The Derivation of the BBGKY-Hierarchy for the Hard Sphere System

by

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Abstract

We study the time evolution of a system of N identical hard spheres in $\mathbb{R}^3$ and present a derivation of the *BBGKY-hierarchy* for the joint distributions of k spheres ($k = 1, \ldots, N$). Previous derivations tacitly assumed that the unknowns had enough regularities for the lower-dimensional integrals appearing in the hierarchy to make sense. A rigorous argument due to Illner & Pulvirenti [7] shows that if the initial measure in phase space is continuous along trajectories and has a suitable decay at space infinity, then a weak version of the *BBGKY-hierarchy* holds. Here a rigorous proof of the uniqueness of the solution of the weak form is given for a sufficiently regular initial value. The uniqueness leads to the equivalence of the weak and mild versions of the hierarchy.

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1

Introduction

It is generally believed that in certain limit situations the dynamics of a gas of particles can be described by the *Boltzmann Equation*. One of the basic problems of kinetic theory is a rigorous proof of the validity of the equation, assuming the laws of classical mechanics as a starting point. The difficulty and appeal of this problem arise from the possibility of linking two evolutions with basically different natures: Newtonian dynamics, deterministic and reversible, and Boltzmann dynamics, stochastic and irreversible.

The Liouville equation is a linear, homogeneous, first-order partial differential equation which describes the time evolution of the N particle distribution function for the N hard sphere dynamics and is a consequence of the law of classical mechanics; the Boltzmann Equation is an integro-differential equation which describes the time evolution of the single particle distribution for the same dynamics. It therefore is highly desirable to derive the Boltzmann Equation from the *Liouville Equation*, that is to derive it from completely deterministic particle dynamics so that the only randomness introduced into the derivation comes through the initial randomness of the particle configuration in phase space. Through arguments (though not of mathematical rigor) based on the careful analysis of the Liouville equation, first we can get a chain of equations to be satisfied by the k particle correlation functions, which are called the BBGKY-hierarchy equations and were introduced in a general setting through works published around 1946 by Bogoliubov, Born & Green, Kirkwood, and Yvon. Subsequently, the Boltzmann equation can be reduced formally from the BBGKY-hierarchy equation for the single particle correlation function in the Boltzmann-Grad limit (see [5]).

The BBGKY-hierarchy for the hard sphere dynamics was brought into an explicit and exact formulation by Cercignani [2] in 1972. The first rigorous result on validity of the Boltzmann equation was obtained by Lanford [9] in 1974. He showed, for the same dynamics, that a series expansion of the single particle correlation function derived from the BBGKY-hierarchy, approaches, in the Grad limit, the expansion corresponding to the Boltzmann equation for a short time (on the order of magnitude of a fraction of the mean free time between collision). His proof begins with a strong form of the BBGKY-hierarchy for finitely many identical hard spheres, and he gave only an outline of his argument and did not rigorously derive the BBGKY-hierarchy. This is the gap in Lanford's proof, because there is a problem with the formal derivation of the hierarchy (see [2]) since the hard sphere dynamics are only defined almost everywhere and the collision integral is over a set of Lebesgue measure zero. This was pointed out by Spohn in his notes [11], where he discussed a more complicated version. By following the general strategy proposed in [9] by Lanford, Illner and Pulvirenti [6,8] consider the same system and prove the validity of the Boltzmann equation for all times in the case of a gas allowed to expand into a vacuum and for large enough mean free paths. Furthermore, they gave an outline of a rigorous derivation of the BBGKY-hierarchy in the appendix of [6]. The details of the derivation were provided in [7], but the crucial fact is that the result was only a weak version of the BBGKY-hierarchy. So the gap revealed in Lanford's argument is still not quite closed. Results concerning the validity of the Boltzmann Equation and BBGKY-hierarchy for a model problem have been proved by Uchiyama [12], but the proofs are very technical, and longer and more complicated than those of Illner and Pulvirenti.

The primary purpose of this thesis is to give a complete and coherent exposition of the Illner-Pulvirenti derivation of the BBGKY-hierarchy, and the derivation of the mild version of the hierarchy under the following hypothesis on the density of the initial distribution function $\mu_0(X^N)$:

1. $\mu_0(X^N)$ is symmetric about the particles and continuous along the trajectories.
2. $\mu_0(X^N)$ has compact support in $(\mathbb{R}^3_x \times \mathbb{R}^3_v)^N$ or decays sufficiently rapidly.

In §2, we introduce the dynamics and fundamental properties of the hard sphere system in $\mathbb{R}^3$. In particular, we show that the flow defined by this dynamics preserves the Lebesgue measure in the phase space. We also mention that the set of configurations which eventually lead to a triple or higher order collisions is Lebesgue null. The BBGKY-hierarchy and its series solution are introduced in §3, and we show that the joint time evolved distribution $t \mapsto f_{k,t}(T_t^k X^k)$ defined by

$$f_{k,t}(T_t^k X^k) = \int_{\Gamma_{N-k}} \mu_0(T_{-t}^N(T_t^k X^k, X^{N-k}))dX^{N-k}$$

is continuous about t and continuous along the $k-$ trajectories. Also we show that if the density of the initial distribution satisfies the above hypothesis, then the series expansion makes sense. In §4, we give the Illner-Pulvirenti derivation of the weak version of the BBGKY-hierarchy. To establish the mild version of the BBGKY-hierarchy from its weak version, we first show that the series expansion satisfies the weak version of the hierarchy, and then we establish a uniqueness theorem for weak solutions of the hierarchy. It follows that the series expansion and the weak solution are equal. We also discuss the difficulty of the direct derivation of the mild version.

2

The Dynamics of the Hard Sphere System

Consider the time evolution of a system of N identical hard spheres with diameter $d > 0$ in the space $\mathbb{R}^3$. A particle (x, v), whose center $x \in \mathbb{R}^3$ represents the position of the particle, will move with constant velocity v until it collides with another particle. All collisions are assumed to be elastic.

To produce a rigorous definition of the flow we introduce the following notation

$$\begin{aligned}\Gamma^{3N} &= \{X \in \mathbb{R}^{3N} \mid \|x_i - x_j\| \geq d, \forall i \neq j, X = (x_1, x_2, \ldots, x_N)\}\\ \Gamma_N &= \{X \mid X = (x_i, v_i)_{i=1}^N, x_i \text{ and } v_i \in \mathbb{R}^3, \|x_i - x_j\| \geq d, \forall\, i \neq j\}\\ \partial\Gamma_N &= \{X \in \Gamma_N \mid \|x_i - x_j\| = d, \text{ for some pair } i, j\}\end{aligned}$$

Where Γ^{3N} denotes the part of $\mathbb{R}^{3N}$ which is admissible for position vectors if the particles are not allowed to overlap each other, $\partial\Gamma_N$ the set of all phase points such that at least two particles touch each other, $\Gamma_N = \Gamma^{3N} \times \mathbb{R}^{3N}$. ($\|x\|$ denotes the Euclidean length of $x \in \mathbb{R}^3$, the symbol $\langle a, b\rangle$ will be used to denote the Euclidean scalar product in either $\mathbb{R}^3$ or $\mathbb{R}^{3N}$.)

Therefore, the system state is explicitly described by the phase point

$$X^N = (\mathbf{q}^N, V) \in \Gamma^{3N} \times \mathbb{R}^{3N}$$

where

$$\mathbf{q}^N = (x_1, x_2, \ldots, x_N) \in \Gamma^{3N} \text{ and } V = (v_1, v_2, \ldots, v_N) \in \mathbb{R}^{3N}$$

are the vectors respresenting the positions and the velocities of the N particles respectively.

We may confine our attention to pair collisions, because phase points X^N leading to triple or higher multiple collisions or grazing collisions form a set of Lebesgue measure zero in Γ_N [1,10,12], and the dynamics of the particle system are in general not well defined past such collisions. Initial configurations, X^N, not leading to multiple or grazing collisions will be called admissible. In order to keep the notation simple, we use Γ_N to denote the admissible phase space and Γ_k the admissible phase space for the k particle system.

Now consider a pair collision between two particles. Let (x_i, v_i) and (x_j, v_j) be positions and pre- collisional velocities of two different particles, then (x_i, v_i, x_j, v_j) is a pre-collisional configuration if and only if the positions and velocities of the two particles satisfy the following conditions:

$$x_j = x_i + d \cdot n,\ \langle n, v_i - v_j \rangle \geq 0,\ n \in S^2 = \{n \mid \|n\| = 1, n \in \mathbb{R}^3\} \quad (2.1)$$

After the collision, the particles fly apart immediately with velocities:

$$v_i' = v_i - n \cdot \langle n, v_i - v_j \rangle,\quad v_j' = v_j + n \cdot \langle n, v_i - v_j \rangle \quad (2.2)$$

Momentum and energy are conserved in the collision:

$$v_i + v_j = v_i' + v_j',\quad v_i^2 + v_j^2 = v_i'^2 + v_j'^2 \quad (2.3)$$

All the collisions we consider have the above properties. Generally, we use the X^* to denote the post- (pre-) collisional configuration if the pre- (post-) collisional configuration is X.

