The set Ω_n is an annulus (see Figure 5.) For n sufficently large but fixed, Xia (see [Xia, 1991]) showed there exists $0 < \delta_1 < \delta_2$ and $0 < \mu_0$ such that for $0 < \mu < \mu_0$ the map $\phi^n (= \phi \circ \phi \circ \ldots \circ \phi$ n times) is a real analytic twist map (see Figure 5.) That is, $\phi_4^n(t) = t + g_n(C)$ where $dg(C)/dC \neq 0$ for all $C \in J$. Since C is a constant of motion, $\phi_3^n(C) = C$, and as $\phi^n(x_n, y_n) = (x_n, y_n)$ then ϕ^n uniformly rotates the points on the circle $C = C^*$ by the "rate of rotation" $g_n(C^*)$. The condition $dg_n(C)/dC \neq 0$ for all $C \in J$ implies that g_n is a strictly increasing or a strictly decreasing function of C, and so the rate of rotation changes monotonically from circle to circle.

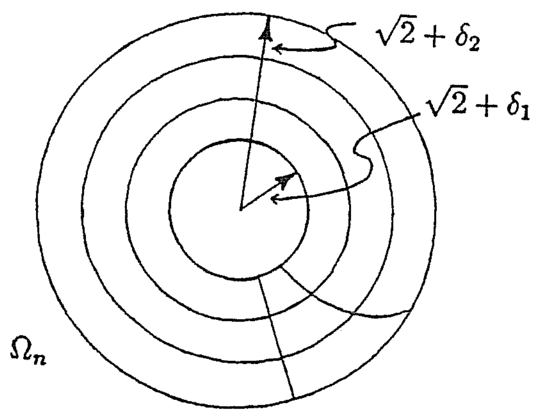

Figure 5
The set Ω_n

5. **Invariant Manifolds.** Since ϕ^n restricted to Ω_n is a twist map, each circle $C = C^*$ in Ω_n is *invariant* under ϕ^n; that is, for any point ξ in the circle $C = C^*$, its iterates $(\phi^n)^m(\xi)$, m a positive number, never leave the circle $C = C^*$. Since each circle $C = C^*, C \in J$, is invariant under ϕ^n, then Ω_n is invariant under ϕ^n; that is, for any point ξ in Ω_n, the point ξ lies in some invariant circle $C = C^*$ and so its iterates $(\phi^n)^m(\xi)$ never leave that circle and hence never leave Ω_n.

Let $\xi \in \Omega_n$. Consider the curve $\alpha(s;\xi)$ in phase space. For every lapse of 2π time units this curve will intesect $\Sigma^{s=\pi}$. Specifically the point ξ lies on some invariant circle $C = C^*$ and with every lapse of 2π time units the curve $\alpha(s;\xi)$ will intersect the circle $C = C^*$ by invariance. Therefore the curve $\alpha(s;\xi)$ lies on some 2-dimensional torus in phase space (see Figure 6).

