Exploring energy extraction from Kerr magnetospheres

by

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ABSTRACT

The aim of this thesis is to reconsider energy extraction from black hole magnetospheres, and more specifically the Blandford-Znajek (BZ) process from an effective field theory (EFT) perspective. Superradiant instabilities of scalar and vector bound states in the presence of a rotating black hole will be reviewed when the inverse mass of the black hole is much smaller than the Compton wavelength of the bound state particle. Two different matching calculations will be described for the vector bound state case and the overall decay rate will be compared. Force-free electrodynamics will be motivated and discussed in the context of the BZ process. Using a perturbation expansion, the Blandford-Znajek process will be reviewed up to second order in the rotation parameter. The absolute-space/universal-time (3+1) viewpoint will be discussed and applied to the BZ process and an EFT-like description will be discussed when the black hole horizon is parametrically small. Using differential forms, a simplified framework for the BZ process will be introduced in the (3+1) formalism and the field strength F will be simplified in the slow-rotation limit up to first-order in the rotation parameter. Finally, the Blandford-Znajek process will be considered as a superradiant process in the massive vector limit and the total energy flux in this (new) regime will be compared to the known BZ energy flux.

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and last but certainly not least,

The Kremlin.

There are things known and things unknown and in between are the doors.

Jim Morrison

DEDICATION

To my dearest friend Paul who encouraged me at a time in my life when I needed it most. I hope you are on a desert island somewhere using your tensor formulation of Hamilton's equations and thinking about quaternions.

Chapter 1

Introduction

This thesis studies energy extraction from black holes, specifically in the presence of large magnetic fields. The basic physical principle is called the Penrose process [1], which is the argument that energy can be classically extracted from a rotating Kerr black hole if it's angular momentum is reduced, because this is interpreted as energy, rather than momentum, in coordinates appropriate for the asymptotic region at large radius. More specifically, the actual mechanism is similar to superradiance (in which you find growing perturbations of fields around a black hole, rather than decaying quasinormal modes), but it utilizes the magnetic field and currents via what is called the Blandford-Znajek (BZ) process [2]. This is the mechanism that is believed to power active galactic nuclei for example. Our particular interest is in a more modern approach to studying this system using effective field theory (EFT) techniques.

The plan is to introduce effective field theories in the context of black holes and apply this EFT method in solving vector bound states around black holes. The Blandford-Znajek process will be looked at perturbatively and then studied in an EFT-like description and some intrinsic properties of the solution will become more apparent. Finally, the BZ process will be approximated as a superradiant instability in the massive vector limit, which takes the solution computed in the vector bound state and transfers it to the force-free regime.

The thesis is organized as follows:

Chapter 1 Introduction

Chapter 2 will begin with some necessary preliminaries for the remainder of the thesis and then will review superradiance from a classical picture by solving the Teukolsky master equation [3] for the Kerr geometry. From there, the EFT for bound states[1] around a black hole will be reviewed in the limit where the Compton wavelength is much larger than the size of the spinning compact object. Two different matching prescriptions will be outlined for the $\ell = j = m = 1$ mode.

Chapter 3 will describe energy extraction from Kerr magnetospheres particularly in the context of the Blandford-Znajek (BZ) process. The solution to the BZ process from classical general relativity will be reviewed and the solution up to second order in the rotation parameter will be worked out using a perturbation expansion. A concise review of the absolute-space/universal-time (3+1) viewpoint will be presented and the BZ process will be briefly described in this framework.

Chapter 4 will first provide an introduction to differential forms and the rotating Michel magnetic monopole [4] solution in flat space will be expressed in this formalism. The next section will cover force-free magnetospheres in the 3+1 viewpoint developed by Gralla and Jacobson [5]. This is then modified to consider the Blandford-Znajek process in the slow rotation limit (consistent with the EFT regime) and the field strength F is written to first order in the rotation parameter.

Chapter 5 will consider the Blandford-Znajek process in the massive vector approximation, where the mass of the field is taken to be equal to the plasma frequency due to a modified dispersion relation. The total energy flux is worked out in this regime and the result is compared to the well-known result obtained by Blandford and Znajek for an externally supported split monopole field [2].

Chapter 6 Conclusion

Therefore the content of the thesis is the following: Chapter 2 and 3 contain review material, Chapter 4 contains a novel analysis and combination of published material and Chapter 5 contains new unpublished results.

[1]Both scalar and vector bound states will be discussed.

Chapter 2

Superradiance

2.1 Preliminaries

Before diving into gravitational systems and superradiance, there are various definitions and quantities that need to be developed and explained in some detail.

2.1.1 Schwarzschild metric

The first black hole solution was discovered by Karl Schwarzschild and is given by the following line element,[1]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (2.1)$$

where M is the mass of the black hole. The Schwarzschild metric describes a non-rotating black hole in an asymptotically flat configuration and since $g_{\mu\nu}$ is invariant under the transformation $t \leftrightarrow -t$, the spacetime is static. This metric has two Killing vectors $k = \partial_t$ and $\ell = \partial_\varphi$. There is a coordinate singularity in the metric when $g_{tt} = 0$. This singularity occurs at the Schwarzschild radius, r_0 given by

$$r_0 = 2GM. \quad (2.2)$$

[1] Using the unit convention that $c = \hbar = 1$.

Thus r_0 is a null hypersurface which divides the manifold into two disconnected components [6],

$$2GM < r < \infty, \quad 0 < r < 2GM. \tag{2.3}$$

Inside the region $0 < r < 2GM$, the coordinates t and r reverse their nature meaning that r becomes timelike and t becomes spacelike [6].

2.1.2 Kerr metric

Unlike the Schwarzschild solution, which is spherically symmetric, the more complicated rotating Kerr geometry is not since it describes a rotating black hole. Instead one looks for solutions that have axial symmetry around the axis of rotation and that are stationary [7].

Boyer-Lindquist Coordinates

The first coordinate system that will be defined for the Kerr metric are the Boyer-Lindquist (BL) coordinates (t, r, θ, φ), where the line element is given by [8],

$$ds^2 = -\left(1 - \frac{2GMr}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{A \sin^2\theta}{\Sigma} d\varphi^2 - \frac{4GMra \sin^2\theta}{\Sigma} dt d\varphi, \tag{2.4}$$

where the dimensionful quantities, Σ, Δ, A, and the angular momentum J are given by the following

$$\Sigma = r^2 + a^2 \cos^2\theta, \tag{2.5}$$
$$\Delta = r^2 - 2GMr + a^2 \equiv (r - r_+)(r - r_-), \tag{2.6}$$
$$A = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta \tag{2.7}$$
$$J = aM, \tag{2.8}$$

and the metric determinant is given by $g = -\Sigma^2 \sin^2\theta$. Due to the $dtd\varphi$ term the metric is not invariant under $t \leftrightarrow -t$ and so it is not static. However, it is axisymmetric and stationary since $g_{\mu\nu}$ is independent of t and φ, therefore it has Killing vectors k and ℓ (just as for the Schwarzschild black hole). The Kerr metric is singular when Σ and Δ vanish [9]. The singularity at $\Sigma = 0$ occurs at $(r, \theta) = (0, \pi/2)$ and is a true singularity corresponding to infinite spacetime curvature [9]. The quantity Δ vanishes at

$$r_{\pm} = GM \pm \sqrt{(GM)^2 - a^2}, \tag{2.9}$$

assuming that $a \leq GM$. These singularities are similar to r_0 in the Schwarzschild metric, since these $r_{\pm}$ correspond to coordinate singularities. The Kerr metric possesses an interesting structure inside the horizon for $r < r_+$. The static surface $g_{tt} = 0$ defines the ergosphere given by [9]

$$g_{tt} = -\left(1 - \frac{2GMr}{\Sigma}\right) = 0 \quad \Rightarrow \quad r^2 - 2GMr + a^2\cos^2\theta = 0 \tag{2.10}$$

$$\Rightarrow \quad r_e^{\pm} = GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta}. \tag{2.11}$$

It is easy to see that at $\theta = 0, \pi$, $r_e^{\pm} = r_{\pm}$ and so these regions can be expressed as

$$r_e^- \leq r_- < r_+ \leq r_e^+. \tag{2.12}$$

The ergosphere is a region between these two surfaces $r_+ < r < r_e^+$ where it is possible to move toward or away from the event horizon and it is even possible to exit the ergosphere [7]. Inside the ergoregion, it is not possible for a test-particle to remain stationary with respect to observers at infinity since everything rotates. Inside the ergoregion, orbits of ∂_φ are not timelike, meaning that one cannot travel along them and remain stationary with respect to an observer at infinity. Therefore any timelike worldline is dragged in the direction of the black hole's rotation. This effect is referred to as frame-dragging.

It is the presence of the ergosphere that allows energy extraction from rotating black holes. This notion of energy extraction can be described by the Penrose process where a particle falls into the ergoregion from infinity and decays into two particles. One of the particles escapes to infinity and the other particle plunges into the black hole horizon. It is possible to configure the situation such that the energy-momentum of the process is conserved in the decay and the emerging particle leaves with more energy than the original particle carried in. This is possible due to the nature of the ergosphere - because inside the ergosphere the timelike vectors become spacelike.

Therefore, the particle can be seen as having carried in negative angular momentum and in the asymptotic region, this is viewed as the particle having "negative energy". The particle having negative angular momentum corresponds to the black hole decreasing its total angular momentum and in this sense, rotational energy is extracted from the black hole [9]. Rotational energy can be extracted in this way until the angular momentum of the black hole is reduced to zero [9].

Kerr-Schild Coordinates

Kerr-Schild (KS) coordinates (t, r, θ, φ) are regular on the black hole horizon and have the following line element [8],

$$ds^2 = -\left(1 - \frac{2GMr}{\Sigma}\right)dt^2 + \left(\frac{4GMr}{\Sigma}\right)drdt + \left(1 + \frac{2GMr}{\Sigma}\right)dr^2 + \Sigma d\theta^2 \tag{2.13}$$

$$+ \ \sin^2\theta\left[\Sigma + a^2\left(1 + \frac{2GMr}{\Sigma}\right)\right]d\varphi^2 \tag{2.14}$$

$$- \ \left(\frac{4GMar\sin^2\theta}{\Sigma}\right)d\varphi dt - 2a\left(1 + \frac{2GMr}{\Sigma}\right)\sin^2\theta d\varphi dr, \tag{2.15}$$

where the length scales, Σ and Δ, and the angular momentum J are given by equations (2.5), (2.6) and (2.8) the determinant is also given by $g = -\Sigma^2\sin^2\theta$. The metric $g_{\mu\nu}$ written in matrix form is the following,

$$g_{\mu\nu} = \begin{bmatrix} -\left(1 - \frac{2GMr}{\Sigma}\right) & \frac{2GMr}{\Sigma} & 0 & -\frac{2GMar\sin^2\theta}{\Sigma} \\ \frac{2GMr}{\Sigma} & 1 + \frac{2GMr}{\Sigma} & 0 & -a\left(1 + \frac{2GMr}{\Sigma}\right)\sin^2\theta \\ 0 & 0 & \Sigma & 0 \\ -\frac{2GMar\sin^2\theta}{\Sigma} & -a\left(1 + \frac{2GMr}{\Sigma}\right)\sin^2\theta & 0 & \sin^2\theta\left[\Sigma + a^2\left(1 + \frac{2GMr}{\Sigma}\right)\right] \end{bmatrix}$$

and the inverse metric $g_{\mu\nu}$ in matrix form is,

$$g^{\mu\nu} = \begin{bmatrix} -\left(1-\frac{2GMr}{\Sigma}\right) & \frac{2GMr}{\Sigma} & 0 & 0 \\ \frac{2GMr}{\Sigma} & \frac{\Delta}{\Sigma} & 0 & \frac{a}{\Sigma} \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ 0 & \frac{a}{\Sigma} & 0 & \frac{\csc^2\theta}{\Sigma} \end{bmatrix}.$$

2.1.3 Curvature

Curvature is physically manifested through something referred to as a "connection" [7] which provides a way to relate vectors in the tangent spaces of nearby points. The Christoffel symbol as given by

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}), \tag{2.16}$$

determines the curvature of the metric. The main use of the connection is to compute the covariant derivative ∇_μ, for instance the covariant derivative of some vector field V^ν is given by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\sigma}V^\sigma. \tag{2.17}$$

Thus, the covariant divergence of V^ν is given by,

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^{\mu}_{\mu\lambda}V^\lambda, \tag{2.18}$$

with

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{\sqrt{-g}}\partial_\lambda\sqrt{-g}. \tag{2.19}$$

2.2 Classical Superradiance

As described in the last section, Penrose [1] demonstrated that there is a way to extract energy and angular momentum from a rotating[2] black hole due to the existence of the ergoregion. Superradiance can be viewed as the wave analogue to the Penrose process [10].

[2]The Penrose process for extracting energy from a rotating black hole can be generalized to any spacetime that is stationary and axisymmetric and possesses an ergoregion [10].

In this section, the theory of superradiant scattering of test fields on a black hole background will be developed. Assume for the remainder of this section that the spacetime is stationary and axisymmetric. Finally, everything in this section is treated from a classical perspective.

2.2.1 Superradiance from rotating black holes

Superradiance is a phenomenon involving radiation amplification and occurs whenever the superradiant condition is met, that is

$$\omega - m\Omega > 0, \tag{2.20}$$

where ω is the angular frequency of the ingoing radiation, m is the angular momentum along the axis of rotation and Ω is the magnitude of the angular velocity for the rotating object. Rotational superradiance is important in the context of energy extraction from compact objects. Later in the thesis, it will be shown that this is a good way to describe the Blandford-Znajek process which is most likely responsible for powering jets of active galactic nuclei and black holes.

The wave equation for linearized fluctuations around the Kerr metric has been studied in great detail by Teukolsky [11] and [3], Press [12], [13] and others[3]. It has been shown that linearized perturbations of the Kerr metric can be described by a single master equation, describing probe scalar ($s = 0$), massless Dirac ($s = \pm 1/2$), electromagnetic ($s = \pm 1$) and gravitational ($s = \pm 2$) fields in a Kerr background. The master equation in vacuum reads

$$\begin{aligned}
&\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4GMar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2} \\
&- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-GM)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi} \\
&- 2s\left[\frac{GM(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 0,
\end{aligned} \tag{2.21}$$

where s is the field's spin weight and the field quantity ψ is directly related to the Newman-Penrose quantities given by Table 1 in [10]. Taking the Fourier transform of ψ and using the

[3] See Brito et al. [10] for an excellent review article on superradiance.

ansatz

$$\psi = \frac{1}{2\pi}\int d\omega e^{i\omega t} e^{im\varphi} S_{s\ell m}(\theta) R_{s\ell m}(r), \tag{2.22}$$

Teukolsky found separated ordinary differential equations (ODE's) for the radial and angular part in [11]. The radial and angular equations read

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR_{s\ell m}}{dr}\right) + \left(\frac{K^2 - 2is(r-GM)K}{\Delta} + 4is\omega r - \lambda\right) R_{s\ell m} = 0, \tag{2.23}$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + A_{s\ell m}\right) S = 0, \tag{2.24}$$

where $K \equiv (r^2 + a^2)\omega - am$ and $\lambda \equiv A_{s\ell m} + a^2\omega^2 - 2am\omega$. Together with the following orthonormality condition,

$$\int_0^\pi |S|^2 \sin\theta d\theta = 1, \tag{2.25}$$

the solutions to the angular equation given by equation (2.24) are referred to as the spin-weighted spheroidal harmonics $e^{im\varphi}S \equiv S_{s\ell m}(a\omega, \theta, \varphi)$. When $a\omega = 0$ they reduce to the spin-weighted spherical harmonics $Y_{s\ell m}$. When $a\omega$ is small, the angular eigenvalues are

$$A_{s\ell m} = \ell(\ell+1) - s(s+1) + O(a^2\omega^2). \tag{2.26}$$

Defining the tortoise coordinate r_* as $dr/dr_* = \Delta/(r^2 + a^2)$, $R_{s\ell m}$ in equation (2.23) has the following asymptotic behaviour

$$R_{s\ell m} \sim \mathcal{T}\Delta^{-s}e^{-ik_H r_*} + \mathcal{O}e^{ik_H r_*}, \quad r \to r_+, \quad R_{s\ell m} \sim \mathcal{I}\frac{e^{-i\omega r}}{r} + \mathcal{R}\frac{e^{i\omega r}}{r^{2s+1}}, \quad r \to \infty \tag{2.27}$$

where $k_H = \omega - m\Omega_H$ and $\Omega_H = a/(2Mr_+)$ is the angular velocity of the horizon. Since there must be purely ingoing waves at the horizon, it must be that $\mathcal{O} = 0$. The perturbation equations (2.23) and (2.24) and their asymptotic behaviour equation (2.27) can be used to compute energy fluxes that the fields carry through the horizon and out to spatial infinity. The full details of these expressions were computed in [13]. The total energy fluxes per unit solid angle for $s = 0, \pm 1$ are given by [10]

$$\frac{d^2E}{dtd\Omega} = \lim_{r\to+\infty} r^2 T^r_t, \tag{2.28}$$

where $T_{\mu\nu}$ is the stress-energy tensor of the test field. For the scalar case,

$$\frac{dE_{\rm out}}{dt} = \frac{\omega^2}{2}|\mathcal{R}|^2, \quad \frac{dE_{\rm in}}{dt} = \frac{\omega^2}{2}|\mathcal{I}|^2. \tag{2.29}$$

For the electromagnetic case where $s = 1$, the total energy fluxes are given by [10]

$$\frac{dE_{\rm out}}{dt} = \frac{4\omega^4}{B^2}|\mathcal{R}|^2, \quad \frac{dE_{\rm in}}{dt} = \frac{1}{4}|\mathcal{I}|^2, \tag{2.30}$$

where $B^2 = Q^2 + 4am\omega - 4a^2\omega^2$ and $Q = \lambda + s(s+1)$.

2.3 Superradiance Using Effective Field Theory Methods

The next step is to look at the process of superradiance using effective field theory techniques (as was done in [14; 15]). This effective field theory (EFT) will be valid at scales where the Compton wavelength of the scalar or vector particle forming the bound state is much larger than the size of the rotating object, which could be a black hole. The main purpose for using an EFT framework in this context is to give a simpler framework to carry out perturbative calculations since the problem simplifies significantly in the limit where the size of the compact object is effectively like a point particle of zero radius. By focusing on the long wavelength limit, this EFT formalism is able to describe complicated objects like black holes [14].

Fundamentally, an EFT is defined by the symmetries and an identification of the degrees of freedom relevant to describe physics across a defined range of distance scales. Effective field theories (EFTs) are an essential tool for treating problems that involve two or more sufficiently separated scales. For instance, in the limit where the black hole is far away, the bound state wave function[4] will appear hydrogen-like and the black hole can ultimately be viewed as a point-particle in this EFT. This effective field theory for black holes (and other

[4]For instance light particles can become gravitationally bound to a black hole.

compact objects) is also known as the worldline approach to black hole dynamics formulated by W. Goldberger and I. Rothstein. In their original paper [16], they constructed an EFT method for systematically calculating gravitational wave observables within a point particle worldline description. In this formalism, the black holes (or any other compact object) are described by generalized worldline actions that encompass all possible terms that are consistent with the general coordinate invariance of general relativity. By including all these operators it is possible to consistently renormalize all short distance divergences that may arise in calculations as well as systematically account for finite size effects. This effective point particle action was sufficient for non-dissipative tidal effects. In [17], they extended their EFT formalism to include dissipation, which is the relevant case for this thesis. This EFT-inspired treatment could prove useful in the context of gravitational systems since in practice it is difficult to solve the force-free equations in general. Using an EFT approach could even allow for a more general treatment beyond the split-monopole solution or using full numerical general relativity.