Now we can define the N-particle flow, T_t^N, formally. First, we define T_t^0 as the flow of N particle free motion, i. e.

$$T_t^0(\mathbf{q}^N, V) = (\mathbf{q}^N + tV,\ V)$$

Let $\alpha(X)$ be the first time after time zero for which the free motion starting at X arrives at $\partial\Gamma_N$, and $\beta(X)$ be the last time before time zero for which the free motion starting at X passed through a collision, i. e.

$$\begin{aligned} \alpha(X) &= \inf\{t > 0 \mid T_t^0 X \in \partial\Gamma_N\} \quad (2.4) \\ \beta(X) &= \sup\{t < 0 \mid T_t^0 X \in \partial\Gamma_N\} \quad (2.5) \end{aligned}$$

We define the flow by

1 If $X \in \Gamma_N \setminus \partial\Gamma$, $T_t^N X = T_t^0 X$ for $0 \leq t \leq \alpha(X)$.

2 If $X \in \partial\Gamma_N$ and X is in the out-going collision configuration then $T_0^N X = X^*$ and $T_t^N X = T_t^0 X$ for $0 < t \leq \alpha(X)$.

3 If $X \in \partial\Gamma_N$ and X is in the in-going collision configuration $T_0^N X = X$ and $T_t^N X = T_t^0 X^*$ for $0 < t \leq \alpha(X^*)$.

In the succeeding steps, repeat the same procedure. Because the system is time-reversible, we can define the flow for $t < 0$ by using $\beta(X)$.

One fundamental result concerning the flow T_t^N is the following [12]:

Theorem 2.1 *For the dynamics of the hard spheres moving in* $\mathbb{R}^3$, *the flow* T_t^N *defined above preserves the Lebesgue measure* $m(\cdot)$ *on* Γ_N, *i. e. for any* $f \in L^1(\Gamma_N)$

$$\int_{\Gamma_N} f(T_t^N X)dX = \int_{\Gamma_N} f(X)dX$$

or for any Borel set $A \subset \Gamma_N$

$$m(T_{-t}^N A) = m(A) \text{ if } m(A) < \infty$$

PROOF: Let $t > 0$ and E_t denote the set of configurations $X \in \Gamma_N$ such that the flow starting from X at least once experiences a collision in the time interval $[0, t)$. Without loss of generality, let the first collision particles be (x_i, v_i) and (x_j, v_j). By introducing the parameter s which stands for the time of the first collision, we see

$$\begin{aligned} E_t &= \{X \in \Gamma_N \mid \exists\, 1 \leq i, j \leq N,\ x_j = x_i + s \cdot (v_i - v_j) + d \cdot n, \\ &\quad \text{and } \langle n, v_i - v_j \rangle > 0 \text{ for some } 0 \leq s < t \text{ and } n \in S^2\} \end{aligned}$$

The velocities after the collision are given by (2.2)

$$v_i' = v_i - n \cdot \langle n, v_i - v_j \rangle, \quad v_j' = v_j + n \cdot \langle n, v_i - v_j \rangle$$

For each fixed n, the mapping $A_n : (v_i, v_j) \mapsto (v_i', v_j')$ is a linear transformation of $\mathbb{R}^6$ with $|\det A_n| = 1$ and $A_n^{-1} = A_n$, the proof for those statements can be found in [3, pages 15–17]. So that A_n preserves the volume $dv_i dv_j$. Hence for any bounded measurable function $f \in L^1(\Gamma_N)$,

$$\int_{E_t} f(T_t^N X)dX = \int_{E_{-t}} f(X)dX \tag{2.6}$$

Here E_{-t} is defined in the same way as E_t but for the time-reversed flow. Since

$$T_t^0(\Gamma_N \setminus E_t) = \Gamma_N \setminus E_{-t}$$

Equation (2.6) yields

$$\int_{\Gamma_N} f(T_t^N X)dX = \int_{\Gamma_N \setminus E_t} f(T_t^0 X)dX + \int_{E_t} f(T_t^N X)dX = \int_{\Gamma_N} f(X)dX$$

Q. E. D.

3

The BBGKY-hierarchy and its Series Solution

C. Cercignani first gave a rigorous formulation of the BBGKY-hierarchy for the hard sphere dynamics in [2]. The BBGKY-hierarchy, which was introduced in a general setting through works published around 1946 by Bogoliubov,Born & Green, Kirkwood, and Yvon, is a chain of equations to be satisfied by a sequence of correlation functions.

Let $\mu_0(X)$ be the density of an absolutely continuous probability measure on Γ_N; because all the particles are indistinguishable, $\mu_0(X^N)$ is symmetric about the particles. For a Borel set $A \subset \Gamma_N$,

$$\int_A \mu_0(X)\, dX$$

is the probability that the state of the system at time 0 is in A.

The joint distribution densities $f_k = f_k(x_1, x_2, \ldots, x_k, v_1, v_2, \ldots, v_k)$ at time 0, dependent only on the first k particles, are defined by:

$$f_k(x_1, x_2, \ldots, x_k, v_1, v_2, \ldots, v_k) = \int_{\Gamma_{N-k}} \mu_0(X) dX^{N-k}, \text{ for } k < N \tag{3.1}$$

and they are also symmetric about the particles.

Since T_t^N preserves Lebesgue measure on Γ_N,

$$\mu_t(X) = \mu_0(T_{-t}^N X)$$

is the time evolved probability density. Generally, $\mu_t(X)$ is not continuous in X (nor in t) even if $\mu_0(X)$ is continuous, because of the interaction between the particles. However, we shall need some continuity property of $\mu_t(X)$ in the following discussion. What we impose on μ_0 is the continuity along the trajectory.

We comment on the assumption that μ_0 be continuous along trajectories. Obviously, this assumption is only serious at collision points, and it is clearly enough to consider just two particles (we exclude multiple collisions).

Let the pre-collisional coordinates and the post-collisional coordinates at a collision point be given by

$$(x_0, u, x_0 + d \cdot n, v) \text{ and } (x_0, u', x_0 + d \cdot n, v').$$

Here n, u' and v' satisfy Equations (2.1) (2.2); then the continuity of μ_0 along the trajectories implies that

$$\mu_0(x_0, u, x_0 + d \cdot n, v) = \mu_0(x_0, u', x_0 + d \cdot n, v').$$

Note that this does not imply that μ_0 be equal to an equilibrium density (or even depend only on collision invariants) at the collision point, because u' and v' are given in terms of u, v and n. The class of μ_0 which are continuous along trajectories is certainly dense in $L^1_+(\Gamma_N)$.

Let $f_{k,t}$ be the associated joint distribution density:

$$f_{k,t}(X^k) = \int_{\Gamma_{N-k}} \mu_t(X^N) dX^{N-k} \tag{3.2}$$

The present thesis aims at rigorously proving that $f_{k,t}$ satisfy the BBGKY-hierarchy equations in mild form,

$$\frac{d}{dt}\left[f_{k,t}(T_t^k X^k)\right] = C_{k+1} f_{k+1,t}(T_t^k X^k) \tag{3.3}$$

Here,

$$X^k = (x_1, x_2, \dots, x_k; v_1, v_2, \dots, v_k)$$

$$f_{k,t} = 0 \text{ if } k > N, \ f_{N,t} = \mu_t \tag{3.4}$$

$$C_{k+1} f_{k+1}(x_1, \dots, x_k; v_1, \dots, v_k) = \sum_{j=1}^{k} (N-k) d^2 \cdot \tag{3.5}$$

$$\int_{S^2} dn \int_{R^3} dv_{k+1} \langle n, v_j - v_{k+1} \rangle \cdot f_{k+1}(x_1, \dots, x_k, x_j + nd; v_1, \dots, v_k, v_{k+1})$$

S^2 is the unit sphere in $\mathbb{R}^3$, elements of S^2 are denoted by n.

Formally, from Equations (3.3) (3.5), we have

$$f_{k,t} = S(t) f_k + \int_0^t d\tau S(t-\tau) C_{k+1} f_{k+1,\tau} \tag{3.6}$$

where $S(t) f_k(x_1, \dots, x_k; v_1, \dots, v_k) = f_k(T^k_{-t}(x_1, \dots, x_k, v_1, \dots, v_k))$ From Equation (3.6), we obtain

$$S(-t) f_{k,t} = f_k + \int_0^t d\tau S(-\tau) C_{k+1} f_{k+1,\tau} \tag{3.7}$$

Substituting $f_{k+1,t}$ by

$$f_{k+1,t} = S(t)f_{k+1} + \int_0^t dt_1 S(t-\tau)C_{k+2}f_{k+2,\tau}$$

into Equation (3.7), we get

$$\begin{aligned} S(-t)f_{k,t} &= f_k + \int_0^t d\tau S(-t_1)C_{k+1}S(t_1)f_{k+1} \\ &+ \int_0^t dt_1 \int_0^{t_1} dt_2 S(-t_1)C_{k+1}S(t_1-t_2)(C_{k+2}f_{k+2,t_2}) \end{aligned} \tag{3.8}$$

Therefore, we can get the formal series solution of BBGKY-hierarchy

$$\begin{aligned} &S(-t)f_{k,t} = f_k + \\ &\sum_{n=1}^{N-k} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n S(-t_1)C_{k+1}S(t_1-t_2)\ldots S(t_n)f_{k+n} \end{aligned} \tag{3.9}$$

by substituting Equation (3.7) successively into itself. For Equation (3.9), the right hand side defines the left hand side.

