SOME MULTIPLE SERIES IDENTITIES

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Abstract

The object of the present paper is to establish three general multiple series identities and to exhibit the connection of each of these identities with various results given recently in the mathematical literature. Some hypergeometric transformation and reduction formulas are also deduced as interesting consequences of the series identities proven here.

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1. Introduction and Definitions

Recently, Srivastava ([9],[10]), Buschman and Srivastava [2], Grosjean and Sharma [4], and Grosjean and Srivastava [5] established a number of double and multiple series identities involving essentially arbitrary coefficients (see also Srivastava and Raina [12]). The aim of this paper is to establish three substantially more general multiple series identities involving similar coefficients.

In terms of the Pochhammer symbol $(\lambda)_n := \Gamma(\lambda+n)/\Gamma(\lambda)$, the generalized hypergeometric ${}_pF_q$ series is defined by [11, pp. 19-20]:

$$ {}_pF_q(\alpha_1,\cdots,\alpha_p;\ \beta_1,\cdots,\beta_q\,;\ z) \equiv {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q;\end{matrix}\ z\right] := \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_n}\,\frac{z^n}{n!}, \tag{1.1}$$

provided that the series in (1.1) converges (or terminates). The generalized Lauricella series in several variables is defined and represented as follows [11, p. 37]:

$$\begin{aligned} &F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\begin{pmatrix} z_1 \\ \vdots \\ z_n\end{pmatrix} \\ &\equiv F^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(n)}]:[(b'):\phi'];\cdots; \\ [(c):\psi',\cdots,\psi^{(n)}]:[(d'):\delta'];\cdots;\end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix}[(b^{(n)}):\phi^{(n)}]; \\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1,\cdots,z_n\right) \\ &:= \sum_{m_1,\cdots,m_n=0}^{\infty} \Lambda(m_1,\cdots,m_n)\,\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!}, \end{aligned} \tag{1.2}$$

where, for convenience,

$$\begin{aligned} &\Lambda(m_1,\cdots,m_n) \\ &:= \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j'+\cdots+m_n\theta_j^{(n)}}\ \sum_{j=1}^{B'}(b_j')_{m_1\phi_j'}\cdots\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j'+\cdots+m_n\psi_j^{(n)}}\ \prod_{j=1}^{D'}(d_j)_{m_1\delta_j'}\cdots\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}]\right)_{m_n\delta_j^{(n)}}}, \end{aligned} \tag{1.3}$$

the coefficients

$$\theta_j^{(k)}\ (j=1,\cdots,A),\quad \phi_j^{(k)}\ (j=1,\cdots,B^{(k)}),\quad \psi_j^{(k)}\ (j=1,\cdots,C),$$

$$\delta_j^{(k)}\ (j=1,\cdots,D^{(k)});\qquad (\forall k\in\{1,\cdots,n\})$$

are real and non-negative, and (a) abbreviates the array of A parameters $a_1,\cdots,a_A$; $\left(b^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}\ (j=1,\cdots,B^{(k)};\quad \forall k\in\{1,\cdots,n\}),$$

with similar interpretations for $\left(c^{(k)}\right)$, *et cetera.*

2. The General Series Identities

Theorem 1. *Let $\omega(m_1,\cdots,m_r)$ be a single-valued, bounded, and real or complex function of r non-negative integer-valued parameters $m_1,\cdots,m_r$.*

Then

$$\sum_{m_1,\cdots,m_r=0}^{\infty}\omega(m_1,\cdots,m_r)\prod_{i=1}^{r}\left\{\left(1-c_i+a_i\,\frac{(a_i+1)_{m_i}}{(c_i)_{m_i}}\right)\frac{x_i^{m_i}}{1-c_i+a_i}\right\}$$
$$=\sum_{m_1,j_1,\cdots,m_r,j_r=0}^{\infty}\Omega(m_1+j_1,\cdots,m_r+j_r)\prod_{i=1}^{r}\left\{\frac{(a_i)_{j_i}}{(c_i)_{j_i}}\,x_i^{m_i+j_i}\right\},\tag{2.1}$$

provided that each of the series involved is absolutely convergent.

Theorem 2. *Let $\Omega(m)$ represent a single-valued, bounded, and real or complex function of the non-negative integer-valued parameter m.*

Then

$$\sum_{m_1,\cdots,m_r=0}^{\infty}\Omega(m_1+\cdots+m_r)\prod_{i=1}^{r}\left\{\left(1-c_i+a_i\,\frac{(a_i+1)_{m_i}}{(c_i)_{m_i}}\right)\frac{x^{m_i}}{1-c_i+a_i}\right\}$$
$$=\sum_{j_1,\cdots,j_r,m=0}^{\infty}\Omega(m+j_1+\cdots+j_r)(r)_m\,\frac{x^m}{m!}\prod_{i=1}^{r}\left\{\frac{(a_i)_{j_i}}{(c_i)_{j_i}}\,x^{j_i}\right\},\tag{2.2}$$

provided that each of the series involved is absolutely convergent.

