

**SPECTRAL PROPERTIES OF THE OPERATOR  
EQUATION  $AX + XB = Y$**

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# Spectral properties of the operator equation $AX + XB = Y$

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### Introduction :

Let  $A$  be a closed operator on a Banach space  $E$  and  $B$  a densely defined operator on  $F$ . We are interested in the operator equation

$$(0.1) \quad AX + XB = Y$$

where  $Y \in \mathcal{L} := \mathcal{L}(F, E)$  is given and  $X \in \mathcal{L}$  the solution.

This equation has been studied extensively for bounded operators (see e.g. [Da Kr], [Lu Ro], [Ro], [Pu]). The case where  $A$  and  $B$  are generators of  $C_0$  semigroups was considered recently by Freeman [F], Lin ; Shaw [LS] and Phóng [Ph].

The purpose of this paper is to study the operator  $\tau_{A,B}$  on  $\mathcal{L}$  given by

$$(0.2) \quad \tau_{A,B}(X) = AX + XB$$

with suitable domain. Then existence and uniqueness of the problem (0.1) is equivalent to saying that  $\tau_{A,B}$  is invertible. Thus it is natural to investigate the spectrum of this operator.

It turns out that always

$$(0.3) \quad \overline{\sigma(A) + \sigma(B)} \subset \sigma(\tau_{A,B})$$

(under the assumption  $\sigma(A) \neq \mathbb{C}$  or  $\sigma(B) \neq \mathbb{C}$ ), so that  $0 \notin \sigma(A) + \sigma(B)$  is a necessary condition for existence and uniqueness of (0.1).

However, the opposite inclusion of (0.3), which is almost trivial in the bounded case, is false, in general. This had been discovered by Phóng [Ph] (even though he does not formulate it that way), see also Section 6 for counterexamples.

We establish the spectral equality

$$(0.4) \quad \sigma(\tau_{A,B}) = \sigma(A) + \sigma(B)$$

in three cases :

1.  $A$  and  $B$  generate eventually norm continuous  $C_0$ -semigroups ;
2. one of the operators is bounded ;
3.  $A$  and  $B$  generate  $C_0$ -semigroups one of which is holomorphic.

The proof of 2. and 3. is based on a complex formula expressing the solution of (0.1) by a contour integral which is due to Rosenblum [Ro]. This formula has also been used by Phong in a particular case and we generalize his proof.

In fact, a separate appendix (Section 7) is devoted to the study of the sum of commuting operators, one of which generates a holomorphic semigroup. This more general context is of interest in the theory of differential equations in Banach spaces (see Da Prato-Grisvard [Da Gr] for a systematic study).

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1. Basic properties of the operator  $X \mapsto AX + XB$  :

Let  $E, F$  be Banach spaces and  $A$  a closed operator on  $E$  and  $B$  a densely defined operator on  $F$ . We define the operator  $\tau_{A,B}$  on  $\mathcal{L} := \mathcal{L}(F, E)$  by

$$D(\tau_{A,B}) = \{X \in \mathcal{L} : XD(B) \subset D(A), \exists Y \in \mathcal{L} \text{ s.t. } AXu + XB u = Yu \ \forall u \in D(B)\}$$

$$\tau_{A,B}(X) = Y.$$

Then  $\tau_{A,B}$  is a closed operator on  $\mathcal{L}$ . For all  $\mu, \lambda \in \mathbb{C}$  we have

$$(1.1) \quad \tau_{A-\lambda, B-\mu} = \tau_{A,B} - \lambda - \mu,$$

in particular

$$(1.2) \quad \tau_{A-\lambda, B+\lambda} = \tau_{A,B}$$

for all  $\lambda \in \mathbb{C}$ . However, this is the only case of coincidence of two such operators. In fact, the following holds.

**Proposition 1.1** : Let  $A_1, A_2$  be closed operators on  $E$  and  $B_1, B_2$  densely defined operators on  $F$  such that  $\tau_{A_1, B_1} = \tau_{A_2, B_2}$ . Then there exists  $\lambda \in \mathbb{C}$  such that  $A_1 = A_2 + \lambda$  and  $B_1 = B_2 - \lambda$ .

For the proof we need two lemmas. If  $\varphi \in F'$  and  $x \in E$ , then we denote by  $\varphi \otimes x \in \mathcal{L}$  the rank-one operator  $y \mapsto \langle y, \varphi \rangle x$ .

**Lemma 1.2** : Let  $\varphi_1, \varphi_2 \in F', x_1, x_2 \in E$  such that  $\varphi_1 \neq 0, x_2 \neq 0$ . If  $\varphi_1 \otimes x_1 + \varphi_2 \otimes x_2 = 0$ , then there exists  $\lambda \in \mathbb{C}$  such that  $x_1 = \lambda x_2$  and  $\varphi_2 = -\lambda \varphi_1$ .

This is easy to see.

**Lemma 1.3** : Let  $\varphi \in F' \setminus \{0\}, x \in E \setminus \{0\}$ . Then  $\varphi \otimes x \in D(\tau_{AB})$  if and only if  $\varphi \in D(B')$  and  $x \in D(A)$ . In that case  $\tau_{A,B}(\varphi \otimes x) = \varphi \otimes Ax + B'\varphi \otimes x$ .

**Proof** : Assume that  $X := \varphi \otimes x \in D(\tau_{A,B})$ . Since  $XD(B) \subset D(A)$  and since  $\varphi \neq 0$  and  $D(B)$  is dense in  $F$  it follows that  $x \in D(A)$ . Let  $Y = \tau_{A,B}(X)$ . Then  $\langle u, \varphi \rangle Ax + \langle Bu, \varphi \rangle x = Yu$  for all  $u \in D(B)$ . Let  $\psi \in E'$  such that  $\langle x, \psi \rangle = 1$ . Then

$$\langle Bu, \varphi \rangle = \langle u, Y'\psi \rangle - \langle u, \varphi \rangle \langle Ax, \psi \rangle$$

for all  $u \in D(B)$ .

It follows that  $\varphi \in D(B')$  and  $B'\varphi = Y'\psi - \langle Ax, \psi \rangle \varphi$  (cf. [Na, B-II Lemma 2.9]). The remaining assertions are obvious.  $\diamond$

**Lemma 1.4** : Let  $C : D(C) \rightarrow G$  be a linear mapping on the vector space  $G$ , where  $D(C)$  is a subspace of  $G$ . Assume that for all  $x \in D(C)$  there exists  $\lambda(x) \in \mathbb{C}$  such that  $Cx = \lambda(x)x$ . Then there exists  $\lambda \in \mathbb{C}$  such that  $Cx = \lambda x$  for all  $x \in D(C)$ .

This is easy to see.

