# HARVEST SCHEDULING IN THE PRESENCE OF THE RISK OF FIRE — FURTHER RESULTS

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## ABSTRACT

Extensions to the model for forest-level harvest scheduling in the presence of risk of fire, presented by the authors in an earlier publication, are discussed. These include the possibility of partial salvage after a fire; the inclusion of multiple timber types and regeneration options; the problem of accessibility; and the inclusion of variable recovery costs.

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# 1. INTRODUCTION

In a previous paper (Reed and Errico 1986), the problem of optimal harvest scheduling at the forest level in the presence of the risk of destruction through fire was investigated. Since fires occur in a random way the problem had to be formulated as one of stochastic control. While an exact solution to the stochastic control problem was not, in general, determined, it was shown how an acceptable, approximately optimal, solution could be obtained by finding the solution to a related deterministic control problem. Over time this solution is applied in a feedback manner. This procedure provided an optimal harvest in any period, given the current state of the forest. Furthermore, the solution to the deterministic control problem provided estimates or predictions of long–run timber supply, under optimal management. It was seen how even modest fire hazards could result in a considerable reduction in long–run yield and in per–period harvests under optimal management.

In a subsequent paper (Reed and Errico, 1988), two distinct numerical methods of solution for the deterministic control problem, both based on linear programming, were discussed and compared. In the first method (LP1) both the areas harvested in each period (the "control vectors") and the areas of standing forest in each period (the "state vectors") were considered as "activities" of the linear program. In the second method (LP2), which is analogous to the method commonly used to solve Model II harvest scheduling problems (Johnson and Scheurman, 1977) only the areas harvested (the control vectors) were considered as activities. The LP1 method was found to be more efficient, computationally, than the LP2 method in problems where the probability of destruction through fire was age–dependent. Other advantages of the LP1 method were also pointed out. These included greater ease of programming and understanding, and the fact that the dual variables calculated incidentally in obtaining an optimal solution, have useful economic interpretations. In light of these various advantages it was suggested that the LP1 method of solution provides an attractive alternative to the LP2 approach.

The example forest considered in these earlier papers was very simple, consisting of a collection of many even–aged stands, each with the same volume–age or value–age relationships, but initially at different ages. The major factor considered which linked the stands, was the imposition of harvest flow constraints to ensure a certain degree of stability in the flow of timber harvests. In the second paper it was shown how constraints on standing forest for non–timber uses (*e.g.* recreation, wildlife, *etc.*) could be incorporated in the model, and how the marginal opportunity costs of these constraints could be determined from their dual variables at the optimum. Also it was shown how the problem of a changing land base for the forest could be handled.

In spite of these extensions the model is still very simple in comparison with existing whole–forest scheduling models (*e.g.* FORPLAN, (Johnson, Jones and Kent, 1980) and MUSYC (Johnson and Jones, 1979)). In this paper we show how the model can be extended to cover more complex problems, but still with the risk of destruction through fire present. Specifically we include:

(a) the possibility of partial salvage after a fire,

(b) multiple timber types and regeneration options,

(c) the problem of accessibility, and

(d) the inclusion of costs of recovery dependent on location, terrain, *etc.*

Each of these extensions is handled separately in the paper and no attempt is made to present a comprehensive example which incorporates all of these aspects simultaneously.

The organization of the paper is as follows. In Section 2 the basic model [as developed in Reed and Errico, (1986, 1988)] is presented and the extension to cover the case of salvage is discussed; numerical examples are given. In Section 3 a model to handle multiple timber types and regeneration options is developed and a numerical example given. Sections 4 and 5 cover the cases of accessibility and variable recovery costs, respectively.

## 2. THE CASE OF SALVAGE

A stochastic dynamic model for the evolution a forest subject to random destructions through fire is developed in Reed and Errico (1986). The model can be written in matrix form as

$$\underset{\sim}{x}_{t+1} = R_t \, \underset{\sim}{x}_t - S_t \, \underset{\sim}{h}_t \tag{1}$$

where:

$$\underset{\sim}{x}_t = \begin{bmatrix} x_1^t \\ x_2^t \\ \vdots \\ x_k^t \end{bmatrix} \quad \text{and} \quad \underset{\sim}{h}_t = \begin{bmatrix} h_1^t \\ h_2^t \\ \vdots \\ h_k^t \end{bmatrix} \tag{2}$$

represent, respectively, the areas in the forest in the various age–classes $(1,2,\cdots,k)$ at the start of period $t$, and the areas harvested in the various age classes in period $t$. The matrices $R_t$ and $S_t$ are *random* matrices of the form

$$R_t = \begin{bmatrix} p_1^t, & p_2^t, & \cdots & p_k^t \\ (1-p_1^t), & 0, & \cdots & 0 \\ 0, & (1-p_2^t), & & 0 \\ \vdots & & \ddots & \\ 0, & & (1-p_{k-1}^t), & (1-p_k^t) \end{bmatrix} \tag{3}$$

and

$$S_t = \begin{bmatrix} (-1+p_1^t), & (-1+p_2^t) & \cdots & (-1+p_k^t) \\ (1-p_1^t), & 0, & \cdots & 0 \\ 0, & (1-p_2^t), & & 0 \\ & & \ddots & \\ & & & (1-p_{k-1}^t)(1-p_k^t) \end{bmatrix} \tag{4}$$

where $p_i^t$ $(i = i,\cdots,k)$ are *random variables* representing the proportions of the areas in the various age–classes destroyed by fire during period $t$.

It is assumed that the value per hectare of timber of age $i$ is $V_i$ $(i = 1,\cdots,k)$. We let

$$\underset{\sim}{V}' = (V_1,V_2,\cdots,V_k). \tag{5}$$

Thus $\underset{\sim}{V}'$ is a value–at–age vector. Corresponding to a harvest (by area) of $\underset{\sim}{h}_t$ in period $t$, the total volume harvested will be $H_t = \underset{\sim}{v}'\,\underset{\sim}{h}_t$, where

$$\underset{\sim}{v}' = (v_1,v_2,\cdots,v_k) \tag{6}$$

is a vector of *volumes* per hectare, $v_i$, by age. It is assumed that there are imposed some constraints on the harvests to ensure a fairly even flow of timber from the forest. For example there could be *sequential flow constraints* of the form

$$(1-\gamma_1)H_{t-1} \le H_t \le (1+\gamma_2)H_{t-1} \qquad t = 2,3,\cdots \tag{7}$$

where $\gamma_1$ is the maximum permitted proportional decrease in volume harvested $(0 \le \gamma_1 \le 1)$ and $\gamma_2$ is the maximum permitted proportional increase in volume harvested $(0 \le \gamma_2)$.

