The Orbit Operator and Invariant Subspaces

by

Robin Deeley
Bachelor of Mathematics, University of Waterloo, 2004

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

The Orbit Operator and Invariant Subspaces

by

Robin Deeley
BMath, University of Waterloo, 2004

Supervisory Committee

Dr. Ahmed R. Sourour (Department of Mathematics and Statistics)
Supervisor

Dr. Heath Emerson (Department of Mathematics and Statistics)
Committee Member

Dr. Ian F. Putnam (Department of Mathematics and Statistics)
Committee Member

Supervisory Committee

Dr. A. R. Sourour (Department of Mathematics and Statistics)

Supervisor

Dr. H. Emerson (Department of Mathematics and Statistics)

Committee Member

Dr. I. F. Putnam (Department of Mathematics and Statistics)

Committee Member

Abstract

The invariant subspace problem is the long-standing question whether every operator on a Hilbert space of dimension greater than one has a non-trivial invariant subspace. Although the problem is unsolved in the Hilbert space case, there are counter-examples for operators acting on certain well-known non-reflexive Banach spaces. These counter-examples are constructed by considering a single orbit and then extending continuously to a bounded linear map on the entire space. Based on this process, we introduce an operator which has properties closely linked with an orbit. We call this operator the orbit operator.

In the first part of the thesis, examples and basic properties of the orbit operator are discussed. Next, properties linking invariant subspaces to properties of the orbit operator are presented. Topics include the kernel and range of the orbit operator, compact operators, dilation theory, and Rota's theorem. Finally, we extend results obtained for strict contractions to contractions.

Contents

Acknowledgements

First I would like to thank my supervisor, Ahmed Sourour, for his assistance in both developing the content of and in the preparation of this thesis. In addition, I would like to thank the other members of my thesis committee, Heath Emerson and Ian Putnam, along with my external examiner, Dale Olesky. I would like to specifically thank Heath for answering many (if not all) of my "random questions". I gratefully acknowledge the financial support received through a Canadian Graduate Scholarship. My family and friends also deserve many thanks as they have always been there for me, both before and during my time as a Master's student. Most of all, I would like to thank Reyna for all her love and support during the writing of this thesis and all her proofreading as well.

Robin Deeley

Chapter 1

Background on Invariant Subspaces

The study of the invariant subspaces of operators acting on a Hilbert space (or more generally a Banach space) is a natural generalization of the same concept for operators acting on a finite dimensional vector space. At its most basic level, it is the theory of eigenvalues and eigenspaces for operators on finite dimensional vector spaces. On any finite dimensional vector space over the complex numbers, every operator has an eigenvalue and hence an eigenvector. This fact leads to the result that any operator on a finite dimensional complex vector space has a basis for which its matrix representation is upper triangular. In this section, we will introduce the basic properties of invariant subspaces using the finite dimensional case as a guide to results which we would like to hold for infinite dimensional Hilbert spaces. For other introductions to invariant subspaces, any of [3], [9] and [19] contain the basic facts.

Throughout, T will be an operator (bounded linear map) acting on $\mathcal{H}$ (a separable complex Hilbert space). We will denote the set of all operators acting on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. We will use $< .,. >$ to denote the inner product of $\mathcal{H}$. A *linear manifold* of $\mathcal{H}$ is a non-empty set which is closed under vector addition and scalar multiplication but may not be closed in the norm topology. A *subspace* of $\mathcal{H}$ will be a (norm) closed linear manifold of $\mathcal{H}$. The *orbit of* $x \in \mathcal{H}$ *under* $T \in \mathcal{B}(\mathcal{H})$ is the set $\{x, Tx, \ldots, T^n x, \ldots\}$. Given a set of vectors, $\mathcal{S}$, in $\mathcal{H}$, we will use *span*$(\mathcal{S})$ and $\overline{span}(\mathcal{S})$ to denote the linear manifold and subspace generated by $\mathcal{S}$ respectively. We will also denote the subspace generated by the orbit of x under T by $\mathcal{M}_x$. That is,

$$\mathcal{M}_x = \overline{span}\{x, Tx, T^2x, \ldots\}$$

Other notation we will use is $\sigma(T)$ for the *spectrum* of T (i.e. $\sigma(T) = \{\lambda \in \mathbb{C} | (\lambda I - T)$ is not invertible$\}$), $r(T)$ for the *spectral radius* of T (i.e. $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$) and $\sigma_p(T)$ for the eigenvalues of T. We will also let $\{T\}' = \{A \in \mathcal{B}(\mathcal{H}) | TA = AT\}$, the *commutant* of T.

Definition 1.1. A subspace $\mathcal{M}$ of $\mathcal{H}$ is an invariant subspace of the operator T if for each $x \in \mathcal{M}$, $Tx \in \mathcal{M}$. We will also refer to $\mathcal{M}$ as T-invariant or that $\mathcal{M}$ is invariant under T.

Definition 1.2. $\mathcal{M}$, an invariant subspace of T, is called reducing if $\mathcal{M}^\perp$ is also an invariant subspace of T.

Example 1.3. Both $\{0\}$ and $\mathcal{H}$ are invariant subspaces for any operator in $\mathcal{B}(\mathcal{H})$. Because of this property, they will be called trivial invariant subspaces. They are also both reducing invariant subspaces for any operator.

Example 1.4. If $x \neq 0$ is an eigenvector of T, that is $Tx = \lambda x$ (for some $\lambda \in \mathbb{C}$), then the one dimensional subspace $\mathcal{M} = \{ax : a \in \mathbb{C}\}$ is an invariant subspace of T. Moreover, unless $\mathcal{H}$ is one dimensional this subspace is non-trivial.

Definition 1.5. Given an operator, T, $Lat(T)$ will denote the collection of invariant subspaces of T.

$Lat(T)$ is closed under taking intersections (that is, if $\mathcal{M}_1$, $\mathcal{M}_2 \in Lat(T)$, then $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2 \in Lat(T)$) and taking spans. Thus, $Lat(T)$ is indeed a lattice of subspaces under these operations. Moreover, given an operator, T, we have that $Lat(T) = Lat(cT)$ for any non-zero $c \in \mathbb{C}$. This implies that we may assume that $\|T\| < 1$ when studying invariant subspaces.

As mentioned above, the study of eigenvalues and eigenvectors allows for nice (upper triangular for example) matrix representations of an operator acting on a finite dimensional complex vector space. More generally, invariant subspaces allow for this type of representation. Precisely, given any operator T and a subspace $\mathcal{M}$ (possibly not invariant), we can write T as a two by two matrix with operators $A : \mathcal{M} \to \mathcal{M}$, $B : \mathcal{M}^\perp \to \mathcal{M}$, $C : \mathcal{M} \to \mathcal{M}^\perp$ and

$D : \mathcal{M}^\perp \to \mathcal{M}^\perp$.

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Now suppose that $\mathcal{M}$ is an invariant subspace of T. Then, by definition, $T\mathcal{M} \subseteq \mathcal{M}$, so that, we have $C = 0$ in the above. That is

$$T = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

If $\mathcal{M}$ is a reducing subspace of T, then we have that $T\mathcal{M} \subseteq \mathcal{M}$ and $T\mathcal{M}^\perp \subseteq \mathcal{M}^\perp$, so that we have $B = 0$ along with $C = 0$. That is

$$T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

An invariant subspace, $\mathcal{M}$, of an operator is also linked with properties of the projection onto $\mathcal{M}$. Precisely, we have the follow result.

Proposition 1.6. *Given T an operator, $\mathcal{M} \subseteq \mathcal{H}$ and P, the projection onto $\mathcal{M}$, then $\mathcal{M}$ is an invariant subspace of T if and only if $TP = PTP$. Moreover, $\mathcal{M}$ is reducing if and only if $TP = PT$.*

Proof: The proof of this result follows from the observation that in the two by two matrix form from above we have

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$TP = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$$

and

$$PTP = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

The first statement now follows.

For the second result, consider

$$PT = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

This gives us the result and also leads to the fact that $\mathcal{M}$ is reducing for T if and only if $\mathcal{M}$ is an invariant subspace of both T and its adjoint (T^*). □

One of the main reasons for studying invariant subspaces is to produce decompositions of the above form. If $\mathcal{M}$ is the span of an eigenvector, then A is an operator on the one dimensional space given by the span of this vector. In general, an operator on an infinite dimensional Hilbert space may not have any eigenvalues or any non-trivial reducing subspaces (the unilaterial shift is an example of such an operator). However, it is unknown whether there exists an operator on an infinite dimensional Hilbert space which has no non-trivial invariant subspaces. This question is known as the invariant subspace problem and is one of the most fundamental problems in operator theory. When studying invariant subspaces, it is useful to consider the orbit of a point or a set of points under the action of an operator. Precisely, given an operator, T, and a set of vectors, $\mathcal{S} \subseteq \mathcal{H}$, we are interested in the smallest T-invariant subspace which contains $\mathcal{S}$. For example, if $\mathcal{S} = \{x\}$, we have that

$$\mathcal{M}_x = \overline{span}\{x, Tx, T^2x, \ldots\}$$

is T-invariant and is also the smallest such subspace (smallest in the sense that if $\mathcal{M}$ is T-invariant and $x \in \mathcal{M}$, then $\mathcal{M}_x \subseteq \mathcal{M}$).

Definition 1.7. $x \in \mathcal{H}$ is said to be a cyclic vector of T (or T-cyclic) if $\mathcal{M}_x$ is all of $\mathcal{H}$, that is if $\overline{span}\{x, Tx, T^2x, \ldots\} = \mathcal{H}$. We will call T a cyclic operator if there exists x which is cyclic under T.

For a general subset, $\mathcal{S}$, the smallest T-invariant subspace containing $\mathcal{S}$ is given by the closed span (taken over $x \in \mathcal{S}$) of the subspaces $\mathcal{M}_x$. In particular, if $\mathcal{M}$ is a non-trivial invariant subspace of T, then each $x \in \mathcal{M}$ is non-cyclic. This follows since $\mathcal{M}_x \subseteq \mathcal{M}$ and $\mathcal{M}$ is the closed span of the subspaces $\mathcal{M}_x$. From this fact, we have the following result.

Proposition 1.8. *T has no non-trivial invariant subspaces if and only if each non-zero $x \in \mathcal{H}$ is cyclic under T.*

Proof: If T has no non-trivial invariant subspaces, then for each $x \neq 0$, $\mathcal{M}_x \neq \{0\}$ and is invariant under T. Hence, $\mathcal{M}_x = \mathcal{H}$. The converse follows from the previous paragraph. □

A cyclic vector of an operator on a finite dimensional vector space leads to a nice matrix representation. Namely, if T is a cyclic operator, then there exists a basis such that the matrix of T with respect to this basis is of the below form.

$$\begin{bmatrix} 0 & 0 & 0 & \dots & a_0 \\ 1 & 0 & 0 & \dots & a_1 \\ 0 & 1 & 0 & \dots & a_2 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a_{k-1} \end{bmatrix}$$

To conclude this section, we give a brief discussion on some important results concerning the existence of invariant subspaces. Details on the results below can be found in [19]. As mentioned above, the problem of finding a non-trivial invariant subspace for an operator on a finite dimensional or non-separable space has been solved. In the separable case, a number of positive results have been discovered. Many of the successful methods used to prove the existence of a non-trivial invariant subspace involve using compactness arguments. One of the most notable is Lomonosov's Theorem which states that an operator that commutes with an operator that commutes with a compact operator has a non-trivial invariant subspace. This result clearly implies that any compact operator has a non-trivial invariant subspace.

A collection of operators acting on a finite dimensional space is triangularizable if there is a basis of the vector space such that each of the matrix representations of each operator in the collection is upper triangular with respect to that basis. In infinite dimensions, this can be generalized to the existence of a chain of subspaces, each of which is an invariant subspace for each operator in the collection. The textbook [18] has a detailed study of triangularization in both finite and infinite dimensional spaces.

Another successful method is to study normal-like operators. That normal operators have non-trivial invariant subspaces follows from the Spectral Theorem. Other classes of operators, related to normal operators, that have invariant subspaces are subnormal operators [5] and a large class of hyponormal operators [6]. Recall that an operator is subnormal if it is the restriction of a normal operator to an invariant subspace (e.g. the unilateral shift is the restriction of the bilateral shift and hence is subnormal). Recall also that an operator, T, is hyponormal if $\|T^*x\| \leq \|Tx\|$ for all x.

Another important result which does not rely on compactness arguments or relationships with normal operators is that if T is power bounded (i.e. there exists $M > 0$ such that $\|T^n\| \leq M$ for all $n = 0, 1, \dots$) and neither of $(T^n)_{n\geq 0}$ and $((T^*)^n)_{n\geq 0}$ converge in the strong operator topology to zero, then T has a non-trivial invariant subspace.

Chapter 2

Basic Properties

2.1 The Definition of $O_T^{e_i}(x)$

2.1.1 Background

The invariant subspace problem is the long-standing question whether every operator on a Hilbert space of dimension greater than one has a non-trivial invariant subspace. This question has been answered in the affirmative for the finite dimensional case and the non-separable case. In addition, the existence of non-trivial invariant subspaces has been proved for many classes of operators, such as compact and normal operators. However, for Banach spaces, counter-examples have been discovered. These counter-examples are constructed by considering a single orbit and then extending continuously to a bounded linear map on the entire space.

Based on this construction, it seems reasonable to attempt to give conditions on an orbit of a single vector which lead to such an operator. As a starting point, given a contraction (an operator with norm less than or equal to one), T, and a vector, x, in the space, we define a new operator closely linked with the orbit of x under T. We call this new operator the orbit operator of T at x. It has properties closely linked with the cyclic behaviour of x. Moreover, if we assume that T is a strict contraction, then the orbit operator will turn out to be trace class and hence easier to work with than an arbitrary operator. At first, this method appears to restate a hard problem about a single operator into a problem about an uncountable number of "nicer" (i.e. trace class) operators. In fact, we can gain information about the action of T on many vectors in the space from a single orbit and hence from a single trace class operator. The operator we will define is similar to one studied by Caradus in [7] (see the remarks following Definition 2.1).

Recall that we will let $\mathcal{H}$ denote an infinite separable complex Hilbert space and T a linear operator acting on it. Also, recall that T is said to be *cyclic* if there is $x \in \mathcal{H}$ such that $span\{T^n x\}_{n\geq 0}$ is dense in $\mathcal{H}$. An x with this property will be called a *cyclic vector* of T. If the orbit of x (i.e. the set $\{x, Tx, \ldots\}$) is itself dense, then T is said to be *hypercyclic* and x is called a *hypercyclic vector* of T. Between these two notions is the notion of *supercyclicity* for which there exists $x \in \mathcal{H}$ such that the set $\{\lambda T^n x : n \geq 0 \text{ and } \lambda \in \mathbb{C}\}$ is dense in $\mathcal{H}$.

It is a well known consequence of the Hahn-Banach theorem that a linear manifold $\mathcal{M}$ contained in $\mathcal{H}$ is dense if and only if the only linear functional in $\mathcal{H}^*$ which vanishes on $\mathcal{M}$ is the zero functional. Thus, a necessary and sufficient condition for $x \in \mathcal{H}$ to be cyclic for T is that

$$\mathcal{M}_x^{\perp} = \{y \in \mathcal{H}| < y, T^n x >= 0 \text{ for } n = 0, 1, \ldots\} = 0$$

As discussed earlier, cyclic vectors are closely linked with the invariant subspace problem, because T has no non-trivial invariant subspaces if and only if every nonzero $x \in \mathcal{H}$ is cyclic for T. This is equivalent to the assertion that $\mathcal{M}_x^{\perp} = 0$ for all nonzero $x \in \mathcal{H}$. We now introduce the operator mentioned in the introduction. The properties of this operator will be the main topic of this thesis.

Definition 2.1. Let $(e_i)_{i=0}^{\infty}$ be an orthonormal basis of $\mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$. Then let $D_x = \{y \in \mathcal{H}| \sum_n | < y, T^n x > |^2 < \infty\}$. We then define a map with domain D_x given by $O_T^{e_i}(x) : D_x \to \mathcal{H}$ mapping

$$y \longmapsto \sum_{i=0}^{\infty} < y, T^i x > e_i$$

We will call $O_T^{e_i}(x)$ the orbit operator of T at x.

Remarks on Definition 2.1

If we assume that $\|T\| < 1$, then for each $x \in \mathcal{H}$ the domain of $O_T^{e_i}(x)$ is all of $\mathcal{H}$. Then we can define $O_T^{e_i}(x)$ for any $T \in \mathcal{B}(\mathcal{H})$ if we rescale T so that its norm is less then one. Moreover, the condition $\|T\| < 1$ is stronger than required. The weakest is that the vector $\sum_{n=0}^{\infty} < y, T^n x > e_i$ be in the

Hilbert space for each $y \in \mathcal{H}$. Another condition which implies that $O_T^{e_i}(x)$ is well-defined is that $\sum_{n=0}^{\infty} \|T^n x\| < \infty$. In [17] (also see [9] pg. 214 exercise 17), it is shown that the condition $\sum_{n=0}^{\infty} \|T^n x\| < \infty$ for each x is equivalent to $r(T) < 1$, which is equivalent to $\|T^m\| < 1$ for some natural number m. To begin, unless otherwise stated, we will always assume that $r(T) < 1$. Later, we will explore the case when this condition does not hold, namely the case when $\|T\| \leq 1$. In this case, we show that $O_T^{e_i}(x)$ is a closed operator (see Proposition 4.1) and give sufficient conditions for it to be densely defined (see Propositions 4.9 and 4.12).

An important property of this construction is that $O_T^{e_i}(x)$ is conjugate linear when considered as a map from $\mathcal{H}$ to $\mathcal{B}(\mathcal{H})$. That is, $O_T^{e_i}(\lambda x + y) = \bar{\lambda} O_T^{e_i}(x) + O_T^{e_i}(y)$ for all $x, y \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

As mentioned in the introduction, this operator is closely related to an operator defined in [7]. Theorem 3.19 implies that if the range of $O_T^{e_i}(x)$ is not dense, then T has an invariant subspace. This result was announced in [7] but the proof there is incomplete. Our approach is quite different from the methods used in [7].

We now discuss the matrix of $O_T^{e_i}(x)$. Usually, we will be interested in the matrix of $O_T^{e_i}(x)$ with respect to the basis $\{e_i\}$. This matrix has the following form

$$\begin{bmatrix} \bar{x} \\ \overline{Tx} \\ \vdots \\ \overline{T^n x} \\ \overline{T^{n+1} x} \\ \vdots \end{bmatrix} \tag{2.1}$$

where we have that if $v \in \mathcal{H}$ is equal to $\sum_k v_k e_k$, then $\bar{v} = (\bar{v_0}, \bar{v_1}, \ldots)$. Given an infinite matrix, $(a_{ij})_{ij}$, one can define a linear map (possibly unbounded) by defining the domain, $D(A)$, of this map to be

$$\begin{aligned} D(A) \ &= \{f \in \mathcal{H} | \ \lim_{m \to \infty} \sum_{k=1}^{m} a_{jk} < f, e_k > \text{ exists for each } j \in \mathbb{N} \\ &\qquad \text{and } \sum_j | \sum_k a_{jk} < f, e_k > |^2 < \infty\} \end{aligned} \tag{2.2}$$

In other words, the domain is defined to be the set of all vectors in l^2 for which the formal multiplication by the matrix $(a_{ij})_{ij}$ yields a vector in l^2. The map, A, is then defined by $f \in D(A) \mapsto \sum_j \sum_k a_{jk} < f, e_k > e_j$ (see [24] Section 6.4). To see that our definition for $O_T^{e_i}(x)$ is equivalent to the definition via the infinite matrix in (2.1), we must show that the domain in these two definitions are the same. If $y \in \mathcal{H}$, then for each $n = 0, 1, \ldots$, we have $< y, T^n x >= \sum_m < y, e_m >< e_m, T^n x >$. Since the matrix of $O_T^{e_i}(x)$ has entries $< e_m, T^n x >$, we have that the first condition in (2.2) is satisfied for any $y \in \mathcal{H}$. Moreover, the second condition in (2.2) is the same as $\sum_n | < y, T^n x > |^2 < \infty$. Hence the domains in both definitions are the same.

