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## Article

# New Riemann–Liouville Fractional-Order Inclusions for Convex Functions via Interval-Valued Settings Associated with Pseudo-Order Relations

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**Abstract:** In this study, we focus on the newly introduced concept of LR-convex interval-valued functions to establish new variants of the Hermite–Hadamard (H-H) type and Pachpatte type inequalities for Riemann–Liouville fractional integrals. By presenting some numerical examples, we also verify the correctness of the results that we have derived in this paper. Because the results, which are related to the differintegral of the  $\frac{Q_1+Q_2}{2}$  type, are novel in the context of the LR-convex interval-valued functions, we believe that this will be a useful contribution for motivating future research in this area.

**Keywords:** convex interval-valued functions; pseudo-order relations; Hermite–Hadamard inequality; Riemann–Liouville fractional integral operators; real vector space; fuzzy interval-valued analysis



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## 1. Introduction

Convex functions have a long and illustrious history in science, and they have been the subject of research for almost a century. Inequalities with distinct convex functions have been an important research problem for several scholars due to the quick growth of the theory and widespread applications of fractional calculus. Mathematical scientists have proposed many types of inequalities or equalities, such as the H-H type, the Ostrowski type, the H-H-Mercer type, the Bullen type, the Opial type, and other types, by using convex functions. Among all of these integral inequalities, the H-H inequality [1] has attracted the interest of most scholars. Since its discovery in 1883, it has been the most popular and useful inequality in mathematical analysis. In addition, as shown in the publications [2–12], other researchers have worked on refining this condition for various classes of convex functions and mappings.

It is worth mentioning here that Leibniz and L'Hôpital (1695) were the ones who first introduced the concept of fractional calculus. However, such other mathematicians as (for example) Riemann, Liouville, Grünwald, Letnikov, Erdéli, and Kober have made valuable contributions to the field of fractional calculus and its widespread applications. Due to its behavior and capability to solve many real-life problems, fractional calculus has

attracted many physical and engineering scientists. In the development of fractional calculus, fractional operators are particularly significant. Fractional calculus is used in a wide range of engineering and science disciplines, including physics [13], epidemiology [14], medicine [15], nanotechnology [16], economy [17], bioengineering [18], and fluid mechanics [19]. Several investigations have shown that fractional operators may accurately explain complex multiscale phenomena that are difficult to model using traditional mathematical calculus. In the last few years, it has become clear that presenting well-known inequalities involving different new notions of fractional integral operators is very popular among mathematicians. In this connection, one can refer to the works presented in [20–29] for various fractional-order integral inequalities.

**Definition 1** (see [30]). Let  $\mathcal{F} : \mathbb{X} \rightarrow \mathbb{R}$  be a function and  $\mathbb{X}$  be a convex subset of a real vector space  $\mathbb{R}$ . Then, we say that the function  $\mathcal{F}$  is convex if and only if the following condition

$$\mathcal{F}(\zeta q_1 + (1 - \zeta)q_2) \leq \zeta \mathcal{F}(q_1) + (1 - \zeta)\mathcal{F}(q_2) \tag{1}$$

holds true for all  $q_1, q_2 \in \mathbb{X}$  and  $\zeta \in [0, 1]$ .

For further discussion, we first present the classical Hermite–Hadamard (H-H) inequality, which states that (see [1]):

If the function  $\mathcal{F} : \mathbb{X} \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is convex in  $\mathbb{X}$  for  $q_1, q_2 \in \mathbb{X}$  and  $q_1 < q_2$ , then

$$\mathcal{F}\left(\frac{q_1 + q_2}{2}\right) \leq \frac{1}{q_2 - q_1} \int_{q_1}^{q_2} \mathcal{F}(x)dx \leq \frac{\mathcal{F}(q_1) + \mathcal{F}(q_2)}{2}. \tag{2}$$

### 2. Preliminaries

Let the collection of all closed and bounded intervals of  $\mathbb{R}$  be defined as follows:

$$K_C = \{[\mathfrak{S}_*, \mathfrak{S}^*] : \mathfrak{S}_*, \mathfrak{S}^* \in \mathbb{R} \text{ and } \mathfrak{S}_* \leq \mathfrak{S}^*\}.$$

We say that the interval  $[\mathfrak{S}_*, \mathfrak{S}^*]$  is a positive interval if  $\mathfrak{S}_* \geq 0$  and it is defined as follows:

$$K_C^+ = \{[\mathfrak{S}_*, \mathfrak{S}^*] : \mathfrak{S}_*, \mathfrak{S}^* \in K_C \text{ and } \mathfrak{S}_* \geq 0\}.$$

The algebraic addition, the algebraic multiplication, and the scalar multiplication for  $[N_*, N^*], [\mathfrak{S}_*, \mathfrak{S}^*] \in K_C$  and  $\zeta \in \mathbb{R}$  are defined as follows:

$$[N_*, N^*] + [\mathfrak{S}_*, \mathfrak{S}^*] = [N_* + \mathfrak{S}_*, N^* + \mathfrak{S}^*],$$

$$[N_*, N^*] \cdot [\mathfrak{S}_*, \mathfrak{S}^*] = [\min\{N_*\mathfrak{S}_*, N^*\mathfrak{S}_*, N_*\mathfrak{S}^*, N^*\mathfrak{S}^*\}, \max\{N_*\mathfrak{S}_*, N^*\mathfrak{S}_*, N_*\mathfrak{S}^*, N^*\mathfrak{S}^*\}]$$

and

$$\zeta \cdot [N_*, N^*] = \begin{cases} [\zeta N_*, \zeta N^*] & (\zeta > 0) \\ \{0\} & (\zeta = 0) \\ [\zeta N^*, \zeta N_*] & (\zeta < 0), \end{cases}$$

respectively.

The Hausdorff–Pompeiu distance between intervals  $[N_*, N^*]$  and  $[\mathfrak{S}_*, \mathfrak{S}^*]$  is defined by

$$d([N_*, N^*], [\mathfrak{S}_*, \mathfrak{S}^*]) = \max\{|N_* - \mathfrak{S}_*|, |N^* - \mathfrak{S}^*|\}.$$

It is well known that  $(K_C, d)$  is a complete metric space.

The inclusion “ $\subseteq$ ” for  $[N_*, N^*], [\mathfrak{S}_*, \mathfrak{S}^*] \in K_C$ , is defined as follows:

$[N_*, N^*] \subseteq [\mathfrak{S}_*, \mathfrak{S}^*]$  if and only if  $\mathfrak{S}_* \leq N_*$  and  $N^* \leq \mathfrak{S}^*$ .

Khan et al. [31] proposed the following developments about the newly developed concept, i.e., LR-convex interval-valued functions.

- Remark 1** (see [31]). 1. The pseudo-order relation " $\leq_p$ " defined on  $\mathcal{K}_C$  by  $[\mathfrak{N}_*, \mathfrak{N}^*] \leq_p [\mathfrak{S}_*, \mathfrak{S}^*]$  holds true if and only if  $\mathfrak{N}_* \leq \mathfrak{S}_*$ ,  $\mathfrak{N}^* \leq \mathfrak{S}^*$ , for all  $[\mathfrak{N}_*, \mathfrak{N}^*], [\mathfrak{S}_*, \mathfrak{S}^*] \in \mathcal{K}_C$ . The relation  $[\mathfrak{N}_*, \mathfrak{N}^*] \leq_p [\mathfrak{S}_*, \mathfrak{S}^*]$  is similar to  $[\mathfrak{N}_*, \mathfrak{N}^*] \leq [\mathfrak{S}_*, \mathfrak{S}^*]$  on  $\mathcal{K}_C$ .
2. It can be seen that " $\leq_p$ " appears the same as that of "left and right" on the real line  $\mathbb{R}$ , so " $\leq_p$ " can also be called "left and right" (or "LR" order in short).

