A Helical Finite Element Applied to the Vibration Analysis of a Mechanical Heart Valve

by

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ABSTRACT

A versatile form of a spatially curved and twisted (or helical) finite element which can model any spatial curve-like structure in terms of helices has been developed. The present formulation can handle static and free vibration problems. The shape functions are based on the solutions of equations governing static equilibrium. The superiority of the element over the existing ones is illustrated through two examples. Later, with a view to developing a non-invasive diagnostic technique for evaluation of implanted artificial heart valves by an analysis of their acoustic signature, the helical element is used to model and analyze the vibration of the struts of a Bjork-Shiley tilting disc valve. The natural frequencies thus determined are compared with the experimentally obtained frequencies.

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&

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Chapter 1

Introduction

1.1 The Topic

A spatially curved and twisted element, otherwise called a helical element, was originally introduced by Mottershead [9] in his paper that dealt with the frequencies of helical springs. This work was confined to rods of circular cross sections. Later Tabarrok *et al.* [17] developed a constant strain helical element. Yap [20] in his masters thesis extended the work done by Mottershead to any doubly symmetric cross section. Since the shape functions of this element are solutions of the governing differential equations for static equilibrium, static analysis problems can be solved exactly using only one element. For dynamic problems, the element outperforms the conventional prismatic beam element.

While Yap [20] has provided the basic formulation for the development of the stiffness and mass matrices for a helical element, his formulation is limited in application to a system of elements having identical local coordinate systems. This occurs, for instance, in helical springs. For more general spatial rods made up of helical elements of different curvatures, twists and local coordinate systems, it is essential to establish a common (global) coordinate system and transform the element matrices to this common coordinate system. This task is undertaken in this report thus generalizing the applicability of the helical element.

The thus formulated versatile helical element is used in the vibration analysis of a Bjork-Shiley tilting disc valve shown in Fig. 1.1. This is carried out to study the diagnostic potential of the frequency spectra emitted by implanted artificial heart valves to detect the presence of valve breakage. Since heart valves open and close in accordance with the heart's pulsation at the rate of approximately 72 times per minute for a lifetime it is evident that an artificial heart valve will ultimately fail due to fatigue. For the type of valve studied

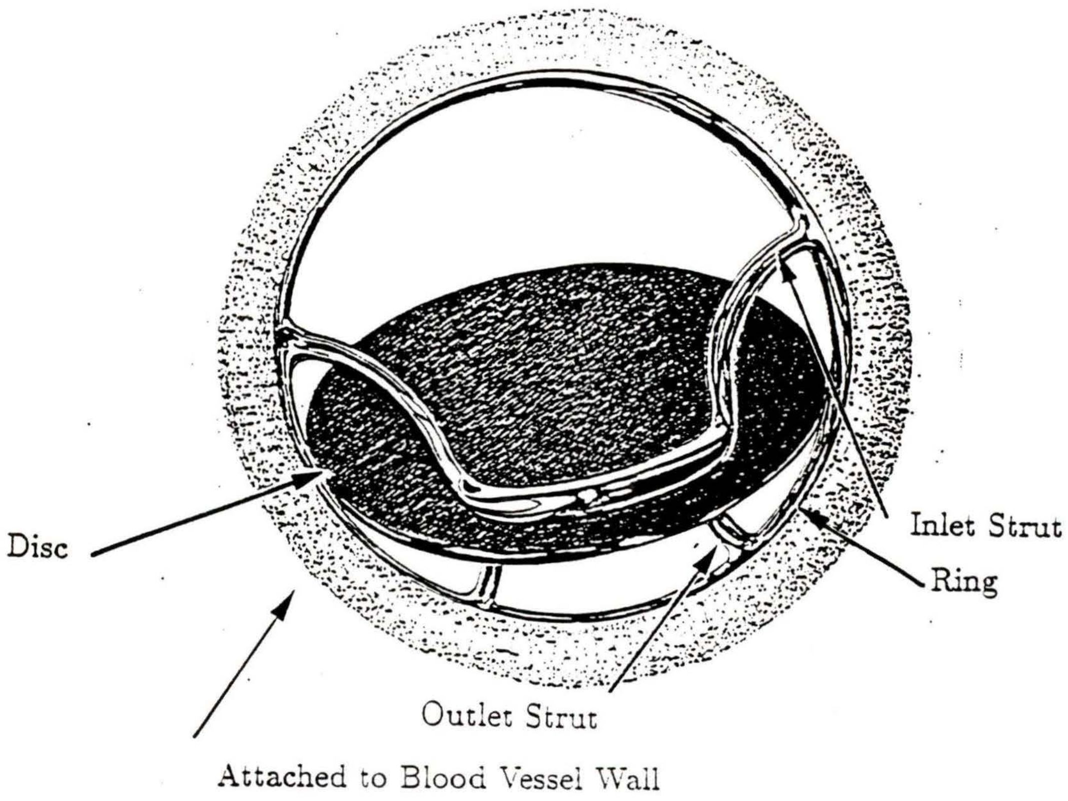

Figure 1.1: Bjork-Shiley Tilting Disc Valve

here one can see that the attachment points of the strut ends with the valve base ring , otherwise called the roots, are stress concentration points and are susceptible to fatigue and fracture.

The repetitive nature of loading brings about progressive failure, through propagation of minute cracks, at stress levels well below the ultimate strength of the joints. This being the case, it is highly improbable for failures to occur at the two ends of the struts simultaneously. That is, when one end has broken off, the other end is likely to remain intact for a while longer. Clearly, under these conditions the stress levels at the unbroken end will increase speeding up its progression to complete fracture. During this period the valve is likely to perform quite well and the patient will remain completely oblivious of the impending danger of the failure of the heart valve. Now, a fracture at one end of the strut will significantly change the natural frequencies of vibrations of that strut. This change in vibration characteristics of the system provides an opportunity for diagnosis. That is, an analysis of the frequency spectrum before and after the fracture of one end of the strut should show marked differences. By using this change for diagnostic purposes the physician could be alerted of the problem and can take corrective measures.

In this thesis the free vibration frequencies of the struts of the mechanical valve shown in Fig.1.1 are computed theoretically and measured experimentally. Initially prismatic beam elements in the ANSYS program [3] are used to compute the natural frequencies of the struts. Because the struts are spatial curves in an arbitrary setting, the present formulation of the helical finite element is subsequently used for their analysis. Finally, the diagnostic feature of the natural frequencies of intact and broken struts is illustrated.

1.2 Approach Outline

The main body of the report begins with the derivation of the governing differential equations for the helical element in Chapter 2. In Chapter 3 these equations are simplified and their solutions are used as shape functions for the helical element. Later on, the stiffness and mass matrices of the element are determined, the former being exact and the latter approximate. Chapter 4 introduces a way to consider a chain of helical elements to model any spatial curve. Here, the properties of the helical element such as the curvature, torsion and pitch are calculated using knowledge of the nodal coordinates and the axial vector. Also, the transformation matrix between the global coordinate system and the one in which stiffness and mass matrices are known is developed. The application of such a helical element is considered in Chapter 5. Two problems, one static and the other dynamic, are dealt with in a more general situation than those in the literature and the results are compared. In chapter 6, an introduction to artificial heart valves is given emphasizing their mechanical problems. Following this, the helical element is used to determine the natural frequencies of the struts of a Bjork-Shiley artificial heart valve and they are compared with the experimentally obtained frequencies and those obtained using ANSYS. In the appendix the computer program which implements the spatially curved and twisted element is listed. As an example, an input file for the outlet strut of the Bjork-Shiley heart valve is given.

Chapter 2

Development

2.1 Overview

This chapter presents the derivation of the equations governing the helix element used in this work. The governing equations of helical rods were developed by Love [7]. An alternative formulation of these equations as given by Wittrick [22] is being used here. The element formulation adopted here follows the work of Tabarrok *et al.* [17], the major difference being in the number of curvatures used to prescribe the geometry of the rod. Tabarrok *et al.* use three while here we employ the principal axes of the cross section and therefore need only two curvatures. In this thesis the final equations are written in scalar form.

2.2 Helical Element Geometry

Consider the spatially curved and twisted rod partially shown in Fig. 2.1. The coordinate system about which the above rod is generally defined, that is, $O'X'Y'Z'$, is defined as the local coordinate system. It can be seen that the locus of the centroids of the cross sections traces a curve in three dimensional space. The arc length measured from the initial point is s and the position vector to any point on the arc is $\mathbf{r}$. There is a coordinate triad moving along the arc called the natural coordinate system which has a normal $\mathbf{n}$, binormal

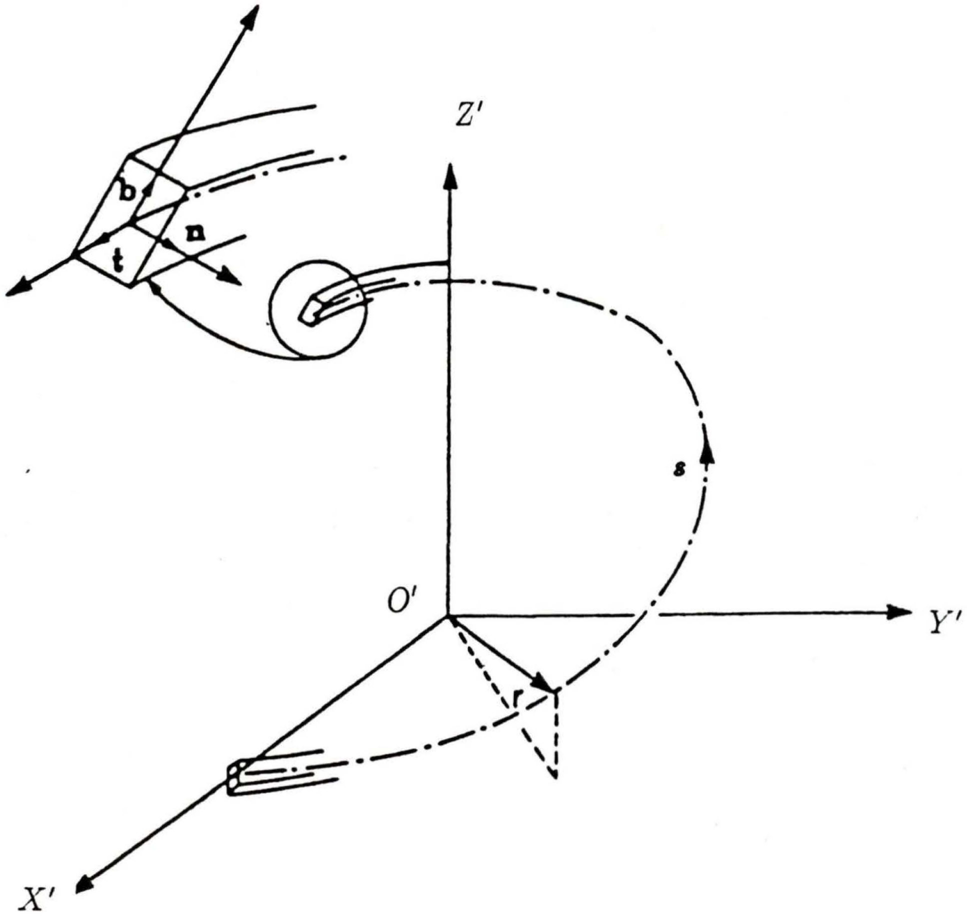

Figure 2.1: Local and Natural Coordinate Systems[17]

b, and tangent t. These bases obey the following frame rate equation called Frenet equations [5].

$$\left\{ \begin{array}{c} \frac{d\mathbf{n}}{ds} \\ \\ \frac{d\mathbf{b}}{ds} \\ \\ \frac{d\mathbf{t}}{ds} \end{array} \right\} = \begin{bmatrix} 0 & \tau & -\kappa \\ -\tau & 0 & 0 \\ \kappa & 0 & 0 \end{bmatrix} \left\{ \begin{array}{c} \mathbf{n} \\ \mathbf{b} \\ \mathbf{t} \end{array} \right\} \tag{2.1}$$

