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Abstract: The main purpose of this article is to find the upper bound of the third Hankel determinant for a family of q-starlike functions which are associated with the Ruscheweyh-type q-derivative operator. The work is motivated by several special cases and consequences of our main results, which are pointed out herein.

Keywords: analytic functions; Hadamard product; starlike functions; q-derivative (or q-difference) operator; Hankel determinant; q-starlike functions

MSC: Primary 05A30, 30C45; Secondary 11B65, 47B38

1. Introduction

We denote by $\mathcal{A}(\mathbb{U})$ the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

where $\mathbb{C}$ is the complex plane. Let $\mathcal{A}$ be the class of analytic functions having the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\forall\, z \in \mathbb{U}) \tag{1}$$

in the open unit disk $\mathbb{U}$, centered at the origin and normalized by the conditions given by

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

In addition, let $\mathcal{S} \subset \mathcal{A}$ be the class of functions which are univalent in $\mathbb{U}$. The class of starlike functions in $\mathbb{U}$ will be denoted by $\mathcal{S}^*$, which consists of normalized functions $f \in \mathcal{A}$ that satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad (\forall\, z \in \mathbb{U}). \tag{2}$$

If two functions f and g are analytic in $\mathbb{U}$, we say that the function f is subordinate to g and write in the form:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w which is analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$f(z) = g\big(w(z)\big).$$

In particular, if the function g is univalent in $\mathbb{U}$, then it follows that (cf., e.g., [1]; see also [2])

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Moreover, for two analytic functions f and g given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\forall\, z \in \mathbb{U})$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (\forall\, z \in \mathbb{U}),$$

the convolution (or the Hadamard product) of f and g is defined as follows:

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

We next denote by $\mathcal{P}$ the class of analytic functions p which are normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{3}$$

such that

$$\Re\big(p(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

We now recall some essential definitions and concept details of the basic or quantum (q-) calculus, which are used in this paper. We suppose throughout the paper that $0 < q < 1$ and that

$$\mathbb{N} = \{1, 2, 3, \cdots\} = \mathbb{N}_0 \setminus \{0\} \qquad (\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}).$$

Definition 1. *Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by*

$$[\lambda]_q = \begin{cases} \dfrac{1-q^{\lambda}}{1-q} & (\lambda \in \mathbb{C}) \\[2ex] \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 2. *Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by*

$$[n]_q! = \begin{cases} 1 & (n=0) \\ \prod\limits_{k=1}^{n-1} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 3. *Let $q \in (0,1)$ and define the generalized q-Pochhammer symbol $[\lambda]_{q,n}$ by*

$$[\lambda]_{q,n} = \begin{cases} 1 & (n=0) \\ \prod\limits_{k=0}^{n} [\lambda+k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 4. *For $\omega > 0$, let the q-gamma function $\Gamma_q(\omega)$ be defined by*

$$\Gamma_q(\omega+1) = [\omega]_q \Gamma_q(\omega) \quad \textit{and} \quad \Gamma_q(1) := 1.$$

Definition 5. (see [3,4]) *The q-derivative (or the q-difference) operator D_q of a function f in a given subset of $\mathbb{C}$ is defined by*

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z=0), \end{cases} \tag{4}$$

provided that $f'(0)$ exists.

We note from Definition 5 that

$$\lim_{q \to 1-} (D_q f)(z) = \lim_{q \to 1-} \frac{f(qz) - f(z)}{(1-q)z} = f'(z),$$

for a differentiable function f in a given subset of $\mathbb{C}$. It is readily deduced from (1) and (4) that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{5}$$

The operator D_q plays a vital role in the investigation and study of numerous subclasses of the class of analytic functions of the form given in Definition 5. A q-extension of the class of starlike functions was first introduced in [5] by using the q-derivative operator (see Definition 6 below). A background of the usage of the q-calculus in the context of Geometric Funciton Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory by Srivastava (see, for details, [6]). Some recent investigations associated with the q-derivative operator D_q in analytic function theory can be found in [7–13] and the references cited therein.

