EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE STEADY BOLTZMANN EQUATION

by

Shahin Ghomeshi
B.Sc., University of Alberta, 1995.

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics.

We accept this thesis as conforming to the required standard.

[redacted]

Dr. R. Illner, Department of Mathematics & Statistics, University of Victoria

[redacted]

Dr. J. Ye, Department of Mathematics & Statistics, University of Victoria

[redacted]

Dr. C. Bose, Department of Mathematics & Statistics, University of Victoria

[redacted]

Dr. A. Watton, Department of Physics, University of Victoria

Supervisor: Dr. R. Illner.

Abstract

In this thesis our main objective is to prove the existence and uniqueness of solutions to the Steady Boltzmann Equation with inflow and diffusive boundary conditions, without truncations of the collision kernel. Due to the singularity at $v = 0$, the non-linear term in the Steady Boltzmann Equation becomes significantly large. In addition to this, another problem with proving existence of steady solutions is that the collision term becomes unbounded as $v \to \infty$ for hard-sphere interactions.

Previous authors have attacked these problems by imposing unphysical truncations on the collision kernel so that collisions between particles having small velocities are ignored. An additional truncation is then made to control the magnitude of the velocities so that it will not grow without bound.

We present and detail some of the work of Maslova and show that the collision operator has the properties which make it possible to avoid truncations. We introduce several properties of the collision operator which include a series of crucial estimates. These estimates are later used to produce function spaces with the contractive property. General existence and uniqueness is then obtained by an application of the Contraction Mapping Principle.

Examiners:

[redacted]

Dr. R. Illner, Department of Mathematics & Statistics, University of Victoria

[redacted]

Dr. J. Ye, Department of Mathematics & Statistics, University of Victoria

[redacted]

Dr. C. Bose, Department of Mathematics & Statistics, University of Victoria

[redacted]

Dr. A. Watton, Department of Physics, University of Victoria

Contents

List of Figures

Acknowledgements

I am deeply grateful for the support and guidance of Dr. Reinhard Illner. This thesis could not have been completed without his help, patience, and valuable advice. In addition, I would like to acknowledge my friend Sean Bohun for giving me his feedback and advice.

to my family

Chapter 1

Introduction

The Classical Boltzmann Equation has the form:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = J(f,f) \qquad t > 0 \;\; x \in \mathbb{R}^3_x, v \in \mathbb{R}^3_v \tag{1.1}$$

in which the function $f(x, v, t)$ satisfying the above equation is a density function describing the distribution of particles in phase space $\Omega \times \mathbb{R}^3_v$, where $x \in \Omega \subset \mathbb{R}^3_x$ denotes a point in physical space and $v \in \mathbb{R}^3_v$ denotes a velocity. We see that for the left hand side of Eq (1.1) we have the rate of change of $f(x, v, t)$ with respect to time, plus a transport term; and on the right hand side we have an operator $J(f, f)$ which describes the rate of change of f due to binary collisions between gas particles. Eq(1.1) is a non-linear transport equation describing the evolution of a rarefied gas. The collision operator is defined at a point (x, v, t) by:

$$J(f,f)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} [f(v')f(v'_*) - f(v)f(v_*)] B(v - v_*, n)\, dn\, dv_*.$$

Here n is a unit vector in the unit sphere $\mathbb{S}^2$ $\{n \in \mathbb{R}^3 \big| |n| = 1\}$ with Lebesgue measure dn, v, v_* are the pre-collisional velocities of the gas particle, and v', v'_* are the post-collisional velocities. In the presence of a collision (Fig. 1.1), any two particles with velocities v, v_* will undergo an instantaneous change in their velocities resulting in the transformation:

$$(v, v_*) \longrightarrow (v', v'_*)$$

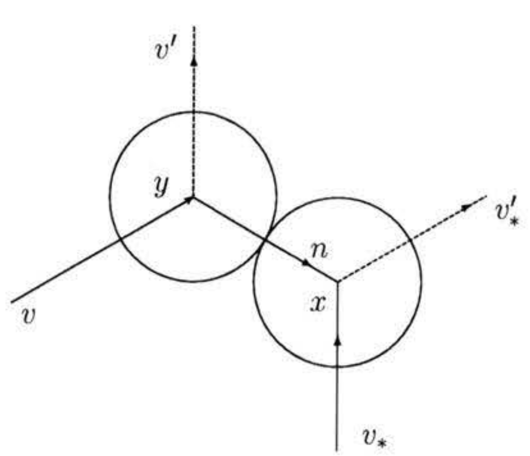

Figure 1.1: Illustration of the collision between two particles.
Here the momentum transfer is in the direction $n = (x-y)/\|x-y\|$.

Actually, in order to arrive at the exact relationship between the post-collisional and the pre-collisional velocities, we insist that the collision should:

1. preserve energy; namely that $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$
2. preserve momentum; $v' + v'_* = v + v_*$
3. momentum transfer should be in the direction n.

With this in mind, we arrive at the relations:

$$v' = v - n(n.(v - v_*)) \qquad v'_* = v_* + n(n.(v - v_*)). \tag{1.2}$$

The configuration of the velocity vectors of each particle just before and after a collision is represented to lie on a collision sphere as shown in Figure 1.2.

B is called the collision kernel and it depends on $|v - v_*|$ and $(n.(v - v_*))$ only. It is uniquely determined by the interaction potential between the particles. In general, after an angular cut-off,[1] it is of the form:

$$B(v - v_*, n) = |v - v_*|^\beta h(\theta)$$

[1]Since B has a dependence on θ, angular cut-off refers to those values of θ, restricted to $\int_0^\pi h(\theta)\, dn < \infty$, for which B is permissible.

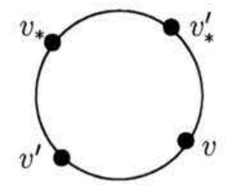

Figure 1.2: Configuration of particles before and after a collision.

where θ is the polar angle of n relative to a polar axis in direction $v - v_*$; and h is assumed to be an integrable function on $[0, \pi]$ with $\int_{\mathbb{S}_+} h(\theta)\, dn = 1$ where $\mathbb{S}_+$ is the hemi-sphere corresponding to $(v - v_*, n) > 0$. The integer β is chosen from the set $\{-1, 0, 1\}$, which describe Maxwellian molecules ($\beta = 0$), a hard sphere gas ($\beta = 1$), and a soft sphere gas ($\beta = -1$). According to H. Grad[8] B satisfies the following conditions:

- $B \in L^\infty_{loc}(\mathbb{R}^3, \mathbb{S}^2)$
- $B(v, n) \leq b_1 \frac{|(n \cdot v)|}{|v|}(1 + |v|^\gamma)$
- $\int_{\mathbb{S}^2} B(v, n)\, dn \geq b_\circ |v|(1 + |v|)^{-1}$

where $\gamma \in [0, 1]$ and $b_\circ$, b_1 are positive constants. Although generalizations are available, we will deal with the case where we have only hard sphere interactions. Then B takes on the particular form:

$$B(v - v_*, n) = |n \cdot (v - v_*)| = |(v - v_*)|\, |\cos\theta|$$

with $h(\theta) = \cos(\theta)$ and $\beta = 1$. In Chapter 2, we will state and prove some properties of the collision operator in the Boltzmann equation neccessary for proving any existence and uniqueness results. In Chapter 3, we will discuss three-dimensional boundary value problems for the Boltzmann equation with inflow boundary conditions in which we denote by Ω a bounded domain with boundary $\partial\Omega$ and outward normal $n(x)$ at a point $x \in \partial\Omega$. We will also denote by $\gamma^+ f$ and $\gamma^- f$ the trace densities of distribution of falling and reflected particles on $\partial\Omega$ respectively. In particular, one sets

$$\gamma^+ f = f\chi(v \cdot n(x)), \qquad \gamma^- f = f(1 - \chi(v \cdot n(x)))$$

χ being the indicator function of $(0, \infty)$:

$$\chi = 1 \quad \text{for} \quad v \cdot n(x) > 0 \;\; x \in \partial\Omega$$

$$\chi = 0 \quad \text{for} \quad v \cdot n(x) \leq 0 \;\; x \in \partial\Omega$$

In Chapter 4, we extend the discussion of Chapter 3 to the diffuse-reflection case, and finally in chapter 5, the particular one-dimensional Couette problem is considered.

Chapter 2

Properties of the Boltzmann Collision Term

2.1 Some Well Known Properties

In this chapter we will describe some properties of the collision operator which are essential in studying boundary value problems later on. Throughout this chapter we consider functions $f : \mathbb{R}^3_v \to \mathbb{R}^1$, since the functional operation of the collision integrals is on v only. The collision integral $J(f,f)$ corresponds to the weighted average of all collisions a particle with velocity v could encounter. For a certain class of molecular interactions, it has the form:

$$J(f,f) = J^+(f,f) - J^-(f,f)$$

where the first term on the right hand side is referred to as the gain term. In the Boltzmann equation, $J^+(f,f)$ is proportional to the probability that a particle after a collision has the velocity v. $J^+(f,f)$ is defined as:

$$J^+(f,f) = \int_{\mathbb{R}^3\times\mathbb{S}^2} f(v')f(v'_*)B(v-v_*,n)\,dn\,dv_*. \tag{2.1}$$

$J^-(f,f)$ is called the loss term and is proportional to the probability that a particle before a collision has velocity v. In general,

$$J^-(f,f) = f(v)\nu(f)$$

where $\nu(f)$ is defined as

$$\nu(f)(v) = \int_{\mathbb{R}^3_v} f(v_*) \left[\int_{\mathbb{S}^2} B(v - v_*, n)\, dn \right] dv_* \tag{2.2}$$

and determines the frequency of collisions associated with the distribution function f. It is often convenient to write equation (2.1) in terms of the more general bilinear expression given by:

$$J^+(f, g) = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{S}^2} [f(v')g(v'_*) + f(v'_*)g(v')]\, dn\, dv_* \tag{2.3}$$

in which one sees that when $g = f$ (2.3) reduces to (2.1) and

$$J^+(f, g) = J^+(g, f). \tag{2.4}$$

We will use the following notations:

$$\begin{aligned} \text{For } \phi = \phi(v, v_*, n), \quad \langle\langle \phi \rangle\rangle &= \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \phi(v, v_*, n)\, dv\, dv_*\, dn \\ \text{For } \phi = \phi(v) \quad \langle \phi \rangle &= \int_{\mathbb{R}^3_v} \phi(v)\, dv \end{aligned}$$

if the integral exists. By applying equation (2.3) and the fact that the Jacobian of the transformation from the unprimed to the primed variables is unity, it is shown [6] that the following identities hold:

$$\langle \phi J^+(f, g) \rangle = \langle\langle \phi' f g_1 \rangle\rangle = \frac{1}{2} \langle\langle (\phi' + \phi'_1) f g_1 \rangle\rangle \tag{2.5}$$

$$\langle \varphi J(f, f) \rangle = \frac{1}{4} \langle\langle (\phi + \phi_1 - \phi' - \phi'_1)[f' f'_1 - f f_1] \rangle\rangle \tag{2.6}$$

in which we have used the traditional notations

$$f' = f(v'), \;\; f'_1 = f(v'_*), \;\; f = f(v), \;\; f_1 = f(v_*).$$

In addition, we use a well known property of the collision operator [6] namely:

$$\langle \psi_j J(f, f) \rangle = 0 \tag{2.7}$$

for the functions

$$\psi^{\circ} = 1, \psi_j = v_j (j = 1, 2, 3), \psi_4 = |v|^2 \tag{2.8}$$

which are called collision invariants. In order to obtain existence and uniqueness of solutions for the Boltzmann equation, one proceeds as Carleman [4] did and introduces function spaces which are invariant under the action of collision operators. To this end, it is neccessary to introduce some further properties of the collision operator. With respect to the kernel B, it is sufficient to assume that B satisfies:

$$B(v, n) \leq b_1 z |v|^{\gamma} \tag{2.9}$$

with $z = \frac{|n.v|}{|v|}$, $\gamma \in [0, 1], b_1 > 0$ consistent with Grad's assumptions mentioned in the introduction.

2.2 Estimates on the Collision Frequency

As stated before, obtaining existence and uniqueness of solutions, relies heavily on the particular estimates produced for the collision operator. In this section we begin by proving a basic property of the collision frequency, namely that it is a function which can be bounded below by a constant $\nu_{\circ}$ and above by a linear function of the absolute value of the velocity.

Lemma 2.1 *Assume that* $f \geq 0, (1+|v|^2)f \in L^1(\mathbb{R}^3_v)$ *and for some positive constant* $b_{\circ}$ *and* $\gamma \in [0, 1]$,

$$\int_{\mathbb{S}^2} B(v, n)\, dn \geq b_{\circ} |v|^{\gamma}. \tag{2.10}$$

Set $a = \langle f \rangle, b = \langle (1 + |v|^2) f \rangle, H = \langle f \ln f \rangle$. *Then exists a* $b_2 > 0$ *such that*

(i) $$b_{\circ}(a|v|^{\gamma} - b) \leq \nu(f) \leq b_2(a|v|^{\gamma} + b)$$

(ii) there exists a positive constant $\nu_{\circ}$, *depending only on* H *and* a *such that:*

$$\inf_v \nu(f) \geq \nu_{\circ}.$$

Proof. To prove *(i)* one first makes an observation using an extension of the Cauchy–Schwarz inequality. For any $\gamma \in [0,1]$ and $v, v_* \in \mathbb{R}^3$,

$$\big|\, |v|^\gamma - |v_*|^\gamma \,\big| \leq |v - v_*|^\gamma.$$

One also requires the estimate

$$|v|^\gamma \leq 1 + |v|^2$$

which is easily justified using the fact the $\gamma \in [0,1]$. We may now estimate $\nu(f)$ from below by employing definition (2.2) and the assumption on the collision kernel (2.10), as follows:

$$\begin{aligned}
\nu(f)(v) &= \int_{\mathbb{R}^3_v} f(v_*) \left[\int_{\mathbb{S}^2} B(v - v_*, n)\, dn\right] dv_* \\
&\geq b_\circ \int_{\mathbb{R}^3_v} f(v_*) |v - v_*|^\gamma \, dv_* \\
&\geq b_\circ \int_{\mathbb{R}^3_v} f(v_*) |v|^\gamma \, dv_* - b_\circ \int_{\mathbb{R}^3_v} f(v_*) |v_*|^\gamma \, dv_* \\
&\geq b_\circ |v|^\gamma \int_{\mathbb{R}^3_v} f(v_*)\, dv_* - b_\circ \int_{\mathbb{R}^3_v} f(v_*)(1 + |v_*|^2)\, dv_* \\
&= b_\circ (a|v|^\gamma - b).
\end{aligned} \tag{2.11}$$

To complete the proof of *(i)*, we must now estimate $\nu(f)$ from above. Therefore, utilizing the modified form of Grad's estimate in (2.9), namely

$$B(v - v_*, n) \leq b_1 \frac{|n.(v - v_*)|}{|v - v_*|} |v - v_*|^\gamma \leq b_1 |v - v_*|^\gamma$$

and applying the trivial inequality $|v - v_*|^\gamma \leq |v|^\gamma + |v_*|^\gamma$ for $\gamma \in [0,1]$, gives the estimate

$$\begin{aligned}
\nu(f)(v) &= \int_{\mathbb{R}^3_v} f(v_*) \left[\int_{\mathbb{S}^2} B(v - v_*, n) dn\right] dv_* \\
&\leq b_1 \int_{\mathbb{R}^3_v} f(v_*) |v - v_*|^\gamma \left[\int_{\mathbb{S}^2} dn\right] dv_* \\
&\leq 4\pi b_1 \int_{\mathbb{R}^3_v} f(v_*)(|v|^\gamma + |v_*|^\gamma)\, dv_*
\end{aligned}$$

$$\begin{aligned} &= b_2|v|^\gamma \int_{\mathbb{R}^3_v} f(v_*)\,dv_* + b_1 \int_{\mathbb{R}^3_v} |v_*|^\gamma f(v_*)\,dv_* \\ &\le b_2|v|^\gamma \int_{\mathbb{R}^3_v} f(v_*)\,dv_* + b_1 \int_{\mathbb{R}^3_v} (1+|v_*|^2) f(v_*)\,dv_* \\ &= b_2(a|v|^\gamma + b) \end{aligned} \tag{2.12}$$

where we have replaced the constant $4\pi b_1$ by b_2. Clearly, estimates (2.11) and (2.12) give the required estimates in *(i)*. Furthermore, we make the important note that since $\nu(f) = \mathrm{O}(|v|^\gamma)$, it is typically unbounded as $|v| \to \infty$. To prove *(ii)* define

$$A = \left\{ v \mid f(v) < e^{-|v|^2} \right\},\ B = \left\{ v \mid e^{-|v|^2} < f < 1 \right\}$$

and $\overline{H} = \int_{\mathbb{R}^3_v} f|\ln f|$. We now make the decomposition:

$$\begin{aligned} \overline{H} &= \int_A f|\ln f|\,dv + \int_B f|\ln f|\,dv + \int_{\mathbb{R}^3\setminus(A\cup B)} f|\ln f|\,dv \\ &:= T_1 + T_2 + T_3 \end{aligned}$$

and proceed by estimating each individual term. For $v \in A$ it is immediate that $\sqrt{f} < \exp(-\frac{|v|^2}{2})$ and $|\ln f| > |v|^2$. Therefore, it is easily verified that the product $\sqrt{f}|\ln f|$ is bounded above by a positive constant C_1. It follows that for $v \in A$

$$\begin{aligned} f|\ln f| &\le \sqrt{f}\sqrt{f}|\ln f| \\ &\le C_1\sqrt{f} \\ &\le C_1 \exp(-\frac{|v|^2}{2}). \end{aligned}$$

Hence

$$T_1 \le C_1 \int_A \exp(-\frac{|v|^2}{2})\,dv \le C.$$

In addition, for $v \in B$, $f|\ln f| \le f|v|^2 \le f(1+|v|^2)$ thus obtaining

$$T_2 \le \int_B f(1+|v|^2)\,dv \le b$$

where one recalls the definition of b as stated in the lemma.

We further recall that

$$\begin{aligned} H &= \int_{\mathbb{R}^3} f \ln f \\ &= \int_{f \geq 1} f|\ln f|\,dv - \int_{f<1} f|\ln f|\,dv \\ &= \int_{\mathbb{R}^3} f|\ln f|\,dv - 2\left(\int_A f|\ln f|\,dv + \int_B f|\ln f|\,dv\right) \\ &\geq T_3 - 2C - 2b \end{aligned}$$

Therefore,

$$\overline{H} \leq H + 2b + 2C.$$

To proceed further, we use a generalization of Young's inequality[3], namely

$$xy \leq x|\ln x| + e^{y-1} \qquad x > 0, y > 0 \tag{2.13}$$

which for the convenience of the reader we prove in Appendix A. By (2.2) and (2.10) we estimate $\nu(f)$ as

$$\begin{aligned} \nu(f)(v) &= \int_{\mathbb{R}^3_v} f(v_*)\left[\int_{\mathbb{S}^2} B(v-v_*,n)dn\right] dv_* > \int_{|v-v_*|>l} f(v_*) b_\circ |v-v_*|^\gamma \, dv_* \\ &> b_\circ l^\gamma \int_{|v-v_*|>l} f(v_*)\,dv_* \\ &= b_\circ l^\gamma \Big(\int_{\mathbb{R}^3_v} f(v_*)dv_* - \int_{|v-v_*|\leq l} f(v_*)\,dv_*\Big) \\ &= b_\circ l^\gamma \left[a - \lambda^{-1} \int_{|v-v_*|\leq l} f(v_*)\lambda\,dv_*\right] \end{aligned}$$

for any $l, \lambda > 0$. An application of the general inequality (2.13), with $x = f(v_*), y = \lambda$ now yields:

$$\begin{aligned} \nu(f)(v) &> b_\circ l^\gamma \left[a - \lambda^{-1}\left(\int_{|v-v_*|\leq l} (f(v_*)|\ln f(v_*)| + e^{\lambda-1})\,dv_*\right)\right] \\ &> b_\circ l^\gamma \left[a - \lambda^{-1}\int_{\mathbb{R}^3_v} f|\ln f| dv_* - \lambda^{-1}\int_{|v-v_*|\leq l} e^{\lambda-1}\,dv_*\right] \\ &= b_\circ l^\gamma \left[a - \lambda^{-1}\overline{H} - \lambda^{-1}e^{\lambda-1}\int_{|v-v_*|\leq l} dv_*\right] \\ &= b_\circ l^\gamma \left[a - \lambda^{-1}\overline{H} - \lambda^{-1}e^{\lambda-1}\frac{4}{3}\pi l^3\right] \end{aligned}$$

Now first choose $\lambda > 0$ such that $\frac{1}{\lambda}\overline{H} < \frac{1}{4}a$. Then choose $l = l(\lambda)$ such that:

$$\lambda^{-1}\left[\overline{H} + e^{\lambda-1}\frac{4}{3}\pi l^3\right] < \frac{1}{4}a + \frac{1}{4}a = \frac{1}{2}a$$

This gives the result

$$\nu(f)(v) \;>\; b_\circ l^\gamma\left(a - \frac{1}{2}a\right) = \frac{1}{2}b_\circ a l^\gamma = \nu_\circ$$

where $\nu_\circ$ is some positive constant. Hence, $\inf_v \nu(f) \geq \nu_\circ$ as required. $\square$

2.3 Estimates on the Gain Term

In this section we produce an important bound on the gain term which will be used later on. This bound, stated in Lemma 2.2 is crucially dependent on the particular choice of norm. As a guess to the form of this norm, assume that the initial and boundary conditions are near a Maxwellian distribution. Then for some $s > 0$, $f \approx \exp(-s|v|^2)$, and there would be exponential decrease for large $|v|$. It is known [5] that for a Maxwellian, $J(f,f) = 0$, and so $J^+(f,f) \approx f\nu(f)$. By Lemma 2.1, the right hand side of this equation will be $O(|v|^\gamma \exp(-s|v|^2))$ for large $|v|$. Hence, $J^+(f,f) = O(|v|^\gamma \exp(-s|v|^2))$for large $|v|$. In order to compensate for the $|v|^\gamma$ terms in $J^+(f,f)$, we consider the space:

$$L^\infty_{s,r} = \{f|\ \varphi_{sr}|f| \in L^\infty(\mathbb{R}^3_v)\} \tag{2.14}$$

with the weight function:

$$\varphi = \varphi_{sr}(v) = (1+|v|^2)^{r/2}e^{s|v|^2} \tag{2.15}$$

and for $f \in L^\infty_{s,r}$ introduce the norm:

$$\|f\|_{s,r} = \varphi(v)\|f\|_{L^\infty}.$$

Now we are in a position to state the next lemma. Before we proceed however, we mention that in the rest of this thesis, C (with or without indices) is used to

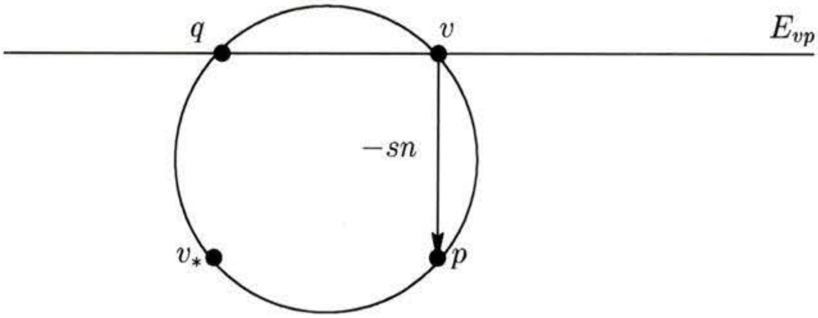

Figure 2.1: Figure representing the Carleman transformation

denote a positive constant independent of the parameters taken into consideration (unless otherwise stated). Moreover, it can represent different constants in different formulas.

Lemma 2.2 *If $f \in L^\infty_{s,r}$, $s > 0$, $r > 4$, then $J^+(f,f) \in L^\infty_{s,r}$. Moreover, there exists a positive constant C such that:*

$$\|J^+(f,f)\|_{s,r} \leq C\|f\|^2_{s,r}$$

Proof. We fix v and make the change of variables

$$(v_*, n) \to (p = v - n(n.(v - v_*)), q = v_* + n(n.(v - v_*)))$$

where it is easy to see that

$$\begin{aligned} (p - v)\cdot(q - v) &= -n(n.(v - v_*))\cdot[(v_* - v) + n(n.(v - v_*))] \\ &= (n.(v - v_*))^2 - (n.(v - v_*))^2 \\ &= 0. \end{aligned}$$

Thus we can obtain a representation of $J^+(f,g)$ in which the integration over $dn\,dv_*$ is replaced by an integration over $dq\,dp$ where q ranges over E_{vp}, the plane which is orthogonal to $v - p$ and contains v, and where p ranges over $\mathbb{R}^3$. This is seen in Figure 2.1. Let $p = v - sn$, where s is the scalar quantity $n.(v - v_*)$. For any fixed n and v, one may write $v_* = q - sn$. We can express p in spherical polar coordinates with the origin at v. Hence we have $p = -sn$ and writing this out in its components yields $(p_1, p_2, p_3) = -s(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ in which we recognize the volume element $dp = s^2 \sin\phi\, ds\, d\theta\, d\phi = s^2\, ds\, dn$. Also since dq is the area element of the plane E_{vp}, and $v_* \in \mathbb{R}^3$, we would have $dv_* = dq\, ds$ (see Figure 2.1). Hence $dv_*\, dn = dq\, ds\, dn = \frac{1}{s^2} dq\, dp$ where $sn = n(n.(v - v_*)) = v - p$. Thus $dv_*\, dn = \frac{1}{|v-p|^2}\, dq\, dp$ and we can now express $J^+(f,g)$ as:

$$J^+(f,g)(v) = 2\int_{\mathbb{R}^3_v} f(p)|v-p|^{-2}\left[\int_{E_{vp}} g(q)B\, dq\right] dp \tag{2.16}$$

where the factor 2 is due to the fact that each plane is represented by two opposite directions n. This representation was first achieved by Carleman [4], by calculating the Jacobian of the transformation; a lengthy calculation which we present in Appendix B. The above simplified method was first introduced by Wennberg[13]. Note that for $q \in E_{vp}$ we can estimate the collision kernel by the earlier stated Grad condition:

$$B(p-q,n) \le b_1 \frac{|n.(p-q)|}{|p-q|}|p-q|^\gamma. \tag{2.17}$$

By employing the equality

$$\begin{aligned} |n.(p-q)| &= |(n.(v - v_* - 2n(n.(v-v_*))))| = |(n.(v-v_*)) - 2(n.(v-v_*))| \\ &= |n.(v-v_*)| \\ &= |v-p| \end{aligned}$$

in equation (2.17) we obtain

$$B(p-q,n) \le b_1 \frac{|v-p|}{|p-q|}|p-q|^\gamma.$$

Now by observing that

$$\begin{aligned} |p-q| &= |(v-v_*) - 2n(n.(v-v_*))| \\ &= |v-v_*| \end{aligned}$$

$$\begin{aligned} &\geq |n.(v-v_*)| \\ &= |v-p| \end{aligned}$$

and making note of the fact that for $\gamma \in [0,1]$

$$\frac{|p-q|^\gamma}{|p-q|} \leq \frac{|v-p|^\gamma}{|v-p|},$$

the collision kernel can simply be estimated as

$$\begin{aligned} B(p-q,n) &\leq b_1|v-p|\frac{|p-q|^\gamma}{|p-q|} \\ &\leq b_1|v-p|\frac{|v-p|^\gamma}{|v-p|} \\ &= b_1|v-p|^\gamma. \end{aligned}$$

Now we define the operators:

$$V_k(f)(v) = 2b_1\int_{\mathbb{R}^3_v}|v-p|^{-k}f(p)dp \tag{2.18}$$

$$G(f)(v,p) = \int_{E_{vp}} f(q)dq \tag{2.19}$$

and so invoking the above estimate on the collision kernel in equation (2.16) and applying the definition of the operators in (2.18) and (2.19), we write

$$\begin{aligned} |J^+(f,g)| &\leq 2b_1\int_{\mathbb{R}^3_v}|f(p)||v-p|^{-2}|v-p|^\gamma\left[\int_{E_{vp}}|g(q)|\,dq\right]dp \\ &= 2b_1\int_{\mathbb{R}^3_v}|f(p)||v-p|^{-2+\gamma}\left[\int_{E_{vp}}|g(q)|dq\right]dp \\ &= V_{2-\gamma}(|f|G(|g|)). \end{aligned} \tag{2.20}$$