The BBGKY-hierarchy (3.3) and its series solution (3.9) were usually established for particles interacting via smooth potentials. From a rigorous point of view the meaning of Equations (3.3) (3.9) is not completely clear, because the flow T_t^N is only almost everywhere defined (e. g. the flow is not defined for the phase points which lead to triple or higher multiple collisions), and therefore $\mu_t(X^N)$ is only almost everywhere defined, even if $\mu_0(X^N)$ has some kind of continuity. We have to show that the collision operator $C_{k+1}f_{k+1,t}(T_t^k X^k)$ in Equation (3.5), which involves an integral over a set of measure zero in Γ_{k+1}, is defined for almost all $X^k \in \Gamma_k$ and for all $t \in \mathbb{R}$ such that the right hand side of Equation (3.9) makes sense. In order to solve our problem we prove the following lemma first.

Lemma 3.1 *Fix $k < N$ and let*

$$\mathcal{F}_{ij}^{+(-)} = \{ \quad X \in \Gamma_N | \exists\, i \in \{1,2,\cdots k\},\ j \in \{k+1, k+2, \cdots, N\} \text{ such that } \|x_i - x_j\| = d,\ \langle x_i - x_j, v_i - v_j\rangle > 0\ (< 0)\}$$

be the set of all phase points in Γ_N which display an out-going (in-going) collision between particle i and particle j. Furthermore, we put

$$\mathcal{F}^{+(-)} = \bigcup_{i=1}^{k} \bigcup_{j=k+1}^{N} \mathcal{F}_{ij}^{+(-)},\ \mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^-$$

then $G = \{(t, X^N) \in \mathbb{R} \times \Gamma_N \mid (T_t^k X^k, X^{N-k}) \notin \mathcal{F}\}$ is an open subset of $\mathbb{R} \times \Gamma_N$ and the map: $(t, X^N) \mapsto (T_t^N X^k, X^{N-k})$ from G into Γ_N is continuous.

PROOF: Let $(t,x) \in \mathbb{R} \times \Gamma_N$ and $t > 0$, $T_t x = (T_t^k X^k, X^{N-k})$. The flow T_t starting at x experiences at most a finite number of collisions between the first k particles and the other $N-k$ particles in the interval $[0,t]$, all of which must be pairwise and proper (i. e. , not grazing, and we will discuss the grazing collision in § 4.3.2). Therefore we can choose a finite sequence $0 = t_0 < t_1 < \ldots < t_m = t$ such that there is at most one collision between t_i and t_{i+1} and $T_{t_i} x \notin \mathcal{F}$, $i = 1, 2, \ldots, m$. The map $T_{t_{i+1}-t_i}$ is continuous at $T_{t_i} x$ for $i = 1, 2, \ldots, m$ and the map $(s,y) \mapsto T_{t_1+s} y$ is continuous at $(0,x)$ These together with the semigroup property of $T_t, t \geq 0$ implies that the map $(\tau, y) \mapsto T_\tau y$ is continuous at (t,x). One can similarly proceed in case when $t \leq 0$. In particular G is open. **Q. E. D.**

Theorem 3.1 *Assume the density of the initial distribution μ_0 has compact support in $(\mathbb{R}_x^3 \times \mathbb{R}_v^3)^N$ or decays sufficiently rapidly, then the function $t \mapsto f_{k,t}(T_t^k X^k)$ is continuous for almost every $X^k \in \Gamma_k$.*

PROOF: The associate $k-$th joint distribution function $t \mapsto f_{k,t}(T_t^k X^k)$ is defined by

$$f_{k,t}(T_t^k X^k) = \int_{\Gamma_{N-k}} \mu_t(T_t^k X^k, X^{N-k}) dX^{N-k}$$

By the definition of the continuity, we consider the difference

$$\begin{aligned} & f_{k,t+s} - f_{k,t} \\ = & \int_{\Gamma_{N-k}} \mu_{t+s}(T_{t+s}^k X^k, X^{N-k}) dX^{N-k} - \int_{\Gamma_{N-k}} \mu_t(T_t^k X^k, X^{N-k}) dX^{N-k} \end{aligned} \tag{3.10}$$

as $s \to 0$

The flow T_s^{N-k} preserves Lebesgue measure on Γ_{N-k}, therefore we can substitute $T_s^{N-k} X^{N-k}$ for X^{N-k} in the first integral of the right hand side of Equation (3.10) and get

$$\begin{aligned} & f_{k,t+s} - f_{k,t} \\ = & \int_{\Gamma_{N-k}} [\mu_{t+s}(T_{t+s}^k X^k, T_s^{N-k} X^{N-k}) - \mu_t(T_t^k X^k, X^{N-k})] dX^{N-k} \end{aligned}$$

Split Γ_{N-k} into two disjoint sets Λ_1 and Λ_2

$$\begin{aligned} \Lambda_1 &= \{X^{N-k} \in \Gamma_{N-k} \mid (T_t^k X^k, X^{N-k}) \text{ is not a k} - \text{interacting state } \} \\ \Lambda_2 &= \Gamma_{N-k} \setminus \Lambda_1 \end{aligned}$$

Therefore, we have

$$\begin{aligned} & f_{k,t+s} - f_{k,t} \\ = & \int_{\Lambda_1} [\mu_{t+s}(T^k_{t+s}X^k, T^{N-k}_s X^{N-k}) - \mu_t(T^k_t X^k, X^{N-k})] dX^{N-k} \\ & + \int_{\Lambda_2} [\mu_{t+s}(T^k_{t+s}X^k, T^{N-k}_s X^{N-k}) - \mu_t(T^k_t X^k, X^{N-k})] dX^{N-k} \\ = & I_1 + I_2 \end{aligned}$$

We study I_1 and I_2 respectively. First, for Λ_2, the set of phase points X^{N-k} in Γ_{N-k} such that $(T^k_t X^k, X^{N-k})$ is a $k-$interacting state, because μ_0 is continuous along the trajectories,

$$\lim_{s\to 0} \mu_{t+s}(T^k_{t+s}X^k, T^{N-k}_s X^{N-k}) = \mu_t(T^k_t X^k, X^{N-k}). \qquad (3.11)$$

By Equation (3.11) and the condition that μ_0 decays sufficiently rapidly, an application of the Lebesgue dominated convergence theorem (see [4, Theorem (2.24)]) shows that

$$\lim_{s\to 0} \int_{\Lambda_2} [\mu_{t+s}(T^k_{t+s}X^k, T^{N-k}_s X^{N-k}) - \mu_t(T^k_t X^k, X^{N-k})] dX^{N-k} = 0.$$

Then for I_1, if the phase point $(T^k_t X^k, X^{N-k})$ is not a $k-$interacting state, by Lemma (3.1), $\exists\ s_0 > 0$ sufficiently small such that if $|s| < s_0$, then $(T^k_{s+t}X^k, T^{N-k}_s X^{N-k})$ is not a $k-$interacting state. Therefore for $X^{N-k} \in \Lambda_1$ and $|s| < s_0$,

$$\begin{aligned} T^N_{-s}(T^k_{s+t}X^k, T^{N-k}_s X^{N-k}) &= (T^k_t X^k, X^{N-k}), \\ \text{and } \mu_{t+s}(T^k_{t+s}X^k, T^{N-k}_s X^{N-k}) &= \mu_t(T^k_t X^k, X^{N-k}). \end{aligned}$$

Therefore $I_2 = 0$ as $|s| < s_0$. Based on the above analysis, we have

$$\lim_{s\to 0} |f_{k,t+s} - f_{k,t}| = 0$$

i. e. $t \mapsto f_{k,t}(T^k_t X^k)$ is continuous of all $t \in \mathbb{R}$ for almost all $X^k \in \Gamma_k$.

Q. E. D.

Corollary 3.1 *The formal sum*

$$f_k + \sum_{n=1}^{N-k} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n S(-t_1) C_{k+1} S(t_1 - t_2) \ldots S(t_n) f_{k+n}$$

makes sense if the continuity assumption on μ_0 is satisfied.