2.1.2 Examples

In this section, we will discuss $O_T^{e_i}(x)$ for certain operators T. First, we will define the finite dimensional counter-part of $O_T^{e_i}(x)$. Throughout, V will be an inner product space over $\mathbb{C}$.

Definition 2.2. Let $(e_i)_{i=0}^n$ be an orthonormal basis of V and $T \in \mathcal{B}(V)$. Given $x \in V$, we define $O_T^{e_i}(x) : V \to V$ by $y \mapsto \sum_{i=0}^n < y, (T^i x) > e_i$.

Example 2.3. Let $J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then, given $x = [x_1, x_2, x_3]^t$, we have

$$O_J^{e_i}(x) = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{x}_2 & \bar{x}_3 & 0 \\ \bar{x}_3 & 0 & 0 \end{bmatrix}$$

where $\{e_i\}$ is the standard basis for $\mathbb{C}^3$. It is easy to check that x is a cyclic vector for J if and only if $x_3 \neq 0$. From the matrix form of $O_J^{e_i}(x)$, we see that $O_J^{e_i}(x)$ has trivial kernel if and only if $x_3 \neq 0$. Therefore, x is a cyclic vector of J if and only if the kernel of $O_J^{e_i}(x)$ is trivial (cf. Proposition 2.13).

Example 2.4. Let $D = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$. Then, given $x = [x_1, x_2, x_3]^t$, we have

$$O_D^{e_i}(x) = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{\alpha}\bar{x}_1 & \bar{\beta}\bar{x}_2 & \bar{\gamma}\bar{x}_3 \\ \bar{\alpha}^2\bar{x}_1 & \bar{\beta}^2\bar{x}_2 & \bar{\gamma}^2\bar{x}_3 \end{bmatrix}$$

where again $\{e_i\}$ is the standard basis for $\mathbb{C}^3$. It is easy to check that $O_D^{e_i}(x)$ is a rank one operator if and only if x is an eigenvector of D. For example, $x = [1, 0, 0]^t$ gives $O_D^{e_i}(x) = \begin{bmatrix} \bar{x}_1 & 0 & 0 \\ \bar{\alpha}\bar{x}_1 & 0 & 0 \\ \bar{\alpha}^2\bar{x}_1 & 0 & 0 \end{bmatrix}$. This is an instance of the result in Proposition 2.17.

Example 2.5. Let $H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then, given $x = [x_1, x_2, x_3, x_4]^t$, we have

$$O_H^{e_i}(x) = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 \\ \bar{x}_2 & 0 & \bar{x}_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\{e_i\}$ is the standard basis for $\mathbb{C}^4$. It is easy to see that H has no cyclic vectors and that $O_H^{e_i}(x)$ never has trivial kernel. Moreover, we see that the rank of $O_H^{e_i}(x)$ is the same as the dimension of the subspace $span\{x, Hx, H^2x, H^3x\}$.

Example 2.6. Given an orthonormal basis, $\{e_i\}$, we define an operator $S : \mathcal{H} \to \mathcal{H}$ given by $Se_i = e_{i+1}$. We call S the *unilateral shift* on $\{e_i\}$. Then $O_S^{e_i}(e_0) = I$. Moreover, given any orthonormal basis $\{f_i\}$, we let U denote the unitary map taking e_i to f_i. Then

$$\begin{aligned} O_S^{f_i}(e_0) &= \sum_n < \cdot, S^n e_0 > f_n \\ &= \sum_n < \cdot, e_n > f_n \\ &= U(\sum_n < \cdot, e_n > e_n) \\ &= U \end{aligned}$$

We will see later in Proposition 3.15 that $O_T^{e_i}(x)$ is unitary if and only if T is a unilateral shift. For a general element of $x \in \mathcal{H}$, we have that

$$O_S^{e_i}(x) = \sum_n < \cdot, S^n x > e_n$$

which, with respect to the basis $\{e_i\}$, has the matrix form

$$\begin{bmatrix} \bar{x}_0 & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \cdots \\ 0 & \bar{x}_0 & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \cdots \\ 0 & 0 & \bar{x}_0 & \bar{x}_1 & \bar{x}_2 & \cdots \\ 0 & 0 & 0 & \bar{x}_0 & \bar{x}_1 & \cdots \\ 0 & 0 & 0 & 0 & \bar{x}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

where $x = [x_0, x_1, \ldots]^t$ with respect to the basis $\{e_i\}$. We note that the above infinite matrix is a Toeplitz matrix.

Example 2.7. The adjoint of the unilateral shift is given by S^* mapping e_0 to zero and e_n to e_{n-1} for $n > 0$. In this case, we get Hankel matrices of the form

$$O_{S^*}^{e_i}(x) = \begin{bmatrix} \bar{x}_0 & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \cdots \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 & \cdots \\ \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 & \bar{x}_6 & \cdots \\ \bar{x}_3 & \bar{x}_4 & \bar{x}_5 & \bar{x}_6 & \bar{x}_7 & \cdots \\ \bar{x}_4 & \bar{x}_5 & \bar{x}_6 & \bar{x}_7 & \bar{x}_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

where, as above, $x = [x_0, x_1, \ldots]^t$ with respect to the basis $\{e_i\}$.

In both cases ($T = S$ or S^*), we have that $O_T^{e_i}(x)$ is densely defined and closed (see [12]). Moreover, we see that $O_T^{e_i}(x)$ can be unbounded for some x while it is bounded for others.

Example 2.8. We define a *weighted shift* on an orthonormal basis $\{e_i\}$ with weights $\{w_i\}$ to be the map given by $e_i \mapsto w_i e_{i+1}$. If T is a weighted shift on an orthonormal basis $\{e_i\}$ with weights $\{w_i\}$, then $O_T^{e_i}(e_0)$ is a diagonal operator with respect to the basis $\{e_i\}$. The diagonal entries of this operator are given by $1, \bar{w}_0, \bar{w}_1\bar{w}_0, \ldots, \Pi_{k=0}^{n}\bar{w}_i, \ldots$. We see that if the product of the weights is unbounded (for example if $w_i = 2$ for all i), then $O_T^{e_i}(e_0)$ is not a bounded operator. However, if the weights satisfy $|w_i| < 1$ (for example if $\|T\| < 1$) then we have that $O_T^{e_i}(e_0)$ is bounded. For a general element $x = [x_0, x_1, \ldots]^t$ with respect to the basis $\{e_i\}$, the matrix of $O_T^{e_i}(x)$ is

$$\begin{bmatrix} \bar{x}_0 & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \cdots \\ 0 & \bar{w}_0\bar{x}_0 & \bar{w}_1\bar{x}_1 & \bar{w}_2\bar{x}_2 & \bar{w}_3\bar{x}_3 & \cdots \\ 0 & 0 & \bar{w}_0\bar{w}_1\bar{x}_0 & \bar{w}_1\bar{w}_2\bar{x}_1 & \bar{w}_2\bar{w}_3\bar{x}_2 & \cdots \\ 0 & 0 & 0 & \bar{w}_0\bar{w}_1\bar{w}_2\bar{x}_0 & \bar{w}_1\bar{w}_2\bar{w}_3\bar{x}_1 & \cdots \\ 0 & 0 & 0 & 0 & \bar{w}_0\bar{w}_1\bar{w}_2\bar{w}_3\bar{x}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

Example 2.9. Assume that x is an eigenvector of T for the eigenvalue λ. In the finite dimensional case ($dim(V) = N$), we have that $O_T^{e_i}(x)$ is given by

$$O_T^{e_i}(x)(y) = \sum_{n=0}^{N} < y, T^n x > e_n = \sum_{n=0}^{N} \bar{\lambda}^n < y, x > e_n$$

This is a rank one operator if $x \neq 0$.

In the infinite dimensional case, if $|\lambda| < 1$, then we notice that $O_T^{e_i}(x)$ is given by

$$O_T^{e_i}(x)(y) = \sum_{n=0}^{\infty} < y, T^n x > e_n = \sum_{n=0}^{\infty} \bar{\lambda}^n < y, x > e_n$$

This defines a bounded operator which is again rank one.

However, if $|\lambda| \geq 1$, then $O_T^{e_i}(x)$ has domain $\{x\}^{\perp}$ and $O_T^{e_i}(x) = 0$ on its domain. Based on the above, we see that if $T = I$, then for each $x \in \mathcal{H}$, $O_T^{e_i}(x)$ has domain equal to its kernel which is $\{x\}^{\perp}$.

Example 2.10. Now suppose that y is an eigenvector of T^* for the eigenvalue λ. If $|\lambda| < 1$, then for any nonzero x and orthonormal basis $\{e_i\}$, we have

$$\begin{aligned} O_T^{e_i}(x)(y) &= \sum_{n=0}^{\infty} < y, T^n x > e_n \\ &= \sum_{n=0}^{\infty} < (T^*)^n y, x > e_n \\ &= \sum_{n=0}^{\infty} \lambda^n < y, x > e_n \end{aligned}$$

We see that $O_T^{e_i}(x)y$ is an element of the Hilbert space, so y is an element of the domain of $O_T^{e_i}(x)$ for every $x \in \mathcal{H}$.

However, if $|\lambda| \geq 1$, then y is not an element of the domain of x unless $x \in \{y\}^{\perp}$. This follows since $\sum_{n=0}^{\infty} \lambda^n < y, x > e_n$ is not an element of the Hilbert space unless $x \in \{y\}^{\perp}$. That is,

$$\sum_{n=0}^{\infty} | < y, T^n x > |^2 = \sum_{n=0}^{\infty} (|\lambda^n|)^2 | < y, x > |^2$$

is infinite unless $< y, x >= 0$.

Example 2.11. Let T be a diagonal operator of the form

$$\begin{bmatrix} \lambda_0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \ddots & \ddots & 0 & 0 & \cdots \\ 0 & 0 & \lambda_k & 0 & 0 & \cdots \\ 0 & 0 & 0 & \mu_0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \end{bmatrix}$$

where $|\lambda_i| = 1$ and $|\mu_i| < 1$. We then have, for $x = [x_0, x_1, \ldots]^t$, that $O_T^{e_i}(x)$ is given by

$$\begin{bmatrix} \bar{x}_0 & \bar{x}_1 & \cdots & \bar{x}_{k+1} & \cdots \\ \bar{\lambda}_0\bar{x}_0 & \bar{\lambda}_1\bar{x}_1 & \cdots & \bar{\mu}_0\bar{x}_{k+1} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ (\bar{\lambda}_0)^n\bar{x}_0 & (\bar{\lambda}_1)^n\bar{x}_1 & \cdots & (\bar{\mu}_0)^n\bar{x}_{k+1} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix}$$

This, in general, defines an unbounded operator with domain $\overline{span}\{e_{k+1}, \ldots\}$; however, if we consider the operator $D_T = (I - T^*T)^{\frac{1}{2}}$, then $O_T^{e_i}(x)D_T$ is given by

$$\begin{bmatrix} 0 & 0 & \cdots & (1-|\mu_0|^2)\bar{x}_{k+1} & \cdots \\ 0 & 0 & \cdots & \bar{\mu}_0(1-|\mu_0|^2)\bar{x}_{k+1} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & (\bar{\mu}_0)^n(1-|\mu_0|^2)\bar{x}_{k+1} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix}$$

This is a bounded operator. This example illustrates how we will generalize to the case when $\|T\| = 1$ in Chapter 4.

2.1.3 Some Properties of $O_T^{e_i}(x)$

We now study some of the basic properties of $O_T^{e_i}(x)$. Firstly, it is clear that $O_T^{e_i}(x)$ is linear, but we will shortly prove that if $r(T) < 1$, then $O_T^{e_i}(x)$ is a compact operator for any choice of $x \in \mathcal{H}$. The condition used in the proof is that $\sum_n \|T^n x\|^2 < \infty$. If $r(T) < 1$, this condition holds for all $x \in \mathcal{H}$. If $\|T\| = r(T) = 1$, then we have seen examples for which $O_T^{e_i}(x)$ is not compact. In fact, we have seen examples where $O_T^{e_i}(x)$ is unbounded.

Theorem 2.12. *If $\sum_n \|T^n x\|^2 < \infty$, in particular if $r(T) < 1$, then $O_T^{e_i}(x)$ is compact.*

Proof: We will give two proofs. For the first, we consider the sequence of finite rank operators given by

$$F_n = \sum_{j=0}^{n} < ., T^j x > e_j$$

We will show that F_n tends to $O_T^{e_i}(x)$ in norm. Consider a unit vector $y \in \mathcal{H}$ and

$$\begin{aligned} \|O_T^{e_i}(x)y - F_k y\|^2 &= \sum_{n>k} | < y, T^n x > |^2 \\ &\leq \|y\|^2 \sum_{n>k} \|T^n x\|^2 \end{aligned}$$

Thus $\|O_T^{e_i}(x) - F_k\|^2 \leq \sum_{n>k} \|T^n x\|^2$. Since $\sum_n \|T^n x\|^2$ converges, we have that F_k tends to $O_T^{e_i}(x)$ in operator norm and hence that $O_T^{e_i}(x)$ is compact. This completes the first proof.

We can also prove the result by showing that if $f_k \to f$ weakly, then $O_T^{e_i}(x) f_k \to O_T^{e_i}(x) f$ in norm. This type of argument will be used again later in the section on limit results. Let $\epsilon > 0$ and f_k tend to f weakly. Then, by the principle of uniform boundedness (see [9] III.14), there exists $M > 0$ such that $sup_k \|f_k - f\| \leq M$. Consider

$$\begin{aligned}
\|O_T^{e_i}(x)(f_k) - O_T^{e_i}(x)(f)\|^2 &= \|O_T^{e_i}(x)(f_k - f)\|^2 \\
&= \|\sum_{n=0}^{\infty} < f_k - f, T^n x > e_n\|^2 \\
&= \sum_{n=0}^{\infty} | < f_k - f, T^n x > |^2
\end{aligned}$$

Now, since $\sum_n^\infty \|T^n x\|^2$ converges, there exists $N > 0$ such that

$$M(\sum_{n=N}^{\infty} \|T^n x\|^2) < \frac{\epsilon}{2}$$

Moreover, since f_k tends to f weakly, there exists $K > 0$ such that

$$\sum_{n=0}^{N-1} | < f_k - f, T^n x > |^2 < \frac{\epsilon}{2} \text{ for } k \geq K$$

These two facts imply that

$$\sum_{n=0}^{\infty} | < f_k - f, T^n x > |^2 < \epsilon \text{ for } k \geq K$$

So we have that $O_T^{e_i}(x) f_k \to O_T^{e_i}(x) f$ in norm. □

The following result links $O_T^{e_i}(x)$ to invariant subspaces of T. Recall that $\mathcal{M}_x = \overline{span}\{x, Tx, \ldots\}$.

Proposition 2.13. *$ker(O_T^{e_i}(x)) = \mathcal{M}_x^{\perp}$. Hence x is T-cyclic if and only if $ker(O_T^{e_i}(x)) = \{0\}$.*

Proof:

$$\begin{aligned} O_T^{e_i}(x)(y) = 0 &\Leftrightarrow \sum_{i=0}^{\infty} < y, (T^i x) > e_i = 0 \\ &\Leftrightarrow < y, (T^i) x >= 0 \text{ for each i} \\ &\Leftrightarrow y \in \mathcal{M}_x^{\perp} \end{aligned}$$

□

We have the following interesting property of $O_T^{e_i}(x)$

Proposition 2.14. *Given $T \in \mathcal{B}(\mathcal{H})$ and $A \in \{T\}'$, the commutant of T, then for each $f \in \mathcal{H}$, we have $f \in dom(O_T^{e_i}(Ax))$ if and only if $A^* f \in dom(O_T^{e_i}(x))$ and, if this is the case, then $O_T^{e_i}(x)(A^* f) = O_T^{e_i}(Ax)(f)$.*

Proof:

$$\begin{aligned} f \in dom(O_T^{e_i}(Ax)) &\Leftrightarrow \sum_n |< f, T^n(Ax) >|^2 < \infty \\ &\Leftrightarrow \sum_n |< f, AT^n x >|^2 < \infty \\ &\Leftrightarrow \sum_n |< A^* f, T^n x >|^2 < \infty \\ &\Leftrightarrow A^* f \in dom(O_T^{e_i}(x)) \end{aligned}$$

Now, if $A^* f \in dom(O_T^{e_i}(x))$, then

$$\begin{aligned} O_T^{e_i}(x)(A^* f) &= \sum_n < A^* f, T^n x > e_n \\ &= \sum_n < f, AT^n x > e_n \\ &= \sum_n < f, T^n Ax > e_n \\ &= O_T^{e_i}(Ax)(f) \end{aligned}$$

□

We now give a proof of a well-known result about invariant subspaces to give an application of the operator $O_T^{e_i}(x)$.

Theorem 2.15. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that there exists $A \in \{T\}'$, the commutant of T, that is not a multiple of the identity and is such that A^* has an eigenvalue. Then T has an invariant subspace.*

Proof: Since A^* has an eigenvalue, λ, then $\lambda I - A^*$ has zero as an eigenvalue (and is nonzero since A is not a constant multiple of the identity). So, by replacing A by $\bar{\lambda}I - A$, we may assume A^* has a zero eigenvalue. It follows that $A^*f = 0$ for some nonzero f. Hence, for all x, we have $O_T^{e_i}(x)(A^*f) = 0$ and since $A \in \{T\}'$, we have, by Proposition 2.14, $O_T^{e_i}(x)(A^*f) = O_T^{e_i}(Ax)(f)$. Now, since A is nonzero, there exists $x \in \mathcal{H}$ such that $Ax = z \neq 0$, but $f \in ker(O_T(z))$ so z is not cyclic; hence T has an invariant subspace. □

We now prove a number of other results relating properties of the operator $O_T^{e_i}(x)$ with properties of invariant subspaces of T.

Proposition 2.16. *$ker(O_T^{e_i}(x))$ is T^* invariant and hence $\overline{range}(O_T^{e_i}(x)^*)$ is T invariant.*

Proof:

$$\begin{aligned} f \in ker(O_T^{e_i}(x)) &\Rightarrow\ < f, T^n x >= 0 \text{ for all n} \\ &\Rightarrow\ < T^* f, T^n x >= 0 \text{ for all n} \\ &\Rightarrow\ T^* f \in ker(O_T^{e_i}(x)) \end{aligned}$$

□

Theorem 2.17. *A vector $x \in \mathcal{H}$ belongs to a finite dimensional invariant subspace of T if and only if $O_T^{e_i}(x)$ is a finite rank operator.*

Proof: First, assume that $O_T^{e_i}(x)$ is a finite rank operator. Then $O_T^{e_i}(x)^*$ is also finite rank. However, $O_T^{e_i}(x)^* = \sum_n < \cdot, e_n > T^n x$ and so $O_T^{e_i}(x)^* e_n =$

$T^n x$ for $n = 0, 1, \ldots$. Thus, the set $\{x, Tx, \ldots\}$ is contained in the range of $O_T^{e_i}(x)^*$. It follows that $\mathcal{M}_x$ is finite dimensional. The first implication now follows since x is clearly an element of $\mathcal{M}_x$ and $\mathcal{M}_x$ is an invariant subspace of T.