To proceed further we define for any fixed v the following sets:

$$\begin{aligned} D_1 &= \left\{p\in\mathbb{R}^3\middle| |p|\leq\tfrac{|v|}{\sqrt{2}}\right\} \\ D_2 &= \left\{p\in\mathbb{R}^3\middle| \tfrac{|v|}{\sqrt{2}}<|p|<|v|\right\} \\ D_3 &= \left\{p\in\mathbb{R}^3\middle| |p|\geq|v|\right\} \end{aligned}$$

and we let

$$f_i(p) = f(p)\chi_i(p) \text{ for } i = 1, 2, 3$$

χ_i being the indicator function of D_i. For any $q \in E_{vp}$ we have:

$$|p - q| \geq |p - v|, \;\; |q|^2 \geq |v|^2 - |p|^2 \tag{2.21}$$

where the second of the two inequalities can readily be seen to hold by applying the expressions for p, and q:

$$\begin{aligned}
|q|^2 &= |v_*|^2 + 2(n.v_*)(n.(v - v_*)) + (n.(v - v_*))^2 \\
|p|^2 &= |v|^2 - 2(n.v)(n.(v - v_*)) + (n.(v - v_*))^2 \\
|v|^2 - |p|^2 &= 2(n.v)(n.(v - v_*)) - (n.(v - v_*))^2 \\
|q|^2 - (|v|^2 - |p|^2) &= |v_*|^2 + 2(n.(v_* - v))(n.(v - v_*)) + 2(n.(v - v_*))^2 \\
&= |v_*|^2 \geq 0.
\end{aligned}$$

Now we proceed with the required estimates. We note that since $J^+(f, f)$ is bilinear with respect to f we can make the decomposition:

$$\begin{aligned}
J^+(f, f) \;\; = \;\; & J^+(f_1, f_1) + J^+(f_1, f_2) + J^+(f_2, f_1) + J^+(f_1, f_3) + J^+(f_3, f_1) \\
& + J^+(f_2, f_2) + J^+(f_2, f_3) + J^+(f_3, f_2) + J^+(f_3, f_2).
\end{aligned}$$

Upon applying the symmetry property of the collision term in equation (2.4), we obtain:

$$\begin{aligned}
J^+(f, f) \;\; = \;\; & J^+(f_1, f_1) + 2J^+(f_1, f_2) + 2J^+(f_2, f_3) \\
& + 2J^+(f_1, f_3) + J^+(f_2, f_2) + J^+(f_3, f_3).
\end{aligned}$$

So we investigate the boundedness in each case separately. By definition,

$$\begin{aligned}
|J^+(f_1, f_1)(v)| \;\; & \leq \;\; 2b_1 \int_{|p| \leq \frac{|v|}{\sqrt{2}}} |v - p|^{-2+\gamma} |f_1| G(|f_1|) \, dp \\
& = \;\; 2b_1 \int_{|p| \leq \frac{|v|}{\sqrt{2}}} |v - p|^{-2+\gamma} f(p) \left[\int_{|q| \geq \frac{|v|}{\sqrt{2}}} |f_1(q)| \, dq \right] dp
\end{aligned}$$

(as $|q|^2 \geq |v|^2 - |p|^2$ we know that if $|p| \leq \frac{|v|}{\sqrt{2}}$, then $|q| \geq \frac{|v|}{\sqrt{2}}$). Hence

$$\int_{|q| \geq \frac{|v|}{\sqrt{2}}} f_1(q) dq = 0$$

since f_1 is by definition supported in $|q| \leq \frac{|v|}{\sqrt{2}}$. It follows that

$$J^+(f_1, f_1) = 0. \tag{2.22}$$

Next, we estimate the term $|J^+(f_1, f_2)|$. We begin with the calculation for the term $G(f_2) = G(f_2)(v, p)$, in which $q \in E_{vp} \cap D_2 : (|v|^2 - |p|^2)^{1/2} \leq |q| \leq |v|$. Therefore,

$$\begin{aligned} G(f_2) &= \int_{E_{vp}} f_2(q)\, dq = \int_{E_{vp}} f(q)\chi_2(q)\, dq \leq \int_{E_{vp}:(|v|^2-|p|^2)^{1/2} \leq |q| \leq |v|} f(q)\, dq \\ &= \int_{E_{vp}:(|v|^2-|p|^2)^{1/2} \leq |q| \leq |v|} f(q)(1+|q|^2)^{r/2} e^{s|q|^2} (1+|q|^2)^{-r/2} e^{-s|q|^2}\, dq \\ &\leq \|f\|_{s,r} \int_{E_{vp}:(|v|^2-|p|^2)^{1/2} \leq |q| \leq |v|} e^{-s|q|^2} (1+|q|^2)^{-r/2}\, dq. \end{aligned}$$

Let $|q| = R$ and writing the above in polar coordinates, we have

$$G(f_2) \leq \|f\|_{s,r} \int_0^{2\pi} \int_{(|v|^2-|p|^2)^{1/2}}^{|v|} (1+R^2)^{-r/2} e^{-sR^2} R\, dR\, d\theta.$$

Since q belongs to the set D_2, $|q| > \frac{|v|}{\sqrt{2}}$ and hence

$$(1+|q|^2)^{-r/2} \leq \left(1 + \frac{|v|^2}{2}\right)^{-r/2} \leq C(1+|v|^2)^{-r/2}$$

for some positive constant C. Hence

$$\begin{aligned} G(f_2) &\leq 2\pi C \|f\|_{s,r} e^{-s|v|^2} e^{s|p|^2} (1+|v|^2)^{-r/2} \int_{(|v|^2-|p|^2)^{1/2}}^{|v|} R\, dR \\ &= 2\pi C \|f\|_{s,r} e^{-s|v|^2} e^{s|p|^2} (1+|v|^2)^{-r/2} \frac{1}{2} R^2 \Big|_{(|v|^2-|p|^2)^{1/2}}^{|v|} \\ &= \pi C \|f\|_{s,r} e^{-s|v|^2} e^{s|p|^2} (1+|v|^2)^{-r/2} |p|^2. \end{aligned} \tag{2.23}$$

By plugging the above estimate on $G(f_2)$ into the inequality (2.20) we arrive at an estimate for $|J^+(f_1, f_2)|$. Thus

$$|J^+(f_1, f_2)| \leq 2b_1 \int_{\mathbb{R}^3_v} |v-p|^{-2+\gamma} f_1(p) G(f_2)\, dp$$

$$\begin{aligned}
&= 2b_1 \int |v-p|^{-2+\gamma} f(p)\chi_1(p)G(f_2)\,dp \\
&= 2b_1 \int_{|p|\le\frac{|v|}{\sqrt{2}}} |v-p|^{-2+\gamma} f(p)G(f_2)\,dp \\
&\le 2\pi C\|f\|_{s,r}\frac{e^{-s|v|^2}}{(1+|v|^2)^{r/2}} \int_{|p|\le\frac{|v|}{\sqrt{2}}} |v-p|^{-2+\gamma} f(p)e^{s|p|^2}|p|^2\,dp.
\end{aligned}$$

Taking terms to the left hand side, yields the estimate

$$\|J^+(f_1,f_2)\|_{s,r} \le C\|f\|^2_{s,r}\int_{|p|\le\frac{|v|}{\sqrt{2}}} |v-p|^{-2+\gamma}(1+|p|^2)^{-r/2}|p|^2\,dp.$$

Now we note that for $|p| \le \frac{|v|}{\sqrt{2}}$ we have the inequality

$$|v| - \frac{|v|}{\sqrt{2}} \le |v| - |p| \le |v-p|$$

which implies

$$|v-p|^{-2+\gamma} \le C|v|^{-2+\gamma}$$

where C is again some positive constant and $\gamma \in [0,1]$. In addition, by using the fact that for $|p| \le |v|$

$$\frac{|p|^2}{1+|p|^2} \le \frac{|v|^2}{1+|v|^2},$$

we obtain

$$\begin{aligned}
\|J^+(f_1,f_2)\|_{s,r} &\le C\|f\|^2_{s,r}|v|^{-2+\gamma}\int_{|p|\le\frac{|v|}{\sqrt{2}}} \frac{|v|^2}{1+|v|^2}\frac{(1+|p|^2)}{(1+|p|^2)^{r/2}}\,dp \\
&\le C\|f\|^2_{s,r}|v|^{-2+\gamma}(1+|v|^2)^{-1}|v|^2\int_{|p|\le\frac{|v|}{\sqrt{2}}} \frac{(1+|p|^2)}{(1+|p|^2)^{r/2}}\,dp.
\end{aligned}$$

The integral in the above expression can be estimated by letting $\rho = |p|$, and changing into polar coordinates. Hence we write

$$\begin{aligned}
\int_{|p|\le\frac{|v|}{\sqrt{2}}} \frac{(1+|p|^2)}{(1+|p|^2)^{r/2}}\,dp &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{|v|}{\sqrt{2}}} \frac{(1+\rho^2)}{(1+\rho^2)^{r/2}}\,\rho^2\sin\theta\,d\rho\,d\theta\,d\phi \\
&= 4\pi\int_0^{\frac{|v|}{\sqrt{2}}} \frac{1+\rho^2}{(1+\rho^2)^{r/2}}\,\rho^2\,d\rho.
\end{aligned}$$

For $r > 4$ the integrand will be less than 1, thus

$$\int_{|p|\leq\frac{|v|}{\sqrt{2}}} \frac{(1+|p|^2)}{(1+|p|^2)^{r/2}}\,dp \leq C\int_0^{\frac{|v|}{\sqrt{2}}} d\rho \leq C|v|$$

and we arrive at the result

$$\|J^+(f_1,f_2)\|_{s,r} \leq C\|f\|^2_{s,r} \underbrace{\frac{|v|^{1+\gamma}}{1+|v|^2}}_{<1} \leq C\|f\|^2_{s,r} \qquad (2.24)$$

for some positive constant C and $\gamma \in [0,1]$.

We continue by estimating the term $|J^+(f_2,f_2)|$. We have from (2.20) that

$$\begin{aligned} |J^+(f_2,f_2)| &\leq 2b_1 \int_{\mathbb{R}^3_v} |v-p|^{-2+\gamma} f_2(p) G(|f_2|)\,dp \\ &= 2b_1 \int_{\frac{|v|}{\sqrt{2}}<|p|<|v|} |v-p|^{-2+\gamma} f(p) G(|f_2|)\,dp. \end{aligned}$$

Using (2.23) we obtain

$$\begin{aligned} |J^+(f_2,f_2)| &\leq C\|f\|_{s,r}\frac{e^{-s|v|^2}}{(1+|v|^2)^{r/2}} \int_{\frac{|v|}{\sqrt{2}}<|p|<|v|} f(p)|v-p|^{-2+\gamma} e^{s|p|^2}|p|^2\,dp \\ &= C\|f\|_{s,r}\frac{e^{-s|v|^2}}{(1+|v|^2)^{r/2}} \int_{\frac{|v|}{\sqrt{2}}<|p|<|v|} \Big[|v-p|^{-2+\gamma} \\ &\quad \times \underbrace{f(p)(1+|p|^2)^{r/2}e^{s|p|^2}}_{\leq \|f\|_{s,r}} (1+|p|^2)^{-r/2}|p|^2 \Big]\,dp. \end{aligned}$$

By taking terms to the left hand side and applying the definition of the $\|.\|_{s,r}$ one further obtains

$$\|J^+(f_2,f_2)\|_{s,r} \leq C\|f\|^2_{s,r}|v|^2(1+|v|^2)^{-r/2} \int_{\frac{|v|}{\sqrt{2}}<|p|<|v|} |v-p|^{-2+\gamma}\,dp.$$

Changing into polar coordinates, let $\rho = |v-p| \leq |v| + |p| \leq 2|v|$. Thus for $r > 4$, we have

$$\|J^+(f_2,f_2)\|_{s,r} \;\leq\; C\|f\|^2_{s,r}\frac{|v|^2}{(1+|v|^2)^{r/2}} \int_0^{2|v|} \rho^{-2+\gamma}\rho^2\,d\rho$$

$$
\begin{aligned}
&\leq \; C\|f\|_{s,r}^2 \underbrace{\frac{|v|^{3+\gamma}}{(1+|v|^2)^{r/2}}}_{\leq 1} \\
&\leq \; C\|f\|_{s,r}^2. \qquad (2.25)
\end{aligned}
$$

In the same manner as before, we estimate the term $G(|f_3|)$:

$$
\begin{aligned}
G(|f_3|) &= \int_{|q|\geq|v|} f(q)(1+|q|^2)^{r/2} e^{s|q|^2}(1+|q|^2)^{-r/2} e^{-s|q|^2}\, dq \\
&\leq \|f\|_{s,r} \int_{|q|\geq|v|} (1+|q|^2)^{-r/2} e^{-s|q|^2}\, dq \\
&\leq 2\pi\|f\|_{s,r} \int_{R\geq|v|} (1+R^2)^{-r/2} e^{-sR^2} R\, dR \\
&\leq C\|f\|_{s,r}(1+|v|^2)^{-r/2} \int_{|v|}^{\infty} Re^{-sR^2}\, dR \\
&\leq C\|f\|_{s,r}(1+|v|^2)^{-r/2} e^{-s|v|^2} \\
&\leq C(\varphi(v))^{-1}\|f\|_{s,r} \qquad (2.26)
\end{aligned}
$$

where we have again changed to polar coordinates, with $|q| = R$; and have used the fact that for $f = f_3$, $|q|$ belongs in the set D_3.

Now we complete the proof by claiming that $\|J^+(f, f_3)\|_{s,r+2-\gamma} \leq C\|f\|_{s,r}^2$. Again from (2.20), and (2.26) we write

$$
\begin{aligned}
|J^+(f_1, f_3)| &\leq 2b_1 \int_{\mathbb{R}_v^3} |v-p|^{-2+\gamma} |f_1| G(|f_3|)\, dp \\
&\leq C(\varphi(v))^{-1}\|f\|_{s,r} \int_{\mathbb{R}_v^3} |v-p|^{-2+\gamma} f(p)\chi_1(p)\, dp \\
&= C(\varphi(v))^{-1}\|f\|_{s,r} \int_{|p|\leq\frac{|v|}{\sqrt{2}}} |v-p|^{-2+\gamma} f(p)\, dp \\
&= C(\varphi(v))^{-1}\|f\|_{s,r} \\
&\quad \times \int_{|p|\leq\frac{|v|}{\sqrt{2}}} |v-p|^{-2+\gamma} f(p)\varphi_{r,s}(p)(\varphi_{r,s}(p))^{-1}\, dp
\end{aligned}
$$

where by taking terms to the left hand side we have

$$
\varphi(v)|J^+(f_1, f_3)| \;\leq\; C\|f\|_{s,r}^2 \int_{|p|\leq\frac{|v|}{\sqrt{2}}} |v-p|^{-2+\gamma}(1+|p|^2)^{-r/2} e^{-s|p|^2}\, dp.
$$

For $|p| \leq \frac{|v|}{\sqrt{2}}$ we note that since

$$|v| - \frac{|v|}{\sqrt{2}} = \left(1 - \frac{1}{\sqrt{2}}\right)|v| \leq |v| - |p| \leq |v-p|$$

one obtains

$$|v-p|^{-2+\gamma} \leq C|v|^{-2+\gamma}$$

and so

$$\varphi(v)|J^+(f_1,f_3)| \leq C\|f\|_{s,r}^2 |v|^{-2+\gamma} \int_{|p|\leq\frac{|v|}{\sqrt{2}}} (1+|p|^2)^{-2}|p|^2|p|^{-2}e^{-s|p|^2}\,dp.$$

Using the estimate

$$|p|^2 \leq \frac{|v|^2}{1+|v|^2}(1+|p|^2)$$

it is easily verified that

$$\begin{aligned}
\varphi(v)|J^+(f_1,f_3)| &\leq C\|f\|_{s,r}^2|v|^{-2+\gamma}\frac{|v|^2}{1+|v|^2}\int_{|p|\leq\frac{|v|}{\sqrt{2}}}\frac{e^{-s|p|^2}}{(1+|p|^2)|p|^2}\,dp \\
&= C\|f\|_{s,r}^2\frac{|v|^\gamma}{1+|v|^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{|v|}{\sqrt{2}}}\frac{e^{-s\rho^2}}{\rho^2(1+\rho^2)}\rho^2\sin\theta\,d\rho\,d\theta\,d\phi \\
&\leq 4\pi C\|f\|_{s,r}^2(1+|v|^2)^{-1+\gamma/2}\int_0^{\frac{|v|}{\sqrt{2}}}e^{-s\rho^2}\,d\rho.
\end{aligned}$$

Where again we have changed to polar coordinates, with $\rho = |p|^2$. Now taking the factor $(1+|v|^2)^{-1+\gamma/2}$ to the left hand side, and making use of the fact that

$$\int_0^{\frac{|v|}{\sqrt{2}}} e^{-s\rho^2}\,d\rho \leq C(s)$$

where $C(s)$ is some positive constant depending on s only, we finally obtain

$$\|J^+(f_1,f_3)\|_{r+2-\gamma} \leq C_1\|f\|_{s,r}^2. \tag{2.27}$$

The term $\|J^+(f_2,f_3)\|_{s,r}$ is estimated in the same way as before, except we take into account the fact that in this case p belongs to the set D_2 rather than D_1 as in the previous estimate. Therefore we have by (2.20) and (2.26) that:

$$\begin{aligned}|J^+(f_2,f_3)| &\le C(\varphi(v))^{-1}\|f\|_{s,r}\int_{\frac{|v|}{\sqrt{2}}<|p|<|v|}|v-p|^{-2+\gamma}f(p)\,dp\\ &\le C(\varphi(v))^{-1}\|f\|_{s,r}^2\int_{\frac{|v|}{\sqrt{2}}<|p|<|v|}|v-p|^{-2+\gamma}(1+|p|^2)^{-r/2}e^{-s|p|^2}\,dp.\end{aligned}$$

Since

$$1+|p|^2 \ge 1+\frac{|v|^2}{2} \ge \frac{1}{2}+\frac{|v|^2}{2} = \frac{1}{2}(1+|v|^2)$$

we have

$$(1+|p|^2)^{-r/2} \le \left(\frac{1}{2}\right)^{-r/2}(1+|v|^2)^{-r/2}.$$

Therefore, using the fact that $\exp(-s|p|^2) < 1$, we write

$$|J^+(f_2,f_3)| \le C2^{r/2}(\varphi(v))^{-1}(1+|v|^2)^{-r/2}\|f\|_{s,r}^2\int_{\frac{|v|}{\sqrt{2}}<|p|<|v|}|v-p|^{-2+\gamma}\,dp$$

By letting $\rho = |v-p| \le 2|v|$ and changing into polar coordinates, the above integral can be evaluated to be bounded by $4\pi(2|v|)^{1+\gamma}$. So we have for $r > 4$,

$$\begin{aligned}|J^+(f_2,f_3)| &\le C2^{r/2}4\pi(2^{1+\gamma})|v|^{1+\gamma}(1+|v|^2)^{-2}(\varphi(v))^{-1}\|f\|_{s,r}^2\\ &\le C_2(1+|v|^2)^{\gamma/2}(1+|v|^2)(1+|v|^2)^{-2}(\varphi(v))^{-1}\|f\|_{s,r}^2\\ &\le C_2(1+|v|)^{\gamma/2}(1+|v|^2)^{-1}(\varphi(v))^{-1}\|f\|_{s,r}^2\\ &\le C_2(1+|v|^2)^{(-r-2+\gamma)/2}\|f\|_{s,r}^2.\end{aligned}$$

Hence, by taking terms to the left hand side of the inequality, one finally obtains the required estimate

$$\|J^+(f_2,f_3)\|_{s,r+2-\gamma} \le C_2\|f\|_{s,r}^2 \tag{2.28}$$

where C_2 is some positive constant and $\gamma \in [0,1]$.

Finally we estimate the term $\|J^+(f_3,f_3)\|_{s,r}$, in which the calculations are exactly the same as before, except that p is in the set D_3, and so by inequalities (2.20) and

(2.26) we get:

$$\begin{aligned}|J^+(f_3, f_3)| &\leq C(\varphi(v))^{-1}\|f\|_{s,r}\int_{|p|\geq|v|}|v-p|^{-2+\gamma}f(p)\,dp\\ &= C(\varphi(v))^{-1}\|f\|_{s,r}^2\int_{|p|\geq|v|}|v-p|^{-2+\gamma}(1+|p|^2)^{-r/2}e^{-s|p|^2}\,dp.\end{aligned}$$

We note that since $1+|p|^2 \geq 1+|v|^2$ for $|p| \geq |v|$ we have $(1+|p|^2)^{-r/2} \leq (1+|v|^2)^{-r/2}$, hence:

$$\begin{aligned}|J^+(f_3, f_3)| &\leq C(\varphi(v))^{-1}\frac{\|f\|_{s,r}^2}{(1+|v|^2)^{r/2}}\int_{|p|\geq|v|}|v-p|^{-2+\gamma}e^{-s|p|^2}\,dp\\ &\leq C(\varphi(v))^{-1}\frac{\|f\|_{s,r}^2}{(1+|v|^2)^{r/2}}\int_{|p|\geq|v|}|v-p|^{-2+\gamma}e^{\left(\frac{-s}{2}|v-p|^2\right)}\,dp\end{aligned}$$

where we have used the fact that for $|p| \geq |v|$, $|v-p| \leq |v|+|p| \leq 2|p|$ and thus:

$$\exp(-s|p|^2) \leq \exp\left(\frac{-s}{2}|v-p|^2\right).$$

And now we evaluate the integral in the above expression by letting $\rho = |v-p|$ and changing into polar coordinates. We note that after a change of variables, the above integral can be rewritten as:

$$4\pi\int_0^\infty \rho^\gamma e^{-s\rho^2/2}\,d\rho$$

since $0 < \rho < \infty$. However this integral is just a constant depending on s, and γ. Thus

$$\begin{aligned}|J^+(f_3, f_3)| &\leq C_3(\varphi(v))^{-1}(1+|v|^2)^{-r/2}(1+|v|^2)^{\gamma/2}\underbrace{(1+|v|^2)^{-\gamma/2}}_{<1}\|f\|_{s,r}^2\\ &\leq C_3(1+|v|^2)^{-r/2}e^{-s|v|^2}(1+|v|^2)^{-1+\gamma/2}\|f\|_{s,r}^2.\end{aligned}$$

By taking terms to the other side of the inequality it is readily observed that

$$\|J^+(f_3, f_3)\|_{s,r+2-\gamma} \leq C\|f\|_{s,r}^2. \tag{2.29}$$

From the bilinearity of the gain term we obtain

$$\begin{aligned}
|J^+(f_1+f_2+f_3,f_3)| &= |J^+(f_1,f_3)|+|J^+(f_2,f_3)|+|J^+(f_3,f_3)| \\
&= |J^+(f,f_3)| \\
&\leq (C_1+C_2+C_3)(\varphi(v))^{-1}(1+|v|^2)^{-1+\gamma/2}\|f\|^2_{s,r} \\
&= C(\varphi(v))^{-1}(1+|v|^2)^{-1+\gamma/2}
\end{aligned}$$

and therefore,

$$\|J^+(f,f_3)\|_{s,r+2-\gamma} \leq C\|f\|^2_{s,r}. \tag{2.30}$$

Finally from (2.24), (2.25), and (2.30) we have the required estimate stated in the Lemma for positive constant C, $\gamma \in [0,1]$, and $r > 4$. □

The above estimate is crucial in proving existence and uniqueness results for the steady Boltzmann Equation with diffuse boundary conditions, as will be seen in Chapter 4. We emphasize the need for the space of functions $L^\infty_{s,r}$ in proving such bounds. In Appendix C, we will consider the properties of the collision operators in the space of functions with only the inverse power decreasing for large $|v|$; and show that we cannot achieve bounds in this norm. We will explain the source of this unboundedness which will further demonstrate the need for a stronger norm.

Remark 2.3 *We emphasize the fact that we are only considering non-negative functions f, and g since they denote the one-particle distribution function. Moreover, due to the symmetry property (2.4), it is sufficient to consider the case $f = g$ as we did in Lemma 2.2.*

Below we show that the collision operators introduce some regularity in the velocity space. The proof of this lemma is used later on in establishing other bounds on the collision operator and producing the appropriate function space in which one can apply the contraction mapping principle.

Lemma 2.4 *Let:*

$$\hat{V}_k(f) = \sup_{\omega} \int_{\mathbb{R}^3_v} |\omega - p|^{-k} f(p)\, dp \qquad \text{for } 0 < k < 2$$

Then for $\gamma \in [0,1]$ *and any positive function* h *satisfying* $h(q) \leq h(v)h(v_*)$ *where* $q = v - n(n \cdot (v - v_*))$, *we have:*

$$\hat{V}_k(hJ^+(f,g)) \leq C(2-k)^{-1}\hat{V}_\circ(fh)\hat{V}_{k-\gamma}(gh)$$

with some positive constant C.

Proof. Let $\bar{\varphi}(v) = h(v)|\omega - v|^{-k}$, and we write by (2.5) :

$$\langle \bar{\varphi}(v) J^+(f,g) \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} f(v) g(v_*) \bar{\varphi}(v') B(v - v_*, n)\, dv\, dv_*\, dn \tag{2.31}$$

We make the change of variables $n \to q = v - n(n \cdot (v - v_*))$ where n is a unit vector which lives in the unit sphere $\mathbb{S}^2$ directed along the vector $v - v'$. q is a vector in $\mathbb{R}^3$ which lives in the sphere K_{vv_*} defined by:

$$K_{vv_*} = \left\{ q \in \mathbb{R}^3 \,\Big|\, |q - \frac{1}{2}(v + v_*)| = \frac{1}{2}|v - v_*| \right\}.$$

We transform the integration variable n on the unit sphere to a new sphere K_{vv_*} of radius $\frac{1}{2}|v - v_*|$ as seen in Figure 2.2.

Writing q in its components we have

$$\begin{aligned} q_1 &= v_1 - \sin\phi \cos\theta\, (n \cdot (v - v_*)) \\ q_2 &= v_2 - \sin\phi \sin\theta\, (n \cdot (v - v_*)) \\ q_3 &= v_3 - \cos\phi\, (n \cdot (v - v_*)) \end{aligned}$$

Looking at the definition of the set K_{vv_*} we notice that q can also be written as:

$$q = \frac{1}{2}(v + v_*) + \frac{1}{2}|v - v_*|e, \qquad e \in \mathbb{S}^2_*$$

where $\mathbb{S}^2_*$ is the sphere with normal vector e having components:

$$e = (\sin\phi_1 \cos\theta_1, \sin\phi_1 \sin\theta_1, \cos\phi_1).$$

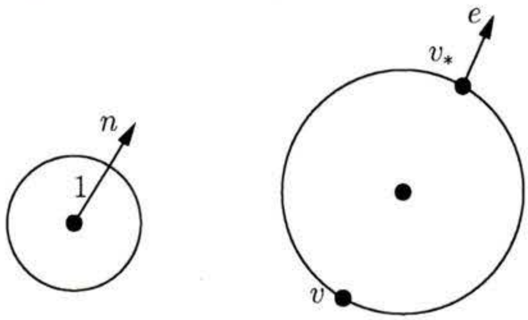

Figure 2.2: A transformation from the unit sphere to the sphere K_{vv_*}.
Here n represents the unit normal to the unit sphere, and e represents the normal vector to the transformed sphere K_{vv_1}.

