THE COEFFICIENTS OF $\dfrac{\sinh xt}{\sin t}$ AND THE BERNOULLI POLYNOMIALS

David J. Leeming

DMS-706-IR **July 1995**

THE COEFFICIENTS OF $\frac{\sinh xt}{\sin t}$ AND THE BERNOULLI POLYNOMIALS

David J. Leeming

Department of Mathematics and Statistics
P.O. Box 3045, MS 7437
University of Victoria
Victoria, British Columbia V8W 3P4
Canada

Summary

For a set of polynomials defined by the generating function $\frac{\sinh xt}{\sin t}$ we are able to show how they are related to the Bernoulli polynomials of odd degree. We also provide an easy method of generating the Bernoulli numbers.

1. Introduction.

Let the polynomial set $\{\beta_{2n}(x)\}$ be defined by the generating function

$$\frac{\sinh xt}{\sin t} = \sum_{n=0}^{\infty} \beta_{2n}(x) \frac{t^{2n}}{(2n)!} \tag{1.1}$$

and set

$$\beta_{2n} \equiv \beta_{2n}(1). \tag{1.2}$$

Several authors (*e.g.* L. Carlitz [2] and J.M. Gandhi [3]) have considered the various properties of the sequence $\{\beta_{2n}\}_{n=0}^{\infty}$. However, confining our attention only to these constants overlooks some interesting properties of the polynomials $\{\beta_{2n}(x)\}_{n=0}^{\infty}$. It is our object here to show the relationship between these polynomials and the classical Bernoulli polynomials $\{B_k(x)\}_{k=0}^{\infty}$ of odd degree. Using this relationship, we are able to obtain a new identity for the Bernoulli polynomials of odd degree. We are also able to give an easy method of generating the Bernoulli numbers. Finally, we obtain a simple representation of $\beta_{2n}(x)$ with coefficients related in a specific way to the Bernoulli numbers.

2. Some Basic Properties

Using the generating function (1.1) we can obtain a relationship that will enable us to determine the polynomials $\{\beta_{2n}(x)\}$.

Lemma 2.1. *For $n = 0, 1, \cdots$ we have*

$$x^{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{2k} (-1)^{n-k} \beta_{2k}(x) \tag{2.1}$$

where $\beta_{2k}(x)$ is defined by (1.1).

Proof. From (1.1) we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^{2n+1}\, t^{2n+1}}{(2n+1)!} &= \sum_{n=0}^{\infty} \frac{(-1)^n\, t^{2n+1}}{(2n+1)!} \sum_{k=0}^{\infty} \beta_{2k}(x) \frac{t^{2k}}{(2k)!} \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \beta_{2k}(x) \frac{t^{2k}}{(2k)!} \frac{(-1)^{n-k}\, t^{2n-2k+1}}{(2n-2k+1)!} \right] \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{2n+1}{2k} (-1)^{n-k} \beta_{2k}(x) \right] \frac{t^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Equating coefficients of t^{2n+1} yields (2.1). ∎

Note: Setting $x = 1$ in (2.1) we get formula (23) of J.M. Gandhi ([3], p. 189) as a special case. The first few polynomials in the set are:

$$\beta_0(x) = x$$

$$\beta_2(x) = \frac{x}{3} (x^2 + 1)$$

$$\beta_4(x) = \frac{x}{15} (x^2 + 1)(3x^2 + 7)$$

$$\beta_6(x) = \frac{x}{21} (x^2 + 1)(3x^4 + 18x^2 + 31)$$

$$\beta_8(x) = \frac{x}{315} (x^2 + 1)(35x^6 + 385x^4 + 1673x^2 + 2667)$$

(We note that $\beta_{2n}(x)$ is actually a polynomial of degree $2n + 1$ but choose to adopt this notation since this conforms to the notation used in [2] and [3] where $\beta_{2n} \equiv \beta_{2n}(1)$. Furthermore each $\beta_{2n}(x)$ contains x as a factor.)