For the converse, we have that the range of $O_T^{e_i}(x)^*$ is contained in $\mathcal{M}_x$ which is finite dimensional by assumption. Hence, $O_T^{e_i}(x)^*$ is a finite rank operator with rank equal to $dim(\mathcal{M}_x)$. By properties of the adjoint, it follows that $rank(O_T^{e_i}(x)) = dim(\mathcal{M}_x)$. □

During the above proof, notice that a formula for the adjoint of the orbit operator was determined.

Proposition 2.18. $O_T^{e_i}(x)^* = \sum_n < \cdot, e_n > T^n x$

2.2 Limit Results

In this section, we will assume that $r(T) < 1$ and will prove certain limit results. This is assumed so that $\sum_n \|T^n\|^2$ is finite.

Theorem 2.19. *If a sequence x_n converges to x in norm, then $O_T^{e_i}(x_n)$ converges to $O_T^{e_i}(x)$ in operator norm.*

Proof: Let $y \in \mathcal{H}$ with $\|y\| = 1$. Then

$$\begin{aligned} \|(O_T^{e_i}(x_n) - O_T^{e_i}(x))(f)\|^2 &= \sum_{k=0}^{\infty} |< y, T^k(x_n - x) >|^2 \\ &\leq \|y\|^2 \sum_{k=0}^{\infty} \|T^k(x_n - x)\|^2 \\ &\leq \|x_n - x\|^2 \sum_{k=0}^{\infty} \|T^k\|^2 \end{aligned}$$

Since $\sum_{k=0}^{\infty} \|T^k\|^2$ converges, the result follows. □

Theorem 2.20. *If a sequence x_n converges to x weakly, then $O_T^{e_i}(x_n)$ converges to $O_T^{e_i}(x)$ in the strong operator topology (SOT).*

Proof: The proof of this result is similar to the proof that $O_T^{e_i}(x)$ is compact. Fix $y \in \mathcal{H}$ and then for each $x \in \mathcal{H}$ we have

$$\begin{aligned} O_T^{e_i}(x)(y) &= \sum_{n=0}^{\infty} < y, T^n x > e_n \\ &= \sum_{n=0}^{\infty} \overline{< x, (T^*)^n y >} e_n \\ &= \overline{O_{T^*}^{e_i}(y)(x)} \end{aligned}$$

For fixed y, we have $O_T^{e_i}(x_n)(y) = \overline{O_{T^*}^{e_i}(y)(x_n)}$ which, using the compactness of $O_{T^*}^{e_i}(y)$, tends to $\overline{O_{T^*}^{e_i}(y)(x)} = O_T^{e_i}(x)(y)$ in norm. That is, $O_T^{e_i}(x_n)$ tends to $O_T^{e_i}(x)$ in the SOT. □

Theorem 2.21. *If a sequence of operators T_n satisfies*

1) $\|T_n\| < 1$ converges in the SOT to T with $\|T\| < 1$

2) $sup_n\{\|T_n\|\} < 1$

3) $T_n \in \{T\}'$ *for all* n

then $O^{e_i}_{T_n}(x)$ *converges to* $O^{e_i}_T(x)$ *in norm.*

Proof: Let $y \in \mathcal{H}$ with $\|y\| = 1$. Then

$$\begin{aligned}
\|(O^{e_i}_{T_n}(x) - O^{e_i}_T(x))(y)\|^2 &= \|\sum_{k=0}^{\infty}(< y, T_n^k x > - < y, T^k x >)e_k\|^2 \\
&= \sum_{k=0}^{\infty} | < y, (T_n^k - T^k)x > |^2 \\
&\leq \|y\|^2 \sum_{k=0}^{\infty} \|(T_n^k - T^k)x\|^2 \\
&= \sum_{k=0}^{\infty} \|(T_n^{k-1} + T_n^{k-2}T + \ldots + T_n T^{k-2} + T^{k-1})(T_n - T)x\|^2 \\
&\leq \|(T_n - T)x\|^2 \sum_{k=0}^{\infty} k^2 (sup_n\{\|T_n\|, \|T\|\})^{2(k-1)}
\end{aligned}$$

Now since T_n converges to T in the SOT and $\sum_{k=0}^{\infty} k^2(sup_n\{\|T_n\|, \|T\|\})^{2(k-1)}$ converges by 2), we have the result by the above inequality. □

Theorem 2.22. *If* x_n *tends to* x *in norm and* y_k *tends to* y *weakly, then* $\lim_{n\to\infty} O^{e_i}_T(x_n)(y_n) = O^{e_i}_T(x)(y)$ *in norm.*

Proof:

$$\begin{aligned}
\|O_T^{e_i}(x_n)(y_n) - O_T^{e_i}(x)(y)\| &= \|O_T^{e_i}(x_n)(y_n) - O_T^{e_i}(x)(y_n) + O_T^{e_i}(x)(y_n - y)\| \\
&\leq \|O_T^{e_i}(x_n - x)(y_n)\| + \|O_T^{e_i}(x)(y_n - y)\| \\
&\leq \sqrt{\sum_s | < y_n, T^s(x_n - x) > |^2} + \|O_T^{e_i}(x)(y_n - y)\| \\
&\leq \|y_k\| \|x_n - x\| \sqrt{\sum_s \|T^s\|^2} + \|O_T^{e_i}(x)(y_n - y)\|
\end{aligned}$$

Since y_k tends to y weakly and $O_T^{e_i}(x)$ is compact, then $\|O_T^{e_i}(x)(y_n - y)\|$ tends to 0. Moreover, $\|y_k\| < M_1$ for some $M_1 > 0$, $\sum_s \|T^s\|^2 < M_2$ for some $M_2 > 0$ and $\|x_n - x\|$ tends to 0. Hence, we have the result from the inequality above. □

Corollary 2.23. *If x_n tends to $x \neq 0$ in norm, y_k tends to $y \neq 0$ weakly and $O_T^{e_i}(x_n)(y_n)$ tends to 0, then T has an invariant subspace.*

Proof: The above theorem implies that $O_T^{e_i}(x)(y) = 0$. Moreover, since x, y are both nonzero, the result follows by Proposition 2.13. □

2.3 Functional Calculus and $O_T^{e_i}(x)$

In this section, we will use the functional calculus of T to gain information about the behaviour of the orbits of points outside the linear span of $\{x, Tx, \ldots\}$. We will again assume that $r(T) < 1$.

Notation 2.24. Given a polynomial, $p(t) = \sum_{j=0}^{n} a_j t^j$, we will let $\hat{p}(t) = \sum_{j=0}^{n} \bar{a}_j t^j$. For the power series of a function f, we will use the same notation. That is, if $f = \sum_n a_j t^j$, then $\hat{f} = \sum_n \bar{a}_j t^j$.

Proposition 2.25. *Let p be a polynomial and S be the unilateral shift on the basis $\{e_i\}$. Then*

$$O_T^{e_i}(p(T)x) = \hat{p}(S^*)O_T^{e_i}(x) = O_T^{e_i}(x)\hat{p}(T^*)$$

where $\hat{p}$ is defined above.

Proof: To prove this result, we consider $O_T^{e_i}(Tx)$. Recall that $e_n = S^* e_{n+1}$ and $S^* e_0 = 0$.

$$\begin{aligned} O_T^{e_i}(Tx) &= \sum_{n\geq 0} < ., T^{n+1}x > e_n \\ &= \sum_{n\geq 0} < ., T^{n+1}x > S^* e_{n+1} \\ &= S^* \sum_{n\geq 0} < ., T^n x > e_n \\ &= S^* O_T^{e_i}(x) \end{aligned}$$

This easily implies that $O_T^{e_i}(T^m x) = (S^*)^m O_T^{e_i}(x)$ for $m = 0, 1, \ldots$. Also, for $a \in \mathbb{C}$, we have $O_T^{e_i}(ax) = \bar{a} O_T^{e_i}(x)$. The first equality now follows for all polynomials since $O_T^{e_i}(.)$ is a conjugate linear map. For the second equality, consider

$$\begin{aligned} O_T^{e_i}(Tx) &= \sum_{n\geq 0} < ., T^{n+1}x > e_n \\ &= \sum_{n\geq 0} < T^*., T^n x > e_n \\ &= O_T^{e_i}(x)T^* \end{aligned}$$

□

From this result, we have the following proposition which links the cyclic vectors in $\mathcal{M}_x$ with the range of $O_T^{e_i}(x)$. Here, as above and in what follows, S will denote unilateral shift on the basis $\{e_i\}$.

Proposition 2.26. *Let x be a cyclic vector of T, p a polynomial and $y = p(T)x$ (i.e. $y \in span\{x, Tx, \ldots\}$). Then y is T-cyclic if and only if $range(O_T^{e_i}(x)) \cap ker(\hat{p}(S^*)) = \{0\}$.*

Proof: First assume that y is T-cyclic. Then, by Proposition 2.13, $ker(O_T^{e_i}(y))$ is trivial. Since $y = p(T)x$, we can apply Proposition 2.25 to get $O_T^{e_i}(y) = \hat{p}(S^*)O_T^{e_i}(x)$. Since $O_T^{e_i}(y)$ has trivial kernel, we have that $range(O_T^{e_i}(x)) \cap ker(\hat{p}(S^*)) = \{0\}$.

For the converse, we again have that $O_T^{e_i}(y) = \hat{p}(S^*)O_T^{e_i}(x)$. Now, since x is T-cyclic, $ker(O_T^{e_i}(x))$ is trivial (again by Proposition 2.13). By assumption, $range(O_T^{e_i}(x)) \cap ker(\hat{p}(S^*)) = \{0\}$ and hence $\hat{p}(S^*)O_T^{e_i}(x)z = 0$ if and only if $O_T^{e_i}(x)z = 0$ which occurs if and only if $z = 0$. That is, the kernel of $\hat{p}(S^*)O_T^{e_i}(x) = O_T^{e_i}(y)$ is trivial. Hence, by Proposition 2.13, y is T-cyclic. □

We can use this method on elements of the Hilbert space which are in $\overline{span}\{x, Tx, \ldots\}$ but are not in $span\{x, Tx, \ldots\}$ by using analytic function theory. Recall that the definition of $\hat{f}$ is the same as the one used for polynomials only on the power series of f. To do this, we need to make sense of $f(T)$. Moreover, we would like to define $f(T)$ in a way which extends the definition of $p(T)$ for a polynomial p. If we consider f which is analytic on an open neighborhood of the closed unit disk, then we have a power series for $f(z)$ given by $f(z) = \sum_n a_n z^n$. Moreover, there exists $\epsilon > 0$ such that this series converges for all $|z| < 1 + \epsilon$. The most natural way of defining $f(T)$ would be to consider $\sum_n a_n T^n$. This sum converges in operator norm since $\|T\| < 1$ so that $f(T)$ is well-defined. Moreover, the mapping from the functions which are analytic on an open neighborhood of the closed unit disk to the bounded linear operators on $\mathcal{H}$ via $f \mapsto f(T)$ respects the algebraic structure of the functions (e.g. $(cf)(T) = cf(T)$, $(f+g)(T) = f(T) + g(T)$, etc). Given $T \in \mathcal{B}(\mathcal{H})$, the process of taking a class of functions and mapping them into $\mathcal{B}(\mathcal{H})$ in this way is a functional calculus for T. The reader can find more on the analytic functional calculus in any of [3], [9], and [19]. The result we will use is that if f is analytic on an open neighborhood of $\sigma(T)$, the spectrum of T, then the analytic functional calculus of T gives meaning to $f(T)$.

Proposition 2.27. *Let f be a complex function which is analytic on an open* *neighbourhood of the closed unit disk (i.e. $\sigma(S^*)$). Then*

$$O_T^{e_i}(f(T)x) = \hat{f}(S^*)O_T^{e_i}(x) = O_T^{e_i}(x)\hat{f}(T^*)$$

.

Proof: Consider a sequence of polynomials p_n where p_n tends to f uniformly on the unit disk. Then $p_n(T)x$ tends to $f(T)x$ and $\hat{p}_n(S^*)$ tends to $\hat{f}(S^*)$. Using the fact that the assertion holds for the polynomials (Proposition 2.25) and that the map $x \mapsto O_T^{e_i}(x)$ is continuous with respect to the norm topology (see Theorem 2.19), we have

$$\begin{aligned} O_T^{e_i}(f(T)x) &= O_T^{e_i}(\lim_{n\to\infty} p_n(T)x) \\ &= \lim_{n\to\infty} O_T^{e_i}(p_n(T)x) \\ &= (\lim_{n\to\infty} \hat{p}_n(S^*))O_T^{e_i}(x) \\ &= \hat{f}(S^*)O_T^{e_i}(x) \end{aligned}$$

The second result follows in the same way from the corresponding result for polynomials (see 2.25). □

We can generalize the previous result (Proposition 2.26) to

Proposition 2.28. *Let x be a cyclic vector of T and f be analytic on an open neighborhood of the closed unit disk such that $f(T)x \neq 0$. Then $f(T)x$ is T-cyclic if and only if $range(O_T^{e_i}(x)) \cap ker(\hat{f}(S^*)) = \{0\}$.*

Proof: Follows in the same way as Propostion 2.26. □

We can actually increase the class of functions for which the above result holds. Let H^∞ be the set of analytic funtions in the unit disk for which the functions $f_r(\theta) = f(re^{i\theta})$ are uniformly bounded in L^∞-norm for $0 \le r < 1$ (see [16]). For S^*, there exists a functional calculus for functions in H^∞ (see [3], [10] or [21]). By using this functional calculus of the backward shift, S^*, we can extend the result above. The analytic functional calculus of T can be applied to any function which is analytic on an open neighbourhood of its spectrum. In particular, since $r(T) < 1$, the analylic functional calculus of T may be applied to any function in H^∞. Recall that the hat notation is defined in Notation 2.24.

Proposition 2.29. *If $\phi \in H^\infty$, then*

$$\hat{\phi}(S^*)O_T^{e_i}(x) = O_T^{e_i}(\phi(T)x) = O_T^{e_i}(x)\hat{\phi}(T^*)$$

.

Proof: We have a sequence of polynomials p_n such that $p_n(S^*) \overset{WOT}{\to} \phi(S^*)$ (see [3] Chapter XI Corollary 1.4). Moreover, since $r(T) < 1$, then, by the analytic functional calculus of T, we have $p_n(T) \to \phi(T)$. Thus,

$$\begin{aligned} O_T^{e_i}(\phi(T)x) &= O_T^{e_i}(\lim_{norm} p_n(T)x) \\ &= \lim_{norm} O_T^{e_i}(p_n(T)x) \\ &= \lim_{WOT} O_T^{e_i}(p_n(T)x) \\ &= \lim_{WOT} \hat{p}_n(S^*)O_T^{e_i}(x) \\ &= \hat{f}(S^*)O_T^{e_i}(x) \end{aligned}$$

Again, the second equality follows in a similar way from the corresponding result for polynomials. □

The above results mean that from the orbit of a single point we can get information on the behaviour of a large set of vectors. We will see later how this can be extended to further restrict the range of $O_T^{e_i}(x)$ in the case when T is assumed to have no non-trivial invariant subspaces.

2.4 Change of Basis and Unitary Transformations

We now prove a result concerning how $O_T^{e_i}(x)$ transforms under a unitary transformation and how, given two bases $\{e_i\}$ and $\{f_i\}$, $O_T^{e_i}(x)$ and $O_T^{f_i}(x)$ are related. Since the cyclic behaviour of x under T is invariant under a change of basis, we should have that the kernel of $O_T^{e_i}(x)$ and $O_T^{f_i}(x)$ are the same. In this section, we need only assume that $O_T^{e_i}(x)$ is compact. In particular, this occurs in the case $r(T) < 1$.

Proposition 2.30. *Let U and V be unitary. Then $UO_T^{e_i}(x)V^* = O_{VTV^*}^{Ue_i}(Vx)$. In particular, $O_T^{Ue_i}(x) = UO_T^{e_i}(x)$.*

Proof: Given $f \in H$, we have

$$\begin{aligned} UO_T^{e_i}(x)V^*(f) &= U(\sum_{n=0}^{\infty} < V^*f, T^n x > e_n) \\ &= \sum_{n=0}^{\infty} < f, VT^n x > Ue_n \\ &= \sum_{n=0}^{\infty} < f, VT^n V^* V x > Ue_n \\ &= O_{VTV^*}^{Ue_i}(Vx)(f) \end{aligned}$$

Since f was arbitrary, we have the result. □

The result linking the cyclic vectors of T to the kernel of $O_T^{e_i}(x)$ (Proposition 2.13) shows that the kernel of $O_T^{e_i}(x)$ does not depend on the choice of orthonormal basis $\{e_i\}$. This result is also clear from part 1) in the above result. It is somewhat unsatisfying that the operator $O_T^{e_i}(x)$ depends on the choice of basis while its kernel does not. Shortly, we will define another operator related to $O_T^{e_i}(x)$ which does not depend on the choice of basis. From this new operator, we will be able to look at objects (like the kernel of $O_T^{e_i}(x)$) associated with $O_T^{e_i}(x)$ which are invariant under the choice of basis. Moreover, as one would hope, many of these objects are linked with properties of invariant subspaces of T.

The eigenvalues of $O_T^{e_i}(x)$ are not independent of the choice of basis, but we would hope that they give information on the nature of the operator T. Specifically, we would like a link between the orbit of x under T and these eigenvalues. To begin, we will construct a natural basis for the orbit of x under T. In this basis, what the eigenvalues of $O_T^{e_i}(x)$ are and their relationship with the orbit of x under T will be clear.

Theorem 2.31. *Given an operator, T, with cyclic vector, x, there exists an orthonormal basis $\{e_i\}$ such that the matrix (with respect to the basis $\{e_i\}$) of the operator $O_T^{e_i}(x)$ is lower triangular.*

Proof: We can construct, using the Gram-Schmidt orthogonalization process, a basis with the following properties.

$$\begin{aligned} span\{e_0\} &= span\{x\} \\ span\{e_0, e_1\} &= span\{x, Tx\} \\ &\vdots \\ span\{e_0, ..., e_n\} &= span\{x, ..., T^n x\} \\ &\vdots \end{aligned}$$

The matrix of $O_T^{e_i}(x)$ has the form

$$\begin{bmatrix} \bar{x} \\ \overline{Tx} \\ \vdots \\ \overline{T^n x} \\ \vdots \end{bmatrix} = \begin{bmatrix} a_{00} & 0 & 0 & \dots & 0 & \dots \\ a_{10} & a_{11} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \dots \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nn} & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{bmatrix}$$

From this construction, we have the result. □

We will call this the natural basis of the orbit of x under T. We now define the basis independent operator mentioned above.