We want to express the polar and azimuthal angles of the unit sphere, in terms of the polar and azimuthal angles of the new sphere with unit normal e. Now choosing the polar axis in direction $v - v_*$, i.e., $n \cdot (v - v_*) = |v - v_*| \cos\phi$, we have the following relations:

$$\begin{aligned} q_1 &= v_1 - \sin\phi\cos\theta\cos\phi|v - v_*| &&= \tfrac{1}{2}(v_1 + v_{*1}) + \tfrac{1}{2}|v - v_*|\sin\phi_1\cos\theta_1 \\ q_2 &= v_2 - \sin\phi\sin\theta\cos\phi|v - v_*| &&= \tfrac{1}{2}(v_2 + v_{*2}) + \tfrac{1}{2}|v - v_*|\sin\phi_1\sin\theta_1 \\ q_3 &= v_3 - \cos^2\phi|v - v_*| &&= \tfrac{1}{2}(v_3 + v_{*3}) + \tfrac{1}{2}|v - v_*|\cos\phi_1 \end{aligned}$$

where $v = (v_1, v_2, v_3)$ and $v_* = (v_{*1}, v_{*2}, v_{*3})$. From the relationship in q_3, we have:

$$\frac{1}{2}|v - v_*|\cos\phi_1 = \frac{1}{2}(v_3 - v_{*3}) - \cos^2\phi|v - v_*|.$$

Now choose Cartesian coordinates (consistent with polar axis in direction $v - v_*$) such that

$$v - v_* = \begin{pmatrix} 0 \\ 0 \\ v_3 - v_{*3} \end{pmatrix}$$

Hence, $|v - v_*| = |v_3 - v_{*3}|$ and so

$$\frac{1}{2}|v - v_*|\cos\phi_1 = \left(\frac{1}{2} - \cos^2\phi\right)|v - v_*|$$

resulting in the equality

$$\cos\phi_1 = 1 - 2\cos^2\phi$$

which leads to the relationship

$$\phi_1 = 2\phi - \pi.$$

Now using the expressions for q_1 and the above relation we can express θ in terms of θ_1. First we observe from the relation in q_1 that

$$\underbrace{\frac{1}{2}(v_1 - v_{*1})}_{=0} - |v - v_*| \sin\phi \cos\theta \cos\phi = \frac{1}{2}|v - v_*| \sin(\phi_1) \cos\theta_1,$$

and then obtain

$$\theta_1 = \pi + \theta.$$

Now we are able to write the relations

$$\sin\phi = \frac{\sin\phi_1}{2\cos\phi} \text{ and } \cos\phi = \frac{(n \cdot (v - v_*))}{|v - v_*|} = \frac{|q - v|}{|v - v_*|}$$

in which we can then express dn in terms of dq. Therefore, we have

$$\begin{aligned}
\sin\phi \, d\phi \, d\theta &= \frac{\sin\phi_1}{2\cos\phi}\left(\frac{1}{2}\, d\phi_1\right) d\theta_1 \\
&= \frac{\sin\phi_1}{4\cos\phi}\, d\phi_1 \, d\theta_1 \\
&= \frac{|v - v_*|}{4|q - v|} \frac{1}{|v - v_*|^2} 4|v - v_*|^2 \sin\phi_1 \, d\phi_1 \, d\theta_1 \\
&= \frac{1}{|q - v||v - v_*|} \underbrace{\frac{1}{4}|v - v_*|^2 \sin\phi_1 \, d\phi_1 \, d\theta_1}_{dq} \\
&= \frac{1}{|q - v||v - v_*|} dq
\end{aligned}$$

Summarizing, we may write

$$\langle \bar{\varphi}(v) J^+(f, g) \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(v) g(v_*) |v - v_*|^{-1} I \, dv \, dv_* \tag{2.32}$$

where

$$I = \int_{K_{vv_*}} \bar{\varphi}(q) B |q - v|^{-1} \, dq.$$

By (2.9) B satisfies:

$$B(v - v_*, n) \leq b_1 |n.(v - v_*))||v - v_*|^{\gamma - 1} \tag{2.33}$$

so that for $q \in K_{vv_*}$

$$|q - v|^{-1} B \leq b_1 |v - v_*|^{\gamma-1}.$$

Therefore, from the definition of $\hat{V}_k(f)$, the above estimate on B, and the fact that $\bar{\varphi}(q) = h(q)|\omega - q|^{-k}$ we write:

$$\begin{aligned}\hat{V}_k(hJ^+(f,g)) &= \sup_\omega \int_{\mathbb{R}^3_v} |\omega - q|^{-k} h(q) J^+(f,g)\, dq \\ &= \sup_\omega \int_{\mathbb{R}^3_v} \bar{\varphi}(q) J^+(f,g)\, dq.\end{aligned}$$

By (2.16) and (2.31) we have:

$$\begin{aligned}\hat{V}_k(hJ^+(f,g)) &\leq \sup_\omega b_1 \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(v) g(v_*) |v - v_*|^{\gamma-2} \left[\int_{K_{vv_*}} \bar{\varphi}(q)\, dq \right] dv\, dv_* \\ &= \sup_\omega b_1 \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(v) g(v_*) |v - v_*|^{\gamma-2} I_1\, dv\, dv_*\end{aligned}$$

where

$$I_1 = \int_{K_{vv_*}} h(q) |\omega - q|^{-k}\, dq.$$

Since $h(q) \leq h(v)h(v_*)$, I_1 is estimated as:

$$I_1 \leq h(v)h(v_*) I_2$$

with

$$I_2 = \int_{K_{vv_*}} |\omega - q|^{-k}\, dq.$$

Hence we are only left with estimating the integral I_2. Taking

$$\omega = |\omega| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and changing into polar coordinates, we get:

$$q = (R \sin\phi \cos\theta, R \sin\phi \sin\theta, R\cos\phi)$$

which then implies

$$\begin{aligned} |\omega - q| &= |R\sin\phi\cos\theta, R\sin\phi\sin\theta, |\omega| - R\cos\phi| \\ &= \left[R^2 - 2R|\omega|\cos\phi + |\omega|^2\right]^{1/2}. \end{aligned}$$

Thus the integral I_2 is written as

$$I_2 = 2\pi R^2 \int_0^\pi (R^2 - 2R|\omega|\cos\phi + |\omega|^2)^{-k/2} \sin\phi \, d\phi$$

where $R = \frac{1}{4}|v - v_*|$. Integrating first for the case where $|\omega|$ lies inside the sphere of radius R, i.e $|\omega| \leq R$, we have, by letting $u = R^2 + |\omega|^2 - 2R|\omega|\cos\phi$:

$$\begin{aligned} I_2 &= \frac{2\pi R^2}{2R|\omega|} \int_{(R-|\omega|)^2}^{(R+|\omega|)^2} u^{-k/2} \, du \\ &= \frac{\pi}{|\omega|} R \frac{2}{2-k} (u^{2-k})^{1/2} \Big|_{(R-|\omega|)^2}^{(R+|\omega|)^2} \\ &= \frac{2\pi}{2-k} R \left[\frac{(R+|\omega|)^{2-k} - (R-|\omega|)^{2-k}}{|\omega|}\right] \\ &= \frac{2\pi}{2-k} R^{2-k} \frac{\left[\left(1 + \frac{|\omega|}{R}\right)^{2-k} - \left(1 - \frac{|\omega|}{R}\right)^{2-k}\right]}{\frac{|\omega|}{R}}. \end{aligned}$$

To simplify matters, let $x = \frac{|\omega|}{R} \leq 1$ for the case $|\omega| \leq R$. Then we immediately notice that:

$$\frac{(1+x)^{2-k} - (1-x)^{2-k}}{x} \leq \frac{(1+x)^2 - (1-x)^2}{x} = 4 \qquad \text{for } 0 \leq k < 2.$$

For the case when $|\omega| > R$ the limits of integration will be from $(|\omega| - R)^2$ to $(|\omega| + R)^2$. After we evaluate the integral in the same manner as before, and let $x = \frac{|\omega|}{R} > 1$ we have:

$$\frac{(x+1)^{2-k} - (x-1)^{2-k}}{x} \leq \frac{(x+1)^2 - (x-1)^2}{x} = 4 \qquad \text{for } 0 \leq k < 2.$$

For the case when $x \leq 1$, we estimate

$$2\pi(2-k)^{-1}R^{2-k}\frac{\left[\left(1+\frac{|\omega|}{R}\right)^{2-k}-\left(1-\frac{|\omega|}{R}\right)^{2-k}\right]}{\frac{|\omega|}{R}} \leq 8\pi(2-k)^{-1}R^{2-k}.$$

The same estimate also holds for the case $|\omega| > R$. Therefore, we have by the above estimates:

$$\begin{aligned} I_1 &\leq h(v)h(v_*)I_2 \\ &\leq Ch(v)h(v_*)(2-k)^{-1}|v-v_*|^{2-k} \end{aligned}$$

and so the required estimate follows:

$$\begin{aligned} \hat{V}_k(hJ^+(f,g)) &\leq C(2-k)^{-1}\sup_{\omega}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)g(v_*)|v-v_*|^{\gamma-k}h(v)h(v_*)\,dv\,dv_* \\ &= C(2-k)^{-1}\sup_{\omega}\int_{\mathbb{R}^3} f(v)h(v)\,dv\int_{\mathbb{R}^3} g(v_*)h(v_*)|v-v_*|^{\gamma-k}\,dv_* \\ &= C(2-k)^{-1}\hat{V}_\circ(fh)\hat{V}_{k-\gamma}(gh) \end{aligned}$$

for some positive constant C, $0 \leq k < 2$, and $\gamma \in [0,1]$. □

Remark 2.5 *In order to fully clarify the results of Lemma 2.4, we give some examples of the type of functions which satisfy the inequality $h(q) \leq h(v)h(v_*)$. For later purposes we look at functions of the type*

$$h_1(q) = (1+|q|^2)^{r/2} \text{ and } h_2(q) = \exp(s|q|^2)$$

for $r \geq 0, s > 0$. We know that for particle velocities q, q_1, v, v_*, lying on the collison sphere, the conservation of energy dictates that $|q|^2 + |q_1|^2 = |v|^2 + |v_*|^2$, where we recall that v, v_* are the pre-collisional velocities and q, q_1 the post-collisional velocities. Therefore,

$$\begin{aligned} (1+|q|^2) &= 1+|v|^2+|v_*|^2-|q_1|^2 \\ &\leq 1+|v|^2+|v_*|^2+|v|^2|v_*|^2 \\ &= (1+|v|^2)(1+|v_*|^2). \end{aligned}$$

It is now evident that $h_1(q)$ *will satisfy,*

$$h_1(q) \leq h_1(v)h_1(v_*).$$

In addition,

$$\exp(s|q|^2 + s|q_1|^2) = \exp(s|v|^2 + s|v_*|^2)$$

and so it is easily verified that

$$h_2(q) \leq h_2(v)h_2(v_*)$$

Therefore, the functions $h_1(q)$, *and* $h_2(q)$ *certainly satisfy the condition for Lemma (2.4) to hold.*

Chapter 3

The Steady Boltzmann Equation with Inflow Boundary Conditions

3.1 Steady Boltzmann Equation

In the rest of this thesis we will deal with the existence and uniqueness of solutions to the Steady Bolzmann Equation in a bounded domain Ω with given boundary conditions. In particular, in this chapter we will deal with the Boundary Value Problem:

$$v \cdot \nabla_x f = \varepsilon J(f, f) \qquad x \in \Omega \tag{3.1}$$

$$\gamma^- f = f^- \qquad x \in \partial\Omega \tag{3.2}$$

in which the size of the domain is bounded in terms of a small positive parameter ε; which is a measure of the inverse mean free path of a particle. Equation (3.1) is the Steady Boltzmann Equation and the boundary condition (3.2) is called inflow boundary condition where a function f^- is prescribed at the boundary. We rewrite the above problem as:

$$\begin{aligned} D_e f &= \frac{\varepsilon}{|v|} J(f, f) \\ \gamma^- f &= f^- \end{aligned}$$

where $D_e f = \vec{e} \cdot \nabla_x f$, $\vec{e} = \frac{\vec{v}}{|v|}$. We see that in a neighborhood of zero in the velocity space, the non-linear term becomes very significant. Also, for hard sphere interactions, where $B = |n \cdot (v - v_*)|$, the collision kernel, and thus the collision term become unbounded as v beomes large. In previous literature [9] general existence and uniqueness results have been proved for a one-dimensional slab. This was done by introducing unphysical truncations on the collision kernel which eliminated collisions between particles having small velocities. The truncated collision kernel took the following form:

$$B_\varepsilon(v, v_*, n) = \begin{cases} C_\varepsilon |v - v_*|^k h(\theta) & \text{if } \min |v|, |v_*|, |v'|, |v'_*| > \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

for $k \in \{-1, 0, 1\}$, $\varepsilon > 0$, and $C_\varepsilon > 0$ such that $\int_{(v-v_*,n)>0} B_\varepsilon(v, v_*, n)\, dn = |v - v_*|$. In addition, a crude truncation was also done on the collision kernel which removed collisions which generate particles with high velocities.

The main task in the remainder of this thesis is to show that the collision operator $J(f, f)(x, v)$ has the properties which make it possible to avoid truncations. For simplicity, we will consider the case of hard sphere collisions, although generalizations are available. The main estimates in Lemmas 2.1 and 2.3 are used here to construct a function space in which contraction mapping arguments are available. The contraction mapping theorem is then used to establish the existence and uniqueness for bounded domains Ω.

3.2 Steady Solution Operators

Solutions of the boundary value problem (3.1),(3.2) can be obtained by the method of characteristics. Recall that $J(f, f) = J^+(f, f) - f\nu(f)$ where the gain term $J^+(f, f)$ and the collision frequency $\nu(f)$ were defined in the last chapter.

Write (3.1), and (3.2) as:

$$\frac{v}{|v|} \cdot \nabla_x f + \frac{\varepsilon}{|v|} f\nu(f) \;=\; \frac{\varepsilon}{|v|} J^+(f, f)$$

$$\gamma^- f \;=\; f^-.$$

Let $x(s) = \frac{v}{|v|}s + x_\circ$, and note that:

$$\frac{d}{ds} f\left(x(s), v\right) = \frac{v}{|v|} \nabla_x f\left(x(s), v\right).$$

Here $x_\circ$ is some initial point, and $|s|$ denotes the 'time' a particle with velocity $\frac{v}{|v|}$ needs to move from a boundary point $x_\circ + s\frac{v}{|v|}$ to a point $x_\circ$. Along the characteristic $s \to (x(s), v)$, the equation reads

$$\frac{df}{ds} + \frac{\varepsilon}{|v|} f\nu(f) = \frac{\varepsilon}{|v|} J^+(f,f).$$

Multiplying through by the integrating factor:

$$\exp\left\{-\frac{\varepsilon}{|v|}\int_s^0 \nu(x_\circ + \tau\frac{v}{|v|})\,d\tau\right\}$$

where $\nu(f)(x,v) = \nu(x,v)$, we have:

$$\frac{d}{ds}\left[f(x + s\frac{v}{|v|}, v)\exp\left\{-\frac{\varepsilon}{|v|}\int_s^0 \nu(x_\circ + \tau\frac{v}{|v|}, v)\,d\tau\right\}\right] =$$
$$\frac{\varepsilon}{|v|} J^+(f,f)\exp\left\{-\frac{\varepsilon}{|v|}\int_s^0 \nu(x_\circ + \tau\frac{v}{|v|}, v)\,d\tau\right\}.$$

Let

$$s^* \;=\; \inf\left\{s : s < 0, x + s\frac{v}{|v|} \in \partial\Omega\right\}$$
$$\Pi(\tau, v) \;=\; \exp\left\{-|v|^{-1}\varepsilon\int_\tau^0 \nu(x + sv|v|^{-1}, v)\,ds\right\}.$$

Integrating from a point $s^* < 0$ to 0, and changing variables we obtain:

$$f(x_\circ, v) = f(x_\circ + s^*\frac{v}{|v|}, v)\exp\left\{-\frac{\varepsilon}{|v|}\int_{s^*}^0 \nu(x_\circ + s\frac{v}{|v|}, v)\,ds\right\} +$$
$$\varepsilon|v|^{-1}\int_{s^*}^0 J^+(f,f)\exp\left\{-\frac{\varepsilon}{|v|}\int_\tau^0 \nu(x_\circ + s\frac{v}{|v|}, v)\,ds\right\}\,d\tau.$$

Applying the boundary condition:

$$\gamma^- f = f(x,v) = f^-(x,v) \text{ for } x \in \partial\Omega \text{ in the half space } v\cdot n(x) \le 0$$

we get, since $x_\circ + s^* \frac{v}{|v|} \in \partial\Omega$:

$$f(x_\circ + s^* \frac{v}{|v|}, v) = f^-(x_\circ + s^* \frac{v}{|v|}, v).$$

Hence, the solution for problem (3.1), (3.2) can be written as:

$$f(x_\circ, v) = f^-(x_\circ + s^* \frac{v}{|v|}, v)\Pi(s^*, v) + \varepsilon |v|^{-1} \int_{s^*}^{0} J^+(f,f)\Pi(\tau, v)\, d\tau$$

To simplify matters further, we define operators

$$(Wf^-)(x,v) = f^-(x + s^* \frac{v}{|v|}, v)\Pi(s^*, v) \tag{3.3}$$

$$(UJ^+(f,f))(x,v) = |v|^{-1} \int_{s^*}^{0} J^+(f,f)(x + \tau \frac{v}{|v|}, v)\Pi(\tau, v)\, d\tau \tag{3.4}$$

and rewrite the solution to the given boundary value problem as:

$$f(x,v) = A(f) \tag{3.5}$$

where

$$A(f) = Wf^-(x,v) + \varepsilon U J^+(f,f)(x,v). \tag{3.6}$$

Thus we have shown that the problem (3.1),(3.2) can be reduced to the integral equation (3.5). We will use the Contraction Mapping Theorem to prove that this equation has a unique solution for sufficiently small ε.

3.3 A Technicality

3.3.1 Norms Defined

We now introduce the weight function:

$$\varphi(v) = \exp(s|v|^2)(1 + |v|^2)^{r/2} \qquad s \geq 0, r \geq 0 \tag{3.7}$$

with the weighted norms:

$$\|f\|_{-1} = \sup_{\omega} \int_{\mathbb{R}^3_v} \varphi(v)|v - \omega|^{-1} \|f(.,v)\|_{L^\infty(\Omega)}\, dv \tag{3.8}$$

$$\|f\| = \int_{\mathbb{R}^3_v} \varphi(v) \|f(.,v)\|_{L^\infty(\Omega)}\, dv. \tag{3.9}$$

In addition, we consider the Banach space:

$$X_\circ = \{f : \varphi f \in L^1(\mathbb{R}^3_v, L^\infty(\Omega))\} \tag{3.10}$$

with the norm $\|.\|$ defined by (3.9). The norm defined in (3.8) implies another estimate on the collision operator which relies mainly on Lemma 2.4. With the aid of this estimate and the above norms, we are able to produce a function set in which we can apply the Contraction Mapping Theorem.

3.3.2 An Estimate on $\|J^+(f,g)\|_{-1}$

Lemma 3.1 *There exists a positive constant C such that:*

$$\|J^+(f,g)\|_{-1} \leq C\|f\|\,\|g\|$$

for all $f, g \in L^\infty$.

Proof. We showed in Lemma 2.4 that functions of the form $\varphi(v)$ satisfy the condition $\varphi(q) \leq \varphi(v)\varphi(v_*)$ for velocities q, v, v_* lying on the collision sphere. Hence, we can apply Lemma 2.4 with $k = 1, \gamma = 1$ and get the estimate:

$$\hat{V}_1(\varphi J^+(f,g)) \leq C\hat{V}_\circ(\varphi f)\hat{V}_\circ(\varphi g).$$

Applying the definition of $\hat{V}_k$, we have:

$$\begin{aligned} \hat{V}_\circ(\varphi f) &= \int_{\mathbb{R}^3_v} \varphi(v) f(v)\,dv \leq \int_{\mathbb{R}^3_v} \varphi(v)\|f\|_{L^\infty(\Omega)}\,dv \;= \|f\| \\ \hat{V}_\circ(\varphi g) &= \int_{\mathbb{R}^3_v} \varphi(v) g(v)\,dv \leq \int_{\mathbb{R}^3_v} \varphi(v)\|g\|_{L^\infty(\Omega)}\,dv_* = \|g\|. \end{aligned}$$

Therefore,

$$\hat{V}_1(\varphi J^+(f,g)) \leq C\|f\|\,\|g\|$$

which will imply

$$\hat{V}_1(\varphi J^+(\sup_x f, \sup_x g)) \leq C\|\sup_x f\|\,\|\sup_x g\| = C\|f\|\,\|g\|.$$

From the definition of $\|.\|_{-1}$ we have:

$$\begin{aligned} \|J^+(f,g)\|_{-1} &= \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1}\|J^+(f,g)(v)\|_{L^\infty(\Omega)}\,dv \\ &= \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1} \sup_x J^+(f,g)(x,v)\,dv \end{aligned}$$

and since

$$\begin{aligned} \sup_x J^+(f,g) &= \sup_x \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,n) f(x,v) g(x,v_*)\,dn\,dv_* \\ &\leq \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,n) \sup_x f(x,v) \sup_x g(x,v_*)\,dn\,dv_* \end{aligned}$$

we have:

$$\begin{aligned} \|J^+(f,g)\|_{-1} &\leq \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1} J^+(\sup_x f, \sup_x g)(v)\,dv \\ &= \hat{V}_1(\varphi J^+(\sup_x f, \sup_x g)) \\ &\leq C\|f\|\,\|g\|. \ \square \end{aligned}$$

To proceed, we assume that the function f^- (the boundary data) satisfies the following conditions:

$$f^- \geq 0, \ \ \varphi f^- \in L^1(\mathbb{R}^3_v, L^\infty(\partial\Omega)) \tag{3.11}$$

$$\sup_\omega \int |\omega - v|^{-1}\varphi(v)\,\|f^-(.,v)\|_{L^\infty(\partial\Omega)} < \infty \tag{3.12}$$

$$\inf_{x,v} \nu(f^-) = \nu_\circ > 0. \tag{3.13}$$

Moreover, let a_j $(j = 1,2,3)$ be some positive constants, and consider sets $\mathcal{A} \subset X_\circ$ defined by

$$\mathcal{A} = \{f \in X_\circ : f \geq 0, \|f\| \leq a_1, \nu(f) \geq a_2, \|f\|_{-1} \leq a_3\} \tag{3.14}$$

Now we are in a position to prove existence and uniqueness of a solution to the steady boundary value problem. We will show that there are ε, a_1, a_2 and a_3 such that the Banach fixed point theorem is applicable in $\mathcal{A}$.

3.4 Existence and Uniqueness

Every set $\mathcal{A}$ is a closed subset of the Banach space $X_\circ$. Hence we apply the Banach fixed point theorem and achieve existence and uniqueness of solutions to problem (3.1)–(3.2).

Lemma 3.2 *There exists positive constants* a_j $(j = 1, ..., 4)$ *such that* $A\mathcal{A} \subset \mathcal{A}$ *if* $\varepsilon \leq a_4$.

Hence we need to show:

$$a)\ Af \geq 0 \qquad b)\ \|Af\| \leq a_1 \qquad c)\ \nu(Af) \geq a_2 \qquad d)\ \|Af\|_{-1} \leq a_3$$

Proof. a) is trivial: Since $f^- \geq 0$ by assumption, we have $Wf^- \geq 0$. Also for $f \geq 0$, $J^+(f,f) \geq 0$. Hence, $Af = Wf^- + \varepsilon UJ^+(f,f) \geq 0$.
To prove b), set

$$a_\circ = \|\varphi f^-\|_{X_1} \qquad X_1 = L^1(\mathbb{R}^3_v, L^\infty(\partial\Omega)) \tag{3.15}$$

$$a_1 = 2a_\circ \qquad d = \text{diam}\ \Omega \tag{3.16}$$

then by the triangle inequality:

$$\begin{aligned} \|Af\| &\leq \|Wf^-\| + \varepsilon\|UJ^+(f,f)\| \\ &\leq \|f^-(x + s^*\frac{v}{|v|}, v)\Pi(s^*)\| + \varepsilon\|UJ^+(f,f)\| \\ &= \int_{\mathbb{R}^3_v} \varphi\|f^-(x + s^*\frac{v}{|v|}, v)\Pi(s^*)\|_{L^\infty(\Omega)}\, dv + \varepsilon\|UJ^+(f,f)\| \end{aligned}$$

where we have used the definition of the norm in (3.9). Since $\Pi(s^*) \leq 1$, we have:

$$\begin{aligned} \|Af\| &\leq \int_{\mathbb{R}^3_v} \varphi\|f^-\|_{L^\infty(\partial\Omega)}\, dv + \varepsilon\|UJ^+(f,f)\| \\ &= \|\varphi f^-\|_{X_1} + \varepsilon\|UJ^+(f,f)\|. \end{aligned}$$

Estimating the term $\|UJ^+(f,f)\|$, we write:

$$\begin{aligned}
\|UJ^+(f,f)\| &= \Big\| |v|^{-1}\int_{s^*}^0 J^+(f,f)(v+\tau\frac{v}{|v|},v)\Pi(\tau,v)\,d\tau\Big\| \\
&= \int_{\mathbb{R}^3_v}\varphi(v)\Big\| |v|^{-1}\int_{s^*}^0 J^+(f,f)(x+\tau\frac{v}{|v|},v)\Pi(\tau,v)\,d\tau\Big\|_{L^\infty(\Omega)} dv \\
&\le \int_{\mathbb{R}^3_v}\varphi(v)|v|^{-1}\int_{s^*}^0 \Big\| J^+(f,f)(x+\tau\frac{v}{|v|},v)\Pi(\tau,v)\Big\|_{L^\infty(\Omega)} dv \\
&\le \int_{\mathbb{R}^3_v}\varphi(v)|v|^{-1}\Big\| J^+(f,f)\Big\|_{L^\infty(\Omega)}\left(\int_{s^*}^0 \Big\|\Pi(\tau,v)\Big\|_{L^\infty} d\tau\right) dv
\end{aligned}$$

where we note that:

$$\int_{s^*}^0 \|\Pi(\tau,v)\|_{L^\infty}\,d\tau \le \int_{s^*}^0 d\tau \le |s^*| \le d$$

which implies

$$\begin{aligned}
\|UJ^+(f,f)\| &\le d\int_{\mathbb{R}^3_v}\varphi(v)|v|^{-1}\Big\|J^+(f,f)\Big\|_{L^\infty(\Omega)} dv \\
&\le d\|J^+(f,f)\|_{-1}
\end{aligned}$$

and so applying Lemma 3.1 we obtain:

$$\begin{aligned}
\|Af\| &\le \|\varphi f^-\|_X + \varepsilon d\|J^+(f,f)\|_{-1} \\
&= \frac{a_1}{2} + \varepsilon d\|J^+(f,f)\|_{-1} \\
&\le \frac{a_1}{2} + \varepsilon dC\|f\|^2.
\end{aligned}$$

By the definition of the set $\mathcal{A}$, we also have that $\|f\| \le a_1$ which gives,

$$\|Af\| \le \frac{a_1}{2} + \varepsilon dC a_1^2.$$

Now by picking $\varepsilon \le \frac{1}{2a_1 Cd} = a_4$ we have proven part $b)$. There exists an $a_4 > 0$ such that:

$$\|Af\| \le a_1 \qquad\qquad \text{if } \varepsilon \le a_4. \tag{3.17}$$

To prove c), we have from the definition of $\nu(f)$:

$$\begin{aligned}
\nu(f)(v) &= \int_{\mathbb{R}^3_v} f(v_*)\left(\int_{\mathbb{S}^2} B(v-v_*,n)\,dn\right)dv_* \\
&= \int_{\mathbb{R}^3_v} f(x,v_*)|v-v_*|\left(\int_{\mathbb{S}^2} |\cos\theta| dn\right)dv_* \\
&= 2\pi\int_{\mathbb{R}^3_v} f(x,v_*)|v-v_*|\,dv_*
\end{aligned}$$

where we have considered only hard sphere interactions in which $B(v,v_*,n) = |(v-v_*,n)|$ and have explicitly evaluated $\int_{\mathbb{S}^2}|\cos\theta|\,dn = 2\pi$. Changing variables $v_* \to w$ we write:

$$\nu(f)(v) = 2\pi\int_{\mathbb{R}^3_v} |v-w| f(x,w)\,dw.$$

In order to prove c) it suffices to show that $\nu(Wf^-)$ is bounded below by some positive constant since

$$\begin{aligned}
\nu(Af) &= 2\pi\int_{\mathbb{R}^3} |v-w|(Wf^- + \varepsilon UJ^+(f,f))\,dw \\
&= 2\pi\int_{\mathbb{R}^3} |v-w| Wf^-\,dw + \varepsilon 2\pi\int_{\mathbb{R}^3}|v-w|UJ^+(f,f)\,dw \\
&= \nu(Wf^-) + \varepsilon\nu(UJ^+(f,f)) \\
&\geq \nu(Wf^-).
\end{aligned}$$

We define for any $b > 0$ the set I as follows:

$$I = \{w|\, b^{-1} < |w| < b\}$$

and we assume that for $\delta > 0$ there exists $b = b(\delta)$ such that

$$2\pi\int_I |v-w| f^-(x,w)\,dw \geq \nu(f^-) - \delta \geq \nu_\circ - \delta. \tag{3.18}$$

Now we have for $f \in \mathcal{A}$,

$$\begin{aligned}
\nu(Wf^-)(x,v) &= 2\pi\int_{\mathbb{R}^3} |v-w| Wf^-(x,w)\,dw \\
&= 2\pi\int_{\mathbb{R}^3} |v-\omega| f^-(x+s^*\frac{\omega}{|w|},w)\Pi(s^*,w)\,dw.
\end{aligned}$$

From the estimate

$$\int_{\mathbb{R}^3_v} |w - w'| f(x + s\frac{w}{|w|}, w')\, dw' \leq |w| \|f\| + \|f\|$$

we conclude that

$$\begin{aligned}
\Pi(s^*, w) &= \exp\left\{ -|w|^{-1} \varepsilon \int_{s^*}^{0} \nu(x + s\frac{w}{|w|}, w)\, ds \right\} \\
&= \exp\left\{ \int_{s^*}^{0} -|w|^{-1} \varepsilon \left[2\pi \int_{\mathbb{R}^3} |w - w'| f(x + s\frac{w}{|w|}, w')\, dw' \right] ds \right\} \\
&\geq \exp\{ -2\pi |w|^{-1} \varepsilon |s^*| (1 + |w|) \|f\| \}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\nu(Wf^-)(x, v) &\geq 2\pi \int_{\mathbb{R}^3} |v - w| f^-(x + s^* \frac{w}{|w|}, w) e^{-\varepsilon 2\pi |s^*| |w|^{-1} (1 + |w|) \|f\|}\, dw \\
&\geq 2\pi \int_{I} |v - w| f^-(x + s^* \frac{w}{|w|}, w) e^{-\varepsilon 2\pi |s^*| |w|^{-1} (1 + |w|) \|f\|}\, dw \\
&\geq 2\pi e^{-\varepsilon 2\pi |s^*| b (1 + b) a_1} \int_{I} |v - w| f^-(x + s^* \frac{w}{|w|}, w)\, dw
\end{aligned}$$

where we have applied the fact that for $f \in \mathcal{A}$, $\|f\| \leq a_1$. We now recall that b, can be chosen such that (3.18) holds. Therefore,

$$\begin{aligned}
\nu(Wf^-)(x, v) &\geq (\nu_\circ - \delta) e^{-\varepsilon 2\pi |s^*| b (1 + b) a_1} \\
&\geq \frac{3}{4} \nu_\circ e^{-\varepsilon 2\pi |s^*| b (1 + b) a_1}
\end{aligned}$$

for the choice of δ, $0 < \delta \leq \frac{1}{4}\nu_\circ$. Now choosing ε small enough such that

$$\nu_\circ \exp(-\varepsilon 2\pi |s^*| b (1 + b) a_1) \geq \frac{2}{3} \nu_\circ$$

we obtain $\nu(Wf^-) \geq \dfrac{\nu_\circ}{2}$, which gives the desired result for c).

