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ABSTRACT

Motivated by certain views of the sign nonsingular matrices, we begin study of several classes of nonsingular matrices naturally intermediate between sign nonsingular matrices and the ordinary nonsingular matrices. These all involve Hadamard products that allow constrained changes in the magnitudes (but not the signs) of entries, or the selection of patterns of subblocks. Among a wide variety of results is the fact that the nonsingular matrices A for which $A \circ A^{-1T}$ is doubly stochastic arise as one of our "strong" forms of nonsingularity.

Strong Forms of Nonsingularity

1. INTRODUCTION.

We say that two matrices $A = [a_{ij}]$ and $B = [b_{ij}] \in M_n(\mathbb{R})$ have the same sign pattern if $\operatorname{sgn} b_{ij} = \operatorname{sgn} a_{ij}$ for all $i, j \in N \equiv \{1,2,\ldots,n\}$. The matrix $A \in M_n(\mathbb{R})$ is called *sign nonsingular* if every matrix with the same sign pattern as A is nonsingular. (It is clear that this is a property of only the arrangement of the signs, +, −, 0, of the entries of A.) Recall that the *Hadamard* (or entrywise) *product* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size is denoted and defined by

$$A \circ B = [a_{ij}b_{ij}],$$

see, for example, [HJ, 7.5]. The set of all matrices with the same sign pattern as $A \in M_n(\mathbb{R})$ may be viewed as

$$\mathscr{A}(A) = \{A \circ R: \ 0 < R \in M_n(\mathbb{R})\}.$$

By "$R > 0$" we mean that every entry of R is positive. Thus A is sign nonsingular if and only if every matrix in $\mathscr{A}(A)$ is nonsingular.

The sign nonsingular matrices, which represent a very strong form of nonsingularity, have received considerable attention, see, for example, [BMQ, C, J88, KLM, LM, T1]. Our purpose here is to begin study of some other strong forms of nonsingularity that are naturally intermediate between sign nonsingularity and ordinary nonsingularity. One sequence of intermediate classes, motivated by the Hadamard product view of sign nonsingularity noted above, was first mentioned in [JvD]. The notion of relating important classes of matrices via Hadamard products dates back to [J74].

2. NOTATION, DEFINITIONS AND BASIC OBSERVATIONS.

We define here several new subclasses of the nonsingular matrices that are our primary focus of interest.

For $k = 1, \ldots, n$, let

$$R_{n,k} \equiv \{R \in M_n(\mathbb{R}): \; R > 0, \text{ rank } R \le k\}.$$

We may then define our first sequence of subclasses of nonsingular matrices, as in [JvD], for each n and for $k = 1, \ldots, n$:

$$L_{n,k} \equiv \{A \in M_n(\mathbb{R}): \; R \in R_{n,k} \Rightarrow \det(A \circ R) \neq 0\}.$$

As noted in the previous section, $L_{n,n}$ is just the set of all n–by–n sign nonsingular matrices. It is also worth noting that $L_{n,1}$ is just the set of all ordinary nonsingular matrices in $M_n(\mathbb{R})$ because $R \in R_{n,1}$ means $R = xy^T$ for positive vectors $x, y \in \mathbb{R}^n$ and $A \circ R$ then is $\text{diag}(x)\, A\, \text{diag}(y)$. Since x and y are positive, A is nonsingular if and only if $\text{diag}(x)\, A\, \text{diag}(y)$ is nonsingular. Since $R_{n,1} \subsetneq R_{n,2} \subsetneq \cdots \subsetneq R_{n,n}$, we may make

Observation 2.1

$\{A \in M_n(\mathbb{R}): \; A$ is nonsingular$\} = L_{n,1} \supseteq L_{n,2} \supseteq \cdots \supseteq L_{n,n} = \{A \in M_n(\mathbb{R}):$ A is sign nonsingular$\}$.

Thus, the intermediate classes $L_{n,k}$, $k = 2, \ldots, n-1$, interpolate between the ordinary nonsingular matrices and the sign nonsingular matrices. It is then natural to ask about the transitions from $L_{n,k}$ to $L_{n,k+1}$, as well as the structure of the intermediate $L_{n,k}$'s.

Given $A \in M_n(\mathbb{R})$, it is generally complicated to determine whether $A \in L_{n,k}$ using the definition. As an aid to studying the $L_{n,k}$'s, we introduce a second sequence of classes by restricting the class of required Hadamard multipliers. Let

$$R'_{n,k} \equiv \{R \in M_n(\mathbb{R}): \ R > 0 \ \text{ and } \ R \ \text{ has at least } \ n-k+1 \ \text{ rows of } \ 1\text{'s}\}.$$

We will often need to refer to the matrix in $M_n(\mathbb{R})$ each of whose entries is 1, and since the dimension will be an important parameter, we use J_n. Similarly I_n denotes the identity matrix in $M_n(\mathbb{R})$. The matrices in $R'_{n,k}$ each differ from J_n in at most $k-1$ rows and thus have rank at most k; thus $R'_{n,k}$ is a special subset of $R_{n,k}$. Analogously to the $L_{n,k}$, we may then define

$$L'_{n,k} \equiv \{A \in M_n(\mathbb{R}): \ R \in R'_{n,k} \Rightarrow \det(A \circ R) \neq 0\}$$

for $k = 1, \ldots, n$. Because $R'_{n,k} \subseteq R_{n,k}$, it is clear that

Observation 2.2

$$L_{n,k} \subseteq L'_{n,k}, \text{ for } \ k = 1, \ldots, n.$$

Since $R'_{n,1} = \{J_n\}$, $L'_{n,1}$ is also the ordinary nonsingular matrices in $M_n(\mathbb{R})$. As each matrix in $R'_{n,n}$ is a (right) positive diagonal multiple of one in

$R_{n,n}$ and the sign nonsingular matrices are closed under positive diagonal multiplications, $L'_{n,n}$ is also the set of sign nonsingular matrices in $M_n(\mathbb{R})$. Because the $R'_{n,k}$'s are also monotone increasing, we have for the $L'_{n,k}$'s the exact analog of observation 2.1.