Notation 2.32. Let $A_T(x)$ denote $O_T^{e_i}(x)^* O_T^{e_i}(x)$

$A_T(x)$ is well-defined (i.e. basis independent) since we have that

$$O_T^{Ue_i}(x)^* O_T^{Ue_i}(x) = O_T^{e_i}(x)^* U^* U O_T^{e_i}(x) = O_T^{e_i}(x)^* O_T^{e_i}(x)$$

Moreover, since $ker(O_T^{e_i}(x)) = ker(O_T^{e_i}(x)^* O_T^{e_i}(x))$, we have the same relationship between the cyclic vectors of T and $A_T(x)$. That is, x is a cyclic vector of T if and only if $ker(A_T(x)) = \{0\}$. Recall that $\sigma_p(A)$ denotes the eigenvalues of A.

Theorem 2.33. *The subspace $\mathcal{M} = \overline{span\{x, Tx, T^2x, ...\}}$ is classified by properties of $A_T(x)$ by the correspondance below.*

1) $dim(\mathcal{M}) < \infty$ if and only if 0 is isolated in $\sigma_p(A_T(x))$.

2) $dim(\mathcal{M})$ is infinite but $\mathcal{M}$ is not all of $\mathcal{H}$ if and only if 0 is a limit of nonzero eigenvalues of $A_T(x)$, and $ker(A_T(x)) \neq 0$.

3) $\mathcal{M} = \mathcal{H}$ if and only if $ker(A_T(x)) = 0$.

Proof: By Theorem 2.17, we have that $\mathcal{M}$ is finite dimensional if and only if $O_T^{e_i}(x)$ is a finite rank operator. Since $O_T^{e_i}(x)$ is compact, this occurs if and only if the range of $O_T^{e_i}(x)$ is closed. Thus, we have 1).

For 2) and 3), the result follows from the proof of 1) along with the fact that x is cyclic if and only if the kernel of $A_T(x)$ is trivial. □

Since $O_T^{e_i}(x)$ is compact, the range of $O_T^{e_i}(x)$ is closed if and only if 0 is not a limit point of the nonzero eigenvalues of $O_T^{e_i}(x)^* O_T^{e_i}(x) = A_T(x)$. Also $ker(O_T^{e_i}(x)) = ker(A_T(x))$. Based on these two results, we can restate the previous theorem in terms of properties of $O_T^{e_i}(x)$.

Theorem 2.34. *The subspace $\mathcal{M} = \overline{span\{x, Tx, T^2x, ...\}}$ is classified by the basis independent properties of $O_T^{e_i}(x)$ by the below correspondance:*

1) $dim(\mathcal{M}) < \infty$ if and only if the range of $O_T^{e_i}(x)$ is closed.

2) $dim(\mathcal{M})$ is infinite but $\mathcal{M}$ is not all of $\mathcal{H}$ if and only if the range of $O_T^{e_i}(x)$ is not closed and $ker(O_T^{e_i}(x)) \neq 0$.

3) $\mathcal{M} = \mathcal{H}$ if and only if $ker(O_T^{e_i}(x)) = 0)$.

Proposition 2.35. *$A_T(x)$ maps $y \in \mathcal{H}$ to $\sum_{n=0}^{\infty} < y, T^n x > T^n x$*

Proof:

$$\begin{aligned}
O_T^{e_i}(x)^* O_T^{e_i}(x)(y) &= O_T^{e_i}(x)^*(\sum_{n=0}^{\infty} < y, T^n x > e_n) \\
&= \sum_{i=0}^{\infty}\sum_{n=0}^{\infty} < y, T^n x >< T^n x, e_i > e_i \\
&= \sum_{n=0}^{\infty} < y, T^n x > \sum_{i=0}^{\infty} < T^n x, e_i > e_i \\
&= \sum_{n=0}^{\infty} < y, T^n x > T^n x
\end{aligned}$$

□

Next, we construct another positive, compact operator by considering $O_T^{e_i}(x)O_T^{e_i}(x)^*$, which we will denote by $G_T^{e_i}(x)$. Firstly, we would like to know how this operator transforms under a change of basis.

Proposition 2.36. *Suppose an orthonormal basis $\{e_i\}$ is mapped to another orthonormal basis via the unitary transformation U. Then $G_T^{Ue_i}(x) = UG_T^{e_i}(x)U^*$*

Proof: Follows from $O_T^{Ue_i}(x) = UO_T^{e_i}(x)$. □

Given a sequence $(f_n)_n$ in a Hilbert space, then the matrix $(< f_j, f_i >)_{ij}$ is called the Gramian of f_n (see [15]). Our next theorem states that $G_T^{e_i}(x)$ is the Gramian of the sequence $\{T^n x\}_n$.

Proposition 2.37. *The infinite matrix of $G_T^{e_i}(x)$ with respect to the basis $\{e_i\}$ is given by $(< T^j x, T^i x >)_{ij}$.*

Proof: We have

$$\begin{aligned}
G_T^{e_i}(x) &= O_T^{e_i}(x)O_T^{e_i}(x)^* \\
&= \sum_n < O_T^{e_i}(x)^*., T^n x > e_n \\
&= \sum_n < ., O_T^{e_i}(x)T^n x > e_n \\
&= \sum_{n,k} < ., e_k >< T^n x, T^k x > e_n
\end{aligned}$$

From this last line, the result follows. □

Example 2.38. Let T be the operator given by multiplication by the independent variable, t, acting on $L^2(0,1)$ (i.e. $(Tg)(t) = tg(t)$) and f be the characteristic function of $(0,1)$. Then, it follows, from $< t^j, t^i >= \frac{1}{i+j+1}$, that $G_T^{e_i}(f)$ is the Hilbert matrix. That is, $G_T^{e_i}(f) = (\frac{1}{i+j+1})_{ij}$ (see [15] Problem and Solution 48).

Proposition 2.39. *The following are equivalent*

1) $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$

2) $G_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$

3) The series $\sum_{n\geq 0} \alpha_n T^n x$ *converges for each* $(\alpha_n)_{n\geq 0} \in l^2(\mathbb{N})$

4) $(T^n x)_{n\geq 0}$ *is a Bessel sequence. That is* $\sum_n | < y, T^n x > |^2 < \infty$ *for each* $y \in \mathcal{H}$.

Proof: That 1) implies 2) is obvious.

The equivalence of 2) and 3) can be found in [15] Solution 49.

The equivalence of 2) and 4) is Exercise 13 of Chapter II.1 in [9] (also see [2] and [23]).

That 4) implies 1) follows from the remarks following the definition of $O_T^{e_i}(x)$.

□

2.5 Trace Class and Hilbert-Schmidt Results

The goal of this section is to find conditions under which $O_T^{e_i}(x)$ is Hilbert-Schmidt or trace class. To begin, we will introduce these classes of operators. Given a compact operator, K, we define the singular values of K, $\{s_j(K)\}$, to be the sequence of eigenvalues of $(K^*K)^{\frac{1}{2}}$ (since K is compact, $(K^*K)^{\frac{1}{2}}$ has countable eigenvalues). For $1 \leq p < \infty$, we define the p-th Schatten class, $\mathcal{S}_p$, to be the set of compact operators such that $\{s_j(K)\}$ is in l_p. The class $\mathcal{S}_p$ is a Banach space with norm given by the l_p-norm of the singular values. We will also refer to $\mathcal{S}_1$ as the trace class operators and $\mathcal{S}_2$ as the Hilbert-Schmidt operators. If K is trace class, then $\sum_n < Ae_n, e_n >$ is independent of the choice of orthonormal basis $\{e_i\}$ and is called the trace of K. The vector space of Hilbert-Schmidt operators ($\mathcal{S}_2$) is a Hilbert space with inner product given by $< A, B >= tr(AB^*)$. This inner product gives the same norm to the Hilbert-Schmidt operators as the l_2-norm of the singular values. Part II of [13] contains an introduction to trace class and Hilbert-Schmidt operators. Also [20] contains results on the Schatten classes. We also note that K is Hilbert-Schmidt if and only if the matrix, (a_{ij}), of K with respect to any (and hence all) orthonormal basis satisfies $\sum_{ij} |a_{ij}|^2 < \infty$.

As in the previous section, we are most interested in finding conditions which are independent of the choice of basis $\{e_i\}$. Since the sets of Hilbert-Schmidt and trace class operators have the structure of a Hilbert space and Banach space respectively, we will be interested in connecting the geometry of these spaces with properties of the invariant subspaces of T. To begin this section, we will assume that $\sum_{i=0}^{\infty} \|T^i x\|^2 < \infty$. This occurs in particular if $r(T) < 1$. As noted above, this condition ensures that $O_T^{e_i}(x)$ is well-defined and a compact operator. In fact, this condition implies more, namely that $O_T^{e_i}(x)$ is a Hilbert-Schmidt operator.

Theorem 2.40. *$O_T^{e_i}(x)$ is Hilbert-Schmidt if and only if $\sum_n \|T^n x\|^2 < \infty$.*

Proof: Consider the infinite matrix representation, (a_{ij}) of $O_T^{e_i}(x)$ with respect to the basis $\{e_i\}$. By the definition of $O_T^{e_i}(x)$, we have that $a_{ij} =< e_j, T^i x >$. So

$$\begin{aligned} \sum_{ij} |a_{ij}|^2 &= \sum_{ij} | < e_j, T^i x > |^2 \\ &= \sum_i \|T^i x\|^2 \end{aligned}$$

This last line implies the result. □

Theorem 2.41. $< O_T^{e_i}(x), O_T^{e_i}(x) >= \sum_k < T^k y, T^k x >$

Proof:

$$\begin{aligned} < O_T^{e_i}(x), O_T^{e_i}(x) > &= tr(O_T^{e_i}(x) O_T^{e_i}(y)^*) \\ &= tr(O_T^{e_i}(y)^* O_T^{e_i}(x)) \\ &= \sum_n < O_T^{e_i}(x)e_n, O_T^{e_i}(y)e_n > \\ &= \sum_{n,k} < e_n, T^k x > \overline{< e_n, T^k y >} \\ &= \sum_k < T^k y, T^k x > \end{aligned}$$

□

From the fact that $O_T^{e_i}(x)$ is Hilbert-Schmidt, it follows that $A_T(x)$ is trace class. In particular, it follows that $A_T(x)$ is Hilbert-Schmidt.

Theorem 2.42. $< A_T(x), A_T(y) >= \sum_{j,k} | < T^j x, T^k y > |^2$

Proof: For an orthonormal basis $\{e_i\}$, we have

$$\begin{aligned} tr(A_T(x)A_T(y)) &= \sum_{i=0}^{\infty} < A_T(x)e_i, A_T(y)e_i > \\ &= \sum_{i=0}^{\infty} < \sum_{j=0}^{\infty} < e_i, T^j x > T^j x >, \sum_{k=0}^{\infty} < e_i, T^k y > T^k y > \\ &= \sum_{i,j,k} < T^j x, T^k y >< e_i, T^j x >< T^k y, e_i > \\ &= \sum_{j,k} < T^j x, T^k y > \sum_i < T^k y, e_i >< e_i, T^j x > \\ &= \sum_{j,k} | < T^j x, T^k y > |^2 \end{aligned}$$

□

Corollary 2.43. $< A_T(x), A_T(y) >= 0$ *if and only if* $\mathcal{M}_x \perp \mathcal{M}_y$

Proof: This follows from the formula for $< A_T(x), A_T(y) >$. □

This result links the invariant subspaces of T to the geometry of the Hilbert-Schmidt operators $A_T(x)$ $(x \in \mathcal{H})$. For example, if $\mathcal{M}$ is a reducing subspace of T and $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$, then $A_T(x)$ and $A_T(y)$ are orthogonal in the Hilbert space of Hilbert-Schmidt operators.

Recall that an integral operator with kernel $k(x, y)$ is defined to be the mapping $f \mapsto \int_a^b k(x, y) f(y) dy$. As discussed in [13], there is a universal model for Hilbert-Schmidt operators as integral operators. Given a Hilbert-Schmidt operator, A, on $\mathcal{H}$ there exists a unitary U from $\mathcal{H}$ to $L_2([0, 1])$ such that UAU^{-1} is an integral operator. The formula for the kernel function of the integral operator associated with A (as is shown in [13]) is the following

$$k(t, s) = \sum_{i,j} < Ae_i, e_j > (Ue_j)(t)\overline{(Ue_i)(s)}$$

where U is the unitary above and e_i is an orthonormal basis for $\mathcal{H}$.

The above proof about $O_T^{e_i}(x)$ gives a necessary and sufficient condition for $O_T^{e_i}(x)$ to be Hilbert-Schmidt and hence for $A_T(x)$ (or $G_T^{e_i}(x)$) to be trace class.

Proposition 2.44. *The following are equivalent*

1) $G_T(x)$ *is trace class*

2) $A_T(x)$ *is trace class*

3) $O_T^{e_i}(x)$ is Hilbert-Schmidt

4) $\sum_n \|T^n x\|^2 < \infty$, that is $(\|T^n x\|)_{n\geq 0}$ is in l^2

In particular, if $O_T^{e_i}(x)$ is Hilbert-Schmidt, then its Hilbert-Schmidt norm is the l^2 norm of the sequence $(\|T^n x\|)_{n\geq 0}$.

Proof: This follows from the formula for the trace of $A_T(x)$. That is,

$$tr(A_T(x)) = \sum_n \|T^n x\|^2$$

□

As mentioned above, a necessary and sufficient condition for $r(T) < 1$ is for $(\|T^n x\|)_{n\geq 0}$ to be in l^1 for each $x \in \mathcal{H}$. This latter result leads to considering $O_T^{e_i}(x)$ for x such that the condition $\sum_n \|T^n x\| < \infty$ holds. We have the following result which, in particular, gives us that $r(T) < 1$ implies that $O_T^{e_i}(x)$ is trace class for all $x \in \mathcal{H}$.

Proposition 2.45. *If $\sum_n \|T^n x\| < \infty$, then $O_T^{e_i}(x)$ is trace class.*

Proof: Firstly, we know that $O_T^{e_i}(x)$ is compact and hence there exist two orthonormal sequences ϕ_i and ψ_i such that $O_T^{e_i}(x)f = \sum_i s_i < f, \phi_i > \psi_i$, where the sequence $(s_i)_{i\geq 0}$ is the singular values of $O_T^{e_i}(x)$. An operator is trace class if and only if its singular values are summable. We have

$$\begin{aligned}
\sum_j s_j &= \sum_j < O_T^{e_i}(x)\phi_j, \psi_j > \\
&= \sum_j < \sum_n < \phi_j, T^n x > e_n, \psi_j > \\
&= \sum_{j,n} < \phi_j, T^n x >< e_n, \psi_j > \\
&\leq \sum_{j,n} | < \phi_j, T^n x > || < e_n, \psi_j > | \\
&\leq \sum_n (\sum_j | < \phi_j, T^n x > |^2 \sum_j | < e_n, \psi_j > |^2)^{\frac{1}{2}} \\
&\leq \sum_n \|T^n x\| \|e_n\| \\
&= \sum_n \|T^n x\|
\end{aligned}$$

By assumption, this last sum is finite. Hence, $O_T^{e_i}(x)$ is trace class. □

Corollary 2.46. *If $r(T) < 1$, then $O_T^{e_i}(x)$ is trace class for each $x \in \mathcal{H}$.*

Proof: This follows from the fact that $r(T) < 1$ if and only if $\sum_n \|T^n x\| < \infty$ for every $x \in \mathcal{H}$. □

We have two results (Theorem 2.40 and Proposition 2.45) linking the Schatten class, $\mathcal{S}_p$, of $O_T^{e_i}(x)$ with the l_p class of the sequence $\{\|T^n x\|\}_n$. The following theorem can be found in [11]. Applying this theorem to $O_T^{e_i}(x)$ leads to a general theorem about the Schatten class of $O_T^{e_i}(x)$.

Theorem 2.47. *([11] page 82) Let $A \in \mathcal{B}(\mathcal{H})$ and let $\{e_i\}$ be an orthonormal basis of $\mathcal{H}$.*

1) If $1 \leq p \leq 2$ *and* $\{\|Ae_i\|\} \in l_p$*, then* $A \in \mathcal{S}_p$ *and*

$$\|A\|_p \leq \|\{\|Ae_i\|\}\|_p$$

2) If $2 \leq p < \infty$ *and* $A \in \mathcal{S}_p$*, then* $\{\|Ae_i\|\} \in l_p$ *and*

$$\|A\|_p \geq \|\{\|Ae_i\|\}\|_p$$

To apply this theorem, note that

$$O_T^{e_i}(x)^* e_j = \sum_n < e_j, e_n > T^n x = T^j x$$

It follows that $\{\|O_T^{e_i}(x)^* e_n\|\} = \{\|T^n x\|\}$. Combining this with the above theorem and the fact that $O_T^{e_i}(x) \in \mathcal{S}_p$ if and only if $O_T^{e_i}(x)^* \in \mathcal{S}_p$, we have the following result.

Corollary 2.48. *Let* $T \in \mathcal{B}(\mathcal{H})$*,* $x \in \mathcal{H}$ *and* $\{e_i\}$ *be an orthonormal basis of* $\mathcal{H}$.

1) If $1 \leq p \leq 2$ *and* $\{\|T^n x\|\} \in l_p$*, then* $O_T^{e_i}(x) \in \mathcal{S}_p$ *and*

$$\|O_T^{e_i}(x)\|_p \leq \|\{\|T^n x\|\}\|_p$$

2) If $2 \leq p < \infty$ *and* $O_T^{e_i}(x) \in \mathcal{S}_p$*, then* $\{\|T^n x\|\} \in l_p$ *and*

$$\|O_T^{e_i}(x)\|_p \geq \|\{\|T^n x\|\}\|_p$$

Chapter 3

Applications to Invariant Subspaces

3.1 Finite Dimensional Case

Here we give a proof of the well known fact that all operators on a complex finite dimensional vector space ($dim > 1$) have a non-trivial invariant subspace. This is done to illustrate properties of $O_T^{e_i}(x)$ in the finite dimensional case. The proof will be based on the finite dimensional counter-part of $O_T^{e_i}(x)$ which was introduced in the Definition 2.2. Throughout, V will be an inner product space over $K = \mathbb{C}$ or $\mathbb{R}$. Recall that given an orthonormal basis $\{e_i\}_{i=0}^n$, $T \in \mathcal{B}(V)$ and $x \in V$, we define the map $O_T^{e_i}(x) : V \to V$ by $y \mapsto \sum_{i=0}^n < y, (T^i x) > e_i$ (see Definition 2.2). As in the infinite dimensional case, there is a link between cyclic vectors of T and the kernel of $O_T^{e_i}(x)$.

However, we can get stronger results using the fact that the dimension of the space is finite.

Theorem 3.1. *The following are equivalent:*

1) $x \in V$ *is cyclic under* T

2) $ker(O_T^{e_i}(x)) = 0$

3) $ker(O_T^{e_i}(x)^*) = 0$

Proof: The equivalence of 1) and 2) follows in the exact same way as the corresponding result in the infinite dimensional case.
The equivalence of 2) and 3) follows from the well-known fact that an operator acting on a finite dimensional space is invertible if and only if its kernel is trivial and the fact that A is invertible if and only if A^* is also invertible. □

Example 3.2. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ with respect to the basis $\{e_0, e_1\}$ acting on $\mathbb{R}^2$. Then it is well known that A has no non-trivial invariant subspaces.
If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then

$$O_T^{e_i}(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{x}_2 & -\bar{x}_1 \end{bmatrix}$$

Hence

$$O_T^{e_i}(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix})^* = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}$$

We have that $det(O_T^{e_i}(x)^*) = x_1^2 + x_2^2$. This homogeneous polynomial is nonzero for all real x_1, x_2 other than $x_1 = 0$, $x_2 = 0$. Hence, T has no non-trivial invariant subspaces. However, if we consider A acting on $\mathbb{C}^2$, then we do have nonzero roots for $x^2 + y^2$ and hence non-trivial invariant subspaces.