In order to prove part d) we have that for $f \in \mathcal{A}$:

$$\begin{aligned}
\|\varepsilon U J^+(f, f)\|_{-1} &= \sup_{w} \int_{\mathbb{R}^3_v} \varphi(v) |v - w|^{-1} \|\varepsilon U J^+(f, f)\|_{L^\infty(\Omega)}\, dv \\
&= \sup_{\omega} \int_{\mathbb{R}^3_v} \varphi(v) |v - w|^{-1}\, dv
\end{aligned}$$

$$
\begin{aligned}
& \times \left[\left\| \varepsilon |v|^{-1} \int_{s^*}^{0} J^+(f,f)(x+\tau\frac{v}{|v|},v)\Pi(\tau,v)\,d\tau \right\|_{L^\infty(\Omega)} \right] dv \\
\leq\ & \sup_w \int_{\mathbb{R}^3_v} \varphi(v)|v-w|^{-1} \\
& \times \left[\|J^+(f,f)\|_{L^\infty(\Omega)} \left\| \varepsilon |v|^{-1} \int_{s^*}^{0} \Pi(\tau,v)\,d\tau \right\|_{L^\infty(\Omega)} \right] dv \\
=\ & \|J^+(f,f)\|_{-1} \left\| \varepsilon |v|^{-1} \int_{s^*}^{0} \Pi(\tau,v)\,d\tau \right\|_{L^\infty(\Omega)} dv.
\end{aligned}
$$

Now from the definition of $\Pi(\tau, v)$ and the fact that for $f \in \mathcal{A}$, $\nu(f) \geq a_2$ we write:

$$
\begin{aligned}
\Pi(\tau,v) &= \exp\{-|v|^{-1}\varepsilon \int_{\tau}^{0} \nu(x+s\frac{v}{|v|},v)\,ds\} \\
&\leq \exp\{-|v|^{-1}\varepsilon a_2 \tau\},
\end{aligned}
$$

and so

$$
\left\| \varepsilon |v|^{-1} \int_{s^*}^{0} \Pi(\tau)\,d\tau \right\|_{L^\infty} \leq \sup_\varepsilon \varepsilon \int_0^d |v|^{-1} e^{\{-a_2\varepsilon|v|^{-1}\tau\}}\,d\tau.
$$

since $|s^*| \leq d$, where d is the diameter of Ω. Therefore,

$$
\begin{aligned}
\|\varepsilon U J^+(f,f)\|_{-1} &\leq \|J^+(f,f)\|_{-1} C \\
\text{where } C &= \sup_\varepsilon \varepsilon \int_0^d |v|^{-1} e^{\{-a_2\varepsilon|v|^{-1}\tau\}}\,d\tau.
\end{aligned}
$$

In addition to the estimate on $\|UJ^+(f,f)\|_{-1}$ we have that:

$$
\begin{aligned}
\|Wf^-\|_{-1} &= \sup_w \int_{\mathbb{R}^3_v} \varphi(v)|v-w|^{-1} \|f^-(x+s^*\frac{v}{|v|},v)\Pi(s^*)\|_{L^\infty(\Omega)}\,dv \\
&\leq \|f^-\|_{-1} \exp\{-|v|^{-1}\varepsilon a_2\tau\} \\
&\leq C_1 \text{ (a positive constant)}
\end{aligned}
$$

Making use of the triangle inequality and the last two estimates, we have:

$$
\|Af\|_{-1} \leq C_1 + \|J^+(f,f)\|_{-1} C
$$

where upon integration:

$$
\begin{aligned}
C &= \sup_\varepsilon \varepsilon \int_0^d |v|^{-1} e^{\{-a_2\varepsilon|v|^{-1}\tau\}}\,d\tau \\
&= \sup_\varepsilon \{-\frac{1}{a_2} e^{\{-a_2\varepsilon|v|^{-1}d\}} + \frac{1}{a_2}\} \\
&\leq \frac{1}{a_2}.
\end{aligned}
$$

From Lemma 3.1, we have for $f \in \mathcal{A}$

$$\begin{aligned} \|J^+(f,f)\|_{-1} &\leq C_2\|f\|^2 \\ &\leq C_2 a_1^2 \end{aligned}$$

which implies

$$\|Af\|_{-1} \leq C_1 + C_2 a_1 a_2^{-1}.$$

Put $a_3 = C_1 + C_2 a_1^2 a_2^{-1}, a_2 = \nu_\circ/2$. Then we have that: $\|Af\|_{-1} \leq a_3$ and $\nu(Af) \geq a_2$ as required. □

Lemma 3.3 *There exists a positive constant C such that:*

$$\|Af - Ag\| \leq C\varepsilon\|f - g\|$$

for all $f, g \in \mathcal{A}$, $\varepsilon \leq a_4$

Proof. We have:

$$\begin{aligned} Af &= Wf^- + \varepsilon U J^+(f,f) \\ Ag &= Wf^- + \varepsilon U J^+(g,g) \end{aligned}$$

Clearly,

$$\|Af - Ag\| = \varepsilon\|UJ^+(f,f) - UJ^+(g,g)\|.$$

Upon applying the definition of the operator U and the norm $\|.\|$, we have:

$$\begin{aligned} \|Af - Af\| &\leq \varepsilon \int_{\mathbb{R}^3_v} \varphi(v) \Big\| |v|^{-1} \left[\int_{s^*}^0 (J^+(f,f) - J^+(g,g))\Pi(\tau)\, d\tau \right] \Big\|_{L^\infty(\Omega)} dv \\ &= \varepsilon \int_{\mathbb{R}^3_v} \varphi(v)|v|^{-1} \Big\| J^+(f,f) - J^+(g,g) \Big\|_{L^\infty} \left(\int_{s^*}^0 \|\Pi(\tau)\|\, d\tau \right) dv \end{aligned}$$

since $|s*| \leq d$ and $\Pi(\tau) \leq 1$, then

$$\|Af - Ag\| \leq \varepsilon d \int_{\mathbb{R}^3_v} \varphi(v)|v|^{-1}\|J^+(f,f) - J^+(g,g)\|_{L^\infty(\Omega)}\, dv.$$

By using the definition of the collision term, we have:

$$J^+(f,f) - J^+(g,g) = \int_{\mathbb{R}^3\times\mathbb{S}^2} (f(v')f(v'_*) - g(v')g(v'_*))B(v-v_*,n)\,dn\,dv_*,$$

moreover, it is readily noticed that

$$[f(v')-g(v')][f(v'_*)+g(v'_*]+[f(v')+g(v')][f(v'_*)-g(v'_*)] = 2[f(v')f(v'_*)-g(v')g(v'_*)]$$

which gives the result

$$\|J^+(f,f) - J^+(g,g)\|_{L^\infty} \leq \|J^+(f-g,f+g) + J^+(f+g,f-g)\|_{L^\infty}.$$

Therefore,

$$\begin{aligned}
\|Af - Ag\| &\leq \varepsilon d \int_{\mathbb{R}^3_v} \varphi(v)|v|^{-1}\|J^+|(f-g,f+g) + J^+(f+g,f-g)\|_{L^\infty}\,dv \\
&\leq \varepsilon d \int_{\mathbb{R}^3_v} \varphi(v)|v|^{-1}\|J^+(f-g,f+g)\|_{L^\infty}\,dv + \\
&\quad \varepsilon d \int_{\mathbb{R}^3_v} \varphi(v)|v|^{-1}\|J^+(f+g,f-g)\|_{L^\infty}\,dv \\
&= \varepsilon d \left(\|J^+(f-g,f+g)\|_{-1} + \|J^+(f+g,f-g)\|_{-1}\right)
\end{aligned}$$

Again by Lemma 3.1 we have:

$$\begin{aligned}
\|Af - Ag\| &\leq \varepsilon d C_1[\|f-g\|\,\|f+g\| + \|f+g\|\,\|f-g\|] \\
&\leq 2\varepsilon d C_1 \|f-g\|(\|f\| + \|g\|).
\end{aligned}$$

Since both $f, g \in \mathcal{A}$, then $\|f\| \leq a_1$, and $\|g\| \leq a_1$ where a_1 is some positive constant. Thus, we have for all $\varepsilon \leq a_4$ and positive constants C and C_1:

$$\begin{aligned}
\|Af - Ag\| &\leq 4\varepsilon d C_1 a_1 \|f-g\| \\
&= C\varepsilon\|f-g\|. \ \square
\end{aligned}$$

Now by choosing ε small enough such that $C\varepsilon < 1$, we have a contraction in $\mathcal{A}$. From Lemma 3.2, and Lemma 3.3 we have an operator A which maps the set $\mathcal{A}$ into itself in a contractive way. Thus, by the contraction mapping principle [11], equation (3.5) has a unique fixed point. Therefore we have the following theorem:

Theorem 3.4 *There exists an a_4 such that the problem (3.1)-(3.2), has a unique solution if $\varepsilon \leq a_4$*

Moreover, the sequence $f^{(0)} = 0, f^{(n)} = Af^{(n-1)}$ converges to this solution in L^∞ and by applying Lemma 3.3, we have for $m \geq n$:

$$\begin{aligned}\|f^{(n)} - f^{(m)}\| &\leq \|f^{(n)} - f^{(n+1)}\| + \|f^{(n+1)} - f^{(n+2)}\| + ... + \|f^{(m-1)} - f^{(m)}\| \\ &= \|Af^{(n-1)} - Af^{(n)}\| + \|Af^{(n)} - Af^{(n+1)}\| + ... + \|Af^{(m-2)} - Af^{(m-1)}\| \\ &\leq C\varepsilon\|f^{(n-1)} - f^{(n)}\| + C\varepsilon\|f^{(n)} - f^{(n+1)}\| + ... + C\varepsilon\|f^{(m-2)} - f^{(m-1)}\| \\ &\leq C\varepsilon\|f^{(n-1)} - f^{(n)}\| + ... + (C\varepsilon)^{(m-n)}\|f^{(n-1)} - f^{(n)}\|.\end{aligned}$$

Continuing in this way, we get the result:

$$\begin{aligned}\|f^{(n)} - f^{(m)}\| &\leq (C\varepsilon)^n\|f^{(0)} - f^{(1)}\| + (C\varepsilon)^{n+1}\|f^{(0)} - f^{(1)}\| + ... + (C\varepsilon)^m\|f^{(0)} - f^{(1)}\| \\ &= \left((C\varepsilon)^n + (C\varepsilon)^{n+1} + ... + (C\varepsilon)^m\right)\|f^{(1)}\|.\end{aligned}$$

If we let $m \to \infty$, we have:

$$\begin{aligned}\|f^{(n)} - f\| &\leq \frac{(C\varepsilon)^n}{1 - C\varepsilon}\|f^{(1)}\| \\ &\leq C_1(\varepsilon)\varepsilon^n\|f^{(1)}\|.\end{aligned}$$

Since $f^{(1)} = Af^{(0)} = Wf^- \leq f^-$, then

$$\|f^{(n)} - f\| \leq C_1(\varepsilon)\varepsilon^n\|\varphi\gamma^- f\|_{X_1}$$

where

$$X_1 = L^1\left(\mathbb{R}^3_v, L^\infty(\partial\Omega)\right)$$

and C_1 is some positive constant depending on $\varepsilon \in (0, a_4)$. Also, we have a similar estimate for the traces:

$$\|\gamma^+ f^{(n)} - \gamma^+ f\| \;=\; \|\gamma^+(f^{(n)} - f)\|$$

$$\begin{aligned} &= \int_{\mathbb{R}^3_v} \varphi \|\gamma^+(f^{(n)} - f)\|_{L^\infty(\partial\Omega)}\, dv \\ &\leq \int_{\mathbb{R}^3_v} \varphi \|f^{(n)} - f\|_{L^\infty(\partial\Omega)}\, dv \\ &\leq \|f^{(n)} - f\| \end{aligned}$$

which gives

$$\|\gamma^+ f^{(n)} - \gamma^+ f\| \leq C_1(\varepsilon)\varepsilon^n \|\varphi\gamma^- f\|_{X_1}.$$

Here the function $f^{(1)}$ describes the flow in a vacuum, and by the convergence of the solutions in L^∞, we have solutions which are near a vacuum.

Chapter 4

Diffuse Reflection

4.1 Classical Results

In this chapter, we extend the results of the previous chapter to the case where we have diffuse boundary conditions. In particular, we want to show existence and uniqueness of solutions for the boundary value problem:

$$\begin{aligned} D_e f &= \frac{\varepsilon}{|v|} J(f,f) & x \in \Omega && (4.1) \\ \gamma^- f &= R\gamma^+ f & x \in \partial\Omega && (4.2) \end{aligned}$$

where again Ω denotes a bounded domain in $\mathbb{R}^3$, with a Lyapunov surface[1] $\partial\Omega$, and $n(x)$ will denote the unit outward normal at a point $x \in \partial\Omega$. Diffuse reflection is determined by the following operator R:

$$R\gamma^+ f = M(x,v) \int_{\omega \cdot n(x) > 0} |\omega \cdot n(x)| \gamma^+ f(x,\omega)\, d\omega \tag{4.3}$$

where $M(x,v)$ is a normalized Maxwellian given by

$$M(x,v) = (2\pi)^{-1} h^2(x) \exp\{-|v|^2 h(x)/2\} \chi \tag{4.4}$$

[1]This is a smooth surface, defined by $F(x) = 0$, on which $n(x) = \frac{\nabla F}{\|\nabla F\|}$ is well defined; in particular, $\nabla F(x) \neq 0$.

where χ is the characteristic function of $\{v : v \cdot n(x) < 0\}$. Physically h in (4.4) is determined by the temperature of the wall $\partial\Omega$. It is assumed that

$$h \in L^\infty(\partial\Omega) \quad \text{and} \inf h = h_\circ > 0. \tag{4.5}$$

In addition we assume hard sphere collisions, i.e.,

$$B(v, n) = |n \cdot v| \tag{4.6}$$

and we recall that:

$$\int_{\mathbb{R}_v^3} J(f, f)\, dv = 0, \tag{4.7}$$

where $J(f, f)$ is the collision operator defined in Chapter 2. Moreover, we note that:

$$\int_{v \cdot n < 0} |v \cdot n| M(x, v)\, dv = 1 \tag{4.8}$$

and it follows from (4.4) that

$$\int_{\mathbb{R}_v^3 \times \partial\Omega} (v \cdot n(x)) f(x, v)\, dv\, d\sigma(x) = 0 \tag{4.9}$$

where $d\sigma$ is the Lebesgue measure on $\partial\Omega$. The proof for (4.8) and (4.9) are done in Appendix D.

4.2 Existence and Uniqueness

As will become clear later on, the boundary value problem (4.1)–(4.2) does not possess a unique solution. There is one degree of freedom. In order to achieve uniqueness, one must impose an additional normalization condition to determine, say, the incoming flux at $\partial\Omega$. Hence, our solutions will be normalized by the condition:

$$\int_{\mathbb{R}_v^3 \times \partial\Omega} |v \cdot n(x)| \gamma^- f(x, v)\, dv\, d\sigma(x) = 1 \tag{4.10}$$

where the value one has no particular meaning; it can be replaced by any positive constant.

4.2.1 Steady Solution Operators

Now we return to the problem (4.1)–(4.2), and we find the appropriate solution operators in this case. Let V be the solution operator to the problem (3.1)–(3.2). Then the function $f = V(\gamma^- f)$ solves (4.1)–(4.2) if the function $\gamma^- f$ solves the equation

$$\gamma^- f = R\gamma^+ V(\gamma^- f) \tag{4.11}$$

We may write equation (4.1) as:

$$\frac{df}{ds} = \frac{\varepsilon}{|v|} J(f, f)$$

Integrating along the characteristic, we have:

$$\begin{aligned}\int_{s^*}^{0} \frac{df}{ds}(x_\circ + \frac{v}{|v|}s, v)\, ds &= \frac{\varepsilon}{|v|}\int_{s^*}^{0} J(f,f)(x_\circ + \frac{v}{|v|}s, v)\, ds, \\ \text{i.e.,} f(x_\circ, v) &= f(x_\circ + \frac{v}{|v|}s^*, v) + \varepsilon|v|^{-1}\int_{s^*}^{0} J(f,f)(x_\circ + \frac{v}{|v|}\tau, v)\, d\tau\end{aligned}$$

where we recall that

$$s^* = \inf\{s : s < 0, x + s\frac{v}{|v|} \in \partial\Omega\}$$

Now applying the boundary condition (4.2), where $x_\circ + \frac{v}{|v|}s^* \in \partial\Omega$, we have:

$$f(x, v) = \gamma^- f(x + s^*\frac{v}{|v|}, v) + \varepsilon|v|^{-1}\int_{s^*}^{0} J(f,f)(x + \frac{v}{|v|}\tau, v)\, d\tau$$

We define solution operators similar to the operators defined in (3.3),(3.4) except with $\Pi = 1$. Therefore we write

$$(W_\circ\gamma^- f)(x, v) = \gamma^- f(x + s^*\frac{v}{|v|}, v) \tag{4.12}$$

$$U_\circ J(f,f)(x, v) = |v|^{-1}\int_{s^*}^{0} J(f,f)(x + \frac{v}{|v|}\tau, v)\, d\tau \tag{4.13}$$

so that the solution to (4.1)–(4.2) is written as:

$$V\gamma^- f = W_\circ\gamma^- f + \varepsilon U_\circ J(V\gamma^- f, V\gamma^- f). \tag{4.14}$$

Upon applying (4.14) to the boundary condition defined in (4.2), $\gamma^- f$ can be expressed as:

$$\gamma^- f = R\gamma^+ W_\circ \gamma^- f + \varepsilon R\gamma^+ U_\circ J(V\gamma^- f, V\gamma^- f) \tag{4.15}$$

where we remind the reader that V is the solution operator to problem (3.1)–(3.2), and R is the diffuse reflection operator defined in (4.3).

4.2.2 Reduction of the Problem

In view of (4.2) and (4.3), we seek a solution of $\gamma^- f$ in the form:

$$\gamma^- f = M(x,v)N(x) \tag{4.16}$$

with some unknown function N. Applying the definition of the diffuse operator in equation (4.15) we observe that:

$$\begin{aligned}\gamma^- f &= M(x,v)\left[\int_{\omega\cdot n(x)>0} |\omega\cdot n(x)|\gamma^+ W_\circ \gamma^- f(x,\omega)\, d\omega\right.\\ &\quad \left. +\varepsilon \int_{\omega\cdot n(x)>0} |\omega\cdot n(x)|\gamma^+ U_\circ J(V\gamma^- f, V\gamma^- f)\, d\omega\right]\\ &= M(x,v)N(x)\end{aligned}$$

by assumption. Hence it follows that

$$\begin{aligned}N(x) &= \int_{\omega\cdot n(x)>0} |\omega\cdot n(x)|\gamma^+ W_\circ \gamma^- f(x,\omega)\, d\omega \\ &\quad +\varepsilon \int_{\omega\cdot n(x)>0} |\omega\cdot n(x)|\gamma^+ U_\circ J(V\gamma^- f, V\gamma^- f)\, d\omega\\ &= I_1 + \varepsilon I_2\end{aligned} \tag{4.17}$$

where we abbreviated the first integral as I_1, and the second as I_2.

Our main strategy now is to find the function N and show that it yields the unique solution to the problem (4.1),(4.2), and (4.10). However, in order to do this, we must express $N(x)$ in terms of fixed points of operators, which will be shown to have the contraction property, thereby proving the necessary results. We begin by

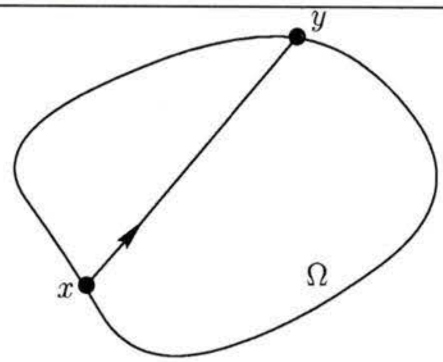

Figure 4.1: Illustration of the corresponding boundary point of $x \in \partial\Omega$.

expressing each term in equation (4.17) in a way which will become useful later on. With this in mind we rewrite I_1 in (4.17) as:

$$\int_{e\cdot n(x)>0} |e \cdot n| \int_0^\infty |\omega|^3 \gamma^+ W_\circ(M(x,\omega)N(x))\, d|\omega|\, de$$

where the velocity vector is $\omega = |\omega| e$ and $e = \omega/|\omega|$ is a unit vector in the direction of ω. Since $\gamma^+ f$ determines the densities of particles impinging on $\partial\Omega$, and because

$$\gamma^+ W_\circ(M(x,\omega)N(x)) = W_\circ(M(x,\omega)N(x)) \qquad \text{for } \omega \cdot n(x) > 0, x \in \partial\Omega$$

the presence of the operator $\gamma^+(.)$ does not affect the value of the integral when $\omega \cdot n(x) > 0$. Also, since the operator $W_\circ$ acts only on the position variable, i.e. $W_\circ M(x,v) = M(x + s\frac{v}{|v|}, v)$, then upon substituting this into (4.4), I_1 takes the form:

$$\int_{e\cdot n(x)>0} |e \cdot n| \gamma^+ W_\circ N(x) \int_0^\infty |\omega|^3 (2\pi)^{-1} h^2(y) e^{-|\omega|^2 h(y)/2}\, d|\omega|\, de \tag{4.18}$$

where $y = x + s\frac{\omega}{|\omega|}$ is the corresponding boundary point of $x \in \partial\Omega$ in direction e (Figure 4.1). The inner integral in (4.18) can easily be calculated to have the value π^{-1}. Hence, the expression for $N(x)$ is written as:

$$N(x) = \pi^{-1} \int_{e.n(x)>0, |e|=1} |e \cdot n(x)| N(y)\, de + \varepsilon \int_{\omega\cdot n(x)>0} |\omega \cdot n(x)| \gamma^+ U_\circ J(V\gamma^- f, V\gamma^- f)\, d\omega$$

We note that $y \in \partial\Omega$ is given by $y = x - \frac{v}{|v|}|x-y|$ which implies:

$$e = \frac{v}{|v|} = \frac{x-y}{|x-y|}.$$

Secondly, de can be expressed in terms of a general surface element $d\sigma(y)$ at a point $y \in \partial\Omega$, where $\partial\Omega$ is the part of the body seen from x. In Appendix C, we show that de is really the solid angle subtended by $d\sigma(y)$ at some fixed point, and is given by:

$$de = \frac{n(y) \cdot e}{|x-y|^2} d\sigma(y) \tag{4.19}$$

where $n(y)$ is the normal to the surface $d\sigma(y)$. Substituting the expressions e, and de in the integral I_1, we can write the equation for $N(x)$ in the form:

$$N(x) = \mathcal{K}N + \varepsilon G(N) \tag{4.20}$$

where the operators $\mathcal{K}$, N are defined as follows:

$$\mathcal{K}N(x) = \int_{\partial\Omega} \mathcal{K}(x,y) N(y)\, d\sigma(y) \tag{4.21}$$

$$\mathcal{K}(x,y) = \pi^{-1}|x-y|^{-4}|(x-y, n(x)||(x-y, n(y)| \tag{4.22}$$

$$G(N)(x) = \int_{\mathbb{R}^3_v} |v \cdot n(x)| \gamma^+ U_\circ J(V\gamma^- f, V\gamma^- f)\, dv \tag{4.23}$$

in which the integral I_1 is now represented by the operator $\mathcal{K}N(x)$, and integral I_2 is represented by the operator $G(N)(x)$. We note that since by definition $\gamma^+ U_\circ J(V\gamma^- f, V\gamma^-) = 0$ in the half space $v \cdot n(x) \leq 0$, we can just as easily integrate over all space without affecting the value of the integral I_2 in (4.23). As mentioned, the above operators have very special properties, which are needed in proving uniqueness. These properties are entailed in the next four lemmas.

4.2.3 Some Properties of the Operators $\mathcal{K}$ and G

The Operator G

Lemma 4.1 *$G(N)$ satisfies:*

$$\int_{\partial\Omega} G(N)\, d\sigma = 0$$

where $d\sigma$ is the surface element at a point $x \in \partial\Omega$.

Proof. As will become clear, this lemma is a direct consequence of (4.7).