The problem of maximizing the expected discounted volume (value) of timber from the forest over an infinite time horizon, and subject to the flow constraints is considered in Reed and Errico (1986). It is shown how an approximately optimal solution can be obtained by finding the optimal solution to the related *deterministic* control problem:

maximize
$$J = \sum_{t=1}^{\infty} \alpha^t \underset{\sim}{V}' \underset{\sim}{h}_t \tag{8}$$

subject to:–

$$\underset{\sim}{x}_{t+1} = \bar{R}\, \underset{\sim}{x}_t - \bar{S}\, \underset{\sim}{h}_t \tag{9}$$

$$\underset{\sim}{h}_t \le \underset{\sim}{x}_t \tag{10}$$

$$\underset{\sim}{h}_t \ge 0 \qquad t = 1,2,\cdots \tag{11}$$

and
$$(1-\gamma_1)\underset{\sim}{v}'\underset{\sim}{h}_{t-1} \le \underset{\sim}{v}'\underset{\sim}{h}_t \le (1+\gamma_2)\underset{\sim}{v}'\underset{\sim}{h}_{t-1} \qquad t = 2,3,\cdots \tag{12}$$

where $\bar{R}$ and $\bar{S}$ are the expected values of the (assumed time–homogeneous) random matrices $R_t$ and $S_t$, and $\alpha$ is the per period discount factor.

Two methods of solution of this problem, both based on linear programming are discussed in Reed and Errico (1988). In one method (LP1) both the $\underset{\sim}{x}_t$ variables and the $\underset{\sim}{h}_t$ variables are regarded as activities, while in the other method (LP2) the $\underset{\sim}{h}_t$ variables are eliminated from the problem and only the $\underset{\sim}{x}_t$ variables remain as activities.

In the above formulation of the problem, it is assumed that when areas are burnt they are completely destroyed and no harvest of value can be obtained from them. In practice, very often, some usable timber can be salvaged after a fire. It is fairly easy to incorporate the possibility of salvage into the model by adjusting the objective function.

Suppose that after a fire in a stand in age–class i in period t, some salvage is possible and that the per hectare value of the salvaged timber is $\theta_i^t \times V_i$, where $(1-\theta_i^t)$ $(0 \le \theta_i^t \le 1)$ reflects the proportional loss in value of timber through burning. If the scheduled harvests in period t are given by the vector $\underset{\sim}{h}_t$, and if the proportions burned in each age–class are $p_1^t, p_2^t, \cdots, p_k^t$, then the value of the total harvest in period t (scheduled harvest plus salvage) is

$$\underset{\sim}{V}\,\underset{\sim}{h}_t + \sum_{i=1}^{k} V_i \theta_i^t p_i^t \left[x_i^t - h_i^t\right]. \tag{13}$$

This quantity is a random variable since the proportions burnt, $p_i^t$, and (quite possibly) the proportions salvaged, $\theta_i^t$, are random variables. As before an approximately optimal solution to the stochastic control problem can be obtained by solving the related deterministic control problems, obtained by replacing the random variables by their expected values. The objective (8) would thus be replaced by

$$J = \sum_{t=1}^{\infty} \alpha^t \left\{ \underset{\sim}{V}' \underset{\sim}{h}_t + \sum_{i=1}^{k} V_i \bar{\theta}_i \bar{p}_i \left[X_i^t - h_i^t\right] \right\} \tag{14}$$

where $\bar{\theta}_i = E(\theta_i^t)$ and $\bar{p}_i = E(p_i^t)$ for $i = 1, \cdots, k$. We want to maximize (14) subject to the constraints (9), (10) and (11) and subject to harvest flow constraints. In formulating these flow constraints we can consider the salvage volume to be either:

(a) extra to the regular harvest and not included in the flow constraints,

or

(b) included in the flow constraints.

In case (a) the harvest flow constraints would simply be the constraints (7). In case (b) they would be of the form (7) but with $H_t$ given by (13) only with the value coefficients $V_i$ replaced by volume coefficients $v_i$.

By setting the time–horizon suitably large, we can reduce the problem to one with a finite time horizon and solve it by linear programming.

As an example we have considered a forest comprising only pure spruce stands of the Fort Nelson area of B.C., with growth characteristics and initial inventory as discussed in Reed and Errico (1986). Value was assumed equal to volume; the per annum fire rate considered was one percent (age independent) and the discount rate used was three percent per annum. Other assumptions were as specified in Reed and Errico (1986). Two distinct salvage scenarios were considered:

(i) expected salvage of 25 percent of volume of stands burnt for stands of ages 70 and older, and zero percent of volume of stands burnt for stands of age less than 70,

(ii) expected salvage of 75 percent of volume of stands burnt for stands of ages 70 and older and zero percent of volume of stands burnt for stands of age less than 70.

The predicted harvest trajectories both with salvage included and not included in the flow constraints, along with that for no salvage are shown in Figures 1 and 2. Also shown is the trajectory for optimal harvests with no fire. Figure 1 shows these predictions for the case (i) of 25% salvage, while Figure 2 shows the case (ii) of 75% salvage. It can be seen that the predicted trajectory when the salvage is included in the flow constraints lies between that predicted when there is fire but no salvage and that predicted when there is no fire. From Figure 2 it can be seen that being able to salvage on average 75% of the timber burnt of age 70 and older, recovers about half of the loss in optimal harvest yield in each period due to fire. When salvage volume is considered to be outside of the flow constraints, there are considerably greater total harvests in early periods before the salvageable old–growth timber present in the initial inventory, has been liquidated. The extra volume harvested each period can be accounted for, more or less completely, through salvage. In both situations the equilibrium optimal cutting age is less than the minimum salvage age of 70 years and hence, in the long run there are no additional volumes obtained through salvage and both schedules converge to the non–salvage case.

One can think of allowing the salvage to be outside the flow constraints as effectively relaxing those constraints. In consequence time–discounting can have a stronger effect, resulting in greater harvests early on, at the expense of reduced harvests later. This explains why the harvest schedule resulting when salvage is considered to be outside the flow constraints converges more rapidly to the no–salvage trajectory (Figures 1(a) and 2(a)) than does the schedule resulting when salvage volumes are part of the flow.

It should be noted that an actual harvest trajectory, obtained by applying the first period harvest of the optimal deterministic policy in a feedback manner over time (using the certainty equivalence procedure (Chow, 1975), will likely deviate more from the predicted harvest trajectory, in the case when salvage is present, than in the case when it is not. The reason for this is that the randomness due to fires enters directly into the total harvest in a given period [see (13)], rather than only indirectly through the state variable $\tilde{x}_t$. The additional variance in actual sample paths could, quite possibly, be considerable. When anticipated salvage is not included in the flow constraints, it is quite possible that for an actual sample path, the period to period fluctuation may be greater than the specified flow limits, because of the variation in actual amounts burnt and salvaged.

