# A GUESSING GAME FOR RADIO

(Amer. Math. Monthly Problem E3448)

B.R. Johnson

DMS–583–IR August 1991

BRUCE R. JOHNSON, DEPARTMENT OF MATHEMATICS AND STATISTICS,
UNIVERSITY OF VICTORIA, VICTORIA, B.C., CANADA, V8W 3P4

# A GUESSING GAME FOR RADIO

(American Mathematical Monthly Problem E3448)

**E3448.** *Proposed by Howard Taylor, University of Delaware, Newark.*

Each day the call–in program on a local radio station conducts the following game. A number is drawn at random from $\{1,2,\cdots,n\}$. Callers are supposed to guess the number drawn. If a caller guesses the correct number, the station awards a prize and the game is finished. If a caller guesses incorrectly, the station announces the number guessed and whether it is too high or too low.

Find the expected number $f(n)$ of calls needed to arrive at the number drawn, assuming that each guess is made at random from the values not already excluded.

*Solution by Bruce R. Johnson, University of Victoria, Victoria, B.C., Canada.* We will show that $f(n)$ satisfies the recurrence relation

$$f(n+1) = 1 - \left[\frac{n}{n+1}\right]^2 + \left[1 - \frac{1}{(n+1)^2}\right] f(n), \quad \text{for} \quad n = 1,2,\cdots.$$

Applying this recursively, together with the boundary value $f(1) = 1$, yields

$$f(n+1) = 1 - \left[\frac{n}{n+1}\right]^2 + \sum_{k=1}^{n} \left[1 - \left[\frac{n-k}{n-k+1}\right]^2\right] \prod_{j=0}^{k-1} \left[1 - \frac{1}{(n-j+1)^2}\right], \quad \text{for} \quad n = 1,2,\cdots.$$

To derive the recurrence relation, we condition on both the value $i$ of the winning number and the value $j$ guessed by the first caller, obtaining

$$f(n) = \sum_{i<j}\sum (1+f(j-1)) \cdot \frac{1}{n^2} + \sum_{i=j}\sum 1 \cdot \frac{1}{n^2} + \sum_{j<i}\sum (1+f(n-j)) \cdot \frac{1}{n^2}$$

$$= 1 + \frac{1}{n^2} \sum_{j=2}^{n} (j-1)\, f(j-1) + \frac{1}{n^2} \sum_{j=1}^{n-1} (n-j) f(n-j) = 1 + \frac{2}{n^2} \sum_{m=1}^{n-1} m\, f(m).$$

Solving for $\sum_{m=1}^{n-1} m\, f(m)$ and plugging the result into the corresponding formula for $f(n+1)$, we find

$$f(n+1) = 1 + \frac{2}{(n+1)^2} \left[\frac{n^2}{2} ( f(n)-1) + n\, f(n)\right],$$

which simplifies to the desired recurrence relation.

As $n \longrightarrow \infty$ the expected number of guesses $f(n)$ needed to win grows ever more slowly toward infinity at the approximate rate $2/n$. A few selected values of $f(n)$ are given in the following table.

| $n$ | 10 | 20 | 30 | 40 | 50 | 100 | 200 | 300 | 400 | 500 | 1000 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(n)$ | 3.44 | 4.56 | 5.26 | 5.77 | 6.18 | 7.48 | 8.81 | 9.61 | 10.17 | 10.61 | 11.99 |