In the above equation τ is the torsion and κ is the curvature and they are geometric properties of the helix. If the origin of the coordinate system moves along the helix at unit speed, κ would be the angular velocity of the normal about the binormal and τ the angular velocity of the normal about the tangent where the right hand rule determines the positive directions. If κ and τ are functions of s then the resulting curve is called a generalized helix. Our formulation assumes them to be constant along the curve. The following finite element formulation is based on the natural coordinate system. There are four vector-valued functions of s to be introduced, viz., displacement $\mathbf{U}$, infinitesimal rotation $\boldsymbol{\Theta}$, internal force $\mathbf{Q}$ and internal moment $\mathbf{M}$. Again these vectors can be resolved into components along the natural coordinate system bases, that is, along the normal, binormal and tangent directions. Thus we can write the four functions as:

$$\mathbf{U}(s) = u\mathbf{n} + v\mathbf{b} + w\mathbf{t} \tag{2.2}$$

$$\boldsymbol{\Theta}(s) = \theta_u\mathbf{n} + \theta_v\mathbf{b} + \theta_w\mathbf{t} \tag{2.3}$$

$$\mathbf{Q}(s) = Q_u\mathbf{n} + Q_v\mathbf{b} + Q_w\mathbf{t} \tag{2.4}$$

$$\mathbf{M}(s) = M_u\mathbf{n} + M_v\mathbf{b} + M_w\mathbf{t} \tag{2.5}$$

In the above equations u, v, and w are the displacements in the normal, binormal and tangent directions respectively. Likewise, θ_u, θ_v and θ_w are infinitesimal rotations about the normal, binormal and tangent directions respectively. Also, unless otherwise specified the subscripts u, v and w refer to the components of the parameter in the context in the normal, binormal and tangent directions respectively.

It is necessary to obtain the derivatives of these vector-valued functions with respect to s. As an example, consider a vector-valued function $\mathbf{V}(s)$, resolved in the natural frame,

$$\mathbf{V} = V_u\mathbf{n} + V_v\mathbf{b} + V_w\mathbf{t} \tag{2.6}$$

Applying the product rule of differentiation,

$$\frac{d\mathbf{V}}{ds} = \frac{dV_u}{ds}\mathbf{n} + V_u\frac{d\mathbf{n}}{ds} + \frac{dV_v}{ds}\mathbf{b} + V_v\frac{d\mathbf{b}}{ds} + \frac{dV_w}{ds}\mathbf{t} + V_w\frac{d\mathbf{t}}{ds} \tag{2.7}$$

Recalling that $\frac{d\mathbf{n}}{ds}$, *etc.*, are given by the frame rate equation 2.1, and substituting this into the above equation yields:

$$\begin{aligned}\frac{d\mathbf{V}}{ds} &= \frac{dV_u}{ds}\mathbf{n} + \frac{dV_v}{ds}\mathbf{b} + \frac{dV_w}{ds}\mathbf{t} + V_u\left[\tau\mathbf{b} - \kappa\mathbf{t}\right] + V_v\left[-\tau\mathbf{n}\right] + V_w\left[\kappa\mathbf{n}\right] \\ &= \left[\frac{dV_u}{ds} - \tau V_v + \kappa V_w\right]\mathbf{n} + \left[\frac{dV_v}{ds} + \tau V_u\right]\mathbf{b} + \left[\frac{dV_w}{ds} - \kappa V_u\right]\mathbf{t} \qquad (2.8)\end{aligned}$$

2.3 Derivation of Governing Equations

2.3.1 Kinematic Equations

Consider a rod of arbitrary length (but small enough for $\mathbf{\Theta}$ to be small), in a deformed state, as shown in Fig. 2.2. The total change of rotation vector $\mathbf{\Theta}$, over the length of the rod is given by [17]:

$$\mathbf{\Theta}_2 - \mathbf{\Theta}_1 = \int_{s_1}^{s_2} \mathbf{K}(s)ds, \qquad (2.9)$$

where $\mathbf{K}(s)$ is the change in curvature-twist along s whose elements are $[k_u\ k_v\ k_w]^T$. Since,

$$\int_{s_1}^{s_2} \frac{d\mathbf{\Theta}}{ds}ds = \mathbf{\Theta}_2 - \mathbf{\Theta}_1, \qquad (2.10)$$

we can rewrite the above equation in an alternative form:

$$\int_{s_1}^{s_2} \left[\frac{d\mathbf{\Theta}}{ds} - \mathbf{K}(s)\right] ds = \mathbf{0} \qquad (2.11)$$

Since the length of the rod is arbitrary, the integrand of this equation must be zero. That is,

$$\mathbf{K}(s) = \frac{d\mathbf{\Theta}}{ds} \qquad (2.12)$$

This is the generalized strain-displacement equation for the rotational degrees of freedom.

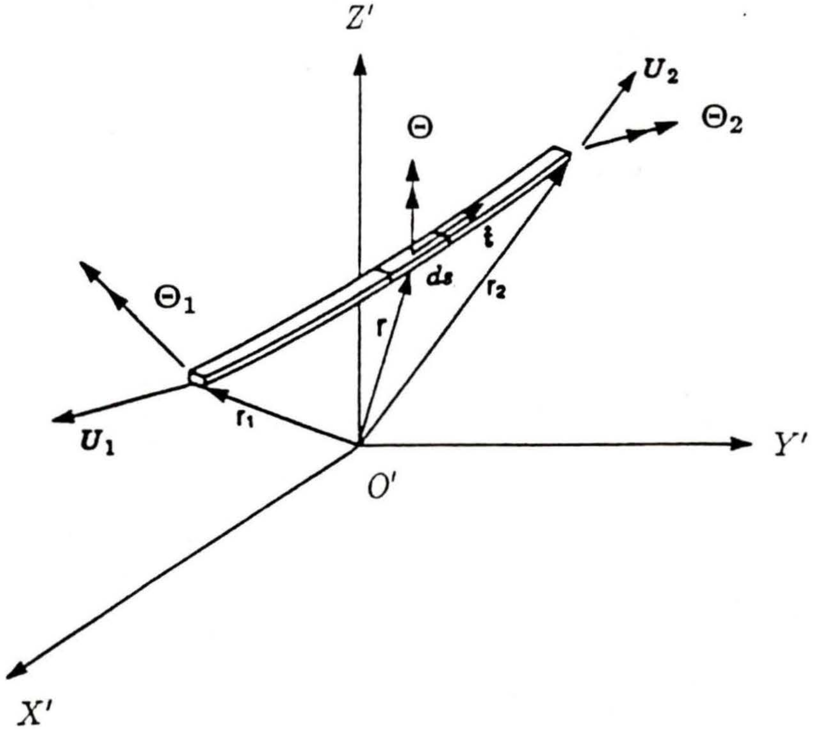

Figure 2.2: Displacements and Rotations of a Spatial Rod[17]

For the translational degrees of freedom, we begin with the following expressions for the change of displacement vector $\mathbf{U}$, over the length of the rod,

$$\mathbf{U}_2 - \mathbf{U}_1 - \int_{s_1}^{s_2} (\Theta \times \mathbf{t}) ds = \int_{s_1}^{s_2} \mathbf{E} ds \tag{2.13}$$

where $\mathbf{E}$ is the strain vector whose elements are $[\epsilon_u \ \epsilon_v \ \epsilon_w]^T$. In this equation the $(\Theta \times \mathbf{t})$ term accounts for the transverse displacements due to rigid body rotations. Subtracting these displacements from the total displacement change, $\mathbf{U}_2 - \mathbf{U}_1$, we obtain those displacements due to elastic deformations, given by the integral of the strain vector $\mathbf{E}$, over the length of the rod. We can express this equation in the form:

$$\int_{s_1}^{s_2} \left[\frac{d\mathbf{U}}{ds} + (\mathbf{t} \times \mathbf{\Theta}) - \mathbf{E} \right] ds = \mathbf{0} \tag{2.14}$$

The integrand must be zero due to the arbitrariness of length and so:

$$\mathbf{E} = \frac{d\mathbf{U}}{ds} + (\mathbf{t} \times \mathbf{\Theta}) \tag{2.15}$$

Thus, now we have the kinematic equations in vector form. It may be noted that the rotations and displacements are independent vectors.

2.3.2 Equilibrium Equations

Considering Fig. 2.3, where $\mathbf{Q}$ and $\mathbf{M}$ are internal forces and moments respectively and $\mathbf{F}$(s) and $\mathbf{G}$(s) are applied forces and moments per unit length respectively, the force equilibrium equation can be written as:

$$\mathbf{Q}_2 - \mathbf{Q}_1 + \int_{s_1}^{s_2} \mathbf{F}(s) ds = \frac{\partial}{\partial t} \int_{s_1}^{s_2} m\dot{\mathbf{U}} ds \tag{2.16}$$

We may rewrite this as:

$$\int_{s_1}^{s_2} \left[\frac{\partial \mathbf{Q}}{\partial s} + \mathbf{F}(s) - \frac{\partial}{\partial t}(m\dot{\mathbf{U}}) \right] ds = 0 \tag{2.17}$$

Since the integrand must be identically zero and the mass per unit length m does not vary with time, the force equilibrium equation becomes:

$$\frac{\partial \mathbf{Q}}{\partial s} + \mathbf{F}(s) = m\ddot{\mathbf{U}} \tag{2.18}$$

To derive the moment equilibrium equations we begin with:

$$\begin{aligned} \mathbf{M}_2 - \mathbf{M}_1 + (\mathbf{r}_2 \times \mathbf{Q}_2) - (\mathbf{r}_1 \times \mathbf{Q}_1) & \\ + \int_{s_1}^{s_2} (\mathbf{r} \times \mathbf{F}(s) + \mathbf{G}(s)) ds &= \frac{\partial}{\partial t} \int_{s_1}^{s_2} m\mathbf{r} \times \dot{\mathbf{U}} ds \\ &+ \frac{\partial}{\partial t} \int_{s_1}^{s_2} \mathbf{j}\dot{\mathbf{\Theta}} ds, \end{aligned} \tag{2.19}$$