We can, using the above example as a guide, prove that every operator on a finite dimensional complex vector space has a non-trivial invariant subspace.

Theorem 3.3. *If A is an operator on a finite dimensional complex vector space, then A has a non-trivial invariant subspace.*

Proof: Fix an orthonormal basis $\{e_i\}$ and consider the function $det : V \to \mathbb{C}$ which maps $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ to $det(O_T^{e_i}(x)^*)$. This map is a homogeneous polynomial in $x_1, \ldots, x_n$. Now, if T has no non-trivial invariant subspaces, then $det(O_T^{e_i}(x)^*)$ is zero only for $x = 0$. However, this is not possible since every homogeneous polynomial over $\mathbb{C}$ has a nonzero root. This implies T must have a non-trivial invariant subspace. $\square$

Notice that, in the standard proof of this theorem, it is the analytic nature of $det(A - \lambda I)$ as a function of λ that is considered. Here we have used the fact that $det(O_T^{e_i}(x)^*)$ is analytic as a funtion of x. In addition, in the finite dimensional setting, we can restate Theorem 3.1 to give a classification of the non-cyclic vectors of an operator.

Theorem 3.4. *If T is a linear operator on V, then x is non-cyclic if and only if $det(O_T^{e_i}(x)^*) = 0$.*

Since $det(O_T^{e_i}(x)^*)$ is a homogeneous polynomial in $x_1, ..., x_n$, finding the zeros of $det(O_T^{e_i}(x)^*)$ is not too difficult.

Example 3.5. Let $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have

$$O_A^{e_i}(x) = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{\lambda}_1\bar{x}_1 & \bar{\lambda}_2\bar{x}_2 \end{bmatrix}$$

It follows that $det(O_A^{e_i}(x)^*) = (\lambda_2 - \lambda_1)x_1x_2$. From this equation, we see that if $\lambda_1 = \lambda_2$, then A has no cyclic vectors (since in this case, for each x, $det(O_A^{e_i}(x)^*) = 0$). While if $\lambda_1 \neq \lambda_2$, then x is a cyclic vector of A if and only if $x_1 \neq 0$ and $x_2 \neq 0$.

3.2 Compact Operators and $O_T^{e_i}(x)$

Using the limit results proved in Section 2.2, we will give a proof of the well-known result that every compact operator has an invariant subspace. To do this, we will deal with the class of quasitriangular operators. The first few results on these operators can be found in Chapter 5 of [19].

Definition 3.6. An operator T is quasitriangular if there exists a sequence of finite rank projections, $\{P_n\}$, which converges to the identity in the strong operator topology and $\|P_nTP_n - TP_n\|$ tends to 0 as n tends to ∞.

Lemma 3.7. *If $T \in \mathcal{B}(\mathcal{H})$, P is a projection and $k \geq 1$, then*

$$\|PT^kP - T^kP\| \leq k\|T\|^{k-1}\|PTP - TP\|$$

Proof: We will use induction on k. The $k = 1$ case is trivial. Assume the result for the exponent k. Then

$$\begin{aligned}
\|PT^{k+1}P - T^{k+1}P\| &= \|(PT^kP - T^kP)(TP) + (T^k - PT^k)(PTP - TP)\| \\
&\leq \|PT^kP - T^kP\|\|TP\| + \|T^k - PT^k\|\|PTP - TP\| \\
&\leq k\|T\|^{k-1}\|PTP - TP\|\|T\| + \|T\|^k\|PTP - TP\| \\
&\leq (k+1)\|T\|^k\|PTP - TP\|
\end{aligned}$$

□

Lemma 3.8. *If T is quasitriangular, then there exists a sequence of finite rank projections, $\{Q_n\}$, which converges in the weak operator topology to an operator, $Q \neq 0$ or I and $\|Q_nTQ_n - TQ_n\|$ tends to 0 as n tends to ∞.*

Proof: (see [19] p.g. 85) □

Recall that T is quasinilpotent if $r(T) = 0$. The Volterra integration operator acting on $L^2(0,1)$ ($f \mapsto \int_0^x f(y)dy$) is an example of a nonzero compact operator which is quasinilpotent (see [9] Chapter II Section 4 or [15] Problem 186).

Lemma 3.9. *If T has a cyclic vector x such that* $\liminf\{\|T^k x\|^{\frac{1}{k}}\} = 0$, *then T is quasitriangular. In particular, if T is quasinilpotent and has a cyclic vector, then T is quasitriangular.*

Proof: We may assume that $\|x\| = 1$. We will consider T and $O_T^{e_i}(x)$ with respect to the natural basis of $O_T^{e_i}(x)$ (see Theorem 2.31). In this case, $e_0 = x$. Then T is given by

$$T = \begin{bmatrix} b_{00} & b_{01} & b_{02} & \dots & b_{0n} & \dots \\ b_{10} & b_{11} & b_{12} & \dots & b_{1n} & \dots \\ 0 & b_{21} & b_{22} & \dots & b_{2n} & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \dots \\ 0 & \dots & 0 & b_{n(n-1)} & b_{nn} & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

while the latter is given by

$$O_T^{e_i}(x) = \begin{bmatrix} \bar{x} \\ \overline{Tx} \\ \vdots \\ \overline{T^n x} \\ \vdots \end{bmatrix} = \begin{bmatrix} a_{00} & 0 & 0 & \dots & 0 & \dots \\ a_{10} & a_{11} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots \\ a_{n0} & a_{n1} & \dots & a_{nn} & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \end{bmatrix}$$

where we have

$$b_{(n+1)n} =< Te_n, e_{n+1} > \text{ and } a_{kk} =< e_k, T^k x >=< e_k, T^k e_0 >$$

Both of these sequences are nonzero since $e_0(= x)$ is cyclic. Moreover, we will begin by showing that

$$b_{(n+1)n} = \frac{< T^{n+1}e_0, e_{n+1} >}{< T^n e_0, e_n >} = \frac{\bar{a}_{(n+1)(n+1)}}{\bar{a}_{nn}} \tag{3.1}$$

To show this, we consider

$$T^n e_0 = f + < T^{n-1}e_0, e_n > e_n$$

where $f \in span\{e_0, \ldots, T^{n-1}\} = span\{e_0, \ldots, e_{n-1}\}$. Hence, $T^{n+1}e_0 = Tf + < T^n e_0, e_n > Te_n$. Then

$$< T^{n+1}e_0, e_{n+1} >=< T^n e_0, e_n >< Te_n, e_{n+1} >=< T^n e_0, e_n > b_{(n+1)n}$$

which can be rearranged to get (3.1).

From this equation, we have that $a_{nn} = \Pi_{j=0}^{n-1} \bar{b}_{(n+1)n}$. Moreover,

$$|a_{nn}|^{\frac{1}{n}} = | < T^n e_0, e_n > |^{\frac{1}{n}} \leq \|T^k e_0\|^{\frac{1}{k}}$$

which implies that a_{nn} tends to 0 and so there exists a subsequence $(n_j)_{j\geq 0}$ such that $b_{(n_j+1)n_j}$ converges to 0. Taking P_j equal to the projection onto

$span\{e_0, \ldots, e_{n_j}\}$, we see that T is quasitriangular since $\|P_jTP_j - TP_j\| = |b_{(n_j+1)n_j}|$. □

Theorem 3.10. *Every compact operator acting on $\mathcal{H}$ has a non-trivial invariant subspace.*

Proof: Let T be a compact operator. We may assume that $\|T\| < 1$ and that T has no eigenvalues. The nonzero elements of the spectrum of a compact operator are eigenvalues. Hence, T is quasinilpotent. Moreover, we may assume that T has a cyclic vector. Then, by Lemma 3.9, T is quasitriangular. Given Q_n and Q as in Lemma 3.8, we will construct a norm convergent sequence of vectors in $\mathcal{H}$ and a weakly convergent sequence of vectors in $\mathcal{H}$ which satisfy the conditions of Corollary 2.23 from which we will have the desired result. For the weakly convergent sequence, consider $y \in \mathcal{H}$ such that $Qy \neq y$ (this can be done since $Q \neq I$). Then $z_n = y - Q_n y$ satisfies

$$z_n \to y - Qy \neq 0 \text{ weakly}$$

$$Q_n z_n = 0 \text{ for each } n$$

For the norm convergent sequence, consider $x \in \mathcal{H}$ such that $Qx \neq 0$ (this can be done since $Q \neq 0$). Since T is compact, we have that $TQ_n x$ converges to TQx in norm. Moreover, since T has no eigenvalues, $TQx \neq 0$. Let $x_n = TQ_n x$. To apply Corollary 2.23, we need to show that the sequence of vectors $O_T^{e_i}(x_n)(z_n)$ converges to 0 in norm. To do this, consider

$$\begin{aligned}
\|O_T^{e_i}(x_n)(z_n)\|^2 &= \sum_{k\geq 0} |< z_n, T^{k+1}Q_n x > |^2 \\
&= \sum_{k\geq 0} |< z_n, (T^{k+1}Q_n - Q_n T^{k+1} Q_n)x > + < z_n, Q_n T^{k+1} Q_n x > |^2 \\
&= \sum_{k\geq 0} |< z_n, (T^{k+1}Q_n - Q_n T^{k+1} Q_n)x > |^2 \text{ (since } Q_n z_n = 0) \\
&\leq \sum_{k\geq 0} \|z_n\|^2 \|T^{k+1}Q_n - Q_n T^{k+1} Q_n\|^2 \|x\|^2 \\
&\leq \|z_n\|^2 \|x\|^2 \|TQ_n - Q_n T Q_n\|^2 (1 + \sum_{k\geq 1} k\|T\|^{2k})
\end{aligned}$$

Since z_n is bounded, $\sum_{k\geq 1} k\|T\|^{2k} < \infty$ and $\|TQ_n - Q_n T Q_n\|$ converges to 0, we have the conditions of Corollary 2.23 satisfied. The result now follows from this Corollary. □

More generally, for any quasitriangular operator, we need only that there exists nonzero $x \in \mathcal{H}$ such that $Q_n x$ converges to $Qx \neq 0$ in norm to conclude that T has a non-trivial invariant subspace. The above method is related to extremal vectors which were introduced in [1].

3.3 Other Relationships Between $O_T^{e_i}(x)$ and Invariant Subspaces

To begin this section, we discuss what happens if T does have an invariant subspace.

Proposition 3.11. *If $\mathcal{M}$ is a T invariant subspace, then for each $x \in \mathcal{M}$, $\mathcal{M}$ is also an invariant subspace of $A_T(x)$.*

Proof: Recall that $A_T(x)z = \sum_n < z, T^n x > T^n x$. Therefore, the range of $A_T(x)$ is contained in $\mathcal{M}_x = \overline{span}\{x, Tx, \ldots\}$. Moreover, $\mathcal{M}_x \subseteq \mathcal{M}$ for each $x \in \mathcal{M}$. These two facts imply the result. □

In general, we do not have that if M is T invariant, then it is $O_T^{e_i}(x)$ invariant also. From Proposition 2.16, we do have that $ker(O_T^{e_i}(x))$ is T^* invariant so that $\overline{range(O_T^{e_i}(x)^*)}$ is T invariant. Hence, $range(A_T(x))$ is T invariant. We now show some connections with dilation and compression theory. Recall the following definitions (which can be found in [14]).

Definition 3.12. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces with $\mathcal{H} \subseteq \mathcal{K}$ and $A \in \mathcal{B}(\mathcal{K})$ and $T \in \mathcal{B}(\mathcal{H})$. Then we call T the compression of A if

$$T = \pi_H A \tau_H$$

where here and in what follows π_H is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ and τ_H is the canonical embedding of $\mathcal{H}$ into $\mathcal{K}$. In matrix form, with respect to $\mathcal{H} \oplus \mathcal{H}^\perp$, we have

$$A = \begin{bmatrix} T & * \\ * & * \end{bmatrix}$$

Definition 3.13. Let $A \in \mathcal{B}(\mathcal{K})$ and $T \in \mathcal{B}(\mathcal{H})$. Then A is said to be a dilation of T if $\mathcal{H}$ is contained in $\mathcal{K}$ and

$$A^n = \pi_H T^n \tau_H, \ n = 1, 2, ...$$

That is, A is a dilation of T if T^n is a compression of A^n for each n.

It is easy to see that A is a dilation of T if and only if

$$< A^n x, y >=< T^n x, y >, \text{ for all } x, y \in \mathcal{H}, \ n = 1, 2, ...$$

Proposition 3.14. *If $T_1 \in \mathcal{B}(\mathcal{K})$ is a dilation of $T_2 \in \mathcal{B}(\mathcal{H})$, then for each $x \in \mathcal{H}$, $A_{T_2}(x)$ is a compression of $A_{T_1}(x)$.*

Proof: Consider

$$\begin{aligned}
\pi_H A_{T_1}(x) \tau_H(f) &= \sum_n < \tau_H(f), T_1^n x > \pi_H(T_1^n x) \\
&= \sum_n < \tau_H(f), T_2^n x > \pi_H T_1^n \tau_H(x) \\
&= \sum_n < f, T_2^n x > T_2^n x \\
&= A_{T_2}(x)
\end{aligned}$$

□

Finally, we discuss unitary equivalence, similarity, and quasi-similarity in connection with $O_T^{e_i}(x)$.

Proposition 3.15. *Given $T \in \mathcal{B}(\mathcal{H})$, we have that*

1) There exists an x such that $O_T^{e_i}(x)$ is unitary if and only if T is the unilateral shift on some basis (i.e. T is unitarily equivalent to the unilateral shift on $\{e_i\}$)

2) There exists an x such that $O_T^{e_i}(x)$ is invertible if and only if T is similar to the unilateral shift

Proof: For 1), given T, $\{e_i\}$, and x, if $O_T^{e_i}(x) = U$ where U is a unitary, then $S^* O_T^{e_i}(x) = O_T^{e_i}(x) T^*$. Thus $T^* = U^* S^* U$ and hence T is the unilateral shift on $\{Ue_i\}$.

We also saw, in Example 2.6, that if T is the unilateral shift on $\{e_i\}$, then $O_T^{e_i}(e_0) = I$. Moreover, by Proposition 2.30, for any basis $\{f_i\}$, $O_T^{f_i}(e_0) = U$ where U is the unitary which maps $\{e_i\}$ to $\{f_i\}$. Hence we have the result.

For 2), if $O_T^{e_i}(x) = P$ an invertible map, then since $O_T^{e_i}(x)T^* = S^* O_T^{e_i}(x)$, we have that $PT^*P^{-1} = S^*$. Hence T is similar to the unilateral shift.

Conversely, if $PTP^{-1} = S$, then

$$I = O_S^{e_i}(e_0) = O_{PTP^{-1}}^{e_i}(e_0) = O_T^{e_i}(P^{-1}e_0)P^*$$

Since the inverse of P^* is unique, we have that $O_T^{e_i}(P^{-1}e_0) = (P^{-1})^*$ and hence the result. □

Given T and R in $\mathcal{B}(\mathcal{H})$, it is easy to see that if T is unitarily equivalent to R, (i.e. if $T = URU^*$ where U is a unitary) then x is cyclic for T if and only if Ux is cyclic for R. We can also see this from Proposition 2.30. From this result, we have

$$O_T^{e_i}(x) = O_{URU^*}^{e_i}(x) = O_R^{e_i}(Ux)U^*$$

Hence, x is cyclic for T if and only if Ux is cyclic for R. More generally, if T is similar to R, that is, if $T = PRP^{-1}$ where P is some invertible operator, then

$$O_T^{e_i}(x) = O_{PRP^{-1}}^{e_i}(x) = O_R^{e_i}(Px)P^*$$

Again, we have that x is cyclic for T if and only if Px is cyclic for R (this is also easy to see directly). An even more general relation than similarity is that of quasi-similarity.

Definition 3.16. Two operators T and R are quasi-similar if there exist operators A and B which are both one-to-one and have dense range with the property that $AT = RA$ and $TB = BR$.

The next result links properties of invariant subspaces to quasi-similarity and can be found in [19]. We note that an invariant subspace of T is called *hyperinvariant* if it is an invariant subspace for every operator which commutes with T.

Theorem 3.17. *If T and R are quasi-similar and T has a non-trivial hyperinvariant subspace, then R has a non-trivial hyperinvariant subspace.*

It is unknown whether this theorem can be weakened to: the existence of a non-trivial invariant subspace of T implies the existence of a non-trivial invariant subspace of R. In relation to the operator $O_T^{e_i}(x)$, we have

Proposition 3.18. *If T and R are quasi-similar (with A and B the operators which implement this relation as in the definition), then $O_T^{e_i}(Bx) = O_R^{e_i}(x)B^*$ and $O_R^{e_i}(Ax) = O_T^{e_i}(x)A^*$*

Proof: By assumption, $BR = TB$ and $RA = TA$. Hence

$$\begin{aligned} O_T^{e_i}(Bx) &= \sum_n < ., T^n Bx > e_n \\ &= \sum_n < ., BR^n x > e_n \\ &= \sum_n < B^*., R^n x > e_n \\ &= O_R^{e_i}(x)B^* \end{aligned}$$

The other case is similar. □

3.4 The Range of $O_T^{e_i}(x)$

3.4.1 Invariant Subspaces and Cyclic Vectors of S^*

To begin this section, we will discuss the H^∞ functional calculus of the backward shift. Recall that a function, f, is in H^∞ if it is analytic in the unit disk and the functions $f_r(\theta) = f(re^{i\theta})$ are uniformly bounded in the L^∞-norm for $0 \leq r < 1$ (see [16]). For $f \in H^\infty$, we would like to determine $f(S^*)$. Let the power series of f be $\sum_n a_n z^n$. Then we can define $f(S^*) = \sum_n a_n (S^*)^n$ where it can be shown that this series converges in the weak operator topology (see [3] or [10]). In matrix form, we have

$$f(S^*) = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & 0 & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

This defines the H^∞ functional calculus for S^* (i.e. the map $f \mapsto f(S^*)$ is an algebra homomorphism see [3] Chapter XI). From Example 2.6, we see that

$$f(S^*) = O_S^{e_i}(a) \text{ where } a = \sum_n \bar{a}_n e_n. \tag{3.2}$$

To prove the next theorem, we need the following two facts. The first is that if $\mathcal{M}$ is a non-trivial invariant subspace of the unilateral shift on $\{e_i\}$, then there exists $a \in \mathcal{M}$ such that $< a, e_i >$ are the Taylor coefficients of an H^∞ function. That is, there exists $f \in H^\infty$ such that $f = \sum_n < a, e_n > z^n$ (see [12] Remark 2.2.2 on page 43). The second is the result that if T is a contraction and there exists $\phi \in H^\infty$ such that $\phi(T) = 0$, then T has an invariant subspace. We note that in our case, since $r(T) < 1$, $\phi(T)$ is defined by the analytic functional calculus of T. The book of Sz-Nagy and Foias [21] contains this result along with many more on this class of operators which are called C_0 contractions. Chapter 2 of [4] also contains many results on C_0

contractions.