2.4 Bound States

There is some physical significance as to why this thesis considers bound states around black holes. For any massive bosonic field in the vicinity of a spinning black hole, there exists a set of gravitationally bound states whose frequency obeys the superradiance condition, $\omega - m\Omega_H < 0$ [10]. Bound states are of interest in the context of superradiance since their existence in the superradiant regime trigger superradiant instabilities [10]. There can be superradiant bound states for any scalar mass[5], however, the growth rates are governed by the Compton wavelength and therefore suppressed for lengths much larger or smaller than the the size of the black hole [15]. For vector Compton wavelengths larger than the size of the black hole it will be shown that the bound states are effectively non-relativistic and appear hydrogen-like. Therefore the following approaches will be valid provided that the Compton wavelength of the particles which forms the bound state is much larger than the size of the spinning object, in which case the spinning object can be treated as a point particle [14]. The next two sections on bound states follow the approach done by Endlich and Penco [14], which involve studying the superradiant instabilities of bound states around spinning objects.

[5]Bound states can only occur if the mass of the field is non-zero.

2.4.1 Scalar bound states

As a first look, consider a spin-0 particle with mass μ. A classical massive scalar field obeys the massive Klein-Gordon equation

$$\nabla^\mu \nabla_\mu \Phi = \mu^2 \Phi. \tag{2.31}$$

The following ansatz is an expansion of a scalar field in terms of annihilation and creation operators and is given by [14]

$$\hat{\Phi} = \sum_{n\ell m} \frac{1}{\sqrt{2E_{n\ell m}}} \left(\hat{a}_{n\ell m} f_{n\ell m}(r,\theta,\varphi) e^{-iE_{n\ell m}t} + \hat{a}^\dagger_{n\ell m} f^*_{n\ell m}(r,\theta,\varphi) e^{iE_{n\ell m}t} \right) + \ldots, \tag{2.32}$$

where the quantum numbers n, ℓ, m label the different bound states, and the dots imply the usual sum over annihilation and creation operators of the asymptotic states with definite momentum. The normalization of the bound states is chosen such that the states are orthonormal. The factor $\sqrt{2E_{n\ell m}}$ in equation (2.32) was inserted such that [14]

$$\int d\Omega dr r^2 f_{n\ell m}(r,\theta,\varphi) f^*_{n'\ell' m'}(r,\theta,\varphi) = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}. \tag{2.33}$$

For $r \gg GM$, assume that the field Φ varies slowly on scales of $1/\mu$, meaning that the components of the momentum are non-relativistic and the metric is approximately flat. Solving the equation of motion given by equation (2.31) .

$$\nabla_\mu \nabla^\mu \Phi - \mu^2 \Phi = (\nabla^2 - \mu^2)\Phi = 0 \tag{2.34}$$

$$g^{\mu\nu}\partial_\mu \partial_\nu \Phi - \mu^2 \Phi = 0 \tag{2.35}$$

$$-g^{00}\partial_0\partial_0\Phi + 2g^{0i}\partial_0\partial_i\Phi + g^{ij}\partial_i\partial_j\Phi - \mu^2\Phi = 0. \tag{2.36}$$

Putting in the ansatz of equation (2.32) yields

$$-g^{00}E^2_{n\ell m} f_{n\ell m} + \nabla^2 f_{n\ell m} - \mu^2 f_{n\ell m} = 0 \tag{2.37}$$

$$(E^2_{n\ell m} - \mu^2) f_{n\ell m} = -\nabla^2 f_{n\ell m} + E^2(1 + g^{00}) f_{n\ell m}. \tag{2.38}$$

In the Kerr metric for $r \gg GM$, $g^{00} \simeq -(1 - 2GM/r)$ and so the above equation resembles a Schrödinger equation describing the motion in a $1/r$ potential, that is

$$\frac{(E_{n\ell m}^2 - \mu^2)}{2\mu} f_{n\ell m}(r, \theta, \varphi) \simeq -\frac{\nabla^2}{2\mu}\Phi - \frac{GM\mu}{r}\Phi. \tag{2.39}$$

Assuming that the function $f_{n\ell m}$ is separable, that is $f_{n\ell m} = R_{n\ell}(r)Y_{\ell m}(\theta, \varphi)$ then this is same problem as the hydrogen atom where the hydrogenic wave functions are related to the $f_{n\ell m}$ by $\psi_{n\ell m} \equiv r f_{n\ell m}$ [14] and the non-relativistic binding energy E is replaced by $(E_{n\ell m}^2 - \mu^2)/2\mu$. The energy levels are discretized in the same fashion as the hydrogen atom and so to lowest order in the interaction, the energy eigenvalue $E_{n\ell m}$ depends only on the quantum number n [18],

$$E_{n\ell m}^2 \simeq \mu^2 \left(1 - \frac{(GM\mu)^2}{n^2}\right), \quad \ell + 1 \leq n, \tag{2.40}$$

where the quantity $GM\mu$ is the strength of the gravitational interaction responsible for the bound states.

The aim is to demonstrate that superradiance is the result of competition between absorption and spontaneous emission and this will be achieved by calculating the probability of each to occur.

Absorption

Begin by calculating the probability for the absorption process,

$$X_i + (n, \ell, m) \rightarrow X_f, \tag{2.41}$$

where X_i and X_f are respectively the initial and final state of the spinning object. Provided the final state X_f is not of interest, then the probability is equal to

$$P_{\text{abs}} = \sum_{X_f} |\langle X_f; 0|S|X_i; n, \ell, m\rangle|^2, \tag{2.42}$$

where the operator S in the above equation is given by the usual,

$$S = T \exp\left(-i \int dt H_{int}(t)\right). \tag{2.43}$$

The interaction Hamiltonian, which describes dissipative processes, contains couplings between the fields interacting with the spinning object and all possible composite operators $\mathcal{O}_{I_1,\ldots I_n}$. These operators exist in the rest frame of the spinning object and encode all the microscopic degrees of freedom of the object itself. In this section, only the coupling between Φ and the composite operator $\mathcal{O}_I$, which carries a single index will be considered[6]. Thus the interaction Hamiltonian can be written as the following [14],

$$H_{\text{int}} = \partial^I \Phi R_I{}^J \mathcal{O}_J, \tag{2.44}$$

where $R_I{}^J$ is a rotation matrix. The rotation here is a necessary frame transformation in order to account for the black hole rotation. Since the object is assumed to be spherically symmetric, the Wightman correlation function $\langle \mathcal{O}_J(t')\mathcal{O}_L(t)\rangle$ must be proportional to δ_{JL}. Therefore it's Fourier transform takes the form

$$\langle \mathcal{O}_J(t')\mathcal{O}_L(t)\rangle = \delta_{JL} \int \frac{d\omega'}{2\pi} \Delta(\omega') e^{i\omega'(t-t')}. \tag{2.45}$$

For simplicity, the spinning black hole is located at $\mathbf{x} = 0$. The following integral will be written in spherical coordinates but using the cartesian basis. Computing the right hand side of equation (2.42) and using the fact that the states $|X_f\rangle$ form a complete set, the absorption probability can be written to first order in perturbation theory as

$$P_{\text{abs}} \simeq \int dtdt' \langle n,\ell,m|\partial^I \Phi R_I{}^J(t')|0\rangle\langle 0|\partial^K \Phi R_K^{\ L}(t)|n,\ell,m\rangle\langle X_i|\mathcal{O}_J(t')\mathcal{O}_L(t)|X_i\rangle \tag{2.46}$$

$$= \int \frac{d\omega}{2\pi}\Delta(\omega) \left| \int dt e^{i\omega t}\langle 0|\partial^I \Phi(t) R_I{}^J(t)|n,\ell,m\rangle \right|^2 \tag{2.47}$$

$$= \int \frac{d\omega}{2\pi}\Delta(\omega) \left| \int e^{i\omega t}\langle 0| \sum_{n\ell m} \frac{1}{\sqrt{2\mu}} \partial^I[\hat{a}_{n\ell m} f_{n\ell m}(r,\theta,\phi) e^{-i\mu t} R_I{}^J]|n,\ell,m\rangle \right|^2$$

$$= \int \frac{d\omega}{2\pi}\Delta(\omega) \frac{1}{2\mu} \left| \int dt e^{i\omega t}\langle 0|\hat{a}_{n\ell m}\partial^I f_{n\ell m}(r,\theta,\phi) e^{-i\mu t} R_I{}^J \hat{a}^\dagger_{n\ell m}|0\rangle \right|^2$$

$$= \int \frac{d\omega}{2\pi}\Delta(\omega) \frac{1}{2\mu} \left| \int dt \partial^I f_{n\ell m}(r,\theta,\phi) e^{i(\omega-\mu)t} R_I{}^J \langle 0|\hat{a}_{n\ell m}\hat{a}^\dagger_{n\ell m}|0\rangle \right|^2. \tag{2.48}$$

[6] Any greek indices $\mu,\nu,\ldots$ will run over $0,1,2,3$ and the capital latin indices $I,J,K,\ldots$ will run over $1,2,3$.

Therefore the absorption probability is given by the following[7]

$$P_{\text{abs}} = \int \frac{d\omega}{2\pi}\frac{\Delta(\omega)}{2\mu}\left|\int dt\ \partial^I f_{n\ell m}(r=0)R_I{}^J(t)e^{i(\omega-\mu)t}\right|^2. \tag{2.49}$$

Using an identity for spherical harmonics given in [19]

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{(2\ell+1)!!}{4\pi\ell!}}V^m_{I_1,\ldots I_\ell}\hat{r}^{I_1\cdots}\hat{r}^{I_\ell} \tag{2.50}$$

to rewrite $\partial^I f_{n\ell m}(r=0)$[8]. The derivative can be computed as

$$\begin{aligned}\partial^I f_{n\ell m}(r=0) &= \partial^I(R_{n\ell}(0)Y_{\ell m}(\theta,\phi)) &(2.51)\\ &= \partial^I\left(R_{n\ell}(r)\sqrt{\frac{(2\ell+1)!!}{4\pi\ell!}}V^m_{I_1,\ldots I_\ell}\hat{r}^{I_1\cdots}\hat{r}^{I_\ell}\right)\Bigg|_{r=0}, &(2.52)\end{aligned}$$

where in cartesian coordinates $\mathbf{V}_m$ is given by [14],

$$\mathbf{V}_m = \left(-\frac{1}{\sqrt{2}}(\delta_1^m - \delta_{-\ell}^m), -\frac{i}{\sqrt{2}}(\delta_1^m - \delta_{-\ell}^m), \delta_0^m\right). \tag{2.53}$$

These complex vectors[9] $\mathbf{V}^m$ are defined to have unit norm, that is $V_n^L(V_L^m)^* = \delta_n^m$. Taking $\partial^I(Y_{\ell m})|_{\ell=1}$ yields,

$$\begin{aligned}\frac{\partial}{\partial x^I}\left(\sqrt{\frac{3}{4\pi}}V_I^m\frac{r_1^I}{r}\right) &= \sqrt{\frac{3}{4\pi}}\left(\frac{V^m_{I_1}}{r}\delta^I_{I_1} - V^m_{I_1}\frac{r^{I_1}}{r^2}\hat{r}^I\right) &(2.54)\\ &= \sqrt{\frac{3}{4\pi}}\left(\frac{V^m_{I_1}}{r}\delta^I_{I_1} - V^m_{I_1}\frac{r^{I_1}}{r^2}\frac{r^I}{r}\right) &(2.55)\\ &= \sqrt{\frac{3}{4\pi}}\frac{V^m_{I_1}}{r}\left(\delta^I_{I_1} - \hat{r}^{I_1}\hat{r}^I\right), &(2.56)\end{aligned}$$

[7]Recall that $\langle 0|aa^\dagger|0\rangle = 1$.

[8]Recall that the mode functions can be factored as $f_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$.

[9]Here $V^m = V_m$ because it is in cartesian coordinates, it will be seen later that this is not true in spherical coordinates.

which agrees with [14]. Computing the full derivative $\partial^I(f_{n\ell m}(r=0) = \partial^I(R_{n\ell}(0)Y_{\ell m}(\theta,\phi)))$ yields,

$$\begin{aligned}
\partial^I(f_{n\ell m}) &= \partial^I(R_{n\ell}(r))Y_{\ell m} + R_{n\ell}\partial^I(Y_{\ell m}(\theta,\phi)) && (2.57)\\
&= \partial_r(R_{n0}Y_{0m} + R_{n1}Y_{1m}) && (2.58)\\
&= \frac{1}{\sqrt{4\pi}}\left\{\partial_r R_{n0}\delta^0_\ell\delta^0_m\hat{r}^I + \frac{\sqrt{3}V_{I_1}}{r}\left[R_{n1}\delta^I_{I_1} - \hat{r}^{I_1}\hat{r}^I R_{n1} + \partial_r R_{n1}\hat{r}^I\hat{r}^{I_1}\right]\right\} && (2.59)\\
&= \frac{1}{\sqrt{4\pi}}\left\{\delta^0_\ell\delta^0_m\hat{r}^I\partial_r R_{n0}(0) + \frac{\sqrt{3}V^{I_1}_m}{r}\left[R_{n1}(0)\delta^I_{I_1} + R_{n1}\left(\frac{\partial_r R_{n1}}{R_{n1}} - 1\right)\hat{r}^I\hat{r}^{I_1}\right]\right\} && (2.60)
\end{aligned}$$

The solution to the hydrogen atom can be found in [20]

$$R_{n\ell} = -\left\{\left(\frac{2Z}{na_0}\right)^3\frac{(n-\ell-1)!}{2n(n+\ell)!}\right\}^{1/2} e^{-\rho/2}\rho^\ell L^{2\ell+1}_{n+\ell}(\rho), \tag{2.61}$$

and making the following variable substitutions

$$\begin{aligned}
Z &= \frac{4\pi\epsilon_0 GM\mu}{e^2}, && (2.62)\\
m &= \frac{\mu}{\hbar^2}, && (2.63)\\
a_0 &= \frac{4\pi\epsilon_0\hbar^2}{me^2} = \frac{4\pi\epsilon_0}{\mu e^2}, && (2.64)\\
\rho &= \frac{2Z}{na_0}r, && (2.65)
\end{aligned}$$

which yields,

$$R_{n\ell} = \left\{\left(\frac{2GM\mu^2}{n}\right)^3\frac{(n-\ell-1)!}{2n(n+\ell)!}\right\}^{1/2} e^{-\frac{GM\mu^2}{n}r}\left(\frac{2GM\mu^2}{n}\right)^\ell r^\ell L^{2\ell+1}_{n-\ell-1}\left(\frac{2GM\mu^2}{n}r\right). \tag{2.66}$$

Thus $\partial^I f_{n\ell m}$ reduces to the following [14]

$$\partial^I(f_{n\ell m}) = \frac{1}{\sqrt{4\pi}}\left\{\delta^0_\ell\delta^0_m\hat{r}^I\partial_r R_{n0}(0) + \frac{\sqrt{3}V^I_m}{r}R_{n1}(0)\delta^1_\ell\right\}. \tag{2.67}$$

From equation (2.67), it is clear that this interaction only affects the modes for $\ell = 0$ and $\ell = 1$[10]. However, since the $\ell = 0$ modes do not exhibit superradiance, it must be that $\ell = 1$, since this is the lowest mode that does exhibit superradiance [14]. Therefore the largest absorption and emission rates will occur for $n = 2$ mode[11]. For small values of r, equation (2.66) for $n = 2$ and $\ell = 1$ yields

$$R_{21}(r) \simeq \left(\frac{GM\mu^2}{2}\right)^{5/2} \frac{2r}{\sqrt{3}}, \tag{2.68}$$

which matches the quantity given in [14]. Putting this result into $\partial^I f_{21m}$ gives,

$$\begin{aligned}
\partial^I f_{21m}(r=0) &= \frac{1}{\sqrt{4\pi}} \left\{ \delta_1^0 \delta_m^0 \hat{r}^I \partial_r R_{20}|_{r=0} + \sqrt{3}\delta_1^1 V_m^I \frac{R_{21}}{r}|_{r=0} \right\} && (2.69)\\
&= \frac{1}{2\sqrt{\pi}} \left(\sqrt{3} V_m^I \left(\frac{GM\mu^2}{2}\right)^{5/2} \frac{2}{\sqrt{3}} \right) && (2.70)\\
&= \frac{1}{\sqrt{\pi}} V_m^I \left(\frac{GM\mu^2}{2}\right)^{5/2}. && (2.71)
\end{aligned}$$

Now putting this result back into P_{abs} yields,

$$P_{\text{abs}} = \int \frac{d\omega}{2\pi} \frac{\Delta(\omega)}{2\mu} \left| \int dt \; \frac{1}{\sqrt{\pi}} V_m^I \left(\frac{GM\mu^2}{2}\right)^{5/2} R_I{}^J(t) e^{i(\omega-\mu)t} \right|^2. \tag{2.72}$$

The rotation matrix $R_K{}^L(t)$ for an object spinning around the z-axis with angular velocity Ω, is given explicitly in cartesian coordinates as the following,

$$R_K{}^L(t) = \begin{bmatrix} \cos(\Omega t) & -\sin(\Omega t) & 0 \\ \sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

[10] Since all other values of ℓ will make the Kronecker delta vanish.
[11] Since n is defined by $n \geq \ell + 1$.

and it is easy to verify that $V^I_m R_I{}^J = V^J_m e^{im\Omega t}$. Putting all of this into equation (2.72) gives,

$$P_{\rm abs} = \int \frac{d\omega}{2\pi}\frac{\Delta(\omega)}{2\mu}\left|\int dt\ \frac{1}{\sqrt{\pi}}V^I_m\left(\frac{GM\mu^2}{2}\right)^{5/2} e^{im\Omega t}e^{i(\omega-\mu)t}\right|^2, \tag{2.73}$$

$$= \int \frac{d\omega}{2\pi}\frac{\Delta(\omega)}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^{5}\left|\int dt\ V^J_m e^{i(\omega+m\Omega-\mu)t}\right|^2 \tag{2.74}$$

$$= \int \frac{d\omega}{2\pi}\frac{\Delta(\omega)}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^{5} |\ V^J_m V^{m*}_J||2\pi\delta(\omega-\mu+m\Omega)|^2 \tag{2.75}$$

$$= \frac{1}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^{5}\Delta(\mu-m\Omega)2\pi\delta(0). \tag{2.76}$$

Spontaneous emission

The interaction Hamiltonian will also give rise to stimulated emission in a very similar way. Consider the emission process,

$$X_i \to X_f + (n,\ell,m), \tag{2.77}$$

where again X_i and X_f are the initial and final states of the spinning black hole. The probability for such an event to occur is given by

$$P_{\rm em} = \sum_{X_f}|\langle X_f;n,\ell,m|S|X_i;0\rangle|^2, \tag{2.78}$$

where the only difference in the probability is the ordering of the Wightman correlation function $\langle\mathcal{O}_L(t)\mathcal{O}_J(t')\rangle$. The correlation function is symmetric about $J \leftrightarrow L$ due to rotational invariance [14]. Therefore, the probability will be identical to the absorption case with $\Delta(\mu - m\Omega)$ replaced with $\Delta(m\Omega-\mu)$. Therefore $P_{\rm em}$ is given by,

$$P_{\rm em} = \frac{1}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^{5}\Delta(m\Omega-\mu)2\pi\delta(0). \tag{2.79}$$