Proof of Proposition 1.1 : Since for  $0 \neq x \in E, 0 \neq \varphi \in F', \varphi \otimes x \in D(\tau_{A_j, B_j})$  if and only if  $\varphi \in D(B'_j)$  and  $x \in D(A_j)$  it follows that  $D(A_1) = D(A_2)$  and  $D(B'_1) = D(B'_2)$ . Moreover, for  $0 \neq x \in D(A_1), 0 \neq \varphi \in D(B'_1), \varphi \otimes A_1x + B'_1\varphi \otimes x = \varphi \otimes A_2x + B'_2\varphi \otimes x$  by Lemma 1.3. Thus  $\varphi \otimes (A_1x - A_2x) + (B'_1\varphi - B'_2\varphi) \otimes x = 0$ . It follows from Lemma 1.2 that there exists  $\lambda \in \mathbb{C}$  such that  $A_1x - A_2x = \lambda x$  and  $B'_1\varphi - B'_2\varphi = -\lambda\varphi$ . Now the conclusion follows from Lemma 1.4.  $\diamond$

Remark 1.5 (density of the domain of  $\tau_{A,B}$ ) :

- a) It follows from Lemma 1.3 that  $\overline{D(\tau_{A,B})}$  contains the compact operators whenever  $D(A)$  is dense,  $E$  has the approximation property and  $F$  is reflexive. Thus, if  $\mathcal{L}(F, E) = K(F, E)$  (the compact operators), then  $\tau_{A,B}$  is densely defined. For example, this is the case if  $F = \ell^p, E = \ell^q, 1 \leq q < p < \infty$ .
- b) Assume now that  $E = F$ . If  $D(A) = E$  and  $D(B) = E$ , then  $D(\tau_{A,B}) = \mathcal{L}$ . If  $D(A) \neq E$  and  $D(B) = E$  or  $D(A) = E$  and  $D(B) \neq E$ , then it is clear that  $D(\tau_{A,B})$  does not contain any invertible operator and so  $\tau_{A,B}$  is not densely defined. However, we do not know whether  $D(\tau_{A,B})$  is dense only if  $D(A) = E$  and  $D(B) = E$ .

## 2. The spectral inclusion :

In this section we assume that  $A$  and  $B$  are both closed and densely defined operators on  $E$ , and  $F$  respectively. By  $\sigma(A), \sigma_p(A), \sigma_{ap}(A), \rho(A)$  we denote the spectrum, point spectrum, approximate point spectrum and resolvent set, respectively, of  $A$  and  $R(\lambda, A) = (\lambda - A)^{-1}$  ( $\lambda \in \rho(A)$ ) is the resolvent. Keeping otherwise the notation of Section 1, we prove the following inclusion.

Theorem 2.1 : Assume that  $\sigma(A) \neq \mathbb{C}$  or  $\sigma(B) \neq \mathbb{C}$ . Then  $\sigma(A) + \sigma(B) \subset \sigma(\tau_{A,B})$ .

At first we prove the following special case.

Lemma 2.2 :

- a)  $\sigma_{ap}(A) + \sigma_{ap}(B') \subset \sigma(\tau_{A,B})$ ;
- b)  $\sigma_{ap}(A') + \sigma_{ap}(B) \subset \sigma(\tau_{A,B})$ .

For this we use

Lemma 2.3 : Let  $C$  be a closed operator on a Banach space  $G$ . Assume that  $\varphi_n \in G'$  satisfying  $\|\varphi_n\| = 1$  ( $n \in \mathbb{N}$ ) such that

$$\lim_{n \rightarrow \infty} \sup_{\substack{x \in D(C) \\ \|x\| \leq 1}} |\langle Cx, \varphi_n \rangle| = 0.$$

Then  $0 \in \sigma(C)$ .

Proof : Assume that  $0 \in \rho(C)$  and let  $\alpha = \|C^{-1}\|$ . Then  $B(0, \alpha^{-1}) := \{y \in G : \|y\| \leq \alpha^{-1}\} \subset \{Cx : x \in D(C), \|x\| \leq 1\}$ . Hence

$$\alpha^{-1} = \sup_{\substack{\|y\| \leq \alpha^{-1} \\ y \in G}} |\langle y, \varphi_n \rangle| \longrightarrow 0 \quad (n \rightarrow \infty),$$

contradiction.  $\diamond$

Proof of Lemma 2.2 : a) Let  $\lambda \in \sigma_{ap}(A), \mu \in \sigma_{ap}(B')$ . There exist  $x_n \in D(A)$  such that  $\|x_n\| = 1$  and  $\|Ax_n - \lambda x_n\| \rightarrow 0$  ( $n \rightarrow \infty$ ),  $\varphi_n \in D(B'), \|\varphi_n\| = 1, \|B'\varphi_n - \mu\varphi_n\| \rightarrow 0$  ( $n \rightarrow \infty$ ).

Let  $X_n = \varphi_n \otimes x_n$ . Then  $\|X_n\| = 1$ , and by Lemma 1.3,  $X_n \in D(\tau_{A,B})$  and  $\tau_{A,B}(X_n) = B'\varphi_n \otimes x_n + \varphi_n \otimes Ax_n$ . Hence

$$\begin{aligned} \|\tau_{A,B}(X_n) - (\lambda + \mu)X_n\| &= \|(B'\varphi_n - \mu\varphi_n) \otimes x_n + \varphi_n \otimes (Ax_n - \lambda x_n)\| \\ &\leq \|B'\varphi_n - \mu\varphi_n\| + \|Ax_n - \lambda x_n\| \longrightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Consequently,  $\lambda + \mu \in \sigma_{ap}(\tau_{A,B})$ .

b) Assume that  $\lambda \in \sigma_{ap}(A'), \mu \in \sigma_{ap}(B)$ . Let  $y_n \in D(B), \varphi_n \in D(A')$  such that  $\|y_n\| = \|\varphi_n\| = 1, \|By_n - \mu y_n\| \rightarrow 0$  and  $\|A'\varphi_n - \lambda\varphi_n\| \rightarrow 0$  ( $n \rightarrow \infty$ ). We define  $\phi_n \in \mathcal{L}'$  by  $\phi_n(T) = \langle Ty_n, \varphi_n \rangle$ . Then  $\|\Phi_n\| = 1$  (as is easy to see); moreover,

$$\sup_{\substack{X \in D(\tau_{A,B}) \\ \|X\| \leq 1}} |\langle \tau_{A,B}(X) - (\lambda + \mu)X, \phi_n \rangle| \longrightarrow 0 \quad (n \rightarrow \infty).$$

In fact, let  $X \in D(\tau_{A,B}), \|X\| \leq 1$ . Then

$$\begin{aligned} |\langle \tau_{A,B}(X) - (\lambda + \mu)X, \phi_n \rangle| &= |\langle AXy_n, \varphi_n \rangle + \langle XBy_n, \varphi_n \rangle - (\lambda + \mu)\langle Xy_n, \varphi_n \rangle| \\ &= |\langle Xy_n, A'\varphi_n - \lambda\varphi_n \rangle + \langle By_n - \mu y_n, X'\varphi_n \rangle| \\ &\leq \|A'\varphi_n - \lambda\varphi_n\| + \|By_n - \mu y_n\| \longrightarrow 0 \quad (n \rightarrow \infty) \end{aligned}$$

It follows from Lemma 2.3 that  $\lambda + \mu \in \sigma(\tau_{A,B})$ .  $\diamond$

Lemma 2.4 : Let  $M, N$  be closed subsets of  $\mathbb{C}$  such that  $M \neq \mathbb{C}$  or  $N \neq \mathbb{C}$ . Then  $M + N \subset (\partial M + N) \cup (M + \partial N)$ .