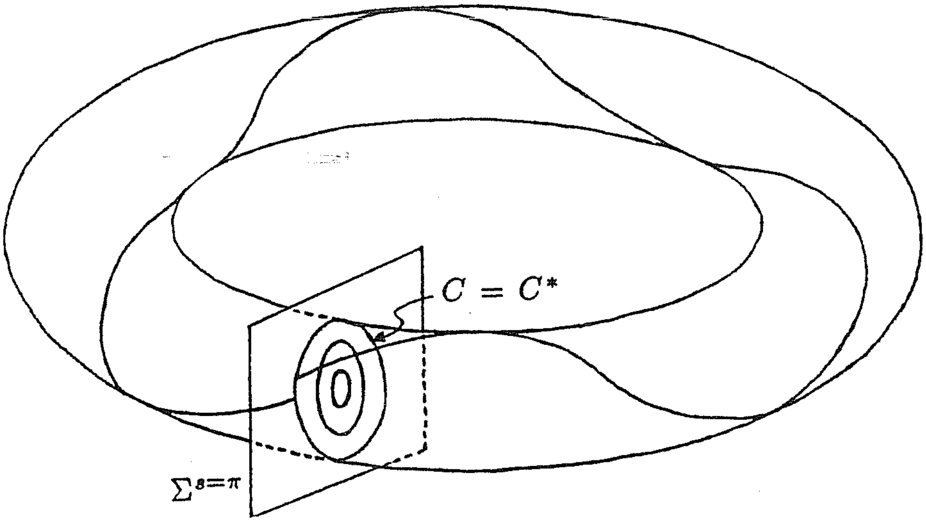

Figure 6
A solution on the torus

So all the curves $\alpha(s;\xi)$ for ξ in the circle $C = C^*$ lie on some 2-dimensional torus in phase space. Thus for each invariant circle $C = C^*$ there is a corresponding 2-dimensional torus, which we denote by T_{C^*}, of curves in phase space. A curve $\alpha(s;\xi)$, for ξ in some circle $C = C^*$, never leaves T_{C^*} and so we say that the torus T_{C^*} is invariant under the flow in phase space. The collection of all the tori T_{C^*} form a family of nested invariant 2-dimensional tori in phase space (see Figure 7). This family of nested invariant tori intersects the section $\Sigma^{s=\pi}$ in the annulus Ω_n.

We now consider the curves in phase space which "approach" or "recede" from a given invariant torus T_{C^*}. Let V be an open neighbourhood of T_{C^*}. The collection of curves $\alpha(s;\xi_0)$ for $\xi_0 \in V$ which satisfy

1) $\alpha(s;\xi_0) \in V \quad$ for all $s > 0$
2) $dist(\alpha(s;\xi_0), T_{C^*}) \to 0 \quad$ as $s \to \infty$

is called the *stable manifold* of the torus The stable manifold of the torus (see Figure 8). Similarly, the collection of curves $\alpha(s;\xi_0)$ which satisfy

$$1') \; \alpha(s;\xi_0) \in V \quad \text{for all } s < 0$$
$$2') \; dist(\alpha(s;\xi_0), T_{C^*}) \to 0 \quad \text{as } s \to -\infty$$

is called the *unstable manifold* of the torus T_{C^*} (see Figure 8). Both of these manifolds are invariant under the flow in phase space by their respective definitions.