Observation 2.3

$\{A \in M_n(\mathbb{R}):\ A$ is nonsingular$\} = L'_{n,1} \supseteq L'_{n,2} \supseteq \cdots \supseteq L'_{n,n} = \{A \in M_n(\mathbb{R}):$ A is sign nonsingular$\}$.

We note that we have chosen L and L' because the term L–matrix is sometimes used for a specially normalized sign nonsingular matrix and the L' classes are closely related to the L classes.

We shall see that $A \in L_{n,k}$ if and only if $\det A \det(A \circ R) > 0$ for all $R \in R_{n,k}$ and similarly for $L'_{n,k}$ and $R'_{n,k}$. A third sequence of classes also arose from the desire to better understand the $L_{n,k}$'s and, in particular, to determine membership in $L_{n,2}$. It appears rather different in that it is based on weakly uniform signedness of certain block determinants made up from A. Because of the emphasis upon blocks, we use B's in place of L's. For this we let

$T_{n,k} \equiv \{T \in M_n(\mathbb{R}):\ T = PUQ$ in which $P, Q \in M_n(\mathbb{R})$ are permutations and $U \in M_n(\mathbb{R})$ is a direct sum of at most k blocks $J_i\}$.

We may then define

$$B_{n,k} \equiv \{A \in M_n(\mathbb{R}):\ \det A \neq 0,\ T \in T_{n,k} \Rightarrow \det A \det (A \circ T) \geq 0\}.$$

Again $T_{n,1} = \{J_n\}$, so that $B_{n,1}$ is the ordinary nonsingular matrices. Since $T_{n,n}$ includes the permutation matrices, the requirements for $B_{n,n}$ are just that the matrix is nonsingular and every term in the determinant is weakly of the same sign; thus, $B_{n,n}$ is also the sign nonsingular matrices. Because the $T_{n,k}$ are again monotone increasing, we have another exact analog of observation 2.1,

Observation 2.4

$\{A \in M_n(\mathbb{R})\colon\ A$ is nonsingular$\} = B_{n,1} \supseteq B_{n,2} \supseteq \cdots \supseteq B_{n,n} = \{A \in M_n(\mathbb{R})\colon$ A is sign nonsingular$\}$.

Just as the $L'_{n,k}$'s were defined by requiring Hadamard multiplication only from a special subset of $R_{n,k}$, it is useful to define another sequence of classes $B'_{n,k}$ similarly related to $B_{n,k}$. Let $D_k \equiv I_{k-1} \oplus J_{n-k+1}$, and let

$$T'_{n,k} \equiv \{T \in M_n(\mathbb{R})\colon\ T = PD_kQ \text{ for permutations } P,\ Q \in M_n(\mathbb{R})\}.$$

We may then define

$$B'_{n,k} \equiv \{A \in M_n(\mathbb{R})\colon\ \det A \neq 0,\ T \in T'_{n,k} \Rightarrow \det A \det(A \circ T) \geq 0\}.$$

Because the $T'_{n,k}$'s are not monotone, it is not as obvious that the analog of observation 2.1 holds for the $B'_{n,k}$'s. However, we shall see (theorem 3.7) that $B'_{n,k} = L'_{n,k}$, for all n and $k = 1, \ldots, n$. Since $T'_{n,k} \subsetneq T_{n,k}$, the analog of observation 2.2 is clear:

Observation 2.5

$$B_{n,k} \subseteq B'_{n,k}, \text{ for } k = 1, \ldots, n.$$

To summarize, we have at this point that for all n and $k = 1, \ldots, n$,

$$L_{n,k} \subseteq L'_{n,k}$$
$$\text{and} \quad B_{n,k} \subseteq B'_{n,k}.$$

In addition, we shall find that $L_{n,k} \subseteq B_{n,k}$ (theorem 3.2) and that $L'_{n,k} = B'_{n,k}$ (theorem 3.7). Examples will show that for some pairs n, k, $L_{n,k} \neq L'_{n,k}$ and $B_{n,k} \neq B'_{n,k}$ (although all sets are the same for a given $n \leq 3$ and a given $k \leq n$).

For all n and $k = 1, \ldots, n$, all matrices in the classes $L_{n,k}$, $L'_{n,k}$, $B_{n,k}$, and $B'_{n,k}$ are nonsingular. Thus, we may define

$$L^{-1}_{n,k},\ L'^{-1}_{n,k},\ B^{-1}_{n,k} \text{ and } B'^{-1}_{n,k}$$

in the obvious way. For example, $L^{-1}_{n,k}$ is the set of all inverses of matrices in $L_{n,k}$. It is obvious that $L^{-1}_{n,1} = L_{n,1}$, etc., but we shall see that there are other interesting relationships involving these inverse classes.

We shall also need to explicitly refer to submatrices. For index sets α, $\beta \subseteq N$, $C = A[\alpha,\beta]$ is the submatrix of $A \in M_n(\mathbb{R})$ lying in the rows indicated by α and the columns indicated by β. Let α^c denote the complement in N of an index set $\alpha \subseteq N$. For $C = A[\alpha,\beta]$, we denote the complementary submatrix $A[\alpha^c,\beta^c]$ as C^c. We shall also encounter the notion of a combinatorially singular matrix. We say that $A \in M_n(\mathbb{R})$ is *combinatorially singular* if it is singular solely by virtue of its zero pattern, i.e. any matrix in $M_n(\mathbb{R})$ with zeros in the same positions as A is singular. It is well known that this happens if and only if A has a submatrix $A[\alpha,\beta] = 0$ with $|\alpha| + |\beta| > n$.

It is often useful to realize that the classes we have defined are closed under various common transformations. For example, for all n and $k = 1, \ldots, n$, $L_{n,k}$ is closed under each of the following:

(2.6) transposition;

(2.7) permutation equivalence;

and

(2.8) diagonal equivalence.