Proof : Let  $m \in \text{int } M, n \in \text{int } N$ . Let  $r, R \in (0, \infty]$  be the largest radii such that  $B(m, r) \subset M, B(n, R) \subset N$  (where  $B(m, r) = \{z \in \mathbb{C} : |m - z| \leq r\}$ ).

Assume that  $r \leq R$ . Since  $M \neq \mathbb{C}$  or  $N \neq \mathbb{C}$  it follows that  $r < \infty$ . Then there exists  $\alpha \in \partial M$  such that  $|m - \alpha| = r$ . Since  $n + m - \alpha \in B(n, r) \subset B(n, R) \subset N$ , it follows that  $m + n = \alpha + (n + m - \alpha) \in \partial M + N$ . Similarly, one obtains  $m + n \in M + \partial N$  in the case  $r \geq R$ .  $\diamond$

Proof of Theorem 2.1 : It is well-known that  $\partial\sigma(A) \subset \sigma_{ap}(A) \cap \sigma_{ap}(A')$ . Consequently by Lemma 2.2 a),  $\partial\sigma(A) + \sigma_{ap}(B') \subset \sigma_{ap}(A) + \sigma_{ap}(B') \subset \sigma(\tau_{A,B})$  and by Lemma 2.2 b),  $\partial\sigma(A) + \sigma_{ap}(B) \subset \sigma_{ap}(A') + \sigma_{ap}(B) \subset \sigma(\tau_{A,B})$ . Since  $\sigma(B) \subset \sigma_{ap}(B) \cup \sigma_{ap}(B')$  we have shown that  $\partial\sigma(A) + \sigma(B) \subset \sigma(\tau_{A,B})$ . One shows similarly that  $\sigma(A) + \partial\sigma(B) \subset \sigma(\tau_{A,B})$ . Now it follows from Lemma 2.4 that  $\sigma(A) + \sigma(B) \subset \sigma(\tau_{A,B})$ .  $\diamond$

We do not know whether  $\sigma(\tau_{A,B}) = \mathbb{C}$  if  $\sigma(A) = \sigma(B) = \mathbb{C}$ .

### 3. Laplace transform methods :

In the present and next two sections we want to show invertibility of  $\lambda - \tau_{A,B}$ . By Theorem 2.1 a necessary condition is that  $\lambda \notin \overline{\sigma(A) + \sigma(B)}$ . But we do not know whether  $\overline{\sigma(A) + \sigma(B)} \neq \mathbb{C}$  implies  $\rho(\tau_{A,B}) \neq \emptyset$ , in general. However, if  $A$  and  $B$  are generators of semigroups, then several methods exist to show invertibility for certain  $\lambda$ . In this section we use the Laplace transform. We show that  $\tau_{A,B}$  is the generator of a (non-strongly continuous) semigroup whose Laplace transform is the resolvent of  $\tau_{A,B}$ .

Assume that  $A$  generates the  $C_0$ -semigroup  $T = (T(t))_{t \geq 0}$  on  $E$  and  $B$  the  $C_0$ -semigroup  $S = (S(t))_{t \geq 0}$  on  $F$ . We define the semigroup  $U : [0, \infty) \rightarrow \mathcal{L} = \mathcal{L}(F, E)$  by

$$U(t)X = T(t)X S(t) \quad (X \in \mathcal{L}).$$

Then  $U(0) = I_{\mathcal{L}}$  (the identity on  $\mathcal{L}$ ) and  $U(t)U(s) = U(t+s)$  ( $t, s \geq 0$ ).

$U$  is not strongly continuous, in general, but for  $X \in \mathcal{L}, f \in F, U(\cdot)Xf$  is strongly continuous. For  $\lambda \in \mathbb{C}$  we define the operator  $\int_0^t e^{-\lambda s} U(s) ds$  on  $\mathcal{L}$  by

$$\left( \int_0^t e^{-\lambda s} U(s) ds \right) (X) = \int_0^t e^{-\lambda s} U(s) X ds \quad (X \in \mathcal{L})$$

where the integral on the right hand side converges strongly in  $E$ .

**Proposition 3.1 :** Let  $\lambda \in \mathbb{C}$ .

a) Let  $X \in \mathcal{L}$ . Then  $\int_0^t e^{-\lambda s} U(s) X ds \in D(\tau_{A,B})$  and

$$\left( \tau_{A,B} - \lambda \right) \int_0^t e^{-\lambda s} U(s) X ds = e^{-\lambda t} U(t) X - X \quad (t \geq 0)$$

b) Let  $X \in D(\tau_{A,B})$ . Then

$$\int_0^t e^{-\lambda s} U(s) \left( \tau_{A,B}(X) - \lambda \right) ds = e^{-\lambda t} U(t) X - X \quad (t \geq 0).$$

The proof is based on Lemma 3.3 which has been shown by Phóng [Ph]. It can be obtained as a consequence of the following general formulation of differentiation of products whose proof we can omit.

**Lemma 3.2 :** Let  $K, L$  be topological vector spaces and let  $\beta : K \times L \rightarrow \mathbb{C}$  be a sequentially continuous bilinear form. Let  $x \in C^1([a, b], K), y \in C^1([a, b], L), f(t) = \beta(x(t), y(t))$ . Then  $f \in C^1([a, b], \mathbb{C})$  and  $\frac{d}{dt} f(t) = \beta\left(\frac{dx}{dt}(t), y(t)\right) + \beta\left(x(t), \frac{dy}{dt}(t)\right)$  ( $t \in [a, b]$ ).

**Lemma 3.3 :** Let  $X \in \mathcal{L}(F, E), v \in D(B), u' \in D(A'), f(t) = \langle T(t)X S(t)v, u' \rangle$ . Then  $f \in C^1([0, \infty), \mathbb{C})$  and  $\frac{d}{dt} f(t) = \langle T(t)X S(t)Bv, u' \rangle + \langle T(t)X S(t)Bv, A'u' \rangle$ .