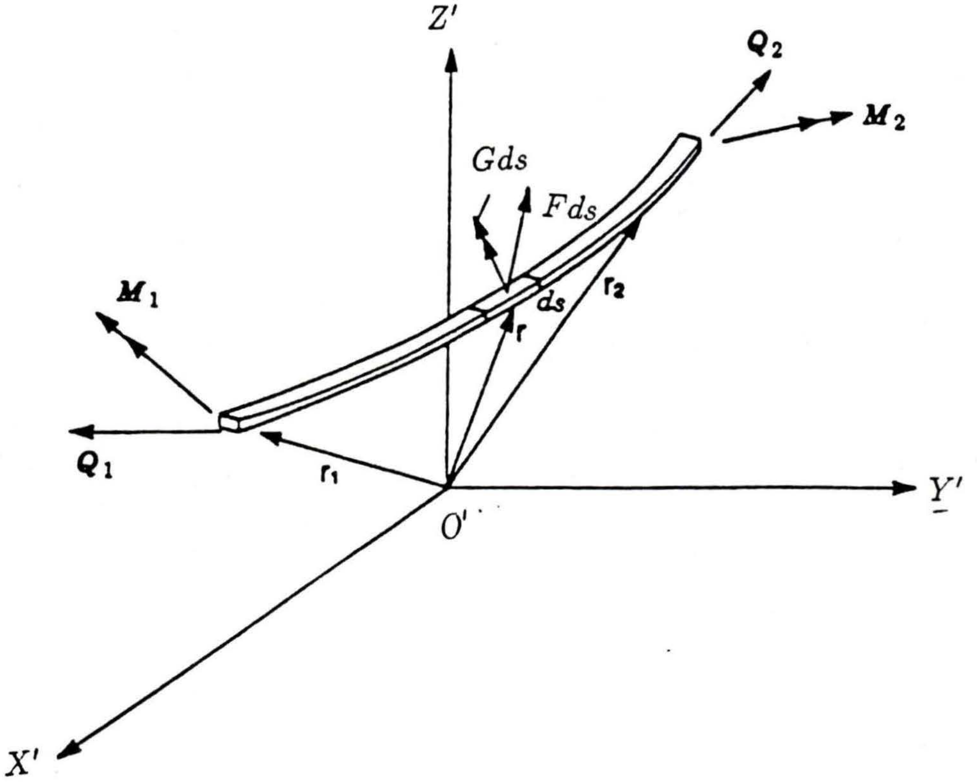

Figure 2.3: Internal and Applied Forces on a Spatial Rod[20]

where $\mathbf{j}$ is the dyadic of mass moments of inertia per unit length. This equation may be expressed as:

$$\begin{aligned} \int_{s_1}^{s_2} \left\{ \frac{\partial}{\partial s}[\mathbf{M} + \mathbf{r} \times \mathbf{Q}] + \mathbf{r} \times \mathbf{F}(s) + \mathbf{G}(s) \right\} ds &= \frac{\partial}{\partial t}\int_{s_1}^{s_2} (\mathbf{j}\dot{\mathbf{\Theta}}) ds \\ &+ \frac{\partial}{\partial t}\int_{s_1}^{s_2} (m\mathbf{r} \times \dot{\mathbf{U}}) ds \end{aligned} \tag{2.20}$$

Because the rod has an arbitrary length we can see that:

$$\begin{aligned} \frac{\partial \mathbf{M}}{\partial s} + \frac{\partial \mathbf{r}}{\partial s} \times \mathbf{Q} + \mathbf{r} \times \frac{\partial \mathbf{Q}}{\partial s} + \mathbf{r} \times \mathbf{F}(s) + \mathbf{G}(s) &= \mathbf{j}\ddot{\mathbf{\Theta}} \\ &+ m(\dot{\mathbf{r}} \times \dot{\mathbf{U}} + \mathbf{r} \times \ddot{\mathbf{U}}) \end{aligned} \tag{2.21}$$

Multiplying equation 2.18 by $\mathbf{r}$, we get:

$$\mathbf{r} \times \frac{\partial \mathbf{Q}}{\partial s} + \mathbf{r} \times \mathbf{F}(s) = m\mathbf{r} \times \ddot{\mathbf{U}} \tag{2.22}$$

We have also:

$$\dot{\mathbf{r}} = \dot{\mathbf{U}} \tag{2.23}$$

and

$$\mathbf{t} = \frac{\partial \mathbf{r}}{\partial s} \tag{2.24}$$

With these results we may simplify the moment equilibrium equation to:

$$\frac{\partial \mathbf{M}}{\partial s} + \mathbf{t} \times \mathbf{Q} + \mathbf{G}(s) = \mathbf{j}\ddot{\mathbf{\Theta}} \tag{2.25}$$

2.3.3 Constitutive Equations

The kinematic equations relate internal forces and moments to the corresponding strains in the rod. To this end, the material and geometric properties are introduced through the constitutive equations. For linearly elastic materials they are:

$$\left\{ \begin{array}{c} Q_u \\ Q_v \\ Q_w \end{array} \right\} = \left[\begin{array}{ccc} GA_u & & \\ & GA_u & \\ & & EA_w \end{array} \right] \left\{ \begin{array}{c} \epsilon_u \\ \epsilon_v \\ \epsilon_w \end{array} \right\}, \tag{2.26}$$

$$\left\{ \begin{array}{c} M_u \\ M_v \\ M_w \end{array} \right\} = \left[\begin{array}{ccc} EI_u & & \\ & EI_v & \\ & & GI_w \end{array} \right] \left\{ \begin{array}{c} k_u \\ k_v \\ k_w \end{array} \right\} \tag{2.27}$$

In the above equations A_w is the cross sectional area while A_u and A_v are the shear areas, that is the cross sectional area times the shear shape factor ; I_u and I_v are the second moments of area about the $\mathbf{n}$ and $\mathbf{b}$ axes respectively and I_w is the torsional constant. Young's modulus and the shear modulus of the material are denoted by E and G respectively.

2.3.4 Governing Equations

Through constitutive equations the equilibrium equations may be expressed in terms of strains and changes in curvature. Finally, through the kinematic relations the six equilibrium equations may be expressed in terms of three displacements and three rotations.

Force-Displacement:

$$\theta_v = \frac{\partial u}{\partial s} - \tau v + \kappa w - \frac{Q_u}{GA_u} \tag{2.28}$$

$$-\theta_u = \frac{\partial v}{\partial s} + \tau u - \frac{Q_v}{GA_v} \tag{2.29}$$

$$0 = \frac{\partial w}{\partial s} - \kappa u - \frac{Q_w}{EA_w} \tag{2.30}$$

Moment-Rotation:

$$\frac{M_u}{EI_u} = \frac{\partial \theta_u}{\partial s} - \tau\theta_v + \kappa\theta_w \tag{2.31}$$

$$\frac{M_v}{EI_v} = \frac{\partial \theta_v}{\partial s} + \tau\theta_u \tag{2.32}$$

$$\frac{M_w}{GI_w} = \frac{\partial \theta_w}{\partial s} + \kappa\theta_u \tag{2.33}$$

Rotational Equilibrium:

$$mI_u\ddot{\theta}_u = \frac{\partial M_u}{\partial s} - \tau M_v + \kappa M_w - Q_v + G_u(s) \tag{2.34}$$

$$mI_v\ddot{\theta}_v = \frac{\partial M_v}{\partial s} + \tau M_u + Q_u + G_v(s) \tag{2.35}$$

$$mI_w\ddot{\theta}_w = \frac{\partial M_w}{\partial s} - \kappa M_u + G_w(s) \tag{2.36}$$

Translational Equilibrium:

$$m\ddot{u} = \frac{\partial Q_u}{\partial s} - \tau Q_v + \kappa Q_w + F_u(s) \tag{2.37}$$

$$m\ddot{v} = \frac{\partial Q_v}{\partial s} + \tau Q_u + F_v(s) \quad (2.38)$$

$$m\ddot{w} = \frac{\partial Q_w}{\partial s} - \kappa Q_u + F_w(s) \quad (2.39)$$

2.4 Summary

In this chapter the equations that relate the displacements and rotations to the internal forces and moments were derived from basic principles for a spatially curved and twisted rod. Additional information in this regard may be obtained from the works by Wittrick [22] and Yap [20]. Since there are twelve degrees of freedom for each element (three translations and three rotations at each end) there are twelve differential equations of second order to be solved. In the next chapter these equations are simplified and a solution procedure is outlined.

Chapter 3

Finite Element Formulation

3.1 Overview

In this chapter the solutions of the differential equations developed in the preceding chapter are used to develop a finite element model for a spatially curved and twisted rod. More precisely, homogeneous solutions of the governing equations are used as shape functions for the element. Also, the element coordinate system is introduced with respect to which the stiffness and mass matrices can be expressed.

3.2 Solution of Differential Equations

It can be seen that the governing differential equations presented in the last chapter have time-dependent quantities. As a first step of simplification we consider the static case for which the time-dependence of variables is removed. Secondly, we drop the distributed forces and moments. These can be accommodated at a later step by superposing a particular solution on the homogeneous solution. Now we have a set of twelve coupled differential equations depending only on s as follows.

Force-Displacement:

$$\theta_v = \frac{du}{ds} - \tau v + \kappa w - \frac{Q_u}{GA_u} \tag{3.1}$$

$$-\theta_u = \frac{dv}{ds} + \tau u - \frac{Q_v}{GA_v} \tag{3.2}$$

$$0 = \frac{dw}{ds} - \kappa u - \frac{Q_w}{EA_w} \tag{3.3}$$

Moment-Rotation:

$$\frac{M_u}{EI_u} = \frac{d\theta_u}{ds} - \tau\theta_v + \kappa\theta_w \tag{3.4}$$

$$\frac{M_v}{EI_v} = \frac{d\theta_v}{ds} + \tau\theta_u \tag{3.5}$$

$$\frac{M_w}{GI_w} = \frac{d\theta_w}{ds} + \kappa\theta_u \tag{3.6}$$

Rotational Equilibrium:

$$0 = \frac{dM_u}{ds} - \tau M_v + \kappa M_w - Q_v \tag{3.7}$$

$$0 = \frac{dM_v}{ds} + \tau M_u + Q_u \tag{3.8}$$

$$0 = \frac{dM_w}{ds} - \kappa M_u \tag{3.9}$$

Translational Equilibrium:

$$0 = \frac{dQ_u}{ds} - \tau Q_v + \kappa Q_w \tag{3.10}$$

$$0 = \frac{dQ_v}{ds} + \tau Q_u \tag{3.11}$$

$$0 = \frac{dQ_w}{ds} - \kappa Q_u \tag{3.12}$$

It can be seen that the degree of coupling amongst the equations is not severe. For example, the translational equilibrium equations are independent of others, and the rotational equilibrium equations are independent of rotations and displacements. The coupling severity increases as we move upwards to the force-displacement equations. The solution technique is to first solve the translational equilibrium equations for the forces, substitute these into the rotational equilibrium equations to obtain the moments and with those at hand we may move to the force-displacement equations. The solutions of these equations are presented explicitly in reference [20].

3.3 Shape Functions

As mentioned in the previous paragraph the above set of differential equations is solved providing four vectors of solutions, viz., displacement $\mathbf{U}$, rotation Θ, force $\mathbf{F}$, and moment $\mathbf{M}$. These solutions may be expressed in matrix form as shown below. Grouping the forces and moments together:

$$\{\mathbf{F}(s)\} = [\mathbf{A}(s)]\{\mathbf{C}_i\} \tag{3.13}$$

where,

$\{\mathbf{F}(s)\} = [Q_u\ Q_v\ Q_w\ M_u\ M_v\ M_w\ ]^T$ and

$\{\mathbf{C}_i\} = [c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7\ c_8\ c_9\ c_{10}\ c_{11}\ c_{12}\ ]^T$

Here $\{\mathbf{C}_i\}$ is a vector of coefficients and $[\mathbf{A}(s)]$ is a 6×12 matrix whose elements are composed of functions of s that relate the forces/moments to the coefficients.

Likewise we may generate another matrix equation relating the generalized displacement vector $\{\mathbf{d}(s)\}$, *i.e.*, combination of displacements and rotations to the coefficient vector. Thus:

$$\{\mathbf{d}(s)\} = [\mathbf{B}(s)]\{\mathbf{C}_i\} \tag{3.14}$$

where,

$\{\mathbf{d}(s)\} = [u\ v\ w\ \theta_u\ \theta_v\ \theta_w\ ]^T$.

Like $[\mathbf{A}(s)]$, $[\mathbf{B}(s)]$ is also a 6×12 matrix constructed from the solution of the differential equations. It should be noted that the components of all the above vectors are, till now, with respect to the natural coordinate system.

