A CERTAIN CLASS OF BIORTHOGONAL POLYNOMIALS ASSOCIATED WITH THE LAGUERRE POLYNOMIALS

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Abstract

The main object of this paper is to present some general families of bilinear, bilateral, or mixed multilateral generating functions for some biorthogonal polynomials which are associated rather closely with the classical Laguerre polynomials. Relevent connections of the results considered here with those that were given in earlier works, especially from the viewpoint of Lie groups and Lie algebras, are also discussed briefly.

1. Introduction and Definitions

Over three decades ago, Konhauser ([3]; see also [2]) introduced and studied two interesting classes of polynomials $Y_n^\alpha(x;\ell)$ and $Z_n^\alpha(x;\ell)$, where $Y_n^\alpha(x;\ell)$ is a polynomial in x and $Z_n^\alpha(x;\ell)$ is a polynomial in x^ℓ, with (in general)

$$\mathrm{Re}(\alpha) > -1 \quad \text{and} \quad \ell \in \mathbb{N} := \{1,2,3,\dots\}.$$

For $\ell = 1$, each of these polynomials reduces to the classical Laguerre polynomials (*cf.*, *e.g.*, Szegö [15]):

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}. \tag{1.1}$$

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In their special case when $\ell = 2$, these polynomials were encountered earlier by Spencer and Fano [7] in their investigation of the penetration of Gamma rays through matter, and were discussed subsequently by Preiser [4]. Furthermore, we have [3, p. 303]

$$\int_0^\infty x^\alpha e^{-x} Y_m^\alpha(x;\, \ell) Z_n^\alpha(x;\, \ell)dx = \frac{\Gamma(\alpha + n\ell + 1)}{n!}\delta_{m,n} \tag{1.2}$$

$$(\mathrm{Re}(\alpha) > -1;\, \ell \in \mathbb{N};\, m, n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}),$$

which exhibits the fact that the polynomial sets $\{Y_n^\alpha(x;\ell)\}_{n=0}^\infty$ and $\{Z_n^\alpha(x;\ell)\}_{n=0}^\infty$ are *biorthogonal* with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0, \infty)$, $\delta_{m,n}$ being the Kronecker delta.

The following explicit representation for the polynomials $Z_n^\alpha(x;\ell)$ was given by Konhauser [3, p. 304, Equation (5)]:

$$Z_n^\alpha(x;\ell) = \frac{\Gamma(\alpha + n\ell + 1)}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^{k\ell}}{\Gamma(\alpha + k\ell + 1)} \tag{1.3}$$

or, equivalently,

$$Z_n^\alpha(x;\ell) = \frac{(\alpha+1)_{n\ell}}{n!}\, {}_1F_\ell\left[-n;\, \frac{\alpha+1}{\ell}, \dots, \frac{\alpha+\ell}{\ell};\, \left(\frac{x}{\ell}\right)^\ell\right] \tag{1.4}$$

in terms of a generalized hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters, defined by

$$\begin{aligned} &{}_pF_q\left[\alpha_1, \dots, \alpha_p;\, \beta_1, \dots, \beta_q;\, z\right] \\ &\qquad := \sum_{k=0}^\infty \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k}\frac{z^k}{k!}, \end{aligned} \tag{1.5}$$

$(\lambda)_k$ being the Pochhammer symbol defined by

$$(\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k = 0) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N}). \end{cases} \tag{1.6}$$

Subsequently, Carlitz pointed out that [1, p. 427, Equation (9)]

$$Y_n^\alpha(x;\ell) = \frac{1}{n!}\sum_{k=0}^{n}\frac{x^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{\alpha+j+1}{\ell}\right)_n. \tag{1.7}$$

A systematic presentation of numerous interesting properties and characteristics of each of the biorthogonal polynomials $Y_n^\alpha(x;\ell)$ and $Z_n^\alpha(x;\ell)$ can be found in the works of (for example) Srivastava ([11]; see also [12]). The main object of the present sequel to these earlier works is to develop several general families of bilinear, bilateral, or mixed multilateral generating functions for the Konhauser polynomials $Y_n^\alpha(x;\ell)$ and to show how the results presented here would relate to those that were considered by earlier workers, especially from the viewpoint of Lie groups and Lie algebras.