Proof : Letting  $K = F$ , and  $L = E'$  with the  $\omega^*$ -topology,  $\beta(y, x') = \langle Xy, x' \rangle$ ,  $x(t) = S(t)v$ ,  $y(t) = T(t)'u'$ , the assertion follows from Lemma 3.2.  $\diamond$

Proof of Proposition 3.1 : Replacing  $A$  by  $A - \lambda$  we can assume  $\lambda = 0$ . Let  $V(t) = \int_0^t U(s)ds$ .

a) Let  $X \in \mathcal{L}$ . Let  $v \in D(B)$ . It follows from Lemma 3.3 that

$$(3.1) \quad \langle U(t)Xv, u' \rangle - \langle Xv, u' \rangle = \langle V(t)XBv, u' \rangle + \langle V(t)Xv, A'u' \rangle$$

for  $t \geq 0$  and all  $u' \in D(A')$ . Consequently, by [Na, B-II Lemma 2.9]  $V(t)Xv \in D(A)$  and  $AV(t)Xv + V(t)XBv = U(t)Xv - Xv$ . This implies that  $V(t)X \in D(\tau_{A,B})$  and  $\tau_{A,B}V(t)X = U(t)X - X$ , which is assertion a).

b) Let  $X \in D(\tau_{A,B})$ ,  $\tau_{A,B}(X) = Y$ . Then by Lemma 3.3, for all  $v \in D(B)$ ,  $u' \in D(A')$ ,

$$\begin{aligned} \langle V(t)Yv, u' \rangle &= \int_0^t \langle T(s)YS(s)v, u' \rangle ds = \int_0^t \{ \langle T(s)AXS(s)v, u' \rangle + \langle T(s)XBS(s)v, u' \rangle \} ds \\ &= \int_0^t \frac{d}{ds} \langle T(s)XS(s)v, u' \rangle ds = \langle T(t)XS(t)v, u' \rangle - \langle Xv, u' \rangle = \langle U(t)(X)v, u' \rangle - \langle Xv, u' \rangle. \end{aligned}$$

Hence  $V(t)Y = U(t)X - X$ . This is assertion b).  $\diamond$

We define the growth bound  $\omega(\tau_{A,B})$  by

$$\omega(\tau_{A,B}) := \inf \{ w \in \mathbb{R} : \sup_{t \geq 0} e^{-wt} \|U(t)\| < \infty \} = \lim_{t \rightarrow \infty} (\log \|U(t)\|) / t$$

(cf. [Hi Ph, Theorem 7.6.1]).

Similarly,  $\omega(A) = \inf \{ w \in \mathbb{R} : \sup_{t \geq 0} e^{-wt} \|T(t)\| < \infty \}$ .

The following proposition is due to Freemann [F]. For the sake of completeness we include the proof.

Proposition 3.4 :  $\omega(\tau_{A,B}) = \omega(A) + \omega(B)$ .

Proof : Choosing  $X = \varphi \otimes u$  with  $\varphi \in F'$ ,  $u \in E$ ,  $\|\varphi\| \leq 1$ ,  $\|u\| \leq 1$  one sees that  $\|U(t)\| = \|T(t)\| \|S(t)\|$  ( $t \geq 0$ ). Hence  $\omega(\tau_{A,B}) = \lim_{t \rightarrow \infty} \frac{1}{t} (\log (\|T(t)\| \|S(t)\|)) = \lim_{t \rightarrow \infty} \frac{1}{t} (\log \|T(t)\| + \log \|S(t)\|) = \omega(A) + \omega(B)$ .  $\diamond$

By  $s(\tau_{A,B}) = \sup \{ \operatorname{Re} \lambda : \lambda \in \sigma(\tau_{A,B}) \}$  we denote the spectral bound of  $\tau_{A,B}$ .

Proposition 3.5 :  $s(A) + s(B) \leq s(\tau_{A,B}) \leq \omega(\tau_{A,B})$ .

Proof : a) It follows from Theorem 2.1 that  $s(A) + s(B) \leq s(\tau_{A,B})$ .

b) Let  $\operatorname{Re} \lambda > \omega(\tau_{A,B}) = \omega(A) + \omega(B)$ . Then  $Q := \lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} U(s) ds$  exists and

$\lim_{t \rightarrow \infty} (\lambda - \tau_{A,B}) \int_0^t e^{-\lambda s} U(s) X ds = \lim_{t \rightarrow \infty} (X - e^{-\lambda t} U(t) X) = X$  by Proposition 3.1.a). Since  $\lambda - \tau_{A,B}$  is closed it follows that  $QX \in D(\tau_{A,B})$  and  $(\lambda - \tau_{A,B})QX = X$  for all  $X \in \mathcal{L}$ , and by Proposition 3.1.b)  $Q(\lambda - \tau_{A,B})X = (\lambda - \tau_{A,B})QX$  if  $X \in D(\tau_{A,B})$ . Hence  $\lambda \in \rho(\tau_{A,B})$  and  $Q = (\lambda - \tau_{A,B})^{-1}$ .  $\diamond$

Remark 3.6 : It follows from the proof that for  $\operatorname{Re} \lambda > \omega(\tau_{A,B})$

$$R(\lambda, \tau_{A,B}) = \int_0^\infty e^{-\lambda t} U(t) dt := \lim_{t \rightarrow \infty} \int_0^t e^{-\lambda s} U(s) ds$$

(with convergence in the operator norm).

Denote by  $\operatorname{abs}(\tau_{A,B}) = \inf \{ \operatorname{Re} \lambda : \int_0^\infty e^{-\lambda t} U(t) dt \text{ converges strongly in } \mathcal{L} \}$  the abscissa of the Laplace transform. Then  $\int_0^\infty e^{-\lambda t} U(t) dt$  converges (even in operator norm) for  $\operatorname{Re} \lambda > \operatorname{abs}(\tau_{A,B})$  and does not converge (even not strongly) if  $\operatorname{Re} \lambda < \operatorname{abs}(\tau_{A,B})$  (see [Hi Ph, Sec. 6.2]). Moreover,  $R(\lambda) = \int_0^\infty e^{-\lambda t} U(t) dt$  is holomorphic in  $[\operatorname{Re} \lambda > \operatorname{abs}(\tau_{A,B})]$ . Since  $R(\lambda)$  coincides with  $(\lambda - \tau_{A,B})^{-1}$  for  $\operatorname{Re} \lambda$  large, one has  $R(\lambda) = (\lambda - \tau_{A,B})^{-1}$  whenever  $\operatorname{Re} \lambda > \operatorname{abs}(\tau_{A,B})$ .

An individual version of the following proposition has been proved by Phóng [Ph ; Theorem 3].

Proposition 3.7 : If  $M := \sup_{t \geq 0} \left\| \int_0^t U(s) ds \right\| < \infty$ , then  $0 \in \rho(\tau_{A,B})$ .

Proof : The hypothesis implies that  $\operatorname{abs}(\tau_{A,B}) \leq 0$  (see [Hi Ph, Sec. 6.2]). It follows from the hypothesis that for  $\lambda > 0$ ,  $\|R(\lambda, \tau_{A,B})\| = \left\| \int_0^\infty e^{-\lambda t} U(t) dt \right\| =$

$$\left\| \lambda \int_0^\infty e^{-\lambda t} \int_0^t U(s) ds dt \right\| \leq M. \text{ Hence}$$

$$[\operatorname{dist}(\sigma(\tau_{A,B}), \lambda)]^{-1} = r(R(\lambda, \tau_{A,B})) \leq \|R(\lambda, \tau_{A,B})\| \leq M$$

(where  $r(K)$  denotes the spectral radius of a bounded operator  $K$ ). Hence  $\operatorname{dist}(\sigma(\tau_{A,B}), 0) > 0$ , i.e.  $0 \notin \sigma(\tau_{A,B})$ .  $\diamond$

Remark 3.8 (integrated semigroups) : a) It is obvious from the preceding that  $V(t) = \int_0^t U(s) ds$  defines a locally Lipschitz continuous once integrated semigroup  $(V(t))_{t \geq 0}$  on  $\mathcal{L}$  and  $\tau_{A,B}$  is its generator (see [Ar], [Ne], [Ke Hi] for this notion). This had been pointed out before by Neubrander [Ne, Example 9.3].

b) The argument given in the proof of Proposition 3.7 shows more generally that the generator of a bounded once integrated semigroup is invertible.