PROOF: We just prove it for $k = N-1$ and $k = N-2$, because the proofs for $k \leq N-3$ are identical with the case $k = N-2$. For $k = N-1$ the formal series has the following form

$$S(-t)f_{N-1,t} = f_{N-1} + \int_0^t d\tau S(-\tau)C_N f_{N,\tau}$$

where $S(t)f_N(X^N) = \mu_0(T_t^N X^N)$, $f_k = \int \mu_0 dX^{N-k}$. Because of the continuity along the trajectories of μ_0, $t \mapsto \mu_0(T_t^N X^N)$ is continuous for all $t \in \mathbb{R}$ and $X^N \in \partial\Gamma_N$, then

$$C_N f_{N,\tau}(x_1, \ldots, x_{N-1}, v_1, \ldots, v_{N-1}) = \sum_{j=1}^{N-1} d^2 \cdot$$

$$\int_{S^2} dn \int_{R^3} dv_N \langle n, v_j - v_N \rangle \cdot f_{N,\tau}(x_1, \ldots, x_{N-1}, x_j + nd; v_1, \ldots, v_{N-1}, v_N)$$

makes sense. So does the formal series (3.9) for $k = N-1$. For $k = N-2$ the series is equivalent to the sequence of equations

$$S(-t)f_{N-1,t} = f_{N-1} + \int d\tau S(-\tau)C_N f_{N,\tau}$$

$$S(-t)f_{N-2,t} = f_{N-2} + \int d\tau S(-\tau)C_{N-1} f_{N-1,\tau}$$

By Theorem (3.1), $t \mapsto f_{N-1,\tau}$ is continuous of all $t \in \mathbb{R}$ for all $X^{N-1} \in \partial\Gamma_{N-1}$, hence $C_{N-1,N-2}f_{N-1,\tau}$ makes sense. So does the formal series for $k = N-2$ by induction. **Q. E. D.**

4

The Derivation of the BBGKY-hierarchy

A derivation of the BBGKY-hierarchy can, e. g. , be found in [2], but the discussion there is largely formal. In particular, because the flow T_t^N is only defined a. e. with respect to the Liouville measure, it is not clear whether the lower-dimensional integration in (3.5) makes sense. This point was disregarded in [2]. Spohn [11] has given a rigorous derivation, but it is very intricate. Here, we follow the Illner-Pulvirenti derivation [7], known as special flow representation, and first obtain a weak version of equation (3.3). In § 4.3, we show that the k particle distribution functions $f_{k,t}$ satisfy the BBGKY-hierarchy (3.3).

4.1 The Special Flow Representation

To obtain the special flow representation, fix $k < N$, for $i \in \{1, 2, \cdots, k\}$ and $j \in \{k+1, \cdots, N\}$, let

$$\mathcal{F}_{ij}^{+(-)} = \{X \in \Gamma_N | \exists i,\ j,\ \|x_i - x_j\| = d,\ \langle x_i - x_j, v_i - v_j\rangle > 0\ (< 0)\} \quad (4.1)$$

be the set of all phase points in Γ_N which display an out-going (in-going) collision between particle i and particle j. Furthermore, we define

$$\begin{aligned} \mathcal{F}^{+(-)} &= \bigcup_{i=1}^{k} \bigcup_{j=k+1}^{N} \mathcal{F}_{ij}^{+(-)},\ \mathcal{F} = \mathcal{F}^{+} \cup \mathcal{F}^{-} \\ \sigma(Y) &= \inf\{t > 0 \mid T_t^N Y \in \mathcal{F}^{+}\},\ \tau(Y) = \sup\{t < 0 \mid T_t^N Y \in \mathcal{F}^{-}\} \end{aligned}$$

Elements of $\mathcal{F}^{+}$ will be referred to as out-going k–interacting states, elements of $\mathcal{F}^{-}$ will be referred to as in-going k–interacting states.

Next we split Γ_N into two disjoint sets Γ^{∞} and $\tilde{\Gamma}$, we define that

$$\begin{aligned} \Gamma^{\infty} = \{X = (X^k, X^{N-k}) \in \Gamma_N \mid X^k \in \Gamma_k, X^{N-k} \in \Gamma_{N-k}, \\ T_t^N X = (T_t^k X^k, T_t^{N-k} X^{N-k}) \text{ for all } t \in (-\infty, \infty)\} \end{aligned}$$

is the set of all phase points which never pass through a k-interacting state, and

$$\tilde{\Gamma} = \Gamma_N \setminus \Gamma^\infty \tag{4.2}$$

is the set of the phase points that do pass through k-interacting states.

The special flow representation is a representation of $\tilde{\Gamma}$. To this end, we split $\mathcal{F}^+$ into the following disjoint subsets, characterized by the properties of the flow:

$\mathcal{F}_0^+$: all k-interacting states Y from $\mathcal{F}^+$ for which $T_{-t}^N Y \notin \mathcal{F}^+$ for all $t > 0$, and $T_t^N Y \notin \mathcal{F}^-$ for all $t > 0$, this means $\sigma(Y) = \infty$, $\tau(Y) = -\infty$.

$\mathcal{F}_1^+$: all k-interacting states Y from $\mathcal{F}^+$ for which $T_{-t}^N Y \notin \mathcal{F}^+$ for all $t > 0$, but $T_s^N Y \in \mathcal{F}^-$ for some $s > 0$, this means $\sigma(Y) < \infty$, $\tau(Y) = -\infty$.

$\mathcal{F}_2^+$: all k-interacting states Y from $\mathcal{F}^+$ for which $T_{-t}^N Y \in \mathcal{F}^+$ for some $t > 0$, and for which $T_s^N Y \in \mathcal{F}^-$ for some $s > 0$, this means $\sigma(Y) < \infty$, $\tau(Y) > -\infty$.

$\mathcal{F}_3^+$: all k- interacting states Y from $\mathcal{F}^+$ for which $T_s^N Y \notin \mathcal{F}^-$ for all $s > 0$, but $T_{-t}^N Y \in \mathcal{F}^+$ for some $t > 0$. this means $\sigma(Y) = \infty$, $\tau(Y) > -\infty$.

This decomposition of $\mathcal{F}^+$ defines a natural decomposition of $\tilde{\Gamma}$. Let

$$\begin{aligned}
\tilde{\Gamma}_0 &= \{X \in \tilde{\Gamma} \mid X = T_t^N Y \text{for some } Y \in \mathcal{F}_0^+ \text{and some } t \in (-\infty, \infty)\} \\
\tilde{\Gamma}_1 &= \{X \in \tilde{\Gamma} \mid X = T_t^N Y \text{for some } Y \in \mathcal{F}_1^+ \text{and } t \in (-\infty, \sigma(Y)\} \\
\tilde{\Gamma}_2 &= \{X \in \tilde{\Gamma} \mid X = T_t^N Y \text{for some } Y \in \mathcal{F}_2^+ \text{and } t \in (0, \sigma(Y)\} \\
\tilde{\Gamma}_3 &= \{X \in \tilde{\Gamma} \mid X = T_t^N Y \text{for some } Y \in \mathcal{F}_3^+ \text{and } t \in (0, \infty)\}
\end{aligned}$$

The natural mapping:

$$\begin{aligned}
\varphi \quad &: \quad [\mathcal{F}_0^+ \times \mathbb{R}] \cup \{(Y,s) | Y \in \mathcal{F}_1^+, t \in (-\infty, \sigma(Y))\} \\
&\cup \quad \{(Y,s) | Y \in \mathcal{F}_2^+, t \in (0, \sigma(Y))\} \cup [\mathcal{F}_3^+ \times (0, \infty)] \\
&\mapsto \quad \tilde{\Gamma}
\end{aligned} \tag{4.3}$$

defined by $\varphi(Y,s) = T_s^N Y$ is one-to-one and bimeasurable. For the sequel, we find it is convenient to extend φ to $\mathcal{F}^+ \times \mathbb{R}$ by setting

$$\varphi(Y,s) = T_s^N Y \tag{4.4}$$

This extension is, of course, not one-to-one. For a function $f \in L^1(\tilde{\Gamma})$, we have

$$\begin{aligned}\int_{\tilde{\Gamma}} f(X)dX &= \sum_{\ell=0}^{3} \int_{\tilde{\Gamma}_\ell} f(X)dX \\ &= \sum_{\ell=0}^{3} \int_{\mathcal{F}_\ell^+} d\kappa^+(Y) \int_{\gamma_\ell}^{\sigma_\ell(Y)} ds f(\varphi(Y,s)) \qquad (4.5)\end{aligned}$$

where:

$$\gamma_0 = \gamma_1 = -\infty, \quad \gamma_2 = \gamma_3 = 0$$
$$\sigma_0(Y) = \sigma_1(Y) = \infty, \quad \sigma_2(Y) = \sigma_3(Y) = \sigma(Y)$$

and $d\kappa^+$ denotes the following measure on $\mathcal{F}^+$:

$$\begin{aligned} d\kappa^+(Y) = dX^k \sum_{i=1}^{k} \sum_{j=k+1}^{N} \quad & dx_{k+1} \dots dx_{j-1} dx_{j+1} \dots dx_N \\ & dv_{k+1} \dots dv_N dy_i \cdot \langle n_{ij},\, v_j - v_i \rangle \qquad (4.6)\end{aligned}$$

Here, dX^k is the Lebesgue measure on Γ_k, dy_i stands for the Lebesgue measure on the sphere with radius d and center x_i and $n_{ij} = \frac{x_j - x_i}{\|x_j - x_i\|}$.

These statements are consequences of the transformation theorem for integrals, and the calculations involved are a little lengthy but straightforward. In [7], the result concerning 3 particles and $k = 2$ was given and was found to be consistent with (4.6). (4.6) defines a measure on $\mathcal{F}^+$. The right hand side also defines a negative measure on $\mathcal{F}^-$, denoted by $d\kappa^-$.