2. The Main Generating Functions

The first of our main results on generating functions for the Konhauser polynomials $Y_n^{\alpha}(x;\ell)$ is contained in

Theorem 1. *Corresponding to a non-vanishing function* $\Omega_\mu(\xi_1,\ldots,\xi_s)$ *of* s *complex variables* $\xi_1,\ldots,\xi_s \quad (s\in\mathbb{N})$ *and of (complex) order* μ*, let*

$$\Lambda^{(1)}_{m,p,q}[x;\xi_1,\ldots,\xi_s;z] = \sum_{n=0}^{\infty} a_n Y_{m+qn}^{\alpha+\lambda qn}(x;\ell)\Omega_{\mu+pn}(\xi_1,\ldots,\xi_s)z^n \tag{2.1}$$

$$(a_n\neq 0;\ m\in\mathbb{N}_0;\ p,q\in\mathbb{N})$$

and

$$\Theta^{\alpha,q,\lambda}_{n,m,p}(x;\xi_1,\ldots,\xi_s;z) = \sum_{k=0}^{[n/q]}\binom{m+n}{n-qk}a_k Y_{m+n}^{\alpha+\lambda qk}(x;\ell)\cdot\Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)z^k, \tag{2.2}$$

where λ *is a suitable (real or complex) parameter.*
Then

$$\sum_{n=0}^{\infty}\Theta^{\alpha,q,\lambda}_{n,m,p}(x;\xi_1,\ldots,\xi_s;z)t^n = (1-t)^{-m-(\alpha+1)/\ell}\exp\left(x\left\{1-(1-t)^{-1/\ell}\right\}\right)\cdot\Lambda^{(1)}_{m,p,q}\left[x(1-t)^{-1/\ell};\xi_1,\ldots,\xi_s;\frac{zt^q}{(1-t)^{(\lambda+\ell)q/\ell}}\right] \tag{2.3}$$

$$(|t|<1;\ \ell\in\mathbb{N}),$$

provided that each member of (2.3) *exists.*

Proof. For the sake of convenience, let $\mathcal{S}$ denote the first member of the assertion (2.3) of Theorem 1. Then, upon substituting for the polynomials

$$\Theta^{\alpha,q,\lambda}_{n,m,p}(x;\xi_1,\ldots,\xi_s;z)$$

from (2.2) into the left-hand side of (2.3), we obtain

$$\begin{aligned}\mathcal{S} :&= \Theta_{n,m,p}^{\alpha,q,\lambda}\left(x;\, \xi_1, \ldots, \xi_s;\, z\right) t^n \\ &= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k Y_{m+n}^{\alpha+\lambda qk}(x;\ell) \\ &\qquad \cdot \Omega_{\mu+pk}\left(\xi_1, \ldots, \xi_s\right) z^k \\ &= \sum_{k=0}^{\infty} a_k\, \Omega_{\mu+pk}\left(\xi_1, \ldots, \xi_s\right) \left(zt^q\right)^k \\ &\qquad \cdot \sum_{n=0}^{\infty} \binom{m+n+qk}{n} Y_{m+n+qk}^{\alpha+\lambda qk}(x;\ell) t^n,\end{aligned} \tag{2.4}$$

by inverting the order of the double summation involved.