Theorem 3.19. *If* $T \in \mathcal{B}(\mathcal{H})$ *with* $r(T) < 1$ *and there exists* $x \in \mathcal{H}$ *such that the range of* $O_T^{e_i}(x)$ *contains a nonzero non-cyclic vector of the backward shift on* $\{e_i\}$*, then* T *has a non-trivial invariant subspace.*

Proof: Suppose that for $x \neq 0$, $y \neq 0$, $w = O_T^{e_i}(x)(y)$ is a non-cyclic vector of the backward shift, S^*, on $\{e_i\}$. If w is the zero vector, then, by Proposition 2.13, T has a non-trivial invariant subspace. Otherwise, it is nonzero and hence $\mathcal{M} = \overline{span}\{w, S^*w, \ldots\}$ is a non-trivial invariant subspace of S^*. By properties of invariant subspaces, $\mathcal{M}^\perp$ is an invariant subspace of S. The first fact (stated in the preceding paragraph) can now be applied to $\mathcal{M}$. It follows that there exists $a \in \mathcal{M}^\perp$ such that $< a, e_i >$ are the Taylor coefficients of an H^∞ function. We will denote this function by f (hence $f(z) = \sum_n < a, e_n > z^n$) and the function with Taylor series $\sum_n < e_n, a > z^n$ by $\hat{f}$ (this is the same notation as in Theorems 2.27 and 2.29). We have that

$$< a, (S^*)^n w >= 0 \text{ for each } n = 0, 1, \ldots$$

That is, $O_S^{e_i}(a)(w) = 0$, but, by Equation 3.2, $O_S^{e_i}(a) = \hat{f}(S^*)$. Using Proposition 2.29, we have that

$$\begin{aligned} 0 &= \hat{f}(S^*)w \\ &= \hat{f}(S^*)O_T^{e_i}(x)(y) \\ &= O_T^{e_i}(f(T)x)(y) \end{aligned}$$

where $f(T)$ is given by the analytic functional calculus of T. Since $y \neq 0$, $O^{e_i}(f(T)x)$ has non-trivial kernel. If $f(T)x \neq 0$, then by Proposition 2.13, T has a non-trivial invariant subspace. Otherwise, $f(T)x = 0$ so that $O_T^{e_i}(f(T)x)(y) = 0$ for all y, but then, by Proposition 2.29, $O_T^{e_i}(x)(\hat{f}(T^*)y) = 0$ for all y. This implies that $O_T^{e_i}(x)$ has non-trivial kernel or $\hat{f}(T^*) = 0$. In the first case, the result follows from Proposition 2.13, while in the second case, it follows from the result on contractions (the second fact above) mentioned just before the statement of the theorem. □

Corollary 3.20. *If $T \in \mathcal{B}(\mathcal{H})$ with $r(T) < 1$ and there exists $x \in \mathcal{H}$ such that the range of $O_T^{e_i}(x)$ is not dense, then T has a non-trivial invariant subspace.*

Proof: Let y be a nonzero element of $\mathcal{H}$. If $O_T^{e_i}(x)y = 0$, then T has a non-trivial invariant subspace by Proposition 2.13. Otherwise, the fact that the range of $O_T^{e_i}(x)$ is not dense implies the set of vectors of the form $\{O_T^{e_i}(x)(p(T^*)y) : p \text{ is a polynomial}\}$ is not dense also. From this fact and Proposition 2.25, it follows that $\{p(S^*)O_T^{e_i}(x)y : p \text{ is a polynomial}\}$ is not dense and hence that $O_T^{e_i}(x)y$ is a non-cyclic vector of S^*. Theorem 3.19 then implies the result. □

Remarks on Corollory 3.20

If $T \in \mathcal{B}(\mathcal{H})$ has no non-trivial invariant subspaces, then a consequence of the above corollary is that there exists an orthonormal basis, $\{f_i\}$, such that $O_T^{f_i}(x)$ is positive. This follows from the polar decomposition of $O_T^{e_i}(x) =$

$V\sqrt{O_T^{e_i}(x)^* O_T^{e_i}(x)}$ and noting that if $O_T^{e_i}(x)$ has dense range and trivial kernel, then V is a unitary and hence we have that $O_T^{V^* e_i}(x) = V^* O_T^{e_i}(x) = \sqrt{O_T^{e_i}(x)^* O_T^{e_i}(x)}$.

Moreover, since in this case the range of $O_T^{e_i}(x)$ is dense and its kernel is trivial, it defines a quasi-affinity (see [21]) between S^* and T^*. This last statement was made by Caradus in a remark following Theorem 1 in [7].

3.4.2 Cyclic Vectors of S^*

Theorem 3.19 in the previous section leads to natural questions about cyclic vectors of the backward shift. Specifically, we would like to study vector spaces of these vectors. Many questions pertaining to these vectors have been answered. For example, in [12], the cyclic vectors of the backward shift are classified in terms of properties of functions in the Hardy space H^2.

The Hardy space, H^2, is defined in similar way as H^∞. Namely, H^2 is the set of analytic functions, f, in the unit disk for which the functions $f_r(\theta) = f(re^{i\theta})$ are uniformly bounded in the L^2-norm for $0 \leq r < 1$ (see [16]). H^2 is a separable complex Hilbert space. The set $\{z^i\}$ forms an orthonormal basis for H^2 and hence the unilateral shift can be realized on H^2 by the mapping $f(z) \mapsto zf(z)$. Its adjoint, the backward shift, can be realized on H^2 by the mapping $f(z) \mapsto P(z^{-1}f(z))$ where P is the projection from L^2 to H^2 (see [16] Chapter 7 or [10]).

Taking $\mathcal{H} = H^2$ allows us to simplify our language. For example, we stated the result that if $\mathcal{M}$ is an invariant subspace of the unilaterial shift, then there exists $x \in \mathcal{M}$ such that $< x, e_i >$ are the Taylor coefficients of an H^∞ function. Taking $\mathcal{H} = H^2$, we can state this result as: if $\mathcal{M}$ is an invariant subspace of the unilaterial shift (realized on H^2 by multipication by z), then there exists $f \in \mathcal{M}$ such that $f \in H^\infty$.

One of the results from [12] is the following, where S^* is the backward shift on the Hardy space H^2.

Theorem 3.21. *If f is holomorphic in $|z| < R$ for some $R > 1$, then f is either S^*-cyclic or is a rational function and hence S^*-noncyclic.*

As is remarked in [12], the assumption on f above can be restated as exponential decay of the Taylor coefficients of f. This result about rational functions and the result of Kronecker that a rational function generates a finite dimensional invariant subspace of S^* can be used to give a proof of Theorem 3.19 which is independent of the result of Sz-Nagy and Foias stated above.

Proof of Theorem 3.19: We may take $\mathcal{H} = H^2$ and $\{e_i\} = \{z^i\}$. To begin, we will show that if $r(T) < 1$, then the Taylor coefficients of any element of the range of $O_T^{z^i}(x)$ decay exponentially. For any $y \in \mathcal{H}$, we have $O_T^{z^i}(x)y = \sum_n < y, T^n x > z^i$. Hence the Taylor coefficients of $O_T^{z^i}(x)y$ are $(< y, T^n x >)_n$. This sequence decays exponentially since $| < y, T^n x > | \leq \|y\| \|T^n x\|$ and $r(T) < 1$.

We now prove the theorem. Assume T has no non-trivial invariant subspaces and that there exists nonzero x and y such that $O_T^{z^i}(x)(y)$ is non-cyclic for S^*.

Then, since the Taylor coefficients of $O_T^{z^i}(x)y$ decay exponentially, we may apply Theorem 3.21 to conclude that $O_T^{z^i}(x)y$ is a rational function. Hence, by the result of Kronecker stated above and Proposition 2.17, $O_{S^*}^{z_i}(O_T^{z_i}(x)(y))$ is a finite rank operator. We have

$$\begin{aligned} O_{S^*}^{z_i}(O_T^{z_i}(x)(y)) &= \sum_n < ., (S^*)^n O_T^{z^i}(x)(y) > z^n \\ &= \sum_n < ., O_T^{z^i}(x)(T^*)^n y > z^n \\ &= O_{T^*}^{z^i}(y) O_T^{z^i}(x)^* \end{aligned}$$

Since T has no non-trivial invariant subspaces, T^* also does not. Hence $ker(O_{T^*}^{z^i}(y))$ is trivial. This result, along with the fact that $O_{S^*}^{z_i}(O_T^{z_i}(x)(y))$ is finite rank, implies $O_T^{z^i}(x)^*$ is finite rank. Hence $O_T^{z^i}(x)$ is also finite rank and hence has non-trivial kernel, by Proposition 2.13, this is a contradiction. □

Shapiro (see [22] or [8]) has proved that if $f = \sum_{n\geq 0} a_n z^n$ with $|a_n|^{\frac{1}{n}} \to 0$ and f is not a polynomial, then f is cyclic for the backward shift on H^2. This will be used to show if T is quasinilpotent and $f \in range(O_T^{e_i}(x))$ is rational, then f must be a polynomial.

Proposition 3.22. *If T is quasinilpotent and x is T-cyclic, then either $range(O_T^{e_i}(x))$ contains only S^*-cyclic vectors or there exists $N > 0$ such that $(S^*)^N O_T^{e_i}(x)(y) = 0$ for some nonzero y (that is, there exists a non-cyclic vector of T in the linear span of x, Tx, T^2x, ...).*

Proof: Consider $z \in \mathcal{H}$ and $O_T^{e_i}(x)(z) = \sum_{n\geq 0} < z, T^n x > e_n$. We will show that if T is quasinilpotent, then the sequence $a_n =< O_T^{e_i}(x)(z), e_n >$ satisfies $|a_n|^{\frac{1}{n}} \to 0$. Consider

$$\begin{aligned} | < O_T^{e_i}(x)(z), e_n > |^{\frac{1}{n}} &= | < z, T^n x > |^{\frac{1}{n}} \\ &\leq \|z\|^2 \|x\|^2 \|T^n\|^{\frac{1}{n}} \end{aligned}$$

Since T is quasinilpotent, we have $|a_n|^{\frac{1}{n}} \to 0$ since $\|T^n\|^{\frac{1}{n}} \to 0$. We can apply the above result of Shapiro to get that $O_T^{e_i}(x)(z)$ is either cyclic under S^* or has only finitely many nonzero terms. If we have a z for which the latter property holds, then there exists $N > 0$ such that $S^N O_T^{e_i}(x)(z) = 0$ and hence by Proposition 2.26, $T^N x$ is a non-cyclic vector of T. Otherwise, all vectors in the range of $O_T^{e_i}(x)$ must be S^* cyclic. □

3.5 The Map $O_T^{e_i}$

Let T be a fixed operator with $r(T) < 1$. In this section, we will consider the map $O_T^{e_i} : \mathcal{H} \to \mathcal{B}(\mathcal{H})$. We have seen that, if $r(T) < 1$, this map is continuous when considered as a map from $\mathcal{H}$ with the norm topology and $\mathcal{B}(\mathcal{H})$ with the operator norm topology (see Proposition 2.19). We have also seen that if $r(T) < 1$, then the range of $O_T^{e_i}$ is contained in the trace class operators and hence are Hilbert-Schmidt operators. We also have mentioned that the Hilbert-Schmidt operators form a Hilbert space, so it is interesting to ask whether the map $O_T^{e_i}$ is a continuous map from $\mathcal{H}$ to the Hilbert-Schmidt operators ($\mathcal{HS}$) with the norm given by $\|A\|_2 = tr(A^*A)$.

Proposition 3.23. *If $r(T) < 1$, then the map $O_T^{e_i} : (\mathcal{H}, \|.\|) \rightarrow (\mathcal{HS}, \|.\|_2)$ is continuous.*

Proof: Suppose in $\mathcal{H}$ we have x_n tending to x in norm. Then

$$\begin{aligned}
\|O_T^{e_i}(x_n - x)\|_2^2 &= \sum_k < O_T^{e_i}(x_n - x)e_k, O_T^{e_i}(x_n - x)e_k > \\
&= \sum_k < \sum_r < e_k, T^r(x_n - x) > e_r, \sum_s < e_k, T^s(x_n - x) > e_s > \\
&= \sum_{k,r} < e_k, T^r(x_n - x) >< T^r(x_n - x), e_k > \\
&= \sum_r \|T^r(x_n - x)\|^2 \\
&\leq \|x_n - x\| \sum_r \|T^r\|
\end{aligned}$$

Since $\sum_r \|T^r\| < \infty$, we have the result. □

Moreover, the map $O_T^{e_i}$ is conjugate linear and its range is a closed subspace of $\mathcal{HS}$. We also have that this map is invertible on its range via the map $\mathcal{HS} \rightarrow \mathcal{H}$ taking A to $A^*(e_0)$. As a result, we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{H} & \xrightarrow{O_T^{e_i}} & \mathcal{HS} \\
T \downarrow & & \downarrow B \\
\mathcal{H} & \xrightarrow{O_T^{e_i}} & \mathcal{HS}
\end{array}$$

where B maps $range(O_T^{e_i})$ to $range(O_T^{e_i})$ via multiplication by the adjoint of the unilateral shift. That is, we have that $T = (O_T^{e_i})^{-1}BO_T^{e_i}$. A logical question to ask is whether there is a more natural way of viewing the operator B.

Let L_{S^*} be the map from $\mathcal{HS}$ to $\mathcal{HS}$ given multiplication by S^* (that is, $A \mapsto S^*A$). This operator's action on the basis of $\mathcal{HS}$ given by the matrix units e_{ij} (the infinite matrix with zero entries except for a 1 for the ij entry) by mapping $e_{ij} = e_{(i-1)j}$. The action of L_{S^*} on this basis implies that L_{S^*} is the unilateral shift of multiplicity $\aleph_0$. We then have that B is the restriction of L_{S^*} to $range(O_T^{e_i})$. This result is similar to Rota's Theorem (Theorem 3.28 on page 54 of [19]) which is stated below. The only difference between our result and Rota's is that $O_T^{e_i}$ is conjugate linear rather than linear. Recall that A is a *part* of B if A is equal to the restriction of B to one of its invariant subspaces.

Theorem 3.24. *If $T \in \mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ separable) with $r(T) < 1$, then T is similar to a part of the adjoint of the unilateral shift of multiplicity $\aleph_0$.*

This theorem states that the adjoint of the unilateral shift of multiplicity $\aleph_0$ is a model for any operator with spectral radius less than one. The previous result gives a nice way of representing that shift (as multiplication by S^* acting on the Hilbert-Schmidt operator). Moreover, this operator acts on the set of Hilbert-Schmidt operators $O_T^{e_i}(x)$ ($x \in \mathcal{H}$). This set forms a Hilbert space (a proper closed subspace of the whole set of Hilbert-Schmidt operators).

In fact, we can recover Rota's result from our result on $O_T^{e_i}(x)$. To do this, realize every member of Hilbert-Schimdt operators as an infinite matrix (the matrix relative to the basis, $\{e_i\}$). Let J be given by $(a_{ij}) \mapsto (\bar{a}_{ij})$ (i.e. the complex conjugate of a_{ij}). Then $JO_T^{e_i}$ is linear and bounded. Next, it is easy to see that $JL_{S^*}J = L_S^*$ and hence that $J(range(O_T^{e_i}))$ is invariant under L_{S^*}. Finally, let C be the restriction of L_{S^*} to the above subspace, namely $J(range(O_T^{e_i}))$; then $C = JBJ$ and so $T = (JO_T)^{-1}CJO_T$. We could also prove Rota's theorem by considering the map $x \mapsto O_T^{e_i}(x)^*$.

Consider the map, $O_T^{e_i}$, as a map from the unit ball of $\mathcal{H}$ with the weak topology to its image with the strong operator topology. In the limit results section, we saw that if x_n tends to x weakly, then $O_T^{e_i}(x_n)$ tends to $O_T^{e_i}(x)$ in the strong operator topology. Since the unit ball is compact in weak topology and for a separable Hilbert space the unit ball is metrizable, the image of $O_T^{e_i}$ is compact in the strong operator topology.

Chapter 4

The Norm One Case

4.1 General Properties

As mentioned, the existence of invariant subspaces is invariant under rescaling. So we can assume that $\|T\| < 1$ when studying invariant subspaces. However, there are theorems which become vacuous when assuming $\|T\| < 1$. For example, we have stated the result that if T is power bounded (i.e. there exists $M > 0$ such that $\|T^n\| \leq M$ for all $n = 0, 1, \dots$) and neither of $(T^n)_{n\geq 0}$ and $((T^*)^n)_{n\geq 0}$ converge in the strong operator topology to zero, then T has a non-trivial invariant subspace. If $\|T\| < 1$, then this theorem does not apply. However, if $\|T\| = 1$, then this theorem applies to many operators. Because of results of this type, we will study $O_T^{e_i}(x)$ in the case when $\|T\| = r(T) = 1$.

In the above sections, we saw that if $r(T) < 1$, then $O_T^{e_i}(x)$ and $A_T(x)$

are compact operators and, moreover, they are trace class operators. If $\|T\| = r(T) = 1$, this need not be the case (for examples consider T equal to the unilateral shift or the identity operator). Indeed, if we choose T to be the identity, then $O_T^{e_i}(x)$ is not even in $\mathcal{B}(\mathcal{H})$. Throughout the section, T will be an operator with norm one.

Proposition 4.1. *$O_T^{e_i}(x)$ is a closed operator.*

Proof: To show this, suppose $y_n \to y$ in the domain of $O_T^{e_i}(x)$ and $O_T^{e_i}(x)y_n \to z$. We must show that $O_T^{e_i}(x)y$ is defined and is equal to z. Consider

$$\begin{aligned} < z, e_k > &= \lim_{n\to\infty} < O_T^{e_i}(x)y_n, e_k > \\ &= \lim_{n\to\infty} < y_n, T^k x > \\ &= < y, T^k x > \end{aligned}$$

Therefore, $\sum_k | < y, T^k x > |^2 = \|z\|^2 < \infty$ (i.e. $y \in dom(O_T^{e_i}(x))$). We also have that $O_T^{e_i}(x)y = \sum_k < y, T^k x > e_k = \sum < z, e_k > e_k = z$. □

Recall that if $\|T\| \leq 1$, then $(I - T^*T)$ is a positive operator. We denote its unique positive square root by D_T (i.e. $D_T = (I - T^*T)^{\frac{1}{2}}$).

Lemma 4.2. *Let $\|T\| \leq 1$ and $D_T = (I - T^*T)^{\frac{1}{2}}$. Then for each $x \in \mathcal{H}$,*

$$\sum_n \|D_T T^n x\|^2 \leq \|x\|^2$$

Proof:

$$\begin{aligned}\sum_n \|D_T T^n x\|^2 &= \sum_n < D_T T^n x, D_T T^n x > \\ &= \sum_n < (I - T^*T)T^n x, T^n x > \\ &= \sum_n (< T^n x, T^n x > - < T^{n+1}x, T^{n+1}x >) \\ &= \|x\|^2 - \lim_{n\to\infty} \|T^n x\|^2 \\ &\leq \|x\|^2\end{aligned}$$

□

Proposition 4.3. *Let $\|T\| \leq 1$ and $D_T = (I - T^*T)^{\frac{1}{2}}$. Then the operator $O_T^{e_i}(x)D_T$ is bounded. Indeed it is Hilbert-Schmidt.*

Proof: To begin, we will show that $O_T^{e_i}(x)D_T$ is everywhere defined. To do this, consider $y \in \mathcal{H}$. We must show that $D_T y$ is in the domain of $O_T^{e_i}(x)$, that is, $\sum_n | < D_T y, T^n x > |^2 < \infty$ (see Definition 2.1). We use Lemma 4.2 to get

$$\begin{aligned}\sum_n | < D_T y, T^n x > |^2 &\leq \|y\|^2 \sum_n \|D_T T^n x\|^2 \\ &\leq \|y\|^2 \|x\|^2\end{aligned}$$

and hence $D_T y$ is in the domain of $O_T^{e_i}(x)$. Thus, $O_T^{e_i}(x)D_T$ is everywhere defined.