The nodal force vector and nodal displacement vector may now be written in terms of the twelve coefficients $\{\mathbf{C}_i\}$, thus :

$$\{\mathbf{F}_n\} = \left\{ \begin{array}{c} \mathbf{F}_{s=0} \\ --- \\ \mathbf{F}_{s=l} \end{array} \right\} = \left[\begin{array}{c} \mathbf{A}_{s=0} \\ --- \\ \mathbf{A}_{s=l} \end{array} \right] \{\mathbf{C}_i\} \tag{3.15}$$

Here, $s = 0$ and $s = l$ represent the initial and final nodes of the element respectively and l is the length of the element. Also, $\{\mathbf{F}_n\}$ is a 12×1 vector, in which n stands for natural coordinate system, $[\mathbf{A}]$ is a 12×12 matrix and $\{\mathbf{C}_i\}$ is a 12×1 vector. Likewise, the nodal displacement vector can be written as:

$$\{\mathbf{d}_n\} = \left\{ \begin{array}{c} \mathbf{d}_{s=0} \\ --- \\ \mathbf{d}_{s=l} \end{array} \right\} = \left[\begin{array}{c} \mathbf{B}_{s=0} \\ --- \\ \mathbf{B}_{s=l} \end{array} \right] \{\mathbf{C}_i\} \tag{3.16}$$

Here again, it may be noted that $\{\mathbf{d}_n\}$ is a 12×1 vector, $[\mathbf{B}]$ is a 12×12 matrix and $\{\mathbf{C}_i\}$ is a 12×1 vector. As mentioned previously, we have components of the nodal vectors with respect to a moving coordinate system which rotates as it moves along s. Thus, in order to express the nodal parameters with respect to a common coordinate system one needs to introduce such a common reference frame. This is necessary because it would not be possible to assemble different elements if their nodal parameters were not expressed with respect to a common system of coordinates. Also, to interpret the results (for example, displacements) meaningfully, it is necessary that they be known with respect to a common reference frame.

3.4 Natural-Elemental Transformation

In the following a special reference frame is introduced and the transformation matrices relating the moving reference frame to this special reference frame are outlined. In this report this frame is called the element coordinate system. This is because it varies from one element to another while remains the same over a given element. Later on, a global coordinate system is introduced with respect to which every element's stiffness and mass matrices are expressed before being assembled. Also, a local coordinate system is introduced with respect to which the helical arc is inherently defined. To differentiate between the four different coordinate systems the following convention is adopted.

- Natural coordinate system is designated by $\mathbf{n}$, $\mathbf{b}$ and $\mathbf{t}$.
- Local coordinate system is designated by X', Y', and Z'.
- Element coordinate system is designated by x, y, and z.
- Global coordinate system is designated by X, Y, and Z.

In Figure 3.1, let A be the starting point of the element and B its end point. Conventionally, its Z' axis runs along the axis of the helix in the direction of increasing 'height', its X' axis would be the vector that runs between point A of the element and the point O', which is the local coordinate system center, that is the point of intersection between the normal $\mathbf{n}$ at A to the helical arc and the Z' axis. The Y' axis would complete the right hand triad between the X' and Z' axes. The element coordinate system y axis points in the direction of the local coordinate system Z' axis and the x axis is orthogonal to the Z' axis and points towards the projection of point B in the $O'X'Y'$ plane. The z axis completes the orthogonal right hand triad between the x and y axes. It can be seen that ϕ is the angle swept by the element about the Z' axis.

We now introduce the rotation matrices that are required for the transformation of the nodal vectors from the natural coordinate system to the element

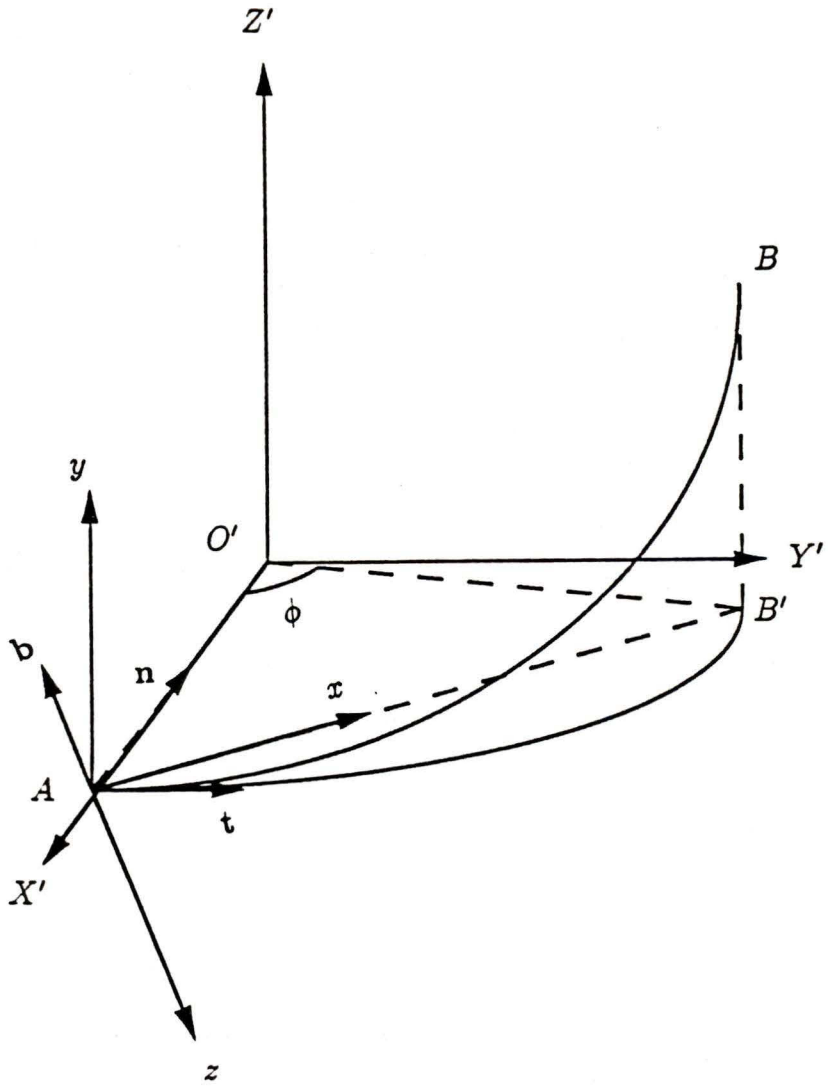

Figure 3.1: Local, Element and Natural Coordinate Systems

coordinate system. We have two rotation matrices; $[\mathbf{\Gamma}_1]$ for the force vector and $[\mathbf{\Gamma}_2]$ for the displacement vector. Their operation is shown below.

$$\{\mathbf{F}_e\} = [\mathbf{\Gamma}_1]\{\mathbf{F}_n\} \tag{3.17}$$

Using equation 3.15 this can be written as:

$$\{\mathbf{F}_e\} = [\mathbf{\Gamma}_1]\begin{bmatrix} \mathbf{A}_{s=0} \\ --- \\ \mathbf{A}_{s=l} \end{bmatrix}\{\mathbf{C}_i\} \tag{3.18}$$

Or, combining matrices $\mathbf{\Gamma}_1$ and $\mathbf{A}$ in the above equation,

$$\{\mathbf{F}_e\} = [\mathbf{Y}]\{\mathbf{C}_i\} \tag{3.19}$$

where the subscript e refers to the element coordinate system. Likewise, we can write for the displacement vector,

$$\{\mathbf{d}_e\} = [\mathbf{\Gamma}_1]\{\mathbf{d}_n\} \tag{3.20}$$

Using equation 3.16 this can be written as:

$$\{\mathbf{d}_e\} = [\mathbf{\Gamma}_2]\begin{bmatrix} \mathbf{B}_{s=0} \\ --- \\ \mathbf{B}_{s=l} \end{bmatrix}\{\mathbf{C}_i\} \tag{3.21}$$

Combining matrices $\mathbf{\Gamma}_2$ and $\mathbf{B}$ in the above equation we have:

$$\{\mathbf{d}_e\} = [\mathbf{X}]\{\mathbf{C}_i\} \tag{3.22}$$

In the following we develop the stiffness and mass matrices of an element with respect to the element coordinate system. In the next chapter ways are presented to transform them to a common global coordinate system. Then they can be assembled allowing representation of any three dimensional curve-like structure by a combination of helical elements.

3.5 The Stiffness Matrix

In any finite element development the stiffness matrix would constitute a relationship between the nodal vector of unknowns, usually called the displacement vector, and the nodal vector of prescribed variables usually called the force vector. Thus in our case,

$$\{\mathbf{F}_e\} = [\mathbf{K}_e]\{\mathbf{d}_e\} \tag{3.23}$$

Using equations 3.19 and 3.22 we can write that:

$$\{\mathbf{F}_e\} = [\mathbf{Y}][\mathbf{X}]^{-1}\{\mathbf{d}_e\} \tag{3.24}$$

From the above two equations we find an expression for the stiffness matrix as:

$$[\mathbf{K}_e] = [\mathbf{Y}][\mathbf{X}]^{-1} \tag{3.25}$$

3.6 The Mass Matrix

The mass matrix is derived using the expression for the complementary kinetic energy,

$$T = \frac{1}{2}\rho \int_0^l \left\{\dot{\mathbf{d}}(s)\right\}^T [\mathbf{H}] \left\{\dot{\mathbf{d}}(s)\right\} ds, \tag{3.26}$$

where ρ is the material density, l is the length of the element, $\left\{\dot{\mathbf{d}}(s)\right\}$ is the velocity vector and $[\mathbf{H}]$ is the following diagonal matrix.

$$[\mathbf{H}] = \begin{bmatrix} A_w & 0 & 0 & 0 & 0 & 0 \\ 0 & A_w & 0 & 0 & 0 & 0 \\ 0 & 0 & A_w & 0 & 0 & 0 \\ 0 & 0 & 0 & I_u & 0 & 0 \\ 0 & 0 & 0 & 0 & I_v & 0 \\ 0 & 0 & 0 & 0 & 0 & I_w \end{bmatrix} \tag{3.27}$$

The displacement vector can be written using equation 3.22 as:

$$\{\mathbf{d}(s)\} = [\mathbf{B}(s)][\mathbf{X}]^{-1}\{\mathbf{d}_e\} \tag{3.28}$$

Differentiating,

$$\left\{\dot{\mathbf{d}}(s)\right\} = [\mathbf{B}(s)][\mathbf{X}]^{-1}\left\{\dot{\mathbf{d}}_e\right\} \tag{3.29}$$

After a little rearrangement substitution of this into the expression for complementary kinetic energy gives:

$$T = \frac{1}{2}\left\{\dot{\mathbf{d}}_e\right\}^T [\mathbf{M}_e]\left\{\dot{\mathbf{d}}_e\right\} \tag{3.30}$$

where $[\mathbf{M}_e]$ is the element mass matrix given by the following expression.

$$[\mathbf{M}_e] = \rho[[\mathbf{X}]^{-1}]^T \left[\int_0^l [\mathbf{B}(s)]^T[\mathbf{H}][\mathbf{B}(s)]ds\right][\mathbf{X}]^{-1} \tag{3.31}$$

The above expression is too tedious to integrate analytically. However, it can easily be integrated numerically to the desired accuracy by using a sufficient number of Gauss points for Guassian quadrature.

3.7 Summary

This chapter was devoted to the development of the stiffness and mass matrices from first principles for the spatially curved and twisted element. This was achieved by solving the twelve differential equations involving the displacements, rotations, internal forces and internal moments and using their solutions to arrive at the shape functions for the displacements and rotations. The complementary kinetic energy expression was used to compute the mass matrix while the stiffness matrix was constructed in an indirect way. In the following chapter ways are presented to transform these matrices, now in the elemental coordinate system, to a common global coordinate system.

Chapter 4

Globalization

4.1 Overview

In engineering practice one might deal with physical structures that can be modelled using finite elements. Often the structure will have different materials and varying geometry. A finite element model must account for these variations for precise solutions. In the present development of a helical element, this means that our finite element formulation should be able to model the structure using different helices with different material and/or geometric properties over the three dimensional curve-like structure. In this chapter this problem is considered and a procedure is presented to enable the user to analyze any three dimensional curve-like structure statically and dynamically. The stiffness and mass matrices developed in the last chapter are with respect to the element coordinate system. Remembering that each of the elements has its own coordinate system, it becomes essential to express these matrices with respect to a common, convenient global coordinate system. This in turn necessitates the following.

