VOICULESCU'S DOUBLE COMMUTANT THEOREM AND THE COHOMOLOGY OF C*–ALGEBRAS

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INTRODUCTION

In a previous paper, Central Cohomology of C*–algebras [5], we outlined a proof of the following result: a separable unital C*–algebra has continuous trace if and only if all of its central cohomology groups for $n \geq 1$ vanish. Unfortunately, as was pointed out to us by Professors A.Ja. Helemskii and B.E. Johnson, the proof we outlined was incorrect. Our appeal to [3, Theorem 3.2] was invalid since the algebras we were interested in were not generally commutative. It is the purpose of this note to give a correct proof of this result as well as other interesting cohomological results. Our main tool will be D. Voiculescu's celebrated double commutant theorem for separable C*–subalgebras of the Calkin algebra [7].

1. PRELIMINARIES

1.1 **Definitions.** Let A be a Banach algebra and X a two–sided A–module. We call X a *Banach A–module* if X is a Banach space and, in addition, we have $\|ax\| \leq \|a\|\|x\|$ and $\|xa\| \leq \|x\|\|a\|$ for all $a \in A$ and $x \in X$. If X is a Banach A–module we define $L^n(A,X)$

to be the Banach space of all bounded n–linear maps from $A^n \to X$, equipped with the norm

$$\|T\| = \sup\{\|T(a_1,\cdots,a_n)\| : a_i \in A \text{ and } \|a_i\| \leq 1 \text{ for each } i = 1,\cdots,n\}.$$

We define maps $\delta^{n+1} : L^n(A,X) \to L^{n+1}(A,X)$ for $n \geq 1$ via

$$(\delta^{n+1} T)(a_0,\cdots,a_n) = a_0 T(a_1,\cdots,a_n) + \sum_{j=0}^{n-1} (-1)^{j+1} T(a_0,\cdots,a_j a_{j+1},\cdots,a_n)$$

$$+ (-1)^{n+1} T(a_0,\cdots,a_{n-1})a_n$$

We define $\delta^1 : X \to L^1(A,X)$ via

$$\delta^1(x)(a) = xa - ax.$$

One checks that $\delta^{n+1} \circ \delta^n = 0$ so that one can define the cohomology groups

$$H^n(A,X) = \ker \delta^{n+1}/\text{Im } \delta^n \text{ for } n \geq 1.$$

If A is a Banach algebra which is also an algebra over the commutative Banach algebra, Z, then one can assume linearity over Z in the above definitions and so obtain the cohomology groups (over Z) $H^n_Z(A,X)$ provided that X is an abelian A–Z–module. See [5] for details. In particular, if A is a C*–algebra and $Z = Z(M(A))$ is the centre of the multiplier algebra of A, then one can define $H^n_Z(A,X)$ provided that X is an abelian A–Z–module (again, see [5] for details).

Finally, if H is a separable infinite dimensional Hilbert space we let $\mathscr{B}(H)$, $\mathscr{K} = \mathscr{K}(H)$, and $Q = Q(H)$ denote the algebra of all bounded operators, the subalgebra of compact operators, and the quotient Calkin algebra, respectively. We let $\pi : \mathscr{B}(H) \to Q$ denote the quotient map. If A is a subalgebra of $\mathscr{B}(H)$ we let A' denote the commutant of A in $\mathscr{B}(H)$ and let $\pi(A)^c$ denote the commutant of $\pi(A)$ in Q. If $\rho : A \to \mathscr{B}(H)$ is a representation of the C*–algebra, A, then $\mathscr{B}(H)$, $\mathscr{K}$, and Q are all A–modules in a natural way, and, in particular, $H^1(A, \mathscr{K})$ is well–defined.

2. COHOMOLOGY OF SEPARABLE C*–ALGEBRAS

The following result is best interpreted as part of the low dimensional (0/1) end of the long exact sequence of cohomology associated to the short exact sequence

$$0 \to \mathscr{K} \to \mathscr{B}(H) \to Q \to 0.$$

However, since the long exact sequence requires restrictive hypotheses on the algebra A which are <u>not</u> necessary in this situation, we give a direct proof.

2.1 **Proposition.** *Let* A *be a subalgebra of* $\mathscr{B}(H)$. *Then there is a natural injection* $\pi(A)^c/\pi(A') \to H^1(A, \mathscr{K})$. *If* A *is a separable, nuclear* C*–*algebra then this map is a bijection.*

Proof. If $\pi(x)$ is in $\pi(A)^c$ then $\delta_x(a) = xa - ax$ maps A into $\mathscr{K}$ and is clearly a derivation. If $\pi(x) = \pi(y)$, then $(x-y) \in \mathscr{K}$ and so $\delta_x - \delta_y = \delta_{(x-y)}$ is inner. That is, $\pi(x) \longmapsto \delta_x : \pi(A)^c \to H^1(A, \mathscr{K})$ is a well–defined homomorphism. Now, if δ_x represents 0 in $H^1(A, \mathscr{K})$ then there is a $y \in \mathscr{K}$ with $\delta_x = \delta_y$ or $\delta_{(x-y)} = 0$. That is, $(x-y) \in A'$ and so $\pi(x) = \pi(x-y) \in \pi(A')$. Hence, the induced map

$$\pi(A)^c/\pi(A') \longrightarrow H^1(A, \mathscr{K})$$

is an injection.