Relating probabilities and flux

Now it is important to understand how these probabilities can be used to compute observable quantities. The link between these probabilities and superradiance is going to be achieved through the definition of the relative change in flux, which is the observable quantity of interest in superradiant calculations. Consider the following simple example to understand how the probabilities are indeed linked to observables in superradiance. Suppose the absorption probability of a particular system was found to be $P_{\text{abs}} = \alpha\Delta(\mu - m\Omega)$ and similarly the emission probability was found to be $P_{\text{em}} = \alpha\Delta(m\Omega - \mu)$, where the value of α is the same in both expressions and has a scaling that directly depends on the system. Now, consider an incoming flux of particles Φ_{in} with some non-zero mass μ, and arbitrary m. The overall flux is then computed by looking at the direct competition between absorption and stimulated emission. Considering looking at absorption alone, the outgoing flux would be equal to $\Phi_{\text{in}}(1 - P_{\text{abs}})$ and for emission the outgoing flux would be $\Phi_{\text{in}}(1 + P_{\text{em}})$, making the total outgoing flux $\Phi_{\text{out}} = \Phi_{\text{in}}(1 + P_{\text{em}} - P_{\text{abs}})$ [14]. However, a more convenient quantity is the relative change in flux which is given by,

$$\frac{\Phi_{\text{out}} - \Phi_{\text{in}}}{\Phi_{\text{in}}} = \frac{\Phi_{\text{in}}(1 + P_{\text{em}} - P_{\text{abs}}) - \Phi_{\text{in}}}{\Phi_{\text{in}}} = P_{\text{em}} - P_{\text{abs}}, \tag{2.80}$$

and putting in the values for the "found" probabilities yields,

$$\frac{\Phi_{\text{out}} - \Phi_{\text{in}}}{\Phi_{\text{in}}} = \alpha(\Delta(m\Omega - \mu) - \Delta(\mu - m\Omega)). \tag{2.81}$$

If the spectral density $\rho(\omega)$ is defined by, [14]

$$\rho(\omega) = \Delta(\omega) - \Delta(-\omega), \tag{2.82}$$

then the relative change in flux can be written as,

$$\frac{\Phi_{\text{out}} - \Phi_{\text{in}}}{\Phi_{\text{in}}} = -\alpha(\rho(\mu - m\Omega)). \tag{2.83}$$

Now going back to the scalar bound state and using equation (2.80), the relative change in flux is given by,

$$\begin{aligned} \frac{\Phi_{\text{out}} - \Phi_{\text{in}}}{\Phi_{\text{in}}} &= \frac{1}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^5 2\pi\delta(0)(\Delta(m\Omega - \mu) - \Delta(\mu - m\Omega)) && (2.84)\\ &= \frac{1}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^5 \rho(m\Omega - \mu)2\pi\delta(0). && (2.85) \end{aligned}$$

In [14], the decay rate for one such process to occur is given by,

$$\Gamma = \frac{P}{T}, \tag{2.86}$$

which in words is the probability of the reaction to occur over the interval of time T. Thus, the expression for the absorption decay rate $\Gamma_{\rm abs}$ is given by,

$$\begin{aligned}\Gamma_{\rm abs} &= \frac{\frac{1}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^5 \Delta(\mu - m\Omega) 2\pi\delta(0)}{2\pi\delta(0)} &(2.87)\\ &= \frac{1}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^5 \Delta(\mu - m\Omega) \quad \ldots \; (n=2, \ell=1), &(2.88)\end{aligned}$$

which matches the quoted result in [14]. Analogously, $\Gamma_{\rm em}$ is given by,

$$\Gamma_{\rm em} = \frac{1}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^5 \Delta(m\Omega - \mu) \quad \ldots (n=2, \ell=1), \tag{2.89}$$

which also exactly matches the result in [14]. The results that have been presented so far can take on a simpler form if some physical assumption about the composite operators $\mathcal{O}$ and the initial state $|X_i\rangle$. The spectral density admits a low-frequency Taylor expansion around $\omega = 0$. The spectral density of bosonic operators is an odd function of ω that is also positive for $\omega > 0$ and so for low energies it can be approximated as [14],

$$\rho(\omega) \simeq \gamma\omega + O(\omega^3), \quad \text{with} \quad \gamma > 0, \tag{2.90}$$

where γ is a coefficient that can depend on the temperature, but, importantly not on the spin of the object since the operators $\mathcal{O}$ live in the rest frame. Thus, in principle γ can be extracted from numerical simulations or analytical calculations in an overly simple, idealized problem and be used directly in more complicated processes. Now going back to compute the difference between the rates given in equations (2.88) and (2.89) for the $n = 2$, $\ell = 1$ mode is given by [14],

$$\begin{aligned}\Delta\Gamma = \Gamma_{\rm em} - \Gamma_{\rm abs} =& \frac{1}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^5 \rho(m\Omega - \mu) &(2.91)\\ \simeq& \frac{1}{2\mu\pi}\left(\frac{GM\mu^2}{2}\right)^5 (m\Omega - \mu)\gamma, &(2.92)\end{aligned}$$

where equation (2.90) has been used for the spectral density $\rho(\omega)$. By putting in the determined value of $\gamma = \frac{2}{3}\pi r_s^4$ which was found in [14], the total decay rate becomes

$$\Delta\Gamma \simeq \frac{(GM\mu)^9}{6}(m\Omega - \mu), \tag{2.93}$$

$$\simeq \frac{(GM\mu)^9}{6}(\Omega - \mu), \quad m = 1. \tag{2.94}$$

A positive value for $\Delta\Gamma$ signals that it is an instability [14] since the rate of production of particles is greater than the rate of their absorption. This instability occurs for $\mu \ll \Omega$. The result for the instability rate of a Kerr black hole due to the production of bound states with spin-0 particles with mass $\mu \ll \Omega$ is given by [14]

$$\Delta\Gamma \simeq \frac{(GM\mu)^9}{6}\Omega \tag{2.95}$$

which exactly agrees with the result found by Detweiler in [21] (via a direct analysis of the scalar equation in the full Kerr geometry). The problem to follow is slightly more complicated since it is a vector bound state with $s = 1$.

2.4.2 Vector bound states

For a massive vector field, the bound states will be labelled by the quantum number n, j, m, ℓ.[12] Using the normal rules of addition for the addition of angular momentum, the allowed values of ℓ for a particular j are given by $\ell = (j-1, j, j+1)$. For simplicity, a collective index $\beta \equiv (n, j, m, \ell)$ can be introduced and the procedure done in the scalar bound state can be followed. This begins by expanding the field in terms of annihilation and creation operators as so, [14]

$$A_\mu = \sum_\beta \frac{1}{\sqrt{2E_\beta}} \left\{ \hat{a}_\beta f_\mu^\beta(r,\theta,\varphi) e^{-iE_\beta t} + \hat{a}_\beta^\dagger f_\mu^{\beta *}(r,\theta,\varphi) e^{iE_\beta t} \right\} + \ldots, \tag{2.96}$$

where the dots stand for the usual sum over creation and annihilation operators of the asymptotic states with definite momentum. The vector field A_μ satisfies the Proca equation, which is equivalent to the following [14],

$$\nabla_\mu A^\mu = 0, \quad \Box A^\mu = \mu^2 A^\mu. \tag{2.97}$$

[12] Instead of using helicity [14].

These equations are valid on an arbitrary gravitational background, but in the limit where μ is small compared to the inverse size of the spinning object, one can work in the Newtonian limit as before. It will be helpful to decompose the mode functions for the bound states of A_μ that appear in equation (2.96) as so

$$f_\mu^{njm\ell}(r,\theta,\varphi) = \big(iS_{nj\ell}(r)Y_{jm}(\theta,\varphi), R_{n\ell}(r)\mathbf{Y}^{\ell,jm}\big(\theta,\varphi)\big), \tag{2.98}$$

where the $\mathbf{Y}^{\ell,jm}$ are the vector spherical harmonics outlined in [19]. Putting this ansatz into $\Box A^\mu - \mu^2 A^\mu = 0$, it can be shown that is indeed separable and the differential equation for $R_{n\ell}(r)$ is given by

$$\frac{2rR'_{n\ell}(r) + r^2 R''_{n\ell}(r)}{R_{n\ell}(r)} + \mu^2 - j(j+1) = 0, \tag{2.99}$$

where the leading order energy levels are hydrogen-like and are given by [15]

$$E_{n\ell m} \simeq \mu \left(1 - \frac{(GM\mu)^2}{2(n+\ell+1)^2}\right), \quad \ell + 1 \leq n. \tag{2.100}$$

The interaction Hamiltonian involving the magnetic field for $j = 1$ is given by

$$H_{\text{int}} = B^I R_I{}^J \mathcal{O}_J^B, \tag{2.101}$$

where $B^I = \frac{1}{2}\epsilon^{IJK}F_{JK}$, and $F_{\mu\nu}$ is the field strength of A_μ, where $B^I = \frac{1}{2}\epsilon^{IJK}\partial_j A_K$. The absorption and emission probabilities will be calculated using the interaction Hamiltonian given by equation (2.101) instead now for a vector bound state, $|n,j,m,\ell\rangle \equiv \hat{a}^\dagger_{njm\ell}|0\rangle$. First consider the probability for the absorption process, [14]

$$X_i + (n,j,m,\ell) \to X_f. \tag{2.102}$$

If the final state X_f of the spinning object is of no concern then the absorption probability is equal to [14],

$$P_{\text{abs}} = \sum_{X_f} |\langle X_f; 0|S|X_i; n,j,m,\ell\rangle|^2 = \sum_{X_f} |\langle X_f; 0|S|X_i; \beta\rangle|^2, \tag{2.103}$$

where $S = T\exp\{-i\int H_{\rm int}(t)dt\}$.[13] The interaction Hamiltonian is given by,

$$H_{\rm int} = \epsilon^{IJK}\partial_J A_K R_I{}^L \mathcal{O}^B_L, \tag{2.104}$$

where the subscript B denotes magnetic field. Using the fact that the states $|X_f\rangle$ form a complete set and rewriting equation (2.103) to first order in perturbation theory gives,

$$P_{\rm abs} \simeq \sum_{X_f}\int dtdt'\langle X_i;\beta|H_{\rm int}(t')|X_f;0\rangle\langle X_f;0|H_{\rm int}(t)|X_i;\beta\rangle, \tag{2.105}$$

$$= \sum_{X_f}\int dtdt'\langle X_i;\beta|\epsilon^{IJK}\partial_J A_K(t')R_I{}^L(t')\mathcal{O}^B_L(t')|X_f;0\rangle\langle X_f;0|\epsilon^{PQR}\partial_Q A_R(t)R_P{}^S(t)\mathcal{O}^B_S(t)|X_i;\beta\rangle, \tag{2.106}$$

$$= \int dtdt'\langle \mathcal{O}^B_L(t')\mathcal{O}^B_S(t)\rangle\langle\beta|\epsilon^{IJK}\partial_J A_K(t')|0\rangle\langle 0|\epsilon^{PQR}\partial_Q A_R(t)|\beta\rangle R_I{}^L(t')R_P{}^S(t). \tag{2.107}$$

The correlation function $\langle \mathcal{O}^B_L(t')\mathcal{O}^B_S(t)\rangle$ can be rewritten as, [14]

$$\langle \mathcal{O}^B_L(t')\mathcal{O}^B_S(t)\rangle = \delta^B_{LS}\int \frac{d\omega}{2\pi}\Delta(\omega')e^{i\omega'(t-t')}. \tag{2.108}$$

Putting this back into equation (2.107) yields,

$$P_{\rm abs} = \int dtdt'\delta^B_{LS}\int\frac{d\omega}{2\pi}\Delta(\omega')e^{iw'(t-t')}\langle\beta|\epsilon^{IJK}\partial_J A_K(t')|0\rangle\langle 0|\epsilon^{PQR}\partial_Q A_R(t)|\beta\rangle R_I{}^L(t')R_P{}^S(t), \tag{2.109}$$

$$= \int\frac{d\omega'}{2\pi}\Delta(\omega')\left|\int dte^{i\omega' t}\langle 0|\epsilon^{PQR}\partial_Q A_R(t)|\beta\rangle R_P{}^S(t)\right|^2, \tag{2.110}$$

$$\simeq \int\frac{d\omega'}{2\pi}\Delta(\omega')\left|\int dte^{i\omega' t}\langle 0|\epsilon^{PQR}\partial_Q\left[\sum_\beta\frac{1}{\sqrt{2E_\beta}}(\hat{a}_\beta f^\beta_R e^{-iE_\beta t} + \hat{a}^\dagger_\beta f^{\beta *}_R e^{iE_\beta t})\right]|\beta\rangle R_P{}^S(t)\right|^2. \tag{2.111}$$

Since $E_\beta \simeq \mu$,[14] equation (2.111) can be rewritten as [15]

$$P_{\rm abs} = \int\frac{d\omega'}{2\pi}\Delta(\omega')\left|\int dte^{i\omega' t}\frac{1}{\sqrt{2\mu}}\langle 0|\frac{\epsilon^{PQR}}{\sqrt{-g}}\left[\hat{a}_\beta\partial_Q f^\beta_R e^{-i\mu t}\right]|\beta\rangle\tilde{R}_P{}^S(t)\right|^2 \tag{2.112}$$

[13] It is assumed that the bound states are normalized, that is $\langle n,j,m,\ell|n,j,m,\ell\rangle = 1$.

[14] Again, $E^2_{n\ell m} \simeq \mu^2\left(1 - \frac{(GM\mu)^2}{n^2}\right)$ since they are hydrogenic wavefunctions.

[15] There is an implied sum over β here.

$$P_{\text{abs}} = \int \frac{d\omega'}{2\pi}\frac{\Delta(\omega')}{2\mu}\left|\int dte^{i\omega' t}\langle 0|\hat{a}_\beta\ \epsilon^{PQR}\partial_Q f_R^\beta e^{-i\mu t}\ \hat{a}_\beta^\dagger|0\rangle\tilde{R}_P{}^S(t)\right|^2 \tag{2.113}$$

$$= \int \frac{d\omega'}{2\pi}\frac{\Delta(\omega')}{2\mu}\left|\int dte^{i\omega' t}\epsilon^{PQR}\partial_Q f_R^\beta e^{-i\mu t}\tilde{R}_P{}^S(t)\langle 0|\hat{a}_\beta\hat{a}_\beta^\dagger|0\rangle\right|^2 \tag{2.114}$$

$$= \int \frac{d\omega'}{2\pi}\frac{\Delta(\omega')}{2\mu}\left|\int dt\frac{\epsilon^{PQR}}{\sqrt{-g}}\partial_Q f_R^\beta(r,\theta,\varphi)\ e^{i(\omega'-\mu)t}\tilde{R}_P{}^S(t)\right|^2, \tag{2.115}$$

where the rotation matrix $\tilde{R}_P^S$ can be rewritten in spherical coordinates as the following,

$$\tilde{R}_P{}^S(t) = \begin{bmatrix} \cos^2\theta + \cos(\Omega t)\sin^2\theta & -\frac{1}{r}\sin 2\theta\sin^2(\frac{\Omega t}{2}) & -\frac{1}{r}\sin(\Omega t) \\ -2r\cos\theta\sin\theta\sin^2(\frac{\Omega t}{2}) & \sin^2\theta + \cos(\Omega t)\cos^2\theta & -\cot\theta\sin(\Omega t) \\ r\sin^2\theta\sin(\Omega t) & \cos\theta\sin\theta\sin(\Omega t) & \cos(\Omega t) \end{bmatrix}.$$

Recall that $f_\mu^{njm\ell}$ is given by

$$f_\mu^{njm\ell}(r,\theta,\varphi) = \left(iS_{nj\ell}(r)Y_{jm}(\theta,\varphi), R_{n\ell}(r)\mathbf{Y}^{\ell,jm}(\theta,\varphi)\right), \tag{2.116}$$

but since the magnetic field is defined by $B^I = \frac{1}{2}\epsilon^{IJK}\partial_J A_K$, only the spatial components will be non-zero. Therefore equation (2.116) is really just $f_R^{njm\ell}(r,\theta,\varphi) = (0, R_{n\ell}(r)\mathbf{Y}^{\ell,jm}(\theta,\varphi))$. In spherical coordinates, the vector spherical harmonics are defined in the following way,

$$\mathbf{Y}^{\ell,jm}(\theta,\varphi) = \sum_{m'=-1}^{1}\sum_{m''=-\ell}^{\ell} C_{1\ell}(j,m;m',m'')\tilde{\mathbf{V}}^{m'}(r,\theta,\varphi)Y_{\ell m''}(\theta,\varphi), \tag{2.117}$$

where $\tilde{\mathbf{V}}_m$ is given in spherical coordinates by,

$$\tilde{\mathbf{V}}_m(r,\theta,\varphi) = \begin{bmatrix} \delta_0^m\cos\theta - \frac{1}{\sqrt{2}}\sin\theta(\delta_1^m e^{i\varphi} - \delta_{-1}^m e^{-i\varphi}) \\ -\delta_0^m r\sin\theta - \frac{1}{\sqrt{2}}r\cos\theta(\delta_1^m e^{i\varphi} - \delta_{-1}^m e^{-i\varphi}) \\ -\frac{i}{\sqrt{2}}r\sin\theta(\delta_1^m e^{i\varphi} + \delta_{-1}^m e^{-i\varphi}) \end{bmatrix}. \tag{2.118}$$

The instability of superradiant modes occurs for $m = 1$ and $\Omega > \mu$ for the scalar bound states and so it is assumed that this holds true for the vector bound state as well. Therefore, setting $\ell = j = m = 1$ yields,

$$\mathbf{Y}^{1,11} = \sum_{m'=-1}^{1} \sum_{m''=-1}^{1} C_{11}(1,1;m',m'')\tilde{\mathbf{V}}^{m'} Y_{\ell m''}(\theta,\varphi), \tag{2.119}$$

$$= \sqrt{\frac{3}{4\pi}}(0, -re^{i\varphi}, -ire^{i\varphi}\sin\theta\cos\theta). \tag{2.120}$$

Finally putting equation (2.120) into f_R^β gives,

$$f_R^\beta = \left(\frac{GM\mu^2}{2}\right)^{5/2} \frac{1}{\sqrt{4\pi}}(0, -r^2 e^{i\varphi}, -ir^2\sin\theta\cos\theta e^{i\varphi}). \tag{2.121}$$