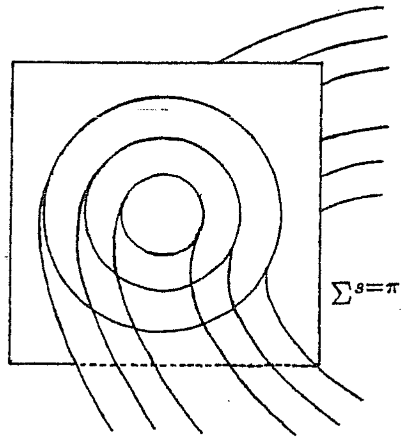

Figure 7
A family of nested invariant tori

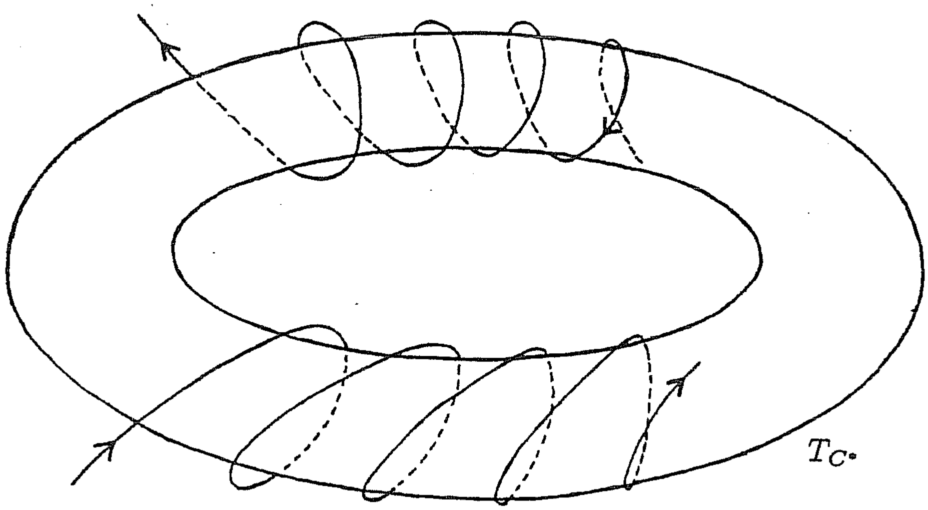

Figure 8
The stable manifold of the torus T_{C^*}

6. **Transversal Intersections and the λ-Lemma.** We have already seen some of the structure of the flow in phase space in the family of nested invariant tori. The stable and unstable manifolds of a torus T_{C^*} also yield information regarding the structure of the flow, especially if they intersect transversally; that is, if there is a point ξ of intersection of the stable and unstable manifolds of the same torus where the unit normal to the stable manifold of the torus at ξ is not a multiple of the unit normal to the unstable manifold of the torus at ξ. Drawing the stable and unstable manifold of the torus T_{C^*} as lines, Figure 9 illustrates a transversal intersection.

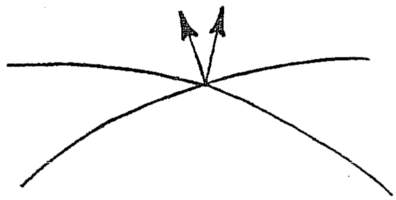

Figure 9
Transversal intersection of the stable and unstable manifolds

When the stable and unstable manifold of an invariant torus T_{C^*} intersect transversally, the resulting structure of the flow in phase space is of almost uncomprehensable complexity. This follows as a consequence of the λ-lemma (see [Palis and de Melo, 1982] and [Xia, 1991]) which states that the stable manifold "accumulates" on itself, and that the unstable manifold "accumulates" on itself. Drawing the stable and unstable manifold of an invariant torus as lines and the torus as a point, Figure 10 illustrates this "accumulation." This fact was first noticed by [Poincaré, 1899]. A nice and clear description of it is to be found in [Ekeland, 1988].

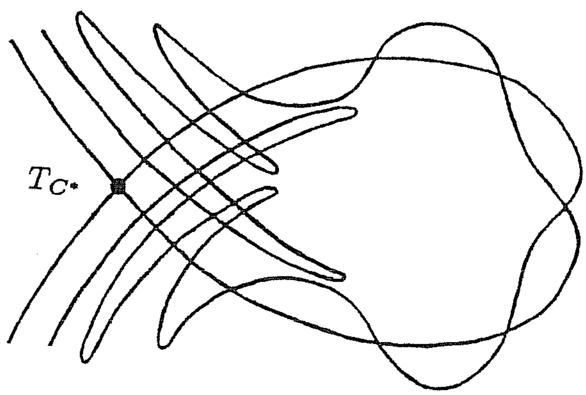

Figure 10
The complicated dynamics of the transversal intersection

7. KAM Theory. So far the discussion has been about the circular restricted three body problem introduced back in Section 3. For this problem we have a family of nested invariant tori in its phase space. Considering the eccentricity of the orbits of the primaries as a perturbation parameter, what happens to this family of nested invariant tori when the eccentricity becomes positive but small? By applying KAM Theory we will answer this question. This will give us some further insight to the structure of the flow in phase space for the elliptic restricted three body problem.