The question of whether salvage should or should not be included in the harvest flow constraints raises the question of the reasons for, and the influence of, such constraints. In forest–level harvest scheduling models, both with and without the presence of the risk of fire, the influence of harvest flow constraints on the optimal solution is considerable and to a large extent overrides the influence of the discount rate. The opportunity cost, in terms of foregone revenue, of imposing flow constraints is fairly easy to assess. The benefits are harder to quantify in simple economic terms. Ultimately the chosen trade–off between evenness of supply and long–run yield will reflect social, political and other preferences, as much as economic criteria. The question of how salvage should be treated should be addressed in this light.

## 3. MULTIPLE TIMBER TYPES AND REGENERATION OPTIONS

In this section we discuss how the model of Reed and Errico (1986, 1988) can be extended to cover the case of multiple timber types and regeneration options, with the risk of destruction through fire present.

We consider firstly the case of two timber types. We suppose that timber type 1 can adequately be described by k age–classes with a value–at–age vector

$$\underset{\sim}{V}' = (V_1,\cdots,V_k) \tag{15}$$

and a volume–at–age vector

$$\underset{\sim}{v}' = (v_1,v_2,\cdots,v_k). \tag{16}$$

Similarly we assume that timber type 2 can be described by $\ell$ age–classes with a value–at–age vector

$$\underset{\sim}{W}' = (W_1,W_2,\cdots,W_\ell) \tag{17}$$

and a volume–at–age vector

$$\underset{\sim}{w}' = (w_2, w_2, \cdots, w_\ell). \tag{18}$$

The state of the forest at the beginning of period $t$ will be described by a $(k+\ell)$–dimensional vector

$$\begin{bmatrix} x_1{}^t \\ \vdots \\ x_k{}^t \\ \hline y_k{}^t \\ \vdots \\ y_\ell{}^t \end{bmatrix}$$

where $x_i^t$ $(i = 1, \cdots, k)$ denotes the area of timber type 1 at age $i$ and $y_i^t$ $(i = 1, \cdots, \ell)$ denotes the area of timber type 2 at age $i$.

We shall suppose that areas, $a_1^t, a_2^t, \cdots, a_k^t$ are harvested from timber type 1 in period $t$, and regenerated as timber type 1 at the start of period $t + 1$, while areas $b_1^t, b_2^t, \cdots, b_k^t$ are harvested from timber type 1 and regenerated as timber type 2. Similarly we shall suppose that areas $c_1^t, c_2^t, \cdots, c_\ell^t$ are harvested from timber type 2 and regenerated as timber type 2, while areas $d_1^t, d_2^t, \cdots, d_\ell^t$ are harvested from timber type 2 and regenerated as timber type 1.

We shall suppose that random proportions $p_1^t, p_2^t, \cdots, p_k^t$ of the areas in timber type 1 are destroyed by fire in period $t$, while random proportions $q_1^t, q_2^t, \cdots, q_\ell^t$ of the areas in timber type 2 are destroyed by fire in the same period.

The dynamics of the evolution of the forest can be described by the equation

$$\begin{bmatrix} x_2^{t+1} \\ x_2^{t+1} \\ \vdots \\ x_k^{t+1} \\ \hdashline y_1^{t+1} \\ y_2^{t+1} \\ \vdots \\ y_\ell^{t+1} \end{bmatrix} = \begin{bmatrix} (a_1^t + \cdots a_k^t) + (d_1^t + \cdots d_\ell^t) + (e_1^t + \cdots + e_k^t) + (h_1^t + \cdots h_\ell^t) \\ (1-p_1^t)(x_1^t - a_1^t - b_1^t) \\ \vdots \\ (1-p_{k-1}^t)(x_{k-1}^t - a_{k-1}^t - b_{k-1}^t) + (1-p_k^t)(x_k^t - a_k^t - b_k^t) \\ \hdashline (c_1^t + \cdots c_\ell^t) + (b_1^t + \cdots b_k^t) + (g_1^t + \cdots g_\ell^t) + (f_1^t + \cdots f_k^t) \\ (1-q_1^t)(y_1^t - c_1^t - d_1^t) \\ \vdots \\ (1-q_{\ell-1}^t)(y_{\ell-1}^t - c_{\ell-1}^t - d_{\ell-1}^t) + (1-q_\ell^t)(y_\ell^t - c_\ell^t - d_\ell^t) \end{bmatrix}$$

where:

$e_i^t$ represents area of timber type 1 of age $i$ burnt in period $t$ and regenerated as timber type 1 in period $t+1$ $(i = 1,\cdots,k)$,

$f_i^t$ represents area of timber type 1 of age $i$ burnt in period $t$ and regenerated as timber type 2 in period $t+1$ $(i = 1,\cdots,k)$,

$g_i^t$ represents area of timber type 2 of age $i$ burnt in period $t$ and regenerated as timber type 2 in period $t+1$ $(i = 1,\cdots,\ell)$,

$h_i^t$ represents area of timber type 2 of age $i$ burnt in period $t$ and regenerated as timber type 1 in period $t+1$ $(i = 1,\cdots,\ell)$.

Clearly if all areas burnt are to be regenerated we have the extra constraints

$$e_i^t + f_i^t = p_i^t\left[x_i^t - a_i^t - b_i^t\right] \qquad (i = 1,\cdots,k) \tag{20}$$

$$g_i^t + h_i^t = q_i^t\left[y_i^t - c_i^t - d_i^t\right] \qquad (i = 1,\cdots,\ell) \tag{21}$$

The problem of maximizing expected discounted revenue from the resource can be expressed as:

$$J = E\left\{ \sum_{t=1}^{N} \alpha^t G^t + \alpha^{N+1}(\underset{\sim}{r}'\underset{\sim}{x}_{N+1} + \underset{\sim}{s}'\underset{\sim}{y}_{N+1}) \right\} \tag{22}$$

where

$$G_t = \underset{\sim}{V}'(\underset{\sim}{a}_t+\underset{\sim}{b}_t) + W'(\underset{\sim}{c}_t+\underset{\sim}{d}_t) \tag{23}$$

is the revenue earned in period $t$, and $\underset{\sim}{r}'$ and $\underset{\sim}{s}'$ are vectors of (stand–level) net present values of single hectares of timber at various ages in timber types 1 and 2. The maximization is subject to the constraints given by the dynamic equation (19) for $t = 1,\cdots,N$, the constraints (20) and (21) and harvest flow constraints