Moore [32] first introduced the concept of the Riemann integral for interval-valued functions, which is given as follows.

**Theorem 1** (see [32]). Let  $\mathcal{F} : [q_1, q_2] \subset \mathbb{R} \rightarrow \mathcal{K}_C$  be an interval-valued function such that

$$\mathcal{F}(x) = [\mathcal{F}_*(x), \mathcal{F}^*(x)].$$

Then,  $\mathcal{F}$  is Riemann-integrable over  $[q_1, q_2]$  if and only if  $\mathcal{F}_*$  and  $\mathcal{F}^*$  are both Riemann-integrable over  $[q_1, q_2]$ .

$$(\mathcal{I}\mathcal{R}) \int_{q_1}^{q_2} \mathcal{F}(x) dx = \left[ (\mathcal{R}) \int_{q_1}^{q_2} \mathcal{F}_*(x) dx, (\mathcal{R}) \int_{q_1}^{q_2} \mathcal{F}^*(x) dx \right]$$

**Definition 2** (see, for details, [33]; see also [34,35]). Let  $\mathcal{F} \in \mathcal{L}[q_1, q_2]$  be the set of all Lebesgue measurable interval-valued functions on  $[q_1, q_2]$ . Then, for the order  $\alpha > 0$ , the left and right Riemann–Liouville R-L fractional integrals are defined as follows:

$$J_{q_1^+}^\alpha \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{q_1}^x (x - \zeta)^{\alpha-1} \mathcal{F}(\zeta) d\zeta, \quad (x > q_1)$$

and

$$J_{q_2^-}^\alpha \mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_x^{q_2} (\zeta - x)^{\alpha-1} \mathcal{F}(\zeta) d\zeta \quad (x < q_2),$$

respectively, where  $\Gamma(\alpha) = \int_0^\infty \zeta^{\alpha-1} e^{-\zeta} d\zeta$  is the Euler gamma function.

**Definition 3** (see [36]). The interval-valued function  $\mathcal{F} : \mathbb{X} \rightarrow \mathcal{K}_C^+$  is said to be LR-convex interval-valued on a convex set  $\mathbb{X}$  if, for all  $q_1, q_2 \in \mathbb{X}$ , and  $\zeta \in [0, 1]$ , we have

$$\mathcal{F}(\zeta q_1 + (1 - \zeta) q_2) \leq_p \zeta \mathcal{F}(q_1) + (1 - \zeta) \mathcal{F}(q_2). \quad (3)$$

If the inequality (3) is reversed, then  $\mathcal{F}$  is said to be LR-concave on  $\mathbb{X}$ . Moreover,  $\mathcal{F}$  is affine on  $\mathbb{X}$  if and only if it is both LR-convex and LR-concave on  $\mathbb{X}$ .

**Theorem 2** (see [36]). Let  $\mathbb{X}$  be a convex set and  $\mathcal{F} : \mathbb{X} \rightarrow \mathcal{K}_C^+$  be an interval-valued function such that

$$\mathcal{F}(q) = [\mathcal{F}_*(q), \mathcal{F}^*(q)] \quad (\forall q \in \mathbb{X})$$

for all  $q \in \mathbb{X}$ . Then,  $\mathcal{F}$  is an LR-convex interval-valued function on  $\mathbb{X}$  if and only if both  $\mathcal{F}_*(q)$  and  $\mathcal{F}^*(q)$  are convex functions on  $\mathbb{X}$ .

In recent years, interval-valued analysis has been utilized in order to prove integral inequalities such as H-H type inequalities, Fejér type inequality, and Ostrowski type inequalities by employing different convexities and different operators. For example, Abdeljawad et al. [37] proved the Hermite–Hadamard inequality for an interval-valued p-convex function and Nwaeze et al. [38] improved the same inequality by introducing the m-polynomial

convex interval-valued function. This inequality was further improved employing the idea of interval-valued analysis for a coordinated convex function [39,40] and quantum calculus [41]. Moreover, many researchers improved the concept of interval-valued analysis to fuzzy interval-valued analysis and LR-convex interval-valued analysis, where a pseudo-order relation is considered. For example, Khan and his collaborators introduced such concepts as LR- $h$ -convex interval-valued functions (see [42]), LR- $\chi$ -preinvex functions (see [43]), LR- $(h_1, h_2)$ -convex interval-valued functions (see [44]), LR- $p$ -convex interval-valued functions (see [45]), and LR-log- $h$ -convex interval-valued functions (see [46]). Several recent developments of the concept of the fuzzy interval-valued analysis of various familiar families of integral inequalities can indeed be found in the works by (for example) Khan et al. [47–49].

Budak et al. [34] provided the following conclusions for interval-valued convex functions by using the R-L fractional integral operator in order to examine the H-H type inequalities and the Pachpatte type inequalities.

**Theorem 3.** Let  $\mathcal{F} : [q_1, q_2] \rightarrow \mathbb{R}_I^+$  be an interval-valued convex function with

$$\mathcal{F}(x) = [\mathcal{F}_*(x), \mathcal{F}^*(x)].$$

Then, the fractional-order H-H inequality of order  $\alpha > 0$  for interval-valued functions is given by

$$\mathcal{F}\left(\frac{q_1 + q_2}{2}\right) \supseteq \frac{\Gamma(\alpha + 1)}{2(q_2 - q_1)^\alpha} \left[ \mathfrak{J}_{(q_1)^+}^\alpha \mathcal{F}(q_2) + \mathfrak{J}_{(q_2)^-}^\alpha \mathcal{F}(q_1) \right] \supseteq \frac{\mathcal{F}(q_1) + \mathcal{F}(q_2)}{2}.$$

**Theorem 4.** If  $\mathcal{F}, \mathcal{G} : [q_1, q_2] \rightarrow \mathbb{R}_I^+$  are two interval-valued convex functions with

$$\mathcal{F}(x) = [\mathcal{F}_*(x), \mathcal{F}^*(x)]$$

and

$$\mathcal{G}(x) = [\mathcal{G}_*(x), \mathcal{G}^*(x)],$$

then the fractional-order H-H type inequality for  $\alpha > 0$  holds true as follows:

$$\begin{aligned} & \frac{\Gamma(\alpha + 1)}{2(q_2 - q_1)^\alpha} \left[ \mathfrak{J}_{(q_1)^+}^\alpha \mathcal{F}(q_2)\mathcal{G}(q_2) + \mathfrak{J}_{(q_2)^-}^\alpha \mathcal{F}(q_1)\mathcal{G}(q_1) \right] \\ & \supseteq \left[ \frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right] \Psi(q_1, q_2) + \left[ \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right] \Omega(q_1, q_2), \end{aligned}$$

where

$$\Psi(q_1, q_2) = [\mathcal{F}(q_1)\mathcal{G}(q_1) + \mathcal{F}(q_2)\mathcal{G}(q_2)]$$

and

$$\Omega(q_1, q_2) = [\mathcal{F}(q_1)\mathcal{G}(q_2) + \mathcal{F}(q_2)\mathcal{G}(q_1)].$$

**Theorem 5.**  $\mathcal{F}, \mathcal{G} : [q_1, q_2] \rightarrow \mathbb{R}_I^+$  are two interval-valued convex functions with

$$\mathcal{F}(x) = [\mathcal{F}_*(x), \mathcal{F}^*(x)]$$

and

$$\mathcal{G}(x) = [\mathcal{G}_*(x), \mathcal{G}^*(x)].$$

Then, the fractional-order H-H type inequality for  $\alpha > 0$  is given by

$$2\mathcal{F}\left(\frac{q_1 + q_2}{2}\right)\mathcal{G}\left(\frac{q_1 + q_2}{2}\right) \supseteq \frac{\Gamma(\alpha + 1)}{2(q_2 - q_1)^\alpha} \left[ \mathfrak{J}_{(q_1)^+}^\alpha \mathcal{F}(q_2)\mathcal{G}(q_2) + \mathfrak{J}_{(q_2)^-}^\alpha \mathcal{F}(q_1)\mathcal{G}(q_1) \right] + \left[ \frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right] \Omega(q_1, q_2) + \left[ \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right] \Psi(q_1, q_2).$$

The above (presumably new) concept was improved by Zhao et al. [50], who introduced the concept of interval-valued coordinated convex functions. An et al. [51] went a step forward by introducing the interval  $(h_1, h_2)$ -convex function. Srivastava et al. [52] presented a new version of the H-H type inequalities via interval-valued preinvex functions. Recently, Khan et al. [23] generalized this concept to fuzzy convex interval-valued functions.