The Bernoulli polynomials $\{B_n(x)\}_{n=0}^{\infty}$ can be defined by the generating function

$$\frac{t\, e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \qquad |t| < 2\pi. \tag{2.2}$$

Also, $B_n \equiv B_n(0)$, $n = 0, 1, \cdots$ is defined to be the nth Bernoulli number. More details on the Bernoulli polynomials can be found in [5] and [6].

The next lemma provides the relationship between the polynomials $\{\beta_{2n}(x)\}$ and the Bernoulli polynomials of odd degree.

Lemma 2.2. *For $n = 0, 1, \cdots$ we have*

$$\beta_{2n}(x) = \frac{(2i)^{2n+1}}{2n+1} B_{2n+1}\left(\frac{1-ix}{2}\right) \tag{2.3}$$

where $\beta_{2n}(x)$ is defined by (1.1) *and $B_{2n+1}(x)$ is defined by* (2.2).

Proof. From (1.1) we have

$$\frac{\sin xt}{\sin t} = \frac{-i\sinh xt}{\sin t} = -i\sum_{n=0}^{\infty} \beta_{2n}(ix)\,\frac{t^{2n}}{(2n)!}. \tag{2.4}$$

It is well known (see *e.g.* Pethe and Sharma [4]) that

$$\frac{\sin xt}{\sin t} = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!}\left[(-1)^n B_{2n+1}\left(\frac{1+x}{2}\right)\right] t^{2n}. \tag{2.5}$$

Equating the coefficients of t^{2n} in (2.4) and (2.5) yields

$$-i\,\beta_{2n}(ix) = \frac{2^{2n+1}}{2n+1}(-1)^n B_{2n+1}\left(\frac{1+x}{2}\right) \tag{2.6}$$

from which (2.3) follows immediately. ∎

From (2.1) and (2.3) we now obtain a new relationship equating the odd powers of x to certain sums of Bernoulli polynomials of odd degree.

Corollary 2.1. *For $n = 0, 1, \cdots$*

$$x^{2n+1} = \frac{2i(-1)^n}{2n+1}\sum_{s=0}^{n}\binom{2n+1}{2s} 2^{2s} B_{2n+1}\left(\frac{1-ix}{2}\right). \tag{2.7}$$

Proof. Follows immediately from (2.1) and (2.3). ∎

The next lemma gives some results on the zeros of $\{\beta_{2n}(x)\}$.

Lemma 2.3.

(a) The only real zero of $\beta_{2n}(x)$, $n = 0, 1, \cdots$, is $x = 0$.

(b) $x^2 + 1 \mid \beta_{2n}(x)$ for $n = 1, 2, \cdots$.

Proof.

(a) From (2.3) it is evident that any zero of $\beta_{2n}(x)$ is necessarily a zero of $B_{2n+1}\left(\frac{1-ix}{2}\right)$. Brillhart ([1], Thm. 13, p. 58) shows that $x = 0$ yields the only zero of $B_{2n+1}\left(\frac{1-ix}{2}\right)$.

(b) Using (2.3) we have

$$\beta_{2n}(i) = \frac{(2i)^{2n+1}}{2n+1} B_{2n+1}(1)$$

$$\beta_{2n}(-i) = \frac{(2i)^{2n+1}}{2n+1} B_{2n+1}(0).$$

Since $B_{2n+1}(0) = B_{2n+1}(1) = 0$, $n \geq 1$ (see *e.g.* [6], p. 125), both i and $-i$ are zeros of $\beta_{2n}(x)$, $n \geq 1$. ∎

3. The Main Results

The following theorem gives us an explicit representation of the polynomials $\{\beta_{2n}(x)\}$. From this we are able to provide a simple method of generating Bernoulli numbers.