Finally, putting equation (2.121) into equation (2.115) yields,

$$P_{\text{abs}} = \int \frac{d\omega'}{2\pi}\frac{\Delta(\omega')}{2\mu}\left(\frac{GM\mu^2}{2}\right)^5 \frac{1}{4\pi}\left|\int dt \frac{\epsilon^{PQR}}{\sqrt{-g}}\partial_Q(0, -r^2e^{i\varphi}, -ir^2\sin\theta\cos\theta e^{i\varphi})\ e^{i(\omega'-\mu)t}\tilde{R}_P^{\ S}(t)\right|^2, \tag{2.122}$$

and in the spherical polar basis, $\sqrt{-g} = r^2\sin\theta$. This allows us to rewrite equation (2.122) as,

$$P_{\text{abs}} = \int \frac{d\omega'}{2\pi}\frac{\Delta(\omega')}{2\mu}\left(\frac{GM\mu^2}{2}\right)^5 \frac{1}{\pi}\left|\int dt \frac{\epsilon^{PQR}}{r^2\sin^2\theta}\partial_Q(0, -r^2e^{i\varphi}, -ir^2\sin\theta\cos\theta e^{i\varphi})\ e^{i(\omega'-\mu)t}\tilde{R}_P^{\ S}(t)\right|^2 \tag{2.123}$$

$$= \int \frac{d\omega'}{2\pi}\frac{\Delta(\omega')}{2\mu}\left(\frac{GM\mu^2}{2}\right)^5 \frac{1}{\pi}\left|\int dt \left(i\sin\theta e^{i\Omega t}, \frac{i\sin 2\theta e^{i\Omega t}}{2r\sin\theta}, -\frac{e^{i\Omega t}}{r\sin\theta}\right) e^{i(\omega'-\mu)t}\right|^2 \tag{2.124}$$

$$= \int \frac{d\omega'}{2\pi}\frac{\Delta(\omega')}{2\mu}\left(\frac{GM\mu^2}{2}\right)^5 \frac{1}{\pi}\left|\int dt \left(i\sin\theta, \frac{i\cos\theta}{r}, -\frac{1}{r\sin\theta}\right) e^{i(\omega'-\mu+\Omega)t}\right|^2 \tag{2.125}$$

$$= \int \frac{d\omega'}{2\pi}\frac{\Delta(\omega')}{2\mu}\left(\frac{GM\mu^2}{2}\right)^5 \frac{1}{\pi}\left|\left(i\sin\theta, \frac{i\cos\theta}{r}, -\frac{1}{r\sin\theta}\right)\right|^2 \left|\int dt e^{i(\omega'-\mu+\Omega)t}\right|^2. \tag{2.126}$$

Computing the vector product inside the above line gives,

$$\left|\left(i\sin\theta, \frac{i\cos\theta}{r}, -\frac{1}{r\sin\theta}\right)\right|^2 = \left(i\sin\theta, \frac{i\cos\theta}{r}, -\frac{1}{r\sin\theta}\right)\cdot g_{\text{polar}}\cdot\left(-i\sin\theta, \frac{-i\cos\theta}{r}, -\frac{1}{r\sin\theta}\right), \tag{2.127}$$

where the spherical metric g_{polar} is the following matrix,

$$g_{\text{polar}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix}.$$

Putting this matrix into equation (2.127) gives,

$$\left|\left(i\sin\theta, \frac{i\cos\theta}{r}, -\frac{1}{r\sin\theta}\right)\right|^2 = -i^2\sin^2\theta - i^2\cos^2\theta + 1 = 2, \tag{2.128}$$

and putting this into P_{abs} yields,

$$P_{\text{abs}} = \left(\frac{GM\mu^2}{2}\right)^5 \int \frac{d\omega'}{2\pi^2}\frac{\Delta(\omega')}{\mu} \int dt \left|e^{i(\omega-\mu-\Omega)t}\right|^2 \tag{2.129}$$

$$= \left(\frac{GM\mu^2}{2}\right)^5 \int \frac{d\omega'}{2\pi^2}\frac{\Delta(\omega')}{\mu} \left|2\pi\delta(\omega-\mu-\Omega)\right|^2 \tag{2.130}$$

$$= 2\left(\frac{GM\mu^2}{2}\right)^5 \frac{\Delta(\Omega-\mu)}{\mu}\delta(0). \tag{2.131}$$

Computing Γ_{abs}[16] yields,

$$\Gamma_{\text{abs}} = \frac{P_{\text{abs}}}{2\pi\delta(0)} = \left(\frac{GM\mu^2}{2}\right)^5 \frac{\Delta(\Omega-\mu)}{\pi\mu}, \quad (n=2, \ell=1, j=1). \tag{2.132}$$

As before, the same procedure can be carried out for emission and the corresponding decay rate Γ_{em} is,

$$\Gamma_{\text{em}} = \frac{P_{\text{em}}}{2\pi\delta(0)} = \left(\frac{GM\mu^2}{2}\right)^5 \frac{\Delta(\mu-\Omega)}{\pi\mu}, \quad (n=2, \ell=1, j=1). \tag{2.133}$$

Using the definition of the spectral function as given by equation (2.90),

$$\Delta\Gamma(j=1, \ell=1) = \gamma_B \left(\frac{GM\mu^2}{2}\right)^5 \frac{\rho(\Omega-\mu)}{\mu}. \tag{2.134}$$

Setting $\rho(\Omega-\mu) \simeq \gamma_B(\Omega-\mu)$ yields,

$$\Delta\Gamma(j=1, \ell=1) \simeq \gamma_B \left(\frac{GM\mu^2}{2}\right)^5 \frac{(\Omega-\mu)}{\pi\mu}. \tag{2.135}$$

[16] Using that $\Gamma = \frac{P}{T}$ as defined in the last section.

The result given in [14] is found by considering the limit where $\mu \ll \Omega$,

$$\Delta\Gamma(j=1,\ell=1) \simeq \gamma_B \left(\frac{GM\mu^2}{2}\right)^5 \frac{\Omega}{\pi\mu}. \tag{2.136}$$

This can be matched to known results (as given in [22])[17] to compute $\gamma_B = \frac{8\pi}{3} r_s^4$ [14]. Thus one obtains,

$$\Delta\Gamma(j=\ell=1) \simeq \frac{8\pi}{3}(2GM)^4 \left(\frac{GM\mu^2}{2}\right)^5 \frac{\Omega}{\pi\mu} \tag{2.137}$$

$$\simeq \frac{4}{3}(GM\mu)^9\Omega. \tag{2.138}$$

For a rotating black hole,[18] $\Omega = \Omega_H = \frac{a}{(2GM)^2}$ so

$$\Delta\Gamma_{EP}(j=\ell=1) \simeq \frac{1}{3}\frac{a}{GM}(GM\mu)^8\mu \simeq \frac{1}{3}a(GM)^7\mu^9, \tag{2.139}$$

where EP stands for the authors, Endlich and Penco [14] and this notation will be used in the sections to follow. Lastly the parametric dependence of these rates on the combination $(GM\mu)$ is consistent with the absorption rates calculated in [23; 24; 15] and confirms the scaling rule proposed in these papers[19],

$$\Delta\Gamma \propto (GM\mu)^{5+2j+2\ell}. \tag{2.140}$$

2.4.3 Alternative EFT matching approach

An alternative classical matching calculation was carried out by Baryakhtar et al. [15] and their approach gives similar results to Endlich and Penco [14][20]. The procedure done by [15] will be looked at for the same $\ell = j = 1$ mode for comparison. In order to find the bound states of a massive vector field around a black hole, it will involve solving the Proca equation (as given in equation (2.97)) in the Kerr background. While this is normally done numerically, in practice there are certain regimes where it can be done analytically. Taking

[17] This is matched to the Schwarzschild black hole in the long wavelength limit.

[18] This will be explained in more depth in the next chapter.

[19] The j here is the ℓ for [23; 24] and vice versa.

[20] The coefficient for the decay rate was found to be off by a factor of two but the overall scaling of the modes matched exactly for the $l = j = 1$ mode.

the same limit as the last section[21] then it is expected that the $\frac{GM}{r}$ piece of the metric will be the most important and so the bound states can also be considered as non-relativistic hydrogenic wavefunctions. These bound states will oscillate with a frequency $\omega \simeq \mu$,[22] [15]

$$A^\mu(t,r,\theta,\varphi) = \frac{1}{\sqrt{2\mu}}(\Psi^\mu(r,\theta,\varphi)e^{-i\omega t} + \Psi^\mu(r,\theta,\varphi)^\dagger e^{i\omega t}), \tag{2.141}$$

which is an identical ansatz as equation (2.96). Due to the presence of the black hole horizon, there is an instability of quasibound states, and these quasibound states will lead to complex frequency ω [22]. Now, looking at the regime $r \gg GM$, and making the assumption that Ψ^μ varies slowly on scales $\frac{1}{\mu}$, this effectively means that the components of the momentum are also non-relativistic and that the metric is approximately flat [15]. The Proca equation $\nabla_\mu F^{\mu\nu} = \mu^2 A^\nu$ becomes,

$$\nabla_\mu(\nabla^\mu A^\nu - \nabla^\nu A^\mu) = \mu^2 A^\nu \tag{2.142}$$

$$(\nabla^2 - \mu^2)A^\nu = \nabla_\mu \nabla^\nu A^\mu. \tag{2.143}$$

Using the fact that the Riemann tensor can be written as $(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)X^\sigma = R^\sigma_{\rho\mu\nu}X^\rho$ from [6], the above expression can be rewritten as

$$(\nabla^2 - \mu^2)A^\nu = \nabla_\mu\nabla^\nu A^\mu + \nabla^\nu\nabla_\mu A^\mu - \nabla^\nu\nabla_\mu A^\mu \tag{2.144}$$

$$= R^\nu{}_{\mu\sigma}{}^\mu A^\sigma + \nabla^\nu\nabla_\mu A^\mu, \tag{2.145}$$

but from the equations of motion $\nabla_\mu A^\mu = 0$ so the second term in the above expression vanishes. Therefore, one obtains

$$(\nabla^2 - \mu^2)A^\nu = R^\nu{}_\sigma A^\sigma = g_{\mu\nu}R^\nu{}_\sigma A^\sigma \tag{2.146}$$

Putting in the ansatz given by equation (2.141) and imposing that the background is approximately flat, and keeping only the terms that are not suppressed by small momenta, the Proca equation becomes [15],

$$(\omega^2 - \mu^2)\Psi^\nu \simeq -\nabla^2\Psi^\nu + \omega^2(1 + g^{00})\Psi^\nu. \tag{2.147}$$

[21] The limit where (in this case) the vectors have a Compton wavelength that is large compared to the size of the rotating object, in this case a black hole.

[22] This is the same result as equation (2.40) where ω replaces $E_{n\ell m}$.

For the Kerr solution, when $r \gg GM$, then $g^{00} \simeq -(1-\frac{2GM}{r})$ and so it is a Schrödinger-type equation which describes motion in a $1/r$ potential [15],

$$(\omega - \mu)\Psi^{\nu} \simeq -\frac{\nabla^2}{2\mu}\Psi^{\nu} - \frac{GM\mu}{r}\Psi^{\nu}. \tag{2.148}$$

Up until this point, this procedure is identical to the previous section on vector bound states, where once again the black hole system is like a gravitational atom. From the Lorentz condition, $\partial_t A_0 \simeq \partial_i A_i$, it is possible to solve for Ψ_0 in terms of Ψ_i and thus determine the bound states by solving equation (2.148). Since the $1/r$ component is spherically symmetric, at leading order Ψ_i can be separated into radial and angular components [15],

$$\Psi_i = R_{n\ell}(r) Y_i^{\ell,jm}(\theta, \varphi), \tag{2.149}$$

where the angular functions are the same vector spherical harmonics as previously, defined in equation (2.117) and given in [19]. The equation for the $R_{n\ell}(r)$ is the same as in equation (2.66) for the scalar Coulomb problem and so the bound state radial wavefunctions are the same as for the hydrogen atom, labelled by the orbital angular momentum ℓ and overtone number n. The leading order energy levels are hydrogen-like and are given by equation (2.100) for $E_{n\ell m} = \omega$.

If a bound state satisfies the superradiance condition given by equation (2.20), and assuming that the gravitational perturbations to the Kerr background are small, then it will grow exponentially as it extracts energy from the black hole. This process is possible for a bound state of any mass μ, provided there is a suitable value of m. To obtain the bound state growth and decay rates, the wave equation must either be solved numerically or a matching calculation between different regimes of validity needs to be performed.

Far away from the black hole, the bound states are hydrogen-like and the treatment has been described above. However, close to the black hole, the mass μ in the Proca equation becomes a sub-leading correction. This means that in the limit of small mass, the Proca field can be separated into two transverse modes, which obey the photon equation of motion and a single longitudinal mode obeying the massless Klein-Gordon equation. The evolution of massless scalars can be solved analytically on the Kerr background from the Teukolsky equation [3][23]. Therefore, for $r \ll \mu^{-1}$, the mass term in the Proca equation can be neglected

[23] This is given by equation (2.21) in the section describing classical superradiance.

and the solutions to the full wave equation are well approximated to the solutions of the massless equations [15]. On the other hand, for $r \gg GM$, it has been shown above that the wave function is hydrogen-like when $GM\mu$ is small. Therefore, in the regime $GM \ll r \ll \mu^{-1}$, the hydrogen-like wavefunction can be approximated by a superposition of massless transverse and scalar modes. The energy flux across the black hole horizon can be then be calculated using the known behaviour of these massless modes.

Massless modes

As stated above, for $r \ll \mu^{-1}$, the mass term can be neglected meaning the Proca equation can be rewritten as,

$$\nabla_\mu F^{\mu\nu} = \nabla_\mu(\nabla^\mu A^\nu - \nabla^\nu A^\mu) = 0 \tag{2.150}$$

$$\nabla_\mu \nabla^\mu A^\nu - \nabla_\mu \nabla^\nu A^\mu = 0 \tag{2.151}$$

which yields

$$\nabla_\mu \nabla^\mu A^\nu = 0, \tag{2.152}$$

where the second term in equation (2.151) vanishes from the Lorentz condition. Putting equation (2.141) into equation (2.152) and using that the Ψ_i are separable yields,[24]

$$\nabla_\mu \nabla^\mu (R_{n\ell}(r) Y_i^{\ell,jm}(\theta,\varphi) e^{-i\omega t}) = 0. \tag{2.153}$$

Solving this yields the following differential equation for the radial part,

$$\frac{2rR'_{n\ell}(r) + r^2 R''_{n\ell}(r)}{R_{n\ell}} + r^2\omega^2 - j(j+1) = 0. \tag{2.154}$$

One linearly independent solution to equation (2.154) occurs when $j = \ell$ and is given by,

$$R_{n\ell}(r) \simeq j_j(\omega r), \tag{2.155}$$

[24] Omitting the complex conjugate term since it leads to the same result.

where $j_j(\omega r)$ is a spherical Bessel function of the first kind. Therefore one solution can be written as,

$$\Psi_i^B(r,\theta,\varphi) \sim j_j(\omega r) Y^{j,jm}(\theta,\varphi), \tag{2.156}$$

which agrees with [15]. The other linearly independent solution occurs when $j = \ell \pm 1$ which yields,

$$\Psi_i^E(r,\theta,\varphi) \sim j_{j-1}(\omega r) Y^{j-1,jm}(\theta,\varphi) - \sqrt{\frac{j}{j+1}} j_{j+1}(\omega r) Y^{j+1,jm}(\theta,\varphi), \tag{2.157}$$

matching [15]. Therefore, Ψ_i^B and Ψ_i^E form a basis for the transverse modes with frequency ω that are regular at the origin [15]. Near the origin, for $r \ll \omega^{-1}$, the spherical Bessel function $j_n(r\omega)$ becomes [15]

$$j_n(r\omega) \simeq \frac{1}{(2n+1)!!}(r\omega)^n + \mathcal{O}(r\omega)^{n+2}. \tag{2.158}$$

The longitudinal solutions of the flat-space wave equation are pure-gauge in the massless limit, meaning that, $-\partial_i A_0 - \partial_t A_i = 0$, and $\nabla \times \vec{A} = 0$ [15]. For modes $j = \ell = 1$, the longitudinal solutions which are regular at the origin are given by, [15]

$$\Psi_i^R \equiv \frac{\sqrt{j} j_{j-1}(\omega r) Y_i^{j-1,jm} + \sqrt{j+1} j_{j+1}(\omega r) Y_i^{j+1,jm}}{\sqrt{2j+1}}, \tag{2.159}$$

$$\Psi_0^R = j_j(\omega r) Y^{jm}. \tag{2.160}$$

These wavefunctions, along with the transverse modes Ψ^E and Ψ^B, form a complete basis for solutions that are regular on the origin to the flat-space wave equation for vanishing mass. We have considered these regular at the origin solutions because they correspond to an ingoing wave that passes through the origin undisturbed to then becoming an outgoing wave. At large radius, $r \gg \omega^{-1}$, we can decompose $j_n(\omega r)$ into ingoing and outgoing pieces [15],

$$j_n(r\omega) \simeq \frac{e^{i\omega r-(n+1)i\pi/2} + e^{-i\omega r+(n+1)i\pi/2}}{2\omega r}. \tag{2.161}$$

The $j = \ell$ hydrogenic bound states have the the following wavefunctions,

$$\Psi_i = R_{nj}(r) Y_i^{j,jm}(\theta, \varphi), \quad \Psi_0 = 0, \tag{2.162}$$

which matches the equation from [14] given in equation (2.98). The magnetic field is defined by the cross product $\mathbf{B}_H = \nabla \times \mathbf{\Psi}$, where this vector $\mathbf{\Psi}$ is given by equation (2.162). The plot of $\mathbf{B}$ in the yz-plane is shown in figure 2.1. In figure 2.2, the modulus squared of this

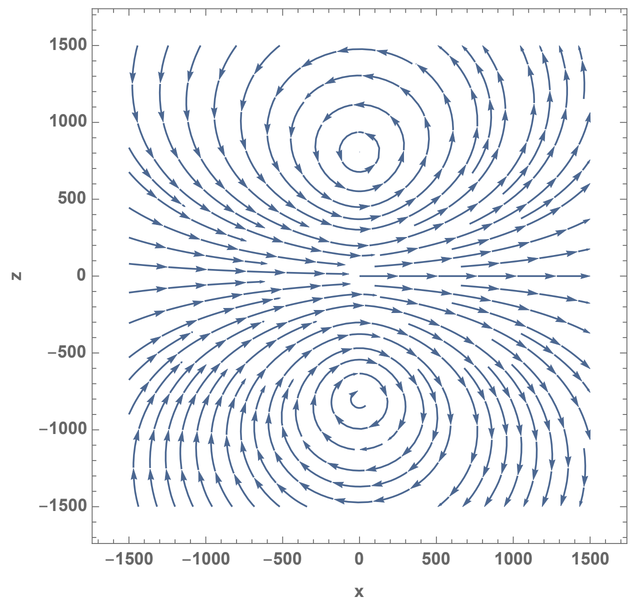

Figure 2.1: Stream vector plot of the vector field $\mathbf{B}_H = \nabla \times \mathbf{\Psi}$ in the $x - z$ plane at some early time t. This clearly represents a magnetic dipole field. Planck units have been used, i.e. $G = c = M = 1$ and so $\mu = 0.1$.