The major goal of this paper is to use some pseudo-order relations in order to combine the concepts of interval-valued analysis and fractional-order integral inequalities. For LR-convex interval-valued functions, we first present a new midpoint type H-H inequality. Then, by using differintegrals of the  $\left(\frac{q_1 + q_2}{2}\right)$  type and the R-L fractional integral operator, we present integral inequalities for the product of two LR-convex interval-valued functions.

Our present investigation is organized as follows. In Section 3, we derive some new versions of the interval-valued H-H type inequalities for LR interval-valued convex functions and for the product of two LR interval-valued convex functions, after having reviewed the pre-requisite and relevant facts regarding the related inequalities and the interval-valued analysis in Section 2. Some examples are also considered to see if the established outcomes are beneficial. A brief conclusion and potential scopes for further research, which are linked to the results presented in this paper, are explored in Section 4.

### 3. New Fractional Inequalities for Interval-Valued Functions

This section focuses on establishing some H-H type interval-valued fractional integral inequalities for LR-convex interval-valued functions, as well as some inequalities of the Pachhpattte type, which involve the product of two LR-convex interval-valued functions.

The family of Lebesgue measurable interval-valued functions is represented here by  $L([q_1, q_2], \mathcal{K}_C^+)$ .

**Theorem 6.** Let  $\mathcal{F} : [q_1, q_2] \rightarrow \mathcal{K}_C^+$  be an LR-convex interval-valued function on  $[q_1, q_2]$ , which is given by

$$\mathcal{F}(\omega) = [\mathcal{F}_*(\omega), \mathcal{F}^*(\omega)]$$

for all  $\omega \in [q_1, q_2]$ . If  $\mathcal{F} \in L([q_1, q_2], \mathcal{K}_C^+)$ , then

$$\begin{aligned} \mathcal{F}\left(\frac{q_1 + q_2}{2}\right) &\leq_p \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(q_2 - q_1)^\alpha} \left( \mathcal{J}_{\left(\frac{q_1 + q_2}{2}\right)^+}^\alpha \mathcal{F}(q_2) + \mathcal{J}_{\left(\frac{q_1 + q_2}{2}\right)^-}^\alpha \mathcal{F}(q_1) \right) \\ &\leq_p \frac{\mathcal{F}(q_1) + \mathcal{F}(q_2)}{2}. \end{aligned}$$

Furthermore, if  $\mathcal{F}(\omega)$  is an LR-concave interval-valued function, then

$$\begin{aligned} \mathcal{F}\left(\frac{q_1 + q_2}{2}\right) &\geq_p \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(q_2 - q_1)^\alpha} \left( \mathcal{J}_{\left(\frac{q_1 + q_2}{2}\right)^+}^\alpha \mathcal{F}(q_2) + \mathcal{J}_{\left(\frac{q_1 + q_2}{2}\right)^-}^\alpha \mathcal{F}(q_1) \right) \\ &\geq_p \frac{\mathcal{F}(q_1) + \mathcal{F}(q_2)}{2}. \end{aligned}$$

**Proof.** Let  $\mathcal{F} : [q_1, q_2] \rightarrow \mathcal{K}_C^+$  be an LR-convex interval-valued function. Then, by hypothesis, we have

$$2\mathcal{F}\left(\frac{q_1 + q_2}{2}\right) \leq_p \mathcal{F}\left(\frac{\zeta}{2}q_1 + \left(\frac{2 - \zeta}{2}\right)q_2\right) + \mathcal{F}\left(\left(\frac{2 - \zeta}{2}\right)q_1 + \frac{\zeta}{2}q_2\right).$$

Therefore, we have

$$2\mathcal{F}_*\left(\frac{q_1 + q_2}{2}\right) \leq \mathcal{F}_*\left(\frac{\zeta}{2}q_1 + \left(\frac{2-\zeta}{2}\right)q_2\right) + \mathcal{F}_*\left(\left(\frac{2-\zeta}{2}\right)q_1 + \frac{\zeta}{2}q_2\right) \tag{4}$$

and

$$2\mathcal{F}^*\left(\frac{q_1 + q_2}{2}\right) \leq \mathcal{F}^*\left(\frac{\zeta}{2}q_1 + \left(\frac{2-\zeta}{2}\right)q_2\right) + \mathcal{F}^*\left(\left(\frac{2-\zeta}{2}\right)q_1 + \frac{\zeta}{2}q_2\right). \tag{5}$$

Multiplying both sides of Equations (4) and (5) by  $\zeta^{\alpha-1}$  and integrating the obtained results with respect to  $\zeta$  over  $(0, 1)$ , we find that

$$\begin{aligned} & 2 \int_0^1 \zeta^{\alpha-1} \mathcal{F}_*\left(\frac{q_1 + q_2}{2}\right) d\zeta \\ & \leq \int_0^1 \zeta^{\alpha-1} \mathcal{F}_*\left(\frac{\zeta}{2}q_1 + \left(\frac{2-\zeta}{2}\right)q_2\right) d\zeta + \int_0^1 \zeta^{\alpha-1} \mathcal{F}_*\left(\left(\frac{2-\zeta}{2}\right)q_1 + \frac{\zeta}{2}q_2\right) d\zeta \end{aligned}$$

and

$$\begin{aligned} & 2 \int_0^1 \zeta^{\alpha-1} \mathcal{F}^*\left(\frac{q_1 + q_2}{2}\right) d\zeta \\ & \leq \int_0^1 \zeta^{\alpha-1} \mathcal{F}^*\left(\frac{\zeta}{2}q_1 + \left(\frac{2-\zeta}{2}\right)q_2\right) d\zeta + \int_0^1 \zeta^{\alpha-1} \mathcal{F}^*\left(\left(\frac{2-\zeta}{2}\right)q_1 + \frac{\zeta}{2}q_2\right) d\zeta, \end{aligned}$$

respectively.

Now, if we let

$$\omega = \left(\frac{2-\zeta}{2}\right)q_2 + \frac{\zeta}{2}q_1 \quad \text{and} \quad \nu = \left(\frac{2-\zeta}{2}\right)q_1 + \frac{\zeta}{2}q_2,$$

then we obtain

$$\begin{aligned} \frac{2}{\alpha} \mathcal{F}_*\left(\frac{q_1 + q_2}{2}\right) & \leq \frac{2^\alpha}{(q_2 - q_1)^\alpha} \int_{q_1}^{\frac{q_1+q_2}{2}} (\nu - q_1)^{\alpha-1} \mathcal{F}_*(\nu) d\nu \\ & + \frac{2^\alpha}{(q_2 - q_1)^\alpha} \int_{\frac{q_1+q_2}{2}}^{q_2} (q_2 - \omega)^{\alpha-1} \mathcal{F}_*(\omega) d\omega \\ & = \frac{2^\alpha \Gamma(\alpha)}{(q_2 - q_1)^\alpha} \left[ J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}_*(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}_*(q_1) \right] \end{aligned}$$

and

$$\begin{aligned} \frac{2}{\alpha} \mathcal{F}^*\left(\frac{q_1 + q_2}{2}\right) & \leq \frac{2^\alpha}{(q_2 - q_1)^\alpha} \int_{q_1}^{\frac{q_1+q_2}{2}} (\nu - q_1)^{\alpha-1} \mathcal{F}^*(\nu) d\nu \\ & + \frac{2^\alpha}{(q_2 - q_1)^\alpha} \int_{\frac{q_1+q_2}{2}}^{q_2} (q_2 - \omega)^{\alpha-1} \mathcal{F}^*(\omega) d\omega \\ & = \frac{2^\alpha \Gamma(\alpha)}{(q_2 - q_1)^\alpha} \left[ J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}^*(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}^*(q_1) \right]. \end{aligned}$$