- Define a global coordinate system.
- Describe the structure in terms of helical elements each having its own coordinate system.

- Determine every element coordinate system with respect to the global coordinate system. This means that all element coordinate system base vectors must be known in terms of the global coordinate system base vectors.
- Determine the transformation matrix between the global coordinate system and every element coordinate system.

4.2 Elemental-Global Transformation

The usefulness of an elemental-global transformation on the stiffness and mass matrices of a helical element can be appreciated with the help of Fig. 4.1. Even though a helix is easily described with respect to its own (local) coordinate system, in practice this coordinate system is not known except when it coincides with the global coordinate system. In Fig. 4.1, $OXYZ$ is the global coordinate system, $O'X'Y'Z'$ is the local coordinate system and $Axyz$ is the element coordinate system. AB is a portion of the spatial curve ABC which is (or can be approximately represented by) a helix with $O'X'Y'Z'$ as the local coordinate system. BC is another part of the spatial curve ABC which can again be considered to be a helix having different local and element coordinate systems. Thus in practice, a spatial curve can be approximated by a series of helices with different coordinate systems. The problem addressed here is one of fitting a series of helices through a series of known points (or nodes) with minimal information on the geometry. This is done in the next section.

4.3 Determining Element Coordinate System

Referring to Fig. 4.2 we wish to consider the initial portion AB of the spatial curve ABC of Fig. 4.1. This is a spatial curve the coordinates of certain points on which can be known (or measured) with respect to the global coordinate system. We wish to have a helix pass through the points A and B such that

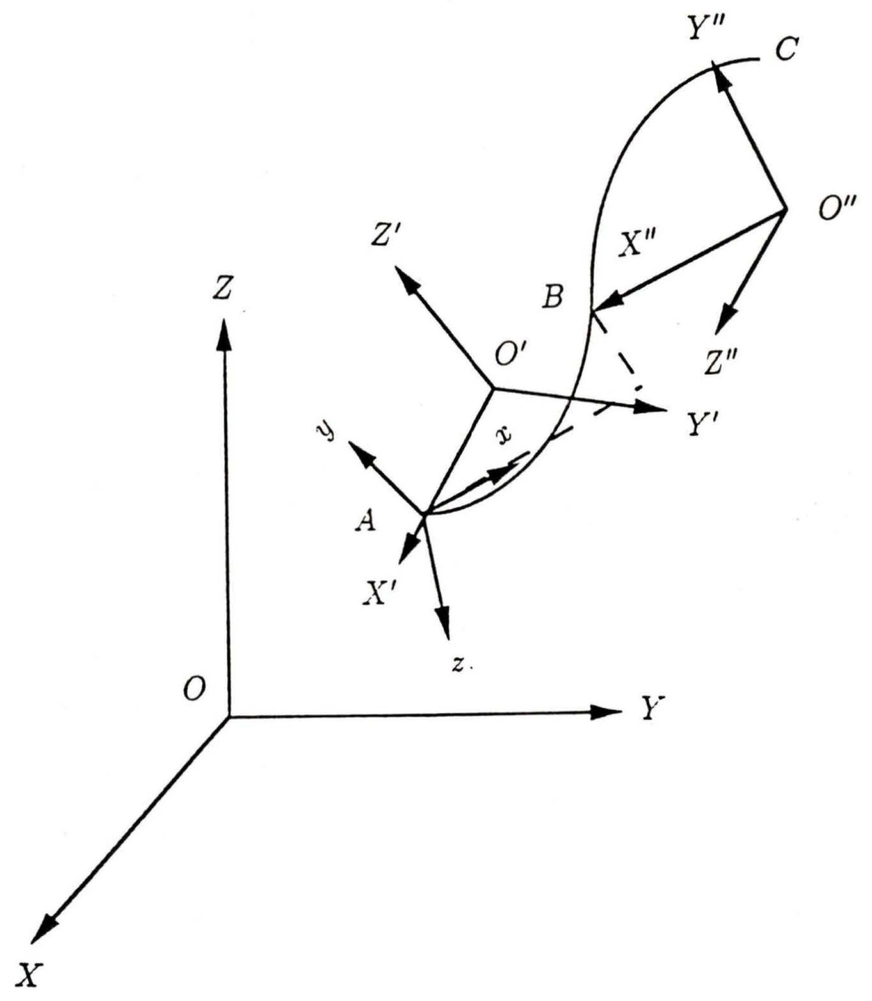

Figure 4.1: Global, Local and Element Coordinate Systems

it approximates the actual curve between those points as closely as possible. The following information on each element is needed.

1. Radius of helix

2. The initial and final points. That is, points A and B with respect to the global coordinate system.

3. The vector $\vec{O'Z'}$ with respect to the global coordinate system.

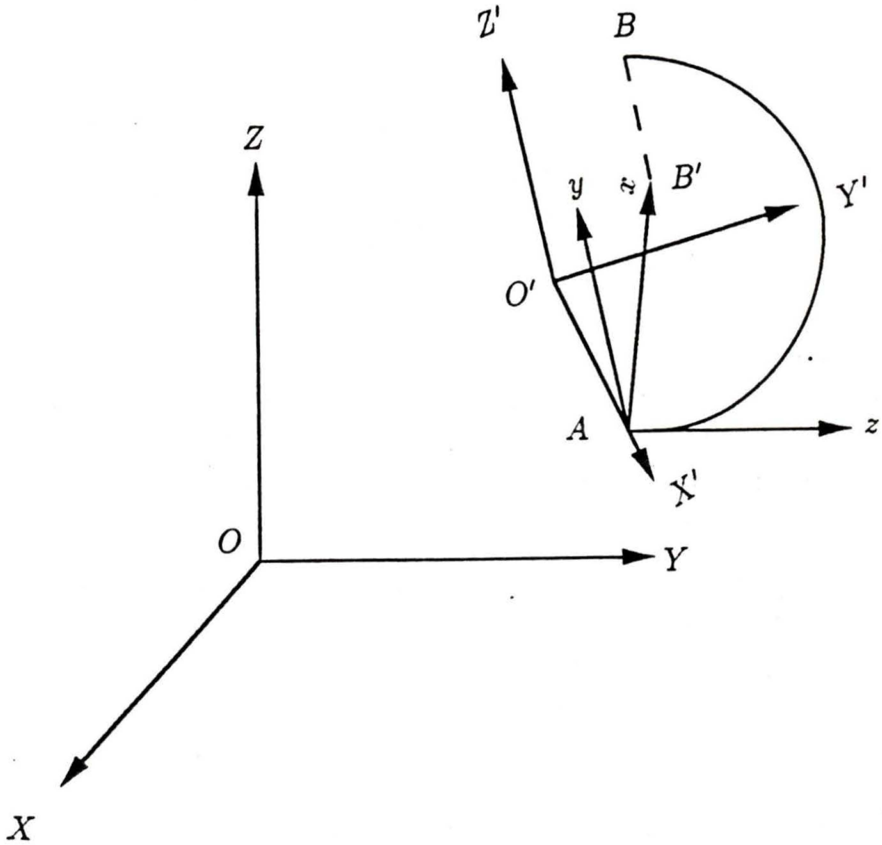

Figure 4.2: Analysis of a Helical Element

4. A sign (positive or negative) to specify if the helix has a positive or negative pitch.

With the above information the following steps will not only describe the helix AB they also will provide a transformation matrix in terms of the element coordinate base vectors which is needed to transform the stiffness and mass matrices developed with respect to the element coordinate system, to the global coordinate system.

1. Fixing the base plane of helix $O'X'Y'$
2. Determination of the local coordinate system origin O'.
3. Determination of the local coordinate system base vectors.

4. Determination of the element coordinate system base vectors.

5. Construction of the transformation matrix

6. Computation of the properties of the helix *viz.*, pitch, curvature and torsion.

The above steps are described mathematically in the following pages of this chapter. The program that implements this technique is listed at the end of this thesis.

Equation for the base plane $O'X'Y'$.

Let $\{n_{31}\ n_{32}\ n_{33}\}^T$ be the unit vector in $O'Z'$ direction. The components n_{31}, etc., are measured (or known) in the global coordinate system. Then the equation for this plane, with respect to the global coordinate system, may be written as [16]:

$$n_{31}x + n_{32}y + n_{33}z + d = 0 \tag{4.1}$$

where d is the distance from the origin O to this plane.
Since point A lies in this plane we can substitute its coordinates into this equation to find d.

Local coordinate system origin O'.

Before we can find O', the distance between point B and the base plane must be computed. Let this be d_1^2. From analytic geometry we have [1]:

$$d_1^2 = (n_{31}b_1 + n_{32}b_2 + n_{33}b_3 + d)^2 \tag{4.2}$$

where $\{b_1\ b_2\ b_3\}^T$ is the position vector of point B with respect to the global coordinate system. Letting B' be the projection of point B on the base plane, now the coordinates of O' are calculated using the following facts.

$$\begin{aligned}
|\vec{O'B}|^2 &= |\vec{O'B'}|^2 + |\vec{BB'}|^2 \\
|\vec{AO'}|^2 &= r^2
\end{aligned}$$

and O' lies in the base plane $O'X'Y'$.

In the above equation r is the radius of the helix. The above mentioned conditions translate to the following three equations.

$$\begin{aligned}(o_1 - b_1)^2 + (o_2 - b_2)^2 + (o_3 - b_3)^2 &= r^2 + d_1^2 && (4.3)\\ (o_1 - a_1)^2 + (o_2 - a_2)^2 + (o_3 - a_3)^2 &= r^2 && (4.4)\\ n_{31}o_1 + n_{32}o_2 + n_{33}o_3 + d &= 0 && (4.5)\end{aligned}$$

where:

$\{o_1\ o_2\ o_3\}^T$ is the position vector to point O' and

$\{a_1\ a_2\ a_3\}^T$ is the position vector to point A, both with respect to the global coordinate system.

Since the coordinates of O' are governed by quadratic equations it can be seen that there are two solutions for the coordinates of O' one of which has to be discarded at a later stage. Indiscriminately, one of the solutions is processed as follows.

Base Vectors.

Obviously at this point $\vec{O'A}$ can be normalized to get one base vector of the local coordinate system. Let the normalized form of $\vec{O'A}$ be $\{n_{11}\ n_{12}\ n_{13}\}^T$. Knowing also $\vec{O'Z'}$ in the normalized form we can compute the third vector by finding the cross product of the above two vectors. That is,

$$\vec{O'Y'} = \vec{O'Z'} \times \vec{O'X'} \qquad (4.6)$$

Let the unit vector in $O'Y'$ direction so determined be $\{n_{21}\ n_{22}\ n_{23}\}^T$.

Transformation of B to Local Coordinate System.

This is done to check if the solution for the local coordinate center is the required one. At this point it should be noted that it is assumed that each helix does not sweep more than π radians about $\vec{O'Z'}$ past A. This is to make sure that point B has, at all times, a positive value with respect to the local coordinate system in the $O'Y'$ direction. The transformation is achieved by the following equation.

$$\begin{Bmatrix} bl_1 \\ bl_2 \\ bl_3 \end{Bmatrix} = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} \begin{Bmatrix} b_1 - o_1 \\ b_2 - o_2 \\ b_3 - o_3 \end{Bmatrix} \tag{4.7}$$

where $\{bl_1\ bl_2\ bl_3\}^T$ is the vector to point B from point O'. If bl_2 is negative then our choice of O' is wrong, in which case the other solution is picked up and processed likewise. Otherwise, our choice is correct and we can proceed to find the element coordinate system base vectors.

Element Coordinate System Base Vectors.

The element coordinate system base vectors are defined as follows:

- $\vec{e_1}$ in the direction of $\vec{AB'}$.
- $\vec{e_2}$ in the direction of $\vec{OZ'}$.
- $\vec{e_3}$ completes the right hand triad.