Theorem 3. *Let* $\Omega(m)$ *represent a single-valued, bounded, and real or complex function of the non-negative integer-valued parameter* m.

Then

$$\sum_{m_1,\cdots,m_r=0}^{\infty} \Omega(m_1+\cdots+m_r)\,\frac{(\alpha)_{m_1+m_2}}{(\alpha)_{m_1}(\alpha)_{m_2}} \prod_{i=1}^{r}\left\{\frac{(\mu_i)_{m_i}}{m_i!}\,x^{m_i}\right\}$$
$$= \sum_{m,n=0}^{\infty} \Omega(m+2n)(\mu_1+\cdots+\mu_r+2n)_m\,\frac{(\mu_1)_n(\mu_2)_n}{(\alpha)_n}\,\frac{x^{m+2n}}{m!\,n!}, \tag{2.3}$$

provided that each of the series involved is absolutely convergent.

Proofs of Theorems 1, 2, and 3. To establish Theorem 1, we first denote the left-hand side of (2.1) by Δ_1 and then use the sum [6, p. 151, Equation (7.1.1)]:

$$\sum_{k=0}^{n} \frac{(a)_k}{(c)_k} = \frac{1}{1-c+a}\left[1-c+a\,\frac{(a+1)_n}{(c)_n}\right]. \tag{2.4}$$

We thus obtain

$$\Delta_1 = \sum_{m_1,\cdots,m_r=0}^{\infty} \omega(m_1,\cdots,m_r) \prod_{i=1}^{r}\left\{\sum_{j_i=0}^{m_i} \frac{(a_i)_{j_i}}{(c_i)_{j_i}}\,x_i^{m_i}\right\}. \tag{2.5}$$

For the sake of clarity, we choose to limit ourselves to the summations with respect to m_1 and j_1 alone, and we have

$$\sum_{m_1=0}^{\infty} \omega(m_1,m_2,\cdots,m_r) \sum_{j_1=0}^{m_1} \frac{(a_1)_{j_1}}{(c_1)_{j_1}}\,x_1^{m_1}$$
$$= \sum_{m_1,j_1=0}^{\infty} \omega(m_1+j_1,m_2,\cdots,m_r)\,\frac{(a_1)_{j_1}}{(c_1)_{j_1}}\,x_1^{m_1+j_1}. \tag{2.6}$$

Repeating the same operations for m_i and j_i $(i=2,3,\cdots,r)$ in (2.5), we arrive at the desired result (2.1).

Theorem 2 would follow from the assertion (2.1) of Theorem 1 when we set

$$\omega(m_1,\cdots,m_r) = \Omega(m_1+\cdots+m_r) \quad \text{and} \quad x_i = x \quad (\forall i \in \{1,\cdots,r\})$$

and make use of the series identity [8, p. 166, Theorem 2]:

$$\sum_{m_1,\cdots,m_r=0}^{\infty} f(m_1+\cdots+m_r)(\mu_1)_{m_1}\cdots(\mu_r)_{m_r}\,\frac{z^{m_1+\cdots+m_r}}{m_1!\cdots m_r!} = \sum_{m=0}^{\infty} f(m)(\mu_1+\cdots+\mu_r)_m\,\frac{z^m}{m!}. \tag{2.7}$$

To establish Theorem 3, we denote the left-hand side of (2.3) by Δ_2 and use the Chu-Vandermonde theorem [1, p. 3]:

$$\frac{(\alpha)_{m+n}}{(\alpha)_m\,(\alpha)_n} = \sum_{j=0}^{\min(m,n)} \binom{m}{j}\binom{n}{j}\frac{j!}{(\alpha)_j} \qquad (m,n\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}), \tag{2.8}$$

and we thus find that

$$\Delta_2 = \sum_{m_1,\cdots,m_r,j=0}^{\infty} \Omega(m_1+\cdots+m_r+2j)\,\frac{(\mu_1+j)_{m_1}}{m_1!}\,\frac{(\mu_2+j)_{m_2}}{m_2!} \cdot \frac{(\mu_3)_{m_3}}{m_3!}\cdots\frac{(\mu_r)_{m_r}}{m_r!}\,\frac{(\mu_1)_j(\mu_2)_j}{(\alpha)_j}\,\frac{x^{2j+m_1+\cdots+m_r}}{j!}. \tag{2.9}$$

We now use (2.7) to arrive at the desired result (2.3).