$\mathcal{F}^-$ can be split in complete analogy to $\mathcal{F}^+$. In fact, $\{(Y, 0-) \mid Y \in \mathcal{F}_0^+\}$ corresponds bijectively to the set of all the in-going k interactions such that $\{T_t^N Y \mid t \in \mathbb{R}\}$ contains one such interaction. $\{(Y, \sigma(Y)-) \mid Y \in \mathcal{F}_1^+\}$ corresponds bijectively to the in-going k interactions which have seen exactly one k interaction before, $\{(Y, 0-) \mid Y \in \mathcal{F}_1^+\}$ corresponds to those which are first k interactions but not last ones, $\{(Y, \sigma(Y)-) \mid Y \in \mathcal{F}_2^+\}$ is simply the rest, i. e. those who have seen at least two k interactions in the past.

4.2 The Weak Version of the Hierarchy

In the previous chapter we have assumed that the initial probability density μ_0 is continuous along trajectories and ordinary continuous at almost all $X \in \Gamma_N$, in particular, with $\mu_t(X) = \mu_0(T_{-t}^N X)$ we want $t \mapsto \mu_t(X)$ to be continuous for almost all X. Then we choose a test function $u_k(x_1, \dots, x_k, v_1, \dots, v_k)$ which satifies the following conditions:

1. u_k is bounded and depends on the first k particles only.

2. $t \mapsto u_k(T_t^k X^k)$ is differentiable a. e. .

3. $\frac{d}{dt}[u_k(T_t^k X^k)]$ is an essentially bounded function.

In order to make sure that all the integrals in the subsequent calculation will exist, we will assume either that μ_0 has compact support in $(\mathbb{R}_x^3 \times \mathbb{R}_v^3)^N$ or decays sufficiently rapidly (e. g. , exponentially in the position and velocity variables). Because energy and momentum are conserved in every collision, we will have compact support (or rapid decay) also at any later time.

Now we finally begin the derivation of the BBGKY-hierarchy. By Equation (4.5),

$$\frac{d}{dt}\int_{\tilde{\Gamma}} u_k(X)\mu_t(X)dX = \frac{d}{dt}\sum_{\ell=0}^{3}\int_{\tilde{\Gamma}_\ell} u_k(X)\mu_t(X)dX \tag{4.7}$$

$$= \frac{d}{dt}\{\sum_{\ell=0}^{3}\int_{\mathcal{F}_\ell^+} d\kappa^+(Y)\int_{\gamma_\ell}^{\sigma_\ell(Y)} ds u_k(\varphi(Y,s))\mu_0(T_{-t}^N \circ \varphi(Y,s)\}$$

$$= \frac{d}{dt}\{\sum_{\ell=0}^{3}\int_{\mathcal{F}_\ell^+} d\kappa^+(Y)\int_{\gamma_\ell}^{\sigma_\ell(Y)} ds u_k(\varphi(Y,s))\mu_0(\varphi(Y,s-t)\}$$

The substitution $\tau := s - t$ leads to

$$\frac{d}{dt}\{\sum_{\ell=0}^{3}\int_{\mathcal{F}_\ell^+} d\kappa^+(Y)\int_{\gamma_\ell - t}^{\sigma_\ell(Y)-t} d\tau\, u_k(\varphi(Y,\tau+t))\mu_0(\varphi(Y,\tau))\} \tag{4.8}$$

where we have used the fact that the total domain of integration is flow-invariant. The assumptions on u_k allow differentiation in the usual way, as is easily checked, but the discontinuity of $t \mapsto u_k(T_t^N X)$ at k interacting states requires the splitting of the time integrals associated with $\ell = 0,\ 1$. We obtain the following four equations:

(I.)

$$\frac{d}{dt}\int_{\mathcal{F}_0^+} d\kappa^+(Y)\left\{\int_{-\infty}^{-t} + \int_{-t}^{\infty}\right\} d\tau\, u_k(\varphi(Y,\tau+t))\mu_0(\varphi(Y,\tau))$$

$$= \int_{\mathcal{F}_0^+} d\kappa^+(Y)[u_k(\varphi(Y,+0)) - u_k(\varphi(Y,0-))]\mu_t(\varphi(Y,0))$$

$$+ \int_{\mathcal{F}_0^+} d\kappa^+(Y)\left\{\int_{-\infty}^{-t} + \int_{-t}^{\infty}\right\} d\tau \frac{\partial}{\partial\tau} u_k(\varphi(Y,\tau+t))\mu_0(\varphi(Y,\tau)) \tag{4.9}$$

(II.)

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{F}_1^+} d\kappa^+(Y)\left\{\int_{-\infty}^{-t}+\int_{-t}^{\sigma_1(Y)-t}\right\} d\tau u_k(\varphi(Y,\tau+t))\mu_0(\varphi(Y,\tau))\\
= &\int_{\mathcal{F}_1^+} d\kappa^+(Y)\left\{\int_{-\infty}^{-t}+\int_{-t}^{\sigma_1(Y)-t}\right\} d\tau\frac{\partial}{\partial\tau}u_k(\varphi(Y,\tau+t))\mu_0(\varphi(Y,\tau))\\
+ &\int_{\mathcal{F}_1^+} d\kappa^+(Y)\{u_k(\varphi(Y,0+))-u_k(\varphi(Y,0-)\}\cdot\mu_t(\varphi(Y,0))\\
- &\int_{\mathcal{F}_1^+} d\kappa^+(Y)u_k\cdot\mu_t(\varphi(Y,\sigma_1(Y)-) \qquad (4.10)
\end{aligned}
$$

(III.)

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{F}_2^+} d\kappa^+(Y)\int_{-t}^{\sigma_2(Y)-t} d\tau u_k(\varphi(Y,\tau+t))\mu_0(\varphi(Y,\tau))\\
= &-\int_{\mathcal{F}_2^+} d\kappa^+(Y)u_k(\varphi(Y,\sigma_2(Y)-))\mu_t(\varphi(Y,\sigma_2(Y))\\
+ &\int_{\mathcal{F}_2^+} d\kappa^+(Y)u_k(\varphi(Y,0+))\mu_t(\varphi(Y,0)) \qquad (4.11)\\
+ &\int_{\mathcal{F}_2^+} d\kappa^+(Y)\int_{-t}^{\sigma_2(Y)-t} d\tau\frac{\partial}{\partial\tau}u_k(\varphi(Y,\tau+t))\mu_0(\varphi(Y,\tau))
\end{aligned}
$$

(IV.)

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{F}_3^+} d\kappa^+(Y)\int_{-t}^{\infty} d\tau u_k(\varphi(Y,\tau+t))\mu_0(\varphi(Y,\tau))\\
= &\int_{\mathcal{F}_3^+} d\kappa^+(Y)u_k(\varphi(Y,0+))\mu_t(\varphi(Y,0)) \qquad (4.12)\\
+ &\int_{\mathcal{F}_3^+} d\kappa^+(Y)\int_{-t}^{\infty} d\tau\frac{\partial}{\partial\tau}u_k(\varphi(Y,\tau+t))\mu_0(\varphi(Y,\tau))\}
\end{aligned}
$$

We collect and discuss all the boundary terms. By Liouville's theorem, we get

$$
\begin{aligned}
- &\int_{\mathcal{F}_0^+} d\kappa^+(Y)u_k\cdot\mu_t(\varphi(Y,0-))\\
- &\int_{\mathcal{F}_1^+} d\kappa^+(Y)u_k\cdot\mu_t(\varphi(Y,\sigma_1(Y)-))\\
- &\int_{\mathcal{F}_1^+} d\kappa^+(Y)u_k\cdot\mu_t(\varphi(Y,0-))\\
- &\int_{\mathcal{F}_2^+} d\kappa^+(Y)u_k\cdot\mu_t(\varphi(Y,\sigma_2(Y)-))\\
= &\int_{\mathcal{F}^-} d\kappa^-(Y)u_k\cdot\mu_t(\varphi(Y,0-)) \qquad (4.13)
\end{aligned}
$$

Here, we have used that $\mathcal{F}^-$ can be split in complete analogy to $\mathcal{F}^+$.