$$(1-\gamma_1)H_{t-1} \leq H_t \leq (1+\gamma_2)H_{t-1} \tag{24}$$

for $t = 2,\cdots,N$, where

$$H_t = \underset{\sim}{v}'(\underset{\sim}{a}_t+\underset{\sim}{b}_t) + \underset{\sim}{w}'(\underset{\sim}{c}_t+\underset{\sim}{d}_t) \tag{25}$$

represents the total volume harvested in period $t$. It should be noted that the control variables are

$$\underset{\sim}{a}_t = \left[a_1^t, a_2^t, \cdots, a_k^t\right]'$$

$$\underset{\sim}{b}_t = \left[b_1^t, b_2^t, \cdots, b_k^t\right]'$$

$$\underset{\sim}{c}_t = \left[c_1^t, c_2^t, \cdots, c_\ell^t\right]'$$

$$\underset{\sim}{d}_t = \left[d_1^t, d_2^t, \cdots, d_\ell^t\right]'.$$

As in the single timber type case this is a problem in stochastic control since the $p_i^t$ and $q_i^t$ are random variables. However if we replace them by their expected values $\bar{p}_i$ $(i = 1,\cdots,k)$ and $\bar{q}_i$ $(i = 1,\cdots,\ell)$, the problem becomes one in deterministic control. As discussed in Section 2 we shall use the deterministic problem to obtain an approximately optimal policy for the stochastic problem. The deterministic problem can be solved by linear programming using the LP1 form since the objective (22) and the constraints (19), (20), (21) and (24) are all linear in the variables $\underset{\sim}{x}_t$, $\underset{\sim}{y}_t$, $\underset{\sim}{a}_t$, $\underset{\sim}{b}_t$, $\underset{\sim}{c}_t$ and $\underset{\sim}{d}_t$. However the problem can be simplified considerably by adding together the equations with $x_1^{t+1}$ and $y_1^{t+1}$ on the left hand side of (19), and defining new variables

$$z_i^t = a_i^t + b_i^t \qquad i = 1,\cdots,k \tag{26}$$

$$u_i^t = c_i^t + d_i^t \qquad i = 1,\cdots,\ell \tag{27}$$

corresponding to total areas harvested in each age class in each timber type in each period.

The objective then becomes

$$J = \sum_{t=1}^{\infty} \alpha^t(\underset{\sim}{V}'\underset{\sim}{z}_t + \underset{\sim}{W}'\underset{\sim}{u}_t) + \alpha^{N+1}(\underset{\sim}{r}'\underset{\sim}{x}_{N+1} + \underset{\sim}{s}'\underset{\sim}{y}_{N+1}) \tag{28}$$

and the constraints corresponding to the dynamic equations (19), and to (20) and (21) become

$$x_1^{t+1} + y_1^{t+1} = \left[z_1^t + \cdots + z_k^t\right] + \left[u_1^t + \cdots + u_\ell^t\right] + \left[\bar{p}_1\left[x_1^t - z_1^t\right] + \cdots + \bar{p}_k\left[x_k^t - z_k^t\right]\right]$$
$$+ \left[\bar{q}_1\left[y_1^t - u_1^t\right] + \cdots + \bar{q}_\ell\left[y_\ell^t - u_\ell^t\right]\right]$$

$$\begin{aligned} x_2^{t+1} &= (1-\bar{p}_1)\left[x_1^t - z_1^t\right] \\ &\vdots \\ x_k^{t+1} &= (1-\bar{q}_1)\left[x_{k-1}^t - z_{k-1}^t\right] + (1-\bar{p}_k\left[x_k^t - z_k^t\right] \\ y_2^{t+1} &= (1-\bar{q}_1)\left[y_1^t - u_1^t\right] \\ &\vdots \\ y_\ell^{t+1} &= (1-\bar{q}_\ell)\left[y_{\ell-1}^t - u_{\ell-1}^t\right] + (1-\bar{q}_\ell)\left[y_\ell^t - u_\ell^t\right]. \end{aligned} \tag{29}$$

The solution will give explicitly only the total harvests from each age-class in each timber type (the $z_i^t$ and $u_i^t$) and the state of the forest (the $x_i^t$ and $y_i^t$) at the beginning of each period for the optimal policy. However information as to how the areas harvested and burnt are regenerated under optimal management can be recovered from equations (19), (20) and (21).

An example of the method was run using the two volume–age relationships given in Table 1. For the sake of the example, value was assumed to be equal to volume. Timber type 1 corresponds to interior B.C. spruce as used in Section 2. Timber type 2 is a hypothetical type with a volume at age relation derived from that of timber type 1, simply by reducing the volume at any given age by fifteen percent. Thus timber type 2 might represent a slower growing species. To offset this we have assumed that type 2 is less susceptible to destruction through fire than type 1. Specifically we have assumed that for all age–classes of type 1 the per annum fire probability is 0.01, while for all age–classes of type 2 it is 0.0065. The fire–adjusted volume rotation curves (VRCs) [see Reed and Errico (1985)] which can be used to determine maximum long–run average yields are shown in Figure 3. For lower rotation ages (below 80 years) the long–run average yield (LRAY) of type 1 is greater than that of type 2, but for higher rotation ages the situation is reversed. The differences, however, are very slight.

In the example it was assumed that the initial inventory contained only stands of timber type 1. The inventory used corresponds to the current inventory of pure spruce, as described in Section 2. The initial inventory is displayed in the top part (period 1) of Figure 4. The remainder of Figure 4 shows how the forest inventory evolves under optimal management using a 5 percent per annum discount rate and sequential flow constraints of ±10 percent of volume per period, and assuming fixed rather than random rates of fire as described above. It can be seen that all volumes harvested are regenerated as type 2. Thus under optimal management the forest is eventually converted to a pure type 2 forest. The gain in yield through switching to type 2, although positive, is very small as the fire–adjusted volume–rotation curves of Figure 3 would suggest. Roughly speaking, in this example, the effects of a reduction in the fire probability from 0.01 per annum to 0.0065 per annum, is equivalent to an increase in the volume growth curve slightly in excess of 15 percent. This would suggest that in fire–prone regions or for fire–prone species, silvicultural activities could be profitably directed towards reducing the risk of fire as much as toward increasing growth in volume.

The problem of multiple (as opposed to dual) timber types or regeneration options can be handled in essentially the same way. For a problem with $m$ timber types, each described by $k$ age–classes, and with a planning horizon of $N$ periods using terminal payoffs, the resulting linear programming problem, if sequential flow constraints of the form (24) are included, would have $((k-1)m+2) \times N$ rows (constraints) and $km$ columns (activities). If there are restrictions on regeneration (*e.g.* that only a certain timber type can be regenerated after a fire) they can be incorporated as extra constraints in the problem, or the appropriate activities can be removed from the formulation.