By an application of the definition of $U_\circ$, we write:

$$\begin{aligned}\int_{\partial\Omega} G(N)\,d\sigma &= \int_{\partial\Omega}\int_{\mathbb{R}^3_v}\frac{|v\cdot n(x)|}{|v|}\int_{s^*}^0 J(V\gamma^-, V\gamma^- f)(x+\tau\frac{v}{|v|}, v)\,d\tau\,dv\,d\sigma(x)\\ &= \int_{\mathbb{R}^3_v}\int_{\partial\Omega}\int_{s^*}^0 \frac{|v\cdot n(x)|}{|v|} J(V\gamma^- f, V\gamma^- f)(y,v)\,d\tau\,d\sigma(x)\,dv\end{aligned}$$

where the transformation $y = x+\tau e$, $e=\frac{v}{|v|}$, $x\in\partial\Omega$, $\tau\in[s^*,0]$, $y\in\Omega\subset\mathbb{R}^3$ with :

$$dy = |e\cdot n(x)|\,d\tau\,d\sigma(x)$$

leads to:

$$\int_{\mathbb{R}^3_v} G(N)\,d\sigma = \int_{\Omega\times\mathbb{R}^3} J(f,f)(y,v)\,dv\,dy = 0$$

from (4.7). □

Lemma 4.2 *For any constant C_1 independent of N, we have:*

$$\|GN\|_{L^\infty(\partial\Omega)} \le C_1\|N\|^2_{L^\infty(\partial\Omega)}.$$

Proof. Since we assume a solution of the form $f^- = \gamma^- f = M(x,v)N(x)$, we may write:

$$\begin{aligned}a_\circ &= \|\varphi f^-\|_{X_1}\\ &= \int_{\mathbb{R}^3_v}\varphi(v)\|M(.,v)N(.)\|_{L^\infty(\partial\Omega)}\,dv\\ &= \frac{1}{2\pi}\int_{\mathbb{R}^3_v}\varphi(v)\,\|h^2(x)e^{-h(x)|v|^2/2}N(x)\|_{L^\infty(\partial\Omega)}\,dv\\ &\le \frac{1}{2\pi}\int_{\mathbb{R}^3_v}\varphi(v)\,\|h^2\|_{L^\infty(\partial\Omega)}\,e^{-h_\circ|v|^2/2}\,\|N(x)\|_{L^\infty(\partial\Omega)}\,dv\\ &= \frac{1}{2\pi}\|h^2\|_{L^\infty(\partial\Omega)}\,\|N(x)\|_{L^\infty}\int_{\mathbb{R}^3_v}(1+|v|^2)^r e^{s|v|^2}e^{-h_\circ|v|^2/2}\,dv\end{aligned}$$

for $h(x)\ge h_\circ$, by (4.5). For $s<\frac{1}{2}h_\circ = \frac{1}{2}\inf h$, the above integral is bounded by a constant. Hence

$$a_\circ \le C\|N(x)\|_{L^\infty}$$

in which we have absorbed $\|h^2(x)\|_{L^\infty}$ into the constant C. Now since $f(x,v) = V(M(x,v)N(x))$ defines the solution operator for the problem (3.1)–(3.2) then by the discussion in Chapter 3, $f \in \mathcal{A}$, and $\|f\| \le a_1 = 2a_\circ$. Therefore,

$$\|f\| \le 2C\|N(x)\|_{L^\infty(\partial\Omega)}. \tag{4.24}$$

We are now able to prove the asserted estimate.

$$\begin{aligned} |G(N)(x)| &= \left|\int_{\mathbb{R}^3_v} |v\cdot n(x)|\gamma^+ U_\circ J(V\gamma^- f, V\gamma^- f)\,dv\right| \\ &\le \int_{\mathbb{R}^3_v} (1+|v|^2)^{r/2} e^{-s|v|^2} e^{s|v|^2} |J^+(f,f)|\,dv \\ &\quad + \int_{\mathbb{R}^3_v} (1+|v|^2)^{r/2} e^{-s|v|^2} e^{s|v|^2} |J^-(f,f)|\,dv. \end{aligned}$$

Estimating the first integral on the right, one gets

$$\begin{aligned} \int_{\mathbb{R}^3_v} (1+|v|^2)^{r/2} e^{-s|v|^2} e^{s|v|^2} |J^+(f,f)|\,dv &\le \int_{\mathbb{R}^3_v} (1+|v|^2)^{r/2} e^{-s|v|^2} e^{s|v|^2} \|J^+(f,f)\|_{L^\infty} \\ &= \int_{\mathbb{R}^3_v} e^{-s|v|^2} \|J^+(f,f)\|_{s,r}\,dv \\ &\le C\int_{\mathbb{R}^3_v} \|f\|^2_{s,r}\,dv \end{aligned}$$

where we applied the definition of $\|.\|_{s,r}$ defined in Chapter 2, and Lemma 2.2. We notice that

$$\begin{aligned} C\varphi(v)\|f\|_{L^\infty} &= C(1+|v|^2)^{r/2} e^{s|v|^2} \|M(.,v)\|_{L^\infty} \|N\|_{L^\infty} \\ &= C(1+|v|^2)^{r/2} e^{s|v|^2} \|h^2\|_{L^\infty} e^{-|v|^2 h/2} \|N\|_{L^\infty} \\ &\le C_1 \|N\|_{L^\infty} \end{aligned}$$

for positive constants C,C_1 and $s < \frac{1}{2}\inf h$. We also recall the definition of the norm $\|.\|$ defined in (3.9) and write

$$\begin{aligned} C\int_{\mathbb{R}^3_v} \|f\|^2_{s,r}\,dv &\le C_1\|N\|_{L^\infty} \int_{\mathbb{R}^3_v} \|f\|_{s,r}\,dv \\ &\le C_1\|N\|_{L^\infty}\|f\|. \end{aligned}$$

The estimate for the integral involving the loss term is done in the same way, and gives the same estimates. Hence, we have from the estimate in (4.24) the required result

$$\|GN\|_{L^\infty(\partial\Omega)} \le C_1 \|N\|^2_{L^\infty(\partial\Omega)}. \quad \square$$

The Operator $\mathcal{K}$

Lemma 4.3 *The operator $\mathcal{K}$ is compact in $L^2(\partial\Omega)$. Moreover, for every $x \in \partial\Omega$*

$$\int_{\partial\Omega} \mathcal{K}(x,y)\, d\sigma(y) = 1.$$

Thus any constant solves the equation $N = \mathcal{K}N$.

Proof. Since $\partial\Omega$ is a Lyapunov surface, there exists a positive constant $\delta > 0$ such that

$$\sup_{x,y} \mathcal{K}(x,y)|x-y|^{3-\delta} < \infty$$

This implies integrability of the operator $\mathcal{K}$. From standard books on real analysis (see for example [7]), any integrable function can be approximated in L^2 by smooth functions having compact support. Thus compactness of the operator $\mathcal{K}$ follows. To continue, let

$$\tilde{N} = \mathcal{K}N(x) + \varepsilon G(N)(x) \tag{4.25}$$

where now we have made the distinction that $\tilde{N}$ represents the out-going flux at the boundary and is defined by

$$\begin{aligned}\tilde{N}(x) &= \int_{\omega\cdot n(x)>0} |\omega\cdot n(x)|\gamma^+ W_\circ \gamma^- f(x,\omega)\, d\omega \\ &\quad +\varepsilon \int_{\omega\cdot n(x)>0} |\omega\cdot n(x)|\gamma^+ U_\circ J(V\gamma^- f, V\gamma^- f)\, d\omega\end{aligned}$$

and $N(x)$ represents the (given) in-going flux and is defined the same way, except the integration is extended to the hemi-sphere $\omega \cdot n(x) \le 0$. Thus using Lemma 4.1, it is readily seen that the total flux exiting the boundary is given by

$$\int_{\partial\Omega} \tilde{N}(x)\, d\sigma(x) = \int_{\partial\Omega}\int_{\omega\cdot n(x)>0} |\omega \cdot n(x)|\gamma^+ W_\circ \gamma^- f(x,\omega)\, d\omega\, d\sigma(x)$$

Equation (4.9) indicates that the total flux entering the domain is equal to the total flux exiting, which is explicitely written as:

$$\int_{\omega\cdot n(x)>0}\int_{\partial\Omega} |\omega \cdot n(x)| f(x,v)\, dv\, d\sigma(x) = \int_{\omega\cdot n(x)<0}\int_{\partial\Omega} |\omega \cdot n(x)| f(x,v)\, dv\, d\sigma(x).$$

Therefore, we see that by this condition,

$$\int_{\partial\Omega} \tilde{N}(x)\, d\sigma(x) = \int_{\partial\Omega} N(x)\, d\sigma(x). \tag{4.26}$$

Now by integrating equation (4.25) over the boundary, and applying Lemma 4.1 one obtains

$$\int_{\partial\Omega} \tilde{N}(x)\, d\sigma(x) = \int_{\partial\Omega} \mathcal{K}N(x)\, d\sigma(x). \tag{4.27}$$

Using (4.21), (4.26) and the fact that $\mathcal{K}(x,y)$ is symmetric, we have

$$\begin{aligned}
\int_{\partial\Omega} N(x)\, d\sigma(x) &= \int_{\partial\Omega}\left(\int_{\partial\Omega} \mathcal{K}(x,y)N(y)\, d\sigma(x)\right) d\sigma(y) \\
&= \int_{\partial\Omega}\left(\int_{\partial\Omega} \mathcal{K}(y,x)N(x)\, d\sigma(y)\right) d\sigma(x) \\
&= \int_{\partial\Omega} N(x)\left(\int_{\partial\Omega} \mathcal{K}(y,x)\, d\sigma(y)\right) d\sigma(x)
\end{aligned}$$

which leads to

$$\int_{\partial\Omega} N(x)\left(1 - \int_{\partial\Omega} \mathcal{K}(y,x)\, d\sigma(y)\right) d\sigma(x) = 0.$$

As this holds for arbitrary $N \geq 0$, we readily observe that this condition holds if and only if

$$\int_{\partial\Omega} \mathcal{K}(x,y)\, d\sigma(y) = 1. \quad \square$$

It is immediate that any constant solves the equation $N = \mathcal{K}N$. The next lemma will show that actually constants are the only possible solution to the equation $N = \mathcal{K}N$.

Lemma 4.4

$$\ker(I-\mathcal{K})\cap\left\{N\in L^2(\partial\Omega),(1,N)_{L^2(\partial\Omega)}=0\right\}=\{0\}$$

Proof. Suppose $\mathcal{M}(y)$ solves the equation $N=\mathcal{K}N$, and that

$$\int_{\partial\Omega}\mathcal{M}(y)\,d\sigma(y)=0$$

If $\mathcal{M}(y)\neq 0$, we make the following decomposition

$$\mathcal{M}(y)=\mathcal{M}_+(y)-\mathcal{M}_-(y)$$

where we define

$$\mathcal{M}_+(y)=\begin{cases}\mathcal{M}(y) & \text{if } \mathcal{M}(y)>0\\ 0 & \text{if } \mathcal{M}(y)\leq 0\end{cases}\quad\text{and}\quad \mathcal{M}_-(y)=\begin{cases}0 & \text{if } \mathcal{M}(y)>0\\ -\mathcal{M}(y) & \text{if } \mathcal{M}(y)<0\end{cases}$$

In addition, we define the sets

$$\begin{aligned} D_+ &= \{x\in\partial\Omega;\ M(x)>0\}\\ D_- &= \{x\in\partial\Omega;\ M(x)\leq 0\}\end{aligned}$$

and note that since $\mathcal{M}(y)$ solves the equation $N=\mathcal{K}N$, then from (4.21) we have

$$\int_{D_+}\mathcal{K}(x,y)\mathcal{M}_+(y)\,d\sigma-\int_{D_-}\mathcal{K}(x,y)\mathcal{M}_-(y)\,d\sigma=\mathcal{M}_+(x)-\mathcal{M}_-(x).$$

Therefore,

$$\begin{aligned}\int_{D_+}\int_{D_+}\mathcal{K}(x,y)\mathcal{M}_+(y)\,d\sigma(y)\,d\sigma(x) &- \int_{D_+}\int_{D_-}\mathcal{K}(x,y)\mathcal{M}_-(y)\,d\sigma(y)d\sigma(x)\\ &= \int_{D_+}\mathcal{M}_+(x)\,d\sigma(x)\end{aligned}$$

and

$$\begin{aligned}\int_{D_-}\int_{D_+}\mathcal{K}(x,y)\mathcal{M}_+(y)\,d\sigma(y)\,d\sigma(x) &- \int_{D_-}\int_{D_-}\mathcal{K}(x,y)\mathcal{M}_-(y)\,d\sigma(y)\,d\sigma(x)\\ &= -\int_{D_-}\mathcal{M}_-(x)\,d\sigma(x).\end{aligned}$$

Hence we have

$$\int_{D_+}\left(\int_{D_+}\mathcal{K}(x,y)\,d\sigma(x)-1\right)\mathcal{M}_+(y)\,d\sigma(y)-\int_{D_-}\left(\int_{D_+}\mathcal{K}(x,y)\,d\sigma(x)\right)\mathcal{M}_-(y)\,d\sigma(y)$$
$$=0. \tag{4.28}$$

But we know from Lemma (4.3) that

$$\int_{D_+}\mathcal{K}(x,y)\,d\sigma(x)\le\int_{\partial\Omega}\mathcal{K}(x,y)\,d\sigma(x)=1$$

and so the left-hand side of equation (4.28) is negative, which is impossible unless both $\mathcal{M}_+=\mathcal{M}_-=0$. So the only solution is if

$$\mathcal{M}_+(y)\equiv\mathcal{M}_-(y)\equiv 0.\ \ \square$$

We have shown that $\mathcal{K}$ is compact, and we have restricted everything to the space orthogonal to the set of constants where one is not an eigenvalue. Hence the operator $(I-\mathcal{K})^{-1}G$ is continuous in $L^\infty(\partial\Omega)$ restricted to the orthogonal complement of constants. Thus we see that the problem (4.1),(4.2),(4.10) reduces to the following equation:

$$N=C+\varepsilon(I-\mathcal{K})^{-1}G(N). \tag{4.29}$$

Remark 4.5 *We make the important observation that the constant in equation (4.29) is in general not unique. However by the normalization condition in (4.10) it is possible to define this constant uniquely.*

Contraction Properties

Lemma 4.6 $N(x)$ *in (4.29) is uniquely defined.*

Proof. From (4.10) and the fact that we are assuming solutions of the form in (4.16), we write:

$$\begin{aligned}\int_{\mathbb{R}^3_v\times\partial\Omega}|v\cdot n(x)|M(x,v)N(x)\,dv\,d\sigma(x) &= \int_{\partial\Omega}N(x)\,d\sigma(x)\int_{\mathbb{R}^3_v}|v\cdot n(x)|M(x,v)\,dv\\ &= 1.\end{aligned}$$

In view of (4.8), and Lemma 4.4 this gives:

$$\begin{aligned}\int_{\partial\Omega} N(x)\, d\sigma(x) &= C\int_{\partial\Omega} d\sigma(x)\\ &= 1\end{aligned}$$

and thus C^{-1} is equal to the measure of boundary $\partial\Omega$. Next we show that the operator $\varepsilon(I-\mathcal{K})G$ is a contraction. Assume that there are two different solutions

$$\begin{aligned}\gamma^- f_1 &= M(x,v)N_1\\ \gamma^- f_2 &= M(x,v)N_2.\end{aligned}$$

Now from (4.23) we write:

$$\begin{aligned}\|G(N_1)-G(N_2)\|_{L^\infty} \leq &\int_{\mathbb{R}^3_v}(1+|v|^2)^{r/2}e^{-s|v|^2}e^{s|v|^2}\|J^+(f_1,f_1)-J^+(f_2,f_2)\|_{L^\infty}\, dv\\ &+\int_{\mathbb{R}^3_v}(1+|v|^2)^{r/2}e^{-s|v|^2}e^{s|v|^2}\|J^-(f_1,f_1)-J^-(f_2,f_2)\|_{L^\infty}\, dv\end{aligned}$$

Again as in Lemma 4.2 we look at the term containing the gain term, since the term involving the loss term is estimated in the same way. We showed in Lemma 3.3 that:

$$\|J^+(f_1,f_1)-J^+(f_2,f_2)\|_{L^\infty} \leq C\|J^+(f_1-f_2,f_1+f_2)\|_{L^\infty}.$$

By Lemma 2.2 the first term in the above inequality is bounded above by:

$$\int_{\mathbb{R}^3_v} e^{-s|v|^2}\|J^+(f_1-f_2,f_1+f_2)\|_{s,r}\, dv \;\leq\; C\int_{\mathbb{R}^3_v}\|f_1-f_2\|_{s,r}\|f_1+f_2\|_{s,r}\, dv$$

and in addition, by a consequence of Lemma 4.2 we have that

$$\|f_1-f_2\|_{s,r}\leq C\|N_1-N_2\|_{L^\infty}$$

for $s<\frac{1}{2}\inf h$. Hence

$$\begin{aligned}C\int_{\mathbb{R}^3_v}\|f_1-f_2\|_{s,r}\|f_1+f_2\|_{s,r} &\leq C_1\|N_1-N_2\|_{L^\infty}\|f_1+f_2\|\\ &\leq C_1\|N_1-N_2\|\end{aligned}$$

for $f_1, f_2 \in \mathcal{A}$. By the bilinearity of the operator $G(N)$, we have:

$$\|\varepsilon(I-\mathcal{K})^{-1}\left(G(N_1)-G(N_2)\right)\|_{L^\infty} \leq C\varepsilon\|N_1-N_2\|_{L^\infty}.$$

Therfore the operator $\varepsilon(I-\mathcal{K})^{-1}G$ is a contraction in a ball

$$\{N|\, \|N\|_{L^\infty} \leq 2C\}$$

if ε is sufficiently small. □

Now we have that the function $f = V\gamma^- f$ with $\gamma^- f = M(x,v)N(x)$ solves (3.5) and (4.10) with N uniquely given in (4.29). Recall that $(1+|v|)^{-1}J(f,f) \in L^\infty$ if $f \in L^\infty$. Thus f belongs to the set:

$$W = \left\{f \in L^\infty : (1+|v|)^{-1}Df \in L^\infty\right\}.$$

We are now in a position to state the main result of this chapter:

Theorem 4.7 *There exists $\varepsilon_\circ$ such that the problem (4.1),(4.2),(4.10) has a unique solution in W if $\varepsilon < \varepsilon_\circ$, $0 \leq s \leq \frac{1}{2}\inf h$, $r \geq 2$.*

Chapter 5

The Couette Problem

In the last chapters, we discussed three-dimensional problems in bounded domains. In this chapter, we consider diffuse reflection at the end points of a one-dimensional slab. Of course, it is important to point out that existence and uniqueness has been done for one-dimensional slabs [1], but for a truncated collison kernel. Here we prove existence and uniqueness, by applying some of the properties of the Boltzmann equation, entailed in the estimates produced on the collision operator. As seen in Chapter 4, we used the estimates from Chapter 3 on the collision term to produce a function set where we applied the contraction mapping principle. However, it is important to realize that for the class of problems involving uniqueness, the role of spatial dimensions plays an important part. For example in the Couette problem discussed here, we are proving existence and uniqueness in a one-dimensional slab where our domain is bounded with respect to the x direction, and infinite in the y and z directions. This makes the problem more difficult to handle, but we will see in this chapter that our method of approach is the same as before, with some modifications. For example the steady solution operators defined in Chapter 3, are used here with a slightly different representation. In addition we introduce another norm which concentrates on a plane and makes it possible to handle the unboundedness in the y and z directions separately. We use this norm to prove an additional property of the collision term which will then be used to produce the suitable function space with the contractive property essential in proving existence and uniqueness in the one-dimensional case. It will become clear in the calculations, how the effects of this norm handles the unboundedness in the y and z directions and make it crucial to the one-dimensional setting.

5.1 Structure of the Problem

In the Couette problem, we consider the steady Boltzmann equation in a slab where our domain is now $\Omega = (0, 1)$, and

$$v_1 \frac{\partial f}{\partial x} = \varepsilon J(f, f) \tag{5.1}$$

where $v \in \mathbb{R}^3$, $v = (v_1, v_2, v_3)$. Our goal is to find a function $f(x, v)$ which solves (5.1) together with the diffuse boundary conditions

$$f(0, v) = M(0, v)N_\circ(f) \qquad \text{if } v_1 > 0 \tag{5.2}$$

$$f(1, v) = M(1, v)N_1(f) \qquad \text{if } v_1 < 0. \tag{5.3}$$

Here $N_\circ, N_1$ represent the flux entering and exiting the slab, respectively; and are given by:

$$\begin{aligned} N_\circ(f) &= \int_{v_1<0} |v_1| f(0, v)\, dv \\ N_1(f) &= \int_{v_1>0} |v_1| f(1, v)\, dv. \end{aligned}$$

The functions $M(i, v)$ are in the one-dimensional case defined to be:

$$M(i, v) = (2\pi)^{-1} h_i^2 \exp(-|v|^2 h_i/2)$$

where i belongs to the set $\{0, 1\}$; which are the positions at the two endpoints of the slab. Here, we also recall that h_i are some positive constants determined by the temperature at the two endpoints. By a simple calculation, we have:

$$\begin{aligned}
\int_{v_1>0} M(0,v)|v_1|\,dv &= \int_{v_1>0} (2\pi)^{-1} e^{-v_1^2 h_i/2} e^{-(v_2^2+v_3^2)h_\circ/2} |v_1| h_\circ^2 \, dv_1\, dv_2\, dv_3 \\
&= \frac{h_\circ^2}{2} \underbrace{\int_{v_1>0} e^{-v_1^2 h_\circ/2} |v_1|\, dv_1}_{=1/h_\circ} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(v_2^2+v_3^2)h_\circ/2}\, dv_2\, dv_1 \\
&= \frac{h_\circ}{2\pi} \left(\int_{-\infty}^{\infty} e^{-v_2^2 h_\circ/2}\, dv_2\right) \left(\int_{-\infty}^{\infty} e^{-v_3^2 h_\circ/2}\, dv_3\right) dv_3 \\
&= 1.
\end{aligned}$$

Similarily,

$$\int_{v_1<0} M(1,v)|v_1|\,dv = 1.$$

Thus, we have, (since $f(0,v) = M(0,v)N_\circ$ for $v_1 > 0$) by multiplying by $|v_1|$ and integrating over all $v_1 > 0$:

$$\begin{aligned}
\int_{v_1>0} |v_1| f(0,v)\,dv &= \int_{v_1>0} |v_1| M(0,v) N_\circ(f)\,dv \\
&= N_\circ(f) \int_{v_1>0} M(0,v)|v_1|\,dv \\
&= N_\circ(f)
\end{aligned}$$

and in the same way,

$$\int_{v_1<0} |v_1| f(1,v)\,dv = N_1(f).$$

As before, the problem (5.1)–(5.3) does not determine a solution in a unique way. We need another condition much like the normalization condition in equation (3.10). Hence our problem reduces to showing that for any constant $C_\circ$, there exists a unique solution of (5.1)–(5.3) which satisfies the condition:

$$N_\circ(f) + N_1(f) = C_\circ. \tag{5.4}$$

Again, we proceed in exactly the same manner as in Chapter 3, by first considering the situation where we have inflow boundary conditions. Thus, we first show that there exists a unique function satisfying:

$$v_1 \frac{\partial f}{\partial x} = \varepsilon J(f,f) \tag{5.5}$$

$$f(0,v) = f^-(0,v) \qquad v_1 > 0 \tag{5.6}$$

$$f(1,v) = f^-(1,v) \qquad v_1 < 0 \tag{5.7}$$

in which the incoming flows are prescribed at the boundary, and are given by the functions $f^-(i,v), i = 0,1$.

5.1.1 Steady Solution Operators

We rewrite equation (5.5) as:

$$\frac{\partial f(x,v)}{\partial x} + \frac{\varepsilon}{v_1} f\nu(f) = J^+(f,f)$$

and holding v fixed we treat this equation as an ordinary differential equation. Hence multiplying through by the integrating factor:

$$\exp\{-\varepsilon v_1^{-1} \int_y^x \nu(z,v)\, dz\}$$

and applying the boundary conditions (5.6)–(5.7), we convert problem (5.5)–(5.7) to the integral equation:

$$f = Af, \qquad A(f) = Wf^- + \varepsilon U J^+(f,f) \tag{5.8}$$

where the operators W,U in one dimension, admit the following representation:

$$(Wf^-)(x,v) = f^-(\chi(v_1),v)\Pi(\chi(v_1)) \tag{5.9}$$

$$(UJ^+(f,f))(x,v) = v_1^{-1} \int_{\chi(v_1)}^x J^+(f,f)(y,v)\Pi(x,y)\, dy \tag{5.10}$$

with

$$\chi(v_1) = \frac{1}{2}(1 - \text{sgn } v_1),$$

$$\Pi(x,y) = \exp\left\{-\varepsilon v_1^{-1} \int_y^x \nu(z,v)\, dz\right\}$$

$$\nu(x,v) = \nu(f)(x,v).$$

Remark 5.1 *We notice that even though v_1 can be positive or negative, the sign of the operator $UJ^+(f,f)(x,v)$ will always be non-negative. This is an important observation for the proof of Lemma 5.4.*

5.2 Further Technicalities

Let E be a plane in $\mathbb{R}^3_v$. Set:

$$\|f\|_2 = \sup_E \int_E \varphi(v)\|f(.,v)\|_{L^\infty((0,1))}\, d\sigma(v) \tag{5.11}$$

where the sup is to be taken over all planes E in $\mathbb{R}^3_v$. It will be seen that the norms, $\|.\|$ and $\|.\|_2$ satisfy certain estimates which are crucial in proving uniqueness in the one-dimensional case. These estimates are proven in the next two lemmas.

5.2.1 An Estimate on $\|J^+(f,f)\|_2$

Lemma 5.2 *There exists a positive constant C_1 independent of f such that:*

$$\|J^+(f,f)\|_2 \leq C_1\|f\|^2 \qquad s \geq 0, r \geq 0$$

where s and r are the parameters in the weight function (3.7).