At this point it is necessary to transform point B' to the global coordinate system because we wish to express the element coordinate system base vectors with respect to the global coordinate system. This is achieved by the following transformation.

$$\begin{Bmatrix} b'g_1 \\ b'g_2 \\ b'g_3 \end{Bmatrix} = \begin{bmatrix} n_{11} & n_{21} & n_{31} \\ n_{12} & n_{22} & n_{32} \\ n_{13} & n_{23} & n_{33} \end{bmatrix} \begin{Bmatrix} bl_1 \\ bl_2 \\ bl_3 \end{Bmatrix} + \begin{Bmatrix} o_1 \\ o_2 \\ o_3 \end{Bmatrix} \tag{4.8}$$

where $\{b'g_1\ b'g_2\ b'g_3\}^T$ is the position vector to point B' from the global coordinate system center O. Based on the above definition the base vectors can be written down as:

$$\vec{e_1} = \{(b'g_1 - a_1)\ (b'g_2 - a_2)\ (b'g_3 - a_3)\ \}^T \tag{4.9}$$

$$\vec{e_2} = \{n_{31}\ n_{32}\ n_{33}\}^T \tag{4.10}$$

$$\vec{e_3} = \vec{e_1} \times \vec{e_2} \tag{4.11}$$

Transformation Matrix for Assembly of Elements.

For a two node element which would have twelve degrees of freedom a 12×12 transformation matrix is necessary between the element coordinate system and the global coordinate system. As noted earlier the stiffness and mass matrices of each element are available with respect to its own element coordinate system. So, before they can be assembled they need to be transformed to a common global coordinate system. This is accomplished by the following transformation matrix, **RT**.

$$\mathbf{RT} = \begin{bmatrix} \mathbf{Q} & 0 & 0 & 0 \\ 0 & \mathbf{Q} & 0 & 0 \\ 0 & 0 & \mathbf{Q} & 0 \\ 0 & 0 & 0 & \mathbf{Q} \end{bmatrix} \tag{4.12}$$

where $[\mathbf{Q}]$ is a 3×3 matrix whose elements are the direction cosines of the element coordinate system base vectors and $[\mathbf{0}]$ is a 3×3 null matrix.

Thus :

$$[\mathbf{Q}] = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \tag{4.13}$$

Parameters of Helix Element.

Having fixed the local coordinate system we can ascertain the parameters of the helix such as the pitch p, curvature κ, and torsion τ, which will be used in the computations of the stiffness and mass matrices. Incidentally, we have to find the helix angle α and the sweep angle ϕ which are involved in the expressions for the above properties. The pitch p is computed by:

$$(p\phi)^2 = d_1^2 \tag{4.14}$$

where ϕ is given by:

$$[2rsin(\phi/2)]^2 = [(a_1 - b'g_1)^2 + (a_2 - b'g_2)^2 + (a_3 - b'g_3)^2] \tag{4.15}$$

and α is given by:

$$tan(\alpha) = p/r \tag{4.16}$$

The curvature and torsion are respectively given by [6]:

$$\kappa = rcos^2\alpha \tag{4.17}$$

and

$$\tau = \kappa tan(\alpha) \tag{4.18}$$

It may be noted that none of the above mentioned properties of the helix element need be input, rather, they are calculated based on the input information outlined before. The above processes are implemented and illustrated in the examples worked out in the following chapter.

4.4 Summary

This chapter was devoted to a technique for determining the element coordinate system and its transformation matrix relative to the global coordinate system. The transformation matrix being unique to each element allows one to transform the stiffness and mass matrices computed using the formulas in the previous chapter. This transformation is necessary for the assembly of all the elements and so this generalizes the use of the helix element to model any arbitrary spatial curve.

Chapter 5

Applications

5.1 Overview

This chapter illustrates the application of the helical element and the globalization discussed in the last chapter. This chapter is devoted to two examples; one on static analysis and the other on free vibration analysis, under a general configuration. The solutions are compared with those obtained under simple configuration [17,18,20]. In the former example a further transformation is needed to compare the solutions. In the static analysis, only the nodal displacements are computed and compared, while in the vibration analysis, natural frequencies are computed and compared. The results are found to be in good agreement.

5.2 Example of Static Analysis

5.2.1 Problem Description

We wish to find the nodal displacements of a spatially curved and twisted rod shown in Fig. 5.1. A moment of 500 N m is applied at one end of the rod, about the local Y' axis, which is free while the other end is fixed. The rod can thus be modelled using a helix element. The rod has a circular cross section of diameter 0.3 m and the coil radius is 5 m. The material properties

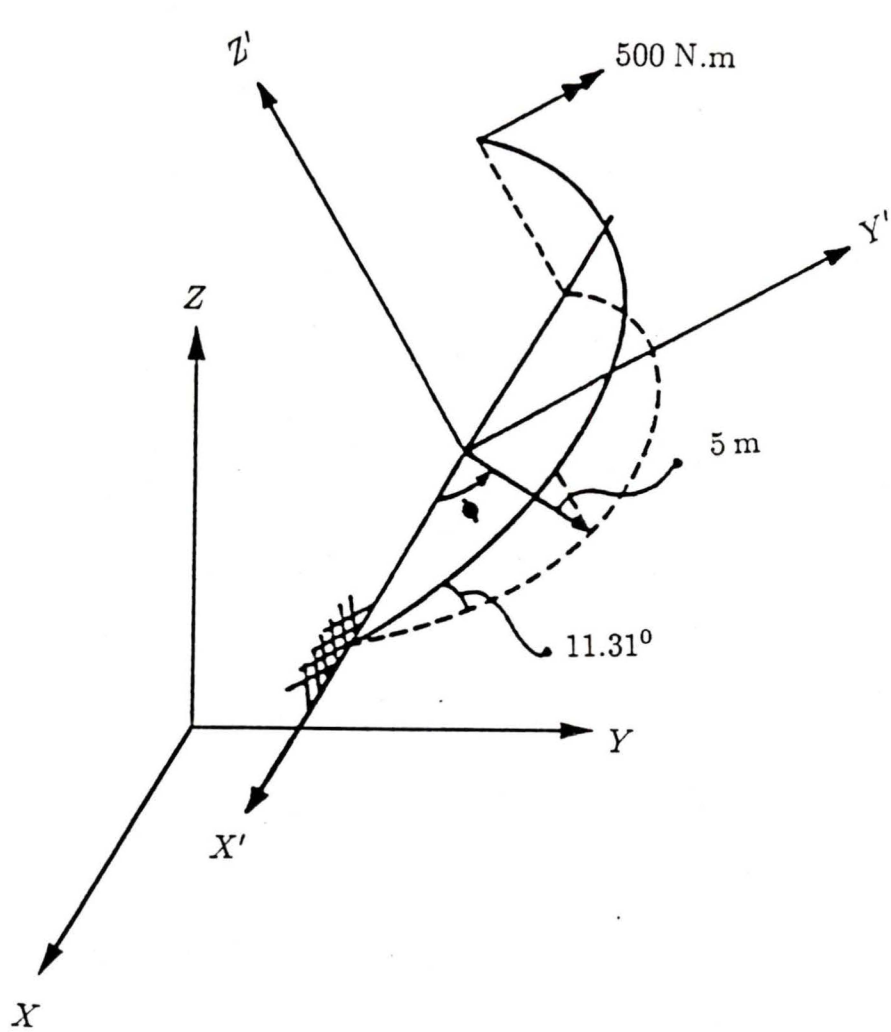

Figure 5.1: Helical Rod with Applied Moment

are: Young's modulus, $E = 207e9$ Pa, and the shear modulus, $G = 95e9$ Pa. The other pieces of input are the nodal coordinates, the vector $O'Z'$ in the global coordinate system and the number of elements. It can be seen that the local coordinate system is tilted with respect to the global coordinate system about the X axis by 30^0. Also, there is a translation of 5 m in the negative direction along the X axis and of 5 m in the positive direction along the other axes. The global tip deflections u_x, u_z, and θ_y were reported in [17], for the case wherein the global coordinate system and local coordinate system were coincident. In order to compare the results of the two methods we need to transform the results obtained for helical elements with respect to the global coordinate system to account for the rotation of the local coordinate system about the global X axis. After this was done the results were entered in Table 5.1.

Table 5.1: Solution Vectors for the Static Analysis Example
u in *meters* and θ in *radians*

D.o.f.	Constant Strain element	Helical element	Prismatic element
u_x	0.1567797e-3	0.15943810e-3	0.15815130e-3
u_y	N.A.	-0.20930570e-5	N.A.
u_z	0.4990427e-3	0.50750732e-3	0.5034091e-3
θ_x	N.A.	0.8399666e-14	N.A.
θ_y	0.9980880e-4	0.10150147e-3	0.1006818e-3
θ_z	N.A.	0.4908000e-11	N.A.

The displacements are given in meters and the rotations in radians for the helical, prismatic and constant strain elements [17]. It may be noted that while only three helical elements were used, seven constant strain elements were needed to model the static analysis problem with reasonable accuracy. It is reported in reference [17] that the results converge from below the exact solution which indicates that the results obtained for the helical element modeling are the exact values to which results of other formulations would converge. The agreement of the results (with an accuracy of within 1.7 percent) is also obvious in the Figures 5.2, 5.3 and 5.4 that show the results obtained using the constant strain and prismatic elements versus the number of elements. Although the present formulation used three elements, a single element would have been enough to produce exact solution to a static analysis problem. It may be noted that in the present method no other properties of the helix such as the pitch, torsion, or curvature were required to be input.

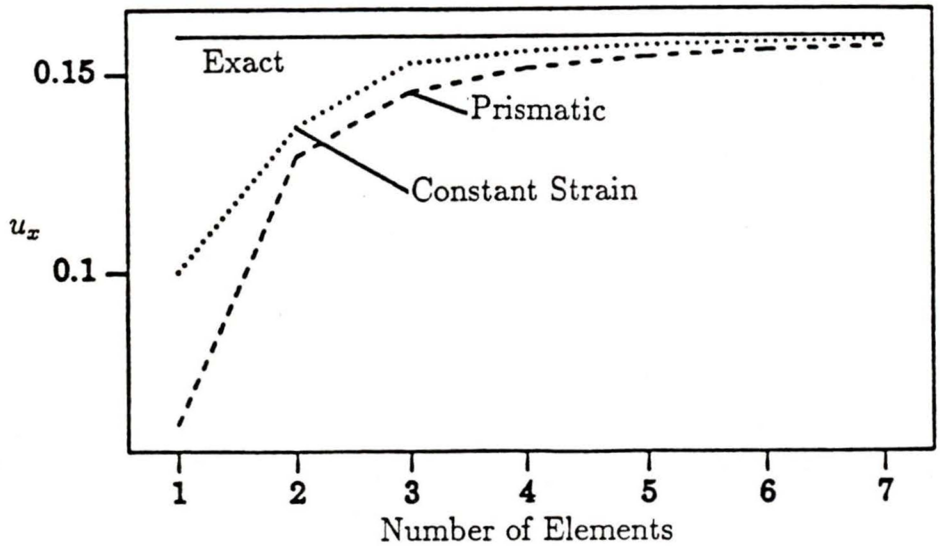

Figure 5.2: Tip Deflection ($m \times 10^{-3}$), u_x of Helical Rod[20]