$\mathbf{B}_H$ is plotted relative to a monopole field for later comparison with the BZ solution. In the interval $GM \ll r \ll 1/\mu$, one can "match" this form onto the Ψ^B transverse solution with [15]

$$\Psi_i^H \simeq C_B j_j(\mu r) Y_i^{j,jm}, \quad C_B = \frac{R_0(2j+1)!!}{(\mu a)^j}. \tag{2.163}$$

Thus, close to the black hole, the wavefunction looks like a superposition of ingoing and outgoing waves, which is characteristic of a black hole event horizon. Thus the energy flux through the horizon is given by the energy flux in the ingoing wave multiplied by the black hole absorption probability given by [22]. Now, the absorption probability for a massless

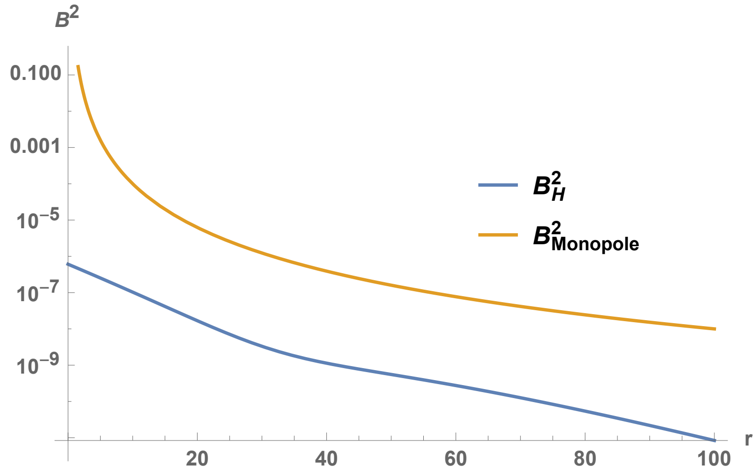

Figure 2.2: Logarithmic plot of the magnitude of the magnetic field squared, for the hydrogenic magnetic field ($\mathbf{B}_H = \nabla \times \mathbf{\Psi}$) against the monopole field $1/r^4$. It can be seen that the solutions have similar asymptotics for large values of r, however the hydrogenic magnetic field squared, B_H^2 decays exponentially unlike $B^2_{Monopole}$. There is an arbitrary normalization for the monopole field and the units of r are suppressed, since the aim is to compare the profiles.

spin-s wave carrying a total angular momentum j was computed in [22] and is given by

$$P_{\rm abs} = \begin{cases} \left(\frac{(j-s)!(j+s)!}{(2j)!(2j+1)!!}\right)^2 \prod_{n=1}^{j} \left[1 + \left(\frac{\omega - m\Omega}{n\kappa}\right)^2\right] 2\left(\frac{\omega - m\Omega}{\kappa}\right)\left(\frac{A_H \kappa}{2\pi}\omega\right)^{2j+1}, & (2s \text{ even}), \\ \left(\frac{(j-s)!(j+s)!}{(2j)!(2j+1)!!}\right)^2 \prod_{n=1}^{j+\frac{1}{2}} \left[1 + \left(\frac{\omega - m\Omega}{n\kappa - \frac{1}{2}\kappa}\right)^2\right] \left(\frac{A_H \kappa}{2\pi}\omega\right)^{2j+1}, & (2s \text{ odd}), \end{cases} \tag{2.164}$$

where A_H is the area of the black horizon, and $\kappa = 4\pi(r_+ - GM)/A_H$. The energy flux across the horizon for the bound state is, [15]

$$\frac{dE}{dt} \sim \frac{|C_B|^2 P_{{\rm abs},s=1,j}}{4\omega} \sim \frac{|C_B|^2 P_{{\rm abs},s=1,j}}{4\mu}, \tag{2.165}$$

where $\omega \simeq \mu$ (up to leading order in $GM\mu$) from equation (2.100). Calculating the analogous quantity for a scalar bound state, the energy flux is given by, [15]

$$\frac{dE}{dt} \sim \frac{|C_B|^2 P_{{\rm abs},s=0}}{4\mu}, \tag{2.166}$$

which is consistent with the decay rate is given by, [15]

$$\Delta\Gamma_{j=\ell} = \frac{P_{\text{abs},s=1,j}}{P_{\text{abs},s=0,j}}\Delta\Gamma_{\text{scalar},j} = \frac{(j+1)^2}{j^2}\Delta\Gamma_{\text{scalar},j} \tag{2.167}$$

$$= \frac{1}{6}a(GM\mu)^7\mu. \tag{2.168}$$

Their result agrees with the expression derived in [24] and is accurate to leading order in $GM\mu$ for any a/GM. Pani et al. [24] gave a more general result for $\Delta\Gamma_{s=j=1}$, which is given by

$$\Delta\Gamma_{s=m=j=1} = \frac{1}{12}\left(\frac{a}{(GM)^2} - \frac{2r_+\mu}{GM}\right)(GM\mu)^9. \tag{2.169}$$

In [15], the energy flux through the horizon can be identified as,

$$\left\langle\frac{dE}{dt}\right\rangle = \dot{E} \simeq \Delta\Gamma\mu, \tag{2.170}$$

where $\Delta\Gamma$ is the total decay rate for the bound state.

2.4.4 Comparison of various results

In the literature, there are many papers that have considered the growth and decay rates of vector bound states around black holes. Rosa and Dolan in [23] were the first to notice the scaling $\Delta\Gamma \sim (GM\mu)^{5+2j+2\ell}$. They performed numerical calculations for bound state decays around a Schwarzschild black hole where the Proca equation is simpler and can be separated. The same decay rate was also found in Endlich and Penco [14], and is given in equation (2.140). In Cardoso et al. [25], they provided a nice summary of the different coefficients that have been computed for different values of j and ℓ but only the $j = \ell = 1$ mode is mentioned here,

$$\Delta\Gamma_{PCGBI} = \frac{1}{12}a(GM)^7\mu^9, \quad \Delta\Gamma_{BLT} \simeq \frac{1}{6}a(GM)^7\mu^9, \quad \Delta\Gamma_{EP} \simeq \frac{1}{3}a(GM)^7\mu^9, \tag{2.171}$$

where $\Delta\Gamma_{PCGBI}$ denotes the rate from [24], $\Delta\Gamma_{BLT}$ denotes the rate from [15] and $\Delta\Gamma_{EP}$ denotes the one from [14] given by equation (2.139).

It is interesting to note that each of these results is at least a factor of 2 apart from one another. The reasoning behind the slight discrepancy of the coefficient between the three results is not entirely clear. One possibility could be from the fact that each group used different methods for performing the matching calculation. In [15], they go through some possible reasons for why the scaling is correct with [14] but the coefficients are not. They argued that one possibility could be that there appears to be no choice for the operator coefficients in the assumed interaction Hamiltonian of [14]. This could lead to correct scaling of growth rates and amplification factors as a function of $GM\mu$ for all the bound and scattering states.

Chapter 3

The Blandford-Znajek Process

This chapter studies energy extraction from black holes, specifically in the presence of large magnetic fields. Our particular interest is in a more modern approach to studying this system using effective field theory techniques.

3.1 Force-Free Electrodynamics

In 1977, Blandford and Znajek [2] developed a mechanism, which is still believed to power the jets of active galactic nuclei and black holes. It involves a rotating black hole, and currents in the accretion disc source the force-free magnetosphere around it. The rotation of the black hole induces an electric field near the black hole, since $\mathbf{E} \cdot \mathbf{B} \neq 0$ [26; 27]. This causes stray charged particles to accelerate along the magnetic field lines and these charged particles will radiate [5; 2]. This radiation in turn produces a cascade of new particles in the form of electron positron pairs [28; 2]. When charges are being created in such abundance, the magnetospheric plasma can support strong electric currents and screen the electric field [27; 2] - in equations this is $\mathbf{E} \cdot \mathbf{B} = 0$ with $B^2 > E^2$. These charged particles eventually diffuse in such a way that they do not exchange energy and momentum with the fields - i.e. $\mathbf{j} \cdot \mathbf{E} = 0$ and $\rho\mathbf{E} + \mathbf{j} \times \mathbf{B} = 0$. Therefore, this is called a force-free process due to the fact that the Lorentz force $F_{\mu\nu}J^{\nu}$ vanishes. The electromagnetic field tensor $F_{\mu\nu}$ is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, [7], and the energy-momentum tensor for electromagnetism written in terms of $F^{\mu\nu}$ is given by [8],

$$T^{\mu\nu}_{EM} = F^{\mu\gamma}F^{\nu}_{\gamma} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}. \tag{3.1}$$

The total energy-momentum tensor is composed of an electromagnetic part and a matter part [29],

$$T^{\mu\nu}_{total} = T^{\mu\nu}_{EM} + T^{\mu\nu}_{matter}, \tag{3.2}$$

but in the force-free approximation near a black hole, the contribution from the matter is assumed to be negligible [8] so

$$T^{\mu\nu}_{total} \approx T^{\mu\nu}_{EM}, \tag{3.3}$$

and from this point forward $T^{\mu\nu}$ will mean $T^{\mu\nu}_{EM}$. For a force-free magnetized plasma, the equations of motion are [8]

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\nu = 0. \tag{3.4}$$

Finally, along with the usual even and odd permutation rules, the following conventions for $\epsilon^{\mu\nu\rho\sigma}$ will be given as [7],

$$\epsilon^{\mu\nu\rho\sigma} = -\frac{1}{\sqrt{-g}}\tilde{\epsilon}^{\mu\nu\rho\sigma}, \quad \text{and} \quad \epsilon_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-g}}\tilde{\epsilon}_{\mu\nu\rho\sigma}, \tag{3.5}$$

where

$$\tilde{\epsilon}_{\mu\nu\rho\sigma} = \begin{cases} +1, & \text{if } \mu\nu\rho\sigma \text{ is an even permutation,} \\ -1, & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

3.2 Blandford-Znajek Process

The following derivation will follow both Grignani et al. [30] and McKinney and Gammie [8]. The equations for force-free electrodynamics are given by the following [30; 31],

$$\nabla_\mu F^{\mu\nu} = J^\nu, \tag{3.7}$$

$$\nabla_{[\mu} F_{\nu\rho]} = 0, \tag{3.8}$$

$$F_{\mu\nu} J^\nu = \nabla^\nu T_{\mu\nu} = 0, \quad J^\nu \neq 0, \tag{3.9}$$

where A_μ is the gauge potential, $F_{\mu\nu}$ is the electromagnetic field strength and J^ν is the electric current. Equations (3.7) and (3.8) are simply Maxwell's equations for classical charged matter, and equation (3.9) is the force-free constraint, and ensures that the current is non-zero. The force-free equations also imply that $\mathbf{E} \cdot \mathbf{B} = 0$. Equation (3.9) describes the energy and momentum exchange between the matter and the electromagnetic field via the Lorentz force [31]. By acting $F^{\mu\nu}$ on equation (3.7), one obtains,

$$F_{\rho\nu} \nabla_\mu F^{\mu\nu} = F_{\rho\nu} J^\nu = 0, \tag{3.10}$$

which means that there is a way to eliminate the current from the force-free equation. The force-free condition, $F_{\mu\nu} J^\nu = 0$ is valid everywhere except at $\theta = \pi/2$ where the equatorial disc is located [2]. These equations are ordinary Maxwell's equations along with the requirement that the electromagnetic energy-momentum tensor be conserved and that the electric current is non-zero. The magnetic source in question is the actual accretion disc that surrounding the black hole. Assuming that the field is stationary and axisymmetric, then the components of $F_{\mu\nu}$ are given by

$$\begin{aligned} F_{tr} &= -\partial_r A_t \\ F_{t\theta} &= -\partial_\theta A_t \\ F_{t\varphi} &= 0 \\ F_{r\theta} &= \partial_r A_\theta - \partial_\theta A_r \\ F_{r\varphi} &= \partial_r A_\varphi \\ F_{\theta\varphi} &= \partial_\theta A_\varphi, \end{aligned} \tag{3.11}$$

where $F_{\mu\nu} = -F_{\nu\mu}$ since it is an antisymmetric tensor. It is easy to see that equation (3.10) must be satisfied,

$$F_{\rho\nu} \nabla_\mu F^{\mu\nu} = \frac{F_{\rho\nu}}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0. \tag{3.12}$$

This equation leads to a system of four equations each for $\mu = \{t, r, \theta, \varphi\}$. First for $\mu = t$,

$$F_{t\nu} \partial_\rho (\sqrt{-g} F^{\rho\nu}) = 0 \tag{3.13}$$

$$F_{tr} \partial_\rho (\sqrt{-g} F^{\rho r}) + F_{t\theta} \partial_\rho (\sqrt{-g} F^{\rho\theta}) + F_{t\varphi} \partial_\rho (\sqrt{-g} F^{\rho\varphi}) = 0 \tag{3.14}$$

$$F_{tr} \partial_\rho (\sqrt{-g} F^{\rho r}) + F_{t\theta} \partial_\rho (\sqrt{-g} F^{\rho\theta}) = 0, \tag{3.15}$$

where equation (3.11) has been used. The $\mu = \varphi$ equation follows the same way,

$$F_{\varphi r}\partial_\rho(\sqrt{-g}F^{\rho r}) + F_{\varphi\theta}\partial_\rho(\sqrt{-g}F^{\rho\theta}) = 0. \tag{3.16}$$

Now, since $F^{\mu\nu} = F^{\mu\nu}(r,\theta)$, equations (3.15) and (3.16) become,

$$F_{tr}\partial_\theta(\sqrt{-g}F^{\theta r}) + F_{t\theta}\partial_r(\sqrt{-g}F^{r\theta}) = 0 \tag{3.17}$$

$$F_{\varphi r}\partial_\theta(\sqrt{-g}F^{\theta r}) + F_{\varphi\theta}\partial_r(\sqrt{-g}F^{r\theta}) = 0, \tag{3.18}$$

and setting $X = \sqrt{-g}F^{r\theta}$, one can rewrite the above two relations as

$$F_{tr}\partial_\theta X = F_{t\theta}\partial_r X, \tag{3.19}$$

$$F_{\varphi r}\partial_\theta X = F_{\varphi\theta}\partial_r X. \tag{3.20}$$

From the above two equations, it is easy to see the following relation,

$$\frac{F_{tr}}{F_{\varphi r}} = \frac{F_{t\theta}}{F_{\varphi\theta}}. \tag{3.21}$$

Using the value of $F_{\mu\nu}$ given in equations (3.11) and (3.21), equation (3.21) can be rewritten as,

$$\frac{\partial_\theta A_t}{\partial_\theta A_\varphi} = \frac{\partial_r A_t}{\partial_r A_\varphi} \equiv -\omega(r,\theta), \tag{3.22}$$

where the function $\omega(r,\theta)$ is some arbitrary function which is usually interpreted as the rotation frequency of the electromagnetic field - which matches the known result in the literature [2; 8; 30]. Using equations (3.21) and (3.22), one obtains the following two key relations,

$$\partial_r A_t = -\omega\partial_r A_\varphi, \quad \partial_\theta A_t = -\omega\partial_\theta A_\varphi. \tag{3.23}$$

By applying ∂_θ to both sides of the first equation and ∂_r to both sides of the second equation, one obtains,

$$\partial_r\omega\partial_\theta A_\varphi = \partial_\theta\omega\partial_r A_\varphi, \tag{3.24}$$

which directly implies that ω is a function of A_φ. Thus equation (3.24) is an integrability condition for ω [30]. One can define the following four quantities,

$$I = \sqrt{-g}F^{\theta r}, \quad B^r = \frac{1}{\sqrt{-g}}F_{\theta\varphi}, \quad B^\theta = -\frac{1}{\sqrt{-g}}F_{r\varphi}, \quad B^\varphi = \frac{1}{\sqrt{-g}}F_{r\theta}, \tag{3.25}$$

where I is the total electric current flowing upward through the (r, θ) loop and B^φ is the toroidal magnetic field. Now looking at $\mu = r$,

$$F_{r\nu}\partial_\rho(\sqrt{-g}F^{\rho\nu}) = 0 \tag{3.26}$$

$$F_{rt}\partial_\rho(\sqrt{-g}F^{\rho t}) + F_{r\theta}\partial_\rho(\sqrt{-g}F^{\rho\theta}) + F_{r\varphi}\partial_\rho(\sqrt{-g}F^{\rho\varphi}) = 0, \tag{3.27}$$

and $\mu = \theta$,

$$F_{\theta\nu}\partial_\rho(\sqrt{-g}F^{\rho\nu}) = 0 \tag{3.28}$$

$$F_{\theta t}\partial_\rho(\sqrt{-g}F^{\rho t}) - F_{r\theta}\partial_\rho(\sqrt{-g}F^{\rho\theta}) + F_{\theta\varphi}\partial_\rho(\sqrt{-g}F^{\rho\varphi}) = 0. \tag{3.29}$$

Following the same procedure as before, and using equation (3.23) one can rewrite these equations as,

$$-\omega\partial_\mu(\sqrt{-g}F^{\mu t}) + \partial_\mu(\sqrt{-g}F^{\varphi\mu}) + F_{r\theta}\frac{dI}{dA_\varphi} = 0, \tag{3.30}$$

with the following integrability equation for I,

$$\partial_r I \partial_\theta A_\varphi = \partial_r A_\varphi \partial_\theta I, \tag{3.31}$$

which directly matches the result from [30]. Using equation (3.25) one can find an expression for B^φ,

$$I = \sqrt{-g}F^{\theta r} = \Sigma^2 \sin^2\theta \left(\frac{\Delta}{\Sigma^2}(\sqrt{-g}B^\varphi) + \frac{2GMr}{\Sigma^2}F_{t\theta} + \frac{a}{\Sigma^2}F_{\varphi\theta} \right) \tag{3.32}$$

$$= \Delta \sin^2\theta B^\varphi - \frac{2GMr}{\Sigma}\sin\theta\partial_\theta A_t - \frac{a}{\Sigma}\sin\theta\partial_\theta A_\varphi. \tag{3.33}$$

Rearranging for B^φ,

$$B^\varphi = \frac{-I\Sigma - 2GMr\sin\theta(\omega\partial_\theta A_\varphi) - a\sin\theta\partial_\theta A_\varphi}{\Delta\Sigma\sin^2\theta} \tag{3.34}$$

$$= -\frac{I\Sigma + (2GMr\omega - a)\sin\theta\partial_\theta A_\varphi}{\Delta\Sigma\sin^2\theta}. \tag{3.35}$$

The radial energy flux as in [8] is computed by,

$$\dot{E} \equiv 2\pi\int_0^\pi d\theta\sqrt{-g}F_E, \tag{3.36}$$

where $F_E \equiv -T^r_t$, with the stress-energy tensor given by

$$T^{\mu\nu} = F^{\mu\gamma}F^\nu_\gamma - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}. \tag{3.37}$$

Now, rewriting T^μ_ν with equation (3.37),

$$T^\mu_\nu = F^{\mu\gamma}F_{\nu\gamma} - \frac{1}{4}g^\mu_\nu F^{\alpha\beta}F_{\alpha\beta} \tag{3.38}$$

$$= g^{\mu\rho}F_\rho{}^\gamma F_{\nu\gamma} - \frac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta}. \tag{3.39}$$

Using the definition for the stress-energy tensor given by equation (3.39) and the results for the components of $F_{\mu\nu}$ from equation (3.11),

$$T^r_t = g^{rr}g^{\theta\theta}F_{r\theta}F_{t\theta} + g^{rt}g^{\theta\theta}F_{t\theta}F_{t\theta} + g^{r\varphi}g^{\theta\theta}F_{\varphi\theta}F_{t\theta} \tag{3.40}$$

$$= (g^{rr}F_{r\theta} + g^{rt}F_{t\theta} + g^{r\varphi}F_{\varphi\theta})g^{\theta\theta}F_{t\theta} \tag{3.41}$$

$$= \left(\frac{\Delta}{\Sigma}\sqrt{-g}B^\varphi + \frac{2GMr}{\Sigma}\omega\sqrt{-g}B^r - \frac{a}{\Sigma}\sqrt{-g}B^r\right)\frac{1}{\Sigma}\omega\sqrt{-g}B^r \tag{3.42}$$

$$= -(\Delta B^\varphi + (2GMr\omega - a)B^r)\frac{g\omega}{\Sigma^2}B^r. \tag{3.43}$$

Recall that the metric determinant is $g = -\Sigma^2\sin^2\theta$, and multiplying the last expression by -1 in order to obtain an expression for $-T^r_t$,

$$-T^r_t = -(2GMr\omega - a)\omega\sin^2\theta(B^r)^2 - \Delta\omega\sin^2\theta B^r B^\varphi \tag{3.44}$$

$$= -2GMr\left(\omega - \frac{a}{2GMr}\right)\omega\sin^2\theta(B^r)^2 - \Delta\omega\sin^2\theta B^r B^\varphi \tag{3.45}$$

$$= -2GM(B^r)^2\omega r\left(\omega - \frac{a}{2GMr}\right)\sin^2\theta - B^r B^\varphi\omega\Delta\sin^2\theta. \tag{3.46}$$

On the horizon, $r = r_+ = GM + \sqrt{(GM)^2 - a^2}$ and $\Delta = 0$, and so the energy flux on the horizon is

$$F_E^{(EM)}|_{r=r_+} = 2GM(B^r)^2\omega r_+(\Omega_H - \omega)\sin^2\theta, \tag{3.47}$$

where $\Omega_H \equiv \frac{a}{2GMr_+} = \frac{a}{(2GM)^2}$ is the rotation frequency of the black hole. This result is equivalent to BZ [2] and still implies that if $0 < \omega < \Omega_H$ and $(B^r)^2 > 0$, then there will be an energy flux directed outwards from the horizon. Since the energy flux has been evaluated in KS coordinates, which are regular on the horizon, there was no need to do any additional work as in [32]. To finish evaluating the energy flux $\dot{E}$ as given in equation (3.36), the solutions for A_φ, ω, and B^φ (or I) need to be determined. Although this can not be solved in general, Blandford and Znajek [2][1] applied the method of a perturbative expansion in powers of $\alpha = \frac{a}{GM}$ in the limit of slow-rotation, when $\alpha \ll 1$. Some other useful forms of α are: $\alpha = \frac{J}{GM^2} = \frac{2a}{r_0}$, where $r_0 = 2GM$. Essentially this will involve finding A_φ, ω, and I which solves equation (3.30), while satisfying the integrability conditions of equations (3.24) and (3.31). It is also necessary that A_φ and B^φ are regular on the event horizon, $r = r_+$.