Proof. We define a weight function $\varphi_\alpha(v)$ in such a way so that it will concentrate on a plane E in $\mathbb{R}^3$ as $\alpha \to \infty$. Without restricting the generality we assume E to be the xy-plane. With this in mind, we choose $\varphi_\alpha(v)$ to be:

$$\varphi_\alpha(v) = \varphi(v)\left(\frac{\alpha}{\pi}\right)^{1/2} \exp(-\alpha v_3^2)$$

where we recall:

$$\varphi(v) = \exp\{s|v|^2\}(1+|v|^2)^{r/2}$$

for $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. Now from the proof of Lemma 2.4, we write:

$$\int_{\mathbb{R}^3_v} \varphi_\alpha(v) J^+(f,f)\, dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)f(v_*) \times \left\{\int_{K_{vv_*}} \varphi_\alpha(q) B(v-v_*,n)\frac{1}{|q-v||v-v_*|}\, d\sigma(q)\right\} dv\, dv_* \tag{5.12}$$

where one recalls K_{vv_*} to be the sphere

$$K_{vv_*} = \left\{ q \in \mathbb{R}^3 \Big| |q - \frac{1}{2}(v + v_*)| = \frac{1}{2}|v - v_*| \right\}.$$

We take $|v_3|$ to be the distance of v from the xy-plane. We see by taking the limit $\alpha \to \infty$, that the only contribution of $\varphi_\alpha(v)$ is when $v_3 = 0$, hence concentrating on the plane E. Now we focus on the inner integral (in brackets) of equation (5.12) which we abbreviate by I. We know from (2.33) that $|B| \leq b_1|q - v||v - v_*|^{\gamma-1}$, hence

$$\begin{aligned} I &\leq b_1 \int_{K_{vv_*}} \varphi_\alpha(q)|v - v_*|^{\gamma-2}\, d\sigma(q) \\ &= b_1|v - v_*|^{\gamma-2} \int_{K_{vv_*}} \varphi_\alpha(q)\, d\sigma(q) \\ &= b_1|v - v_*|^{\gamma-2} \int_{K_{vv_*}} \varphi(q) \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha q_3^2}\, d\sigma(q). \end{aligned}$$

From Lemma (3.1), we have:

$$I \leq b_1|v - v_*|^{\gamma-2} \left(\frac{\alpha}{\pi}\right)^{1/2} \varphi(v)\varphi(v_*) \int_{K_{vv_*}} e^{-\alpha q_3^2}\, d\sigma(q)$$

so now we need to estimate the integral:

$$\int_{K_{vv_*}} e^{-\alpha q_3^2}\, d\sigma(q).$$

To do this, we exploit the geometry of the problem. We consider the case where K_{vv_*} does not intersect the plane E, (as seen in Figure 5.1) and that its origin is situated at the centre of the sphere. Therefore we write:

$$\int_{K_{vv_*}} e^{-\alpha q_3^2}\, d\sigma(q) = \frac{|v - v_*|^2}{4} \int_0^{2\pi} \int_0^{\pi} e^{-\alpha q_3^2} \sin\phi\, d\phi d\theta$$

where $|q_3| = z_\circ - \dfrac{|v - v_*|}{2} \cos\phi$. This gives:

$$I \leq b_1|v - v_*|^{\gamma-2} \left(\frac{\alpha}{\pi}\right)^{1/2} \varphi(v)\varphi(v_*) 2\pi \frac{|v - v_*|^2}{4} \int_0^{\pi} e^{-\alpha q_3^2} \sin\phi\, d\phi.$$

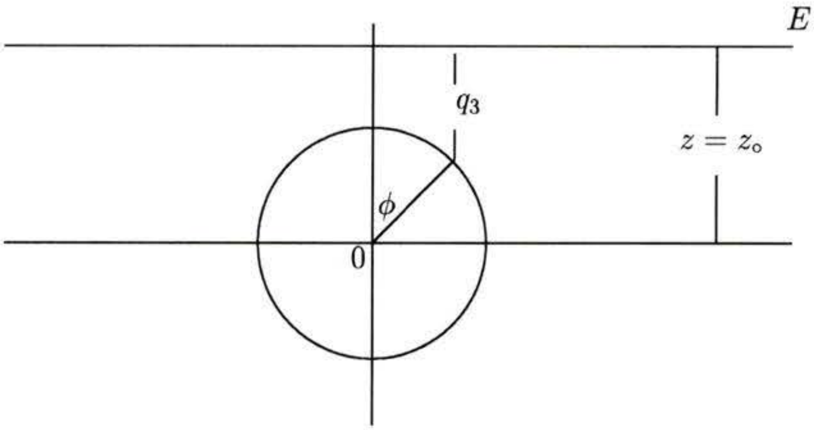

Figure 5.1: Relative distance between a plane $E \subset \mathbb{R}_v^3$ and sphere K_{vv_*}.

$$\text{When } \phi = 0, \qquad |q_3| = z_\circ - \frac{|v - v_*|}{2}$$
$$\text{When } \phi = \pi, \qquad |q_3| = z_\circ + \frac{|v - v_*|}{2}$$

Therefore,

$$\frac{|v - v_*|^2}{4} \int_0^\pi e^{-\alpha q_3^2} \sin\phi \, d\phi = \frac{|v - v_*|}{2} \int_{z_\circ - \frac{|v-v_*|}{2}}^{z_\circ + \frac{|v-v_*|}{2}} e^{-\alpha q_3^2} \, dq_3$$

and has two possible estimates:

$$\begin{aligned} \frac{|v - v_*|}{2} \int_{z_\circ - \frac{|v-v_*|}{2}}^{z_\circ + \frac{|v-v_*|}{2}} e^{-\alpha q_3^2} \, dq_3 &\leq \frac{|v - v_*|}{2} \int_{-\infty}^{\infty} e^{-\alpha q_3^2} \, dq_3 \\ &\leq C|v - v_*| \end{aligned}$$

or

$$\begin{aligned} \frac{|v - v_*|}{2} \int_{z_\circ - \frac{|v-v_*|}{2}}^{z_\circ + \frac{|v-v_*|}{2}} e^{-\alpha q_3^2} \, dq_3 &\leq \frac{|v - v_*|}{2} \int_{z_\circ - \frac{|v-v_*|}{2}}^{z_\circ + \frac{|v-v_*|}{2}} dq_3 \\ &\leq C|v - v_*|^2 \end{aligned}$$

which yields:

$$I \leq C|v - v_*|^{\gamma-2} \min\{|v - v_*|, |v - v_*|^2\}\varphi(v)\varphi(v_*).$$

The proof is done if we can show that

$$C|v - v_*|^{\gamma-2} \min\{|v - v_*|, |v - v_*|^2\} \leq C_1$$

where C, C_1 are some positive constants, and $\gamma \in [0,1]$. For γ=0 or 1 this estimate is clear. For $0 < \gamma < 1$, consider two cases:

$$\begin{array}{lll} \text{if } |v-v_*|<1 & \text{we have} & |v-v_*|^2 < |v-v_*|^{2-\gamma} < |v-v_*| \\ \text{if } |v-v_*|>1 & \text{we have} & |v-v_*|^2 > |v-v_*|^{2-\gamma} > |v-v_*|. \end{array}$$

In either case, $\min\{|v-v_*|, |v-v_*|^2\} < |v-v_*|^{2-\gamma}$ so we have the required result. Thus,

$$\begin{aligned} \int_{\mathbb{R}^3_v} \varphi_\alpha J^+(f,f)\,dv &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)f(v_*)I\,dv\,dv_* \\ &\leq C_1 \int_{\mathbb{R}^3_v} \varphi(v)\|f(x,v)\|_{L^\infty}\,dv \int_{\mathbb{R}^3_v} \varphi(v_*)\|f(x,v_*)\|_{L^\infty}\,dv_* \\ &\leq C_1\|f\|^2. \end{aligned}$$

By taking the limit as $\alpha \to \infty$ the integral on the left hand side of the inequality will concentrate on the plane E, and thus one obtains:

$$\sup_E \int_E \varphi(v)\|J^+(f,f)\|_{L^\infty}\,d\sigma(v) \leq C_1\|f\|^2$$

or:

$$\|J^+(f,f)\|_2 \leq C_1\|f\|^2. \quad \square$$

5.2.2 An Estimate on $\|J^+(f,f)\|$

Lemma 5.3 *For the norms $\|.\|_{-1}$, and $\|.\|$ defined in Chapter 3, we have:*

$$\|J^+(f,f)\| \leq \pi s^{-1}\|f\|_{-1}\|f\| \qquad s>0.$$

Proof. In order to estimate $\|J^+(f,g)\|$ we first evaluate the integral

$$\int_{K_{vv_*}} \exp\{s|q|^2\}\,d\sigma(q)$$

in which K_{vv_*} was defined in Chapter 3, with a unit vector e, given by the components:

$$e = (\sin\phi_1\cos\theta_1, \sin\phi_1\sin\theta_1, \cos\phi_1)$$

and where it was also shown that:

$$q = \frac{1}{2}(v+v_*) + \frac{1}{2}|v-v_*|e \quad \text{and} \quad d\sigma(q) = \frac{1}{4}|v-v_*|^2 \sin\phi \, d\phi \, d\theta.$$

Let $r = \frac{1}{2}|v-v_*|$ be the radius of the sphere K_{vv_*}. Then

$$\begin{aligned}\int_{K_{vv_*}} \exp\{s|q|^2\} \, d\sigma(q) &= r^2 \int_0^{\pi}\int_0^{2\pi} e^{s(v+v_*)^2/4 + sr(v+v_*)\cdot e + s|v-v_*|^2/4} \sin\phi_1 \, d\phi_1 \, d\theta_1 \\ &= 2\pi r^2 e^{s(v+v_*)^2/4} \int_0^{\pi} e^{sr(v+v_*)\cdot e + s|v-v_*|^2/4} \sin\phi_1 \, d\phi_1\end{aligned}$$

Note that

$$re = \frac{1}{2}|v-v_*|(\cos\theta_1 \sin\phi_1, \sin\phi_1 \sin\theta_1, \cos\phi_1)$$

and if we take the polar direction as $v+v_*$, then

$$r(v+v_*).e = |v+v_*| r \cos\phi_1.$$

Therefore, with the substitution $u = s|v-v_*| r\cos\phi_1$

$$\begin{aligned}\int_{K_{vv_*}} \exp\{s|q|^2\} \, d\sigma(q) &= 2\pi r^2 e^{s(v+v_*)^2/4} e^{s|v-v_*|^2/4} \int_0^{\pi} e^{s|v+v_*| r\cos\phi_1} \sin\phi_1 \, d\phi_1 \\ &= -2\pi s^{-1} r^2 r^{-1} |v+v_*|^{-1} e^{s(v+v_*)^2/4} e^{s|v-v_*|^2/4} \int_{sr|v+v_*|}^{-sr|v+v_*|} e^u \, du.\end{aligned}$$

By integrating and cancelling out some terms we obtain:

$$\begin{aligned}\int_{K_{vv_*}} \exp\{s|q|^2\} \, d\sigma(q) &= -\pi s^{-1} \frac{|v-v_*|}{|v+v_*|} e^{s(v+v_*)^2/4 + s|v-v_*|^2/4} \left[e^{-sr|v+v_*|} - e^{sr|v+v_*|}\right] \\ &= \pi s^{-1} \frac{|v-v_*|}{|v+v_*|} \left[e^{s(|v+v_*|+|v-v_*|)^2/4} - e^{s(|v+v_*|-|v-v_*|)^2/4}\right].\end{aligned}$$

Now we let $\varphi(q) = \exp\{s|q|^2\}$, where we recall from Lemma 2.4 that for q, v, v_* lying on the collision sphere, $\varphi(q) \leq \varphi(v)\varphi(v_*)$. The above identity is now written as:

$$\begin{aligned}\int_{K_{vv_*}} \varphi(q) \, d\sigma(q) &= \pi s^{-1} \frac{|v-v_*|}{|v+v_*|} \left[e^{s(|v+v_*|^2+|v-v_*|^2)/4} \left(e^{s|v+v_*||v-v_*|/2} - e^{-s|v+v_*||v-v_*|/2}\right)\right] \\ &\leq \pi s^{-1} \frac{|v-v_*|}{|v+v_*|} \left[e^{s(|v+v_*|^2+|v-v_*|^2)/4} \cdot e^{s|v+v_*||v-v_*|/2}\right].\end{aligned}$$

The right hand side of the above inequality can be further estimated, by making the following observations:

$$|v+v_*|^2 + |v-v_*|^2 = 2(|v|^2 + |v_*|^2)$$

and

$$\begin{aligned} 2|v+v_*||v-v_*| &\leq |v+v_*|^2 + |v-v_*|^2 \\ &= 2(|v|^2 + |v_*|^2). \end{aligned}$$

Therefore,

$$\begin{aligned} e^{s(|v+v_*|^2+|v-v_*|^2)/4} \cdot e^{s|v+v_*||v-v_*|/2} &\leq e^{s(|v|^2+|v_*|^2/2)} \cdot e^{s(|v|^2+|v_*|^2)/2} \\ &= e^{s|v|^2} \cdot e^{s|v_*|^2} \end{aligned}$$

which leads to the estimate

$$\begin{aligned} \int_{K_{vv_*}} \varphi(q)\, d\sigma(q) &\leq \pi s^{-1} \frac{|v-v_*|}{|v+v_*|} (e^{s|v|^2} \cdot e^{s|v_*|^2}) \\ &= \pi s^{-1} \frac{|v-v_*|}{|v+v_*|} \varphi(v)\varphi(v_*). \end{aligned} \tag{5.13}$$

From Lemma 2.4 we have for any function $\varphi(v) \in \mathrm{C}(\mathbb{R}^3_v)$, $\varphi \geq 0$ that

$$\langle \varphi J^+(f,f)\rangle = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v) f(v_*) |v-v_*|^{-1} \left\{ \int_{K_{vv_*}} \varphi(q) B |q-v|^{-1}\, d\sigma(q) \right\} dv\, dv_*.$$

Considering only hard sphere collisions where $B = |q-v||\cos\theta| \leq |q-v|$, we obtain:

$$\langle \varphi(v) \sup_x J^+(f,f)\rangle \leq \int_{\mathbb{R}^3\times\mathbb{R}^3} \sup_x f(v) \sup_x f(v_*) |v-v_*|^{-1} \left\{ \int_{K_{vv_*}} \varphi(q)\, d\sigma \right\} dv\, dv_*$$

and after applying the estimate in (5.13), and using the norms defined in (3.8) and (3.9) we have:

$$\begin{aligned} \|J^+(f,f)\| &\leq \pi s^{-1} \int_{\mathbb{R}^3\times\mathbb{R}^3} \sup_x f(v) \sup_x f(v_*) |v-v_*|^{-1} \frac{|v-v_*|}{|v+v_*|} \varphi(v)\varphi(v_*)\, dv\, dv_* \\ &\leq \pi s^{-1} \sup_v \int_{\mathbb{R}^3} \varphi(v) \sup_x f(v)\, dv \left\{ \int_{\mathbb{R}^3} \varphi(v_*) |v+v_*|^{-1} \sup_x f(v_*)\, dv_* \right\} \end{aligned}$$

$$= \pi s^{-1} \sup_{v} \int_{\mathbb{R}^3} \varphi(v) \|f(.,v)\|_{L^\infty(0,1)}\, dv$$
$$\times \left\{ \int_{\mathbb{R}^3} \varphi(v_*) |v+v_*|^{-1} \|f(.,v_*)\|_{L^\infty(0,1)}\, dv_* \right\}$$
$$= \pi s^{-1} \|f\|_{-1} \|f\| \quad \square$$

5.3 Existence and Uniqueness Results

The estimate used in Lemma 5.3 was not needed in the existence and uniqueness proof of the three-dimensional problems of Chapters 3 and 4. However, we will see that both estimates proved in the last two lemmas are used in proving uniqueness in the one-dimensional case.

5.3.1 The Inflow Case

Let $X_\circ$ be the Banach space defined by (3.10) with $\Omega = (0,1)$, and consider the following set in $X_\circ$:

$$\mathcal{A} = \{f \in X_\circ : f \geq 0, \|f\| \leq a_1, \nu(f) \geq a_2, \|f\|_{-1} \leq a_3, \|f\|_2 \leq a_4\}.$$

Assume that the conditons (3.11)–(3.13) are satisfied with $\partial\Omega = \{0,1\}$. Suppose in addition that:

$$\sup_{E} \int_{E} \varphi f^{-}(i,v)\, d\sigma(v) < \infty \quad \text{for} \quad i = \{0,1\} \tag{5.14}$$

then we have the following lemma:

Lemma 5.4 *Let $s \geq 0, r \geq 1$. There exists positive constants $a_j (j = 1, ..., 5)$ such that $A\mathcal{A} \subset \mathcal{A}$ if $\varepsilon < a_5$.*

Hence we need to show that for suitable constants $a_j (j = 1...5)$ and $f \in \mathcal{A}$,

a) $Af \geq 0$, *b)* $\|Af\| \leq a_1$, *c)* $\nu(Af) \geq a_2$, *d)* $\|Af\|_{-1} \leq a_3$ *e)* $\|Af\|_2 \leq a_4$.

Proof. To prove a), we recall that since both $f^- \geq 0$ and $\Pi(\chi(v_1)) \geq 0$, it is immediate that $Wf^-(x,v) = f^-(\chi(v_1),v)\Pi(\chi(v_1)) \geq 0$. Also, $f \geq 0$ will imply

$J^+(f,f) \geq 0$ and by Remark 5.1 $UJ^+(f,f)(x,v) \geq 0$. Hence, it follows that

$$Af \geq 0 \qquad \text{if } f \geq 0.$$

To prove b), set:

$$a_1 = 2[\|Wf^-\| + \|Wf^-\|_{-1} + \|Wf^-\|_2].$$

We need to show that for $f \in \mathcal{A}$, $\|Af\| \leq a_1$. By the triangle inequality we write:

$$\|Af\| \leq \|Wf^-\| + \|\varepsilon UJ^+(f,f)\|$$

where we see from the definition of a_1, that $\|Wf^-\| \leq a_1/2$. In order to estimate $\|\varepsilon UJ^+(f,f)\|$, we have that:

$$\begin{aligned} \|\varepsilon UJ^+(f,f)\| &= \int_{\mathbb{R}^3_v} \varphi(v)\varepsilon|v_1|^{-1} \left\| \int_{\chi(v_1)}^x J^+(f,f)(y,v)\Pi(x,y)\,dy \right\|_{L^\infty} dv \\ &\leq \int_{\mathbb{R}^3_v} \varepsilon|v_1|^{-1}\varphi(v) \left\| \int_0^1 J^+(f,f)(y,v)\Pi(x,y)\,dy \right\|_{L^\infty} dv \end{aligned}$$

Now we make the following observation:

$$\begin{aligned} \|J^+(f,f)\|_{L^\infty} &= \sup_x J^+(f,f) \\ &\leq J^+(\sup_x f, \sup_x f) = \bar{J}^+(f,f) \end{aligned}$$

and we write:

$$\|\varepsilon UJ^+(f,f)\| \leq \int_{\mathbb{R}^3_v} \varphi(v)\varepsilon|v_1|^{-1}\bar{J}^+(f,f) \left\{ \int_0^1 \|\Pi(x,y)\|_{L^\infty}\,dy \right\} dv.$$

For $f \in \mathcal{A}$, we recall that

$$\nu(f) \geq a_2,$$

therefore,

$$\begin{aligned} \Pi(x,y) &\leq \exp\{-\varepsilon v_1^{-1} \int_y^x a_2\,dz\} \\ &= \exp\{-\varepsilon v_1^{-1}(x-y)a_2\}. \end{aligned}$$

By making the change of variables, $\tau = a_2(x-y)$ one may write:

$$\|\varepsilon J^+(f,f)\| \le a_2 \int_{\mathbb{R}^3_v} \varphi(v)\bar{J}^+(f,f)\|h\|_{L^\infty}\, dv \tag{5.15}$$

where

$$h = \varepsilon|v_1|^{-1}\int_0^1 \exp\{-\varepsilon v_1^{-1}\tau\}\, d\tau$$

Our strategy is to estimate the integral over v_1, and bound the integral over the variable v_2, v_3 by the norm $\|.\|_2$. Hence we break up the integration in (5.15) corresponding to the v_1 variable into a sum of two integrals. The first integral is an integral over the set $S_1 = \{|v_1| < \varepsilon^{-1}\}$, and the second, is an integral over the set $S_2 = \{|v_1| > \varepsilon^{-1}\}$. The part containing an integration over the set S_1, we will denote by I_1; and the part containing the integration over S_2, we denote by I_2. So we write:

$$\begin{aligned}\int_{\mathbb{R}^3_v} \varphi(v)\bar{J}^+ h\, dv &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{S_1} \varphi(v)\bar{J}^+ h\, dv_1\, dv_2\, dv_3 + \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{S_2} \varphi(v)\bar{J}^+ h\, dv_1\, dv_2\, dv_3 \\ &= I_1 + I_2. \end{aligned} \tag{5.16}$$

In the domain where $|v_1| > \varepsilon^{-1}(|v_1|^{-1} < \varepsilon)$, we have :

$$h \le \varepsilon^2 + \mathrm{O}(\varepsilon^3)$$

because

$$\int_0^1 e^{-\varepsilon|v_1|^{-1}\tau}\, d\tau = 1 + \mathrm{O}(\varepsilon).$$

Hence we have:

$$\begin{aligned} I_2 &\le \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{S_2} \left(\varepsilon^2 + \mathrm{O}(\varepsilon^3)\right)\varphi(v)\|\bar{J}^+\|_{L^\infty}\, dv + \mathrm{O}(\varepsilon^2) \\ &\simeq \varepsilon^2\|\bar{J}^+\| \end{aligned}$$

We now rewrite the integral I_1 as:

$$I_1 = \int_{|v_1|<\varepsilon^{-1}} \varepsilon|v_1|^{-1}\int_0^1 \exp\{-\varepsilon|v_1|^{-1}\tau\}\, d\tau \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(v)\|\bar{J}^+\|_{L^\infty}\, dv_2\, dv_3\right) dv_1$$

and note that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(v)\|\bar{J}^{+}(f,f)\|_{L^\infty}\, dv_2\, dv_3 \le \sup_E \int_E \varphi(v)\|\bar{J}^{+}(f,f)\|_{L^\infty(0,1)}\, d\sigma(v)$$

where $d\sigma$ is the surface element of the plane E. Continuing with the original estimate:

$$\begin{aligned} I_1 &\le \int_{S_1} \varepsilon|v_1|^{-1}\int_0^1 \exp\{-\varepsilon|v_1|^{-1}\tau\}\, d\tau \left(\sup_E \int_E \varphi(v)\|\bar{J}^{+}(f,f)\|_{L^\infty}\, d\sigma(v)\right) \\ &\le \|\bar{J}^{+}\|_2 \int_{S_1} \varepsilon|v_1|^{-1}\int_0^1 \exp\{-\varepsilon|v_1|^{-1}\tau\}\, d\tau\, dv_1. \end{aligned}$$

By explicitly evaluating the inner integral in the above estimate, we have

$$\int_0^1 \exp\{-\varepsilon|v_1|^{-1}\tau\}\, d\tau = \frac{|v_1|}{\varepsilon}(1 - e^{-\varepsilon|v_1|^{-1}})$$

which gives

$$\begin{aligned} I_1 &\le \|\bar{J}^{+}\|_2 \left[\int_{|v_1|<\varepsilon} (1 - e^{-\varepsilon|v_1|^{-1}})\, dv_1 + \int_{\varepsilon}^{1/\varepsilon} (1 - e^{-\varepsilon|v_1|^{-1}})\, dv_1\right] \\ &\le \|\bar{J}^{+}\|_2 \left[\int_{|v_1|<\varepsilon} dv_1 + \int_{\varepsilon}^{1/\varepsilon} (1 - e^{-\varepsilon|v_1|^{-1}})\, dv_1\right] \\ &\le \|\bar{J}^{+}\|_2 \left[2\varepsilon + \int_{\varepsilon}^{1/\varepsilon} (1 - e^{-\varepsilon|v_1|^{-1}})\, dv_1\right]. \end{aligned} \tag{5.17}$$

Our last task, in the proof of b), is to estimate the integral in the inequality in (5.17). By making a change of variables, $z = \frac{\varepsilon}{|v_1|}$, and doing a Taylor expansion about the point $z = 0$, we have:

$$\begin{aligned} \int_{\varepsilon}^{1/\varepsilon} (1 - e^{-\varepsilon/|v_1|})\, dv_1 &= -\varepsilon \int_1^{\varepsilon^2} \frac{(1 - e^{-z})}{z^2}\, dz \\ &= \varepsilon \int_{\varepsilon^2}^1 (\frac{1}{z} - \frac{1}{2} + \frac{z}{3} - ...)\, dz \\ &= \varepsilon(\ln 1 - \ln \varepsilon^2) - \varepsilon\frac{1}{2}(1 - \varepsilon^2) + \mathrm{O}(\varepsilon). \end{aligned} \tag{5.18}$$

Estimate (5.17) now becomes

$$I_1 \le \|\bar{J}^{+}\|_2 \left[-2\varepsilon \ln \varepsilon + \mathrm{O}(\varepsilon)\right].$$

In view of (5.15) and (5.16) one gets

$$\begin{aligned}\|\varepsilon UJ^{+}(f,f)\| &\leq a_2(I_1+I_2)\\ &\leq a_2\|\bar{J^{+}}\|_2\left[-2\varepsilon\ln\varepsilon+\mathrm{O}(\varepsilon)\right]+\varepsilon^2 a_2\|\bar{J^{+}}\|\end{aligned}$$

Using the results of Lemma 5.2, and Lemma 5.3, we have for $f\in\mathcal{A}$

$$\begin{aligned}\|\varepsilon UJ^{+}(f,f)\| &\leq C_1\|f\|^2\left[-2\varepsilon\ln\varepsilon\right]+\varepsilon^2 C_2\|f\|\|f\|_{-1}+\mathrm{O}(\varepsilon)\|f\|^2\\ &\leq C_1 a_1^2(-2\varepsilon\ln\varepsilon)+\varepsilon^2 C_2 a_1 a_3+\mathrm{O}(\varepsilon)a_1^2\end{aligned}$$

where we see that as $\varepsilon\to 0, -2\varepsilon\ln\varepsilon\to 0$. The other terms are even smaller. Hence we can pick ε small enough, i.e. $\varepsilon<a_5$ such that:

$$\|\varepsilon UJ^{+}(f,f)\|\leq\frac{a_1}{2}$$

and,

$$\begin{aligned}\|Af\| &\leq \|Wf^{-}\|+\|\varepsilon UJ^{+}(f,f)\|\\ &\leq a_1\end{aligned}$$

as required for the proof of b).

To prove c) we define for any $b>0$ the set

$$I=\{\omega=(\omega_1,\omega_2,\omega_3)\in\mathbb{R}_v^3:b^{-1}<|\omega|<b,|\omega_1|>b^{-1}\}.$$

From the definition of $\nu(f^{-})$ and $Wf^{-}(x,v)$ defined in (5.9) for the one-dimensional case, we write

$$\begin{aligned}\nu(Wf^{-})(x,v) &= 2\pi\int_{\mathbb{R}_v^3}|v-w|f^{-}(\chi(w_1),w)\,dw\\ &\geq 2\pi\int_I|v-w|f^{-}(\chi(w_1),w)\Pi(x,\chi(w_1))\,dw\end{aligned}$$

where we recall that the constant ν_1 which appears in the definition of $\nu(f^{-})$ was evaluated to be 2π in Chapter 3. Now applying the definition of $\Pi(x,\chi(w_1))$, one obtains the estimate:

$$\begin{aligned}\nu(Wf^-)(x,v) &\geq 2\pi \int_I |v-w| f^-(\chi(w_1),w) \exp\left\{-\varepsilon w_1^{-1}\int_{\chi(w_1)}^{x} \nu(z,w)\,dz\right\} dw \\ &= 2\pi \int_I |v-w| f^-(\chi(w_1),w) \\ &\quad \times \exp\left\{-\varepsilon 2\pi w_1^{-1}\int_{\chi(w_1)}^{x} \left(\int_{\mathbb{R}^3} |w-w'| f(z,w')\,dw'\right) dz\right\} dw.\end{aligned}$$

Now estimating the integral in the exponential, we have for $f \in \mathcal{A}$,

$$\begin{aligned}\int_{\mathbb{R}^3} |w-w'| f^-(z,w')\,dw' &\leq |w| \int_{\mathbb{R}^3} f^-(z,w')\varphi(w')\,dw' + \int_{\mathbb{R}^3} f^-(z,w')\varphi(w')\,dw' \\ &= (1+|w|)\|f\| \\ &\leq (1+b)a_1.\end{aligned}$$

Therefore,

$$\begin{aligned}\nu(Wf^-)(x,v) &\geq 2\pi \int_I |v-w| f^-(\chi(w_1),w) e^{-\varepsilon 2\pi b(1+b)a_1}\,dw \\ &= 2\pi e^{-\varepsilon 2\pi b(1+b)a_1} \int_I |v-w| f^-(\chi(w_1),w)\,dw.\end{aligned}$$

Let $\delta > 0, 0 < \delta \leq \frac{1}{4}\nu_\circ$. As in Chapter 3, we can choose b such that for all v

$$\inf_v \int_I |v-w| f^-(\chi(w_1),w)\,dw \geq \nu_\circ - \delta \geq \frac{3}{4}\nu_\circ.$$

Then choose ε such that

$$2\pi e^{-\varepsilon 2\pi b(1+b)a_1}(\nu_\circ - \delta) \geq \frac{\nu_\circ}{2}.$$

Letting $a_2 = \frac{\nu_\circ}{2}$, it follows that $\nu(Wf^-) \geq a_2$, and hence

$$\nu(Af) \geq a_2$$

which is the desired result.

Now we have to bound $\|Af\|_{-1}$. From the triangle inequality, and the definition of a_1, one has

$$\|Af\|_{-1} \leq \frac{a_1}{2} + \|\varepsilon U J^+(f,f)\|_{-1}.$$

For $f \in \mathcal{A}$, $\nu(f) \geq a_2$, and hence

$$\Pi(x, y) \leq \exp\{-\varepsilon |v_1|^{-1} a_2 (x - y)\}.$$

By applying the definition of the operator $UJ^+(f, f)$ in (5.10), and the norm $\|.\|_{-1}$ in (3.8) we have

$$\begin{aligned}
\|UJ^+(f,f)\|_{-1} &\leq \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1}|v_1|^{-1} \int_0^1 \|J^+(f,f)\|_{L^\infty} \|\Pi(x,y)\|_{L^\infty}\, dy\, dv \\
&\leq \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1}|v_1|^{-1} \|J^+(f,f)\|_{L^\infty} \left\| \int_0^1 e^{-\varepsilon|v_1|^{-1}a_2\tau}\, d\tau \right\| dv
\end{aligned}$$

where we made the change of variables $\tau = x - y$. Explicitly evaluating the integral in the last estimate, we see that

$$\int_0^1 \exp\{-\varepsilon|v_1|^{-1}\tau a_2\} d\tau \leq \frac{|v_1|}{\varepsilon a_2}$$

and so

$$\begin{aligned}
\varepsilon \|UJ^+(f,f)\|_{-1} &\leq \varepsilon \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1}|v_1|^{-1} \|J^+(f,f)\|_{L^\infty} \left(\frac{|v_1|}{\varepsilon a_2}\right) dv \\
&= a_2^{-1} \int_{\mathbb{R}^3} \varphi(v)|v-\omega|^{-1} \|J^+(f,f)\|_{L^\infty}\, dv \\
&= \|J^+(f,f)\|_{-1} a_2^{-1}.
\end{aligned}$$

Again for $f \in \mathcal{A}$ we have by Lemma 3.1 that

$$\begin{aligned}
\|\varepsilon UJ^+(f,f)\|_{-1} &\leq C\|f\|^2 a_2^{-1} \\
&\leq C a_1^2 a_2{}^{-1}.
\end{aligned}$$

Thus, by letting $a_3 = a_1/2 + C a_1^2 a_2{}^{-1}$, we have the required estimate for part d) of the proof.