#### 4. Eventually norm continuous semigroups :

The spectral equality  $\sigma(\tau_{A,B}) = \sigma(A) + \sigma(B)$  will be established in special cases in the present and next section.

Theorem 4.1 : Let  $A$  and  $B$  be generators of eventually norm continuous semigroups on  $E$  and  $F$ , respectively. Then

$$\sigma(\tau_{AB}) = \sigma(A) + \sigma(B).$$

A semigroup  $T = (T(t))_{t \geq 0}$  on  $E$  is called eventually norm continuous if  $T : [t_0, \infty) \rightarrow \mathcal{L}(E)$  is continuous for the operator norm for some  $t_0 > 0$ . Such a semigroup has two remarkable spectral properties :

$$(4.1) \quad \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq b\} \text{ is bounded for all } b \in \mathbb{R}$$

and the spectral mapping theorem holds, i.e.,

$$(4.2) \quad \sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \quad (t > 0)$$

(see [Na, A-II Theorem 1.20 and A-III.6.6] for example).

For the proof of Theorem 4.1 we will use the following property which holds without assumptions on the semigroup.

Proposition 4.2 : Let  $\lambda \in \sigma(\tau_{A,B})$ . Then  $e^{t\lambda} \in \sigma(U(t)) \quad (t \geq 0)$ .

Proof : Let  $t \geq 0, \lambda \in \mathbb{C}$  such that  $e^{t\lambda} \in \rho(U(t))$ . Let  $R = \int_0^t e^{-\lambda s} U(s) ds, (I - e^{-\lambda t} U(t))^{-1}$ . It follows from Proposition 3.1 that  $R = (\lambda - \tau_{A,B})^{-1}$ .  $\diamond$

Proof of Theorem 4.1 : a) One has  $\sigma(U(t)) \subset \sigma(T(t)) \cdot \sigma(S(t)) \quad (t \geq 0)$ . In fact, let  $L(t)X = T(t)X$  and  $R(t)X = XS(t) \quad (X \in \mathcal{L})$ . Then  $U(t) = L(t)R(t) = R(t)L(t)$ . It follows from Gelfand's theorem that  $\sigma(U(t)) \subset \sigma(R(t)) \cdot \sigma(L(t)) = \sigma(S(t)) \cdot \sigma(T(t))$ .

b) Let  $\lambda \in \sigma(\tau_{A,B})$ . Then  $e^{t\lambda} \in \sigma(U(t))$  by Proposition 4.2. It follows from a) and (4.2) that  $e^{t\lambda} \in e^{t(\sigma(A) + \sigma(B))}$  for all  $t \geq 0$ . Thus, for all  $t > 0$  there exist  $\alpha_t \in \sigma(A), \beta_t \in \sigma(B), k_t \in \mathbb{Z}$  such that  $t(\lambda - \alpha_t - \beta_t) = 2\pi i k_t$ . In particular,  $\operatorname{Re} \lambda = \operatorname{Re} \alpha_t + \operatorname{Re} \beta_t$ . Since  $\sigma(A)$  and  $\sigma(B)$  are situated in left half-planes, there exists  $b \in \mathbb{R}$  such that  $\operatorname{Re} \alpha_t \geq b$  and  $\operatorname{Re} \beta_t \geq b$  for all  $t > 0$ . It follows from (4.1) that  $\sup_{t > 0} |\alpha_t + \beta_t| < \infty$ . Consequently, for  $t > 0$  sufficiently small one has  $k_t = 0$ , i.e.  $\lambda = \alpha_t + \beta_t$ .  $\diamond$

Remark : Note that under the hypothesis of Theorem 4.1  $\sigma(A) + \sigma(B)$  is closed because of (4.1).

### 5. Holomorphic semigroups :

Throughout this section  $A$  is a closed operator on  $E$  and  $B$  a densely defined operator on  $F$ . If one of the operators is bounded, then the spectral equality holds without any further assumptions on the other operator.

Theorem 5.1 : Let  $A$  be a closed operator on  $E$ , and  $B$  a densely defined operator on  $F$ . If  $A$  or  $B$  is bounded, then

$$\sigma(\tau_{A,B}) = \sigma(A) + \sigma(B).$$

Theorem 5.2 : Assume that there exist  $R \geq 0$ ,  $0 < \theta' < \theta < \frac{\pi}{2}$ ,  $M \geq 0$  such that

$$\lambda \in \rho(A) \text{ and } \|\lambda R(\lambda, A)\| \leq M \text{ whenever } |\lambda| \geq R, |\arg \lambda| \leq \pi/2 + \theta$$

and

$$\lambda \in \rho(B) \text{ and } \|\lambda R(\lambda, B)\| \leq M \text{ whenever } |\lambda| \geq R, |\arg \lambda| \leq \pi/2 - \theta'.$$

Then

$$\sigma(\tau_{A,B}) = \sigma(A) + \sigma(B).$$

Remark : If  $D(A)$  is dense, the assumption on  $A$  is equivalent to saying that  $A$  generates a holomorphic  $C_0$ -semigroup (cf. [Na, A-II, Theorem 1.14]).

Remark 5.3 : Theorem 5.2 remains true if the hypotheses on  $A$  and  $B$  are exchanged.

Corollary 5.4 : Assume that  $A$  and  $B$  generate  $C_0$ -semigroups. If one of the semigroups is holomorphic, then

$$\sigma(\tau_{A,B}) = \sigma(A) + \sigma(B).$$

For the proofs denote by  $L_A$  the operator on  $\mathcal{L} = \mathcal{L}(F, E)$  given by

$$\begin{aligned} L_A X &= AX \\ D(L_A) &= \{X \in \mathcal{L} : XF \subset D(A)\} \end{aligned}$$

and by  $R_B$  the operator on  $\mathcal{L}$  defined by

$$\begin{aligned} D(R_B) &= \{X \in \mathcal{L} : \exists Y \in \mathcal{L} \quad XB u = Y u \quad (u \in D(B))\} \\ R_B X &= Y. \end{aligned}$$

Then  $L_A$  and  $R_B$  are commuting operators on  $\mathcal{L}$ . In fact,  $\rho(A) \subset \rho(L_A)$  and  $R(\lambda, L_A)X = R(\lambda, A)X$  ( $\lambda \in \rho(A)$ ); similarly,  $\rho(B) \subset \rho(R_B)$  and  $R(\lambda, R_B)X = XR(\lambda, B)$  for all  $\lambda \in \rho(B)$ .

It is clear that  $\tau_{A,B}$  is an extension on  $L_A + R_B$ , and that  $\tau_{A,B} = L_A + R_B$  whenever one of the operators is bounded.

Now the proofs of Theorem 5.1, 5.2, Remark 5.3 and Corollary 5.4 follow from the results on commuting operators in the Appendix.