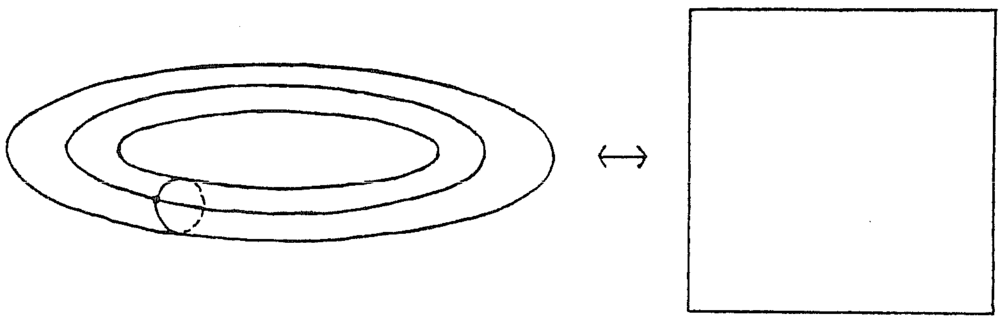

Figure 11
The equivalence between a torus and a rectangle

Consider a torus T_{C^*}. So far we have identified this torus by C^*. We now show that there is another way to identify this torus by two frequencies ω_1, ω_2. Any 2-dimensional torus is equivalent to a rectangle with its opposite edges identified (see Figure 11). This rectangle allows us to "breakdown" a curve on the torus into two motions: one along the vertical edge, the other along the horizontal edge. The latter we associate with the motion in the x, y coordinates of the curve. Recall that the circle $C = C^*$ consists of the set of points $x = x_n, y = y_n, C = C^*, t \in S^1, s = \pi$. In the x, y coordinates the curve is periodic; that is, after a lapse of $2\pi n$ time units, the x, y coordinates return to their initial values. So this motion of the curve along the horizontal edge of the rectangle has a frequency of $\omega_1 = 1/n$. We associate the motion along the vertical edge of the rectangle with the t coordinate of the curve. In this coordinate, the curve is periodic and its period is related to the "rate of rotation" $g_n(C^*)$. Recall that g_n is a strictly monotonic function of C, and so for different values of $C^* \in J$, the frequency ω_2 of the motion in the t coordinate of the curve, takes different values. So we can identify the torus T_{C^*} by the two frequencies ω_1, ω_2 and write $T(\omega_1, \omega_2) = T_{C^*}$.

If the ratio ω_1/ω_2 is rational, a curve on $T(\omega_1, \omega_2)$ will repeat, and hence is periodic. In this case we refer to $T(\omega_1, \omega_2)$ as *resonant*. However, if the ratio ω_1/ω_2 is irrational, a curve on the torus will never repeat but will "fill the torus densely." In this case we refer to the torus $T(\omega_1, \omega_2)$ as *nonresonant*. A curve on a nonresonant torus is called *quasiperiodic*. It is quasiperiodic motion that corresponds to real planetary motion. For example, we know that the perihelion of Earth is slowly advancing: its orbit is not truly periodic but is quasiperiodic.

Now we can state KAM Theory as it applies to our situation. (For a more general disucussion of KAM Theory see [Tabor, 1989].) If $e > 0$ is sufficently small, then "most of" the nonresonant tori $T(\omega_1, \omega_2)$ do not vanish but are only deformed to invariant tori in the phase space of the perturbed problem (that is, the elliptic restricted three body problem) with the curves on the perturbed invariant tori being quasiperiodic. In particular, for a torus $T(\omega_1, \omega_2)$ which perturbs to an invariant torus (that is, $T(\omega_1, \omega_2)$ *persists* under the perturbation) the frequencies of the quasiperiodic curves on the perturbed torus are the same as those of the unperturbed torus. So we denote the perturbed torus by $T^e(\omega_1, \omega_2)$. Since we can identify $T(\omega_1, \omega_2)$ by T_{C^*} we also write $T^e_{C^*}$ for $T^e(\omega_1, \omega_2)$.

Now there are many more irrational than rational numbers, and so it follows that "most of" the invariant tori in the family of nested tori are nonresonant. So, applying KAM Theory, "most of" these nonresonant tori persist under the perturbation, provided it is small enough.