The inner series in (2.4) can be summed in a closed form by appealing to the known generating function (*cf.*, *e.g.*, [11, p. 238, Equation (2.17)]):

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha}(x;\ell) t^n &= (1-t)^{-m-(\alpha+1)/\ell} \\ &\quad \cdot \exp\left(x\left\{1-(1-t)^{-1/\ell}\right\}\right) Y_m^{\alpha}\left(x(1-t)^{-1/\ell};\, \ell\right)\end{aligned} \tag{2.5}$$

$$\left(|t| < 1;\; m \in \mathbb{N}_0\right),$$

with m and α replaced by $m + qk$ and $\alpha + \lambda qk$, respectively $(q \in \mathbb{N};\; k \in \mathbb{N}_0)$. We thus find from (2.4) and (2.5) that

$$\begin{aligned}\mathcal{S} &= (1-t)^{-m-(\alpha+1)/\ell} \exp\left(x\left\{1-(1-t)^{-1/\ell}\right\}\right) \\ &\quad \cdot \sum_{k=0}^{\infty} a_k Y_{m+qk}^{\alpha+\lambda qk}\left(x(1-t)^{-1/\ell};\, \ell\right) \\ &\quad \cdot \Omega_{\mu+pk}\left(\xi_1, \ldots, \xi_s\right) \left(\frac{zt^q}{(1-t)^{(\lambda+\ell)q/\ell}}\right)^k \quad \left(|t| < 1\right).\end{aligned} \tag{2.6}$$

If we now interpret this last infinite series in (2.6) by means of the definition (2.1), we shall arrive readily at the second member of the assertion (2.3).

This evidently completes the proof of Theorem 1 under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, Theorem 1 holds true (at least as a relation between *formal* power series) for those values of the various parameters and variables involved for which each member of the assertion (2.3) exists.

The proof of Theorem 1, which we have detailed above fairly adequately, can be applied *mutatis mutandis* with a view to deriving each of the following results (Theorem 2 and Theorem 3 below). Indeed, in place of the generating function (2.5) used in proving Theorem 1, the proof of Theorem

2 would make use of the known result (*cf.*, *e.g.*, [11, p. 238, Equation (2.19)]):

$$\sum_{n=0}^{\infty}\binom{m+n}{n}Y_{m+n}^{\alpha-n\ell}(x;\,\ell)t^n=(1+t)^{-1+(\alpha+1)/\ell}\cdot\exp\left(x\left\{1-(1+t)^{1/\ell}\right\}\right)Y_m^{\alpha}\left(x(1+t)^{1/\ell};\,\ell\right)\tag{2.7}$$

$$\left(|t|<1;\,m\in\mathbb{N}_0\right),$$

and we shall require yet another known generating function (*cf.*, *e.g.*, [11, p. 240, Equation (2.35)]):

$$\sum_{k=0}^{\infty}Y_n^{\alpha+k}(x;\,\ell)\frac{t^n}{n!}=e^t\,Y_n^{\alpha}(x-t;\,\ell)\quad(n\in\mathbb{N}_0)\tag{2.8}$$

in proving Theorem 3 below.

Theorem 2. *Under the hypotheses of Theorem* 1, *let*

$$\Lambda_{m,p,q}^{(2)}[x;\,\xi_1,\ldots,\xi_s;\,z]=\sum_{n=0}^{\infty}a_nY_{m+qn}^{\alpha+(\lambda-\ell)qn}(x;\,\ell)\Omega_{\mu+pn}(\xi_1,\ldots,\xi_s)\,z^n\tag{2.9}$$

$$(a_n\neq 0;\,m\in\mathbb{N}_0;\,p,q\in\mathbb{N})$$

and

$$\Phi_{n,m,p}^{\alpha,q,\lambda}(x;\,\xi_1,\ldots,\xi_s;z)=\sum_{k=0}^{[n/q]}\binom{m+n}{n-qk}a_kY_{m+n}^{\alpha-n\ell+\lambda qk}(x;\,\ell)\cdot\Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)\,z^k,\tag{2.10}$$

where λ *is a suitable (real or complex) parameter.*

Then

$$\sum_{n=0}^{\infty}\Phi_{n,m,p}^{\alpha,q,\lambda}(x;\,\xi_1,\ldots,\xi_s;\,z)t^n=(1+t)^{-1+(\alpha+1)/\ell}\exp\left(x\left\{1-(1+t)^{1/\ell}\right\}\right)\cdot\Lambda_{m,p,q}^{(2)}\left[x(1+t)^{1/\ell};\,\xi_1,\ldots,\xi_s;\,zt^q(1+t)^{(\lambda-\ell)q}\right]\tag{2.11}$$