Consequently, we have

$$\begin{aligned} & \frac{2}{\alpha} \left[ \mathcal{F}_* \left( \frac{q_1 + q_2}{2} \right), \mathcal{F}^* \left( \frac{q_1 + q_2}{2} \right) \right] \\ & \leq_p \frac{2^\alpha \Gamma(\alpha)}{(q_2 - q_1)^\alpha} \left[ \left( J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}_*(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}_*(q_1) \right), \right. \\ & \qquad \left. \left( J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}^*(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}^*(q_1) \right) \right], \end{aligned}$$

that is,

$$\mathcal{F} \left( \frac{q_1 + q_2}{2} \right) \leq_p \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(q_2 - q_1)^\alpha} \left( J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}(q_1) \right). \tag{6}$$

In a similar way as above, we also have

$$\frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(q_2 - q_1)^\alpha} \left( J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}(q_1) \right) \leq_p \frac{\mathcal{F}(q_1) + \mathcal{F}(q_2)}{2}. \tag{7}$$

Next, from Equations (6) and (7), we obtain

$$\begin{aligned} \mathcal{F} \left( \frac{q_1 + q_2}{2} \right) & \leq_p \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(q_2 - q_1)^\alpha} \left( J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}(q_1) \right) \\ & \leq_p \frac{\mathcal{F}(q_1) + \mathcal{F}(q_2)}{2}. \end{aligned}$$

This completes the proof of Theorem 6.  $\square$

**Remark 2.** It can be clearly seen that if we put  $\alpha = 1$ , then Theorem 6 reduces to the following result given in [53]:

$$\mathcal{F} \left( \frac{q_1 + q_2}{2} \right) \leq_p \frac{1}{q_2 - q_1} \int_{q_1}^{q_2} \mathcal{F}(\omega) d\omega \leq_p \frac{\mathcal{F}(q_1) + \mathcal{F}(q_2)}{2}$$

If we take  $\mathcal{F}_*(\omega) = \mathcal{F}^*(\omega)$  in Theorem 6, then the following fractional integral inequality of the H-H type obtained by Sarikaya and Yildirim [22] is recaptured.

$$\begin{aligned} \mathcal{F} \left( \frac{q_1 + q_2}{2} \right) & \leq \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(q_2 - q_1)^\alpha} \left( J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}(q_1) \right) \\ & \leq \frac{\mathcal{F}(q_1) + \mathcal{F}(q_2)}{2}. \end{aligned}$$

Let  $\alpha = 1$  and  $\mathcal{F}_*(\omega) = \mathcal{F}^*(\omega)$ . Then, the classical H-H type inequality (2) results from Theorem 6.

**Example 1.** If we choose  $\alpha = \frac{1}{2}$ ,  $\omega \in [0, 2]$ , and the following interval-valued function:  $\mathcal{F}(\omega) = [1, 2] \left( 2 - \omega^{\frac{1}{2}} \right)$ . Then,  $\mathcal{F}(\omega)$  is an LR-convex interval-valued function as the left and right endpoints  $\mathcal{F}_*(\omega) = 2 - \omega^{\frac{1}{2}}$ ,  $\mathcal{F}^*(\omega) = 2 \left( 2 - \omega^{\frac{1}{2}} \right)$  are LR-convex interval-valued functions. We thus obtain

$$\mathcal{F}_* \left( \frac{q_1 + q_2}{2} \right) = \mathcal{F}_*(1) = 1,$$

$$\mathcal{F}^* \left( \frac{q_1 + q_2}{2} \right) = \mathcal{F}^*(1) = 2,$$

$$\frac{\mathcal{F}_*(q_1) + \mathcal{F}_*(q_2)}{2} = \frac{4 - \sqrt{2}}{2}$$

and

$$\frac{\mathcal{F}^*(q_1) + \mathcal{F}^*(q_2)}{2} = 4 - \sqrt{2}.$$

We note that

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(q_2 - q_1)^\alpha} \left( J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}_*(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}_*(q_1) \right) \approx \frac{4.42920}{4}$$

and

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(q_2 - q_1)^\alpha} \left( J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}^*(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}^*(q_1) \right) \approx \frac{4.42920}{2}.$$

Therefore, we obtain

$$[1, 2] \leq_p \left[ \frac{4.42920}{4}, \frac{4.42920}{2} \right] \leq_p \left[ \frac{4 - \sqrt{2}}{2}, 4 - \sqrt{2} \right],$$

which evidently verifies Theorem 6.

The major goal of the next two theorems is to prove the H-H type interval fractional integral inequalities using the product of two LR-convex interval-valued functions.

**Theorem 7.** Let  $\mathcal{F}, \mathcal{G} : [q_1, q_2] \rightarrow \mathcal{K}_C^+$  be two LR-convex interval-valued functions on  $[q_1, q_2]$  such that

$$\mathcal{F}(\omega) = [\mathcal{F}_*(\omega), \mathcal{F}^*(\omega)] \quad \text{and} \quad \mathcal{G}(\omega) = [\mathcal{G}_*(\omega), \mathcal{G}^*(\omega)]$$

for all  $\omega \in [q_1, q_2]$ . If  $\mathcal{F} \cdot \mathcal{G} \in L([q_1, q_2], \mathcal{K}_C^+)$ , then

$$\begin{aligned} & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(q_2 - q_1)^\alpha} \left( J_{\left(\frac{q_1+q_2}{2}\right)^+}^\alpha \mathcal{F}(q_2)\mathcal{G}(q_2) + J_{\left(\frac{q_1+q_2}{2}\right)^-}^\alpha \mathcal{F}(q_1)\mathcal{G}(q_1) \right) \\ & \leq_p \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) \Psi(q_1, q_2) + \frac{\alpha}{4} \left( \frac{2}{\alpha+1} - \frac{1}{\alpha+2} \right) \Omega(q_1, q_2), \end{aligned}$$

where

$$\Psi(q_1, q_2) = \mathcal{F}(q_1)\mathcal{G}(q_1) + \mathcal{F}(q_2)\mathcal{G}(q_2),$$

$$\Omega(q_1, q_2) = \mathcal{F}(q_1)\mathcal{G}(q_2) + \mathcal{F}(q_2)\mathcal{G}(q_1),$$

$$\Psi(q_1, q_2) = [\Psi_*(q_1, q_2), \Psi^*(q_1, q_2)]$$

and

$$\Omega(q_1, q_2) = [\Omega_*(q_1, q_2), \Omega^*(q_1, q_2)].$$

**Proof.** Using  $\mathcal{F}, \mathcal{G}$  as LR-convex interval-valued functions, we have

$$\mathcal{F}_* \left( \frac{\zeta}{2}q_1 + \left( \frac{2-\zeta}{2} \right)q_2 \right) \leq \frac{\zeta}{2}\mathcal{F}_*(q_1) + \frac{(2-\zeta)}{2}\mathcal{F}_*(q_2),$$

$$\mathcal{F}^* \left( \frac{\zeta}{2}q_1 + \left( \frac{2-\zeta}{2} \right)q_2 \right) \leq \frac{\zeta}{2}\mathcal{F}^*(q_1) + \frac{(2-\zeta)}{2}\mathcal{F}^*(q_2),$$

$$\mathcal{G}_* \left( \frac{\zeta}{2}q_1 + \left( \frac{2-\zeta}{2} \right)q_2 \right) \leq \frac{\zeta}{2}\mathcal{G}_*(q_1) + \frac{(2-\zeta)}{2}\mathcal{G}_*(q_2)$$

and

$$\mathcal{G}^* \left( \frac{\zeta}{2} \varrho_1 + \left( \frac{2-\zeta}{2} \right) \varrho_2 \right) \leq \frac{\zeta}{2} \mathcal{G}^*(\varrho_1) + \frac{(2-\zeta)}{2} \mathcal{G}^*(\varrho_2).$$