Since $(A \circ R)^T$ is nonsingular if and only if $A \circ R$ is and since $A^T \circ R^T = (A \circ R)^T$, while $R \in R_{n,k}$ if and only if $R^T \in R_{n,k}$, (2.6) holds. Since, for permutations P and Q, $PAQ \circ PRQ = P(A \circ R)Q$, $\det P(A \circ R)Q = \pm \det(A \circ R)$ and $PRQ \in R_{n,k}$ if and only if $R \in R_{n,k}$, (2.7) holds. For nonsingular diagonal matrices D, E, $\det(DAE \circ R) = \det D(A \circ R)E = \det D \det E \det(A \circ R)$; thus, (2.8) holds. Entirely analogous closure statements hold for each of the other classes $L'_{n,k}$, $B_{n,k}$ and $B'_{n,k}$. The only one of these that is not similarly obvious is the closure of $L'_{n,k}$ under transposition. Note that $R'_{n,k}$ is *not* closed under transposition. (We could have defined an $R''_{n,k}$ via *columns* of 1's and then an analogous $L''_{n,k}$.) The definition of $L'_{n,k}$ is not symmetric in rows and columns. However, since $L'_{n,k} = B'_{n,k}$ (theorem 3.7 to follow), we could have equivalently defined $L'_{n,k}$ in terms of columns and $L'_{n,k}$ is closed under transposition!

We also note that each of the classes $L_{n,k}$, $L'_{n,k}$ and $B_{n,k}$ is closed under direct summation in the following sense. If $A_1 \in L_{n_1,k}$ (resp. $L'_{n_1,k}$ or $B_{n_1,k}$) and $A_2 \in L_{n_2,k}$ (resp. $L'_{n_2,k}$ or $B_{n_2,k}$), then

$$A = A_1 \oplus A_2 \in L_{n_1+n_2,k} \text{ (resp. } L'_{n_1+n_2,k} \text{ or } B_{n_1+n_2,k})$$

for any n_1, n_2 with $k \leq n_1$, n_2. In each case, this may be verified directly from the definition. For example if

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \in R_{n_1+n_2,k},$$

with $R_{ii} \in M_{n_i}(\mathbb{R})$, $i = 1, 2$, then $R_{ii} \in R_{n_i,k}$, $i = 1, 2$. Thus,

$$A \circ R = (A_1 \circ R_{11}) \oplus (A_2 \circ R_{22}),$$

which is nonsingular, as $A_1 \in L_{n_1,k}$ and $A_2 \in L_{n_2,k}$. Since we shall see that $B'_{n,k} = L'_{n,k}$, this closure holds also for $B'_{n,k}$. We finally note that the same statements are true also for subdirect sums, i.e. matrices A of the form $A = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}$ or $A = \begin{bmatrix} A_1 & 0 \\ * & A_2 \end{bmatrix}$.

In section 3 we present various relations among the sets $L_{n,k}$, $L'_{n,k}$, $B_{n,k}$ and $B'_{n,k}$ and give examples showing that certain containments are strict. In particular for $n \leq 3$ all classes are the same for a given pair n, k. This does not remain true for $n \geq 4$. In section 4 we include a variety of statements about particular classes, including a nice characterization of $L'_{n,2}$. These lead to several observations including statements about inverse classes and a table displaying information related to observation 2.1 and its analogs (transitions that are strict containments or equalities). Section 5 summarizes several open questions that should lead to further work.

3. RELATIONS AMONG CLASSES

As noted in section 2, $L_{n,k} \subseteq L'_{n,k}$. We are interested here in exhibiting other relationships between classes in our four interpolating sequences. We begin with an observation useful for subsequent results.

Lemma 3.1

For $A \in M_n(\mathbb{R})$, we have

(a) $A \in L_{n,k}$ if and only if $\det A \det(A \circ R) > 0$ for all $R \in R_{n,k}$;

and

(b) $A \in L'_{n,k}$ if and only if $\det A \det(A \circ R) > 0$ for all $R \in R'_{n,k}$.

Proof. The proof consists of showing that for each $R \in R_{n,k}$ (resp. $R'_{n,k}$) there is a continuous path $R(t) \in R_{n,k}$ (resp. $R'_{n,k}$) connecting $R = R(0)$ to $J_n = R(1)$. Since $\det$ is a continuous, real valued function and $\det(A \circ R(t)) \neq 0$ for $A \in L_{n,k}$ (resp. $L'_{n,k}$), it follows that $\det A = \det(A \circ R(1))$ and $\det(A \circ R) = \det(A \circ R(0))$ have the same (nonzero) sign. In each case the converses are clear.

To connect R to J_n along a continuous path in $R_{n,k}$, identify a maximal linearly independent set of at most k rows $r_{i_1}, \ldots, r_{i_\ell}$ of R.

The remaining rows of R are linear combinations of these with not necessarily nonnegative coefficients. If a negative coefficient appears in the linear combination giving row r_j of R, then continuously change this coefficient to 0. This amounts to continuously adding multiples of one row of R to another. Positivity is retained and rank does not change, so that we stay in $R_{n,k}$. In this manner, we may sequentially and continuously move to a matrix $\hat{R} \in R_{n,k}$ for which all rows are nonnegative linear combinations of the independent rows $r_{i_1}, \ldots, r_{i_\ell}$. Now, linearly change each row $r_{i_1}, \ldots, r_{i_\ell}$, in turn, to the vector $e^T = [1,1,\ldots,1]$, while keeping the coefficients, in the linear combinations that the remaining rows are of $r_{i_1}, \ldots, r_{i_\ell}$, fixed. Thus, we continuously move within $R_{n,k}$ to a matrix in $R_{n,k}$ for which each row is a positive multiple of e^T. By continuously scaling each row, we may complete a continuous path to J_n.

For $R'_{n,k}$, we may use the same argument applied only to the rows that are not already equal to e^T. ■

Theorem 3.2

$L_{n,k} \subseteq B_{n,k}$, for $k = 1, \ldots, n$.

Proof. Take $A \in L_{n,k}$. Let $T = [t_{ij}] \in T_{n,k}$ and define $R(\epsilon) = [r_{ij}]$ by $r_{ij} = t_{ij}$ for entries (i,j) where $t_{ij} = 1$, $r_{ij} = \epsilon > 0$ otherwise; thus $R(\epsilon)$ is continuous in ϵ, $R(\epsilon) \in R_{n,k}$ and $R(1) = J_n$. Since $A \in L_{n,k}$, $\det A \det(A \circ R(\epsilon)) > 0$, by lemma 3.1. But $R(0) = T$; thus $\det A \det(A \circ T) \geq 0$, showing $A \in B_{n,k}$. ■

As seen in section 2, for a fixed n each of our four sequences is identical at the extreme values of k, namely $k = 1$ and $k = n$. Thus for $n = 2$, $L_{2,k} = L'_{2,k} = B_{2,k} = B'_{2,k}$ for each value of k. We now set $n = 3$ and show that $L_{3,k} = L'_{3,k}$ for each value of k. In view of the above, we need only to prove the result for $k = 2$, which we do after a preliminary lemma.