## 4. ACCESSIBILITY WHEN RISK OF FIRE PRESENT

In this section we consider the case of accessing roadless areas where the rate of access or road construction is determined prior to the harvest scheduling exercise Most forest–level optimization models treat the accessibility problem through the simple application of a series of constraints which denote fixed amounts of hectares available for harvest in a given period. In these models, the only manner in which hectares in the first age–class can be created is after a harvest which logically can only take place in accessible areas. Thus, inaccessible timber undergoes an aging process only. When the risk of fire is present the problem is not as simple. In this case there is the situation in which hectares in the first age–class may be created through harvest or fire where fire can occur in inaccessible as well as accessible areas. Therefore the inaccessible timber undergoes a dynamic process which includes aging and regeneration, and fewer hectares than anticipated may actually be harvestable after roading, due to fire.

One way of viewing the problem of accessibility, which makes it relatively easy to model, is as a problem with a changing land base — extra hectares are added to the land base as new areas of the forest become accessible. As has been discussed in Reed and Errico (1988) the problem of a changing land base is easily handled by appropriate modifications to the dynamic equation for the system.

If we let $\underset{\sim}{x}_t$ denote the areas (by age) of the forest accessible in period t, then the dynamics of the system are described by the equation

$$\underset{\sim}{x}_{t+1} = R_t(\underset{\sim}{x}_t + \underset{\sim}{M}_t) - S_t \underset{\sim}{h}_t \qquad (30)$$

where $M_t' = (M_1^t, M_2^t, \cdots, M_k^t)$ is a vector of areas (by age) assumed to be newly roaded during period t, and $R_t$ and $S_t$ are random matrices as given in (3) and (4). To estimate the future age distribution of those areas to be roaded, the effects of aging and fire must be considered.

Suppose that at the beginning of the problem (in period 1) the total extent of the forest can be partitioned into regions corresponding to those parts currently accessible, those parts to be roaded during period 1, those parts to be roaded during period 2, $\cdots$, *etc.* Let $\underset{\sim}{a}_t$ describe the areas (by current age) which are to be roaded in period t, $t = 1,2,3,\cdots$. Long range plans for road development may be available for this. It follows that the actual future areas by age (in period t) which will be newly roaded in period t, will be

$$M_t = R_{t-1} R_{t-2} \cdots R_1 \underset{\sim}{a}_t \qquad (31)$$

where $R_1, R_2, \cdots, R_{t-1}$ are random matrices of the form (3). Using the expected values of these matrices, one obtains predictions of the areas, by period–t age, to be roaded in period t:

$$\hat{M}_t = \bar{R}^{t-1} \underset{\sim}{a}_t. \qquad (32)$$

Substituting this in (30), and again using expected values of $R_t$ and $S_t$, gives a deterministic dynamic equation of the form

$$\underset{\sim}{x}_{t+1} = \bar{R}\underset{\sim}{x}_t - \bar{S}\underset{\sim}{h}_t + \bar{R}^t \underset{\sim}{a}_t, \qquad t = 1,2,\cdots \qquad (33)$$

The optimization problem to maximize discounted value of the stream of future harvests subject to flow constraints would be the same as the problem discussed in Section 2 (*i.e.* maximize (8) subject to (9), (10), (11) and (12)), only with the dynamic equation (9) replaced by (33). The LP1 method of solution using linear programming could be applied.

## 5. RECOVERY COSTS

In the previous sections the costs of harvesting timber of a given type have been ignored, or at least assumed to be equal for all stands of a given age, so that they can be subsumed into the net stumpage value. Costs of harvesting timber will depend on many factors [see Williams and Morrison (1985)] including location, terrain, piece size *etc.* Suppose that the per hectare costs of harvesting of a stand can be decomposed into two parts, one corresponding to biological characteristics of the stand (age, size *etc.*) and the other corresponding to geographical and topographical characteristics of the site on which the stand is growing. Let

$$\underset{\sim}{w}' = (w_1, w_2, \cdots, w_k)$$

denote, for each age class, the value of timber harvested, net of costs corresponding to biological characteristics. Assume that for a given timber type these costs are related only to age, so that $\underset{\sim}{w}'$ is the same for all stands of the given timber type.

Suppose that the other component of cost (geographic/topographic) can be made to assume one of the discrete values, $c_1, c_2, \cdots, c_m$. We can partition the forest into $m$ regions corresponding to these costs. Suppose the initial inventory can be described by area by age vectors

$$\underset{\sim}{x}_1^1, \underset{\sim}{x}_1^2, \cdots, \underset{\sim}{x}_1^m$$

(each of form (2)), corresponding to each of the m cost regions. For each cost region there is a dynamic equation of the form (1);

$$\underset{\sim}{x}_{t+1}^j = R_t^j \underset{\sim}{x}_t^j - S_t^j \underset{\sim}{h}_t^j \qquad j = 1,\cdots,m \tag{34}$$

where $\underset{\sim}{h}_t^j$ is a vector of areas harvested (by age) in period $t$ in the $j$th cost region, and $\underset{\sim}{x}_t^j$ is a vector of total areas (by age) at the beginning of period $t$ in the $j$th cost region.

In period $t$, the total value net of costs of both types of harvesting areas $\underset{\sim}{h}_t^1, \underset{\sim}{h}_t^2, \cdots, \underset{\sim}{h}_t^m$ will be

$$V_t = \sum_{j=1}^{m} \left\{ \underset{\sim}{w}' \underset{\sim}{h}_t^j - c_j \underset{\sim}{e}' \underset{\sim}{h}_t^j \right\} \tag{35}$$

where $\underset{\sim}{e}' = (1,1,1,\cdots,1)$, (so that $\underset{\sim}{e}' \underset{\sim}{h}_t^j$ denotes the total area harvested in cost region j).

Over an infinite time horizon the expected present net value of a given harvest sequence will be

$$J = \sum_{t=1}^{\infty} \alpha^t V_t. \tag{36}$$

This is linear in the harvest variables $\underset{\sim}{h}_t^j$. To maximize expected present value we will have to maximize (36) subject to constraints corresponding to the dynamic equations (34), constraints of the form

$$\underset{\sim}{0} \le \underset{\sim}{h}_t^j \le \underset{\sim}{x}_t^j \qquad j = 1,\cdots,M \tag{37}$$

and possibly flow constraints on the *volume* of timber harvested of the form

$$(1-\gamma_1)H_{t-1} \le H_t \le (1+\gamma_1)H_{t-1} \tag{38}$$

where $H_t$ is the total *volume* harvested in period $t$ and is

$$H_t = \sum_{j=1}^{m} \underset{\sim}{v}'\underset{\sim}{h}_t^j \tag{39}$$

where $\underset{\sim}{v}' = (v_1,\cdots,v_k)$ is a volume at age vector, presumed the same for all sites in the forest. This problem is linear in the $\underset{\sim}{h}_t^j$ and $\underset{\sim}{x}_t^j$ and so can be solved by the linear programming using the LP1 method.