Now, by the definition of LR-convex interval-valued functions, we obtain

$$0 \leq_p \mathcal{F}(\omega) \quad \text{and} \quad 0 \leq_p \mathcal{G}(\omega),$$

so that

$$\begin{aligned} & \mathcal{F}_* \left( \frac{\zeta}{2} \varrho_1 + \left( \frac{2-\zeta}{2} \right) \varrho_2 \right) \mathcal{G}_* \left( \frac{\zeta}{2} \varrho_1 + \left( \frac{2-\zeta}{2} \right) \varrho_2 \right) \\ & \leq \left( \frac{\zeta}{2} \mathcal{F}_*(\varrho_1) + \frac{(2-\zeta)}{2} \mathcal{F}_*(\varrho_2) \right) \left( \frac{\zeta}{2} \mathcal{G}_*(\varrho_1) + \frac{(2-\zeta)}{2} \mathcal{G}_*(\varrho_2) \right) \\ & = \frac{\zeta^2}{4} \mathcal{F}_*(\varrho_1) \mathcal{G}_*(\varrho_1) + \frac{(2-\zeta)^2}{4} \mathcal{F}_*(\varrho_2) \mathcal{G}_*(\varrho_2) \\ & \quad + \frac{\zeta(2-\zeta)}{4} [\mathcal{F}_*(\varrho_1) \mathcal{G}_*(\varrho_2) + \mathcal{F}_*(\varrho_2) \mathcal{G}_*(\varrho_1)] \end{aligned}$$

and

$$\begin{aligned} & \mathcal{F}^* \left( \frac{\zeta}{2} \varrho_1 + \left( \frac{2-\zeta}{2} \right) \varrho_2 \right) \mathcal{G}^* \left( \frac{\zeta}{2} \varrho_1 + \left( \frac{2-\zeta}{2} \right) \varrho_2 \right) \\ & \leq \left( \frac{\zeta}{2} \mathcal{F}^*(\varrho_1) + \frac{(2-\zeta)}{2} \mathcal{F}^*(\varrho_2) \right) \left( \frac{\zeta}{2} \mathcal{G}^*(\varrho_1) + \frac{(2-\zeta)}{2} \mathcal{G}^*(\varrho_2) \right) \\ & = \frac{\zeta^2}{4} \mathcal{F}^*(\varrho_1) \mathcal{G}^*(\varrho_1) + \frac{(2-\zeta)^2}{4} \mathcal{F}^*(\varrho_2) \mathcal{G}^*(\varrho_2) \\ & \quad + \frac{\zeta(2-\zeta)}{4} [\mathcal{F}^*(\varrho_1) \mathcal{G}^*(\varrho_2) + \mathcal{F}^*(\varrho_2) \mathcal{G}^*(\varrho_1)]. \end{aligned}$$

Analogously, we have

$$\begin{aligned} & \mathcal{F}_* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) \mathcal{G}_* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) \\ & \leq \left( \frac{2-\zeta}{2} \mathcal{F}_*(\varrho_1) + \frac{\zeta}{2} \mathcal{F}_*(\varrho_2) \right) \left( \frac{2-\zeta}{2} \mathcal{G}_*(\varrho_1) + \frac{\zeta}{2} \mathcal{G}_*(\varrho_2) \right) \\ & = \frac{(2-\zeta)^2}{4} \mathcal{F}_*(\varrho_1) \mathcal{G}_*(\varrho_1) + \frac{\zeta^2}{4} \mathcal{F}_*(\varrho_2) \mathcal{G}_*(\varrho_2) \\ & \quad + \frac{\zeta(2-\zeta)}{4} [\mathcal{F}_*(\varrho_1) \mathcal{G}_*(\varrho_2) + \mathcal{F}_*(\varrho_2) \mathcal{G}_*(\varrho_1)] \end{aligned} \tag{8}$$

and

$$\begin{aligned} & \mathcal{F}^* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) \mathcal{G}^* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) \\ & \leq \left( \frac{2-\zeta}{2} \mathcal{F}^*(\varrho_1) + \frac{\zeta}{2} \mathcal{F}^*(\varrho_2) \right) \left( \frac{2-\zeta}{2} \mathcal{G}^*(\varrho_1) + \frac{\zeta}{2} \mathcal{G}^*(\varrho_2) \right) \\ & = \frac{(2-\zeta)^2}{4} \mathcal{F}^*(\varrho_1) \mathcal{G}^*(\varrho_1) + \frac{\zeta^2}{4} \mathcal{F}^*(\varrho_2) \mathcal{G}^*(\varrho_2) \\ & \quad + \frac{\zeta(2-\zeta)}{4} [\mathcal{F}^*(\varrho_1) \mathcal{G}^*(\varrho_2) + \mathcal{F}^*(\varrho_2) \mathcal{G}^*(\varrho_1)]. \end{aligned} \tag{9}$$

Adding (8) and (9), we have

$$\begin{aligned} & \mathcal{F}_* \left( \frac{\zeta}{2} \varrho_1 + \frac{2-\zeta}{2} \varrho_2 \right) \mathcal{G}_* \left( \frac{\zeta}{2} \varrho_1 + \frac{2-\zeta}{2} \varrho_2 \right) + \mathcal{F}_* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) \mathcal{G}_* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) \\ & \leq \left( \frac{\zeta^2 + (2-\zeta)^2}{4} \right) [\mathcal{F}_*(\varrho_1) \mathcal{G}_*(\varrho_1) + \mathcal{F}_*(\varrho_2) \mathcal{G}_*(\varrho_2)] \\ & \quad + \frac{\zeta(2-\zeta)}{2} [\mathcal{F}_*(\varrho_2) \mathcal{G}_*(\varrho_1) + \mathcal{F}_*(\varrho_1) \mathcal{G}_*(\varrho_2)] \end{aligned} \tag{10}$$

and

$$\begin{aligned}
 & \mathcal{F}^* \left( \frac{\zeta}{2} \varrho_1 + \frac{2-\zeta}{2} \varrho_2 \right) \mathcal{G}^* \left( \frac{\zeta}{2} \varrho_1 + \frac{2-\zeta}{2} \varrho_2 \right) + \mathcal{F}^* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) \mathcal{G}^* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) \\
 & \leq \left( \frac{\zeta^2 + (2-\zeta)^2}{4} \right) [\mathcal{F}^*(\varrho_1) \mathcal{G}^*(\varrho_1) + \mathcal{F}^*(\varrho_2) \mathcal{G}^*(\varrho_2)] \\
 & + \frac{\zeta(2-\zeta)}{2} [\mathcal{F}^*(\varrho_2) \mathcal{G}^*(\varrho_1) + \mathcal{F}^*(\varrho_1) \mathcal{G}^*(\varrho_2)].
 \end{aligned} \tag{11}$$