Lemma 3.3

Let $R \in M_{m,n}(\mathbb{R})$, with rows $r_1, r_2, \ldots, r_m$, be a positive matrix of rank 2 or less. Then there exists a pair of rows r_p, r_q such that for each i, $1 \leq i \leq m$, there are coefficients $\alpha_i \geq 0$, $\beta_i \geq 0$ with $r_i = \alpha_i r_p + \beta_i r_q$.

Proof. If rank $R = 1$, the statement is obvious. If rank $R = 2$, then, for each pair of independent rows of R, count the total number of rows of R that are nonnegative linear combinations of the pair, and choose a pair r_p, r_q for which this count is a maximum. If there were another row, say r_j, such that $r_j = sr_p + tr_q$ with $t < 0$, then the pair r_j, r_q would contradict the maximality of the count. Thus, the identified pair fulfills the assertion. ∎

Remark: Geometrically, the vectors $r_1, r_2, \ldots, r_m$ all lie in a two dimensional plane in the first orthant of $\mathbb{R}^n$. The pair r_p, r_q is any pair for which the intervening angle is maximized. A maximizing pair is unique up to scalar multiples.

Theorem 3.4

$L_{3,2} = L'_{3,2}$.

Proof. Because of observation 2.2, we need only prove $L'_{3,2} \subseteq L_{3,2}$. Suppose that $A \in L'_{3,2}$ and compute $\det(A \circ R)$ for all $R \in R_{3,2}$. One need only consider the case in which rank $R = 2$. In this event one row of R must be a nonnegative linear combination of the other two rows by lemma 3.3. By permuting rows, assume $\tilde{R} = [r_{ij}]$, $r_{ij} > 0$, has $r_{3j} = \alpha_1 r_{1j} + \beta_1 r_{2j}$ for $j = 1, 2, 3$, with $\alpha_1 \geq 0$, $\beta_1 \geq 0$, and $\alpha_1 + \beta_1 > 0$. Taking the same permutation on A, giving $\tilde{A}$, we observe $\det(A \circ R) = \pm \det(\tilde{A} \circ \tilde{R})$ and we compute

$$\det(\tilde{A} \circ \tilde{R}) = \alpha_1 \det\left[\tilde{A} \circ \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{11} & r_{12} & r_{13} \end{bmatrix}\right] + \beta_1 \det\left[\tilde{A} \circ \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{21} & r_{22} & r_{23} \end{bmatrix}\right]$$

$$= \alpha_1 r_{11} r_{12} r_{13} \det(\tilde{A} \circ \tilde{R}_2) + \beta_1 r_{21} r_{22} r_{23} \det(\tilde{A} \circ \tilde{R}_1)$$

in which $\tilde{R}_2$ has 2 rows of ones (rows 1 and 3) and $\tilde{R}_1$ also has 2 rows of ones (rows 2 and 3). Thus $\tilde{R}_i \in R'_{3,2}$ and satisfies the conditions of lemma 3.1. As we are assuming $A \in L'_{3,2}$, the signs of $\det(\tilde{A} \circ \tilde{R}_2)$ and $\det(\tilde{A} \circ \tilde{R}_1)$ are both sgn(det A) or both $-$sgn(det A). Thus $\det(A \circ R) \neq 0$, and so $A \in L_{3,2}$. ■

We use an extension of this argument to prove the following result which is valid for all n. This will complete the proof of the statement that $L_{3,k} = L'_{3,k} = B_{3,k} = B'_{3,k}$ for each of $k = 1, 2, 3$.

Theorem 3.5

$L_{n,2} = B_{n,2}$.

Proof. As $L_{n,2} \subseteq B_{n,2}$ (theorem 3.2), we need to prove only that $B_{n,2} \subseteq L_{n,2}$. Let $A \in B_{n,2}$, and let R be any positive matrix with rows $r_1, \ldots, r_n$ and rank 2 or less. Thus, by lemma 3.3, there exist $\alpha_i \geq 0$, $\beta_i \geq 0$ with $\alpha_i + \beta_i > 0$ and rows r_p, r_q such that $r_i = \alpha_i r_p + \beta_i r_q$ for all i, $1 \leq i \leq n$. By the multilinearity of the determinant, $\det(A \circ R)$ can be written as the sum of products of α's, β's and determinants of matrices of the form $A \circ C$, where C is a matrix with each row equal to either r_p or r_q. If all rows of C are of one type, clearly $\det A \det(A \circ C) \geq 0$. If not, calculating $\det(A \circ C)$ by using the Laplace expansion on the rows of C equal to r_p, all terms can be written as products of entries from r_p, r_q and determinants of the form $A \circ T$, where $T \in T_{n,2}$. Since $A \in B_{n,2}$, $\det A \det(A \circ T) \geq 0$. Thus the computation of $\det(A \circ R)$ as outlined above produces a sum of terms consisting of products of positive constants and determinants which agree weakly with the sign of $\det A$. Some of these terms must be nonzero, otherwise a similar Laplace expansion of $\det A$ would yield all zero terms. Hence $\det(A \circ R) \neq 0$, and $A \in L_{n,2}$. ■

The following example shows that $L_{4,2} \neq L'_{4,2}$; thus our sequences are not all identical for $n = 4$. In addition, this example shows that $B_{4,2} \neq B'_{4,2}$.