8. **Arnold Diffusion.** Fix n sufficiently large, and let $\lambda_1, \lambda_2, \mu_0$ be as before. Let $0 < \mu < \mu_0$. Then we have the following result of [Xia, 1991] regarding the intersection of the stable and unstable manifolds of a perturbed invariant torus $T^e_{C^*}$.

Theorem (Xia, 1991): *There exists $e_0 > 0$ such that for all $0 < e < e_0$, if T_{C^*} perturbs to $T^e_{C^*}$, then the stable and unstable manifold of $T^e_{C^*}$ intersect transversally. Moreover there exists $\epsilon > 0$ such that if $| C_1 - C_2 | < \epsilon$ and T_{C^1} and T_{C^2} perturb to $T^e_{C^1}$ and $T^e_{C^2}$ respectively, then the stable manifold of $T^e_{C^1}$ intersects transversally with the unstable manifold of $T^e_{C^2}$.*

Note that the last part of Xia's Theorem, by interchanging C_1 and C_2, implies that the unstable manifold of $T^e_{C^1}$ intersects transversally with the stable manifold of $T^e_{C^2}$.

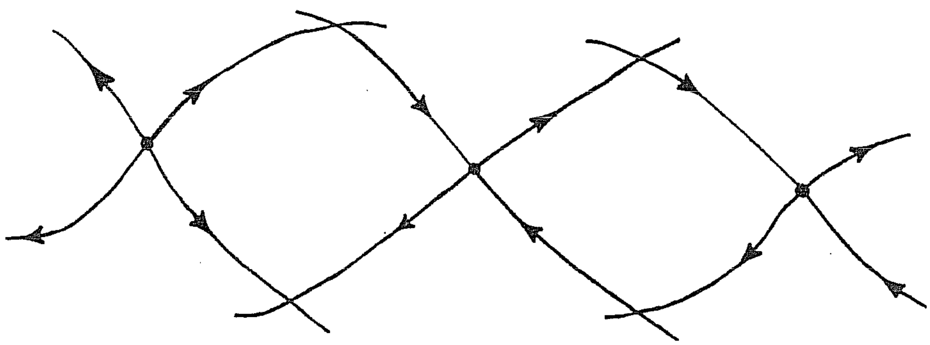

Figure 12
A chain of three tori

Recall that "most of" of the invariant tori in the unperturbed problem persist under a small enough perturbation. So applying Xia's Theorem we find that there are many tori in the perturbed problem whose stable and unstable manifolds intersect transversally. A torus $T^e_{C^*}$ with its stable and unstable manifolds intersecting tranversally is called a *transition torus*. We want to construct a finite sequence of transition tori $T^e_{C^i}, i = 1, \ldots, k$, which satisfies the following *chain condition*: for $1 \le i < k$, the stable manifold of $T^e_{C^i}$

intersects transversally with the unstable manifold of T^e_{Ci+1}, and the unstable manifold of T^e_{Ci} intersects transversally with the stable manifold of T^e_{Ci+1}. Again, drawing the tori as points and the stable and unstable manifolds as lines, Figure 12 illustrates the chain condition for three tori.

The construction of a transition chain is accomplished by using Xia's Theorem and KAM Theory. We begin with a torus T^e_{C1}. By Xia's Theorem and KAM Theory we can find a transition torus T^e_{C2} such that the stable (unstable) manifold of T^e_{C1} intersects transversally with the unstable (stable) manifold of T^e_{C2}. Again, by Xia's Theorem and KAM Theory, we can find a transition torus T^e_{C3} such that its stable (unstable) manifold intersects transversally with the unstable (stable) manifold of T^e_{C2}. Continuing in this way we construct a finite sequence of transition tori which satisfy the chain condition. Hence we have a transition chain of invariant tori in the phase space of the elliptic restricted three body problem for e sufficently small.