Remark 5.5 : Assume that  $E = F$ . Under the hypothesis of Theorem 5.2  $D(\tau_{A,B})$  is not dense in  $\mathcal{L}$  unless both operators are bounded. In fact, assume that  $A$  is unbounded. It follows from Remark 7.4 that  $D(\tau_{A,B}) \subset \overline{D(L_A)}$ . It follows from the definition that  $D(L_A)$  does not contain any invertible operator. Hence  $D(L_A)$  is not dense in  $\mathcal{L}$  and  $D(\tau_{A,B})$  is not either. Assume now that  $A$  is bounded and  $B$  unbounded. Then  $D(\tau_{A,B})$  is not dense by Remark 1.5.

It is of interest to know under which condition  $\tau_{A,B}$  has compact resolvent. In that case Fredholm's alternative holds for equation (0.1).

The following result is due to Voigt [Vo].

Proposition 5.6 : Let  $G$  be a Banach space and let  $K : [a, b] \rightarrow \mathcal{L}(G)$  be strongly continuous and let  $K_0 x = \int_a^b K(s)x ds$  ( $x \in G$ ). If  $K(s)$  is compact for all  $s \in [a, b]$  then  $K_0$  is compact.

Using this, we obtain

Proposition 5.7 : Assume that the hypotheses of Theorem 5.2 are satisfied . If  $A$  and  $B$  have compact resolvent, then  $\tau_{A,B}$  has compact resolvent.

Proof : The operator  $X \mapsto R(\lambda, A)XR(\lambda, -B) : \mathcal{L} \rightarrow \mathcal{L}$  is compact by [Bo Du, §33 Theorem 3]. Assuming  $0 \notin \sigma(A) + \sigma(B)$  we have by the proof of Theorem 7.3,  $R(0, \tau_{A,B})X = 1/2\pi i \int_{\Gamma} R(\lambda, A)XR(\lambda, -B)d\lambda$ . So  $R(0, \tau_{A,B})$  is compact by Proposition 5.6.  $\diamond$

Remark 5.8 : One sees in a similar way that  $R(\lambda, \tau_{A,B}) = \int_0^{\infty} e^{-\lambda t}U(t)dt$  is compact if  $A$  and  $B$  generate compact  $C_0$ -semigroups (i.e.  $T(t)$  and  $S(t)$  are compact for  $t > 0$ ).

## 6. Counterexamples :

In the particular cases considered in Sections 4 and 5  $\sigma(A) + \sigma(B)$  was closed. Of course, this is not always the case :

Example 6.1 : Let  $E = F = \ell^2$  and  $Ax = ((-\frac{1}{n} + in)x_n)_{n \in \mathbb{N}}$  with maximal domain. Let  $B = -A$ . Then  $(\sigma(A) + \sigma(B)) \cap i\mathbb{R} = \emptyset$  but  $i\mathbb{Z} \subset \overline{\sigma(A) + \sigma(B)}$ . Hence  $i\mathbb{Z} \subset \sigma(\tau_{A,B})$ .

In the following we show by several examples that in general

$$\sigma(\tau_{A,B}) \not\subset \overline{\sigma(A) + \sigma(B)}.$$

Example 6.2 : Let  $A$  be the generator of a  $C_0$ -group such that  $\sigma(A) = \emptyset$ . Then  $\sigma(A) + \sigma(-A) = \emptyset$  but  $0 \in \sigma(\tau_{A,-A})$  since  $I \in D(\tau_{A,-A})$  and  $\tau_{A,-A}(I) = 0$ . For a concrete example see [Hi Ph, 23.16, p. 665].

Example 6.3 : Let  $A$  be the generator of a  $C_0$ -semi-group on a Hilbert space  $H$ . Then, by a result of Groh and Neubrander [Gr Ne, 4.1 Bem.3], [Na, D-IV Remark 2.1 b)],  $s(\tau_{A^*,A}) = \omega(\tau_{A^*,A})$ . Now choose  $A$  such that  $s(A) < \omega(A)$  (see [Na, A-III Example 1.4]). Since  $s(A) = s(A^*)$  and  $\omega(A) = \omega(A^*)$ , it follows that  $\overline{s(A) + s(A^*)} < \omega(A) + \omega(A^*) = \omega(\tau_{A^*,A}) = s(\tau_{A^*,A})$ . This shows that  $\sigma(\tau_{A^*,A}) \not\subset \overline{\sigma(A) + \sigma(A^*)}$ .

Remark 6.4 : If  $A$  generates a  $C_0$ -semigroup on a Hilbert space and  $s(\tau_{A^*,A}) < 0$ , then the above mentioned result of Groh and Neubrander implies that  $\omega(A) = \frac{1}{2}\omega(\tau_{A^*,A}) < 0$ .

Example 6.5 (cf. Phóng [Ph, Example 10]) : Let  $A$  be the generator of a contraction semigroup on a Hilbert space  $H$  such that  $s(A) < \omega(A) = 0$  (see e.g. [Na, A-III Example 1.4]). Moreover, consider the translation group on  $F = C_u(\mathbb{R}, H)$ , the space of all uniformly continuous bounded  $H$ -valued functions on  $\mathbb{R}$  with supremum norm, with generator  $B = \frac{d}{dt}$ . Then  $0 \in \sigma(\tau_{A,B})$  (see [Ph, Example 10]). Thus,  $s(A) + s(B) < s(\tau_{A,B}) = \omega(\tau_{A,B}) = 0$ ; i.e.,  $\sigma(\tau_{A,B}) \not\subset \overline{\sigma(A) + \sigma(B)}$ .

## 7. Appendix : The sum of commuting operators :

Let  $A$  and  $B$  be operators on a Banach space  $G$  with non-empty resolvent set.

Proposition 7.1 : The following are equivalent.

- (i)  $R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$  for some (all)  $\lambda \in \rho(A), \mu \in \rho(B)$ .
- (ii)  $x \in D(A)$  implies  $R(\mu, B)x \in D(A)$  and  $AR(\mu, B)x = R(\mu, B)Ax$  for some (all)  $\mu \in \rho(B)$ .

This is well-known and easy to prove.

We say that  $A$  and  $B$  commute if the equivalent conditions of Proposition 7.1. are satisfied. The operator  $A+B$  is defined by  $(A+B)x = Ax+Bx$  with domain  $D(A+B) = D(A) \cap D(B)$ .

Theorem 7.2 : Assume that  $A$  and  $B$  commute and that one of the operators is bounded. Then  $\sigma(A+B) \subset \sigma(A) + \sigma(B)$ .

Proof : We assume that  $B$  is bounded. Let us assume that  $0 \notin \sigma(A) + \sigma(B)$  ; i.e.  $\sigma(A) \cap (-\sigma(B)) = \emptyset$ . We have to show that  $0 \in \rho(A+B)$ .