Similarly, the other boundary terms add up to

$$\int_{\mathcal{F}^+} d\kappa^+(Y)u_k \cdot \mu_t(\varphi(Y,0+)) \tag{4.14}$$

and by (4.6) and $d\kappa^-$, the expressions (4.13) and (4.14) add up to

$$\int_{\mathcal{F}} dX^k u_k(X^k) \sum_{i=1}^{k} \sum_{j=k+1}^{N} dx_{k+1} \dots dx_{j-1} dx_{j+1} \dots dx_N \tag{4.15}$$
$$dy_j \langle n_{ij},\ v_j - v_i \rangle dv_{k+1} \dots dv_N\ \mu_t(x_1 \dots x_{j-1}\ y_j\ x_{j+1} \dots x_N, v_1 \dots v_N)$$

By the continuous property of μ_0 along the trajectories, μ_t is $d\kappa^+$ ($d\kappa^-$) a. e. defined in $\mathcal{F}^+(\mathcal{F}^-)$. In the last chapter, we have in particular proved (see Theorem 3.1) that the joint distribution $t \mapsto f_{k+1,t}(T_t^k X^k)$ is continuous for almost all $X^k \in \Gamma_k$ and $t \mapsto f_{k+1,t}(x_1, x_2, \dots, x_k, x_k + d \cdot n, v_1, v_2, \dots, v_{k+1})$ is defined for almost all $X^k \in \Gamma_k$

Because $\mu_0(x_1, x_2, \dots, x_N; v_1, v_2, \dots, v_N)$ is symmetric about the particles, we can rewrite Equation (4.15) as

$$\begin{aligned}
&\sum_{i=1}^{k}(N-k)\int_{\mathcal{F}} dX^k u_k(X^k) dx_{k+2}\dots dx_N dv_{k+1}\dots dv_N \\
&\cdot dy_{k+1}\langle n_{i,k+1}, v_{k+1} - v_i\rangle \mu_t(x_1,\dots,x_k,y_{k+1},x_{k+2},\dots,x_N;v_1,\dots,v_N) \\
= &\sum_{i=1}^{k}(N-k)d^2 \int dX^k u_k(X^k) \int_{S^2} dn \int_{R^3} dv_{k+1}\langle n, v_{k+1} - v_i\rangle \\
&\cdot f_{k+1,t}(x_1,\dots,x_k,x_i + d\cdot n; v_1,\dots,v_k,v_{k+1}) \\
= &\int dX^k u_k(X^k)(C_{k+1} f_{k+1,t})(X^k)
\end{aligned} \tag{4.16}$$

The operator C_{k+1} is the collision operator given by (3.5).

The remaining terms in the identities (4.9) (4.10)(4.11) (4.12) add up to

$$\sum_{\ell=0}^{3} \int_{\mathcal{F}_\ell^+} d\kappa^+(Y) \int_{\gamma_\ell}^{\sigma_\ell(Y)} d\tau \frac{d}{d\tau}[u_k(\varphi(Y,\tau))]\mu_t(\varphi(Y,\tau)) \tag{4.17}$$

We set

$$\frac{d}{d\tau}[u_k(\varphi(Y,\tau))] = 0$$

at the discontinuity points of $\tau \mapsto u_k(\varphi(Y,\tau))$ and $\mathcal{L}_k$ as the generator associated with the flow T_t^k and defined by

$$(\mathcal{L}_k u_k)(X^k) = \lim_{t\to 0} \frac{u_k(T_t^k X^k) - u_k(X^k)}{t}$$

Hence (4.17) is equal to

$$\int_{\tilde{\Gamma}} dX(\mathcal{L}_k u_k)(X)\cdot\mu_t(X) \tag{4.18}$$

Summarizing, we have the following equations:

$$\begin{aligned}
&\frac{d}{dt}\int dX^k u_k(X^k) f_{k,t}(X^k) \qquad (4.19)\\
&= \frac{d}{dt}\int_{\Gamma^\infty} dX^N u_k(X^N)\mu_t(X^N) + \frac{d}{dt}\int_{\tilde{\Gamma}} dX^N u_k(X^N)\mu_t(X^N)\\
&= \frac{d}{dt}\int_{\Gamma^\infty} dX^N u_k(T_t^k X^k, T_t^{N-k}X^{N-k})\mu_0(X^N) + \frac{d}{dt}\int_{\tilde{\Gamma}} dX^N u_k(X^N)\mu_t(X^N)
\end{aligned}$$

We have used that Γ^∞ is invariant under T_t^N and for $X\in\Gamma^\infty$ i. e.

$$T_t^N X = (T_t^k X^k,\ T_t^{N-k}X^{N-k})$$

Therefore, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma^\infty} dX u_k(T_t^k X^k)\mu_0(X) &= \int_{\Gamma^\infty} dX \frac{d}{dt}[u_k(T_t^k X^k)]\mu_0(X)\\
&= \int_{\Gamma^\infty} dX(\mathcal{L}_k u_k)(X)\mu_t(X)
\end{aligned}$$

From (4.16), (4.18) and (4.19) we have

$$\begin{aligned}
&\frac{d}{dt}\int dX^k u_k(X^k) f_{k,t}(X^k) \qquad (4.20)\\
&= \int dX^k(\mathcal{L}_k u_k)(X^k) f_{k,t}(X^k) + \int dX^k u_k(X^k) C_{k+1} f_{k+1,t}(X^k)
\end{aligned}$$

An equivalent formulation is

$$\begin{aligned}
&\frac{d}{dt}\int dX^k u_k(T_t^k X^k) f_{k,t}(T_t^k X^k)\\
&= \int dX^k \frac{d}{dt}[u_k(T_t^k X^k)] f_{k,t}(T_t^k X^k) \qquad (4.21)\\
&+ \int dX^k u_k(T_t^k X^k)(C_{k+1} f_{k+1,t})(T_t^k X^k)
\end{aligned}$$

and this is exactly what one obtains by multiplying the mild form of the k-th hierarchy equation (3.3)

$$\frac{d}{dt}[f_{k,t}(T_t^k X^k)] = (C_{k+1} f_{k+1,t})(T_t^k X^k) \tag{4.22}$$

by the test function $u_k(T_t^k X^k)$ and integrating.

We have therefore shown that the $k-$ particle distribution functions satisfy the BBGKY-hierarchy equations in the weak form (4.20) or (4.21).

4.3 Uniqueness, The Mild Version of the Hierarchy

In Section 4.2 we proved that the joint distribution functions $f_{k,t}$ $1 \leq k < N$ satisfy the weak version of the BBGKY-hierarchy

$$\begin{aligned} &\frac{d}{dt}\int dX^k u_k(X^k) f_{k,t}(X^k) \\ = &\int dX^k (\mathcal{L}_k u_k)(X^k) f_{k,t}(X^k) + \int dX^k u_k(X^k) C_{k+1} f_{k+1,t}(X^k) \end{aligned} \tag{4.23}$$

under the assumptions :

1. The density of the initial distribution μ_0 is continuous along the trajectories, and μ_0 is symmetric in the N particles, i. e. , invariant under any permutation of the variables $(x_1, v_1), \ldots, (x_N, v_N)$.

2. μ_0 has compact support in $(\mathbb{R}^3_X \times \mathbb{R}^3_V)^N$ or decays sufficiently rapidly, e. g. exponentially in the position and velocity.

4.3.1 Uniqueness Method

We take two steps to show that the functions $f_{k,t}$ are actually a mild solution of the BBGKY-hierarchy.

1. We show the uniqueness of the weak solution of the BBGKY-hierarchy in the sense of the previous section.

2. We show that the series solution given by (3.9), which is certainly a mild solution, is also a weak solution of the BBGKY-hierarchy.

It is easy to verify that a mild solution of the BBGKY-hierarchy (3.3) is a solution of Equation (4.21). We can verify this by multipling the mild form of the k-th BBGKY-hierarchy equation (3.3)

$$\frac{d}{dt}\left[f_{k,t}(T_t^k X^k)\right] = C_{k+1} f_{k+1,t}(T_t^k X^k)$$

by a test function $u_k(T_t^k X^k)$ and integrating over Γ_k.

Theorem 4.1 *Assume that the density of the initial distribution μ_0 is continuous along the trajectories. Then there is at most one weak solution $f_{k,t}$ of the BBGKY-hierarchy (4.23) such that $t \mapsto f_{k,t}$, for $1 \leq k \leq N-1$, is continuous for almost all $X^k \in \Gamma_k$.*

PROOF: In fact we just need to prove the uniqueness of $f_{N-1,t}$, for the proofs are identical for $k \leq N-1$. Assume that there are two functions $f_{N-1,t}$, $g_{N-1,t}$ such that

$$\begin{aligned}
&\frac{d}{dt}\int dX^{N-1}u_{N-1}(X^{N-1})f_{N-1,t}(X^{N-1}) \qquad (4.24)\\
&= \int dX^{N-1}(\mathcal{L}_{N-1}u_{N-1})(X^{N-1})f_{N-1,t}(X^{N-1})\\
&\quad + \int dX^{N-1}u_{N-1}(X^{N-1})C_N\mu_t(X^{N-1})\\
&\frac{d}{dt}\int dX^{N-1}u_{N-1}(X^{N-1})g_{N-1,t}(X^{N-1}) \qquad (4.25)\\
&= \int dX^{N-1}(\mathcal{L}_{N-1}u_{N-1})(X^{N-1})g_{N-1,t}(X^{N-1})\\
&\quad + \int dX^{N-1}u_{N-1}(X^{N-1})C_N\mu_t(X^{N-1})
\end{aligned}$$

and the initial condition

$$f_{N-1,t}\mid_{t=0} = g_{N-1,t}\mid_{t=0} = \int \mu_0 dx_{N-1}dv_{N-1}$$

Let $h_{N-1,t} = f_{N-1,t} - g_{N-1,t}$, then $h_{N-1,t}\mid_{t=0} = 0$ at a. e. $X^{N-1} \in \Gamma_{N-1}$. So we have to prove that

$$\begin{aligned}
&\frac{d}{dt}\int dX^{N-1}u_{N-1}(X^{N-1})h_{N-1,t}(X^{N-1}) \qquad (4.26)\\
&= \int dX^{N-1}(\mathcal{L}_{N-1}u_{N-1})(X^{N-1})h_{N-1,t}(X^{N-1})\\
&h_{N-1,t}\mid_{t=0} = 0
\end{aligned}$$

has only the zero solution, where $u_{N-1}(X^{N-1})$ is the test function which satisfies the following conditions:

1. u_{N-1} is bounded and depends on the first $N-1$ particles only.