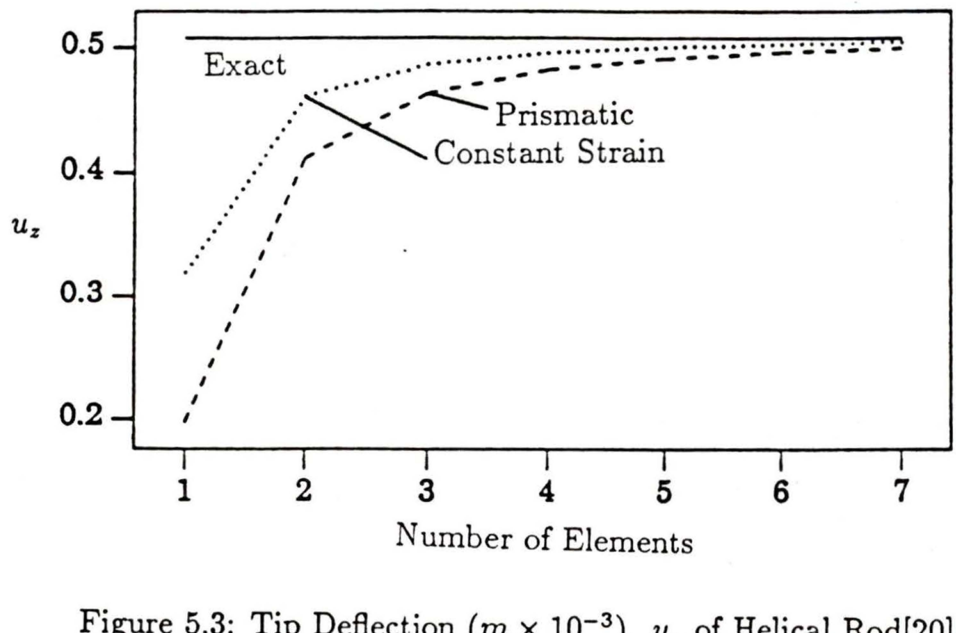

Figure 5.3: Tip Deflection ($m \times 10^{-3}$), u_z of Helical Rod[20]

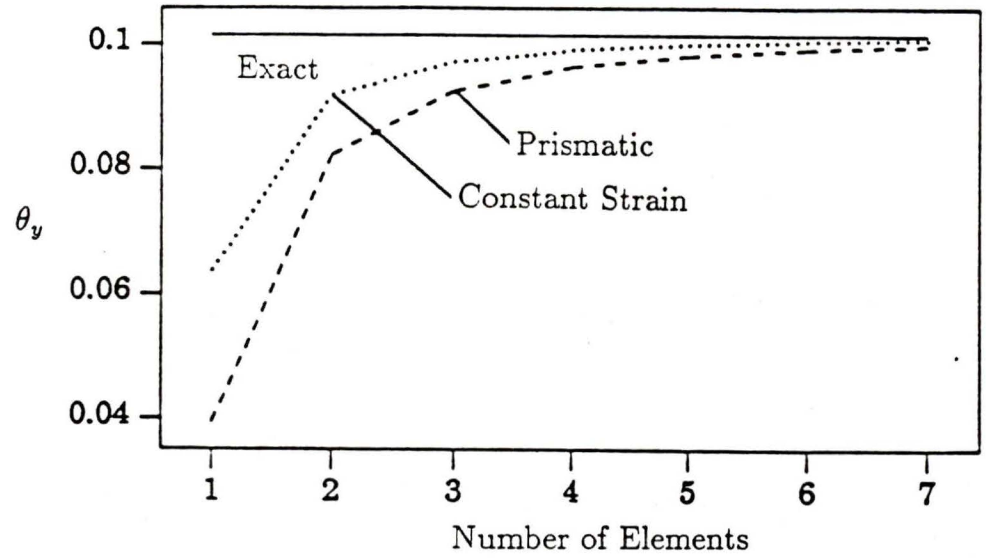

Figure 5.4: Tip Deflection ($radians \times 10^{-3}$), θ_y of Helical Rod[20]

5.3 Example of Free Vibration Analysis

5.3.1 Problem Description

In this example we compute the free vibration frequencies of an arch shown in Fig. 5.5. This was done and the frequencies were reported in references [18,20] along with the frequencies measured experimentally. However, the problem dealt with here is different in that the arch has its local axes different from the global ones. This is shown in Fig. 5.5. The arch has a rectangular cross section of area 0.0189 m$\times$ 0.0062 m and the arch radius is 0.305 m. One end of the arch is fixed and the other is free. The material properties are: Young's modulus, E = 68.13e9 Pa., the shear modulus, G = 25.61e9 Pa., and density, $\rho = 2882\ kg/m^3$.

Here, unlike in the static analysis, one element would not give results of sufficient accuracy. So, the larger the number of elements the better the accuracy of the computed frequencies. However, we have used only three elements to model the arch. Even so, the results are accurate to within 2.8

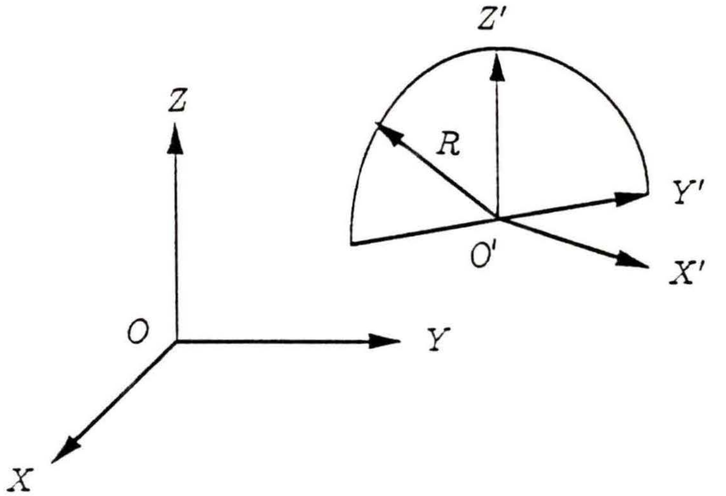

Figure 5.5: Free Vibration of Arch

percent of the experimental values. It should be noted that since the cross section is not circular the Coulomb theory of torsion is inadequate and so to account for the warping of the cross section, as it twists the following expression was used [8] for the torsional constant.

$$I_w = \frac{Ht^3}{16}\left[\frac{16}{3} - 3.6\frac{t}{H}\left(1 - \frac{1}{12}\left(\frac{t}{H}\right)^4\right)\right] \tag{5.1}$$

where H is the larger, and t is the smaller of the cross sectional dimensions. The results are shown in Table 5.2 and compare well with the reported values in references [18,20]. Since the frequencies obtained using the helical elements are lower than those for the constant strain elements the former are closer to the exact values because the actual system is always less stiff than the finite element model. This is supported by the closeness of the former with the experimental frequencies.

Table 5.2 : Free Vibration Frequencies of the Arch in Hz

Mode	Experimental Results	Helical Elements (3 elements)	Constant Strain Elements (7 elements)
1	6.525358	6.478626	7.05057
2	8.800000	8.551624	9.46
3	N.A.	20.467540	N.A.
4	41.2	41.504370	N.A.
5	N.A.	70.16936	N.A.
6	150.0	151.0140	N.A.

5.4 Summary

The technique of generating helical elements with different local coordinate systems and their assembly with respect to a convenient global coordinate system presented in the last chapter, were applied to two problems previously analyzed in special settings. One of the problems was on static analysis while the other on free vibration analysis. These results were then compared with those available in literature and found to be in good agreement. This demonstrates the versatility of the current formulation. In the next chapter the same approach is used to model the struts of an artificial heart valve for free vibration analysis.

Chapter 6

Application to an Artificial Heart Valve

6.1 Overview

The efficiency of the helical element in modelling three dimensional spatial rods, illustrated through examples in the preceding chapter, is utilized in this chapter to determine the free vibration frequencies of the struts of a Bjork-Shiley tilting disc artificial heart valve. In addition to discussing the importance of the problem, some introduction to the physiological aspects is given in the initial part of the chapter. The results obtained for the helical element model are compared with those obtained for the prismatic elements using the ANSYS finite element package [3] as well as the experimental results. The analytically computed results are in good agreement with each other while the experimental results, considering the uncertainties in some input parameters, are reasonably close to the analytical results.

6.2 Artificial Heart Valves

The human heart has four valves that normally allow blood to flow only in one direction and are opened or closed by virtue of the differential pressure across them. Two of them are between the *atria* and the *ventricles* which are called atrioventricular valves, Figure 6.1 [13]. In this figure, RA labels the right

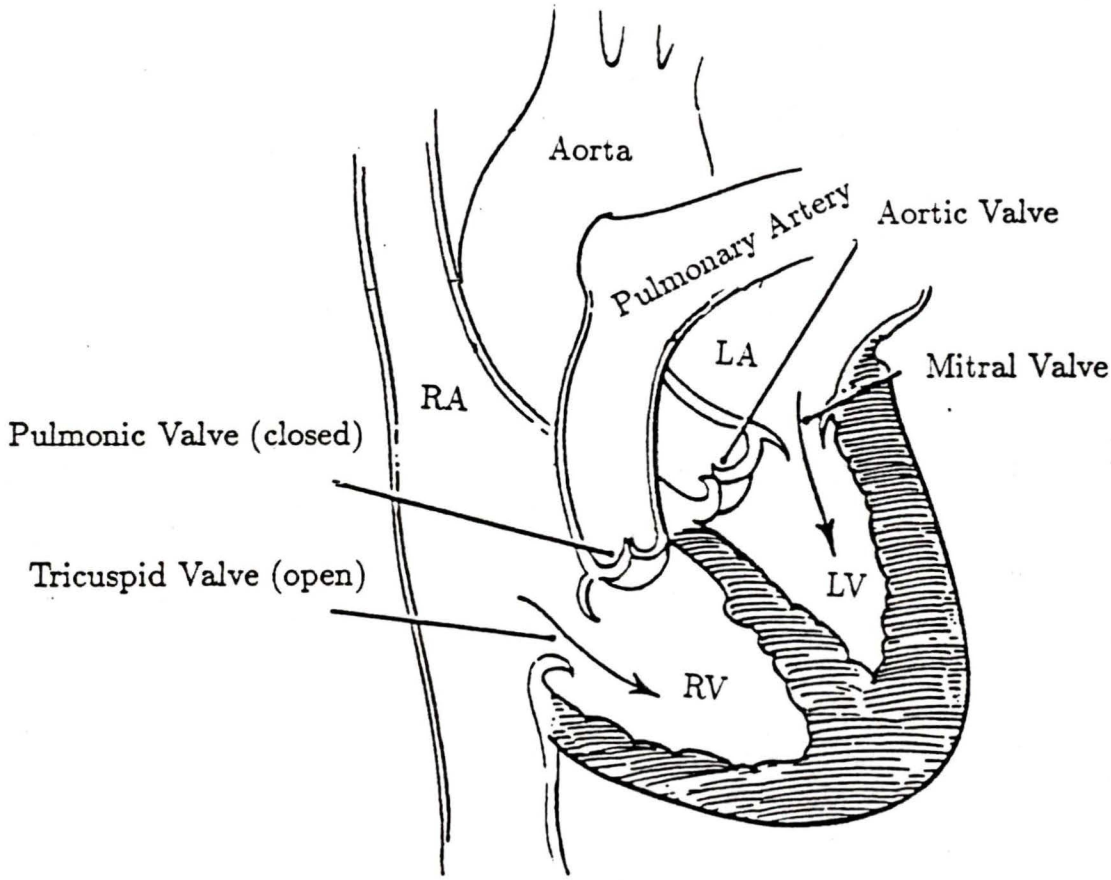

Figure 6.1: Cutaway view of Heart During Relaxation[13]

atrium, RV labels the right ventricle, LA labels the left atrium and LV labels the left ventricle. The valve between the left atrium and the left ventricle is called the *mitral valve.* The right atrioventricular valve is called the *tricuspid valve.* Of the other two valves, one leads to the lungs from the right ventricle through the pulmonary artery called the *pulmonic valve* and the other, the *aortic valve*, opens to the *aorta* from the left ventricle and through which the reoxygenated blood reaches the different parts of the body.