Definition 6. (see [5]) *A function $f \in \mathcal{A}(\mathbb{U})$ is said to belong to the class $\mathcal{S}_q^*$ if*

$$f(0) = f'(0) - 1 = 0 \tag{6}$$

and

$$\left| \frac{z}{f(z)} (D_q f)(z) - \frac{1}{1-q} \right| \leqq \frac{1}{1-q} \qquad (\forall\, z \in \mathbb{U}). \tag{7}$$

The notation $\mathcal{S}_q^*$ was first used by Sahoo et al. (see [14]).

It is readily observed that, as $q \to 1-$, the closed disk given

$$\left| w - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}$$

becomes the right-half plane and the class $\mathcal{S}_q^*$ reduces to $\mathcal{S}^*$. Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (6) and (7) as follows (see [15]):

$$\frac{z}{f(z)} \left(D_q f \right)(z) \prec \widehat{p} \qquad \left(\widehat{p} = \frac{1+z}{1-qz} \right).$$

Definition 7. (see [16]) *For a function $f \in \mathcal{A}(\mathbb{U})$, the Ruscheweyh-type q-derivative operator is defined as follows:*

$$\mathcal{R}_q^\delta f(z) = \phi(q, \delta+1; z) * f(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} a_n z^n \quad (z \in \mathbb{U};\ \delta > -1), \tag{8}$$

where

$$\phi(q, \delta+1; z) = z + \sum_{n=2}^{\infty} \psi_{n-1} z^n \tag{9}$$

and

$$\psi_{n-1} = \frac{\Gamma_q(\delta+n)}{[n-1]_q! \Gamma_q(\delta+1)} = \frac{[n+1]_{n-1,q}}{[n-1]_q!}. \tag{10}$$

From (8) it can be seen that

$$\mathcal{R}_q^0 f(z) = f(z) \quad \text{and} \quad \mathcal{R}_q^1 f(z) = z D_q f(z),$$

$$\mathcal{R}_q^m f(z) = \frac{z D_q^m f(z) \left(z^{m-1} f(z) \right)}{[m]_q!} \quad (m \in \mathbb{N}),$$

$$\lim_{q \to 1-} \phi(q, \delta+1; z) = \frac{z}{(1-z)^{\delta+1}}$$

and

$$\lim_{q \to 1-} \mathcal{R}_q^\delta f(z) = f(z) * \frac{z}{(1-z)^{\delta+1}}.$$

This shows that, in case of $q \to 1-$, the Ruscheweyh-type q-derivative operator reduces to the Ruscheweyh derivative operator $D^\delta f(z)$ (see [17]). From (8) the following identity can easily be derived:

$$z D_q \mathcal{R}_q^\delta f(z) = \left(1 + \frac{[\delta]_q}{q^\delta} \right) \mathcal{R}_q^{\delta+1} f(z) - \frac{[\delta]_q}{q^\delta} \mathcal{R}_q^\delta f(z). \tag{11}$$

If $q \to 1-$, then

$$z \left(\mathcal{R}^\delta f(z) \right)' = (1+\delta) \mathcal{R}^{\delta+1} f(z) - \delta \mathcal{R}^\delta f(z).$$

Now, by using the Ruscheweyh-type q-derivative operator, we define the following class of q-starlike functions.

Definition 8. *For $f \in \mathcal{A}(\mathbb{U})$, we say that f belongs to the class $\mathcal{RS}_q^*(\delta)$ if the following inequality holds true:*

$$\left| \frac{z D_q \mathcal{R}_q^\delta f(z)}{f(z)} - \frac{1}{1-q} \right| \leqq \frac{1}{1-q} \qquad (z \in \mathbb{U};\ \delta > -1)$$

or, equivalently, we have (see [15])

$$\frac{zD_q\mathcal{R}_q^\delta f(z)}{f(z)} \prec \frac{1+z}{1-qz} \tag{12}$$

by using the principle of subordination.