An alternative approach would be to regard the problem as a multi–type problem (see Section 3) where a cost region would be treated as a separate type. Constraints would have to be modified to ensure that the areas harvested or burned remain within a cost region after regeneration.

## 6. DISCUSSION

In this paper we have described how the technique of optimal harvest scheduling in the presence of the risk of fire of Reed and Errico (1986) can be extended in a number of directions. The following extensions have been considered:

(a) the possibility of partial salvage after a fire,

(b) multiple timber types and regeneration options,

(c) the problem of accessibility, and

(d) the inclusion of costs dependent on terrain, location, *etc.*

Each extension has been considered on its own. A fully operational whole–forest scheduling model would incorporate all of these aspects together. While conceptually not difficult, the mathematical description of such a model would be a nightmare of subscripts and superscripts *etc.*, and we have not attempted it here.

The equivalence of the harvest scheduling model discussed in this paper when the fire probability is set to zero, and the basic Model II form of FORPLAN and MUSYC has been proved in Reed and Errico (1988). While current operational forms of FORPLAN and MUSYC incorporate many of the extensions discussed in this paper, they cannot, at present, handle the problem of catastrophic loss, which appears to have a considerable impact on volume yields. (A formulation has been developed [Johnson and Stuart (1985)] but has not been implemented.)

Although several of the extensions we have presented here are common to many operational forest–level optimization models, we do not purport that this extended form is sufficiently complete to be able to handle many of the harvest scheduling analyses typically being carried out at present. Rather, we wish to demonstrate the potential the present approach has for extension and the ease with which such extensions may be made. While some small scale trials have been performed comparing solution characteristics of the LP1 and LP2 forms with the more familiar Model II form of MUSYC [Reed and Errico (1988)], the problem of determining whether or not the approach described lends itself to operational use cannot be completely resolved without the testing of large–scale examples to compare the performance of the current approach with alternatives such as that described by Johnson and Stuart (1985).

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**Table 1**

**Volume–Age Relationships for Two Timber Types**

| | Volume ($m^3$/ha) | |
|---|---|---|
| Age (years) | Timber Type 1 | Timber Type 2 |
| 10 | 0 | 0 |
| 30 | 0 | 0 |
| 50 | 16 | 13.60 |
| 70 | 107 | 90.95 |
| 90 | 217 | 184.45 |
| 110 | 275 | 233.75 |
| 130 | 298 | 253.30 |
| 150 | 306 | 260.10 |

# FIGURE CAPTIONS

Figure 1 Projected optimal harvest trajectories when there is an average rate of fires of one percent per annum, and a discount rate of 3 percent per annum. The trajectories correspond to: (a) no salvage, (b) 25% salvage for stands of age 70 or over which are burnt, with salvage considered as part of the harvest flow, and (c) 25% salvage for stands of age 70 or over which are burnt, with salvage considered extra to the harvest flow. Also shown (d) is the optimal trajectory when there is no fire risk.

Figure 2 Projected optimal harvest trajectories when there is an average rate of fires of one percent per annum, and a discount rate of 3 percent per annum. The trajectories correspond to: (a) no salvage, (b) 75% salvage for stands of age 70 or over which are burnt, with salvage considered as part of the harvest flow, (c) 75% salvage for stands of age 70 or over which are burnt with salvage considered extra to the harvest flow. Also shown (d) is the optimal trajectory when there is no fire risk.

Figure 3 Fire–adjusted volume rotation curves [Reed and Errico (1986)] for the two timber types discussed in Section 5.

Figure 4 Projected age–distributions of timber types 1 and 2 under optimal management, with a discount rate of 5 percent per annum. Note how areas of timber type 1 are harvested and regenerated with timber type 2.