There exists a compact set  $K$  with oriented (piecewise  $C^1$ -) boundary  $\Gamma$  such that

$$(7.1) \quad \sigma(-B) \subset \text{int } K \subset K \subset \mathbb{C} \setminus \sigma(A)$$

(where  $\text{int } K$  denotes the interior of  $K$ ), see e.g. [Bo Du, Chapter I § 6]. Let

$$(7.2) \quad Q = 1/2\pi i \int_{\Gamma} R(\lambda, A)R(\lambda, -B)d\lambda$$

Then  $Q \in \mathcal{L}(G), QG \subset D(A)$ . Since  $A$  and  $B$  commute we have

$$(A+B)R(\lambda, A)R(\lambda, -B) = R(\lambda, A) - R(\lambda, -B)$$

(use  $AR(\lambda, A) = \lambda R(\lambda, A) - I$ ,  $BR(\lambda, -B) = I - \lambda R(\lambda, -B)$ ).

Since by Cauchy's theorem,  $\int_{\Gamma} R(\lambda, A)d\lambda = 0$ , it follows that

$$(A+B)Qx = -1/2\pi i \int_{\Gamma} R(\lambda, -B)x d\lambda = -x \quad (x \in G),$$

by Dunford's spectral calculus. Since  $Q$  commutes with  $A+B$ , it follows that  $Q = -(A+B)^{-1}$ .  $\diamond$

In certain cases, even if both operators are unbounded, formula (7.2) can still be used for suitable contours. However, it will represent the resolvent of a certain extension of  $A+B$ .

For  $\theta \in (0, \pi), R > 0$  we let  $\Sigma(\theta, R) = \{z \in \mathbb{C} : |z| \geq R, |\arg z| \leq \theta\}$ .

Theorem 7.3 : Let  $A$  and  $B$  be commuting operators on  $G$ . Assume that there exist  $R > 0$  and  $\theta \in (0, \pi/2)$  such that

$$\Sigma(\theta + \pi/2, R) \subset \rho(A) \text{ and } \sup_{\lambda \in \Sigma(\theta + \pi/2, R)} \|\lambda R(\lambda, A)\| < \infty$$

and there exists  $0 < \theta' < \theta$  such that

$$\Sigma(\pi/2 - \theta', R) \subset \rho(B) \text{ and } \sup_{\lambda \in \Sigma(\pi/2 - \theta', R)} \|\lambda R(\lambda, B)\| < \infty.$$

Then there exists a unique extension  $(A + B)^\sim$  of  $A + B$  such that  $(w, \infty) \subset \rho((A + B)^\sim)$  for some  $w \in \mathbb{R}$  and  $(A + B)^\sim$  commutes with  $A$ .

Moreover

$$(7.3) \quad \sigma((A + B)^\sim) \subset \sigma(A) + \sigma(B).$$

If  $D(A)$  is dense, then  $(A + B)^\sim$  is the closure of  $A + B$ .

Remark : During the work on this manuscript the authors learnt that J. Prüss proved the spectral inclusion (7.3) by the same arguments in a different context (see [Pr, Section 8.3]).

The operator  $(A + B)^\sim$  will be defined as the closure of  $A + B$  for a certain topology.

We define the topology  $\mathcal{T}_A$  on  $G$  induced by the norm  $\|x\|_{\mathcal{T}_A} := \|R(\lambda, A)x\|$ , where  $\lambda \in \rho(A)$ . It is easy to see that different  $\lambda$  yield equivalent norms. Note that  $(G, \|\cdot\|_{\mathcal{T}_A})$  is not complete, in general.

Let  $C$  be an operator on  $G$ . We say that  $C$  is  $A$ -closable if  $x_n \rightarrow 0, x_n \in D(C), Cx_n \rightarrow y$  for  $\mathcal{T}_A$  implies  $y = 0$ . In that case, the  $A$ -closure  $\overline{C}^A$  of  $C$  is the operator on  $G$  defined by

$$D(\overline{C}^A) = \{x \in G : \exists x_n \in D(C), x_n \rightarrow x, \exists y \in G \text{ s.t. } Cx_n \rightarrow y \text{ for } \mathcal{T}_A\},$$

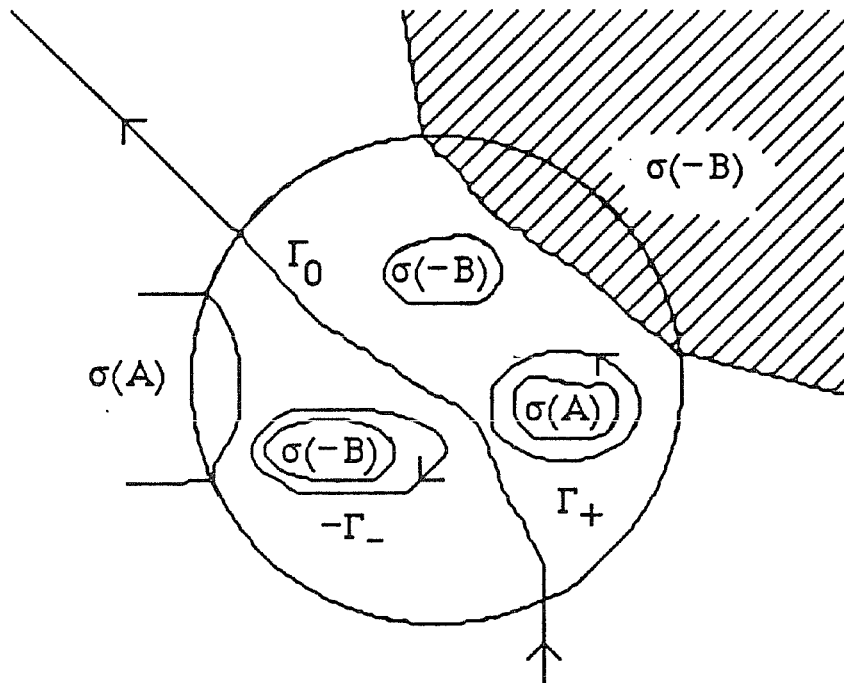
$$\overline{C}^A x = y.$$

$C$  is called  $A$ -closed if  $\overline{C}^A = C$ . It is obvious that every  $A$ -closed operator is closed. Moreover, if  $C$  is  $A$ -closable, then  $C$  is closable and  $\overline{C} \subset \overline{C}^A$ .

We will show that under the hypotheses of Theorem 7.3  $A + B$  is  $A$ -closable and  $\overline{A + B}^A$  is the unique extension commuting with  $A$ .

Proof of Theorem 7.3 : Let  $\mu \in \mathbb{C} \setminus (\sigma(A) + \sigma(B))$ . We show that  $A + B$  is  $A$ -closable and  $\mu \in \rho(\overline{A + B}^A)$ . Replacing  $A$  by  $A - \mu$  we can assume that  $\mu = 0$ . In fact,  $A - \mu$  satisfies the same condition as  $A$  with  $\theta$  replaced by any  $\theta'' < \theta$ . Thus  $\sigma(A) \cap \sigma(-B) = \emptyset$ .