3.2.1 Perturbation expansion in slow rotation limit

One possible (and simplest) form of the solution is that of a (split-)monopole field. Recall that the physical configuration involves a uniform magnetic field with a rotating black hole at the centre. In order to make the solution analytically tractable, one must approximate the physical configuration. A slowly rotating black hole is essentially spherically symmetric since it resembles a Schwarzschild black hole when $\alpha \ll 1$. However, the uniform magnetic field breaks this symmetry and makes it axisymmetric. Therefore by modifying the uniform field by bending it in the middle, it resembles a monopole but with the lower hemisphere flipped so there is uniform direction in the field lines. Then immediately $\omega = I = 0$, since the background and the force-free fields are not rotating, by defintion. Since $A_\varphi(r,\theta)$ is related to the magnetic field lines, the perturbative expansion will be in even powers of α since the sign of the angular momentum does not change the shape of the field lines. However ω, B^φ and I must be in odd powers of α since a reversal in the rotation direction will change the sign of these quantities. This means that the expansions of ω, B^φ and I must start at order

[1] As well as many others, i.e. [8], [30], [29] to name a few.

α. Thus the expansion of the four fields is given by the following,[2]

$$
\begin{aligned}
A_\varphi(r,\theta) &= A_\varphi^{(0)} + \alpha^2 A_\varphi^{(2)} + \mathcal{O}(\alpha^4),\\
r_0\omega(r,\theta) &= \alpha\omega^{(1)} + \alpha^3\omega^{(3)} + \mathcal{O}(\alpha^5),\\
r_0 I(r,\theta) &= C(\alpha I^{(1)} + \alpha^3 I^{(3)} + \mathcal{O}(\alpha^5)),\\
B^\varphi(r,\theta) &= \alpha B^{\varphi(1)} + \alpha^3 B^{\varphi(3)} + \mathcal{O}(\alpha^5),
\end{aligned}
\tag{3.48}
$$

where $A_\varphi^{(m)}, \omega^{(m)}$ and $I^{(m)}$ are all functions of r and θ[3]. The next thing to do is to expand equation (3.30) in terms of the expansions given by equation (3.48). Schematically this gives the following relation to be satisfied [2]

$$
\mathcal{L}A_\varphi^{(2m)}(r,\theta) = S_{(2m)}(r,\theta), \tag{3.49}
$$

where $m = 0, 1, 2\ldots$, and $\mathcal{L}$ is the following differential operator

$$
\mathcal{L} \equiv \frac{1}{\sin\theta}\partial_r\left(1 - \frac{r_0}{r}\right)\partial_r + \frac{1}{r^2}\partial_\theta\frac{1}{\sin\theta}\partial_\theta, \tag{3.50}
$$

and $S_{(2m)}$ is the source-term with $S_0 = 0$ [2]. The solution of interest is the static (split-)monopole [2],

$$
A_\varphi^{(0)} = -C\cos\theta, \tag{3.51}
$$

where C is an arbitrary constant. Inserting the solution for $A_\varphi^{(0)}$ into equation (3.24) gives (up to order $\mathcal{O}(\alpha)$),

$$
(\partial_r\omega^{(1)})\sin\theta = \partial_\theta\omega^{(1)}(0), \tag{3.52}
$$

which implies that $\partial_\theta\omega^{(1)} \neq 0$ and $\partial_r\omega^{(1)} = 0$, thus $\omega^{(1)} \equiv \omega^{(1)}(\theta)$ only. The same argument can be made for $I^{(1)}$ using equation (3.31) and so $I^{(1)} \equiv I^{(1)}(\theta)$. Therefore the first order monopole solution is

$$
\begin{aligned}
A_\varphi^{(0)} &= -C\cos\theta,\\
\omega^{(0)} &= I^{(0)} = B^{\varphi(0)} = 0.
\end{aligned}
\tag{3.53}
$$

[2]The value C given in the equation for $I(r,\theta)$ is simply an arbitrary constant found in the monopole solution, which will be seen below.

[3] Equation (3.24) and equation (3.31) should be imposed order by order to ensure that $\omega = \omega(A_\varphi)$ and $I = I(A_\varphi)$.

The next step is to find the solution to order α.

$$B^{\varphi} = \alpha B^{\varphi(1)} = -\left[\frac{I\Sigma + (2GMr\omega - a)\sin\theta\partial_{\theta}A_{\varphi}}{\Delta\Sigma\sin^2\theta}\right], \tag{3.54}$$

where to order α, $I = \frac{C\alpha}{2GM}I^{(1)}(\theta)$, $\omega = \frac{\alpha}{2GM}\omega^{(1)}(\theta)$, $A_{\varphi} = -C\cos\theta$, $\Sigma = r^2$, and $\Delta = r^2 - 2GMr$. Putting these relations in yields,

$$B^{\varphi(1)} = -\left(\frac{\frac{C\alpha}{2GM}I^{(1)}(\theta)r^2 + \left(\alpha r\omega^{(1)} - GM\alpha\right)\sin\theta(C\sin\theta)}{r^2(r^2 - 2GMr)\sin^2\theta}\right) \tag{3.55}$$

$$= -\left(\frac{C\frac{r^2 I^{(1)}}{2GM\sin^2\theta} + C(r\omega^{(1)} - GM)}{r^3(r - 2GM)}\right) \tag{3.56}$$

$$= -C\left(\frac{\frac{2r^2 I^{(1)}}{2GM\sin^2\theta} + 2r\omega^{(1)} - 2GM}{2r^3(r - 2GM)}\right) \tag{3.57}$$

$$= C\left(\frac{r_0 - 2r\omega^{(1)}(\theta) - \frac{2r^2 I^{(1)}(\theta)}{r_0\sin^2\theta}}{2r^3(r - r_0)}\right), \tag{3.58}$$

which agrees with [30] up to an arbitrary constant. The next requirement is that B^{φ} must be regular on the event horizon, which requires equating the numerator of equation (3.58) equal to zero while setting $r = r_0$,[4]

$$\left.\left(r_0 - 2r\omega^{(1)}(\theta) - \frac{2r^2}{r_0\sin^2\theta}I^{(1)}(\theta)\right)\right|_{r=r_0} = 0, \tag{3.59}$$

and solving this equation for $I^{(1)}$ yields

$$I^{(1)}(\theta) = \frac{r_0 - 2r_0\omega^{(1)}(\theta)}{2r_0}\sin^2\theta, \tag{3.60}$$

$$= \frac{1 - 2\omega^{(1)}(\theta)}{2}\sin^2\theta, \tag{3.61}$$

which agrees with [30], and further yields the following

$$B^{\varphi(1)} = -C\left(\frac{1 - 2\omega^{(1)}(\theta)}{2r_0 r^2} + \frac{1}{2r^3}\right). \tag{3.62}$$

[4] Recall that for a Schwarzschild black hole $r_+ = r_0 = 2GM$.

To second order in α, there is a non-zero source term S_2 which depends on $\omega^{(1)}(\theta)$ and the dominant terms for $r \to \infty$ are

$$S_2 = \frac{4\omega^{(1)} - 1}{2r_0^2} \sin\theta\cos\theta + \frac{\frac{d}{d\theta}\omega^{(1)}}{2r_0^2} \sin^2\theta + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{3.63}$$

Imposing that $S_2 \to 0$ as $r \to \infty$ sets

$$\omega^{(1)} = \frac{1}{4}, \quad I^{(1)} = \frac{1}{4}\sin^2\theta. \tag{3.64}$$

To produce the largest power output, Macdonald and Thorne analytically [33] showed that $\omega \simeq \frac{1}{2}\Omega_H$, so $\Omega_H^{(1)} = 2\omega^{(1)}$.[5] Thus, equation (3.50) becomes

$$\mathcal{L}A_\varphi^{(2)} = -\frac{r_0}{2r^3}\left(1 + \frac{r_0}{r}\right)\sin\theta\cos\theta, \tag{3.65}$$

which is solved by

$$A_\varphi^{(2)} = Cf(r)\sin^2\theta\cos\theta, \tag{3.66}$$

and

$$\begin{aligned} f(r) = &\frac{r_0^2 + 6r_0 r - 24r^2}{12r_0^2}\ln\left(\frac{r}{r_0}\right) + \frac{11}{72} + \frac{r_0}{6r} + \frac{r}{r_0} - \frac{2r^2}{r_0^2} \\ &+ \left(\mathrm{Li}_2\left(\frac{r_0}{r}\right) - \ln\left(\frac{r}{r_0}\right)\ln\left(1 - \frac{r_0}{r}\right)\right)\frac{r^2(4r - 3r_0)}{2r_0^3}, \end{aligned} \tag{3.67}$$

where

$$\mathrm{Li}_2(x) = -\int_0^1 dt\frac{\ln(1 - tx)}{t}. \tag{3.68}$$

At the event horizon, this solution is indeed finite. In the asymptotic limit $r \to \infty$, the solution becomes

$$A_\varphi^{(2)} = C\frac{GM}{4r}\sin^2\theta\cos\theta + \mathcal{O}\left(\frac{(GM)^2}{r^2}\ln\frac{r}{2GM}\right), \tag{3.69}$$

which agrees with the result found by BZ [2] and McKinney and Gammie [8]. In [2] it was also concluded that the solution matches the flat-space rotating Michel monopole [4] up to second order in the spin parameter α at large radius.

[5] This is something that will be better motivated in Chapter 4 when the Blandford-Znajek process is looked at in the language of differential forms.

The last computation to do will be to evaluate the total energy flux of the electromagnetic field $\dot{E}$ using the quantities found for $\omega^{(1)}$ and $B^{\varphi(1)}$. First, computing F_E^{EM},

$$F_E^{EM} = (2GM(B^r)\omega(\Omega_H - \omega)r\sin^2\theta)|_{r=r_+} \tag{3.70}$$

$$= \left(\frac{2GMC^2}{r^4}\frac{\Omega_H}{2}\left(\Omega_H - \frac{\Omega_H}{2}\right)r\sin^2\theta\right)\bigg|_{r=r_+} \tag{3.71}$$

$$= \left(\frac{2GMC^2}{r^4}\left(\frac{\Omega_H}{2}\right)^2 r\sin^2\theta\right)\bigg|_{r=r_+}. \tag{3.72}$$

Putting this into equation (3.36) yields,

$$\dot{E} = 2\pi\int_0^\pi d\theta(r^2\sin\theta)|_{r=r_+}\left(\frac{2GMC^2}{r^3}\left(\frac{\Omega_H}{2}\right)^2\sin^2\theta\right)\bigg|_{r=r_+} \tag{3.73}$$

$$= GM\pi C^2\Omega_H^2\int_0^\pi d\theta\,\frac{\sin^3\theta}{r}\bigg|_{r=r_+} \tag{3.74}$$

$$= \frac{GM\pi C^2\Omega_H^2}{r_+}\int_0^\pi d\theta\sin^3\theta \tag{3.75}$$

$$= \frac{2\pi C^2\Omega_H^2}{3}. \tag{3.76}$$

Finally using that $\Omega_H = 2\omega$,

$$\dot{E} = \frac{2\pi}{3}a^2(B^r)^2, \tag{3.77}$$

which matches the result given in [29].

3.2.2 BZ summary

In Kerr-Schild coordinates, the magnetic field components are

$$B^r = \frac{C}{r^2} + a^2\frac{C}{2r^4}[-2\cos^2\theta + \frac{r^2}{(GM)^2}(1+3\cos 2\theta)f(r)], \tag{3.78}$$

$$B^\theta = -a^2\frac{C}{(GM)^2r^2}\cos\theta\sin\theta f'(r), \tag{3.79}$$

$$B^\varphi = -a\frac{C}{8r^2}\left(\frac{1}{GM} + \frac{4}{r}\right), \tag{3.80}$$

where all terms are accurate to second order in a.

3.3 Absolute-Space/Universal-Time Viewpoint

In [34], Thorne and Macdonald reformulated curved-spacetime electrodynamics into a 3+1 (space + time) split, where the three-dimensional vectors are defined to lie in hypersurfaces of constant time t. Their prescription involves rewriting the curved-spacetime as follows:

1. At each event in the spacetime, choose a fiducial reference frame[6], and therefore split the spacetime into three space components and one uniquely chosen time τ. This time τ[7] can be thought of as a Galilean-type time coordinate, which ticks away in this absolute space [35].

2. In the frame of the FIDO, the electromagnetic field tensor $\mathbf{F}$ is split into electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, where again these fields are measured in terms of the proper time τ, and the electric field is given by the usual time-space part of $\mathbf{F}$ and the magnetic field is given by the space-space part [35].

3. Again, in the frame of the FIDO, split the four-current vector j into a time part ρ_e, given by the charge density and a three component space part $\mathbf{j}$, which is the current density.

4. Finally, using $\mathbf{E}, \mathbf{B}, \rho_e$ and $\mathbf{j}$ rewrite Maxwell's equations in curved-spacetime, the Lorentz force law and the law of charge conservation [34].

[6]It is useful to think of this absolute space being filled with fiducial observers (FIDOs). At each point (r, θ, φ) there is a FIDO taking measurements of the physical processes in their neighbourhood [35].

[7]The relationship between the FIDO's proper time τ and the universal time t is given by $\frac{d\tau}{dt} = \alpha = (-g_{00})^{1/2}$.

These equations in both differential and integral form are summarized below[8] [34]

$$\nabla \cdot \mathbf{B} = 0, \tag{3.81}$$

$$\nabla \cdot E = 4\pi\rho_e, \tag{3.82}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\alpha \mathbf{E}) \tag{3.83}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla \times (\alpha \mathbf{B}) - 4\pi\alpha\mathbf{j}, \tag{3.84}$$

$$\frac{\partial \rho_e}{\partial t} = -\nabla \cdot (\alpha \mathbf{j}), \tag{3.85}$$

$$\oint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0 \tag{3.86}$$

$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = 4\pi \int_V \rho_e dV \tag{3.87}$$

$$\frac{d}{dt}\int_A \mathbf{B} \cdot d\mathbf{A} = -\oint_{\partial A} \alpha \mathbf{E} \cdot d\mathbf{l} \tag{3.88}$$

$$\frac{d}{dt}\int_A \mathbf{E} \cdot d\mathbf{A} = -\oint_{\partial A} \alpha \mathbf{B} \cdot d\mathbf{l} - 4\pi \int_A \alpha \mathbf{j} \cdot d\mathbf{A}, \tag{3.89}$$

$$\frac{d}{dt}\int_V \rho_e dV = -\oint_{\partial V} \alpha \mathbf{j} \cdot d\mathbf{A}, \tag{3.90}$$

where ∂V is the boundary of the volume V, and ∂A is the curve bounding the area A. The usual right hand rules on the orientations of $d\mathbf{l}, d\mathbf{A}$ and dV apply. The volume V, the area A and the boundary ∂A are all assumed to be at rest in absolute space meaning that they are at rest with respect to the FIDOs and (r, θ, φ) [35].

Electric and magnetic field lines

Maxwells equations given by equations (3.81) and (3.82) demonstrate that there are no magnetic monopoles, while the electric field has non-zero divergence in the presence of charge, meaning that electric charges exist. These equations show nothing new. Magnetic field lines are closed curves and the electric field lines must begin and end on charges.

Voltage drops and EMFs around closed curves

Faraday's law, equation (3.88) demonstrates that a time-changing magnetic field will produce an electromotive force (EMF) around a closed curve. This is just as in flat space-time, except that $\alpha\mathbf{E}$ enters the definition of the EMF, meaning that the electric field drives the

[8] These relations are written in Gaussian units.

motion of charges as measured locally by FIDOs [35].

Consider a circuit composed of a long superconducting wire, part near the horizon and part far away with various circuit components (resistors, inductors, batteries, etc.) at certain points. Assuming that the circuit is in a steady state, then the integral law of charge conservation, given by equation (3.90) defines the current on a "per-unit-universal-time" basis [35],

$$I \equiv \frac{dQ}{dt} = \int \alpha \mathbf{j} \cdot d\mathbf{A}, \tag{3.91}$$

where Q is the charge and this integral is circuit independent. The resistance is a local quantity and is given by

$$R \equiv \frac{\int \mathbf{E} \cdot d\mathbf{l}}{dQ/d\tau}, \tag{3.92}$$

and since $d\tau = \alpha dt$, the voltage drop across the resistor is given by

$$\Delta V \equiv \int \alpha \mathbf{E} \cdot d\mathbf{l} = \alpha \frac{dQ}{d\tau} R = IR, \tag{3.93}$$

which is just the familiar Ohm's law.

Frozen boundary layer

Far away from the horizon, the absolute time t describes the evolution of the fields well. However, close to the horizon it leads to a distorted picture. If the electromagnetic field is dynamical, the fields near the horizon will form a layered structure encoding and storing their former evolutionary history [36]. Thorne and Macdonald [34] first proposed a way to correct for the sluggishness of the near-horizon fields, but Macdonald and Suen fully developed how this is done in [36]. Their method consists of defining some interesting boundary conditions on a closed two-dimensional surface just outside the horizon, and applying the boundary conditions on this surface rather than the event horizon. This surface is referred to as the stretched horizon, and for mathematical convenience it is assumed to be at some fixed value of $\alpha = \alpha_H \ll 1$. It is also chosen such that no interesting physics occurs between the true event horizon and the stretched horizon.