A corollary of the λ-lemma states that if the stable (unstable) manifold of T^e_{Ci} intersects transversally with the unstabe (stable) manifold of T^e_{Ci+1}, and the stable (unstable) manifold of T^e_{Ci+1} intersects with the unstable (stable) manifold of T^e_{Ci+2}, then the stable (unstable) manifold of T^e_{Ci} intersects transversally with the unstable (stable) manifold of T^e_{Ci+2} (see Figure 13).

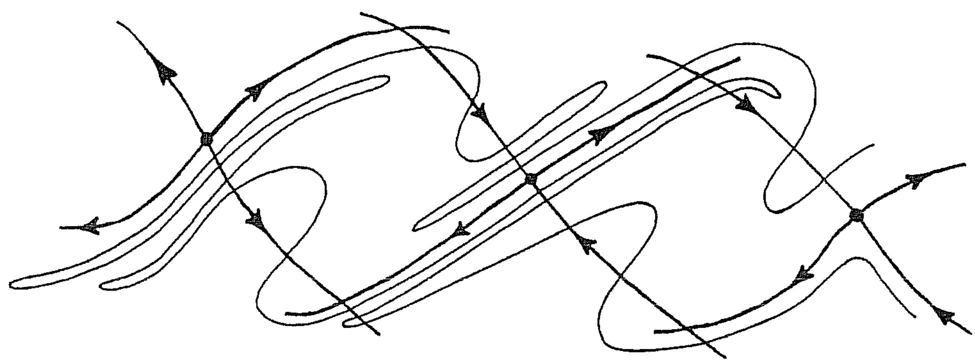

Figure 13
The Arnold diffusion

Applying this to a transition chain, we have that for $1 \leq i < j \leq k$ the stable (unstable) manifold of T^e_{Ci} intersects transversally with the unstable (stable) manifold of T^e_{Cj}. All these transversal intersections imply that all the stable (unstable) manifolds of the transition tori in the transition chain "accumulate" on each other, thus leading to a very complicated structure of the flow in the phase space of the elliptic restricted three body problem. A curve in a neighbourhood of a transition chain will "wander freely" between the invariant tori in the chain. This wandering is called *Arnold Diffusion.* This type of phenomenon was first noticed by [Arnold, 1964]. Curves which wander freely in a neighbourhood of the transition chain we call *Arnold curves.* In Figure 13 we try to suggest what an Arnold curve looks like in a transition chain of three tori. Again we draw the tori as points, and the stable and unstable manifolds as lines. It's real behavior is too complicated to draw accurately, and the number of transition tori in the chain can be much larger than three. The presence of Arnold diffusion implies that there are curves in phase space going from any torus T^e_{Ci} in the transition chain to any other torus T^e_{Cj}, $i \neq j$, in the transition chain. Thus the quasiperiodic solutions on T^e_{Ci} are unstable.

9. **Applications and Conclusion.** A restricted problem is often used to model the motion of an asteroid, such as in the Sun-Jupiter-asteroid system. If we think of P_1 as the Sun and P_2 as Jupiter, we can use the planar elliptic restricted three body problem to model the motion of an asteroid with a reasonable approximation. Here the mass of Jupiter relative to the Sun is approximately $\mu = 0.001$. If we account for the rest of the mass in the solar system by the eccentricity e of the orbits of the Sun and Jupiter, then the question is 'How large can e become before KAM Theory no longer applies?' The eccentricity of the orbit of Jupiter is approximately $e = 0.05$ which is essentially circular. If e is small enough here for KAM Theory to apply, then Arnold diffusion exists in the model. Actually the possibility to apply KAM Theory to the solar system was already investigated by Arnold himself. An important condition to be fulfilled is that the system is *isoenergetic* (see [Arnold,1978]), which is shown to be true for the solar system.