Choose a rectifiable path  $\gamma_0$  lying in  $\{z \in \rho(A) \cap \rho(B) : |z| \leq R\}$  with initial point  $Re^{-i(\pi/2+\theta)}$  and end point  $Re^{i(\pi/2+\theta)}$ . Consider the oriented contour  $\Gamma_0$  consisting of  $\{re^{-i(\pi/2+\theta)} : r \geq R\}$ ,  $\Gamma_0$  and  $\{re^{i(\pi/2+\theta)} : r \geq R\}$ . Then there exists a partition  $\mathbb{C} = \Omega_- \cup \Gamma_0 \cup \Omega_+$  where  $\Omega_+, \Omega_-$  are open such that  $\text{int}(\Sigma(\theta + \pi/2, R)) \subset \Omega_+$  and  $\{re^{i\alpha} : r \geq R, \alpha \in (\theta + \pi/2, 3\pi/2 - \theta)\} \subset \Omega_-$ . Thus  $\{\lambda \in \sigma(A) : |\lambda| \geq R\} \subset \Omega_-$  and  $\{\lambda \in \sigma(-B) : |\lambda| \geq R\} \subset \Omega_+$ .



There exist compact sets  $K_+, K_-$  with oriented boundary  $\Gamma_+$  and  $\Gamma_-$  respectively (piecewise of class  $C^1$ ), and such that

$$\Omega_- \cap \sigma(-B) \subset \text{int } K_- \subset K_- \subset \Omega_- \setminus \sigma(A)$$

and

$$\Omega_+ \cap \sigma(A) \subset \text{int } K_+ \subset K_+ \subset \Omega_+ \setminus \sigma(-B)$$

(see e.g. [Bo Du, I. § 6]).

Let  $\Gamma = \Gamma_0 \cup (-\Gamma_-) \cup \Gamma_+$ .

Since  $\sup_{\lambda \in \Gamma} |\lambda|^2 \|R(\lambda, A)R(\lambda, -B)\| < \infty$ ,

$$Q := 1/2\pi i \int_{\Gamma} R(\lambda, A)R(\lambda, -B)d\lambda$$

$$Q_t := 1/2\pi i \int_{\Gamma} e^{\lambda t} R(\lambda, A)R(\lambda, -B)d\lambda \quad (t > 0)$$

define bounded operators on  $G$  such that  $\lim_{t \downarrow 0} Q_t = Q$  strongly. Denote by

$$T(t) = 1/2\pi i \int_{\Gamma} e^{\lambda t} R(\lambda, A)d\lambda$$

the semigroup generated by  $A$ . Then

$$(7.4) \quad T_A - \lim_{t \rightarrow 0} T(t)x = x$$

for all  $x \in G$  (cf. [Si, Prop. 1.1, Prop. 1.2]) and  $\lim_{t \rightarrow 0} T(t)x = x$  for  $x \in \overline{D(A)}$ .

Since  $A$  and  $B$  commute we have  $(A + B)R(\lambda, A)R(\lambda, -B) = R(\lambda, A) - R(\lambda, -B)$ . Hence  $Q_t x \in D(A + B)$  for all  $x \in G$  and

$$(A + B)Q_t x = 1/2\pi i \int_{\Gamma} e^{\lambda t} R(\lambda, A)x d\lambda - 1/2\pi i \int_{\Gamma} e^{\lambda t} R(\lambda, -B)x d\lambda.$$

Since  $\int_{\Gamma} e^{\lambda t} R(\lambda, -B)d\lambda = 0$  by Cauchy's theorem, we conclude that

$$(7.5) \quad (A + B)Q_t x = T(t)x \quad (t > 0, x \in G)$$

Moreover, for  $x \in D(A + B)$ ,  $Q_t(A + B)x = (A + B)Q_t x$ . Letting  $t \rightarrow 0$ , we obtain  $Q(A + B)x = x$ . Since  $Q$  commutes with  $A$  and  $B$  we conclude that

$$(7.6) \quad (A + B)Qx = Q(A + B)x = x \quad (x \in D(A + B))$$

Next we show that  $Q$  is injective. In fact, let  $Qx = 0$ . Then for  $\mu \in \rho(A) \cap \rho(B)$ , by (7.6),  $R(\mu, A)R(\mu, B)x = (A + B)QR(\mu, A)R(\mu, B)x = 0$ . Hence  $x = 0$ .

We show that  $A + B$  is  $A$ -closable. Let  $\lambda \in \rho(A)$ . Let  $x_n \in D(A + B)$  such that  $x_n \rightarrow 0$  and  $R(\lambda, A)(A + B)x_n \rightarrow R(\lambda, A)y$ . Then

$$R(\lambda, A)Qy = QR(\lambda, A)y = \lim_{n \rightarrow \infty} QR(\lambda, A)(A + B)x_n = \lim_{n \rightarrow \infty} R(\lambda, A)x_n = 0.$$

Since  $Q$  and  $R(\lambda, A)$  are injective it follows that  $y = 0$ .

Now let  $x \in G$ . Then  $Q_t x \rightarrow Qx$  ( $t \downarrow 0$ ),  $Q_t x \in D(A + B)$  and  $T_A - \lim_{t \rightarrow 0} (A + B)Q_t x = T_A - \lim_{t \rightarrow 0} T(t)x = x$ . Hence  $Qx \in D(\overline{A + B^A})$  and  $(\overline{A + B^A})Qx = x$ .

Conversely, let  $x \in D(\overline{A + B^A})$ . Let  $x_n \in D(A + B)$  such that  $x_n \rightarrow x$  and  $R(\lambda, A)(A + B)x_n \rightarrow R(\lambda, A)(\overline{A + B^A})x$ . Then

$$R(\lambda, A)Q(\overline{A + B^A})x = \lim_{n \rightarrow \infty} QR(\lambda, A)(A + B)x_n = \lim_{n \rightarrow \infty} R(\lambda, A)x_n = R(\lambda, A)x.$$

Hence  $Q(\overline{A + B^A})x = x$ . We have shown that  $(\overline{A + B^A})^{-1} = Q$ .

If  $\overline{D(A)} = X$ , then  $\lim_{t \rightarrow 0} (A + B)Q_t x = \lim_{t \rightarrow 0} T(t)x = x$  for  $x \in G$ . So  $Qx \in D(\overline{A + B})$ . Hence  $\overline{A + B} = (\overline{A + B^A})^A$  in that case.

Finally, we show uniqueness. Let  $C$  be an extension of  $A + B$  commuting with  $A$  such that  $(w, \infty) \subset \rho(C)$ . Replacing  $A$  by  $A - \mu$  and  $C$  by  $C - \mu$  we can assume that  $0 \in \rho(C)$ ,  $0 \notin \sigma(A) + \sigma(B)$ .

Let  $x \in G$ . Then  $CQ_t x = (A + B)Q_t x = T(t)x$ . Hence  $Q_t x = C^{-1}T(t)x$  ( $t > 0$ ). Consequently,  $Qx = \lim_{t \downarrow 0} Q_t x = \lim_{t \rightarrow 0} C^{-1}T(t)x = C^{-1}x$  since  $T_A - \lim_{t \downarrow 0} T(t)x = x$  and  $C$  commutes with  $A$ . Thus  $-R(0, \overline{A + B^A}) = Q = C^{-1}$ . Consequently,  $C = \overline{A + B^A}$ .  $\diamond$

**Remark 7.4** : It follows from the proof that  $D((A + B)^\sim) = \text{range } Q \subset \overline{D(A) \cap D(B)}$ .

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