Electrical properties of the stretched horizon membrane

It is convenient to express the electromagnetic fields as "renormalized" parallel fields on the stretched horizon [36]

$$\mathbf{E}_H \equiv (\alpha \mathbf{E}_\parallel)|_{\alpha=\alpha_H}, \quad \mathbf{B}_H = (\alpha \mathbf{B}_\parallel)|_{\alpha=\alpha_H}. \tag{3.94}$$

These non-divergent fields are useful since they are independent of the location chosen for α_H. In fact, these horizon fields live in, and are tangent to the two-dimensional stretched horizon [35]. The normal electric and magnetic fields given by

$$E_n \equiv \mathbf{E} \cdot \hat{\mathbf{n}}, \quad B_n \equiv \mathbf{B} \cdot \hat{\mathbf{n}}, \tag{3.95}$$

where $\hat{\mathbf{n}}$ is the unit radial vector, along with equation (3.94) completely describe the electromagnetic fields at the stretched horizon [35]. From [35], the electromagnetic boundary conditions at the stretched horizon are given by

$$E_n, B_n, \mathbf{E}_H, \text{ and } \mathbf{B}_H \text{ all finite}, \tag{3.96}$$

$$\mathbf{E}_H = \hat{\mathbf{n}} \times \mathbf{B}_H, \quad \mathbf{B}_H = -\hat{\mathbf{n}} \times \mathbf{E}_H. \tag{3.97}$$

Equation (3.97) can be viewed as an ingoing boundary condition on the stretched horizon because the FIDOs near the horizon are moving outward at nearly the speed of light relative to physical observers and so in their frame of reference it appears as though all electromagnetic fields appear as ingoing electromagnetic waves [35]. There will be fractional errors to these boundary conditions of order α_H^2 [36], but by choosing α_H to be very small these errors can be made to be practically negligible. It was first shown by Znajek in [37] and then by Damour in [38] that these boundary conditions arise from physical properties of a fictitious membrane located at the stretched horizon. More specifically, Hanni and Ruffini [39] were the first to demonstrate that it is beneficial to pretend that there is a surface charge density σ_H on the stretched horizon. By construction, it terminates the normal components of the electric fields that hit the stretched horizon,

$$\sigma_H \equiv \left.\frac{E_n}{4\pi}\right|_H, \tag{3.98}$$

which is simply the charge per unit area on the stretched horizon. Similarly Znajek [37] and Damour [38] showed that it is convenient to assume that there is a surface current on the stretched horizon. This surface current $\mathbf{j}_H$ will terminate any tangential magnetic field lines by the use of Ampères law. The surface current must be written in terms of the universal time t because if it were written in terms of the FIDO proper time τ, the clock would tick far too slowly depending on where the stretched horizon is located. Therefore, the surface current $\mathbf{j}_H$ must be defined as the charge crossing a unit length per unit universal time t in the stretched horizon [35]. This surface current will be smaller than the surface current as measured by a FIDO, by a factor of $\alpha_H = d\tau/dt$. Using Ampère's law, the horizon magnetic field can be written as [35]

$$\mathbf{B}_H \equiv 4\pi \mathbf{j}_H \times \hat{\mathbf{n}}. \tag{3.99}$$

An important consequence of this definition of the horizons surface current is that the boundary condition given by $\mathbf{B}_H = \mathbf{E}_H \times \hat{\mathbf{n}}$ from equation (3.97) can be reinterpreted as Ohm's law [37; 38]. Therefore the ingoing boundary condition dictates that the surface current on the stretched horizon must be proportional to the electric field $\mathbf{E}_H$,

$$\mathbf{j}_H = \frac{\mathbf{E}_H}{R_H}, \tag{3.100}$$

where the surface resistivity in gaussian units is

$$R_H = \frac{4\pi}{c} = \frac{\sqrt{\epsilon_0 \mu_0}}{\mu_0} = \sqrt{\frac{\epsilon_0}{\mu_0}} \simeq 376.6\ \Omega, \tag{3.101}$$

which is simply the impedance Z_0 of the vacuum [40], where waves propagate freely.

The last section has provided the foundations for considering the membrane paradigm for black holes has outlined the fundamental equations of the paradigm. In the chapter to follow (more specifically in the section titled "Impedance matching"), these equations will be used to study the relationship between a black hole's rotation and the electromagnetic fields surrounding it. More specifically, following the work done by [33] the BZ process will be described in this formalism and shown to behave as an electrical circuit similar to a voltage divider with a load resistance.

Chapter 4

A Simplified Limit of the BZ Process

In [5], Gralla and Jacobson developed a convenient description of the Blandford-Znajek (split-)monopole solution by relating it more closely to a rotating (Michel) monopole in flat space. In this section an alternate form to the Blandford-Znajek solution[1] will be derived for the slow rotation limit (up to order a) using the language of differential forms. In this description, certain aspects of the BZ solution will be more clear, for instance a better argument for explaining why $\omega = \frac{1}{2}\Omega_H$.

4.1 Differential Forms

Differential forms can be thought of as a special class of tensors which are antisymmetric and covariant. Under the operations of scalar multiplication, addition and the wedge product $\wedge$, differential forms form a graded algebra [5]. The link with tensor notation is given by the wedge product [7] Given a p-form A and a q- form B, a $(q+p)$-form known as the wedge product $A \wedge B$ can be formed by taking the antisymmetrized tensor product [7],

$$(A \wedge B)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_q]}, \tag{4.1}$$

where the square brackets denote antisymmetrization. As an example, the wedge product of two 1-forms is

$$(A \wedge B)_{\mu\nu} = 2(A_\mu B_\nu - A_\nu B_\mu). \tag{4.2}$$

[1]This will be done using techniques given in [5].

Note that [7]

$$A \wedge B = (-1)^{pq} B \wedge A, \tag{4.3}$$

meaning the wedge product can be rearranged. The exterior derivative d provides a way to differentiate p-form fields to $(p+1)$-form fields. Another useful property is that the exterior derivative is in fact a tensor, unlike the partial derivative. It is defined as a normalized, antisymmetrized partial derivative, [7]

$$(dA)_{\mu_1 \dots \mu_p} = (p+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}. \tag{4.4}$$

A simple example of this is the gradient, which is the exterior derivative of a 0-form,

$$(d\phi)_\mu = \partial_\mu \phi. \tag{4.5}$$

Exterior derivatives obey a modified Leibniz rule when it is applied to the product of a p-form α and a q-form β,

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta. \tag{4.6}$$

Another useful property of exterior derivatives is for any form A,

$$d(dA) = 0, \tag{4.7}$$

which is often written as $dd = d^2 = 0$. Therefore, a p-form A is closed if $dA = 0$ and is exact if $A = dB$ for some $(p-1)$-form B.[2] The last operation on differential forms that will be used in the next sections is the Hodge duality. The Hodge star operator is defined on an n-dimensional manifold that maps p-forms to $(n-p)$-forms,

$$(\star A)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \dots \nu_p}{}_{\mu_1 \dots \mu_{n-p}} A_{\nu_1 \dots \nu_p}, \tag{4.8}$$

mapping A to A dual [7]. Clearly the Hodge dual depends on the metric of the manifold (this is easy to see from equation (3.5)). Electrodynamics is a very nice use of differential forms since it involves antisymmetric tensors. From the definition of the exterior derivative,

[2] All exact forms are closed, but all closed forms are not exact.

it is clear that equation (3.8) can be rewritten as the closure of the two-form $F_{\mu\nu}$,

$$dF = 0. \tag{4.9}$$

This also implies that F is exact,[3] and so there must be a one-form A_μ such that

$$F = dA. \tag{4.10}$$

This one-form is the familiar vector potential of electrodynamics, with the 0 component given by the scalar potential. The other one of Maxwell's equations, equation (3.7) can be written as an equation between three-forms,

$$d(\star F) = \star J, \tag{4.11}$$

where the current one-form J is the current four-vector with index lowered [7]. Conservation of current is immediately realized by acting the exterior derivative d on equation (4.11), since $dd = 0$. Lastly, in order to perform calculations, the volume element $d^n x$ can be defined as an antisymmetric tensor duality constructed with wedge products [7],

$$d^n x = dx^0 \wedge \ldots dx^{n-1}. \tag{4.12}$$

Lastly as an example, the electromagnetic 2-form F can be rewritten as follows

$$F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{4.13}$$

4.2 Rotating Magnetic (Michel) Monopoles

First, begin by introducing the non-rotating magnetic monopole,

$$F_{monopole} = C \sin\theta d\theta \wedge d\varphi. \tag{4.14}$$

This is clearly a monopole field since it is proportional to the area element on the sphere and has a flux integral equal to $4\pi C$ for any radius. Traditionally, the monopole charge is defined to equal the flux integral and so the C given in equation (4.14) is simply $1/4\pi$ times the usual monopole charge [5]. Verifying Maxwell's equations are satisfied, taking the exterior

[3] All closed forms are exact in Minkowski space [7].

derivative of equation (4.14) yields,

$$dF_{monopole} = C\cos\theta d\theta \wedge d\theta \wedge d\varphi = 0, \tag{4.15}$$

since $d\theta \wedge d\theta = 0$. The other equation to be satisfied is equation (4.11), so taking the Hodge dual of $F_{monopole}$,

$$\begin{aligned}
\star F_{monopole} &= \frac{C\sin\theta}{r^2\sin\theta} \star (rd\theta \wedge r\sin\theta d\varphi) &(4.16)\\
&= \frac{C}{r^2}\left(\left(1-\frac{2GM}{r}\right)^{1/2} dt \wedge \left(1-\frac{2GM}{r}\right)^{-1/2} dr\right) &(4.17)\\
&= \frac{C}{r^2}(dt \wedge dr). &(4.18)
\end{aligned}$$

Therefore, $d(\star F)$ is given by

$$d(\star F_{monopole}) = -\frac{2C}{r^3} dr \wedge dt \wedge dr = 0, \tag{4.19}$$

since again $dr \wedge dr = 0$. The $3+1$ version of the magnetic monopole is $\mathbf{B} = \frac{C}{r^2}\hat{r}$ and $\mathbf{E} = 0$ [5]. Before discussing the BZ solution in this context, it is necessary to summarize the Michel monopole solution [4] for direct comparison. It consists of a rotating radial field in a flat spacetime. The rotating magnetic monopole in flat-space is a vacuum solution and although magnetic monopoles do not appear to exist in nature, it still plays an important role in understanding force-free magnetospheres. The Michel solution in flat-space is given by

$$F_{Michel} = -Cd(\cos\theta) \wedge (d\varphi - \omega du), \tag{4.20}$$

which is rotating at constant angular velocity ω and u is the retarded time, which in flat space is given by $u = t - r$. Alternatively, this can be summarized in spherical coordinates as the following [41]

$$B^r = \frac{C}{r^2} \tag{4.21}$$

$$B^\theta = 0 \tag{4.22}$$

$$B^\varphi = -\frac{C\omega\sin\theta}{r} \tag{4.23}$$

$$E^r = 0 \tag{4.24}$$

$$E^\theta = -\frac{C\omega\sin\theta}{r} \tag{4.25}$$

$$E^\varphi = 0. \tag{4.26}$$

The poloidal magnetic field is unchanged from the normal monopole solution but the rotation creates the additional B^φ term and there is a non-zero electric field given by $E^\theta = B^\varphi$. Also, this solution clearly satisfies $\mathbf{E}\cdot\mathbf{B} = 0$.

Recall from Chapter 3, that BZ found a perturbative monopole solution up to order a^2 which describes a stationary, axisymmetric outgoing flux of energy from a rotating black hole. Following Gralla and Jacobson [5], the solution may be re-derived to first order by promoting the flat-space magnetic (Michel) monopole solution to the Schawarzschild background. This is done by taking the retarded time u to the Schwarzschild retarded time $u = t - r_*$, where r_* is the usual tortoise coordinate. Therefore the Michel monopole solution in Schwarzschild can be written as

$$F_{Michel} = C\sin\theta d\theta \wedge (d\varphi - \omega(dt - dr_*)). \tag{4.27}$$

4.3 Simplified Description of BZ process

The goal is to rewrite the Blandford-Znajek process in differential forms, so begin by rewriting the components of $F_{\mu\nu}$ in terms of A_φ and B^φ,

$$\begin{aligned}
F_{tr} &= \omega \partial_r A_\varphi \\
F_{t\theta} &= \omega \partial_\theta A_\varphi \\
F_{t\varphi} &= 0 \\
F_{r\theta} &= \sqrt{-g} B^\varphi \\
F_{r\varphi} &= \partial_r A_\varphi \\
F_{\theta\varphi} &= \partial_\theta A_\varphi ,
\end{aligned} \tag{4.28}$$

where for the Kerr metric $\sqrt{-g} = \Sigma \sin\theta = (r^2 + a^2 \cos\theta) \sin\theta$. Now using equation (4.10) to write F in differential forms,

$$F = \omega \partial_\theta A_\varphi dt \wedge d\theta + \omega \partial_r A_\varphi dt \wedge dr + \sqrt{-g} B^\varphi dr \wedge d\theta + \partial_r A_\varphi dr \wedge d\varphi + \partial_\theta A_\varphi d\theta \wedge d\varphi \tag{4.29}$$

It is easy to see that the result above reduces to the following,

$$F = \omega dt \wedge dA_\varphi + \sqrt{-g} B^\varphi dr \wedge d\theta + dA_\varphi \wedge d\varphi, \tag{4.30}$$

$$= dA_\varphi \wedge (d\varphi - \omega dt) + \sqrt{-g} B^\varphi dr \wedge d\theta. \tag{4.31}$$

The goal of this section is to write an equivalent formulation to section 3.2.1 except present the results using differential forms. To zeroth-order in a, the only piece that is non-zero is $A_\varphi^{(0)} = C \sin\theta d\theta$, which means

$$F^{(0)} = C \sin\theta d\theta \wedge d\varphi. \tag{4.32}$$

So up to order a one gets the following,

$$F^{(1)} = C \sin\theta d\theta \wedge (d\varphi - \omega dt) + r^2 \sin\theta B^\varphi dr \wedge d\theta, \tag{4.33}$$

where up to order α, $\omega = \frac{\alpha}{r_0}\omega^{(1)}$, with $\omega^{(1)} = \frac{1}{4}$, and $B^{\varphi(1)} = -C\left(\frac{1-2\omega^{(1)}}{2r_0 r^2} + \frac{1}{2r^3}\right)$. Rewriting $F^{(1)}$ as

$$F^{(1)} = C\sin\theta d\theta \wedge \left(d\varphi - \frac{\alpha}{4r_0}dt\right) - Cr^2\sin\theta\left(\frac{1}{2r^3} + \frac{1-\frac{2}{4}}{2r_0 r^2}\right)\alpha dr \wedge d\theta \tag{4.34}$$

$$= C\sin\theta d\theta \wedge \left(d\varphi - \frac{\alpha}{4r_0}dt\right) + \alpha C\sin\theta\left(\frac{1}{2r} + \frac{1}{4r_0}\right)d\theta \wedge dr \tag{4.35}$$

Since $\omega = \frac{\alpha}{4r_0} = \frac{a}{8r_0^2}$, the result to order a is

$$F^{(1)} = C\sin\theta d\theta \wedge \left(d\varphi - \omega dt - \omega\left(1 + \frac{2r_0}{r}\right)dr\right). \tag{4.36}$$

Up until this point only Kerr-Schild coordinates have been used and so a way to transform from KS to Boyer-Lindquist coordinates will be necessary. These transformation rules are the following,

$$dt_{KS} = dt_{BL} + \frac{2Mr}{\Delta}dr_{BL} \tag{4.37}$$

$$d\varphi_{KS} = d\varphi_{BL} + \frac{a}{\Delta}dr_{BL} \tag{4.38}$$

$$d\theta_{KS} = d\theta_{BL} \equiv d\theta \tag{4.39}$$

$$dr_{KS} = dr_{BL} \equiv dr. \tag{4.40}$$

Thus, $F^{(1)}$ written in Boyer-Lindquist coordinates is

$$F^{(1)} = C\sin\theta d\theta \wedge \left(d\varphi_{BL} + \frac{a}{\Delta}dr - \omega\left(dt_{BL} + \frac{r_0 r}{\Delta}dr\right) + \omega\left(1 + \frac{2r_0}{r}\right)dr\right) \tag{4.41}$$

$$= C\sin\theta d\theta \wedge \left(d\varphi_{BL} - \omega dt_{BL} + a\left(\frac{1}{\Delta} - \frac{r}{8r_0\Delta} + \frac{1}{8r_0^2}\left(1 + \frac{2r_0}{r}\right)\right)dr\right). \tag{4.42}$$

Recall that to order a, $\Delta = r^2 - r_0 r = r^2\left(1 - \frac{r_0}{r}\right)$ and for $r_0 \ll r$, $\frac{1}{\Delta} \simeq \frac{1}{r^2}\left(1 + \frac{r_0}{r}\right)$.