Now, if A is separable and nuclear, then $H^1(A, \mathscr{B}(H)) = \{0\}$ and so if $\delta : A \longrightarrow \mathscr{K}$ is a derivation, then it can be regarded as a derivation : $A \longrightarrow \mathscr{B}(H)$ and so there is an $x \in \mathscr{B}(H)$ with $\delta(a) = xa - ax \in \mathscr{K}$ for all $a \in A$. But then $\pi(x) \in \pi(A)^c$ and $\delta = \delta_x$ so that the map $\pi(A)^c/\pi(A') \longrightarrow H^1(A, \mathscr{K})$ is onto. ■

2.2 **Theorem.** *If* A *is a separable infinite–dimensional* C*–*subalgebra of* $\mathscr{B}(H)$ *then* $H^1(A, \mathscr{K}) \neq \{0\}$.

Proof. By Theorem 1.1, it suffices to see that $\pi(A)^c \neq \pi(A')$. So, suppose $\pi(A') = \pi(A)^c$. Then, letting $A^1 = C^*(A,1)$ we have by Voiculescu's double commutant theorem [7],

$$\pi(A^1) = (\pi(A^1)^c)^c = (\pi(A)^c)^c = (\pi(A'))^c \supseteq \pi(A'').$$

But, then $\pi(A'')$ and hence A'' is a norm separable infinite–dimensional von Neumann algebra in $\mathscr{B}(H)$, which is impossible. ■

We observe that this technique is sufficient to prove the separable case of a theorem of Lazar, Tsui, and Wright [4].

2.3 **Theorem.** *Let* A *be a separable* C*–*algebra, then* $H^1(A,B) = \{0\}$ *for every* C*–*algebra* B *containing* A *if and only if* A *is finite–dimensional.*

Proof. If A is finite–dimensional, it is well–known that $H^1(A,X) = \{0\}$ for any Banach A–module, X.

On the other hand, assume A is infinite–dimensional and let A be faithfully represented on a separable Hilbert space, H, in such a way that $A \cap \mathscr{K} = \{0\}$. This can easily be done by taking an infinite multiple of a faithful representation. Let $B = C^*(A, \mathscr{K}) = A + \mathscr{K}$ and the sum is clearly direct as A–modules. Then

$$H^1(A,B) = H^1(A,A) \oplus H^2(A, \mathscr{K}) \neq \{0\}$$

by Theorem 2.2. ■

<u>Remark</u>. We thank the referee for the very short proof above which replaces our longer, far less elegant one.

Finally we prove the theorem mentioned in the introduction. We let $Z = Z(A)$ denote the centre of the C*–algebra, A.

2.4 **Theorem.** *Let* A *be a separable unital* C*–*algebra. Then,* $H^n_Z(A,X) = \{0\}$ *for all abelian* A–Z–*modules* X *and all* $n \geq 1$ *if and only if* A *has continuous trace.*

Proof. If A has continuous trace, then $H^n_Z(A,X) = \{0\}$ for all X and all $n \geq 1$ by Theorem 4.20 of [6].

On the other hand, if $H^1_Z(A,A) = \{0\}$ then $H^1(A,A) = \{0\}$ since any bounded derivation $\delta : A \longrightarrow A$ is automatically Z–linear. By the deep result of Elliott, Akemann and Pedersen [1], A is the direct sum of a continuous trace algebra together with a finite number of simple C*–algebras. If one of the simple summands say A_k is infinite–dimensional then by representing A_k on H we see by Theorem 2.2 that

$H^1(A_k, \mathscr{K}) \neq \{0\}$. Projecting A down onto A_k makes $\mathscr{K}$ into an abelian A–Z–module and we have $H^1_Z(A, \mathscr{K}) = H^1(A_k, \mathscr{K}) \neq \{0\}$. Thus, if $H^n_Z(A,X) = \{0\}$ for all X and all $n \geq 1$ then all simple summands of A would be finite–dimensional and so A would have continuous trace. ∎

<u>Remark</u>. After distributing preprints of an earlier version of this manuscript, we learned from Professor Helemskii that his former student, Dr. Z.A. Lykova, had already proven Theorem 2.4 (without the separability assumption). She had not published it as they were (vainly) waiting for a reply from us concerning our error in [5]. We apologize to them for our extreme tardiness. Upon receiving our paper, she was stimulated into writing up her own work, and we have received a preprint [8] (in Russian) with the title "The Structure of Banach Algebras with Zero Central Bidimension". Her methods are very different from ours, more along the lines of classical homological algebra. Her focus is thus very different from ours, and so, while she does not obtain Theorems 2.2 or 2.3, she does obtain many other results which are not in this short note. We thank Professors Helemskii and Lykova for the preprints and their interest in our work.

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