Example 3.6

$$A = \begin{bmatrix} -3 & 2 & 2 & 3 \\ 4 & -1 & 3 & 3 \\ 5 & 3 & -4 & 5 \\ 2 & 2 & 1 & -1 \end{bmatrix}$$

$A \in L'_{4,2}$ because $A \circ A^{-1T}$ is doubly stochastic (theorem 4.9 to follow). However, $A \notin B_{4,2}$ as $\det(A \circ (I_2 \oplus I_2)) = 5$, and $I_2 \oplus I_2 \in T_{4,2}$, whereas $\det A = -696$. As $L_{4,2} = B_{4,2}$ (theorem 3.5), $A \notin L_{4,2}$; and as $L'_{n,k} = B'_{n,k}$ (theorem 3.7 to follow), $A \in B'_{4,2}$. ■

THEOREM 3.7

$L'_{n,k} = B'_{n,k}$.

Proof. First we show $L'_{n,k} \subseteq B'_{n,k}$. Let $A \in L'_{n,k}$ and $T \in T'_{n,k}$. We must show that $\det A \det(A \circ T) \geq 0$. Since we are done if $\det(A \circ T) = 0$, we now assume $\det(A \circ T) \neq 0$. Define the n-by-n matrix $R(x) = [r_{ij}(x)]$ in which $r_{ij}(x) = x$ if $t_{ij} = 1$ and $t_{ik} = 0$ for all $k \neq j$, and $r_{ij}(x) = 1$ otherwise. Note that for $x > 0$, $R(x) > 0$ and has $n-k+1$ rows of ones, and thus $R(x) \in R'_{n,k}$. By lemma 3.1, $\det A \det(A \circ R(x)) > 0$, for all $x > 0$. But since $\det(A \circ R(x)) = x^{k-1} \det(A \circ T)$ + lower order terms in x, $\det(A \circ R(x))$ and $\det(A \circ T)$ must agree in sign for sufficiently large x. Hence $\det A \det(A \circ T) > 0$. Thus $A \in B'_{n,k}$.

We now show $B'_{n,k} \subseteq L'_{n,k}$. Let $A \in B'_{n,k}$ and $R \in R'_{n,k}$. We must show that $\det(A \circ R) \neq 0$. Choose $n-k+1$ rows of A which consist of all ones. Let $C = \{i_1, \ldots, i_{n-k+1}\}$ be the set of indices of these rows. By considering the Laplace expansion of $\det(A \circ R)$ on these rows, we can write $\det(A \circ R)$ as a sum of determinants of the form $\det(A \circ R \circ T)$, where $T \in T'_{n,k}$. But $\det(A \circ R \circ T)$ can be written as $\prod_{i \in C^c} r_{i\sigma(i)} \det(A \circ T)$. Since $\det A \det(A \circ T) \geq 0$ for all T, we conclude that $\det A \det(A \circ R \circ T) \geq 0$ for all T. Hence $\det(A \circ R)$ can be written as a sum of weakly similarly signed determinants. Some of these determinants must be nonzero, otherwise a similar Laplace expansion of $\det A$ would yield all zero terms. Therefore $\det(A \circ R) \neq 0$ and $A \in L'_{n,k}$. ∎

In view of the above result, we will no longer refer to $B'_{n,k}$. Note that membership of A in $L'_{n,k}$ may then be checked by computing $\det(A \circ T)$ for

$\binom{n}{k-1}^2$ matrices $T \in T'_{n,k}$. We note that membership in $B_{n,k}$ may also be checked via finitely many trials, which, in part, motivates interest in $B_{n,k}$.

4. INCLUSIONS FOR SEQUENCES

For matrices of sizes 2, 3 and 4, we are easily able to prove that each inclusion in observation 2.1 is strict. We state and prove the result for the sequence $L_{n,k}$, after stating a useful lemma [G], see also [T]. We note that Gibson's paper is concerned only with the conversion of the permanent into the determinant (sign nonsingularity is not mentioned), but [G, Cor. 2] is equivalent to the following.

Lemma 4.1 [G]

If $A \in M_n(\mathbb{R})$ is sign nonsingular, then A has at least $(n-1)(n-2)/2$ zero entries with equality if and only if A is permutation equivalent to a Hessenberg matrix.

Theorem 4.2

(a) $L_{2,1} \neq L_{2,2}$

(b) $L_{3,2} \neq L_{3,3}$

(c) $L_{4,3} \neq L_{4,4}$

(d) If $L_{n,k} \neq L_{n,k+1}$, then $L_{n+1,k} \neq L_{n+1,k+1}$.

Proof. For (a) consider the example $\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix} \in L_{2,1} \backslash L_{2,2}$, as it is nonsingular but not sign nonsingular.

For (b) consider the example $A = \begin{bmatrix}1 & 1 & 1\\ -1 & 1 & 1\\ -1 & -1 & 1\end{bmatrix} \in L_{3,2} \backslash L_{3,3}$. If $R = \begin{bmatrix}1 & 1 & 1\\ 1 & a & b\\ 1 & c & d\end{bmatrix}$ with $\det R = 0$, then $\det(A \circ R) = 2(ad+c) > 0$. For $R \in R_{3,2}$

it suffices to consider R of the above form; thus $A \in L_{3,2}$. By lemma 4.1, matrix A is not sign nonsingular.

For (c) consider the Hadamard matrix $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \in L_{4,3} \backslash L_{4,4}$, by an argument similar to (b).

For (d) assume $A \in L_{n,k} \backslash L_{n,k+1}$, then $A \oplus [1] \in L_{n+1,k} \backslash L_{n+1,k+1}$. ∎

Remark: The Hadamard matrix identified in part (c) above is, up to transformations (2.6), (2.7), and (2.8), the only matrix in $L'_{4,3} \backslash L'_{4,4}$. Thus $L_{4,3} = B_{4,3} = L'_{4,3}$.