Using this,

$$F^{(1)} = C\sin\theta d\theta \wedge \left(d\varphi_{BL} - \omega dt_{BL} + a\left(\frac{1}{r^2}\left(1+\frac{r_0}{r}\right) - \frac{1}{8r^2} + \frac{1}{8r_0^2}\left(1+\frac{r_0}{r}\right)\right)dr\right) \tag{4.43}$$

$$= C\sin\theta d\theta \wedge \left(d\varphi_{BL} - \omega dt_{BL} + a\left(\frac{1}{r^2}\left(1+\frac{r_0}{r} - \frac{1}{8}\right) + \frac{1}{8r_0^2}\left(1+\frac{r_0}{r}\right)\right)dr\right) \tag{4.44}$$

$$= C\sin\theta d\theta \wedge \left(d\varphi_{BL} - \omega dt_{BL} + \omega\left(1+\frac{r_0}{r}\right)dr + \frac{r_0^2}{r^2}\left(7-\frac{8r_0}{r}\right)dr\right) \tag{4.45}$$

But the last term in the above equation goes to zero since $r_0 \ll r$ since $r >> r_+$[4], so this yields,

$$F^{(1)} \simeq C\sin\theta d\theta \wedge \left(d\varphi_{BL} - \omega\left(dt_{BL} - \left(1+\frac{r_0}{r}\right)dr\right)\right). \tag{4.46}$$

Now up to first order in a, the BZ solution is simply the Michel monopole from equation (4.27) exported from Schwarzschild to Kerr by identifying the Schwarzschild coordinates with the Boyer-Lindquist coordinates [5]. Writing $F^{(1)}$ in terms of the Eddington-Finklestein coordinate $u = t_{BL} - r_*$, where r_* is the usual tortoise coordinate,

$$du = dt_{BL} - dr_* \tag{4.47}$$

$$= dt_{BL} - \frac{dr}{1-\frac{r_0}{r}} \tag{4.48}$$

$$\simeq dt_{BL} - \left(1+\frac{r_0}{r}\right)dr. \tag{4.49}$$

Thus equation (4.46) can be rewritten as the following

$$F^{(1)} \simeq C\sin\theta d\theta \wedge (d\varphi_{BL} - \omega du) \tag{4.50}$$

which is the same equation as was found in [5] denoted by F^{BZ}_{ansatz}. This solution must be force-free since it is simply the same solution that was computed perturbatively, which is known to be force-free. Boyer-Lindquist coordinates are not inherently regular on the horizon and so a regularization on the horizon must be done. Now, this requirement can only be met for the Michel monopole solution with a particular value of ω [5]. The 1-forms $d\varphi_{BL}$ and du are singular on the Kerr horizon but there is a particular value of ω where the singularities cancel in $d\varphi_{BL} - \omega du$. The ingoing Kerr coordinates are related to the Boyer-Lindquist

[4] In the far-field limit - very far away from the black hole.

coordinates as before,

$$dt_{BL} = dt_{KS} - \frac{2Mr}{\Delta}dr = dt_{KS} - \frac{(r^2 + a^2)}{\Delta}dr \tag{4.51}$$

$$d\varphi_{BL} = d\varphi_{KS} - \frac{a}{\Delta}dr \tag{4.52}$$

and the outgoing Kerr coordinates are given by

$$dt_{BL} = dt_{KS} + \frac{2Mr}{\Delta}dr = dt_{KS} + \frac{(r^2 + a^2)}{\Delta}dr \tag{4.53}$$

$$d\varphi_{BL} = d\varphi_{KS} + \frac{a}{\Delta}dr, \tag{4.54}$$

where as before, $a = 2Mr_+\Omega_H = \Omega_H(r_+^2 + a^2)$. The relation $d\varphi_{BL} - \omega du$ can be rewritten as

$$d\varphi_{BL} - \omega du = d\varphi_{KS} - \frac{a}{\Delta}dr - \omega\left(dt_{BL} - \frac{r^2 + a^2}{\Delta}dr\right) \tag{4.55}$$

$$= d\varphi_{KS} - \frac{a}{\Delta}dr - \omega\left(dt_{KS} - \frac{r^2 + a^2}{\Delta}dr\right) + \omega\left(\frac{r^2 + a^2}{\Delta}\right) \tag{4.56}$$

$$= d\varphi_{KS} - \omega dt_{KS} - \frac{\Omega_H(r_+^2 + a^2)}{\Delta}dr + \frac{2\omega(r^2 + a^2)}{\Delta}dr \tag{4.57}$$

$$= \varphi_{KS} - \omega dt_{KS} + \frac{2\omega(r^2 + a^2) - \Omega_H(r_+^2 + a^2)}{\Delta}dr. \tag{4.58}$$

Therefore, the singularity at the horizon is avoided if and only if ω is equal to one-half of the horizon angular velocity,

$$\omega = \frac{1}{2}\Omega_H. \tag{4.59}$$

This matches the result obtained using locality of the source in the force-free equations from the previous section. The BZ solution can be written in a very simple form,

$$F_{BZ}^{(1)} = C \sin\theta d\theta \wedge \left(d\varphi_{BL} - \frac{1}{2}\Omega_H du\right), \tag{4.60}$$

which encompasses the condition that $\omega = \frac{1}{2}\Omega_H$. There is a striking similarity at order $\mathcal{O}(a)$ to the rotating monopole solution as given by equation (4.20). This solution is simply the Michel monopole solution written in Boyer-Lindquist coordinates but with a locked rotation frequency of $\omega = \frac{1}{2}\Omega_H$. Now we are going to briefly revisit the black hole membrane paradigm and once again find that $\omega = \frac{1}{2}\Omega_H$.

4.4 BZ Process in the Membrane Paradigm and Impedance Matching

In [35], Thorne et al. considered how the membrane paradigm could be used to describe the Blandford-Znajek process. Starting with the electric field for an axisymmetric, stationary, force-free magnetosphere they presented a method of setting up a circuit composed of a rotating black hole, a stretched horizon around the black hole, a toroidal[5] and poloidal magnetic field, a poloidal electric field and a load resistance and can be seen in figure 4.1.

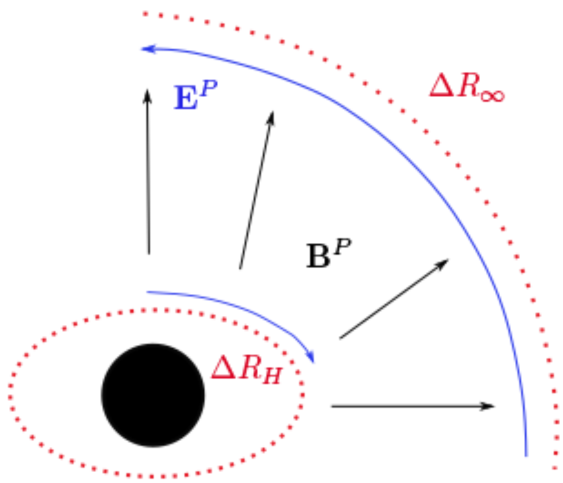

Figure 4.1: Circuit diagram for the force-free black hole magnetosphere. The black hole is represented as the large black object. The red dotted lines are the membranes at the stretched horizon and at stretched infinity, which are the resistor drops ΔR_H and ΔR_∞ respectively. The black arrows are the poloidal magnetic field lines and the blue arrows are the poloidal electric field lines.

The resistor drops across the horizon and infinity will be proportional to the surface resistance $R_H = R_\infty = 376.6\Omega$ as given in equation (3.101). In [42], this load resistance was replaced with a membrane at stretched infinity. The black hole effectively acts as the voltage source, or battery of the circuit. In the magnetosphere, the current follows the poloidal magnetic field $\mathbf{B}^P$, and on the membranes the current follows the poloidal electric field $\mathbf{E}^P$. The membranes themselves are the resistors and are given by the values computed in the last section for $R_H = R_\infty \simeq 376.7\ \Omega$ times the angular distance that is travelled along the

[5]The toroidal magnetic field is responsible for the spiralling [35].

membrane,

$$\Delta R_H \sim R_H \Delta\theta_H \tag{4.61}$$

$$\Delta R_\infty \sim R_\infty \Delta\theta_\infty, \tag{4.62}$$

and for radial fields the angular distance must be the same so $\Delta R_H = \Delta R_\infty$. The full calculation can be viewed in [35] and in [42] but it simply follows from the equations that were worked out in section 3.3. The voltage drop across the membranes on the horizon and infinity are given by

$$\Delta V_H = I\Delta R_H = \frac{1}{2\pi}(\Omega_H - \omega)\Delta\Psi, \tag{4.63}$$

$$\Delta V_\infty = I\Delta R_\infty = \frac{1}{2\pi}\omega\Delta\Psi, \tag{4.64}$$

where Ψ is the magnetic flux function. Taking ratios of the above expressions yields [42]

$$\frac{\omega}{\Omega_H - \omega} = \frac{\Delta V_\infty}{\Delta V_H} = \frac{\Delta R_\infty}{\Delta R_\infty}, \tag{4.65}$$

and solving for ω/Ω_H gives [42]

$$\frac{\omega}{\Omega_H} = \frac{\Delta R_\infty/\Delta R_H}{1 + \Delta R_\infty/\Delta R_H}. \tag{4.66}$$

This demonstrates that the ratio of ω/Ω_H is simply the circuit efficiency [42]. Just as for regular electrical circuits, the maximum power output will be achieved at the load when the efficiency is equal to 50%. Therefore this yields the familiar result,

$$\omega \simeq \frac{1}{2}\Omega_H. \tag{4.67}$$

If the efficiency is high, this indicates that the power generated by the black hole is dissipated in the horizon, and if the efficiency is low then the load power is small since the overall resistance of the circuit must be large [42]

4.5 Membrane Paradigm in an EFT Limit

Consider the EFT limit where the radius of the black hole is much smaller than the scale of electromagnetic field variations in the spacetime formulation. This would make the stretched horizon over the black hole an infinitesimal surface, indicating that in this limit the black hole can be thought of as a point-particle. The new circuit contains the following: a rotating point-particle black hole which acts as a battery, an infinitesimal stretched membrane around the point-particle and a stretched membrane at infinity, with the same poloidal electric and magnetic field $\mathbf{E}^P$ and $\mathbf{B}^P$. This updated circuit can be seen in figure 4.2.

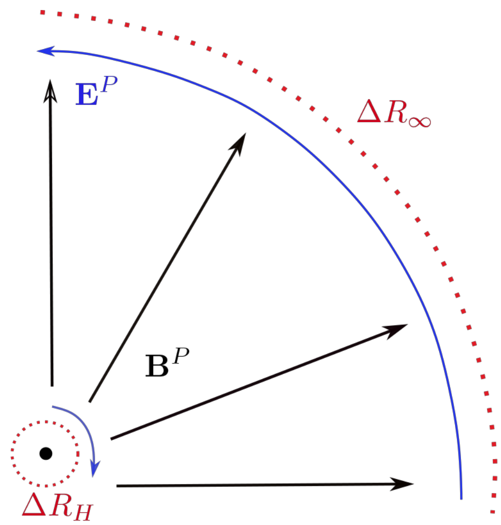

Figure 4.2: Circuit diagram for the force-free black hole magnetosphere in the limit where the inverse mass of the black hole is smaller than the gradient of the fields. The black hole is represented as the small point-like black object. The red dotted lines are the membranes at the stretched horizon and at stretched infinity, and ΔR_H and ΔR_∞ are the resistance drops across the horizon and infinity respectively. The black arrows are the poloidal magnetic field lines and the blue arrows are the poloidal electric field lines. It is also assumed that the membrane around the black hole is a ball of radius ϵ.

The voltage of the circuit has not changed since the voltage only depends on the angular size of the horizon, which stays the same regardless of the relative size of the horizon. The resistance drops across the stretched membranes should not have changed either since they do not depend on the size of the black hole. They are simply a consequence of the "impedance of a vacuum" or the impedance of an open wave guide [35]. Since nothing has changed, the efficiency of the circuit would stay the same, meaning that $\omega \simeq \frac{1}{2}\Omega_H$.

Chapter 5

BZ Process in the Massive Vector Limit

In this chapter, an approximation to the force-free magnetosphere will be considered by treating the impact of the plasma as inducing a photon mass determined by the plasma frequency. This will allow the superradiance discussion to be applied to this approximation of the force-free regime. The end goal will be to relate the BZ process to the superradiance of a massive vector in the EFT approach.

5.1 Superradiant Instabilities Triggered by the Presence of Plasma

Since a plasma is made up of mobile charged particles, when an electromagnetic wave propagates through it, the plasma has coherent vibrations of both the electromagnetic field and the density of charged particles [43]. These coherent vibrations will behave differently than for electromagnetic waves in vacuum since there are longitudinal waves and transverse waves propagating slower than the speed of light [43].

Due to the quantization of the electromagnetic waves in plasma, a spin-1 particle with one longitudinal polarization and two transverse spin polarizations will arise. These three polarizations are referred to as plasmons [43]. The longitudinal mode exists solely because of the plasma but the transverse mode has only had its dispersion relation modified for low frequencies. The dispersion relation for photons in vacuum is given by[1] [44]

$$\omega^2 = k^2. \tag{5.1}$$

In [43], Braaten and Segel showed that the simplified dispersion relations for the transverse and longitudinal modes,

$$\omega_t^2 = k^2 + \omega_p^2 \frac{3\omega_t^2}{2v_F}\left(1 - \frac{\omega_t^2 - v_f^2 k^2}{\omega_t^2}\frac{\omega_t}{2v_f k}\ln\frac{\omega_t + v_f k}{\omega_t - v_F k}\right), 0 \leq k < \infty, \tag{5.2}$$

$$\omega_\ell^2 = \omega_p^2 \frac{3\omega_\ell^2}{v_F^2 k^2}\left(\frac{\omega_\ell}{2v_F k}\ln\frac{\omega_\ell + v_F k}{\omega_\ell - v_F k} - 1\right), \quad 0 \leq k < k_{max}, \tag{5.3}$$

are valid for all k, where $v_F = p_F/E_F$ is the Fermi velocity, E_F is the Fermi energy and $p_F = \sqrt{E_F^2 - m_e^2}$ is the Fermi momentum. In [43], it was shown as $k \to 0$, the dispersion relations for the transverse frequency $\omega_t(k)$ and the longitudinal frequency $\omega_\ell(k)$ both approach the plasma frequency ω_p. In other words, the plasma frequencies of the transverse and longitudinal modes are equal to the mass scale only in the non-relativistic limit. In the relativistic limit the mass scales are proportional to ω_p but are not equal as can be seen in equations (5.2) and (5.3).

For the remainder of the section, only the non-relativistic regime ($v \ll 1$) will be studied and so the transverse photon mass can be approximated as the plasma frequency ω_p. Therefore, when a black hole is surrounded by a hot plasma, photons should acquire a mass given by the following plasma frequency,

$$\omega_p = \sqrt{\frac{4\pi e^2 n}{m_e}}, \tag{5.4}$$

where n is the electron density and m_e and e are the mass and charge of the electron. In the

[1] It is really $\omega^2 = c^2 k^2$, but this thesis has been using the convention that $c = 1$.

force-free magnetosphere, $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, and $\nabla \cdot \mathbf{E} = \rho_e$, so

$$\begin{aligned} \rho_e &= \nabla \cdot (-\mathbf{v} \times \mathbf{B}) && (5.5) \\ &= -\mathbf{B} \cdot (\nabla \times \mathbf{v}) + \mathbf{v} \cdot (\nabla \times \mathbf{B}) && (5.6) \\ &= -\mathbf{B} \cdot \mathbf{\Omega} + \mathbf{v} \cdot \mathbf{j} && (5.7) \\ &= -\mathbf{B} \cdot \mathbf{\Omega} + \mathbf{v} \cdot (\rho_e \mathbf{v}) && (5.8) \\ &= -\mathbf{B} \cdot \mathbf{\Omega} + \rho_e \mathbf{v} \cdot \mathbf{v}, && (5.9) \end{aligned}$$

which becomes

$$\rho_e \left(1 - v^2\right) = -B^r \Omega_H. \tag{5.10}$$

Since we are working in the the non-relativistic limit , $v \ll 1$ and so ρ_e becomes

$$\rho_e \simeq -\Omega_H B^r = \frac{aB^r}{4(GM)^2}, \tag{5.11}$$

which agrees with [2]. The charge density can also be expressed as $\rho_e = en$, and putting this into equation (5.4) yields the following dispersion relation,

$$\omega_p = \sqrt{\frac{4\pi e \rho_e}{m_e}} = \sqrt{\frac{\pi e a B^r}{(GM)^2 m_e}}. \tag{5.12}$$

In the BZ process, a black hole is surrounded by plasma but its energy density is negligible compared to background spacetime. Therefore, photons that interact with the plasma around the black hole will gain a mass equal to equation (5.12). As noted in a somewhat different context in [45], as a consequence of the different dispersion relation, the Maxwell's equations can be written such that,

$$\nabla_\mu F^{\mu\nu} = \omega_p^2 A^\nu, \tag{5.13}$$

where $F_{\mu\nu}$ and A^ν are the usual quantities. Recall that equation (5.13) was studied in detail for constant ω_p in chapter 2.

In general, for arbitrary $\omega_p \neq 0$, the equation will not be separable in the Kerr background but it has been shown that at linear order, the system will develop a superradiant instability, which is similar to what occurs for a massive scalar field around a Kerr black hole [45]. This means that the frequency ω_R of the unstable modes can be approximated in the limit of small $GM\omega_p$ as

$$\omega_R^2 \simeq \omega_p^2 \left(1 - \left(\frac{GM\omega_p}{\ell + n + 1}\right)^2\right) + \mathcal{O}(\omega_p^4), \tag{5.14}$$

which is the same result as equation (2.100) with μ replaced with ω_p. The total decay rate for this mode is given by translating the result of equation (2.169)

$$\Delta\Gamma(\omega_p) \simeq \frac{1}{12}\left(\frac{a}{(GM)^2} - \frac{2r_+\omega_p}{GM}\right)(GM\omega_p)^9. \tag{5.15}$$

Even though ω_p is not a constant, since it depends on r through the magnetic field B^r, this will still serve as a good approximation since it is slowly varying.

The idea now is to compare the BZ energy flux given in equation (3.77) to $\dot{E}$, since this is the energy flux in the force-free limit[2]. Note that the full BZ solution is axisymmetric and stationary and it is supported by an accretion disc, while this is not true of the vector bound state. The energy flux becomes,

$$\begin{aligned} \dot{E} &\simeq \frac{1}{12}\left(\frac{a}{GM} - 4GM\omega_p\right)(GM)^8\omega_p^{10} && (5.16)\\ &\simeq \frac{a}{12GM}(GM\omega_p)^8\omega_p^2 + \frac{1}{3}(GM\omega_p)^9\omega_p^2, && (5.17)\end{aligned}$$

but $GM\omega_p \ll 1$ so the second term in the above expression is negligible. Therefore the total energy flux is given by,

$$\dot{E}(\omega_p) \simeq \frac{1}{12}\frac{a^6}{(GM)^3}\left(\frac{\pi e a B^r}{m_e}\right)^5. \tag{5.18}$$

Comparing with equation (3.77), in this approximation the scaling is very different between the two. In BZ, the dependence on the rotation parameter is a^2 and in the vector bound state it is a^6. As expected, the energy flux computed in this section is much lower than for BZ. This makes physical sense since in the BZ case, the magnetic field is externally supported by currents in the accretion disc. In contrast, the vector bound state is internally supported.

[2]Essentially this is just taking $\mu \to \omega_p$ in equation (2.170).

Chapter 6

Conclusions

In this thesis, several EFT-inspired approaches were utilized in the context of black holes. First, in the context of superradiance, where the radius of the black hole event horizon was taken to be much smaller than the particle's Compton wavelength, that is $GM\mu \ll 1$. In this description it was shown that superradiance is a feature of the dissipative rotating object.

The Blandford-Znajek solution was developed using general relativity and solved perturbatively up to order a^2 in the rotation parameter. An EFT-like description was also developed following [5], but instead in the limit of slow-rotation. In the slow-rotation limit, F was found up to order a as the following,

$$F^{(1)}_{BZ} = C \sin\theta d\theta \wedge \left(d\varphi - \frac{a}{32(GM)^2} dt - \frac{a}{32(GM)^2} \left(1 + \frac{4GM}{r} \right) dr \right). \tag{6.1}$$

This description is EFT-like since it is the Michel monopole solution in flat space, plus corrections to account for the black hole in the point-particle (Newtonian) description, all in the limit of slow-rotation. From the language of differential forms following [5] it is much simpler to see why the rotation of the field around the black hole, ω is given by $\omega = \frac{1}{2}\Omega_H$, since it occurs as a manifestation of ensuring that the solution is regular at the event horizon.

The membrane paradigm of black holes was first described to demonstrate that black hole systems can be viewed as dissipative circuits. This was then considered in the limit where the wavelengths of all the electromagnetic modes are much larger than the black hole horizon. It was found that the picture is still entirely valid and the parameters remain exactly the same.

Finally, the Blandford-Znajek process was examined in the approximation that the force-free plasma induces a vector mass and thus a superradiant instability and the energy flux determined here is new. This total energy flux was found to be of order a^6 in the rotation parameter, which was significantly smaller (by a^4) than the total energy flux computed by BZ in the case where a (split)-magnetic monopole is externally supported by an accretion disc. In future work it would be interesting to see if an effective field theory formalism could be developed for the complete Blandford-Znajek process.

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