2. $t \mapsto u_{N-1}(T_t^{N-1}X^{N-1})$ is differentiable a. e. .

3. $\frac{d}{dt}[u_k(T_t^{N-1}X^{N-1})]$ is an essentially bounded function.

Hence, if $u_{N-1}(X^{N-1})$ is a test function, then for all $\tau \in \mathbb{R}$

$$u_{N-1,\tau}(X^{N-1}) = u_{N-1}(T_{-\tau}^{N-1}X^{N-1})$$

is a test function. In particular, we have

$$\begin{aligned}
&\frac{d}{dt}\int dX^{N-1}u_{N-1,\tau}(X^{N-1})h_{N-1,t}(X^{N-1})\\
&= \int dX^{N-1}(\mathcal{L}_{N-1}u_{N-1,\tau})(X^{N-1})h_{N-1,t}(X^{N-1}) \qquad (4.27)
\end{aligned}$$

By the definition of the generator associated with flow T_t^{N-1}, we have

$$\begin{aligned}\frac{d}{d\tau}u_{N-1,\tau}(X^{N-1}) &= \lim_{s\to 0}\frac{u_{N-1,s+\tau}(X^{N-1}) - u_{N-1,\tau}(X^{N-1})}{s} \\ &= \lim_{s\to 0}\frac{u_{N-1,\tau}(T_{-s}^{N-1}X^{N-1}) - u_{N-1,\tau}(X^{N-1})}{s} \\ &= -(\mathcal{L}_{N-1}u_{N-1,\tau})(X^{N-1})\end{aligned}$$

$$\begin{aligned}&\frac{\partial}{\partial\tau}\int dX^{N-1}u_{N-1,\tau}(X^{N-1})h_{N-1,t}(X^{N-1}) \\ &= -\int dX^{N-1}(\mathcal{L}_{N-1}u_{N-1,\tau})(X^{N-1})h_{N-1,t}(X^{N-1}) \qquad (4.28)\end{aligned}$$

By Equations (4.27) (4.28), we obtain

$$\begin{aligned}&\frac{d}{dt}\int dX^{N-1}u_{N-1,t}(X^{N-1})h_{N-1,t}(X^{N-1}) \\ &= \frac{\partial}{\partial t}\int dX^{N-1}u_{N-1,\tau}(X^{N-1})h_{N-1,t}(X^{N-1})|_{\tau=t} \\ &\quad +\frac{\partial}{\partial\tau}\int dX^{N-1}u_{N-1,\tau}(X^{N-1})h_{N-1,t}(X^{N-1})|_{\tau=t} \\ &= \int dX^{N-1}(\mathcal{L}_{N-1}u_{N-1,t})(X^{N-1})h_{N-1,t}(X^{N-1}) \\ &\quad -\int dX^{N-1}(\mathcal{L}_{N-1}u_{N-1,t})(X^{N-1})h_{N-1,t}(X^{N-1}) \\ &= 0 \qquad (4.29)\end{aligned}$$

By the initial condition $h_{N-1,t}\ |_{t=0} = 0$, we have

$$\int dX^{N-1}u_{N-1,t}(X^{N-1})h_{N-1,t}(X^{N-1}) = 0$$

Therefore $h_{N-1,t} = 0$ for almost all $(t, X^{N-1}) \in \mathbb{R}\times\Gamma_{N-1}$.

Because $t \mapsto h_{N-1,t}(T_t^{N-1}X^{N-1})$ is continuous a. e. , it follows that

$$f_{N-1,t}(T_t^{N-1}X^{N-1}) = g_{N-1,t}(T_t^{N-1}X^{N-1})$$

for almost all X^{N-1} and all t. This contains that

$$C_{N-1}f_{N-1,t}(X^{N-1}) = C_{N-1}g_{N-1,t}(X^{N-1}),$$

we can repeat the above argument to obtain

$$f_{N-2,t}(T_t^{N-2}X^{N-2}) = g_{N-2,t}(T_t^{N-2}X^{N-2}) \quad \textbf{Q.E.D.}$$

The uniqueness method can be applied to the direct proof of that the joint distribution functions $f_{k,t}(T_t^k X^k)$, $1 \le k < N$ satisfy the BBGKY-hierarchy equation (3.3). We give an outline of the proof here. By the same method in the above proof, from Equation (4.20) we obtain:

$$\frac{d}{dt}\int dX^k u_{k,\tau}(X^k) f_{k,t}(X^k) \tag{4.30}$$
$$= \int dX^k (\mathcal{L}_k u_{k,\tau})(X^k) f_{k,t}(X^k) + \int dX^k u_{k,\tau}(X^k) C_{k+1} f_{k+1,t}(X^k)$$

By the definition of the generator associated with flow T_t^k, we have

$$\frac{\partial}{\partial \tau}\int dX^k u_{k,\tau}(X^k) f_{k,t}(X^k) = -\int dX^k (\mathcal{L}_k u_{k,\tau})(X^k) f_{k,t}(X^k) \tag{4.31}$$

By Equations (4.30) (4.31), we obtain

$$\frac{d}{dt}\int dX^k u_{k,t}(X^k) f_{k,t}(X^k) = \int dX^k u_{k,t}(X^k) C_{k+1} f_{k+1,t}(X^k) \tag{4.32}$$

Because the flow T_t^k preserves the Lebesgue measure on Γ_k, from Equation (4.32) we have

$$\frac{d}{dt}\int dX^k u_k(X^k) f_{k,t}(T_t^k X^k) = \int dX^k u_k(X^k) C_{k+1} f_{k+1,t}(T_t^k X^k) \tag{4.33}$$

Equation (4.33) is equivalent to Equation (3.3).

4.3.2 Why A Direct Derivation Is Difficult

The technically most difficult part in the direct derivation of the hierarchy is to show that $t \mapsto f_{k,t}(T_t^k X^k)$ is differentiable with respect to time t for almost all $X^k \in \Gamma_k$ for the sufficiently regular initial value $\mu_0(X^N)$. If we can solve this problem, then we can obtain the BBGKY-hierarchy (3.3) directly from its weak version (4.21). We now explain why this is difficult to verify directly, even though it seems highly plausible.

We impose one more condition on the density of the initial distribution $\mu_0(X^N)$ as follows :

3. $\mu_0(X^N)$ has a bounded gradient $\nabla_{\mathbf{q}^N}\mu_0(X^N)$ off $\partial\Gamma_N$ and satisfies the boundary condition $\mu_0(Y^*) = \mu_0(Y)$ for $Y \in \partial\Gamma_N$; here Y and Y^* are the pre- and post- collisional coordinates respectively, and $\mathbf{q}^{\mathbf{N}}$ is the position vector of the phase point $X^N = (\mathbf{q}^{\mathbf{N}}, \mathbf{v}^{\mathbf{N}})$.

But this condition does not trivially imply that

$$t \mapsto f_{k,t}(T_t^k X^k) = \int_{\Gamma_{N-k}} \mu_t(T_t^k X^k, X^{N-k}) dX^{N-k}$$

is differentiable. To explain this point more closely, we decompose $t \mapsto \mu_t(T_t^k X^k, X^{N-k})$ into two maps

$$\mu_t(T_t^k X_k, X^{N-k}) = \mu_0(T_{-t}^N(T_t^k X^k, X^{N-k})) = \mu_0 \circ \Phi(t, X^N)$$

where

$$\Phi(t, X^N) = T_{-t}^N(T_t^k X^k, X^{N-k}), \ \Phi : \mathbb{R} \times \Gamma_N \to \Gamma_N, \ \mu_0 : \Gamma_N \to \mathbb{R}^+$$

If we can show that

1. $\partial\Phi/\partial t$ exists.
2. For almost all $X^k \in \Gamma_k$, there is a function $g(X^{N-k}) \in \mathcal{L}^1(\Gamma_{N-k})$ such that
$$|\langle \nabla_{\mathbf{q}^N} \mu_0, \frac{\partial \Phi}{\partial t} \rangle| \leq g(X^{N-k})$$

then by the chain rule and the dominated convergence theorem (see [4, Theorem (2.27)]), we can calculate

$$\frac{d}{dt} f_{k,t}(T_t^k X^k) = \int_{\Gamma_{N-k}} \langle \nabla_{\mathbf{q}^N} \mu_0, \frac{\partial \Phi}{\partial t} \rangle dX^{N-k}$$

Because of the condition 3, we only need to show that $\partial\Phi/\partial t$ exists and is bounded by some function $g(X^{N-k}) \in \mathcal{L}^1(\Gamma_{N-k})$ for almost all $X^k \in \Gamma_k$.