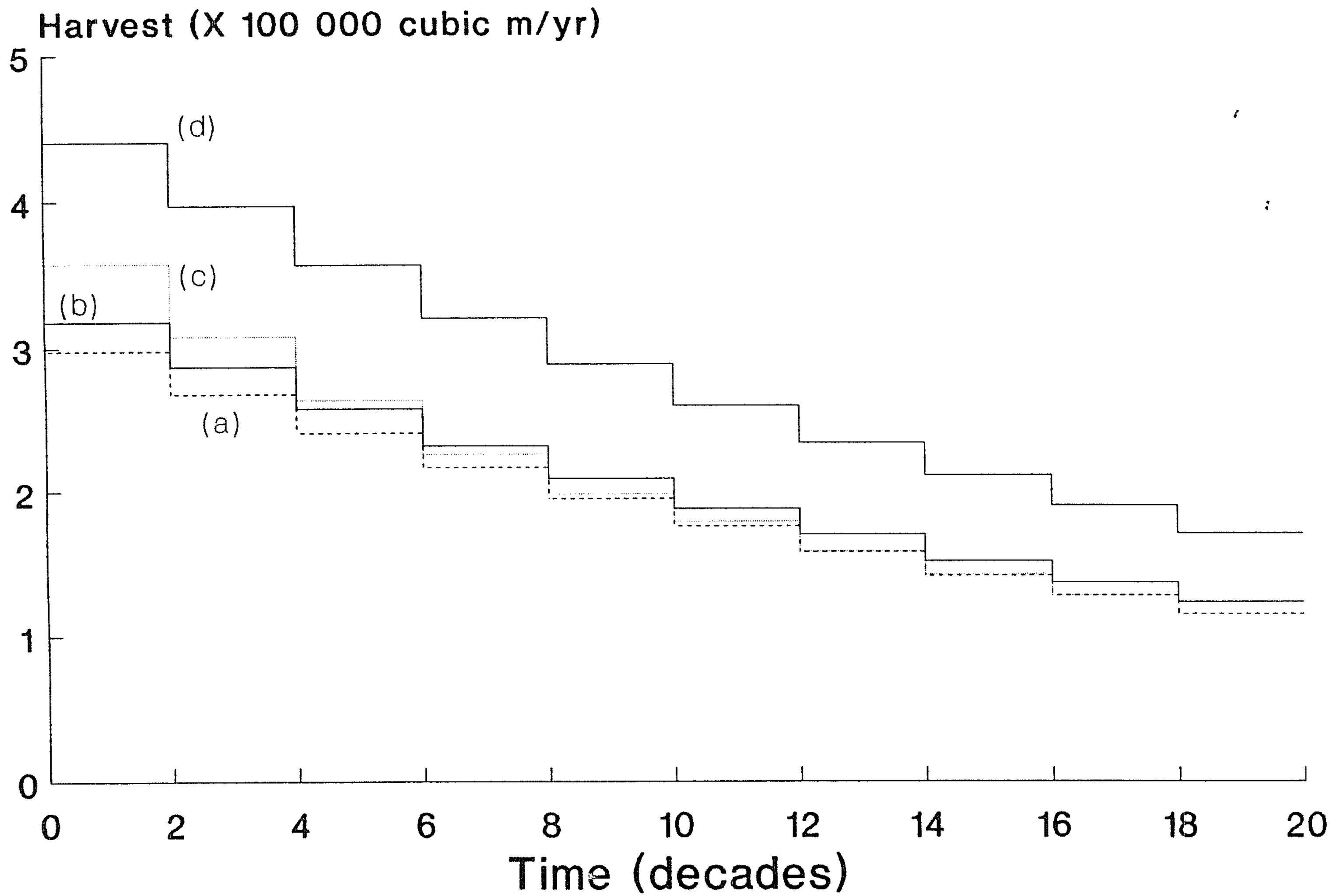

Harvest (X 100 000 cubic m/yr)
5
4
3
2
1
0
(d)
(c)
(b)
(a)
0
2
4
6
8
10
12
14
16
18
20
Time (decades)

FIGURE 1

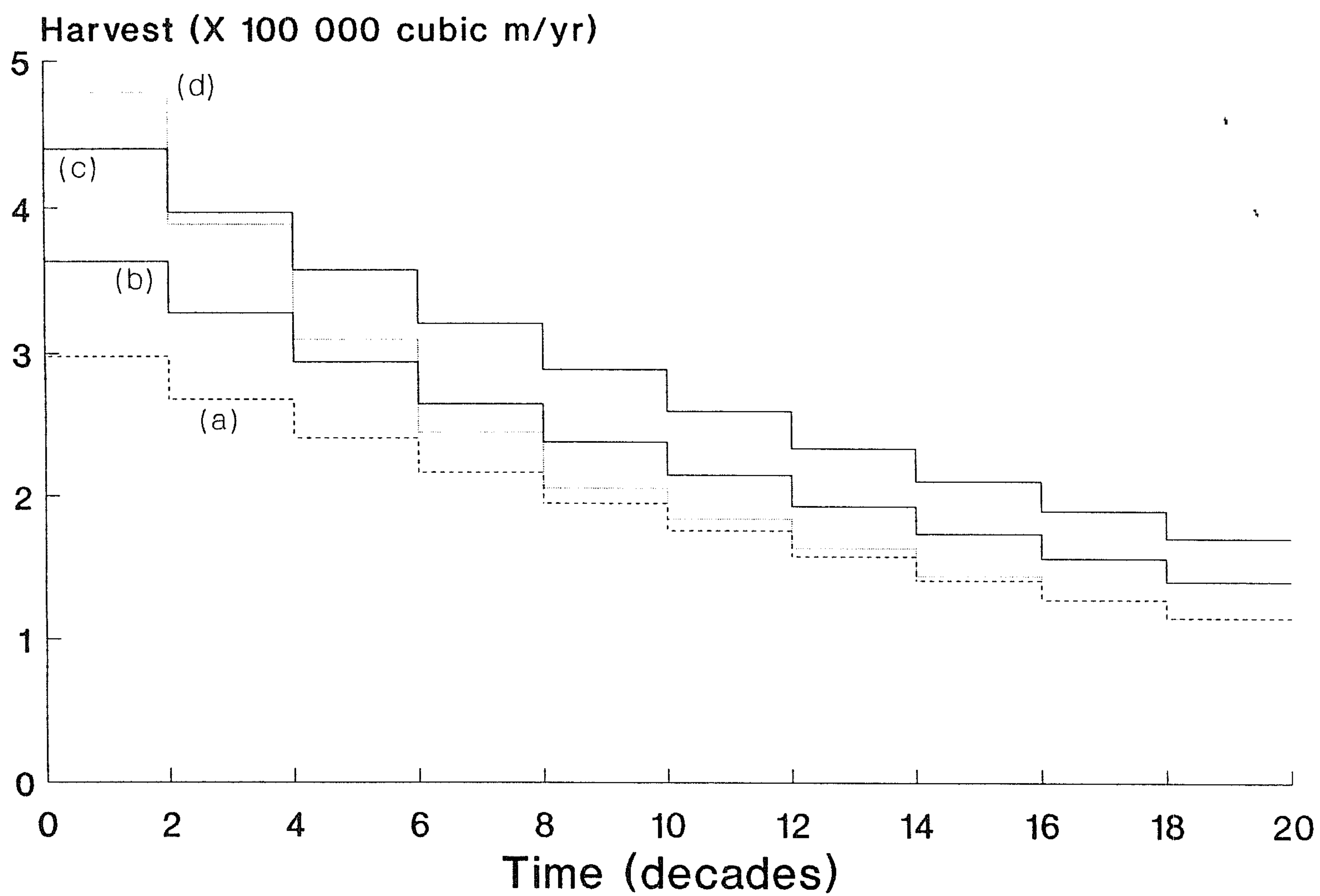

Harvest (X 100 000 cubic m/yr)
5
4
3
2
1
0
(d)
(c)
(b)
(a)
0
2
4
6
8
10
12
14
16
18
20
Time (decades)