Proof: Let $A \in L'_{4,3} \backslash L'_{4,4}$. By theorem 4.3 (to follow), A can have no zero entry. Let F be any 2–by–2 submatrix of A. Since $A \in B'_{4,3}$ by theorem 3.7, $\det(A \circ T)$ must be weakly of the same sign as det A for all matrices T that are permutation equivalent to $J_2 \oplus J_1 \oplus J_1$. By positioning the elements of J_2 in T to coincide with the elements of F in A, we see that if F is nonsingular, F^c must be sign nonsingular since both terms of $\det F^c$ must be similarly signed. It follows that both F and F^c must be sign nonsingular or both must be singular. Since not every 2–by–2 submatrix of the first two rows of A can be sign nonsingular, at least one must be singular. By permutation equivalence and diagonal scaling we may assume that A has the form $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, in which A_{11} and A_{22} are 2–by–2 singular submatrices with positive entries. If A_{12} and A_{21} were both sign nonsingular, the sign pattern of A would reveal another 2–by–2 sign nonsingular submatrix that has a singular complement; we conclude that A_{12} and A_{21} must also be singular. It is straightforward to observe that any two rows or any two columns of A must contain a sign nonsingular submatrix. Applying this to the first two rows and the last two columns of A, it follows that A_{12} has an alternating sign pattern, either $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ or $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$. Similarly A_{21} must have an alternating sign pattern. By permutation equivalence, we may assume A has negative diagonal entries and positive off-diagonal entries. By diagonal scaling, we may assume $a_{ii} = -1$ for $i \leq i \leq 4$, and by diagonal similarly $a_{i,i+1} = +1$ for $1 \leq i \leq 3$. Since the sign pattern of A precludes any 2-by-2 principal submatrix from being sign nonsingular, all are singular, yielding $a_{i,i-1} = 1$ for $2 \leq i \leq 4$. Similarly $a_{14} = a_{41} = 1$. Setting $a_{13} = x$ and noting that A_{12} is singular, we find $a_{24} = 1/x$, but then a_{31} and a_{42} must both equal x and $1/x$. Hence $x = 1$ and A is the claimed Hadamard matrix. Since $A \in L_{4,3} \backslash L_{4,4}$ and $L_{n,n-1} \backslash L_{n,n} \subseteq B_{n,n-1} \backslash B_{n,n} \subseteq L'_{n,n-1} \backslash L'_{n,n}$, we have that $L_{4,3} = B_{4,3} = L'_{4,3}$. ∎

Combining results of theorem 4.2 gives that for $n = 2, 3, 4$ each class of the sequence $L_{n,k}$ is distinct. The statement of theorem 4.2 holds with $L_{n,k}$ replaced with $L'_{n,k}$ or $B_{n,k}$. The proofs are identical. For $n = 5$, theorem 4.2 gives $L_{5,1} \neq L_{5,2} \neq L_{5,3} \neq L_{5,4}$. We will see that in fact $L_{5,4}$ is equal to $L_{5,5}$, the class of sign nonsingular matrices of size 5. We prove this by a series of results which also give more information about the sequences for a fixed value of n. Recall that $L_{n,n} = L'_{n,n} = B_{n,n}$.

Theorem 4.3

If $A \in L'_{n,n-1} \backslash L'_{n,n}$, $A \in L_{n,n-1} \backslash L_{n,n}$, or $A \in B_{n,n-1} \backslash B_{n,n}$, then A can have no zero entry.

Proof. We prove only the first result; the others follow because $L_{n,n-1} \subseteq L'_{n,n-1}$ and $B_{n,n-1} \subseteq L'_{n,n-1}$. Since A is not sign nonsingular, at least one summand in $\det A$ must have sign opposite to the sign of $\det A$. Since membership in $L'_{n,n-1}$ is closed under permutation equivalence, this summand can be taken as $\prod_{i=1}^{n} a_{ii}$. Assume that A has a zero entry, which can be taken as $a_{12} = 0$ by the closure. Let $R(\epsilon) = [r_{ij}]$ with $r_{ij} = 1$ if $i \leq 2$ or $i = j$, and $r_{ij} = \epsilon$ otherwise, so $R(\epsilon) \in R'_{n,n-1}$. Then $\det(A \circ R(\epsilon)) = \prod_{i=1}^{n} a_{ii} + \mathcal{O}(\epsilon)$. If the $\mathcal{O}(\epsilon)$ term is zero, there is a contradiction. If not, then there exists $\epsilon_0 > 0$ such that $\det A$ and $\det(A \circ R(\epsilon_0))$ differ in sign, again giving a contradiction (by lemma 3.1). Hence A can have no zero entry. ∎

We now consider a submatrix and prove the following qualitative result.

Theorem 4.4

Let $A \in L'_{n,k}$, $A \in L_{n,k}$ or $A \in B_{n,k}$, $k = 2, 3, 4, \ldots, n$ and, for $m \leq k-1$, let C be an m-by-m submatrix of A which is not combinatorially singular. Then C must be sign nonsingular or have a singular complement.

Proof. We prove the first result only; the others follow since $L_{n,k}$ and $B_{n,k}$ are both subsets of $L'_{n,k}$. The case $k = 2$ gives $m = 1$, and the result is obviously true as C is a nonzero matrix of size 1. By the permutation closure property, we may assume that $C = [c_{ij}]$ is the leading m-by-m principal submatrix of A. Take $A \in L'_{n,k}$, $k = 3, \ldots, n$, and suppose that C is not sign nonsingular and C^c is nonsingular. These assumptions mean that at least one of the summands in $\det C$ has sign opposite that of the sign of $(\det A/\det C^c)$. By permutation closure, we may assume one such summand is $\prod_{i=1}^{m} c_{ii} = \prod_{i=1}^{m} a_{ii}$. Let $R(\epsilon) = [r_{ij}]$ with $r_{ij} = 1$ if $i > m$ or if $i = j$, and $r_{ij} = \epsilon$ otherwise. Then $R(\epsilon)$ has at least $n-k+1$ rows of ones, and so $R(\epsilon) \in R'_{n,k}$. Also $\det(A \circ R(\epsilon)) = (\det C^c) \prod_{i=1}^{m} a_{ii} + \mathcal{O}(\epsilon)$.

So there exists $\epsilon_0 > 0$ such that $\det A$ and $\det(A \circ R(\epsilon_0))$ differ in sign, again (by lemma 3.1) giving a contradiction. Thus either C must be sign nonsingular or C^c must be singular. ■

Observe that when $k = n$ and $m = n-1$, theorem 4.6 states that if A is sign nonsingular and $a_{ij} \neq 0$, then $A[\{i\}^c, \{j\}^c]$ is sign nonsingular or combinatorially singular, which is known. Thus if A is sign nonsingular and $a_{ij} \neq 0$, then the i, j entry of A^{-1} has a determined sign.