Recall that we considered the planar elliptic restricted three body problem as a perturbation of the planar circular restricted three body problem. Applications of the latter have produced some results on the presence of the Kirkwood gaps which correspond to certain low-order resonances (see [Moser, 1955] and [Brjuno, 1970].) The hypothesis that Arnold diffusion is a mechanism by which the Kirkwood gaps formed has been investigated by [Chirikov, 1971]. He tentatively concluded that Arnold diffusion would be sufficent, over the lifetime of the solar system, to account for the emptying of the Kirkwood gaps. Now that we know that Arnold diffusion exists in the planar three body problem (see [Xia, 1992]) it becomes more plausible that Arnold diffusion could be the mechanism for the formation of the Kirkwood gaps. However instead of applying our planar formulation of the elliptic restricted three body problem to an investigation of the Kirkwood gaps, we will consider its application to nonresonant motion. After all, it is the majority of the nonresonant tori $T^e_{C^*}$, for $C^* \in J = (\sqrt{2}+\delta_1, \sqrt{2}+\delta_2)$ which persist under a small enough perturbation of e. The location of the transition chain constructed in Section 8 is not quantitatively known, and so the whereabouts of the Arnold diffusion is likewise. However, we can expect, from a qualitative perspective, to find evidence of Arnold diffusion in action.

In the distribution of the mean motion of the asteroids in the asteroid belt, there are some mean motions which are not necessarily resonant with Jupiter but in which are found only a few asteroids (see Figure 15 which is taken from [Brower, 1963].) Between the Eos and Themis group is one such place. Also between the 4/1 and the 3/1 resonances are found other such places. These *quasigaps*, as we call them, may exist because of the presence of Arnold diffusion. That is, if a quasigap corresponds of a nonresonant motion and if Arnold diffusion is present near this motion, then the nonresonant motion is unstable, as noted at the end of Section 8. Thus over long periods of time, a quasigap would tend to empty out, becoming scarely populated. Asteroids that leave a quasigap because of Arnold diffusion will generally either become captured by larger bodies or escape the solar system.

Returning to the title of this paper, the presence of Arnold curves, those curves on which the motion is random, implies that there can be small celestial bodies, namely asteroids, whose motion is unpredictable in that it is highly sensitive to initial conditions. The probability that such objects still exist in the solar system is low considering that most of them have, over the lifetime of the solar system, either been captured by larger bodies or have escaped. We note that our conclusions are tentative in that we have made some assumptions along the way. Namely, that the motion of an asteroid can be modelled effectively by the planar formulation of the elliptic restricted three body problem, that the mass of Jupiter relative to the Sun is small enough, and that the eccentricities of the orbits of Sun and Jupiter are small enough. However, all these approximations are reasonable enough to conclude that the mathematical theory can be applied to the solar system.

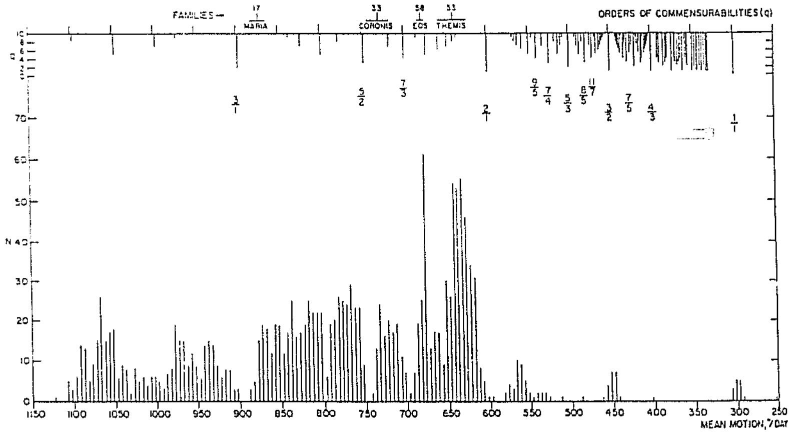

Figure 14
The distribution of asteroids

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