Let $n \geqq 0$ and $j \geqq 1$. The jth Hankel determinant is defined as follows:

$$\mathcal{H}_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+j-1} \\ a_{n+1} & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n+j-1} & \cdot & \cdot & \cdot & \cdot & a_{n+2(j-1)} \end{vmatrix}$$

The above Hankel determinant has been studied by several authors. In particular, sharp upper bounds on $\mathcal{H}_2(2)$ were obtained by several authors (see, for example, [18–21]) for various classes of normalized analytic functions. It is well-known that the Fekete-Szegö functional $|a_3 - a_2^2| = \mathcal{H}_2(1)$. This functional is further generalized as $|a_3 - \mu a_2^2|$ for some real or complex μ. In fact, Fekete and Szegö gave sharp estimates of $|a_3 - \mu a_2^2|$ for real μ and $f \in \mathcal{S}$, the class of normalized univalent functions in $\mathbb{U}$. It is also known that the functional $|a_2a_4 - a_3^2|$ is equivalent to $\mathcal{H}_2(2)$. Babalola [22] studied the Hankel determinant $\mathcal{H}_3(1)$ for some subclasses of analytic functions. In the present investigation, our focus is on the Hankel determinant $\mathcal{H}_3(1)$ for the above-defined function class $\mathcal{RS}_q^*(\delta)$.

2. A Set of Lemmas

Each of the following lemmas will be needed in our present investigation.

Lemma 1. (see [23]) *Let*

$$p(z) = 1 + c_1z + c_2z^2 + \cdots$$

be in the class $\mathcal{P}$ of functions with positive real part in $\mathbb{U}$. Then, for any complex number v,

$$|c_2 - vc_1^2| \leqq \begin{cases} -4v+2 & (v \leqq 0) \\ 2 & (0 \leqq v \leqq 1) \\ 4v-2 & (v \geqq 1). \end{cases} \tag{13}$$

When $v < 0$ or $v > 1$, the equality holds true in (13) if and only if

$$p(z) = \frac{1+z}{1-z}$$

or one of its rotations. If $0 < v < 1$, then the equality holds true in (13) if and only if

$$p(z) = \frac{1+z^2}{1-z^2}$$

or one of its rotations. If $v = 0$, the equality holds true in (13) if and only if

$$p(z) = \left(\frac{1+\rho}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\rho}{2}\right)\frac{1-z}{1+z} \qquad (0 \leqq \rho \leqq 1)$$

or one of its rotations. If $v = 1$, then the equality in (13) holds true if $p(z)$ is a reciprocal of one of the functions such that the equality holds true in the case when $v = 0$.

Lemma 2. (see [24,25]) *Let*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

be in the class $\mathcal{P}$ of functions with positive real part in $\mathbb{U}$. Then

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right)$$

for some x, $|x| \leqq 1$ and

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z$$

for some z $(|z| \leqq 1)$.

Lemma 3. (see [26]) *Let*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

be in the class $\mathcal{P}$ of functions positive real part in $\mathbb{U}$. Then

$$|p_k| \leqq 2 \qquad (k \in \mathbb{N})$$

and the inequality is sharp.

3. Main Results

In this section, we will prove our main results. Throughout our discussion, we assume that

$$q \in (0,1) \quad \text{and} \quad \delta > -1.$$

Our first main result is stated as follows.

Theorem 1. *Let $f \in \mathcal{RS}_q^*(\delta)$ be of the form (1). Then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \dfrac{\left(1+q+q^2\right)\psi_1^2 - \mu\left(1+q\right)^2\psi_2}{q^2\psi_2\psi_1^2} & \left(\mu < \dfrac{\left(q^2+1\right)\psi_1^2}{\left(1+q\right)^2\psi_2}\right) \\[3ex] \dfrac{1}{q\psi_2} & \left(\dfrac{\left(q^2+1\right)\psi_1^2}{\left(1+q\right)^2\psi_2} \leqq \mu \leqq \dfrac{\psi_1^2}{\psi_2}\right) \\[3ex] \dfrac{\mu\left(1+q\right)^2\psi_2 - \left(1+q+q^2\right)\psi_1^2}{q^2\psi_2\psi_1^2} & \left(\mu > \dfrac{\psi_1^2}{\psi_2}\right), \end{cases}$$

where ψ_{n-1} is given by (10).