To see that it is also bounded, consider a unit vector $y \in \mathcal{H}$. We again use

Lemma 4.2 to get

$$\begin{aligned}
\|O_T^{e_i}(x)D_T y\|^2 &= \sum_n |< D_T y, T^n x >|^2 \\
&\leq \|y\|^2 \sum_n \|D_T T^n x\|^2 \\
&\leq \|x\|^2
\end{aligned}$$

Hence $O_T^{e_i}(x)D_T$ is bounded.

To show $O_T^{e_i}(x)D_T$ is Hilbert-Schmidt, we consider

$$\begin{aligned}
\sum_k \|O_T^{e_i}(x)D_T e_k\|^2 &= \sum_{k,n} |< O_T^{e_i}(x)D_T e_k, e_n >|^2 \\
&= \sum_{k,n} |< D_T e_k, T^n x >|^2 \\
&= \sum_n \|D_T T^n x\|^2 \\
&\leq \|x\|^2
\end{aligned}$$

where the last line follows by Lemma 4.2. □

Proposition 4.4. *If $\|T\| \leq 1$, then the operator $O_T^{e_i}(x)D_{T^*}$ is bounded.*

Proof: We will use the fact that $D_{T^*}T = TD_T$ (see [14]) and the previous result. Consider a unit vector $y \in \mathcal{H}$. Then

$$\begin{aligned}
\|O_T^{e_i}(x)D_{T^*}y\|^2 &= \sum_n |< D_{T^*}y, T^n x >|^2 \\
&= \sum_n |< y, D_{T^*}T^n x >|^2 \\
&= |< y, D_{T^*}x >|^2 + \sum_n |< y, TD_T T^n x >|^2 \\
&= |< y, D_{T^*}x >|^2 + \sum_n |< D_T T^* y, T^n x >|^2 \\
&= |< y, D_{T^*}x >|^2 + \|O_T^{e_i}(x)D_T T^* y\|^2 \\
&\leq \|D_{T^*}x\|^2 + \|O_T^{e_i}(x)D_T\|^2
\end{aligned}$$

where the last line follows since $\|T^*\| \leq 1$ and $\|y\| = 1$. This proves that $O_T^{e_i}(x)D_{T^*}$ is bounded. □

If T is an isometry, then $O_T^{e_i}(x)D_T = 0$, as is the case for the shift operator and the identity operator. Moreover, if D_T is not invertible, then we have in some sense lost information on how $O_T^{e_i}(x)$ acts on the entire Hilbert space. However, the behaviour of the operator $O_T^{e_i}(x)$ is much improved by multiplying by D_T. The map $x \longmapsto O_T^{e_i}(x)D_T$ is also better behaved than the map $x \longmapsto O_T^{e_i}(x)$.

Proposition 4.5. *Let $\|T\| \leq 1$. Then the map taking $(\mathcal{H}, \|.\|) \rightarrow (\mathcal{HS}, \|.\|_2)$ given by $x \longmapsto O_T^{e_i}(x)D_T$ is continuous.*

Proof: As in the above proofs, Lemma 4.2 is key to this proof. Suppose

$x_n \to x$ in norm. Then

$$\begin{aligned}
\|O_T^{e_i}(x_n - x)D_T\|_2^2 &= \sum_{m,k} | < O_T^{e_i}(x_n - x)D_T e_m, e_k > |^2 \\
&= \sum_{m,k} | < D_T e_m, T^k(x_n - x) > |^2 \\
&= \sum_{m,k} | < e_m, D_T T^k(x_n - x) > |^2 \\
&= \sum_{k} \|D_T T^r(x_n - x)\|^2 \\
&\leq \|x_n - x\|^2
\end{aligned}$$

from which we have the result. □

The next result implies that unless T is a co-isometry, then there exists nonzero x such that $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$. Both this result and the one that follows it give sufficient conditions for $O_T^{e_i}(x)$ to be bounded.

Proposition 4.6. *If $x \in \mathcal{H}$ is in the range of D_{T^*}, then $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$.*

Proof: Recall that, by Lemma 4.2, for any contraction T, $\sum_{n \geq 0} \|D_T T^n z\| \leq \|z\|^2$. We will first show that the domain of $O_T^{e_i}(x)$ is all of $\mathcal{H}$; hence we must show that for any $f \in \mathcal{H}$, $\sum_n | < f, T^n x > |^2 < \infty$. By assumption, there exists $y \in \mathcal{H}$ such that $D_{T^*}y = x$.

$$\begin{aligned}
\|O_T^{e_i}(x)f\|^2 &= \sum_{n\geq 0} |< f, T^n x >|^2 \\
&= \sum_{n\geq 0} |< f, T^n D_{T^*} y >|^2 \\
&= \sum_{n\geq 0} |< D_{T^*}(T^*)^n f, y >|^2 \\
&\leq \sum_{n\geq 0} \|y\|^2 \|D_{T^*}(T^*)^n f\|^2 \\
&\leq \|y\|^2 \|f\|^2
\end{aligned}$$

Hence the domain of $O_T^{e_i}(D_{T^*}y) = O_T^{e_i}(x)$ is all of $\mathcal{H}$.

To see that $O_T^{e_i}(x)$ is bounded, we note that the above calculation gives $\|O_T^{e_i}(x)\| \leq \|y\|$. That is, $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$. □

If we assume that there exists x which is cyclic under T such that $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$, then, using $O_T^{e_i}(p(T)x) = p(S^*)O_T^{e_i}(x)$, we have a dense set of vectors y such that $O_T^{e_i}(y) \in \mathcal{B}(\mathcal{H})$. A natural question is whether $O_T^{e_i}(x)$ is a Hilbert-Schmidt operator when x is in the range of D_{T^*}. This is false, as the next example shows.

Example 4.7. Consider S, the unilateral shift on the basis $\{e_i\}$. Then $O_S^{e_i}(D_{S^*}e_0) = O_s^{e_i}(e_0)$ equals the identity operator. Moreover, since e_0 is a cyclic vector of S, we have a dense set of vectors such that $O_S^{e_i}(x)$ are in $\mathcal{B}(\mathcal{H})$. The operators associated with these vectors are $p(S^*)$ where p is a polynomial. Operators of this form are never compact unless $p = 0$.

Next we show that if x is in the range of D_T, then $O_T^{e_i}(x)$ is in $\mathcal{B}(\mathcal{H})$.

Proposition 4.8. *If $x = D_T y$, then $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$.*

Proof: Consider

$$\begin{aligned}
\|O_T^{e_i}(x)f\|^2 &= \sum_{n\geq 0} |< f, T^n x >|^2 \\
&= \sum_{n\geq 0} |< f, T^n D_T y >|^2 \\
&= |< f, D_T y >|^2 + \sum_{n\geq 0} |< D_{T^*}(T^*)^n f, y >|^2 \\
&\leq \|f\|^2\|D_T y\|^2 + \sum_{n\geq 0} \|y\|^2\|D_{T^*}(T^*)^n f\|^2 \\
&\leq \|y\|^2(\|D_T\|^2\|f\|^2 + \|f\|)
\end{aligned}$$

This shows that the domain of $O_T^{e_i}(x)$ is all of $\mathcal{H}$ and the operator norm of $O_T^{e_i}(x)$ is bounded by $(\|D_T\|^2 + 2)\|y\|^2$. Hence, $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$. □

The domain of $O_T^{e_i}(x)$ need not be dense. For example, if $T = I$, then we saw in Example 2.9 that the domain of $O_I^{e_i}(x)$ was $\{x\}^\perp$ which is clearly not dense. We now give a sufficient condition for the domain to be dense. This result will be generalized in Corollary 4.12.

Proposition 4.9. *If $T \in \mathcal{B}(\mathcal{H})$ and $\|Th\| < \|h\|$ for all nonzero $h \in \mathcal{H}$, then for each $x \in \mathcal{H}$, $O_T^{e_i}(x)$ is a closed and densely defined operator.*

Proof: Above, it was shown that $O_T^{e_i}(x)$ is, in general, closed. To show that it is densely defined, we will show that the range of D_T is dense. This follows since

$$ker(D_T) = \{h \in \mathcal{H} : \|Th\| = \|h\|\}$$

and, by assumption, this is trivial and hence the range of D_T is dense. The result now follows since $O_T^{e_i}(x)$ is well-defined on the range of D_T. □

4.2 Completely Non-unitary Contractions

Recall that a contraction, T, is *completely non-unitary* if there is no nonzero reducing subspace, $\mathcal{M}$, such that $T|_{\mathcal{M}}$ is a unitary operator. The goal of this section is to prove that if T is a completely non-unitary contraction, then for all $x \in \mathcal{H}$, $O_T^{e_i}(x)$ is a closed and densely defined operator. Moreover, for a dense set of $x \in \mathcal{H}$, we will prove $O_T^{e_i}(x)$ is bounded. To do this, we will study properties of the operators $D_{T^n} = (I - (T^*)^n T^n)^{\frac{1}{2}}$ and $D_{(T^*)^n} = (I - T^n (T^*)^n)^{\frac{1}{2}}$ for $n = 1, 2, 3, \ldots$.

Proposition 4.10. *For each $k \in \mathbb{N}$, $O_T^{e_i}(x) D_{T^k}$ is bounded. It is moreover Hilbert-Schmidt.*

Proof: The proof is very similar to the proof for the $k = 1$ case. We will only prove the first statement. To do this, consider a unit vector $y \in \mathcal{H}$.

Then

$$\begin{aligned}
\|O_T^{e_i}(x)D_{T^k}y\|^2 &= \sum_n |< D_{T^k}y, T^n x >|^2 \\
&\leq \sum_n \|y\|^2 \|D_{T^k}T^n x\|^2 \\
&= \sum_n < D_{T^k}T^n x, D_{T^k}T^n x > \\
&= \sum_n < (I - (T^*)^k T^k)T^n x, T^n x > \\
&= \sum_n (< T^n x, T^n x > - < T^{n+k}x, T^{n+k}x >) \\
&= (\sum_{n=0}^{k} \|T^n x\|^2) - \lim_{n\to\infty} \|T^n x\|^2 \\
&\leq (k+1)\|x\|^2
\end{aligned}$$

Hence, $O_T^{e_i}(x)D_{T^k}$ is bounded. □

Proposition 4.11. *For each $k \in \mathbb{N}$, $O_T^{e_i}(x)D_{(T^*)^k}$ is bounded.*

Proof: Again, this is similar to the proof for the $k = 1$ case. We will use the fact that $D_{(T^*)^k}T^k = T^k D_{T^k}$ and the previous result. Consider a unit vector

$y \in \mathcal{H}$. Then

$$\begin{aligned}
\|O_T^{e_i}(x)D_{(T^*)^k}y\|^2 &= \sum_n |< D_{(T^*)^k}y, T^n x >|^2 \\
&= \sum_n |< y, D_{(T^*)^k}T^n x >|^2 \\
&= \sum_{n=0}^{k-1} |< y, D_{(T^*)^k}T^n x >|^2 + \sum_n |< y, T^k D_{T^k} T^n x >|^2 \\
&= \sum_{n=0}^{k-1} |< y, D_{(T^*)^k}T^n x >|^2 + \sum_n |< D_{T^k}(T^*)^k y, T^n x >|^2 \\
&= \sum_{n=0}^{k-1} |< y, D_{(T^*)^k}T^n x >|^2 + \|O_T^{e_i}(x)D_{T^k}(T^*)^k y\|^2 \\
&\leq k\|D_{(T^*)^k}\|^2\|x\|^2 + \|O_T^{e_i}(x)D_{T^k}\|^2
\end{aligned}$$

where the last line follows since $\|T^*\| \leq 1$ and $\|y\| = 1$. This proves that $O_T^{e_i}(x)D_{(T^*)^k}$ is bounded. □

Corollary 4.12. *If T is a completely non-unitary contraction, then for all $x \in \mathcal{H}$, $O_T^{e_i}(x)$ is a closed densely defined linear map.*

Proof: The above two results imply that if for some $k \in \mathbb{N}$, $h \in range(D_{T^k})$ or $range(D_{(T^*)^k})$, then $O_T^{e_i}(x)h$ is in the Hilbert space, that is, $O_T^{e_i}(x)$ is well-defined for such vectors. We now show that these vectors form a dense linear manifold. By assumption, T is completely non-unitary, which implies that $\bigcap_{n=1}^{\infty}(ker(D_{T^n}) \cap ker(D_{(T^*)^n})) = \{h \in \mathcal{H} : \|T^n h\| = \|(T^*)^n h\| = \|h\|$ for each $n = 1, 2, \ldots\} = 0$ (see [21] Theorem 3.2 on page 9). It follows that $span\{range(D_{(T^*)^n}), range(D_{T^n})\}$ is dense, which implies the result. □

Proposition 4.13. *If $x \in range(D_{(T^*)^k})$ for some k, then $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$.*

Proof: Let y be a unit vector in $\mathcal{H}$ and $D_{(T^*)^n}h = x$. Then consider

$$\begin{aligned} \|O_T^{e_i}(D_{(T^*)^k}h)y\|^2 &= \sum_n |< y, T^n D_{(T^*)^k}h >|^2 \\ &= \sum_n |< (T^*)^n y, D_{(T^*)^k}h >|^2 \\ &= \|O_{T^*}^{e_i}(y)D_{(T^*)^k}h\|^2 \\ &\leq \|O_{T^*}^{e_i}(y)D_{(T^*)^k}\|^2\|h\|^2 \end{aligned}$$

Since h is fixed and the map $y \mapsto O_{T^*}^{e_i}(y)D_{(T^*)^k}$ is continuous, we have the result. □

Proposition 4.14. *If $x \in range(D_{T^k})$ for some k, then $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$.*

Proof: Let y be a unit vector and $D_{T^k}h = x$. We will again use the fact that $T^k D_{T^k} = D_{(T^*)^k}T^k$. Consider

$$\begin{aligned} \|O_T^{e_i}(D_{T^k}h)y\|^2 &= \sum_n |< y, T^n D_{T^k}h >|^2 \\ &= \sum_{n=0}^{k-1} |< y, T^n D_{T^k}h >|^2 + \sum_n |< (T^*)^n y, D_{(T^*)^n}T^k h >|^2 \\ &\leq k\|D_{T^k}h\|^2 + \|O_{T^*}^{e_i}(y)D_{(T^*)^n}T^k h\|^2 \end{aligned}$$

The result now follows since the map y to $O_T^{e_i}(y)D_{T^n}$ is continuous. □

Corollary 4.15. *If T is a completely non-unitary contraction, then for a dense set of $x \in \mathcal{H}$, $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$*

Proof: The result follows from Propositions 4.13 and 4.14 since the set $span\{range(D_{(T^*)^n}), range(D_{T^n})\}$ is dense in the case of a completely non-unitary contraction. □

We can extend part of Theorem 3.19 to the $\|T\| = 1$ case. Later, we will show that every nonzero element of the range of $O_T^{e_i}(x)$ must be S^*-cyclic.

Theorem 4.16. *Let $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \leq 1$. If there exists nonzero $x \in \mathcal{H}$ such that the range of $O_T^{e_i}(x)$ does not contain a cyclic vector of the backward shift on $\{e_i\}$, then T has a non-trivial invariant subspace. In particular, if there exists nonzero $x \in \mathcal{H}$ such that the range of $O_T^{e_i}(x)$ is not dense, then T has a non-trivial invariant subspace.*

Proof: Assume that T has no non-trivial invariant subspaces. Since every isometry has a non-trivial invariant subspace, T is not an isometry, i.e., $T^*T \neq I$ and so $D_T \neq 0$. Similarly, T cannot have a nonzero reducing subspace on which it is unitary. In other words, T is completely non-unitary so the H^∞ functional calculus is well defined (see [3] Chapter XI or [21]). Let x be a nonzero vector. The kernel of $O_T^{e_i}(x)$ is trivial, since otherwise T has a non-trivial invariant subspace by Proposition 2.13. Let $y \in \mathcal{H}$ such that $D_T y \neq 0$ and let $z = O_T^{e_i}(x) D_T y$. We will show this vector is cyclic under S^* from which the result will follow. Suppose it is not cyclic. If z is the zero vector, then $ker(O_T^{e_i}(x))$ is non-trivial which is not possible. Hence, it is nonzero, so there exists (see [12] Remark 2.2.2 on page 43) $f \in H^\infty$ with Fourier coefficients $(a_n)_{n\geq 0}$ such that, if we let $a = \sum_n a_n e_n$, then

$$< a, (S^*)^n z >= 0 \text{ for each } n$$

That is, $O_S^{e_i}(a)(z) = 0$, but, by Equation 3.2 in Section 3.4.1, $O_S^{e_i}(a) = \hat{f}(S^*)$ where $\hat{f}$ is as in Notation 2.24. Therefore,

$$\begin{aligned} 0 &= \hat{f}(S^*)z \\ &= \hat{f}(S^*)O_T^{e_i}(x)(D_T y) \\ &= O_T^{e_i}(f(T)x)(D_T y) \end{aligned}$$

If $f(T)x$ is nonzero, then $O_T^{e_i}(f(T)x)$ has non-trivial kernel. Since we have assumed T has no non-trivial invariant subspaces, this is not possible (see Proposition 2.13). Hence $f(T)x = 0$ so that $O_T^{e_i}(f(T)x)(y) = 0$ for all y. Then, by Proposition 2.29, $O_T^{e_i}(x)(\hat{f}(T^*)y) = 0$ for all y. Since $O_T^{e_i}(x)$ has trivial kernel, this implies that $\hat{f}(T^*) = 0$. From the result of Sz-Nagy and Foias (see [21] page 133) that contractions, which satisfy $\phi(T) = 0$ for some $\phi \in H^\infty$, have non-trivial invariant subspaces, it follows that T^* has a non-trivial invariant subspace. This is a contradiction. Hence $O_T^{e_i}(x)D_T y$ cannot be a non-cyclic vector of S^*.

The proof of the second statement is the same as the proof given for Corollary 3.20. □

Next, we extend this result to generalize Theorem 3.19 to the $\|T\| \leq 1$ case. That is, if T has no nontrivial invariant subspaces, then Theorem 4.16 implies that the range of $O_T^{e_i}(x)$ contains a S^*-cyclic vector. In the next theorem, we show that, in the case of an invariant subspace free operator, every nonzero vector in the range of $O_T^{e_i}(x)$ is S^*-cyclic.