Finally, the last part of the proof is done in the same manner as before. From the definition of a_1

$$\|Af\|_2 \leq \frac{a_1}{2} + \varepsilon \|UJ^+(f,f)\|_2$$

where, by applying the definition of $UJ^+(f,f)$ and the norm $\|.\|_2$, we have the following estimate:

$$\begin{aligned}
\varepsilon\|UJ^+(f,f)\|_2 &\leq \varepsilon \sup_E \int_E \varphi(v)\|J^+(f,f)\|_{L^\infty}|v_1|^{-1}\left\|\int_0^1 e^{-\varepsilon|v_1|^{-1}a_2\tau}d\tau\right\| dv \\
&\leq \sup_E \int_E \|J^+(f,f)\|_{L^\infty} a_2^{-1}\, dv \\
&= a_2^{-1}\|J^+(f,f)\|_2.
\end{aligned}$$

Upon applying Lemma 5.2, for $f \in \mathcal{A}$

$$\begin{aligned}
\|\varepsilon J^+(f,f)\|_2 &\leq Ca_2^{-1}\|f\|^2 \\
&\leq Ca_1^2 a_2^{-1}.
\end{aligned}$$

Again by letting $a_1/2 + Ca_1^2 a_2^{-1} = a_3$

$$\|Af\|_2 \leq a_3. \quad \square$$

In the next lemma we prove the contraction property of the operator A which is a result similar to Lemma 3.3. However, we notice that the result of the following lemma gives contraction in a stronger sense.

Lemma 5.5 *There exists a constant C such that*

$$\|\psi(Af - Ag)\| \leq C\varepsilon \ln\frac{1}{\varepsilon}\|f-g\|$$

if $\psi = (1+|v|^2)^{1/2}$, $f, g \in \mathcal{A}$.

Proof. Since $f^-(x,v)$, and $g^-(x,v)$ are prescribed functions at the boundary, it follows that $Wf^- = Wg^-$; and applying the fact that $Af = Wf^- + \varepsilon UJ^+(f,f)$ we have

$$\begin{aligned}
\|\psi(Af - Ag)\| &= \varepsilon\left\|\psi\left(UJ^+(f,f) - UJ^+(g,g)\right)\right\| \\
&\leq \varepsilon \int_{\mathbb{R}_v^3} \psi(v)\varphi(v)|v_1|^{-1} \\
&\quad \times \left(\int_0^1 \|(J^+(f,f) - J^+(g,g))\|_{L^\infty}\|\Pi(x,y)\|_{L^\infty}\, dy\right) dv.
\end{aligned}$$

From Lemma 3.3, we know that

$$\|J^+(f,f) - J^+(g,g)\|_{L^\infty} \le \|J^+(f-g,f+g) + J^+(f+g,f-g)\|_{L^\infty}$$

Therefore, applying the definition of $\Pi(x,y)$, and using the symmetry property of $J^+(f,f)$, we have

$$\|\psi(Af - Ag)\| \le 2\varepsilon \int_{\mathbb{R}^3_v} \varphi(v)\psi(v)\|J^+(f-g,f+g)\|_{L^\infty}|v_1|^{-1}\left(\int_0^1 e^{-\varepsilon|v_1|^{-1}\tau}\,d\tau\right)\,dv \tag{5.19}$$

where $\tau = a_2(x-y)$. It is easily seen that

$$\int_0^1 e^{-\varepsilon|v_1|^{-1}\tau}\,d\tau = \frac{|v_1|}{\varepsilon}(1 - e^{-\varepsilon|v_1|^{-1}}).$$

Using this, and the fact that we can break up the integration in (5.19) in the same way as we did in the last lemma; we are able to explicitly evaluate the integration over the v_1 variables, and estimate the part which defines the plane E by the norm $\|.\|_2$. We see this more clearly in the following estimate

$$\begin{aligned}\|\psi(Af - Ag)\| \;\le\; & 2\varepsilon \int_{|v_1|<\varepsilon^{-1}} |v_1|^{-1}\frac{|v_1|}{\varepsilon}(1 - e^{-\varepsilon|v_1|^{-1}})dv_1 \\ & \times \int_{\mathbb{R}}\int_{\mathbb{R}} \psi(v)\varphi(v)\|J^+(f-g,f+g)\|_{L^\infty}\,dv_2\,dv_3 \\ & +2\varepsilon^2 \int_{|v_1|>\varepsilon^{-1}} dv_1 \int_{\mathbb{R}}\int_{\mathbb{R}} \psi(v)\varphi(v)\|J^+(f-g,f+g)\|_{L^\infty}\,dv_2\,dv_3\end{aligned}$$

where we have used the fact that in the domain where $|v_1| > \varepsilon^{-1}$, $|v_1|^{-1} < \varepsilon$, and

$$\frac{|v_1|}{\varepsilon}(1 - e^{-\varepsilon|v_1|^{-1}}) = 1 + \mathrm{O}(\varepsilon).$$

We can estimate the integral in this domain by the norm $\|.\|$ defined in (3.9). Hence we have

$$\begin{aligned}\|\psi(Af - Ag)\| \;\le\; & 2\int_{|v_1|<\varepsilon^{-1}} (1 - e^{-\varepsilon|v_1|^{-1}})\,dv_1 \\ & \times \left\{\sup_E \int_E \psi(v)\varphi(v)\|J^+(f-g,f+g)\|_{L^\infty}\,d\sigma(v)\right\} \\ & +2\varepsilon^2\|J^+(f-g,f+g)\|.\end{aligned}$$

By (5.18) the integral in the domain where $|v_1| < \varepsilon^{-1}$ can be approximated to be

$$\begin{aligned}\int_{|v_1|<\varepsilon^{-1}} (1 - e^{-\varepsilon|v_1|^{-1}})\, dv_1 &= \int_{|v|<\varepsilon} (1 - e^{-\varepsilon|v_1|^{-1}})\, dv_1 + \int_{\varepsilon}^{1/\varepsilon} (1 - e^{-\varepsilon|v_1|^{-1}})\, dv_1 \\ &\leq \int_{|v_1|<\varepsilon} dv_1 + 2\varepsilon \ln\frac{1}{\varepsilon} - \frac{1}{2}\varepsilon + \mathrm{O}(\varepsilon) \\ &\leq 2\varepsilon \ln\frac{1}{\varepsilon} + \mathrm{O}(\varepsilon).\end{aligned}$$

Going back to the original estimate, we have

$$\begin{aligned}\|\psi(Af - Ag)\| &\leq 2\left(\mathrm{O}(\varepsilon) + 2\varepsilon\ln\frac{1}{\varepsilon}\right)\left\{\sup_E \int_E \varphi(v)\|J^+\|_{L^\infty}\, d\sigma(v)\right\} + \nu_\circ\varepsilon^2\|J^+\| \\ &\leq C_1\left[\mathrm{O}(\varepsilon) + \varepsilon\ln\frac{1}{\varepsilon}\right]\|J^+\|_2 + \varepsilon^2\|J^+\|\end{aligned}$$

By taking ε to be small, one can ignore the second order epsilon term and focus only on the first term. Now applying Lemma 5.2, we have for $f, g \in \mathcal{A}$

$$\begin{aligned}\|\psi(Af - Ag)\| &\leq C_1\left[\varepsilon + \varepsilon\ln\frac{1}{\varepsilon}\right]\|f-g\|\|f+g\| \\ &\leq C_1\left[\varepsilon + \varepsilon\ln\frac{1}{\varepsilon}\right]\|f-g\|\left(\|f\| + \|g\|\right) \\ &\leq C\left[\varepsilon + \varepsilon\ln\frac{1}{\varepsilon}\right]\|f-g\| \\ &\leq C\varepsilon\ln\frac{1}{\varepsilon}\|f-g\|. \quad \square\end{aligned}$$

It is immediate that if we choose ε small enough so that $C\varepsilon\ln\frac{1}{\varepsilon} < 1$ we have a contraction; and as in the three-dimensional problem of Chapter 3, we have a unique solution in $\mathcal{A}$ for the problem (5.5)–(5.7).

5.3.2 Diffuse Reflective Case

In view of problem (5.5)–(5.7), in order to find unique solutions for the problem (5.1)–(5.3), we seek the functions $f^-(i, v)$ in the form:

$$f^-(i, v) = M(i, v)N_i$$

where N_i are constants we need to find. Let V be the solution operator for the equation (5.8); and as in Chapter 4 we denote by $W_\circ, U_\circ$ the operators (5.9),(5.10) with $\Pi = 1$. i.e we have the representation

$$(W_\circ f^-)(x,v) = f^-(\chi(v_1), v) \tag{5.20}$$

$$(U_\circ J)(x,v) = v_1^{-1} \int_{\chi(v_1)}^{x} J(f,f)(y,v)\,dy \tag{5.21}$$

where the solution to the problem (5.1)–(5.3) is given by

$$f = Vf^-, \quad Vf^- = W_\circ f^- + \varepsilon U_\circ J(Vf^-, Vf^-) \tag{5.22}$$

Remark 5.6 *We see that Wf^- prescribes the function at the boundary. As a particle of gas begins to emerge, say at the boundary $x = 1$, its total number of collisions from $x = 1$ to $x = 0$ would be determined by $\varepsilon UJ(Vf^-, Vf^-)$.*

In view of the above remark, if we multiply equation (5.22) by $|v_1|$ and integrate we obtain

$$\int_{v_1<0} |v_1| f(0,v)\,dv = \int_{v_1>0} |v_1| f^-(1,v)\,dv + \varepsilon \int_{v_1>0} \int_1^0 J(Vf^-, Vf^-)(y,v)\,dy\,dv,$$

and applying the definition of $N_\circ, N_1$, we have the following relations:

$$N_\circ = N_1(f) - \varepsilon G(N), \quad N_1(f) = N_\circ + \varepsilon G(N) \tag{5.23}$$

where

$$G(N) = \int_{v_1>0} \int_0^1 J(Vf^-, Vf^-)\,dy\,dv \tag{5.24}$$

and N is the vector $(N_\circ, N_1)$. By substituting (5.23) into (5.4), the following conditions are satisfied:

$$N_\circ(f) = \frac{1}{2}(C_\circ - \varepsilon G(N)), \tag{5.25}$$

$$N_1(f) = \frac{1}{2}(C_\circ + \varepsilon G(N)). \tag{5.26}$$

In the next lemma we will prove an estimate on $|G|$, which is the one-dimensional form of the bound in Lemma 4.2. This will provide the neccessary estimate to prove contraction of the operator $\varepsilon G(N)$.

Lemma 5.7 *For some positive constant C_1 independent of N, the following estimate is true*

$$|G| \leq C_1 \left(\max\{N_\circ, N_1\}\right)^2.$$

Proof. For $f \in \mathcal{A}$ we know that

$$\|f\| \leq a_1 = 2\left[\|Wf^-\| + \|Wf^-\|_{-1} + \|Wf^-\|_2\right].$$

We estimate each of the norms above, in order to find an estimate on a_1, in terms of $N_\circ, N_1$. Hence

$$\begin{aligned}
\|Wf^-\| &= \int_{\mathbb{R}^3_v} \varphi(v) \left\|f^-(i,v)\Pi(i)\right\|_{L^\infty} dv \\
&= \int_{\mathbb{R}^3_v} \varphi(v) \left\|M(i,v)N_i\Pi(i)\right\|_{L^\infty} dv \\
&\leq \frac{1}{2\pi}\|h_i^2\|_{L^\infty} \int_{\mathbb{R}^3_v} \varphi(v) e^{-|v|^2 h_i/2} N_i \, dv \\
&= \frac{N_i}{2\pi} \int_{\mathbb{R}^3_v} (1+|v|^2)^r e^{(s-h_i/2)|v|^2} \, dv.
\end{aligned}$$

For $s < \min_i h_i/2$, the integral in the above expression will be bounded by a constant and so

$$\|Wf^-\| \leq C_1 N_i \leq C_1 \left(\max\{N_\circ, N_1\}\right).$$

Estimating $\|Wf^-\|_{-1}$, we have from the definition of the norm in (3.8)

$$\begin{aligned}
\|Wf^-\|_{-1} &= \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1} \left\|M(i,v)N_i\right\|_{L^\infty} dv \\
&= \frac{N_i}{2\pi}\|h_i^2\|_{L^\infty} \sup_\omega \int_{\mathbb{R}^3_v} (1+|v|^2)^r e^{(s-h_i)|v|^2/2} |v-\omega|^{-1} \, dv \\
&= CN_i \|e^{-|v|^2 h_i/2}\|_{-1}
\end{aligned}$$

for $s < \min_i h_i/2$, and $f(.,v) = e^{-|v|^2 h_i/2}$ (a Maxwellian). Hence

$$\|Wf^-\|_{-1} = C_2 N_i \left(\max\{N_\circ, N_1\}\right).$$

Similarily, the same estimate for $\|Wf^-\|_2$ follows, namely that

$$\|Wf^-\|_2 \leq C_3 \left(\max\{N_\circ, N_1\}\right).$$

Therefore it is clear that

$$\|f\| \leq a_1 \leq C \max\{N_\circ, N_1\} \tag{5.27}$$

for a positive constant C independent of N, and $f \in \mathcal{A}$. To estimate G, we have

$$\begin{aligned} |G| &= \int_{v_1>0} \int_0^1 |J(Vf^-, Vf^-)| \, dy \, dv \\ &\leq \int_{v_1>0} \int_0^1 (1+|v|^2) e^{s|v|^2} e^{-s|v|^2} |J^+(Vf^-, Vf^-)| \, dy \, dv \\ &\quad + \int_{v_1>0} \int_0^1 (1+|v|^2) e^{s|v|^2} e^{-s|v|^2} |J^-(Vf^-, Vf^-)| \, dy \, dv. \end{aligned}$$

Estimating the part in the above inequality involving the gain term, we have by applying Lemma 2.2; that for $s < \min_i h_i/2$

$$\begin{aligned} \int_{v_1>0} \int_0^1 (1+|v|^2) e^{s|v|^2} e^{-s|v|^2} |J^+(Vf^-, Vf^-)| &\leq \int_{\mathbb{R}^3_v} \|J^+(Vf^-, Vf^-)\|_{s,1} e^{-s|v|^2} \, dv \\ &\leq C \int_{\mathbb{R}^3_v} \|f\|^2_{s,1} \, dv \\ &\leq C \int_{\mathbb{R}^3_v} \|M(i,v)N\|_{s,r} \, \|f\|_{s,r} \, dv \\ &\leq C|N| \, \|f\|. \end{aligned}$$

The estimate for the loss term is even simpler and is done in the same way. Now applying inequality (5.27) we have the required estimate

$$|G| \leq C_1 \left(\max\{N_\circ, N_1\}\right)^2. \ \square$$

Lemma 5.8 *The operator $\varepsilon G(N)$ is a contraction .*

This is a direct consequence of Lemma 5.7. We assume two different solutions of the form:

$$\begin{aligned} f^-(i,v) &= M(i,v)I_i \\ f^-(i,v) &= M(i,v)J_i \end{aligned}$$

where I_i and J_i are components of I,J respectively, and satisfy (5.4). In the previous lemma, we showed:

$$\begin{aligned} |G(I)| &\leq C_1(\max\{I_\circ, I_1\})^2 \\ |G(J)| &\leq C_2(\max\{J_\circ, J_1\})^2 \end{aligned}$$

for positive constants C_1, and C_2. Thus we have:

$$\begin{aligned} G(I-J) &\leq C\left(\max\{I_\circ - J_\circ, I_1 - J_1\}\right)^2 \\ &\leq C\left(\max(I_\circ - J_\circ, I_1 - J_1)\right)\left(\max(I_\circ + J_\circ, I_1 + J_1)\right) \\ &\leq C|I-J|(I_\circ + I_1 + J_\circ + J_1). \end{aligned}$$

By (5.4) we know that for any given constants $C_\circ$, C_1 there exists a unique solution satisfying the conditions:

$$\begin{aligned} I_\circ + I_1 &= C_\circ \\ J_\circ + J_1 &= C_1. \end{aligned}$$

Therefore,

$$\varepsilon G(I-J) \leq C\varepsilon|I-J|. \quad \square$$

So we have that the system in (5.24) has a unique solution for small ε. Now let $\mathcal{W}^-$ be defined by:

$$\mathcal{W}^- = \{f \in L^\infty : ((1+|v|^2)^{-1}Df \in L^\infty\}$$

with Df given by:

$$Df = v_1\frac{\partial f}{\partial x}.$$

We are now ready to state the main result of this chapter:

Theorem 5.9 *There exists a positive constant* $\varepsilon_\circ$ *such that the problem (5.1)–(5.4) has a unique solution in* $\mathcal{W}^-$*, if* $\varepsilon < \varepsilon_\circ$, $s \in (0, \min_i \frac{1}{2}h_i)$.

Chapter 6

Concluding Remarks

In this thesis we have examined different kinds of boundary value problems for the steady Boltzmann equation. Our main goal was to prove the existence and uniqueness of solutions to these problems without truncations of the collision kernel.

The main difficulties in producing local solutions is that the operators generated by the collision integrals are unbounded. The unboundedness is caused not only by the singularity at $v = 0$, but also by the growth of the operators $J^{\pm}(f, f)$ for large velocities. We addressed these problems by introducing certain weighted norms which allowed us to produce bounds on the collision term in terms of the density function f, which became useful in producing the appropriate function spaces with the contractive property.

The introduction of such norms relied partly on the spatial dimensions of our problem. For example, we recall that for bounded domains in the three-dimensional case, we introduced the norms $||.||$, and $||.||_{-1}$ which enabled us to produce a function set where it is possible to apply the contraction mapping principle. In addition, the the norm $||.||_{s,r}$ used in Lemma 2.2 was useful in proving the contractive nature of the operator G in Chapter 4. However, the nature of the problems in Chapter 5, had to do with proving uniqueness results for a one-dimensional slab where our domain was bounded from 0 to 1 in the x direction but unbounded in the remaining two directions. This made the treatment of the problem more difficult, because the norms defined previously were not enough to ensure a unique solution. This led the way to the introduction of the norm $\|.\|_2$ which was used later in producing the appropriate function set with the contractive property to ensure uniqueness for the one-dimensional problem.

We notice that these problems were solved with the inclusion of a parameter ε in front of the collision term. This factor is due to a rescaling of the spatial variables. If we introduce new variables:

$$r = \frac{x}{\varepsilon}$$

where ε^{-1} is the mean free path of the particle, then in terms of this new variable, the steady Boltzmann equation:

$$v \cdot \frac{\partial f}{\partial x} = J^{+}(f, f)$$

becomes

$$v \cdot \frac{\partial \tilde{f}}{\partial r} = \varepsilon J^{+}(f, f)$$

where $\tilde{f}(r, v) = f(x, v)$. We make the important note that if the right hand side of this equation was equal to zero, then we would have Liouville's equation and we are faced with free molecular flow. However, with a small parameter ε infront of the collision operator, we have not neglected collisions completely. This almost free molecule flow, suggests that the above problems are restricted to particles having large mean free paths. By the rescaling process this is equivalent to saying the boundary value problems discussed are restricted to particles in small bounded domains. The reason for this is that for large mean free paths (or for small ε) we can always choose our spatial coordinates small enough, say $O(\varepsilon)$ so that we could obtain unique solutions.

Bibliography

[1] Arkeryd, L., Cercignani, C. and Illner, R.(1991). *Measure Solutions of the Steady Boltzmann Equation in a slab*, Communications Mathematical Physics 142, 285-296

[2] Arkeryd, L. and Nouri A. (1994). *Asymptotics of the Boltzmann Equation with Diffuse Reflection Boundary Conditions.*

[3] Beckenbach E. F. and Bellman R. (1961) *Inequalities.* Berlin: Springer–Verlag.

[4] Carleman, T. (1957). *Problèmes Mathématiques dans la Théorie Cinétique des Gaz.* Almqvist & Wiksells Boktryckeri AB: Uppsala

[5] Cercignani, C., Illner, R. & Pulvirenti, M. (1994). *Applied Mathematical Sciences. Vol. 106: The Mathematical Theory of Dilute Gases.* New York: Springer–Verlag.

[6] Cercignani, C. (1987). *Applied Mathematical Sciences. Vol. 67: The Boltzmann Equation and Its Applications.* New York: Springer–Verlag.

[7] Folland, G. (1984) *Real Analysis Modern Techniques and their Applications.* New York: John Wiley and Sons

[8] Grad H. (1963) *Asymptotic theory of the Boltzmann Equation.* Rarefied Gas Dynamics I(J.A.Laurman, ed.), Acad. Press, New York.

[9] Illner R. and Struckmeier (1995) *Boundary Value Problems for the Steady Boltzmann Equation.* Journal of Statistical Physics 85: 427-454.

[10] Maslova, N. B. (1993) *Series on Advances in Mathematics for Applied Sciences. Vol. 10: Nonlinear Evolution Equations.* Singapore: World Scientific

[11] Smart D. R. (1974) *Fixed Point Theorems.* New York: Cambridge University Press

[12] Taylor A. E. and Mann R. W (1972) *Advanced Calculus Second Edition.* Lexington: Xerox College Publishing

[13] Wennberg, B (1994). *Regularity in the Boltzmann Equation and the Radon Transform.* Communications in Partial Differential Equations 19: 2057-2074

Appendix A

A Generalization of Young's Inequality

In this section we pove the inequality in (2.13). We note that this is a famous inequality, and it can be referred to in [3], however for the convenience of the reader we present the proof here. If $y = \psi(x)$ is a strictly increasing function of x for $x \geq 0$ with $\varphi(0) = 0$ then by examining the areas in Figure A.1 we have the generalized one-dimensional form of Young's inequality, namely:

$$ab \leq \int_0^a \psi(x)\,dx + \int_0^b \psi^{-1}(y)\,dy \tag{A.1}$$

where $\psi^{-1}(y)$ is the inverse function to $\psi(x)$. So let $y = \psi(x) = \ln(x+1)$, which gives $\psi^{-1}(y) = e^y - 1$. Therefore:

$$ab \leq \int_0^a \ln(x+1)\,dx + \int_0^b e^y - 1\,dy$$

Evaluating the integrals we have:

$$\int_0^a \psi(x)\,dx = \int_0^a \ln(x+1)\,dx = (a+1)\ln(a+1) - a$$

and

$$\int_0^b \psi^{-1}(y)\,dy = \int_0^b (e^y - 1)\,dy = e^b - b - 1$$

So we have:

$$ab \leq (a+1)\ln(a+1) - a + e^b - b - 1$$

By taking b to the right hand side gives

$$(a+1)b \leq (a+1)\ln(a+1) + e^b - a - 1$$

Now replacing a by $a-1$ and b by $b-1$ the inequality becomes:

$$ab \leq a \ln a + e^{b-1}$$

for $a, b > 0$, which is the inequality in (2.13).

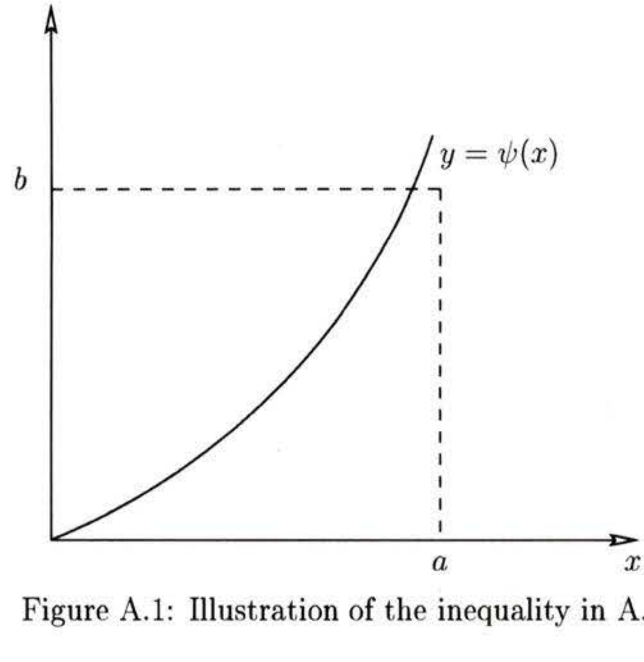

Figure A.1: Illustration of the inequality in A.1

Appendix B

The Carleman Transformation

The purpose of this appendix is the verification of the calculations leading to the result in equation (2.16). We recall that this equation is a representation of the gain term, after a coordinate transformation. The usual form of the gain term is written as

$$J^{+}(f,g)(v) = \int_{\mathbb{R}^3\times\mathbb{S}^2} f(v')g(v'_*)B(v-v_*,n)\,dn\,dv_* \tag{B.1}$$

as defined in Chapter 2.

We make the change of variables:

$$v_*, n \longrightarrow p = v - n(n\cdot(v-v_*)),\ q = v_* + n(n\cdot(v-v_*)).$$

Breaking p, q, v, v_* into their components, we introduce the following rectangular coordinates:

$$p = (\xi', \eta', \zeta'),\ q = (\xi'_*, \eta'_*, \zeta'_*),\ n = (s,t,u)$$

of p, q, n respectively, to obtain:

$$\xi' = \xi - sW,\ \ \xi'_* = \xi_* + sW$$

$$\eta' = \eta - tW,\ \ \eta'_* = \eta_* + tW$$

$$\zeta' = \zeta - uW,\ \ \zeta'_* = \zeta_* + uW$$

where W is used to represent the collision kernel for hard spheres, i.e.,

$$W = (n\cdot(v-v_*)) = s(\xi-\xi_*) + t(\eta-\eta_*) + u(\zeta-\zeta_*).$$

We introduce the coordinates θ, ϕ on the sphere $s^2 + t^2 + u^2 = 1$ by

$$\begin{aligned} s &= \sin\theta\cos\phi \\ t &= \sin\theta\sin\phi \\ u &= \cos\theta. \end{aligned}$$

For simplicity, we write:

$$\begin{aligned} \frac{\partial s}{\partial\theta} = s', \ \frac{\partial t}{\partial\theta} = t', \ \frac{\partial u}{\partial\theta} = u' \\ \frac{\partial s}{\partial\phi} = s'', \ \frac{\partial t}{\partial\phi} = t'', \ \frac{\partial u}{\partial\phi} = u''. \end{aligned}$$

We focus on the transformation

$$\xi_*, \eta_*, \zeta_*, \theta, \phi \longrightarrow \xi', \eta', \zeta', \xi'_*, \eta'_*$$

hence we need to calculate the Jacobian determinant

$$\frac{\partial(\xi', \eta', \zeta', \xi'_*, \eta'_*)}{\partial(\xi_*, \eta_*, \zeta_*, \theta, \phi)} = \frac{\partial(\xi' + \xi'_*, \eta' + \eta'_*, \zeta', \xi'_*, \eta'_*)}{\partial(\xi_*, \eta_*, \zeta_*, \theta, \phi)}$$

where I have used the fact that by adding to any row of a matrix a multiple of another row, the determinant of the matrix will remain unchanged. In addition we note that:

$$\xi' + \xi'_* = \xi + \xi_*, \ \text{and} \ \eta' + \eta'_* = \eta + \eta_*$$

which brings us to the problem of calculating

$$\frac{\partial(\xi + \xi_*, \eta + \eta_*, \zeta', \xi'_*, \eta'_*)}{\partial(\xi_*, \eta_*, \zeta_*, \theta, \phi)}.$$

However we see that since the only non-zero terms in the first two rows, are

$$\frac{\partial(\xi + \xi_*)}{\partial\xi_*} \qquad \text{and} \qquad \frac{\partial(\eta + \eta_*)}{\partial\eta_*}$$

the above 5×5 determinant is equal to a 3×3 determinant as shown below.