$$\left(|t|<1;\,\ell\in\mathbb{N}\right),$$

provided that each member of (2.11) *exists.*

Theorem 3. *Under the applicable hypotheses of Theorem* 1 (*and Theorem* 2), *let*

$$\Lambda^{(3)}_{p,q,\lambda}\left[x;\ \xi_1,\ldots,\xi_s;\ z\right] = \sum_{k=0}^{\infty} a_k Y_n^{\alpha+(\lambda+1)qk}(x;\ell)\Omega_{\mu+pk}\left(\xi_1,\ldots,\xi_s\right)\frac{z^k}{(qk)!} \tag{2.12}$$

$$(a_k \neq 0;\ n \in \mathbb{N}_0;\ p,q \in \mathbb{N})$$

and

$$\Psi^{\alpha,\mu,\lambda}_{k,p,q}\left(x;\ \xi_1,\ldots,\xi_s;\ z\right) = \sum_{j=0}^{[k/q]}\binom{k}{qj} a_j Y_n^{\alpha+\lambda qj+k}(x;\ \ell) \cdot \Omega_{\mu+pj}\left(\xi_1,\ldots,\xi_s\right) z^j, \tag{2.13}$$

where λ *is a suitable* (*real or complex*) *parameter.*

Then

$$\sum_{k=0}^{\infty}\Psi^{\alpha,\mu,\lambda}_{k,p,q}\left(x;\ \xi_1,\ldots,\xi_s;\ z\right)\frac{t^k}{k!} = e^t\Lambda^{(3)}_{p,q,\lambda}\left[x-t;\ \xi_1,\ldots,\xi_s;\ zt^q\right], \tag{2.14}$$

provided that each member of (2.14) *exists.*

3. Concluding Remarks and Observations

For *each* suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if we express the multivariable function

$$\Omega_\mu\left(\xi_1,\ldots,\xi_s\right) \quad (s \in \mathbb{N}\setminus\{1\})$$

as an appropriate product of several *simpler* functions, each of our results (Theorems 1, 2, and 3 above) can be shown to yield various families of *mixed* multilateral generating functions for the Konhauser biorthogonal polynomials $Y_n^\alpha(x;\ \ell)$.

Since

$$Y_n^\alpha(x;\ 1) = L_n^{(\alpha)}(x) = Z_n^\alpha(x;\ 1), \tag{3.1}$$

as we pointed out in Section 1, the special case $\ell = 1$ of each of our results (Theorems 1, 2, and 3 above) would yield the corresponding family of bilinear, bilateral, or mixed multilateral generating functions for the classical Laguerre polynomials $L_n^{(\alpha)}(x)$, and we are thus led to the *main* results of Rassias and Srivastava [5], which (in turn) contain numerous assertions made in the recent (as well as current) literature on bilateral generating functions for the Laguerre (or the *modified* Laguerre) polynomials, especially from the viewpoint of Lie groups and Lie algebras (see, for details, [5] *and* the references cited therein).

Finally, we turn to a very special case of Theorem 1 when

$$m = \lambda = 0 \quad \text{and} \quad s = p = q = 1,$$

which was given, over two decades ago, by Srivastava [8, p. 489, Equation (1)] and which was *red-erived* very recently by Shreshtha and Bajracharya [6] by using the aforementioned group-theoretic technique. In fact, much more general results for the Konhauser biorthogonal polynomials $Y_n^\alpha(x;\,\ell)$ than that given earlier by Srivastava [8] (and, very recently, by Shreshtha and Bajracharya [6]) can be found in the works of Srivastava and Lavoie [13, p. 315, Corollary 14], Srivastava ([10, p. 197, Equation (1)], [9, Part II, p. 241, Corollary 18], and [11, p. 241, Equation (2.37)]), and Srivastava and Manocha [14, p. 433, Corollary 18].

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