It is also asserted that, for

$$\frac{\left(q^2+1\right)\psi_1^2}{\left(1+q\right)^2\psi_2} \leqq \mu \leqq \frac{\left(1+q+q^2\right)\psi_1^2}{\left(1+q\right)^2\psi_2},$$

$$|a_3 - \mu a_2^2| + \left(\mu - \frac{\left(q^2+1\right)\psi_1^2}{\left(1+q\right)^2\psi_2}\right)|a_2|^2 \leq \frac{1}{q\psi_2}$$

and that , for

$$\frac{\left(1+q+q^2\right)\psi_1^2}{\left(1+q\right)^2\psi_2} \leqq \mu \leqq \frac{\psi_1^2}{\psi_2},$$

$$|a_3-\mu a_2^2|+\left(\frac{\psi_1^2-\mu\psi_2}{\psi_2}\right)|a_2|^2\leqq\frac{1}{q\psi_2}.$$

Proof. If $f\in\mathcal{RS}_q^*(\delta)$, then it follows from (12) that

$$\frac{zD_q\mathcal{R}_q^\delta f(z)}{f(z)}\prec\phi(z), \tag{14}$$

where

$$\phi(z)=\frac{1+z}{1-qz}.$$

We define a function $p(z)$ by

$$p(z)=\frac{1+w(z)}{1-w(z)}=1+p_1z+p_2z^2+p_3z^3+\cdots.$$

It is clear that $p\in\mathcal{P}$. From the above equation, we have

$$w(z)=\frac{p(z)-1}{p(z)+1}.$$

From (14), we find that

$$\frac{zD_q\mathcal{R}_q^\delta f(z)}{f(z)}=\phi(w(z)),$$

together with

$$\phi(w(z))=\frac{2p(z)}{(1-q)p(z)+1+q}.$$

Now

$$\begin{aligned}
&\frac{2p(z)}{(1-q)p(z)+1+q}\\
&\quad=1+\frac{1}{2}(1+q)p_1z+\left\{\frac{1}{2}(q+1)p_2-\frac{1}{4}(1-q^2)p_1^2\right\}z^2\\
&\qquad+\left\{\frac{1}{2}(1+q)p_3-\frac{1}{2}(1-q^2)p_1p_2+\frac{1}{8}(1+q)(1-q)^2p_1^3\right\}z^3\\
&\qquad+\left\{\frac{1}{2}(1+q)p_4=\frac{1}{4}\left(1-q^2\right)p_2^2-\frac{1}{2}\left(1-q^2\right)p_1p_3\right.\\
&\qquad\left.+\frac{3}{8}(1+q)(q-1)^2p_1^2p_2+\frac{1}{16}(1+q)(1-q)^3p_1^4\right\}z^4+\cdots.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\frac{zD_q\mathcal{R}_q^\delta f(z)}{\mathcal{R}_q^\delta f(z)}={}&1+qa_2\psi_1z+\left\{\left(q+q^2\right)\psi_2a_3-q\psi_1^2a_2^2\right\}z^2+\left\{\left(q+q^2+q^3\right)\psi_3a_4\right.\\
&\left.-\left(2q+q^2\right)\psi_1\psi_2a_2a_3+q\psi_1^3a_2^3\right\}z^3+\left\{\left(q+q^2+q^3+q^4\right)\psi_5a_5\right.\\
&-\left(2q+q^2+q^3\right)\psi_2\psi_3a_2a_4-\left(q+q^2\right)\psi_2^2a_3^2\\
&\left.+\left(3q+q^2\right)\psi_1^2\psi_2a_2^2a_3-q\psi_1^4a_2^4\right\}z^4+\cdots,
\end{aligned}$$

Therefore, we have

$$a_2 = \frac{(1+q)}{2q\psi_1} p_1, \tag{15}$$

$$a_3 = \frac{1}{2q\psi_2} p_2 + \frac{(q^2+1)}{4q^2\psi_2} p_1^2 \tag{16}$$

and

$$\begin{aligned} a_4 &= \frac{(1+q)}{2q(1+q+q^2)\psi_3} p_3 - \frac{(1+q)(q-2)(2q+1)}{4q^2(1+q+q^2)\psi_3} p_1 p_2 \\ &+ \frac{(1+q)(q^2+1)(q^2-q+1)}{8q^3(1+q+q^2)\psi_3} p_1^3. \end{aligned} \tag{17}$$

We thus obtain

$$\left|a_3 - \mu a_2^2\right| = \frac{1}{2q\psi_2} \left| p_2 - \left(\frac{\mu (1+q)^2 \psi_2 - (1+q^2)\psi_1^2}{2q\psi_1^2} \right) p_1^2 \right|. \tag{18}$$

Finally, by applying Lemma 1 and Equation (13) in conjunction with (18), we obtain the result asserted by Theorem 1. □

We now state and prove Theorem 2 below.