3. Transformation and Reduction Formulas

From the assertion (2.3) of Theorem 3, with

$$\Omega(n) = \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \qquad (n\in\mathbb{N}_0),$$

we arrive at the following multiple hypergeometric identity involving the generalized

Lauricella function defined by (1.2):

$$F^{p+1:1;1;1;\cdots;1}_{q:1;1;0;\cdots;0}\left(\begin{matrix}[(a_p):1,\cdots,1],\ [\alpha:1,1,0,\cdots,0]: \\ [(b_q):1,\cdots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}(\mu_1:1); & (\mu_2:1); & (\mu_3:1); & \cdots; & (\mu_r:1); \\ (\alpha:1); & (\alpha:1); & \text{———}; & \cdots; & \text{———};\end{matrix}\ x,\cdots,x\right) \tag{3.1}$$

$$=\sum_{n=0}^{\infty}\frac{(a_1)_{2n}\cdots(a_p)_{2n}}{(b_1)_{2n}\cdots(b_q)_{2n}}\ \frac{(\mu_1)_n(\mu_2)_n}{(\alpha)_n}\ \frac{x^{2n}}{n!}$$

$$\cdot\ {}_pF_q\left[\begin{matrix}\mu_1+\cdots+\mu_r+2n,\ a_1+2n,\cdots,a_p+2n; \\ b_1+2n,\cdots,b_q+2n;\end{matrix}\ x\right].$$

For $p=q=1$, (3.1) reduces at once to

$$F^{2:1;1;1;\cdots;1}_{1:1;1;0;\cdots;0}\left(\begin{matrix}(a:1,\cdots,1), & (\alpha:1,1,0,\cdots,0): \\ & (b:1,\cdots,1):\end{matrix}\right.$$

$$\left.\begin{matrix}(\mu_1:1); & (\mu_2:1); & (\mu_3:1); & \cdots; & (\mu_r:1); \\ (\alpha:1); & (\alpha:1); & \text{———}; & \cdots; & \text{———};\end{matrix}\ x,\cdots,x\right) \tag{3.2}$$

$$=\sum_{n=0}^{\infty}\frac{(a)_{2n}(\mu_1)_n(\mu_2)_n}{(b)_{2n}(\alpha)_n}\ \frac{x^{2n}}{n!}\ {}_2F_1\left[\begin{matrix}\mu_1+\cdots+\mu_r+2n,\ a+2n; \\ b+2n;\end{matrix}\ x\right].$$

Making use of the known result [3, p. 108, Equation 2.10(1)]:

$${}_2F_1\left[\begin{matrix}-N,\beta; \\ \gamma;\end{matrix}\ z\right]=\frac{(\gamma-\beta)_N}{(\gamma)_N}\ {}_2F_1\left[\begin{matrix}-N,\beta; \\ \beta-\gamma-N+1;\end{matrix}\ 1-z\right]\quad (N\in\mathbb{N}_0) \tag{3.3}$$

to transform the Gaussian hypergeometric ${}_2F_1$ function in (3.2) when

$$x=\frac{1}{2}\qquad\text{and}\qquad \mu_i=-M_i\quad(\forall i\in\{1,\cdots,r\}),$$

and interpreting the resulting sum by means of (3.2) itself, we arrive at the following transformation formula:

$$
\begin{aligned}
&F^{2:1;1;1;\cdots;1}_{1:1;1;0;\cdots;0}\left(\begin{matrix}(a:1,\cdots,1), & (\alpha:1,1,0,\cdots,0): \\ & (b:1,\cdots,1):\end{matrix}\right.\\
&\qquad\left.\begin{matrix}(-M_1:1); & (-M_2:1); & (-M_3:1) & ;\cdots; & (-M_r:1); \\ (\alpha:1); & (\alpha:1); & \text{———}; & ;\cdots; & \text{———};\end{matrix}\ \frac{1}{2},\cdots,\frac{1}{2}\right)\\
&= \frac{(b-a)_{M_1+\cdots+M_r}}{(b)_{M_1+\cdots+M_r}}\ F^{2:1;1;1;\cdots;1}_{1:1;1;0;\cdots;0}\left(\begin{matrix}(a:1,\cdots,1), \quad (\alpha:1,1,0,\cdots,0): \\ (1-b+a-M_1-\cdots-M_r:1,\cdots,1):\end{matrix}\right.\\
&\qquad\left.\begin{matrix}(-M_1:1); & (-M_2:1); & (-M_3:1) & ;\cdots; & (-M_r:1); \\ (\alpha:1); & (\alpha:1); & \text{———}; & ;\cdots; & \text{———};\end{matrix}\ \frac{1}{2},\cdots,\frac{1}{2}\right).
\end{aligned}
\tag{3.4}
$$