FIGURE 2

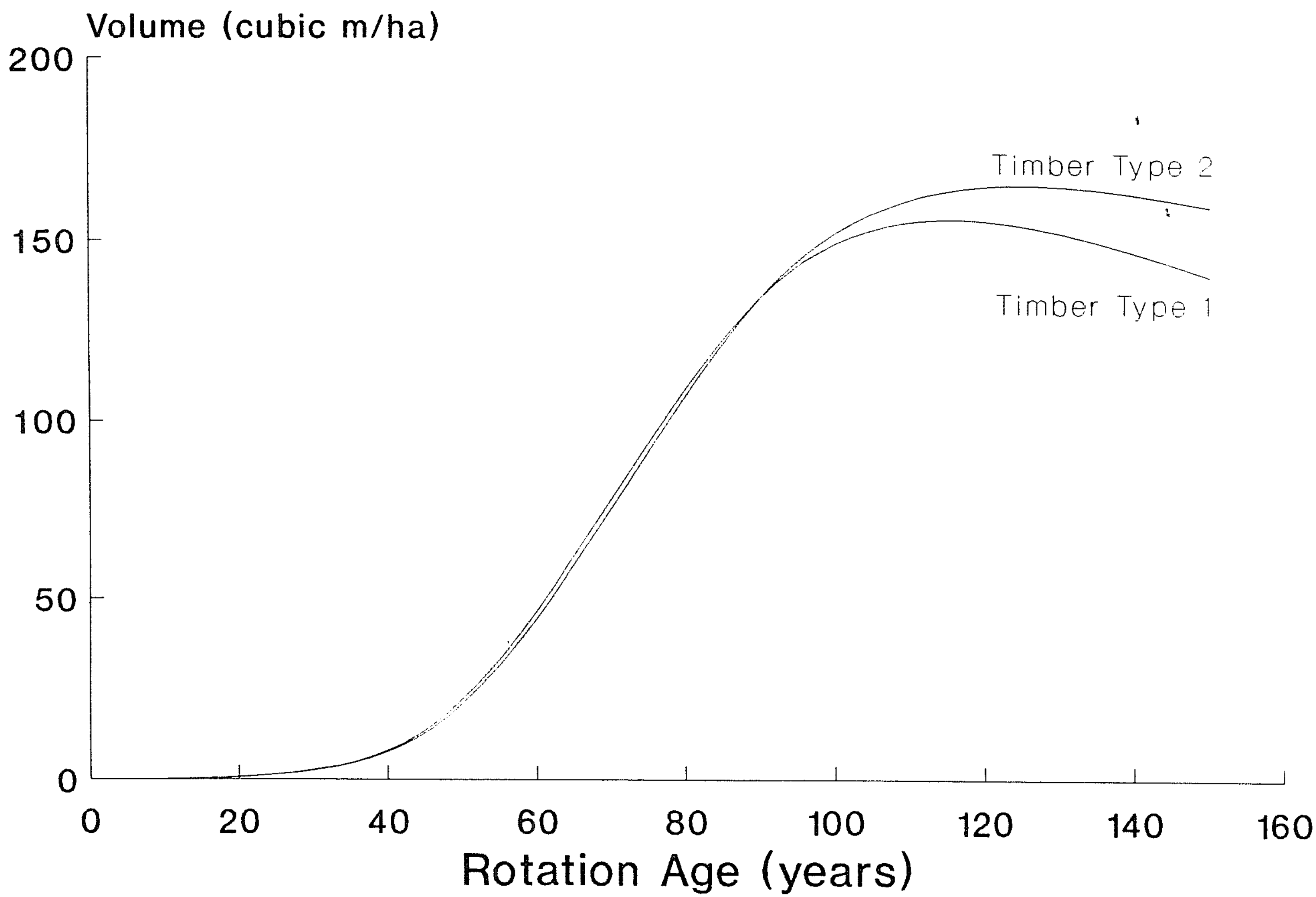
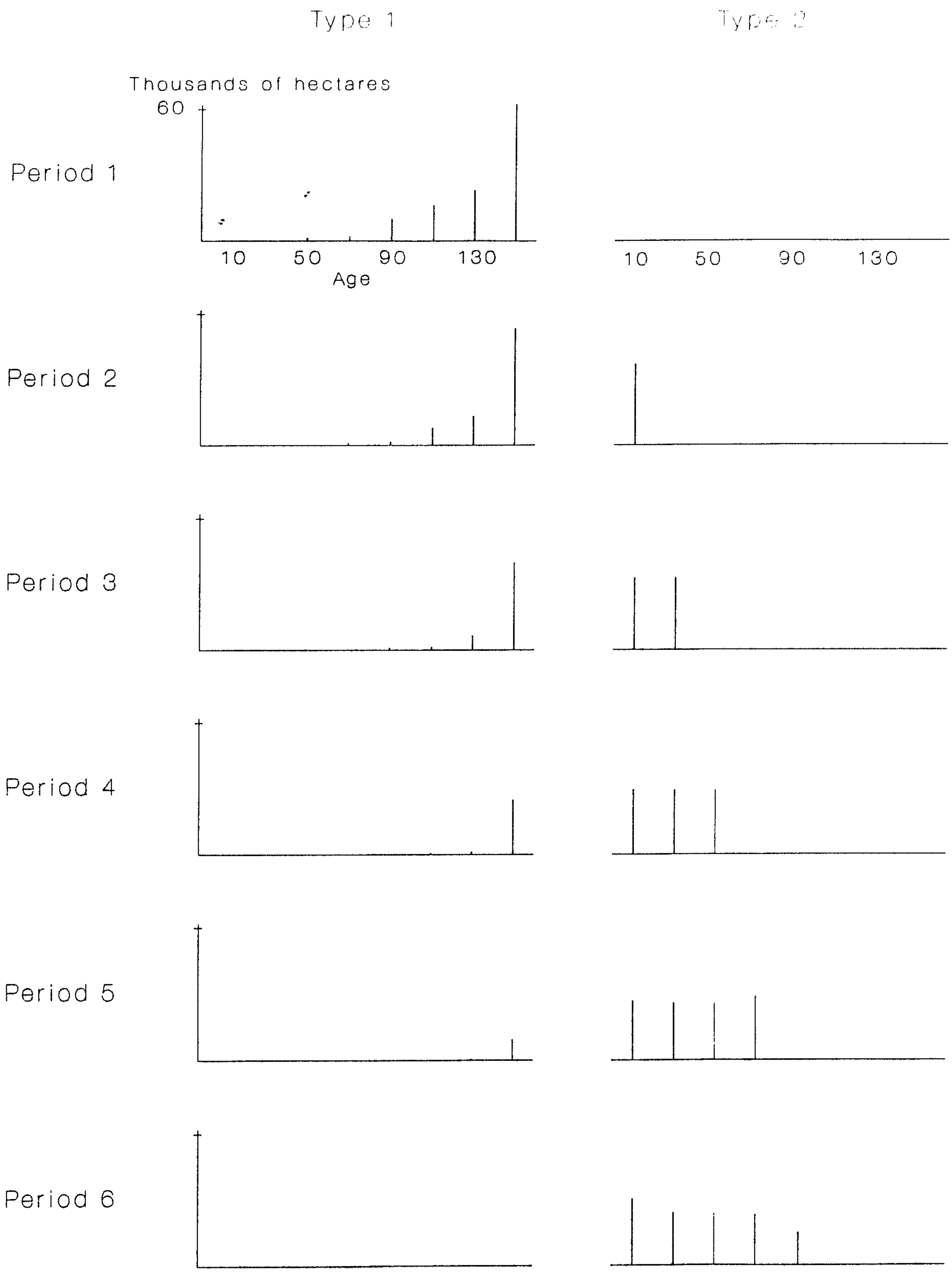

Volume (cubic m/ha)
200
150
100
50
0
0
20
40
60
80
100
120
140
160
Rotation Age (years)
Timber Type 2
Timber Type 1

FIGURE 3



Type 1
Type 2
Thousands of hectares
60
Period 1
10
50
90
130
Age
10
50
90
130
Period 2
Period 3
Period 4
Period 5
Period 6


FIGURE 4