Theorem 2. *Let $f \in \mathcal{RS}_q^*(\delta)$ be of the form (1). Then*

$$\left|a_2 a_4 - a_3^2\right| \leqq \frac{1}{q^2\psi_2^2}.$$

Proof. From (15)–(17), we obtain

$$\begin{aligned} a_2 a_4 - a_3^2 = & \left(\frac{(1+q)^2}{4q^2(1+q+q^2)\psi_1\psi_3} \right) p_1 p_3 - \left(\frac{(1+q)^2(q-2)(2q+1)}{8q^3(1+q+q^2)\psi_1\psi_3} + \frac{(q^2+1)}{4q^3\psi_2^2} \right) p_1^2 p \\ & - \left(\frac{1}{4q^2\psi_2^2} \right) p_2^2 + \left(-\frac{(q^2+1)^2}{16q^4\psi_2^2} + \frac{(1+q)^2(q^2+1)(q^2-q+1)}{16q^3(1+q+q^2)\psi_1\psi_3} \right) p_1^4. \end{aligned}$$

By using Lemma 2, we have

$$\begin{aligned} a_2 a_4 - a_3^2 = & \left(\frac{(1+q)^2(q^2+1)(q^2-q+1)}{16q^3(1+q+q^2)\psi_1\psi_3} - \frac{(q^2+1)^2}{16q^4\psi_2^2} \right) p_1^4 \\ & + \left(\frac{(1+q)^2}{16q^2(1+q+q^2)\psi_1\psi_3} \right) p_1 \left\{ p_1^3 + 2p_1\left(4-p_1^2\right)x \right. \\ & \left. - p_1\left(4-p_1^2\right)x^2 + 2\left(4-p_1^2\right)\left(1-|x|^2\right)z \right\} + \left(\frac{(q^2+1)}{8q^3\psi_2^2} \right. \\ & \left. \cdot \frac{(1+q)^2(q-2)(2q+1)}{16q^3(1+q+q^2)\psi_1\psi_3} \right) p_1^2 \left\{ \left(p_1^2 + \left(4-p_1^2\right)x \right) \right\} \\ & - \left(\frac{1}{16q^2\psi_2^2} \right) \left\{ p_1^4 + \left(4-p_1^2\right)^2 x^2 + 2p_1^2\left(4-p_1^2\right)x \right\}. \end{aligned}$$

Now, taking the moduli and replacing $|x|$ by ρ and p_1 by p, we have

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &\leqq \frac{1}{\Lambda(q)}\Big[\omega(q)\,p^4+2q(1+q)^2\psi_2^2p\left(4-p^2\right)\\&\quad+\Omega(q)\left(4-p^2\right)p^2\rho+\Big(q(q+1)^2\psi_2^2p^2+q\left(4-p^2\right)\\&\quad\cdot\left(1+q+q^2\right)\psi_1\psi_3-2q(1+q)^2\psi_2^2p\Big)\left(4-p^2\right)\rho^2\Big]\\&=F(p,\rho),\end{aligned} \tag{19}$$

where

$$\Lambda(q)=16q^3\left(1+q+q^2\right)\psi_1\psi_3\psi_2^2,$$

$$\begin{aligned}\omega(q)=\Big|&\left(3+3q-q^3+q^4\right)(1+q)^2\psi_2^2-\left(1+3q+2q^2+2q^3+q^4\right)\\&\cdot\left(1+q+q^2\right)\psi_1\psi_3\Big|\end{aligned}$$

and

$$\Omega(q)=\left|(1+q)^2\left(2q^2-5q-2\right)\psi_2^2+2q\left(q^2+2\right)\left(1+q+q^2\right)\psi_1\psi_3\right|.$$

Upon differentiating both sides (19) with respect to ρ, we have

$$\begin{aligned}\frac{\partial F(p,\rho)}{\partial\rho}=\left(\frac{1}{\Lambda(q)}\right)\Big[&\Omega(q)\left(4-p^2\right)p^2+2\Big(q(q+1)^2\psi_2^2p^2+q\left(4-p^2\right)\\&\cdot\left(1+q+q^2\right)\psi_1\psi_3-2q(1+q)^2\psi_2^2p\Big)\left(4-p^2\right)\rho\Big].\end{aligned}$$