Theorem 4.17. *If $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \leq 1$ is invariant subspace free, then every nonzero vector in the range of $O_T^{e_i}(x)$ is a cyclic vector of the backward shift on $\{e_i\}$.*

Proof: Assume that T is invariant subspace free. Then, by Proposition 2.13, $O_T^{e_i}(x)$ has trivial kernel and, by Theorem 4.16, dense range. Furthermore, T is completely non-unitary so, by Corollary 4.12, $O_T^{e_i}(x)$ is a closed and densely defined operator. Hence, $O_T^{e_i}(x)^*$ has trivial kernel and dense range. Moreover, since T has no non-trivial invariant subspaces, so does T^* and hence for each $y \neq 0$, $O_{T^*}^{e_i}(y)$ has trivial kernel (by Proposition 2.13).
Now, suppose that for x and y nonzero, $O_T^{e_i}(x)(y)$ is S^* non-cyclic. Then, by Proposition 2.13, there exists nonzero $z \in \mathcal{H}$ such that

$$\begin{aligned} 0 &= O_{S^*}^{e_i}(O_T^{e_i}(x)(y))z \\ &= O_{T^*}^{e_i}(y)O_T^{e_i}(x)^*z \end{aligned}$$

Now, since $O_{T^*}^{e_i}(y)$ and $O_T^{e_i}(x)^*$ both have trivial kernels, we must conclude that $z = 0$, which is a contradiction. □

4.3 $C_{1\cdot}$ and $C_{0\cdot}$ Contractions

Recall that a contraction T is called $C_{1\cdot}$ if for each $x \neq 0$, $\|T^n x\| \not\to 0$, and is called $C_{0\cdot}$ if for each x, $\|T^n x\| \to 0$ (see [21] page 72). In this section, we link properties of $O_T^{e_i}(x)$ to the type of contraction T is.

Proposition 4.18. *If T is a $C_{1\cdot}$ contraction, then $O_T^{e_i}(x)$ is not compact.*

Proof: Suppose not, that is, assume $O_T^{e_i}(x)$ is compact. In particular, $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$. Hence, for each $y \in \mathcal{H}$, $\sum_n |<y, T^n x>|^2 < \infty$. It follows that $T^n x$ converges weakly to 0. Since $O_T^{e_i}(x)$ is compact, $O_T^{e_i}(x)(T^k x)$ converges in norm to 0. However,

$$\begin{aligned} \|O_T^{e_i}(x)(T^k x)\|^2 &= \sum_n |<T^k x, T^n x>|^2 \\ &\geq \|T^k x\|^2 \end{aligned}$$

From which it follows that $\|T^k x\| \to 0$. This is a contradiction. □

Using this last result and Theorem 4.17, we have the following result.

Corollary 4.19. *If T is an invariant subspace free $C_{1\cdot}$ contraction, then the range of $O_T^{e_i}(x)$ contains an infinite dimensional closed subspace whose nonzero elements are cyclic vectors of the backward shift operator.*

4.4 Closed Range

In Theorem 2.33, we saw that if $O_T^{e_i}(x)$ was compact and had closed range, then $\overline{span\{x, Tx, ...\}}$ is finite dimensional and therefore T has a non-trivial finite dimensional invariant subspace. Moreover, closed range is one of the properties which is invariant under a change of basis. We would hope for a result linking this property with properties of invariant subspaces. That is, we would ask the natural question, whether for the case $\|T\| \leq 1$ it follows that if the range of $O_T^{e_i}(x)$ is closed, then T has a non-trivial invariant subspace. We will show that it does, but we must be sure that $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$. We will give two proofs. For the first, we will use the following two results.

Proposition 4.20. *([19] page 107) If $I - T^*T$ is Hilbert Schmidt, then T has a non-trivial invariant subspace.*

Proposition 4.21. *(cf. [10] page 85) If $\mathcal{H}$ is a separable Hilbert space and R is a non-compact operator in $\mathcal{B}(\mathcal{H})$, then there exists operators A, B such that $ARB = I$.*

Theorem 4.22. *Assume $O_T^{e_i}(x)$ is a non-compact operator in $\mathcal{B}(\mathcal{H})$ and, hence there exists A and B such that $AO_T^{e_i}(x)B = I$. If $BD_T = D_TC$ for some operator $C \in \mathcal{B}(\mathcal{H})$, then T has an invariant subspace. In particular, if $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$ has closed range, then T has a non-trivial invariant subspace.*

Proof: We have $AO_T^{e_i}(x)B = I$ so that $AO_T^{e_i}(x)BD_T = D_T$. By assumption, $D_T = AO_T^{e_i}(x)BD_T = AO_T^{e_i}(x)D_TC$, but, since $O_T^{e_i}(x)D_T$ is Hilbert-Schmidt,

$AO_T^{e_i}(x)D_TB = D_T$ is also Hilbert-Schmidt. This implies that $I - T^*T$ is Hilbert-Schmidt (actually trace class). So, the result follows by Proposition 4.20.

If $O_T^{e_i}(x)$ has closed range and is compact, then it is finite-rank and hence has non-trivial kernel. Thus, T has an invariant subspace by Proposition 2.13. If $O_T^{e_i}(x)$ is non-compact, has closed range and has trivial kernel, then there exists $A \in \mathcal{B}(\mathcal{H})$ such that $AO_T^{e_i}(x) = I$; that is, B (and C) in the above can be taken to be I. The first statement of the theorem now applies so that we have the result. □

We now give the second proof. It uses Theorem 4.16, which asserts that if the range of $O_T^{e_i}(x)$ is not dense, then T has a non-trivial invariant subspace.

Theorem 4.23. *If $O_T^{e_i}(x) \in \mathcal{B}(\mathcal{H})$ has closed range, then T has a non-trivial invariant subspace.*

Proof: Assume that range of $O_T^{e_i}(x)$ is closed. If it is not dense, then, by Theorem 4.16, T has an invariant subspace. We are left with the case when the range of $O_T^{e_i}(x)$ is both closed and dense; that is, the case when the range of $O_T^{e_i}(x)$ is $\mathcal{H}$. Hence, $O_T^{e_i}(x)$ is onto $\mathcal{H}$. Thus, there exists $y \in \mathcal{H}$ such that $O_T^{e_i}(x)y = e_0$ and, hence, $0 = S^*O_T^{e_i}(x)y = O_T^{e_i}(Tx)y$ from which it follows that $O_T^{e_i}(Tx)$ has a non-trivial kernel. If $Tx = 0$, then $ker(T)$ is a non-trivial invariant subspace of T. Otherwise $Tx \neq 0$ and, by Proposition 2.13, it follows that T has a non-trivial invariant subspace. □

Recall that an operator, $A \in \mathcal{B}(\mathcal{H})$, is Fredholm if $range(A)$ is closed and both $ker(A)$ and $ker(A^*)$ are finite dimensional. For such A, the index is defined by $dim(ker(A)) - dim(ker(A^*)$ and will be denoted by $ind(A)$. Both [13] (see Chapter XI) and [9] (see Chapter XI) contain discussions on Fredholm operators. We will use Theorem 3.2 in [13], which states that if A and B are Fredholm, then BA is Fredholm and $ind(BA) = ind(B) + ind(A)$. We also note that the backward shift is Fredholm with index one. In the next theorem, we prove directly (i.e. without using Theorem 4.23) that under certain conditions on $O_T^{e_i}(x)$, T has an invariant subspace.

Proposition 4.24. *a) If $O_T^{e_i}(x)$ is Fredholm, then T has a non-trivial invariant subspace.*
b) If $O_T^{e_i}(x)$ is onto $\mathcal{H}$, then T has non-trivial invariant subspace.

Proof: For a) Let $n = ind(O_T^{e_i}(x))$.
Case 1: $n > 0$ It then follows from the definition of the index that $ker(O_T^{e_i}(x))$ is non-trivial and hence that T has a non-trivial invariant subspace by Proposition 2.13.
Case 2: $n \leq 0$ Consider $(S^*)^{-n+1}O_T^{e_i}(x)$. From the property of the index discussed above, we know that $ind((S^*)^{-n+1}O_T^{e_i}(x))$ =(-n+1)$ind(S^*) + ind(O_T^{e_i}(x)$. It follows, using the fact that $ind(S^*) = 1$, that $ind((S^*)^{-n+1}O_T^{e_i}(x)) =$ 1 and hence has non-trivial kernel. By Proposition 2.25, $O_T^{e_i}(T^{-n+1}x) = (S^*)^{-n+1}O_T^{e_i}(x)$ and so, by Proposiiton 2.13, $T^{-n+1}x$ is T-non-cyclic.
Part (b) follows from Theorem 4.23 also, we will give a direct proof. If $O_T^{e_i}(x)$ is surjective, then there exists $y \in \mathcal{H}$ such that $O_T^{e_i}(x)y = e_0$. Hence,

$S^*O_T^{e_i}(x)y = 0$ where S^* is the backward shift on $\{e_i\}$ and so $O_T^{e_i}(Tx)y = 0$. Thus, Tx is non-cyclic and we have the result. □

In the proof of the above proposition, we gained information about the cyclic behaviour of vectors in the orbit of x (i.e. $x, Tx, \ldots$).

Corollary 4.25. *a) If $O_T^{e_i}(x)$ is Fredholm with positive index, then x is T-non-cyclic.*

b) If $O_T^{e_i}(x)$ is Fredholm with $ind(O_T^{e_i}(x)) = -n$ ($n \in \mathbb{N}$), then $T^{n+1}x$ is T-non-cyclic.

c) If $O_T^{e_i}(x)$ is surjective, then Tx is T-non-cyclic.

Proof: Follows from the proof of Proposition 4.24. □

Unilateral shift and its adjoint provide examples of operators for which the previous theorem can be applied (see Example 2.6 and 2.7). We can also use Proposition 4.20 to generalize the result that if $O_T^{e_i}(x)$ is finite rank, then T has an invariant subspace. This theorem can be applied to any contraction with a finite dimensional invariant subspace or finite rank defect operator (i.e. D_T).

Proposition 4.26. *If $O_T^{e_i}(x)D_T$ is a finite rank operator, then T has an invariant subspace.*

Proof: If $O_T^{e_i}(x)$ has non-trivial kernel, then, by Proposition 2.13, we are done. So assume $ker(O_T^{e_i}(x))$ is trivial. We will show that D_T must be a finite rank operator. $O_T^{e_i}(x)D_T$ being finite rank implies that kernel of

$O_T^{e_i}(x)D_T$ has finite codimension. Since $O_T^{e_i}(x)$ is one-to-one, it follows that $ker(D_T) = ker(O_T^{e_i}(x)D_T)$. It follows that $ker(D_T)$ has finite codimension and hence that D_T is a finite rank operator. Now, by applying Proposition 4.20, we have the result. □

4.5 Spectral Properties

In previous sections, we have proved a number of conditions on the spectrum of $O_T^{e_i}(x)$ which lead to the existence of invariant subspaces for T. Here we will summarize these results, but first, we would like to mention that the spectrum of $O_T^{e_i}(x)$ is not invariant under a change of basis. That is, $O_T^{Ue_i}(x)$ and $O_T^{e_i}(x)$ may have different spectrum. However, from the construction of $O_T^{e_i}(x)$, we have been most interested in whether zero is an eigenvalue of $O_T^{e_i}(x)$. This property is however independent of the choice of basis. This is evident since, by Proposition 2.30, $O_T^{Ue_i}(x) = UO_T^{e_i}(x)$ for every unitary U. More generally, if zero is an approximate eigenvalue of $O_T^{e_i}(x)$, then zero is an approximate eigenvalue of $O_T^{Ue_i}(x)$ for any unitary U. To see this, by the definition of an approximate eigenvalue, there exists a sequence $(y_n)_{n\geq 0} \in \mathcal{H}$ with $\|y_n\| = 1$ such that $O_T^{e_i}(x)y_n \to 0$. It then follows that

$$O_T^{Ue_i}(x)y_n = UO_T^{e_i}(x)y_n \to 0$$

and hence that zero is an approximate eigenvalue of $O_T^{Ue_i}(x)$.

Theorem 4.27. *If T has no non-trivial invariant subspaces, then for each nonzero x, zero is an approximate eigenvalue of $O_T^{e_i}(x)$ (but is not an eigenvalue).*

Proof: Firstly, zero must be in the spectrum of $O_T^{e_i}(x)$, since otherwise $O_T^{e_i}(x)$ is surjective and, by Proposition 4.24, T has a non-trivial invariant subspace. If zero is an eigenvalue of $O_T^{e_i}(x)$, then, by Proposition 2.13, T has a non-trivial invariant subspace. Finally, if $O_T^{e_i}(x)$ does not have dense range, then, by Theorem 4.16, T has a non-trivial invariant subspace. The only remaining possiblity is that zero is an approximate eigenvalue of $O_T^{e_i}(x)$. □

Chapter 5

Conclusion

We have proved a number of results linking the properties of invariant subspaces of T with the operator $O_T^{e_i}(x)$. The most basic is the result that x is T cyclic if and only if the kernel of $O_T^{e_i}(x)$ is trivial. However, we have seen that the range of $O_T^{e_i}(x)$ also plays an important role in the structure of the invariant subspaces of T. Indeed, the result that the range is dense or T has a non-trivial invariant subspace, along with the generalization that the range must contain only S^*-cyclic vectors and the zero vector, puts strong conditions on the range of $O_T^{e_i}(x)$ for an invariant subspace free operator T. Another interesting property of $O_T^{e_i}(x)$ is its connection with Rota's Theorem (see Section 3.5). This allows us to view the set of Hilbert-Schmidt operators $\{O_T^{e_i}(x)|x \in \mathcal{H}\}$ as a Hilbert space and the action of T^* on $\mathcal{H}$ as multipication by the adjoint of the backward shift on $\{e_i\}$.

A number of additional questions may be asked about $O_T^{e_i}(x)$. For example, what are necessary and sufficient conditions on T for $O_T^{e_i}(x)$ to be bounded for each $x \in \mathcal{H}$? Conditions that imply the compactness of $O_T^{e_i}(x)$ are also of interest. Such conditions would be of particular interest if it could be shown that T having non-trivial invariant subspaces implies they hold. The spectrum of a number of the counter-examples to the invariant subspaces problem in the Banach space setting have been calculated; for each, the spectral radius is strictly less than the norm of the operator. Given T with norm one we have seen how the behaviour of $O_T^{e_i}(x)$ differs greatly between the $r(T) < 1$ and $r(T) = 1$ cases. Thus properties of $O_T^{e_i}(x)$ in the case $r(T) = \|T\| = 1$ may lead to results on the structure of invariant subspaces of operators in this case.

Given a Banach space, X, with Schauder basis, e_n, there is a natural generalization of the operator $O_T^{e_i}(x)$. Namely, we can define a mapping from its dual into itself via the formula $f \longmapsto \sum_n f(T^n x)e_n$. More generally, given a separable Banach space, X, and a Banach space, Y, with Schauder basis, e_n, we can define $O_T^{e_i}(x) : X^* \longrightarrow Y$ via $f \longmapsto \sum_n f(T^n x)e_n$. The kernel of this map is independent of the choice of Y and, as in the Hilbert space case, is trivial if and only if x is a cyclic vector of T. It would be of interest to compare results we have obtained for the case when Y is a Hilbert space with corresponding results when Y is a sequence space (e.g. c_0 or l_p).

Appendix A

Notation

We list here the notation of the thesis.
Standard
$\mathcal{H}$ - a separable complex Hilbert space
$\mathcal{M}$ - a subspace (closed linear manifold) of $\mathcal{H}$
$\mathcal{B}(\mathcal{H})$ - operators from $\mathcal{H}$ to $\mathcal{H}$
$Lat(T)$ - Lattice of invariant subspaces of T
$span$ - linear span
$dim(\mathcal{M})$ - dimension of $\mathcal{M}$
$ker(T)$ - $x \in \mathcal{H}$ such that $Tx = 0$
$ran(T)$ - $y \in \mathcal{H}$ such that there exists x with $Tx = y$
$\sigma(T)$ - spectrum of T
$r(T)$ - spectral radius of T
$tr(T)$, $det(T)$ - trace of T and determinant of T
$\{T\}'$ - $R \in \mathcal{B}(\mathcal{H})$ such that $RT = TR$
$\mathcal{S}_p$ - Schatten classes
$\mathcal{HS}$ - the Hilbert-Schmidt operators (equal to $\mathcal{S}_2$)
$C_{1\cdot}$ - set of contractions such that $T^n x \not\to 0$ for all $x \neq 0$
$C_{0\cdot}$ - set of contractions such that $T^n x \to 0$ for all x
$\mathcal{H}^p$ - Hardy spaces
D_T - defect operator of a contration T $(D_T = (I - T^*T)^{\frac{1}{2}})$
Non-standard
$\mathcal{M}_x$ - $\overline{span}\{x, Tx, \ldots\}$
$O_T^{e_i}(x)$ - the orbit operator of T at x
$A_T(x)$ - $O_T^{e_i}(x)^* O_T^{e_i}(x)$
$G_T(x)$ - $O_T^{e_i}(x) O_T^{e_i}(x)^*$

Bibliography

[1] S. Ansari and P. Enflo, *Extremal Vectors and Invariant Subspaces*, Trans. Amer. Math. Soc. **350** no. 2 (1998) 539-558

[2] S. K. Bari, *Biorthogonal systems and bases in Hilbert space*, Moskov Gos Univ Ucenye Zapinski **148**, Matematika **4**, (1951) 69-107. (Russian)

[3] B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North Holland 1988

[4] H. Bercovici, *Operator Theory and Arithmetic in H^∞*, AMS 1988

[5] S. W. Brown, *Some invariant subspaces for subnormal operators*, Integral Equations Operator Theory **1** (1978), 310-333

[6] S. W. Brown, *Hyponormal operators with thick spectra have invariant subspaces* Ann. of Math (2) **125** (1987) 93-103

[7] S. R. Caradus, *Invariant subspaces of operators related to the unilateral shift*, Collection of articles dedicated to the memory of Hanna Neumann, II. J. Austral. Math. Soc. 16 (1973), 210–213.

[8] J. A. Cima and W. T. Ross, *The Backward Shift on the Hardy Space* Amer. Math. Soc. 2000

[9] J. B. Conway, *A Course in Functional Analysis, Second ed.* New York: Springer-Verlag. 1990

[10] J. B. Conway, *A Course in Operator Theory* Amer. Math. Soc. 1991

[11] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators* Cambridge studies in advanced mathematics 1995.

[12] R. G. Douglas, H. S. Shapiro, and A. L. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator*, Ann. Inst. Fourier (Grenoble), **20** (1970) 37-76.

[13] I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of Linear Operators Vol. 1* Operator Theory: Advances and Applications Vol. 49, 1990

[14] I. Gohberg, S. Goldberg, and A. Kaashoek, *Classes of Linear Operators Vol. 2* Operator Theory: Advances and Applications Vol. 63, 1993

[15] P. R. Halmos, *A Hilbert Space Problem Book, Second ed.* New York: Springer-Verlag. 1982

[16] K. Hoffman, *Banach Spaces of Analytic Functions* Prentice-Hall, inc. 1962

[17] T. F. McCabe *A note on iterates that are contractions.* J. math. Anal. Appl., **104**, (1984) 64-66. Basel: Birkhauser

[18] H. Radjavi and P. Rosenthal, *Simultaneous Triangularization* Springer 2000

[19] H. Radjavi and P. Rosenthal, *Invariant Subspaces, Second ed.* Dover 2003

[20] J. R. Ringrose, *Compact Non-Self-Adjoint Operators*New York: Van Nostrand Reinhold Co. 1971

[21] B. Sz-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space* North-Holland 1970

[22] H. S. Shapiro, *Weakly invertible elements in certain function spaces and generators in* l_1, Michigan Math. J. **11** (1964) 161-165.

[23] H. S. Shapiro and A. L. Shields, *On some interpolation problems for analytic funtions*, Amer. J. Math., **83**, (1961) 513-532.

[24] J. Weidmann, *Linear Operators in Hilbert Spaces* Springer-Verlag 1980