Now we are able to prove the result for $n = 5$ stated before theorem 4.3, and in fact we prove that for $n \geq 5$, the two strongest nonsingular classes are equal.

Theorem 4.5

For $n \geq 5$, $L'_{n,n-1} = L'_{n,n}$. In particular, we also have $L_{n,n-1} = B_{n,n-1} = L'_{n,n}$.

Proof. Suppose $A \in L'_{n,n-1} \setminus L'_{n,n}$. Then, by theorem 4.4, every $(n-2)$-by-$(n-2)$ submatrix of A is combinatorially singular, sign nonsingular or has a singular complement. Since, for $n \geq 5$, every sign nonsingular matrix of size $n-2$ must have at least one zero entry (by lemma 4.1), but A contains no zero entry (by theorem 4.3), no submatrix of this size can be sign nonsingular. Thus each 2-by-2 submatrix of A must be singular, as a combinatorially singular complement is ruled out by the absence of zero entries in A. But this implies that $\det A = 0$, which contradicts our assumption. Hence $L'_{n,n-1} \setminus L'_{n,n} = \emptyset$, proving the first equality. The other equalities follows immediately since $L_{n,n-1} \subseteq L'_{n,n-1}$ and $B_{n,n-1} \subseteq L'_{n,n-1}$. ■

We summarize our complete results for $n \le 5$ as follows and indicate our incomplete results for $n > 5$ by including $n = 6, 7$.

$$
\begin{array}{l}
L_{1,1} \\
L_{2,1} \supsetneq L_{2,2} \\
L_{3,1} \supsetneq L_{3,2} \supsetneq L_{3,3} \\
L_{4,1} \supsetneq L_{4,2} \supsetneq L_{4,3} \supsetneq L_{4,4} \\
L_{5,1} \supsetneq L_{5,2} \supsetneq L_{5,3} \supsetneq L_{5,4} = L_{5,5} \\
L_{6,1} \supsetneq L_{6,2} \supsetneq L_{6,3} \supsetneq L_{6,4} \supset L_{6,5} = L_{6,6} \\
L_{7,1} \supsetneq L_{7,2} \supsetneq L_{7,3} \supsetneq L_{7,4} \supset L_{7,5} \supset L_{7,6} = L_{7,7} \\
\qquad \vdots \qquad \vdots \qquad \vdots \qquad\qquad\qquad\qquad \ddots
\end{array}
$$

It is unresolved whether all inclusions for $n \ge 6$ are sharp; unresolved cases are indicated by $\supset$. This table remains valid if each $L_{n,k}$ is replaced by $L'_{n,k}$ or each $L_{n,k}$ is replaced by $B_{n,k}$.

For larger values of n we can show that more of the strongest nonsingular $L_{n,k}$ classes are in fact equal. We again focus on zero entries.

LEMMA 4.6

If $A \in M_n(\mathbb{R})$, $n \ge 5$, $A = [a_{ij}]$ with $a_{ii} \ne 0$ for $i = 1, \ldots, n$ has more than $4n-6$ zero entries in off-diagonal positions, then there must be two zero entries which share no common indices; that is there exist p, q, r, s distinct such that $a_{pq} = a_{rs} = 0$.

Proof. Given $a_{ii} \neq 0$, assume $a_{pq} = 0$ with p, q distinct. If A has more than 4n–6 off–diagonal zero entries, at least one zero entry must lie outside of rows p, q and columns p, q, since 4n–6 zeros could totally fill the off–diagonal entries of rows p, q and columns p, q. ∎

Note that if an $A \in M_n(\mathbb{R})$ has more than 4n–6 zeros in off–diagonal positions, there exists a permutation matrix Q such that $C = Q^TAQ$ with $c_{12} = c_{34} = 0$. We use this to prove our next result (cf. theorem 4.3).

Theorem 4.7

If $A \in L_{n,n-2} \backslash L_{n,n}$ with $n \geq 5$, then A has less than 4n–5 zero entries.

Proof. Suppose $A \in L_{n,n-2} \backslash L_{n,n}$ and A has more than 4n–6 zero entries. We may assume $\left[\prod_{i=1}^{n} a_{ii}\right](\det A) < 0$ and $a_{12} = a_{34} = 0$ (by lemma 4.6). Let $R(\epsilon) = [r_{ij}]$ with $r_{ij} = 1$ for $i = j$ or $(i,j) = (1,2), (2,1), (3,4)$ or $(4,3)$, $r_{ij} = \epsilon$ otherwise; thus $R \in R_{n,n-2}$. Then $\det(A \circ R(\epsilon)) = \prod_{i=1}^{n} a_{ii} + \mathcal{O}(\epsilon)$, and by the now familiar argument using lemma 3.1, we arrive at a contradication. ∎

This result on zero entries now gives us another equality between classes.

Theorem 4.8

For $n \geq 15$, $L_{n,n-2} = L_{n,n}$.

Proof. Suppose $A \in L_{n,n-2} \backslash L_{n,n}$. Then by theorem 4.4, every (n–3)–by–(n–3) submatrix of A is combinatorially singular, sign nonsingular or has a singular complement. Since for $n \geq 15$, every sign nonsingular matrix of size $n-3$ must have at least 55 zero entries (lemma 4.1), but A has fewer than 55 zero entries (by theorem 4.7), no submatrix of this size can be sign nonsingular. Thus each 3-by-3 submatrix of A must be singular or have a combinatorially singular complement. Hence $\det A = 0$, contradicting our assumption. Thus $L_{n,n-2} \backslash L_{n,n} = \emptyset$. ■