Multiplying both sides of Equations (10) and (11) by  $\zeta^{\alpha-1}$  and then integrating with respect to  $\zeta$  over (0,1), we have

$$\begin{aligned}
 & \int_0^1 \zeta^{\alpha-1} \mathcal{F}_* \left( \frac{\zeta}{2} \varrho_1 + \frac{2-\zeta}{2} \varrho_2 \right) \mathcal{G}_* \left( \frac{\zeta}{2} \varrho_1 + \frac{2-\zeta}{2} \varrho_2 \right) d\zeta \\
 & + \int_0^1 \zeta^{\alpha-1} \mathcal{F}_* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) \mathcal{G}_* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) d\zeta \\
 & \leq \Psi_*(\varrho_1, \varrho_2) \int_0^1 \zeta^{\alpha-1} \left( \frac{\zeta^2 + (2-\zeta)^2}{4} \right) d\zeta + \Omega_*(\varrho_1, \varrho_2) \int_0^1 \zeta^{\alpha-1} \frac{\zeta(2-\zeta)}{2} d\zeta
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^1 \zeta^{\alpha-1} \mathcal{F}^* \left( \frac{\zeta}{2} \varrho_1 + \frac{2-\zeta}{2} \varrho_2 \right) \mathcal{G}^* \left( \frac{\zeta}{2} \varrho_1 + \frac{2-\zeta}{2} \varrho_2 \right) d\zeta \\
 & + \int_0^1 \zeta^{\alpha-1} \mathcal{F}^* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) \mathcal{G}^* \left( \frac{2-\zeta}{2} \varrho_1 + \frac{\zeta}{2} \varrho_2 \right) d\zeta \\
 & \leq \Psi^*(\varrho_1, \varrho_2) \int_0^1 \zeta^{\alpha-1} \left( \frac{\zeta^2 + (2-\zeta)^2}{4} \right) d\zeta + \Omega^*(\varrho_1, \varrho_2) \int_0^1 \zeta^{\alpha-1} \frac{\zeta(2-\zeta)}{2} d\zeta.
 \end{aligned}$$

It follows from the above developments that

$$\begin{aligned}
 & \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(\varrho_2 - \varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}^\alpha \mathcal{F}_*(\varrho_2) \mathcal{G}_*(\varrho_2) + J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}^\alpha \mathcal{F}_*(\varrho_1) \mathcal{G}_*(\varrho_1) \right) \\
 & \leq \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) \Psi_*(\varrho_1, \varrho_2) + \frac{\alpha}{4} \left( \frac{2}{\alpha+1} - \frac{1}{\alpha+2} \right) \Omega_*(\varrho_1, \varrho_2)
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(\varrho_2 - \varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}^\alpha \mathcal{F}^*(\varrho_2) \mathcal{G}^*(\varrho_2) + J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}^\alpha \mathcal{F}^*(\varrho_1) \mathcal{G}^*(\varrho_1) \right) \\
 & \leq \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) \Psi^*(\varrho_1, \varrho_2) + \frac{\alpha}{4} \left( \frac{2}{\alpha+1} - \frac{1}{\alpha+2} \right) \Omega^*(\varrho_1, \varrho_2).
 \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha} \left( \left[ J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}^\alpha \mathcal{F}_*(\varrho_2)\mathcal{G}_*(\varrho_2) + J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}^\alpha \mathcal{F}^*(\varrho_1)\mathcal{G}^*(\varrho_1) \right], \right. \\ & \left. \left[ J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}^\alpha \mathcal{F}^*(\varrho_2)\mathcal{G}^*(\varrho_2) + J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}^\alpha \mathcal{F}_*(\varrho_1)\mathcal{G}_*(\varrho_1) \right] \right) \\ & \leq_p \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) [\Psi_*(\varrho_1, \varrho_2), \Psi^*(\varrho_1, \varrho_2)] \\ & \quad + \frac{\alpha}{4} \left( \frac{2}{\alpha+1} - \frac{1}{\alpha+2} \right) [\Omega_*(\varrho_1, \varrho_2), \Omega^*(\varrho_1, \varrho_2)], \end{aligned}$$

that is,

$$\begin{aligned} & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}^\alpha \mathcal{F}(\varrho_2)\mathcal{G}(\varrho_2) + J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}^\alpha \mathcal{F}(\varrho_1)\mathcal{G}(\varrho_1) \right) \\ & \leq_p \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) \Psi(\varrho_1, \varrho_2) + \frac{\alpha}{4} \left( \frac{2}{\alpha+1} - \frac{1}{\alpha+2} \right) \Omega(\varrho_1, \varrho_2). \end{aligned}$$

This completes the proof of Theorem 7.  $\square$

**Example 2.** Let  $\alpha = \frac{1}{2}$  and  $[\varrho_1, \varrho_2] = [0, 2]$ . Moreover, let the interval-valued functions be given by

$$\mathcal{F}(\omega) = [\omega, 2\omega] \quad \text{and} \quad \mathcal{G}(\omega) = \left[ \frac{\omega}{2}, \omega \right].$$

Since both the left and right end-points  $\mathcal{F}_*(\omega) = \omega$ ,  $\mathcal{F}^*(\omega) = 2\omega$ ,  $\mathcal{G}_*(\omega) = \frac{\omega}{2}$  and  $\mathcal{G}^*(\omega) = \omega$  are LR-convex functions,  $\mathcal{F}(\omega)$  and  $\mathcal{G}(\omega)$  are LR-convex interval-valued functions. We then clearly see that  $\mathcal{F}(\omega)\mathcal{G}(\omega) \in L([\varrho_1, \varrho_2], \mathcal{K}_C^+)$  and that

$$\begin{aligned} & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}^\alpha \mathcal{F}_*(\varrho_2)\mathcal{G}_*(\varrho_2) + J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}^\alpha \mathcal{F}^*(\varrho_1)\mathcal{G}^*(\varrho_1) \right) \\ & = \frac{\Gamma\left(\frac{3}{2}\right)}{2} \frac{1}{\sqrt{\pi}} \int_1^2 (2-\omega)^{-\frac{1}{2}} \left( \frac{1}{2} \cdot \omega^2 \right) d\omega + \frac{\Gamma\left(\frac{3}{2}\right)}{2} \frac{1}{\sqrt{\pi}} \int_0^1 (\omega)^{-\frac{1}{2}} \left( \frac{1}{2} \cdot \omega^2 \right) d\omega \\ & \approx 0.7666 \end{aligned}$$

and

$$\begin{aligned} & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}^\alpha \mathcal{F}^*(\varrho_2)\mathcal{G}^*(\varrho_2) + J_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}^\alpha \mathcal{F}_*(\varrho_1)\mathcal{G}^*(\varrho_1) \right) \\ & = \frac{\Gamma\left(\frac{3}{2}\right)}{2} \frac{1}{\sqrt{\pi}} \int_1^2 (2-\omega)^{-\frac{1}{2}} \cdot 2\omega^2 d\omega + \frac{\Gamma\left(\frac{3}{2}\right)}{2} \frac{1}{\sqrt{\pi}} \int_0^1 (\omega)^{-\frac{1}{2}} \cdot 2\omega^2 d\omega \\ & \approx 3.0667. \end{aligned}$$