Similarly, if we take $x=\frac{1}{2}$ and $a=-N$ in (3.2), and proceed as above, we obtain the following transformation formula:

$$
\begin{aligned}
&F^{2:1;1;1;\cdots;1}_{1:1;1;0;\cdots;0}\left(\begin{matrix}(-N:1,\cdots,1), & (\alpha:1,1,0,\cdots,0): \\ & (b:1,\cdots,1):\end{matrix}\right.\\
&\qquad\left.\begin{matrix}(\mu_1:1); & (\mu_2:1); & (\mu_3:1) & ;\cdots; & (\mu_r:1); \\ (\alpha:1); & (\alpha:1); & \text{——}; & ;\cdots; & \text{——};\end{matrix}\ \frac{1}{2},\cdots,\frac{1}{2}\right)\\
&= \frac{(b-\mu_1-\cdots-\mu_r)_N}{(b)_N}\ F^{2:1;1;1;\cdots;1}_{1:1;1;0;\cdots;0}\left(\begin{matrix}(-N:1,\cdots,1), \quad (\alpha:1,1,0,\cdots,0): \\ (1-b+\mu_1+\cdots+\mu_r-N:1,\cdots,1):\end{matrix}\right.\\
&\qquad\left.\begin{matrix}(\mu_1:1); & (\mu_2:1); & (\mu_3:1) & ;\cdots; & (\mu_r:1); \\ (\alpha:1); & (\alpha:1); & \text{——}; & ;\cdots; & \text{——};\end{matrix}\ \frac{1}{2},\cdots,\frac{1}{2}\right).
\end{aligned}
\tag{3.5}
$$

Finally, if we use the familiar result [1, p. 11, Equation 2.4(2)]:

$$
{}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \frac{1}{2}(\alpha+\beta+1);\end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left[\frac{1}{2}(\alpha+\beta+1)\right]}{\Gamma\left[\frac{1}{2}(\alpha+1)\right]\,\Gamma\left[\frac{1}{2}(\beta+1)\right]}
$$

in (3.2) when

$$
x=\frac{1}{2} \qquad \text{and} \qquad b=\frac{1}{2}(a+\mu_1+\cdots+\mu_r+1),
$$

we obtain the following reduction formula:

$$\begin{aligned}
&F^{2:1;1;1;\cdots;1}_{1:1;1;0;\cdots;0}\left(\begin{array}{c}(a:1,\cdots,1),\quad(\alpha:1,1,0,\cdots,0):\\ \left(\frac{1}{2}(a+\mu_1+\cdots+\mu_r+1):1,\cdots,1\right):\end{array}\right.\\
&\qquad\left.\begin{array}{cccccc}(\mu_1:1); & (\mu_2:1); & (\mu_3:1); & \cdots; & (\mu_r:1); & \frac{1}{2},\cdots,\frac{1}{2}\\ (\alpha:1); & (\alpha:1); & \underline{\qquad}; & \cdots; & \underline{\qquad}; & \end{array}\right)\\
&\qquad\qquad=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left[\frac{1}{2}(a+\mu_1+\cdots+\mu_r+1)\right]}{\Gamma\left[\frac{1}{2}(\mu_1+\cdots+\mu_r+1)\right]\Gamma\left[\frac{1}{2}(a+1)\right]}\\
&\qquad\qquad\cdot\ {}_3F_2\left[\begin{array}{c}\frac{1}{2}a,\mu_1,\mu_2;\\ \alpha,\ \frac{1}{2}(\mu_1+\cdots+\mu_r+1);\end{array}1\right].
\end{aligned}\tag{3.6}$$

Remarks. Setting

$$x_2=x_3=\cdots=x_r=0$$

in the assertion (2.1) of Theorem 1, we readily obtain the known result [7, p. 159, Equation (II.3)]. On the other hand, in view of the series identity (2.7), Theorem 2 with

$$c_i=1\quad(\forall i\in\{1,\cdots,r\})$$

yields another known result [9, p. 297, Equation (16)]. Finally, in their special cases when $r=2$, our formulas (3.4), (3.5), and (3.6) would correspond to the known results [4, p. 102, Equation (9); p. 108, Equation (15)] and [5, p. 297, Equation (46)].

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