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{\partial \zeta'}{\partial \xi'_*} & \frac{\partial \zeta'}{\partial \eta_*} & \frac{\partial \zeta'}{\partial \xi_*} & \frac{\partial \zeta'}{\partial \theta} & \frac{\partial \zeta'}{\partial \phi} \\ \frac{\partial \xi'_*}{\partial \xi_*} & \frac{\partial \xi'_*}{\partial \eta_*} & \frac{\partial \xi'_*}{\partial \zeta_*} & \frac{\partial \xi'_*}{\partial \theta} & \frac{\partial \xi'_*}{\partial \phi} \\ \frac{\partial \eta'_*}{\partial \xi_*} & \frac{\partial \eta'_*}{\partial \eta_*} & \frac{\partial \eta'_*}{\partial \zeta_*} & \frac{\partial \eta'_*}{\partial \theta} & \frac{\partial \eta'_*}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \frac{\partial \zeta'}{\partial \zeta_*} & \frac{\partial \zeta'}{\partial \theta} & \frac{\partial \zeta'}{\partial \phi} \\ \frac{\partial \xi'_*}{\partial \zeta_*} & \frac{\partial \xi'_*}{\partial \theta} & \frac{\partial \xi'_*}{\partial \phi} \\ \frac{\partial \eta'_*}{\partial \zeta_*} & \frac{\partial \eta'_*}{\partial \theta} & \frac{\partial \eta'_*}{\partial \phi} \end{vmatrix}$$

Calculating all the partial derivatives we obtain:

$$\begin{aligned} \frac{\partial(\zeta', \xi'_*, \eta'_*)}{\partial(\zeta_*, \theta, \phi)} &= \begin{vmatrix} u^2 & -u'W - u\frac{\partial W}{\partial \theta} & -u''W - u\frac{\partial W}{\partial \phi} \\ -su & s'W + s\frac{\partial W}{\partial \theta} & s''W + s\frac{\partial W}{\partial \phi} \\ -tu & t'W + t\frac{\partial W}{\partial \theta} & t''W + t\frac{\partial W}{\partial \phi} \end{vmatrix} \\ &= uW^2 \begin{vmatrix} s & s' & s'' \\ t & t' & t'' \\ u & u' & u'' \end{vmatrix} = uW^2 \sin\theta. \end{aligned}$$

Therefore,

$$\left| \frac{\partial(\xi_*, \eta_*, \zeta_*, \theta, \phi)}{\partial(\xi', \eta', \zeta', \xi'_*, \eta'_*)} \right| = \frac{1}{|u|W^2 \sin\theta}.$$

We note that the domain in which $\xi_*, \eta_*, \zeta_*, \theta, \phi$ vary, corresponds to the space $(\xi', \eta', \zeta', \xi'_*, \eta'_*)$ twice. Therefore, as for hard spheres $B = W$ and $dn = \sin\theta \, d\theta \, d\phi$, we have by substitution into (B.1):

$$J^+(f, f) = 2 \iiiiint_{-\infty}^{\infty} f(\xi', \eta', \zeta') f(\xi'_*, \eta'_*, \zeta'_*) \frac{W}{|u|W^2} \, d\xi' \, d\eta' \, d\zeta' \, d\xi'_* \, d\eta'_*. \qquad \text{(B.2)}$$

When ξ', η', ζ' are fixed, the point $\xi'_*, \eta'_*, \zeta'_*$ describes a plane given by the equation:

$$(\xi' - \xi)(\xi'_* - \xi) + (\eta' - \eta)(\eta'_* - \eta) + (\zeta' - \zeta)(\zeta'_* - \zeta) = 0$$

which passes through the point $v = (\xi, \eta, \zeta)$ and is perpendicular to $v - p$ (see Figure 2.1). We denote this plane by E_{vp}. Let q be a point ranging over E_{vp}, then one has:

$$d\xi'_* d\eta'_* = |u| d\sigma(q)$$

where $d\sigma(q)$ is the surface element of E_{vp}. Replacing this in (B.2), and remembering that

$$|W| = \sqrt{(\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2} = |v - p|$$

we get

$$J^+(f, f) = 2 \int_{\mathbb{R}^3} f(p)|v - p|^{-2} \left\{ \int_{E_{vp}} f(q) B \, d\sigma(q) \right\} dp,$$

as required.

Appendix C

The r-norm

The purpose of this section, is to demonstrate the need for the space $L^\infty_{s,r}$, as was introduced in Chapter 2. We recall that the choice of this particular norm was important for the neccessary bounds used in the existence and uniqueness results. We will begin this discussion by looking at another norm first introduced by Carleman[4], and try to produce similar estimates as in Lemma 2.2. It will be shown that the collision operators are actually unbounded with respect to this norm. However, by examining the results more carefully, it is possible to spot the source of this unboundedness. We note that many of the calculations in this section are analogous to the lengthy calculations of Lemma 2.2, and will not be done in detail. I will only produce detailed estimates on the parts which are important for the intention of this section.

Following Carleman, we consider the properties of the collision operators in the space of functions with only the inverse power decreasing for large $|v|$. So we set

$$L^\infty_r = \{f|\varphi_r f \in L^\infty\}\,, \varphi = \varphi_r = (1+|v|^2)^{r/2}$$

and denote by $\|f\|_r$, the norm of $f \in L^\infty_r$. Then we have the following lemma:

Lemma C.1 *If $f, g \in L^\infty_r, r > 5$, then $J^\pm(f,g) \in L^\infty_{r-\gamma}$ and*

$$\|J^\pm(f,g)\|_{r-\gamma} \leq C\|f\|_r\,\|g\|_r.$$

Proof. By definition we have

$$\|J^{\pm}(f,g)\|_{r-\gamma} = \|(1+|v|^2)^{(r-\gamma)/2} J^{\pm}(f,g)\|_{L^\infty}$$

where we recall from Lemma 2.1 that

$$J^{-}(f,g) = f\nu(g) \leq f b_2(a|v|^\gamma + b)$$

and $a = \langle f \rangle$, $b = \langle (1+|v|^2)g \rangle$. Hence

$$\begin{aligned} \|J^{-}(f,g)\|_{r-\gamma} &\leq \|(1+|v|^2)^{r/2} f (1+|v|^2)^{-\gamma/2} b_2(a|v|^\gamma + b)\|_{L^\infty} \\ &\leq b_2 \|f\|_r \, \|(1+|v|^2)^{-\gamma/2}(a|v|^\gamma + b)\|_{L^\infty}. \end{aligned}$$

Now we need to show

$$b_2\|(1+|v|^2)^{-\gamma/2}(a|v|^\gamma + b)\|_{L^\infty} \leq C\|g\|_r.$$

which follows directly from the definitions of a, b and the r-norm, since

$$a|v|^\gamma + b = |v|^\gamma \int g(v_*)\, dv_* + \int (1+|v_*|^2) g(v_*)\, dv_*.$$

By multiplying inside the integral with $(1+|v|^2)^{r/2}(1+|v|^2)^{-r/2}$ we arrive at the result

$$a|v|^\gamma + b \leq \|g\|_r \left[|v|^\gamma \int_{\mathbb{R}^3_v} (1+|v_*|^2)^{-r/2}\, dv_* + \int_{\mathbb{R}^3_v} (1+|v_*|^2)^{1-r/2}\, dv_* \right].$$

Mutiplying through by $b_2(1+|v|^2)^{-\gamma/2}$, we notice that

$$\frac{|v|^\gamma}{(1+|v|^2)^{\gamma/2}} \leq 1$$

and so we have

$$b_2\|(1+|v|^2)^{-\gamma/2}(a|v|^\gamma + b)\|_{L^\infty} \leq b_2\|g\|_r \left\| \int_{\mathbb{R}^3_v} (1+|v_*|^2)^{-r/2}\, dv_* + \int_{\mathbb{R}^3_v} (1+|v_*|^2)^{1-r/2}\, dv_* \right\|.$$

It is easy to check that the above integrals are bounded by a constant depending on r, for $r > 5$. Hence

$$b_2 \| (1 + |v|^2)^{-\gamma/2} (a|v|^\gamma + b) \|_{L^\infty} \le C \|g\|_r.$$

Therefore we have shown that

$$\|J^-(f,g)\|_r \le C \|f\|_r \|g\|_r$$

The estimate for the gain term is done the same way as in Chapter 2. Using the Carleman representation of the gain term (2.16), we arrive at the result in (2.20). We further consider the post-collisional velocities in the different regions D_1, D_2, and D_3 (see section 2). From a look at section 2, it becomes immediate that

$$J^+(f_1, f_1) = 0.$$

In addition, we also have the following estimates

$$\begin{aligned} G(f_2 + f_3) &\le C_1 (\varphi_{r-2}(v))^{-1} \|f\|_r \\ J^+(f, f_2 + f_3) &\le C_2 (\varphi_{r-\gamma}(v))^{-1} \|f\|_r^2 \end{aligned}$$

for some positive constants C_1, C_2. □

The bound stated in the lemma cannot be improved. Therefore, the operators $J^\pm$ are unbounded in L_r^∞. However we show in the next lemma that it is possible to control the unboundedness,

Lemma C.2 *There exists a positive constant C such that*

$$\varphi_r J^+(f,f) \le (r-2)^{-1} 8\pi (1 + |v|)^\gamma \|f\|_r^2 + C \|f\|_r^2$$

for $f, g \in L_r^\infty$ with $r > 5 + \gamma$.

Proof. We split the density function f into

$$f = f_1 + f_2 + f_3$$

where f_1, f_2, and f_3 are defined by

$$f_i(v) = f(v)\chi_i, \quad i = 1, 2, 3,$$

where χ_i is the indicator function of D_i. Then we use the bilinearity of the collision operator, together with the symmetry property (2.4) to write:

$$J^+(f,f){=}J^+(f_1,f_1){+}2J^+(f_1,f_2){+}2J^+(f_1,f_3){+}2J^+(f_2,f_3){+}J^+(f_2,f_2){+}J^+(f_3,f_3)$$

The various contributions $J^+(f_i, f_j)$ for $i = 1, 2, 3, j = 1, 2, 3$ will now be estimated separately. We write the above as:

$$\begin{aligned}
J^+(f,f) &= J^+(f, f_1{+}f_2) + J^+(f, f_3) \\
&= J^+(f_3, f_1 + f_2){+}J^+(f_1 + f_2, f_1{+}f_2) + J^+(f, f_3) \\
&= J^+(f_3, f_1 + f_2){+}J^+(f_3, f_3) + J^+(f_1{+}f_2, f_1 + f_2{+}J^+(f, f_3) - J^+(f_3, f_3) \\
&\le 2J^+(f, f_3){+}J^+(f_1 + f_2, f_1 + f_2) \\
&= 2J^+(f, f_3){+}J^+(f_1{+}f_2, f_2){+}J^+(f_1 + f_2, f_1)
\end{aligned}$$

where upon using the fact that $J^+(f_1, f_1) = 0$, we have

$$\begin{aligned}
J^+(f_1 + f_2, f_1) &= J^+(f_1, f_1){+}J^+(f_2, f_1) \\
&= J^+(f_2, f_1 + f_2){-}J^+(f_2, f_2)
\end{aligned}$$

which gives the estimate

$$\begin{aligned}
J^+(f,f) &\le 2J^+(f, f_3){+}J^+(f_1 + f_2, f_2){+}J^+(f_2, f_1 + f_2){-}J^+(f_2, f_2) \\
&\le 2J^+(f_1 + f_2, f_2){+}2J^+(f, f_3). \qquad \text{(C.1)}
\end{aligned}$$

Now we want to estimate each term on the right hand side of (C.1), separately. We proceed with the first term. Hence,

$$\begin{aligned}
G(f_2) &= \int_{\frac{|v|}{\sqrt{2}}<|q|<|v|} f(q)\,dq \\
&= \int_{\frac{|v|}{\sqrt{2}}<|q|<|v|} f(q)(1+|q|^2)^{r/2}(1+|q|^2)^{-r/2}\,dq \\
&\le \|f\|_r \int_{\frac{|v|}{\sqrt{2}}<|q|<|v|} (1+|q|^2)^{-r/2}\,dq.
\end{aligned}$$

Changing to polar coordinates and integrating, the right-hand side is further estimated to be

$$\begin{aligned} &\leq 2\pi\|f\|_r \int_{\frac{|v|}{\sqrt{2}}}^{|v|} (1+R^2)^{-r/2} R\, dR \\ &\leq C(1+|v|^2)^{-r/2}|p|^2\|f\|_r. \end{aligned}$$

From (2.18) and the above estimate on $G(f_2)$ we have

$$\begin{aligned} |J^+(f_1+f_2,f_2)| &\leq 2b_1 \int_{|p|<|v|} |v-p|^{-2+\gamma} G(f_2)\, dp \\ &\leq C(1+|v|^2)^{-r/2} \int_{|p|<|v|} |v-p|^{-2+\gamma}|p|^2 f(p)\, dp. \end{aligned}$$

Using the fact that for $|p| \leq |v|$

$$\frac{|p|^2}{1+|p|^2} \leq \frac{|v|^2}{1+|v|^2} \tag{C.2}$$

we write:

$$\begin{aligned} |J^+(f_1+f_2,f_2)| &\leq C\|f\|_r|v|^2(1+|v|^2)^{-1-r/2} \int_{|p|<|v|} |v-p|^{-2+\gamma} f(p)\, dp \\ &\leq C\|f\|_r^2(1+|v|^2)^{-1-r/2}|v|^2|v|^{-2+\gamma} \int_{|p|<|v|} \frac{dp}{(1+|p|^2)^{r/2}}. \end{aligned}$$

By changing into polar coordinates one can easily verify that the above integral is bounded by a positive constant depending on r only. Hence we have

$$\begin{aligned} \|J^+(f_1+f_2,f_2)\|_r &\leq C\|f\|_r^2 \frac{|v|^\gamma}{1+|v|^2} \\ &\leq C\|f\|_r^2. \end{aligned}$$

So the above result gives boundedness for the first term on the right-hand side of (C.1). Thus the real source of the problem must come from the second term in (C.1). Estimating the term $G(f_3)$, we write:

$$\begin{aligned} G(f_3) &= \int_{|p|>|v|} f(q)\, dq \\ &\leq 2\pi\|f\|_r \int_{|v|}^{\infty} \frac{R}{(1+R^2)^{r/2}}\, dR \\ &= 2\pi\|f\|_r(1+|v|^2)^{-r/2}(1+|v|^2)^{-2} \int_{|v|}^{\infty} \frac{R}{(1+R^2)^2}. \end{aligned}$$

The integral in the above is evaluated to be

$$\frac{1}{r-2}(1+|v|^2)^{-1}$$

which gives

$$G(f_3) \leq 2\pi(r-2)^{-1}(1+|v|^2)^{(-r+2)/2}\|f\|_r.$$

Plugging this into the expression (2.18), we obtain

$$J^+(f, f_3) \leq 2\pi(r-2)^{-1}\varphi_{-r+2}V_{2-\gamma}(f)\|f\|_r \tag{C.3}$$

where we recall $\varphi(v) = (1+|v|^2)^{r/2}$ and $V_k(f)$ being the operator defined in (2.18). To complete the proof it is neccessary to estimate $V_{2-\gamma}$. So we write:

$$\begin{aligned} V_{2-\gamma} &= (1+|v|^2)^{-1}V_{-\gamma}(f) + (1+|v|^2)^{-1}\int_{\mathbb{R}^3}|v-p|^{-2+\gamma}\left[|v|^2+1-|v-p|^2\right]f(p)\,dp \\ &= \mathcal{U} + \mathcal{W} \end{aligned}$$

where we represent the first term by $\mathcal{U}$, and the second term by $\mathcal{W}$. We estimate $V_{2-\gamma}$ by separating the integrations into two parts: the first part will be integrated over the domain $S_1 = \{|p| \leq |v|\}$, and the second part integrated over $S_2 = \{|p| > |v|\}$. To avoid lengthy formulas, I will divide the terms $\mathcal{U}$, and $\mathcal{W}$ over the different sets S_1, and S_2 and write $\mathcal{U} = \mathcal{U}_{S_1} + \mathcal{U}_{S_2}$ and $\mathcal{W} = \mathcal{W}_{S_1} + \mathcal{W}_{S_2}$. So estimating $\mathcal{U}_{S_1}$ we have:

$$\begin{aligned} \mathcal{U}_{S_1} &\leq (1+|v|^2)^{-1+\gamma/2}\int_{|p|\leq|v|} f(p)\,dp \\ &\leq (1+|v|^2)^{-1+\gamma/2}\|f\|_r\int_{|p|\leq|v|}(1+|p|^2)^{-r/2}\,dp \\ &\leq C(1+|v|^2)^{-1+\gamma/2}\|f\|_r \end{aligned}$$

where it is easily verified by a change in polar coordinates that the above integral is bounded by a constant for $r > 5$. In addition we have that

$$\begin{aligned} \mathcal{U}_{S_2} &\leq (1+|v|^2)^{-1}\|f\|_r\int_{|p|>|v|}\frac{|p|^\gamma}{(1+|p|^2)^{r/2}}\,dp \\ &\leq C(1+|v|^2)^{-1}\|f\|_r. \end{aligned}$$

Turning to $\mathcal{W}$, we have:

$$\begin{aligned}\mathcal{W}_{S_1} &\leq (1+|v|^2)^{-1}|v|^{-2+\gamma}\int_{|p|\leq|v|}(1+|v|^2)|p|^2|p|^{-2}f(p)\,dp \\ &\leq (1+|v|^2)^{-1}|v|^{\gamma}\|f\|_r\int_{|p|\leq|v|}\frac{(1+|p|^2)}{|p|^2(1+|p|^2)^{r/2}}\,dp\end{aligned}$$

where we have used inequality (C.2) and the fact that for $|p|\leq|v|$,

$$|v-p|^{-2+\gamma}\leq C|v|^{-2+\gamma}$$

Since the integral

$$\int_{|p|\leq|v|}\frac{(1+|p|^2)}{|p|^2(1+|p|^2)^{r/2}}\leq C \text{ (a positive constant)}$$

we have

$$\mathcal{W}_{S_1}\leq C(1+|v|^2)^{-1+\gamma/2}\|f\|_r$$

Estimating $\mathcal{W}_{S_2}$, we write:

$$\mathcal{W}_{S_2}\leq(1+|v|^2)^{-1}\|f\|_r\int_{|p|>|v|}|v-p|^{-2+\gamma}(1+|p|^2)(1+|p|^2)^{-r/2}\,dp.$$

Using the fact that for $|p|>|v|$, $|v-p|\leq 2|p|$, one gets:

$$(1+|p|^2)^{1-r/2}\leq(1+|v-p|^2)^{1-r/2}$$

and so we obtain the estimate

$$\mathcal{W}_{S_2}\leq(1+|v|^2)^{-1}\|f\|_r\int_{|v-p|\leq 2|p|}\frac{|v-p|^{-2+\gamma}(1+|v-p|^2)}{(1+|v-p|^2)^{r/2}}.$$

If we let $\rho=|v-p|$ and change to polar coordinates, we will observe that for $r>5+\gamma$, the above integral is bounded by a positive constant. So we then have

$$\mathcal{W}_{S_2}\leq C(1+|v|^2)^{-1}\|f\|_r$$

and therefore,

$$\begin{aligned}V_{2-\gamma} &\leq \mathcal{U}_{S_1}+\mathcal{U}_{S_2}+\mathcal{W}_{S_1}+\mathcal{W}_{S_2} \\ &\leq 2(1+|v|^2)^{-1+\gamma/2}\|f\|_r+C(1+|v|^2)^{-1}\|f\|_r.\end{aligned}$$

Substituting this result into (C.3), we obtain :

$$J^+(f, f_3) \leq 4\pi(r-2)^{-1}(1+|v|^2)^{-r/2}(1+|v|^2)^{\gamma/2}\|f\|_r^2 + C\|f\|_r^2,$$

and in view of (C.1) the main result of this lemma follows, namely that:

$$\|J^+(f,f)\|_r \leq (r-2)^{-1}8\pi(1+|v|)^{\gamma}\|f\|_r^2 + C\|f\|_r^2. \quad \square$$

Thus, we see that the source of unboundedness is due to collisions generating particles with high velocities. However, in Chapter 2 we introduced the space $L^\infty_{s,r}$ which made it possible to bound the gain term in terms of the density functions, even for high velocities. Of course, it is important to take note of the fact that we can control the unboundedness here, if we choose $\gamma = 0$.

Appendix D

Outstanding Proofs

Here we prove the results (4.8) and (4.9). Hence, from the definition of $M(x,v)$ we write:

$$\begin{aligned}
\int_{v\cdot n(x)\leq 0}(v\cdot n(x))M(x,v)\,dv &= \int_{v\cdot n(x)\leq 0}(v\cdot n(x))\frac{1}{2\pi}h^2(x)e^{-v^2h(x)/2}\,dv \\
&= -\frac{h^2}{2\pi}\iiint_0^\infty v_3e^{-(v_1^2+v_2^2)h/2}e^{-v_3^2h/2}\,dv_3\,dv_2\,dv_1 \\
&= -\frac{h^2}{2\pi}\iint e^{-h(v_1^2+v_2^2)}\,dv_1\,dv_2\left(\int_0^\infty v_3e^{-v_3^2h/2}\,dv_3\right)
\end{aligned}$$

for $n=(0,0,-1): v_3>0$. By simple calculations, we have:

$$\begin{aligned}
\int_0^\infty v_3e^{-v_3h/2}\,dv_3 &= \frac{1}{h} \\
\frac{h^2}{2\pi}\iint e^{-h(v_1^2+v_3^2)/2}\,dv_1\,dv_2 &= \frac{h^2}{2\pi}\left(\frac{2\pi}{h}\right)=h
\end{aligned}$$

Therefore,

$$\int_{v\cdot n(x)\leq 0}(v\cdot n(x))M(x,v)\,dv=-1 \tag{D.1}$$

Remark D.1 *We note here that it is clear from the above calculation that the result in (4.8) is seen to be true if we had the absolute value signs. However, because the result (D.1) is used in the proof of (4.9) the above calculation was done without absolute value signs.*

We write:

$$\begin{aligned}\int_{\mathbb{R}^3_v}\int_{\partial\Omega}(v\cdot n(x))f(x,v)\,dv\,d\sigma(x) &= \int_{\partial\Omega}\int_{v\cdot n(x)>0}(v\cdot n(x))\gamma^- f(x,v)\,dv\,d\sigma(x)\\ &\quad+\int_{\partial\Omega}\int_{v\cdot n(x)>0}(v\cdot n(x))\gamma^+ f(x,v)\,dv\,d\sigma(x).\end{aligned}$$

By (4.2) and the definition of the diffuse reflection operator, the above equality becomes:

$$\begin{aligned}&= \int_{\partial\Omega}\int_{v\cdot n(x)\leq 0}(v\cdot n(x))M(x,v)\int_{\omega\cdot n(x)>0}(\omega\cdot n(x))\gamma^+(x,\omega)\,d\omega\,dv\,d\sigma(x)\\ &\quad+\int_{\partial\Omega}\int_{v\cdot n(x)>0}(v\cdot n(x))\gamma^+ f(x,v)\,dv\,d\sigma(x)\end{aligned}$$

$$\begin{aligned}&= \int_{\partial\Omega}\int_{\omega\cdot n(x)>0}(\omega\cdot n(x))\gamma^+(x,\omega)\,d\omega\left(\int_{v\cdot n(x)\leq 0}(v\cdot n(x))M(x,v)\,dv\right)d\sigma(x)\\ &\quad+\int_{\partial\Omega}\int_{\omega\cdot n(x)>0}(\omega\cdot n(x))\gamma^+ f(x,\omega)\,d\omega\,d\sigma(x)\end{aligned}$$

By (D.1) the integral in brackets is equal to -1, and so the equality (4.9) follows, namely that

$$\int_{\mathbb{R}^3_v}\int_{\partial\Omega}(v\cdot n(x))f(x,v)\,dvd\sigma(x)=0.$$

Appendix E

Solid Angle Formula

The purpose of this appendix is to verify the result in (4.19) namely that

$$de = \frac{n(y) \cdot (x-y)}{|x-y|^3}\, d\sigma(y)$$

where $n(y)$ is the outward unit normal vector at y on the boundary. Let Σ be the surface element of the unit sphere formed by the rays drawn from a point O at the center of the unit sphere to a point P which varies over a general surface element S, as seen in Figure E. 1. Let r be the distance OP, and F be the vector function $\frac{\vec{r}}{r^3}$. In the figure we denote by D the solid region formed by the bundle of rays cut off between Σ and S. Now we can establish the proof by an application of the divergence theorem to the region D. We have that $\frac{\vec{r}}{r^3}$ has components $\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}$. Hence the divergence of F can easily be calculated:

$$\begin{aligned}
\nabla \cdot F &= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) \\
&= -3\frac{x^2}{r^5} + \frac{1}{r^3} - 3\frac{y^2}{r^5} + \frac{1}{r^3} - 3\frac{z^2}{r^5} + \frac{1}{r^3} \\
&= 0.
\end{aligned}$$

Therefore, since $\nabla \cdot F = 0$, we have that

$$\iiint_D \nabla \cdot F \, dV = 0$$

where dV is the volume element of this region. Now by the divergence theorem, the integral of $F \cdot n$ over the entire surface of D is equal to zero. We see from the figure, that the surface of D consists of the surfaces Σ ,S, and the lateral portions formed by the rays joining O to S. However, since F is in the direction $\vec{r}$, it is perpendicular to n on the lateral portion. Hence, $F \cdot n = 0$ on this portion. Finally we have that:

$$\iint_{\Sigma} F \cdot n \, d\sigma + \iint_{S} F \cdot n \, d\sigma = 0$$

Thus we will be done if we show that

$$\iint_{\Sigma} F \cdot n \, d\sigma = -e$$

where e is the area of the surface Σ. We notice that on Σ, the outer normal points towards the center of the unit sphere at O. However, the F points away from O, and has a magnitude of $\frac{1}{r^2}$. On Σ, $r = 1$. This, together with the fact that F and n point in the opposite direction, gives $F \cdot n = -1$, and so:

$$\iint_{\Sigma} F \cdot n \, d\sigma = -\iint_{\Sigma} d\sigma = -(\text{area of } \Sigma)$$

Therefore,

$$\iint_{S} F \cdot n \, d\sigma = e$$

and for the distance $r = |x - y|$, we would have:

$$\iint_{S} \frac{n \cdot (x - y)}{|x - y|^3} \, d\sigma = e.$$

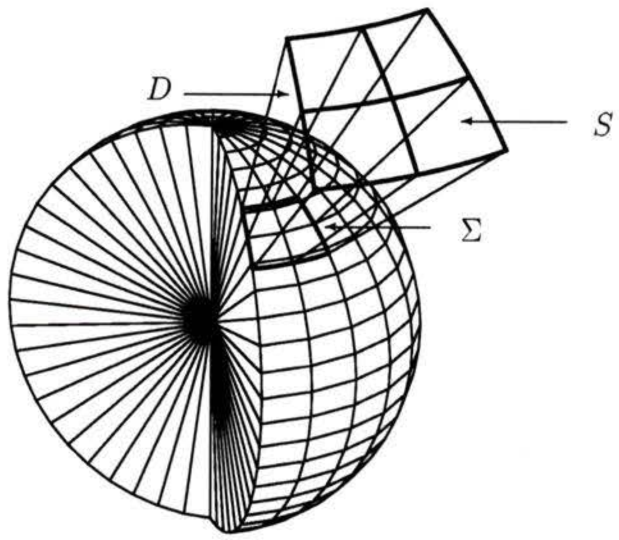

Figure E.1: Depiction of a solid angle e.

Σ is a surface patch of the unit sphere with center at O. The area of this surface is the solid angle e. S represents a general surface, however in the above diagram it is the surface of a sphere with radius 2.5 times the radius of the unit sphere. Finally D is the solid region formed by Σ, the lateral surfaces, and S.

VITA

Surname: Ghomeshi   Given Names: Shahin

Place of Birth: Tehran, Iran

Educational Institution Attended:

University of Victoria	1995 to 1998
University of Alberta	1988 to 1995

Degress Awarded:

B.Sc.	University of Alberta	1995

Honours and Awards:

University of Victoria Fellowship	1995-1997

Title of Thesis:

Existence and Uniqueness of Solutions to the Steady Boltzmann Equation

Author [redacted]

Shahin Ghomeshi

July 21, 1998