We also note that

$$\begin{aligned} & \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) \Psi_*(\varrho_1, \varrho_2) \\ & = \frac{23}{60} [\mathcal{F}_*(\varrho_1) \cdot \mathcal{G}_*(\varrho_1) + \mathcal{F}_*(\varrho_2) \cdot \mathcal{G}_*(\varrho_2)] = \frac{23}{30}, \end{aligned}$$

$$\begin{aligned} & \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) \Psi^*(\varrho_1, \varrho_2) \\ &= \frac{23}{60} [\mathcal{F}^*(\varrho_1) \cdot \mathcal{G}^*(\varrho_1) + \mathcal{F}^*(\varrho_2) \cdot \mathcal{G}^*(\varrho_2)] = \frac{92}{30}, \end{aligned}$$

$$\begin{aligned} & \frac{\alpha}{4} \left( \frac{2}{\alpha+1} - \frac{1}{\alpha+2} \right) \Omega_*(\varrho_1, \varrho_2) \\ &= \frac{7}{60} [\mathcal{F}_*(\varrho_1) \cdot \mathcal{G}_*(\varrho_2) + \mathcal{F}_*(\varrho_2) \cdot \mathcal{G}_*(\varrho_1)] = 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{\alpha}{4} \left( \frac{2}{\alpha+1} - \frac{1}{\alpha+2} \right) \Omega_*(\varrho_1, \varrho_2) \\ &= \frac{7}{60} [\mathcal{F}^*(\varrho_1) \cdot \mathcal{G}^*(\varrho_2) + \mathcal{F}^*(\varrho_2) \cdot \mathcal{G}^*(s\varrho_1)] = 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) \Psi(\varrho_1, \varrho_2) + \frac{\alpha}{4} \left( \frac{2}{\alpha+1} - \frac{1}{\alpha+2} \right) \Omega(\varrho_1, \varrho_2) \\ &= \left[ \frac{23}{30}, \frac{92}{30} \right] + \frac{7}{60} [0, 0] = \left[ \frac{23}{30}, \frac{92}{30} \right]. \end{aligned}$$

It follows that

$$[0.7666, 3.0667] \leq_p \left[ \frac{23}{30}, \frac{92}{30} \right].$$

Hence, Theorem 7 has been demonstrated.

**Theorem 8.** Let  $\mathcal{F}, \mathcal{G} : [\varrho_1, \varrho_2] \rightarrow \mathcal{K}_C^+$  be two LR-convex interval-valued functions such that

$$\mathcal{F}(\omega) = [\mathcal{F}_*(\omega), \mathcal{F}^*(\omega)] \quad \text{and} \quad \mathcal{G}(\omega) = [\mathcal{G}_*(\omega), \mathcal{G}^*(\omega)]$$

for all  $\omega \in [\varrho_1, \varrho_2]$ . If

$$\mathcal{F} \cdot \mathcal{G} \in L([\varrho_1, \varrho_2], \mathcal{K}_C^+),$$

then each of the following interval-valued fractional inequalities holds true:

$$\begin{aligned} & {}_2 \mathcal{F} \left( \frac{\varrho_1 + \varrho_2}{2} \right) \mathcal{G} \left( \frac{\varrho_1 + \varrho_2}{2} \right) \\ & \leq_p \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(\varrho_2 - \varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^+}^\alpha \mathcal{F}(\varrho_2) \mathcal{G}(\varrho_2) + J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^-}^\alpha \mathcal{F}(\varrho_1) \mathcal{G}^*(\varrho_1) \right) \\ & \leq_p \frac{\alpha}{2} \left( \frac{1}{\alpha+1} - \frac{1}{2(\alpha+2)} \right) \Psi(\varrho_1, \varrho_2) + \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) \Omega(\varrho_1, \varrho_2), \end{aligned}$$

where

$$\Psi(\varrho_1, \varrho_2) = \mathcal{F}(\varrho_1) \mathcal{G}(\varrho_1) + \mathcal{F}(\varrho_2) \mathcal{G}(\varrho_2),$$

$$\Omega(\varrho_1, \varrho_2) = \mathcal{F}(\varrho_1) \mathcal{G}(\varrho_2) + \mathcal{F}(\varrho_2) \mathcal{G}(\varrho_1)$$

and

$$\Psi(\varrho_1, \varrho_2) = [\Psi_*(\varrho_1, \varrho_2), \Psi^*(\varrho_1, \varrho_2)]$$

and

$$\Omega(\varrho_1, \varrho_2) = [\Omega_*(\varrho_1, \varrho_2), \Omega^*(\varrho_1, \varrho_2)].$$

**Proof.** Suppose that  $\mathcal{F}, \mathcal{G} : [q_1, q_2] \rightarrow \mathcal{K}_C^+$  are LR-convex interval-valued functions. Then, by hypothesis, we have

$$\begin{aligned} & 4\mathcal{F}_* \left( \frac{q_1 + q_2}{2} \right) \mathcal{G}_* \left( \frac{q_1 + q_2}{2} \right) \\ & \leq \mathcal{F}_* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \mathcal{G}_* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \\ & \quad + \mathcal{F}_* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \mathcal{G}_* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \\ & \quad + \mathcal{F}_* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \mathcal{G}_* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \\ & \quad + \mathcal{F}_* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \mathcal{G}_* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \end{aligned}$$

and

$$\begin{aligned} & 4\mathcal{F}^* \left( \frac{q_1 + q_2}{2} \right) \mathcal{G}^* \left( \frac{q_1 + q_2}{2} \right) \\ & \leq \mathcal{F}^* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \mathcal{G}^* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \\ & \quad + \mathcal{F}^* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \mathcal{G}^* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \\ & \quad + \mathcal{F}^* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \mathcal{G}^* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \\ & \quad + \mathcal{F}^* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \mathcal{G}^* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right). \end{aligned}$$

We thus find that

$$\begin{aligned} & 4 \mathcal{F}_* \left( \frac{q_1+q_2}{2} \right) \mathcal{G}_* \left( \frac{q_1+q_2}{2} \right) \\ & \leq \mathcal{F}_* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \mathcal{G}_* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \\ & \quad + \mathcal{F}_* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \mathcal{G}_* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \\ & \quad + \left( \frac{\zeta(2-\zeta)}{2} \right) \Psi_*(q_1, q_2) + \left( \frac{\zeta^2+2-2\zeta}{2} \right) \Omega_*(q_1, q_2) \end{aligned} \tag{12}$$

and

$$\begin{aligned} & 4 \mathcal{F}^* \left( \frac{q_1+q_2}{2} \right) \mathcal{G}^* \left( \frac{q_1+q_2}{2} \right) \\ & \leq \mathcal{F}^* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \mathcal{G}^* \left( \frac{\zeta}{2} q_1 + \frac{2-\zeta}{2} q_2 \right) \\ & \quad + \mathcal{F}^* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \mathcal{G}^* \left( \frac{2-\zeta}{2} q_1 + \frac{\zeta}{2} q_2 \right) \\ & \quad + \left( \frac{\zeta(2-\zeta)}{2} \right) \Psi^*(q_1, q_2) + \left( \frac{\zeta^2+2-2\zeta}{2} \right) \Omega^*(q_1, q_2). \end{aligned} \tag{13}$$

Multiplying Equations (12) and (13) by  $\varsigma^{\alpha-1}$  and then integrating over  $(0, 1)$ , we obtain

$$\begin{aligned} & 4 \mathcal{F}_* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \mathcal{G}_* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \int_0^1 \varsigma^{\alpha-1} d\varsigma \\ & \leq \int_0^1 \varsigma^{\alpha-1} \mathcal{F}_* \left( \frac{\varsigma}{2} \varrho_1 + \frac{2-\varsigma}{2} \varrho_2 \right) \mathcal{G}_* \left( \frac{\varsigma}{2} \varrho_1 + \frac{2-\varsigma}{2} \varrho_2 \right) d\varsigma \\ & \quad + \int_0^1 \varsigma^{\alpha-1} \mathcal{F}_* \left( \frac{2-\varsigma}{2} \varrho_1 + \frac{\varsigma}{2} \varrho_2 \right) \mathcal{G}_* \left( \frac{2-\varsigma}{2} \varrho_1 + \frac{\varsigma}{2} \varrho_2 \right) d\varsigma \\ & \quad + \Psi_*(\varrho_1, \varrho_2) \int_0^1 \varsigma^{\alpha-1} \frac{\varsigma(2-\varsigma)}{2} d\varsigma + \Omega_*(\varrho_1, \varrho_2) \int_0^1 \varsigma^{\alpha-1} \frac{\varsigma^2 + 2 - 2\varsigma}{2} d\varsigma \end{aligned}$$