For simplicity, we only consider the case $N = 2$. Let

$$X = (x, v), \ Y = (y, w), \ X^2 = (X, Y) = (x, v; y, w) \text{ and } \mathbf{q}^2 = (x, y),$$

be the coordinates of the two particles, the phase point and the position vector respectively. We have

$$f_{2,t}(T_t^2 X^2) = \mu_0(X^2), \ f_{1,t}(T_t(x, v)) = \int_{\Gamma_1} \mu_0 \circ \Phi(t, X, Y) dY$$

Suppose that $T_t^1(x, v)$ and (y, w) are such that the backward flow experiences exactly one collision which is proper (i. e. not a grazing collision) in the time interval $[0, \ t]$, then there is a time instant τ such that

$$y - \tau \cdot w = x + (t - \tau) \cdot v + d \cdot n, \ \langle n, \ v - w \rangle < 0 \qquad (4.34)$$

$$\Phi(t, X^N) = (x + (t-\tau)\cdot(v-v'),\ v';\ y - \tau\cdot w - (t-\tau)\cdot w',\ w') \quad (4.35)$$

where $v' = v - n\cdot\langle n, v-w\rangle$, $w' = v + n\cdot\langle n, v-w\rangle$ are the post-collisional velocities and $n \in S^2 = \{n \mid \|n\| = 1,\ n \in \mathbb{R}^3\}$ is the collision parameter.

Because $\|n\| = 1$, from Equation (4.34) we calculate

$$\|x + t\cdot v - y - \tau\cdot(v-w)\|^2 = d^2$$

$$\|v-w\|^2\tau^2 - 2\langle v-w, x+t\cdot v-y\rangle\tau + \|x+t\cdot v-y\|^2 - d^2 = 0 \quad (4.36)$$

We want to determine τ from Equation (4.36), which has at least one positive solution provided that t, y and w satisfy the following inequalities

$$\langle v-w,\ x+t\cdot v-y\rangle > 0,\ \langle m, x+tv-y\rangle^2 - \|x+tv-y\|^2 + d^2 \geq 0$$

where $m = (v-w)/\|v-w\|$.

We solve the quadratic equation (4.36) under the condition that the above inequalities are true, and obtain

$$\tau_1 = \frac{\langle m, x+tv-y\rangle - \sqrt{\langle m, x+tv-y\rangle^2 - \|x+tv-y\|^2 + d^2}}{\|v-w\|}$$

$$\tau_2 = \frac{\langle m, x+tv-y\rangle + \sqrt{\langle m, x+tv-y\rangle^2 - \|x+tv-y\|^2 + d^2}}{\|v-w\|}$$

If $\|x+tv-y\|^2 = d^2$, we obtain $\tau = 0$ from Equation (4.34). Therefore the solution τ_2 is not physically feasible and we delete it.

For convenience, we define

$$\begin{aligned} P(t,y,w) &= \langle m, x+tv-y\rangle \\ R(t,y,w) &= \langle m, x+tv-y\rangle^2 - \|x+tv-y\|^2 + d^2 \end{aligned}$$

Based on the above calculation, we determine τ and n as follows

$$\begin{aligned} \tau &= \frac{\langle m, x+tv-y\rangle - \sqrt{\langle m, x+tv-y\rangle^2 - \|x+tv-y\|^2 + d^2}}{\|v-w\|} \\ &= \frac{P(t,y,w) - \sqrt{R(t,y,w)}}{\|v-w\|} \qquad (4.37) \\ n &= \frac{y - x - t\cdot v + \tau\cdot(v-w)}{d} \\ &= \frac{y - x - t\cdot v + (P(t,y,w) - \sqrt{R(t,y,w)})\cdot m}{d} \qquad (4.38) \end{aligned}$$

If the collision is proper (i. e. not grazing), then $R(t, y, w) > 0$. By Equation (4.37), we obtain

$$\frac{\partial \tau}{\partial t} = \frac{Q(t, y, w)}{\sqrt{R(t, y, w)}}.$$

$Q(t, y, w)$ is a smooth function of t, y and w and defined if t, y and w are such that a collision happens at all. If $\partial \tau / \partial t$ were bounded, Equation (4.38) would enable us to calculate a bounded derivative $\partial n / \partial t$, and by Equation (4.35) we could calculate the bounded derivative $\partial \Phi / \partial t$. Finally, we would have

$$\frac{d}{dt}[f_{1,t}(T_t^1 X)] = \int_{\Gamma_1} \langle \nabla_{\mathbf{q}^2} \mu_0, \frac{\partial \Phi}{\partial t} \rangle dY$$

Unfortunately, $\partial \tau / \partial t$ is not bounded as the proper collision becomes the grazing collision. The condition for a grazing collision is

$$y - \tau \cdot w = x + (t - \tau) \cdot v + d \cdot n, \quad \langle n, \ v - w \rangle = 0$$

and from these equations, we obtain

$$\langle x + t \cdot v - y, \ v - w \rangle = \|v - w\|^2 \cdot \tau \text{ and } \tau = \frac{\langle x + t \cdot v - y, \ m \rangle}{\|v - w\|}$$

which is equivalent to

$$R(t, y, w) = \langle m, x + tv - y \rangle^2 - \|x + tv - y\|^2 + d^2 = 0$$

For any fixed X, assume that $Y_0 = (y_0, w_0) \in \Gamma_1$ is a phase point which leads to a grazing collision, i. e. $R(t, y_0, w_0) = 0$; therefore, as $Y \to Y_0$, a proper collision becomes the grazing collision, and

$$\frac{\partial \tau}{\partial t} = \frac{Q(t, y, w)}{\sqrt{R(t, y, w)}}$$

becomes singular.

There is an easy way out of this dilemma, we could just set $\mu_0 = 0$ on a small neighbourhood of the phase points (y, w) which lead to the grazing collisions, then $\nabla_{\mathbf{q}^2} \mu_0$ would vanish near the singularity. However, this is not satisfactory for physical reasons.

Based on the above discussion, we can conclude that the hardest problem in the direct derivation of the BBGKY-hierarchy is to prove that the function $\partial \Phi / \partial t$, which is defined by

$$\Phi(t, X^N) = T_{-t}^N (T_t^k X^k, X^{N-k}),$$

is integrable about dX^{N-k} on Γ_{N-k}, as the proper collision becomes the grazing collision. Therefore if we assume the integrability of $\partial\Phi/\partial t$, we can calculate the derivative

$$\frac{d}{dt}f_{k,t}(T_t^k X^k) = \int_{\Gamma_{N-k}} \langle \nabla_{\mathbf{q}^N}\mu_0, \frac{\partial\Phi}{\partial t}\rangle dX^{N-k}$$

and finish the direct derivation of the hierarchy as follows.

Theorem 4.2 *Assume that the integral*

$$\int_{\Gamma_{N-k}} \frac{d}{dt}\mu_t(T_t^k X^k, X^{N-k})dX^{N-k}$$

exists, then the time evolved joint distribution function $t \mapsto f_{k,t}(T_t^k X^k)$ *satisfies the BBGKY-hierarchy (3.3) under the assumptions* **1, 2, 3** *(see pages 20 and 23).*

PROOF: In Section 4.2, we proved that $f_{k,t}(T_t^k X^k)$ satisfies the weak version BBGKY-hierarchy

$$\frac{d}{dt}\int dX^k u_k(T_t^k X^k) f_{k,t}(T_t^k X^k) \tag{4.39}$$
$$= \int dX^k \frac{d}{dt}[u_k(T_t^k X^k)] f_{k,t}(T_t^k X^k) + \int dX^k u_k(T_t^k X^k)(C_{k+1}f_{k+1,t})(T_t^k X^k)$$

By the assumption, the left side of Equation (4.39) can be reduced to

$$\frac{d}{dt}\int dX^k u_k(T_t^k X^k) f_{k,t}(T_t^k X^k) \tag{4.40}$$
$$= \int dX^k \frac{d}{dt}[u_k(T_t^k X^k)] f_{k,t}(T_t^k X^k) + \int dX^k u_k(T_t^k X^k)\frac{d}{dt}[f_{k,t}(T_t^k X^k)]$$

by exchanging the differentiation and integration. By Equations (4.39) and (4.40) we obtain the following equation:

$$\int dX^k u_k(T_t^k X^k)\frac{d}{dt}[f_{k,t}(T_t^k X^k)] = \int dX^k u_k(T_t^k X^k)(C_{k+1}f_{k+1,t})(T_t^k X^k)$$

Because the test function $u_k(T_t^k X^k)$ is arbitrary, it follows that

$$\frac{d}{dt}f_{k,t}(T_t^k X^k) = (C_{k+1}f_{k+1,t})(T_t^k X^k)$$

this is the BBGKY-hierarchy.

Q. E. D.

The result in this thesis together with those in [6,8] gives a rigorous proof of the complete transition from Liouville equation to the BBGKY-hierarchy to the Boltzmann equation. But the results are restricted to the Cauchy problem in $\mathbb{R}^3$, and do not apply in the more relevant context of bounded domains. This is a technical difficulty.

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