So for $n \geq 15$, the three strongest nonsingular classes are all equal. Unfortunately our method of proof does not indicate that $n = 15$ is the minimum value of n for which this is true. Recall that we have proved that $n = 5$ is the minimum value of n for which the *two* strongest nonsingular classes are equal. As n is increased, it can be shown that further classes become equal by using the same technique as above. For example, to show that $L_{n,n-3} = L_{n,n}$ for sufficiently large n, we use the fact that if an n-by-n matrix A has more than $8n-20$ zero entries in off-diagonal positions, then there must be three zero entries which share no common indices. It follows that if $A \in L_{n,n-3} \backslash L_{n,n}$, then A has less than $8n-19$ zero entries. By lemma 4.1 an $(n-4)$-by-$(n-4)$ submatrix of A would have at least $(n-6)(n-5)/2$ zero entries if it were sign nonsingular. Thus, by theorem 4.4, if $(n-6)(n-5)/2 > 8n-20$, $\det A = 0$ and $L_{n,n-3} \backslash L_{n,n} = \emptyset$. Similarly, if $(n-7)(n-6)/2 > 12n-42$, then $L_{n,n-4} \backslash L_{n,n} = \emptyset$. In general, if $(n-k-3)(n-k-2)/2 > 4(k-1)n-(2k-1)(2k-2)$, then $L_{n,n-k} \backslash L_{n,n} = \emptyset$. Or, more explicitly, if $2n > -3 + 10k + \sqrt{64k^2 - 32k - 31}$, then $L_{n,n-k} \backslash L_{n,n} = \emptyset$.

We next investigate some properties of inverses. Observe that $L_{2,2} = L_{2,2}^{-1}$, but this is not in general true for sign nonsingular matrices of larger size. In fact, for $n \geq 3$, some entries in the inverse of an n-by-n sign nonsingular matrix may not even have a determined sign. (Matrices for which every entry in the inverse has a determined sign have been studied, see for example [T2].) However, we are able to prove an interesting fact about inverses of sign nonsingular matrices, namely that they are contained in $L'_{n,2}$. We begin this with a theorem that characterizes $L'_{n,2}$ in terms of the *relative gain array* $A \circ A^{-1T}$ [JS]. Recall that a nonnegative matrix is *doubly stochastic* if each row and column sum is 1. For $A \in M_n(\mathbb{R})$ nonsingular, let $A^{-1} = [\alpha_{ij}]$. The i-th row sum of $A \circ A^{-1T} = \sum_{j=1}^{n} a_{ij}\alpha_{ji} = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A[\{i\}^c,\{j\}^c]/\det A = 1$; similarly, each column sum is also 1. Thus, $A \circ A^{-1T}$ has all row and column sums 1 and is doubly stochastic if and only if it is nonnegative.

Theorem 4.9

The matrix $A \in L'_{n,2}$ if and only if $A \in M_n(\mathbb{R})$ is nonsingular and $A \circ A^{-1T}$ is doubly stochastic.

Proof. Let $R = [r_{ij}] \in R'_{n,2}$ have all rows, other than row i, equal to e^T. Expansion along row i then gives

$$\det(A \circ R) = \sum_{j=1}^{n} r_{ij}a_{ij}(-1)^{i+j} \det A[\{i\}^c,\{j\}^c].$$

We have $A \in L'_{n,2}$ if and only if this sum is nonzero for all positive vectors $r_i = (r_{i1}, r_{i2}, \ldots, r_{in}) > 0$ if and only if the sum has the same sign as $\det A$ for all $r_i > 0$. This is equivalent to the i-th row of $A \circ A^{-1T}$ being component-wise nonnegative. (We know that it is nonzero.) Since $1 \leq i \leq n$ was arbitrary, we conclude the assertion of the theorem. ∎

1. For a given p, what is the minimum value of n, such that $L_{n,n-p} = L_{n,n}$? For $p = 1$, we have proved that $n = 5$, and for $p = 2$ we know $n \in [6,15]$.

There are some open questions relating the intermediate classes of sequences.

2. Given a fixed n, for what values of $k \in [2,n-1]$ is $L_{n,k} = L'_{n,k}$? We have proved that for $n \leq 3$ there is equality for all k; but $L_{4,2} \neq L'_{4,2}$.

3. Is there an $n > 4$ and a $k \in [3,n-1]$ such that $L_{n,k} \neq B_{n,k}$? We have proved that for $n \leq 4$ there is equality for all k; also we have proved $L_{n,2} = B_{n,2}$ (theorem 3.5).

Related sequences of classes may be defined. For example, let

$$R'''_{n,k} = \{R \in M_n(\mathbb{R}) : R \text{ is the sum of } k \text{ positive rank } 1 \text{ matrices}\}$$

and

$$L'''_{n,k} = \{A \in M_n(\mathbb{R}) : \det (A \circ R) \neq 0 \text{ for all } R \in R'''_{n,k}\}.$$

We have not investigated the classes $L'''_{n,k}$ here because it was not necessary for our understanding of $L_{n,k}$.

4. What analogous results are true for the classes $L'''_{n,k}$?

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COROLLARY 4.10

(i) For a given $n \geq 2$, $L'^{-1}_{n,2} = L'_{n,2}$

(ii) For a given $n \geq 2$, $L^{-1}_{n,n} \subseteq L'_{n,2}$

(iii) $L^{-1}_{3,3} \subseteq L_{3,2}$

(iv) $L^{-1}_{3,2} = L_{3,2}$.

Proof. (i) By theorem 4.9, $A \in L'_{n,2}$ if and only if $A \circ A^{-1T}$ is doubly stochastic. But this implies that $(A \circ A^{-1T})^T = A^{-1} \circ A^T$ is doubly stochastic, which is equivalent to $A^{-1} \in L'_{n,2}$.

(ii) If $A^{-1} \in L_{n,n} = L'_{n,n}$, then $A^{-1} \in L'_{n,2}$, and so $A \in L'_{n,2}$.

(iii) Combining (ii) with theorem 3.4, we obtain the result that the inverse of a 3-by-3 sign nonsingular matrix belongs to $L_{3,2}$.

(iv) Combining (i) with theorem 3.4 gives (iv), thus the class $L_{3,2}$ is closed under inversion. ∎

Theorem 4.9 also shows that if A is an n-by-n orthogonal matrix, then $A \in L'_{n,2}$, since for A orthogonal, $A^{-1T} = A$ and (for any matrix) $A \circ A$ is nonnegative. We remark that our proof that $L'_{n,2}$ is closed under inversion (corollary 4.10(i)) relies on the characterization in theorem 4.9.

5. OPEN QUESTIONS

Several open problems have already been posed; we collect these and others here. As noted in section 4, for $n \geq 6$, there are some inclusions in the sequences left uncertain as to equality.