It is clear that

$$\frac{\partial F(p,\rho)}{\partial\rho}>0,$$

which show that $F(p,\rho)$ is an increasing function of ρ on the closed interval $[0,1]$. This implies that the maximum value occurs at $\rho=1$. This implies that

$$\max\{F(p,\rho)\}=F(p,1)=:G(p).$$

We now observe that

$$\begin{aligned}G(p)&=\left(\frac{1}{\Lambda(q)}\right)\Big[\Big(\omega(q)-\Omega(q)-q(q+1)^2\psi_2^2+\left(q+q^2+q^3\right)\psi_1\psi_3\Big)p^4\\&\quad+\Big(4\Omega(q)+4q(q+1)^2\psi_2^2-8\left(q+q^2+q^3\right)\psi_1\psi_3\Big)p^2\\&\quad+16\left(q+q^2+q^3\right)\psi_1\psi_3\\&=G(p).\end{aligned} \tag{20}$$

By differentiating both sides of (20) with respect to p, we have

$$\begin{aligned}G'(p)=\left(\frac{1}{\Lambda(q)}\right)\Big[&4\Big(\omega(q)-\Omega(q)-q(q+1)^2\psi_2^2+\left(q+q^2+q^3\right)\psi_1\psi_3\Big)p^3\\&+2\Big(4\Omega(q)+4q(q+1)^2\psi_2^2-8\left(q+q^2+q^3\right)\psi_1\psi_3\Big)p\Big].\end{aligned}$$

Differentiating the above equation once again with respect to p, we get

$$G''(p) = \left(\frac{1}{\Lambda(q)}\right)\left[12\left(\omega(q) - \Omega(q) - q(q+1)^2\psi_2^2 + \left(q+q^2+q^3\right)\psi_1\psi_3\right)p^2 \right.$$
$$\left. +2\left(4\Omega(q) + 4q(q+1)^2\psi_2^2 - 8\left(q+q^2+q^3\right)\psi_1\psi_3\right)\right].$$

For $p = 0$, this shows that the maximum value of $(G(p))$ occurs at $p = 0$. Hence, we obtain

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{1}{q^2\psi_2^2}.$$

The proof of Theorem 2 is thus completed. □

If, in Theorem 2, we let $q \longrightarrow 1-$ and put $\delta = 1$, then we are led to the following known result.

Corollary 1. (see [18]) *Let $f \in \mathcal{S}^*$. Then*

$$\left|a_2a_4 - a_3^2\right| \leqq 1,$$

and the inequality is sharp.

Theorem 3. *Let $f \in \mathcal{RS}_q^*(\delta)$. Then*

$$|a_2a_3 - a_4| \leqq \frac{(1+q)\kappa(q)}{\psi_1\psi_2\psi_3(q^2+q^3+q^4)},$$

where

$$\kappa(q) = \left|\left(1+q+q^2\right)^2\psi_3 - \left(q^4 - 3q + 6q^2 + q + 1\right)\psi_1\psi_2\right|. \tag{21}$$

Proof. Using the values given in (15) and (16) we have

$$\begin{aligned} a_2a_3 - a_4 &= \left(\frac{(1+q)(q^2+1)}{8q^3\psi_1\psi_2} - \frac{(1+q)(q^2+1)(q^2-q+1)}{8\psi_3(q^2+q^3+q^4)}\right)p_1^3 \\ &+ \left(\frac{(1+q)}{4q^2\psi_1\psi_2} - \frac{(q-2)(2q+1)(1+q)}{4\psi_3(q^2+q^3+q^4)}\right)p_1p_2 \\ &- \left(\frac{(1+q)}{2(q+q^2+q^3)\psi_3}\right)p_3. \end{aligned} \tag{22}$$