Theorem 3.1. *The polynomials $\beta_{2n}(x)$, $n \geq 0$, are given by*

$$\beta_{2n}(x) = \frac{x}{2n+1} \sum_{s=0}^{n} \binom{2n+1}{2s} (-1)^s D_{2s}\, x^{2n-2s} \tag{3.1}$$

where $D_s = 2(1-2^{s-1})\, B_s$, $s \geq 0$, and B_s is the sth *Bernoulli number.*

Proof. From ([1], p. 58) we have

$$B_{2n+1}(x) = \sum_{s=0}^{2n+1} \binom{2n+1}{s} 2^{-s} \left(x - \frac{1}{2}\right)^{2n+1-s} D_s. \tag{3.2}$$

Thus,

$$B_{2n+1}\left(\frac{1}{2} - \frac{ix}{2}\right) = \sum_{s=0}^{2n+1} \binom{2n+1}{s} 2^{-s} \left(\frac{-ix}{2}\right)^{2n+1-s} D_s$$

$$= 2^{-2n-1} \sum_{s=0}^{2n+1} \binom{2n+1}{s} (-1)^{-s+1} (ix)^{2n+1-s} D_s.$$

Since $D_s = 0$ for s odd, we have

$$B_{2n+1}\left(\frac{1}{2}-\frac{ix}{2}\right) = 2^{-2n-1}(-1)^{n+1}\, i \sum_{s=0}^{n} \binom{2n+1}{2s} (-1)^s D_{2s}\, x^{2n+1-2s}. \tag{3.3}$$

Now, using (2.3) we have

$$\beta_{2n}(x) = \frac{2^{2n+1}(-1)^n\, i}{2n+1}\, B_{2n+1}\left(\frac{1}{2}-\frac{ix}{2}\right). \tag{3.4}$$

Finally, using (3.3) in (3.4), and simplifying we get (3.1). ∎

Corollary 3.1. *$x^{-1}\beta_{2n}(x)$ contains only even powers of x.*

Proof. Immediate from (3.1). ∎

Theorem 3.2. *Given $B_0 = 1$ (and $B_1 = -\frac{1}{2}$) the remaining sequence of nonzero Bernoulli numbers can be generated by*

$$\sum_{s=0}^{n} \binom{2n+1}{2s} (1-2^{2s-1})\, B_{2s} = 0. \tag{3.5}$$

Proof. Setting $x = i$ in (3.1) and using Lemma 2.3(b), we have, upon simplifying

$$\frac{i}{2n+1} \sum_{s=0}^{n} \binom{2n+1}{2s} D_{2s} = 0$$

which implies that

$$\sum_{s=0}^{n} \binom{2n+1}{2s} D_{2s} = 0. \tag{3.6}$$

Using $D_{2s} = 2(1-2^{2s-1})\, B_{2s}$ in (3.6) yields (3.5). ∎

Finally, we note that it is not necessary in the proof of Theorem 3.2 to know that $B_1 = -\frac{1}{2}$. However B_1 is the only nonzero Bernoulli number with an odd subscript.

References

[1] BRILLHART, J., 1969, On the Euler and Bernoulli polynomials, *J. für die reine und ang. Math.*, **234**, 45–64.

[2] CARLITZ, L., 1956, The coefficients of $\frac{\sinh x}{\sin x}$, *Mathematics Magazine*, **29**, 193–197.

[3] GANDHI, J.M., 1958, The coefficients of $\frac{\cosh x}{\cos x}$ and a note on Carlitz's coefficients of $\frac{\sinh x}{\sin x}$, *Mathematics Magazine*, **31**, 185–191.

[4] PETHE, S.P. and SHARMA, A., 1972, Modified Abel expansions and a subclass of completely convex functions, *SIAM J. Math. Anal.*, **3**, 546–558.

[5] DILCHER, K., 1988, Zeros of Bernoulli, generalized Bernoulli and Euler polynomials, *AMS Memoirs*, No. 386, Providence, Rhode Island.

[6] LEEMING, D.J., 1989, The real zeros of the Bernoulli polynomials, *J. Approx. Theory*, (2), **58** 124–150.