and

$$\begin{aligned} & 4 \mathcal{F}^* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \mathcal{G}^* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \int_0^1 \varsigma^{\alpha-1} d\varsigma \\ & \leq \int_0^1 \varsigma^{\alpha-1} \mathcal{F}^* \left( \frac{\varsigma}{2} \varrho_1 + \frac{2-\varsigma}{2} \varrho_2 \right) \mathcal{G}^* \left( \frac{\varsigma}{2} \varrho_1 + \frac{2-\varsigma}{2} \varrho_2 \right) d\varsigma \\ & \quad + \int_0^1 \varsigma^{\alpha-1} \mathcal{F}^* \left( \frac{2-\varsigma}{2} \varrho_1 + \frac{\varsigma}{2} \varrho_2 \right) \mathcal{G}^* \left( \frac{2-\varsigma}{2} \varrho_1 + \frac{\varsigma}{2} \varrho_2 \right) d\varsigma \\ & \quad + \Psi^*(\varrho_1, \varrho_2) \int_0^1 \varsigma^{\alpha-1} \frac{\varsigma(2-\varsigma)}{2} d\varsigma + \Omega^*(\varrho_1, \varrho_2) \int_0^1 \varsigma^{\alpha-1} \frac{\varsigma^2 + 2 - 2\varsigma}{2} d\varsigma. \end{aligned}$$

In view of the above equations, we find that

$$\begin{aligned} & \frac{4}{\alpha} \mathcal{F}_* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \mathcal{G}_* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \\ & \leq \frac{2^\alpha \Gamma(\alpha)}{(\varrho_2 - \varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^+}^\alpha \mathcal{F}_*(\varrho_2) \mathcal{G}_*(\varrho_2) + J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^-}^\alpha \mathcal{F}_*(\varrho_1) \mathcal{G}_*(\varrho_1) \right) \\ & \leq \left( \frac{1}{\alpha + 1} - \frac{1}{2(\alpha + 2)} \right) \Psi_*(\varrho_1, \varrho_2) + \frac{1}{2} \left( \frac{1}{\alpha + 2} - \frac{2}{\alpha + 1} + \frac{2}{\alpha} \right) \Omega_*(\varrho_1, \varrho_2) \end{aligned}$$

and

$$\begin{aligned} & \frac{4}{\alpha} \mathcal{F}^* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \mathcal{G}^* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \\ & \leq \frac{2^\alpha \Gamma(\alpha)}{(\varrho_2 - \varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^+}^\alpha \mathcal{F}^*(\varrho_2) \mathcal{G}^*(\varrho_2) + J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^-}^\alpha \mathcal{F}^*(\varrho_1) \mathcal{G}^*(\varrho_1) \right) \\ & \leq \left( \frac{1}{\alpha + 1} - \frac{1}{2(\alpha + 2)} \right) \Psi^*(\varrho_1, \varrho_2) + \frac{1}{2} \left( \frac{1}{\alpha + 2} - \frac{2}{\alpha + 1} + \frac{2}{\alpha} \right) \Omega^*(\varrho_1, \varrho_2). \end{aligned}$$

Consequently, we have

$$\begin{aligned} & 2 \mathcal{F}_* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \mathcal{G}_* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \\ & \leq \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(\varrho_2 - \varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^+}^\alpha \mathcal{F}_*(\varrho_2) \mathcal{G}_*(\varrho_2) + J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^-}^\alpha \mathcal{F}_*(\varrho_1) \mathcal{G}_*(\varrho_1) \right) \\ & \leq \frac{\alpha}{2} \left( \frac{1}{\alpha + 1} - \frac{1}{2(\alpha + 2)} \right) \Psi_*(\varrho_1, \varrho_2) + \frac{\alpha}{4} \left( \frac{1}{\alpha + 2} - \frac{2}{\alpha + 1} + \frac{2}{\alpha} \right) \Omega_*(\varrho_1, \varrho_2) \end{aligned}$$

and

$$\begin{aligned} & 2 \mathcal{F}^* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \mathcal{G}^* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \\ & \leq \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(\varrho_2 - \varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^+}^\alpha \mathcal{F}^*(\varrho_2) \mathcal{G}^*(\varrho_2) + J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^-}^\alpha \mathcal{F}^*(\varrho_1) \mathcal{G}^*(\varrho_1) \right) \\ & \leq \frac{\alpha}{2} \left( \frac{1}{\alpha+1} - \frac{1}{2(\alpha+2)} \right) \Psi^*(\varrho_1, \varrho_2) + \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) \Omega^*(\varrho_1, \varrho_2). \end{aligned}$$

It follows from the above developments that

$$\begin{aligned} & 2 \left[ \mathcal{F}_* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \mathcal{G}_* \left( \frac{\varrho_1 + \varrho_2}{2} \right), \mathcal{F}^* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \mathcal{G}^* \left( \frac{\varrho_1 + \varrho_2}{2} \right) \right] \\ & \leq_p \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(\varrho_2 - \varrho_1)^\alpha} \left[ \left( J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^+}^\alpha \mathcal{F}_*(\varrho_2) \mathcal{G}_*(\varrho_2) + J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^-}^\alpha \mathcal{F}_*(\varrho_1) \mathcal{G}_*(\varrho_1) \right), \right. \\ & \quad \left. \left( J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^+}^\alpha \mathcal{F}^*(\varrho_2) \mathcal{G}^*(\varrho_2) + J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^-}^\alpha \mathcal{F}^*(\varrho_1) \mathcal{G}^*(\varrho_1) \right) \right] \\ & \leq_p \frac{\alpha}{2} \left( \frac{1}{\alpha+1} - \frac{1}{2(\alpha+2)} \right) [\Psi_*(\varrho_1, \varrho_2), \Psi^*(\varrho_1, \varrho_2)] \\ & \quad + \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) [\Omega_*(\varrho_1, \varrho_2), \Omega^*(\varrho_1, \varrho_2)], \end{aligned}$$

which readily yields

$$\begin{aligned} & 2 \mathcal{F} \left( \frac{\varrho_1 + \varrho_2}{2} \right) \mathcal{G} \left( \frac{\varrho_1 + \varrho_2}{2} \right) \\ & \leq_p \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(\varrho_2 - \varrho_1)^\alpha} \left( J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^+}^\alpha \mathcal{F}(\varrho_2) \mathcal{G}(\varrho_2) + J_{\left(\frac{\varrho_1 + \varrho_2}{2}\right)^-}^\alpha \mathcal{F}(\varrho_1) \mathcal{G}(\varrho_1) \right) \\ & \leq_p \frac{\alpha}{2} \left( \frac{1}{\alpha+1} - \frac{1}{2(\alpha+2)} \right) \Psi(\varrho_1, \varrho_2) + \frac{\alpha}{4} \left( \frac{1}{\alpha+2} - \frac{2}{\alpha+1} + \frac{2}{\alpha} \right) \Omega(\varrho_1, \varrho_2). \end{aligned}$$

This leads us to the desired result asserted by Theorem 8.  $\square$

#### 4. Conclusions

The use of fractional calculus for finding various integral inequalities via convex functions has skyrocketed in recent years. This paper addresses a novel type of interval-valued convex function of a pseudo-order relation, as well as the associated integral inequalities. In order to generalize some H-H (Hermite–Hadamard) type inequalities, the interval-valued R-L (Riemann–Liouville) fractional integral operator is employed. The concept of LR-convex interval-valued functions and fuzzy interval-valued functions will be highly fascinating to apply to the Hadamard–Mercer type and other related integral inequalities in a future study.

We choose to conclude our present investigation by remarking that, in many recent publications, fractional-order analogues of various families of familiar integral inequalities have been routinely derived by using some obviously trivial or redundant parametric variations of known as well as widely and extensively studied operators of fractional integrals and fractional derivatives (see, for details, [54]).

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