We now use Lemma 2 and assume that $p_1 \leqq 2$. In addition, by Lemma 3, we let $p_1 = p$ and assume without restriction that $p \in [0,2]$. Then, by taking the moduli and applying the trigonometric inequality on (22) with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| \leqq &\left(\frac{(1+q)}{8(q^3+q^4+q^5)\psi_1\psi_2\psi_3}\right)\left[\kappa(q)p^3 + \eta(q)p(4-p^2)\rho\right. \\ &\left.+2q^2\psi_1\psi_2(4-p^2) + q^2\psi_1\psi_2(p-2)(4-p^2)\rho^2\right] \\ =: &F(\rho), \end{aligned}$$

where

$$\eta(q) = \left|\left(q+q^2+q^3\right)\psi_3 + \left(2q^3 - q^2 - 2q\right)\psi_1\psi_2\right|$$

and $\kappa(q)$ is given by (21). Differentiating $F(\rho)$ with respect to ρ, we have

$$F'(\rho) = \left(\frac{(1+q)}{8(q^3+q^4+q^5)\psi_1\psi_2\psi_3}\right)\left[\eta(q)\,p(4-p^2) + 2q^2\psi_1\psi_2(p-2)(4-p^2)\rho\right] > 0.$$

This implies that $F(\rho)$ is an increasing function of ρ on the closed interval $[0,1]$. Hence, we have

$$F(\rho) \leqq F(1) \qquad (\forall\, \rho \in [0,1]),$$

that is,

$$\begin{aligned} F(\rho) \leqq & \left(\frac{(1+q)}{8(q^3+q^4+q^5)\psi_1\psi_2\psi_3}\right)\left[\left(\kappa(q) - \eta(q) - q^2\psi_1\psi_2\right)p^3 \right. \\ & \left. + \left(4\eta(q) + 4q^2\psi_1\psi_2\right)p\right] \\ =: & \; G(p). \end{aligned}$$

Since $p \in [0,2]$, $p = 2$ is a point of maximum. We thus obtain

$$G(p) \leqq \frac{(1+q)\,\kappa(q)}{(q^3+q^4+q^5)\,\psi_1\psi_2\psi_3},$$

which corresponds to $\rho = 1$ and $p = 2$ and it is the desired upper bound. □

For $\delta = 1$ and $q \to 1-$, we obtain the following special case of Theorem 3.

Corollary 2. (see [22]) *Let $f \in S^*$. Then*

$$|a_2a_3 - a_4| \leqq 2.$$

Finally, we prove Theorem 4 below.

Theorem 4. *Let $f \in \mathcal{RS}_q^*(\delta)$. Then*

$$\mathcal{H}_3(1) \leqq \left[\frac{(1+q+q^2)}{q^4\psi_2^3} + \frac{\varkappa(q)\,\kappa(q)}{q^5(1+q+q^2)^2\psi_1\psi_2\psi_3^2} + \frac{\tau(q)}{q^5(1+q+q^2+q^3)(1+q+q^2)\psi_2\psi_4}\right],$$

where

$$\varkappa(q) = (1+q)^2\left(q^4 - 3q^3 + 6q^2 + q + 1\right), \tag{23}$$

$$\tau(q) = (1+q)\left(4q^7 + 2q^6 + 6q^5 + 7q^4 + 13q^3 - q - 1\right) \tag{24}$$

and $\kappa(q)$ is given by (21).

Proof. Since

$$\mathcal{H}_3(1) \leqq |a_3|\left|a_2a_4 - a_3^2\right| + |a_4|\,|a_2a_3 - a_4| + |a_5|\left|a_3 - a_2^2\right|,$$

by using Lemma 3, we have

$$|a_4| \leqq \frac{(1+q)(1+q+6q^2-3q^3+q^4)}{q^3(1+q+q^2)\,\psi_3}$$

and

$$|a_5| \leqq \frac{\tau(q)}{q^4(1+q+q^2+q^3)(1+q+q^2)\,\psi_4},$$

where $\tau(q)$ is given by (24). Now, by applying Theorems 1–3, we have the required result asserted by Theorem 4. □

4. Conclusions

By making use of the basic or quantum (q-) calculus, we have introduced a Ruscheweyh-type q-derivative operator. This Ruscheweyh-type q-derivative operator is then applied to define a certain subclass of q-starlike functions in the open unit disk $\mathbb{U}$. We have successfully derived the upper bound of the third Hankel determinant for this family of q-starlike functions which are associated with the Ruscheweyh-type q-derivative operator. Our main results are stated and proved as Theorems 1–4. These general results are motivated essentially by their several special cases and consequences, some of which are pointed out in this presentation.

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