In a normal human heart all these valves function properly opening and

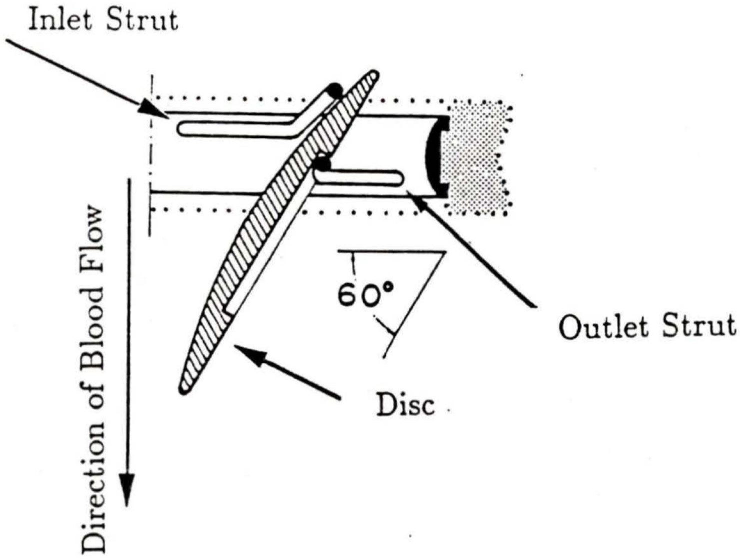

Figure 6.2: Schematic of a Bjork-Shiley Tilting Disc Valve

closing synchronously and sequentially. However, due to various reasons they may fail to do so in which case the proper heart functions are not performed and the results may be fatal. As a solution to this, artificial heart valves have been used since the 1960's. They are implanted in place of the malfunctioning natural valve.

Unfortunately, artificial heart valves have their limitations. Some of them have broken in the implanted condition and led to the patient's death. There exist many kinds of prosthetic valves , some are metallic and others are made of plastic or tissue. For our purposes, only metallic valves, in particular, Bjork-Shiley tilting disc valves will be studied.

The basic parts of a Bjork-Shiley tilting disc valve are shown in Figure 6.2. Experience shows that one of the causes of failure of such a valve is breakage at the root of the inlet and/or outlet strut. Breakage of the outlet strut is more common than that of the inlet strut and such a failure leads to the other end of the same strut breaking and eventual malfunctioning of the valve which may lead to the patient's death.

6.2.1 Statement of the Problem

As mentioned above the breakage at the root of a strut is a cause of failure of Bjork-Shiley tilting disc valves. The breakage occurs for several possible reasons ranging from excessive stress during valve handling and implantation to fatigue failure. Ritchie and Lubock [12] in their paper suggest that under 'hostile' physiological environment subcritical growth of inherent material flaws, driven by the presence of alternating stresses (approximately 38 million cycles per year) can be a principal cause of mechanical failure. In these instances the life of the valve will be limited by material fatigue. Other reasons suggested include production and quality control techniques [2]. Examination of explanted valves by various groups has shown that the majority of broken prostheses have fractured near the outlet strut weld. As mentioned before, it is hypothesized that after one weld has broken the second weld is likely to fracture at a later time resulting in the catastrophic failure of the valve. The period of time between the breakage of the first weld and that of the second varies from valve to valve. However, laboratory simulated studies by Health and Welfare Canada, the US Food and Drug Administration and Shiley Inc. have demonstrated that fractured valves have operated for extended periods of time without failure [21]. Hence, if a reliable method were available to detect a valve with one weld broken, patients could be screened and the valve replaced.

In this context the work done by Walker and Scotten [21] has been useful in classifying in vivo valves as broken or intact by looking at the frequency spectra of those valves. They are planning to develop a non-invasive technique for screening patients at risk using inexpensive recording equipment. The frequency spectra emitted by the valves are processed using discriminant analysis [4] to generate two functions that are characteristic of the spectra. The values of these functions indicate whether the valve is broken or intact. However, due to limited data processing power, a range of frequencies had to be set for the discriminant analysis. The decision on the range of frequencies was made after determining the neighborhood of the frequencies associated with emission from broken and normal valves. This was accomplished by performing a vibration analysis on both the broken and intact valves.

Mechanical prosthetic heart valves operating in air, water and blood emit sound over a wide range of frequencies including ultrasound. The implanted valve frequencies are also dependent on their biological environment. However, those frequencies associated with the anatomy are in the frequency band 100-500 Hz. So, this range was not considered for scanning. The intact valve outlet strut would have different frequency spectra than a fractured or broken outlet strut. This was shown experimentally by Walker and associates. They found that a broken minor strut has a fundamental frequency of between 2500 Hz and 3000 Hz. This could also be shown by a finite element analysis of the strut. Thus the major application in this thesis of the helical element is the free vibration analysis of broken and intact inlet and outlet struts of the Bjork-Shiley tilting disc valve.

Some important aspects of finding the fundamental frequencies of such a valve are described in the following paragraphs. The models of the inlet strut and outlet strut of a Bjork-Shiley valve used for the analysis are shown in Figures 6.3 and 6.4 respectively with nodes, extracted from references [14,15].

The struts are welded to the outer ring at their roots. Both the inlet and outlet struts are made of a special alloy called Haynes 25, which is cobalt-based. They are spatial curves and could be modeled accurately using helical elements. Such a modeling was used for the determination of their frequencies. Since each of their regions has a different local coordinate system the special technique of generating helical elements in different local coordinate systems and the assembly routine outlined in chapter 4 was used for modelling the struts.

Even though clinical experience has shown that breakage of outlet struts

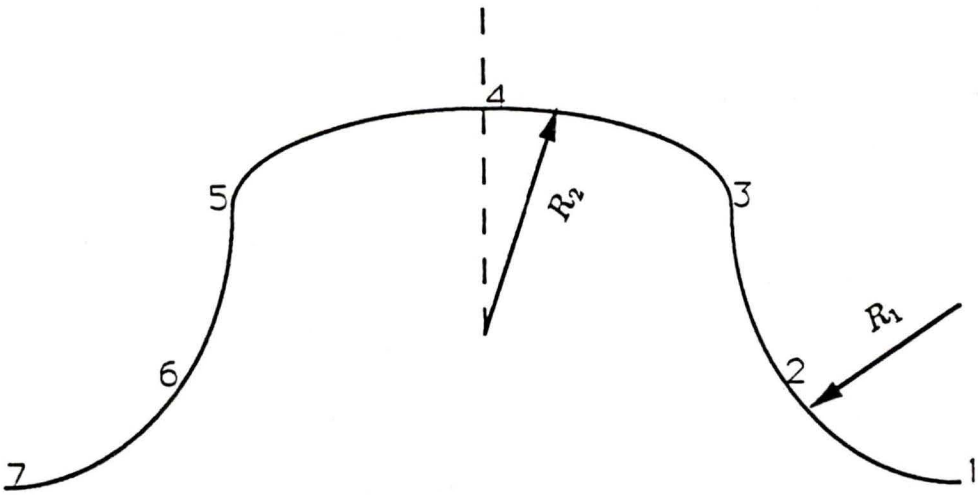

Fig. 6.3 Inlet Strut Model

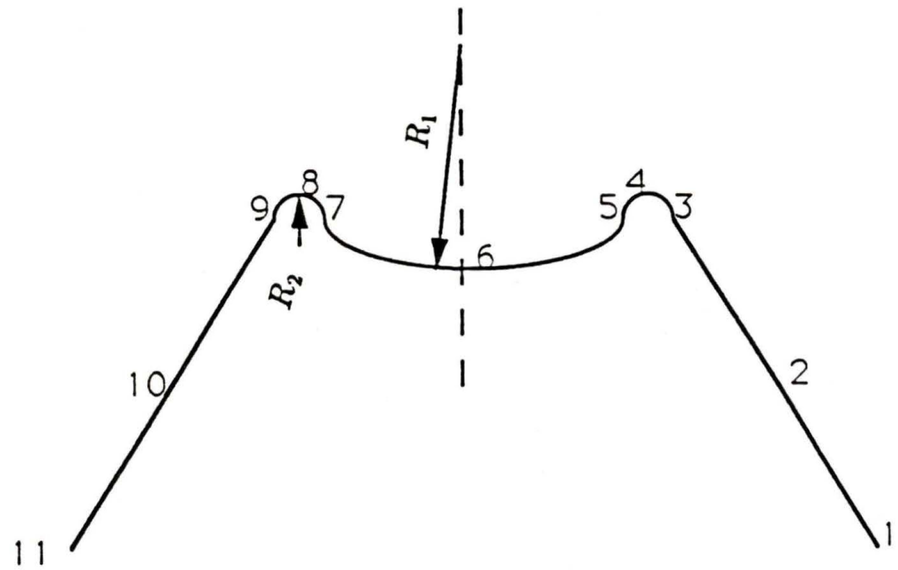

Fig. 6.4 Outlet Strut Model

is more common the same analysis was carried out on the inlet strut too. The input required for this analysis consists of information on the helical axial vector of each element, radius of curvature of the element, and the coordinates of the two nodes that constitute an element. Besides this the material properties and cross sectional properties of each element are also required. Since in the case of the inlet and outlet struts the cross section varied from region to region individual input of elemental cross sectional properties was necessary. Other properties of the elements such as the curvature and pitch are automatically computed by the program. The accuracy of the model was solely dependent upon the accuracy of the input axial vector information between any two nodes for a given radius. So, a graphics program could be used in conjunction with this procedure to overlay the projected model on the actual drawing and a correction on the axial vector might reduce the descrepecies. In the present case the coordinates of the points between two nodes of an element developed by the program were used to get a picture of the struts and it was found to approximate closely the drawing supplied by the manufacturer.

The material properties of the inlet and outlet struts are :

Young's modulus = 225 GPa

Shear modulus = 86 GPa

Density = 9130 kg/m^3.

While Young's modulus and density were reported in reference [19], the shear modulus was calculated using an assumed Poisson's ratio of 0.3.

The computed natural frequencies of the inlet strut and outlet strut of the 29mm Bjork-Shiley tilting disc valve are given in Tables 6.1 and 6.2 respectively. The input file for the outlet strut is given in the Appendix. Also, the same problem was solved using prismatic beam elements and was run on the ANSYS package [3]. Those results are also presented for comparison. Analysis by the ANSYS package was done using eighty elements for the inlet strut and thirty elements for the outlet strut, while only ten helical elements were used to model the outlet strut and six helical elements for the inlet strut. It may be noted that two cases were considered for analysis : one in which both ends of the struts were constrained representing an intact strut, and the second in which one end of the strut was free which represented a broken strut. Since the struts tend to break at their roots (because the stresses are highest at these points) only such cases were considered.

Table 6.1: Natural Frequencies in Hz of Inlet Strut

	Intact Strut			Broken Strut	
Mode	Helical Element	ANSYS	Experiment	Helical Element	ANSYS
1	3385	3399	3544	855	857
2	8715	8795	8848	910	910
3	9399	9492	9760	2416	2423
4	17150	17587	16896	2430	2438
5	20371	20750	20224	8625	8681

Table 6.2: Natural Frequencies in Hz of Outlet Strut

	Intact Strut			Broken Strut		
Mode	Helical Element	ANSYS	Experiment	Helical Element	ANSYS	Experiment
1	9580	9658	9504	2460	2469	2896
2	18132	18590	N.A.	2575	2600	3056
3	22258	22629	N.A.	6740	6815	N.A.

In addition to the analytical approaches, the above two cases of the struts were studied experimentally and the natural frequencies were measured. The valve was secured between the upper and lower fixtures and the appropriate strut was plucked. The emitted frequencies were recorded using a microphone and analyzed by a frequency analyzer. A dual channel, type 2034, Bruel & Kjaer analyzer was used for the experiment. The experimental results are given both in tabular(Tables 6.1 and 6.2) and graphical forms (Fig. 6.5, 6.6 and 6.7).

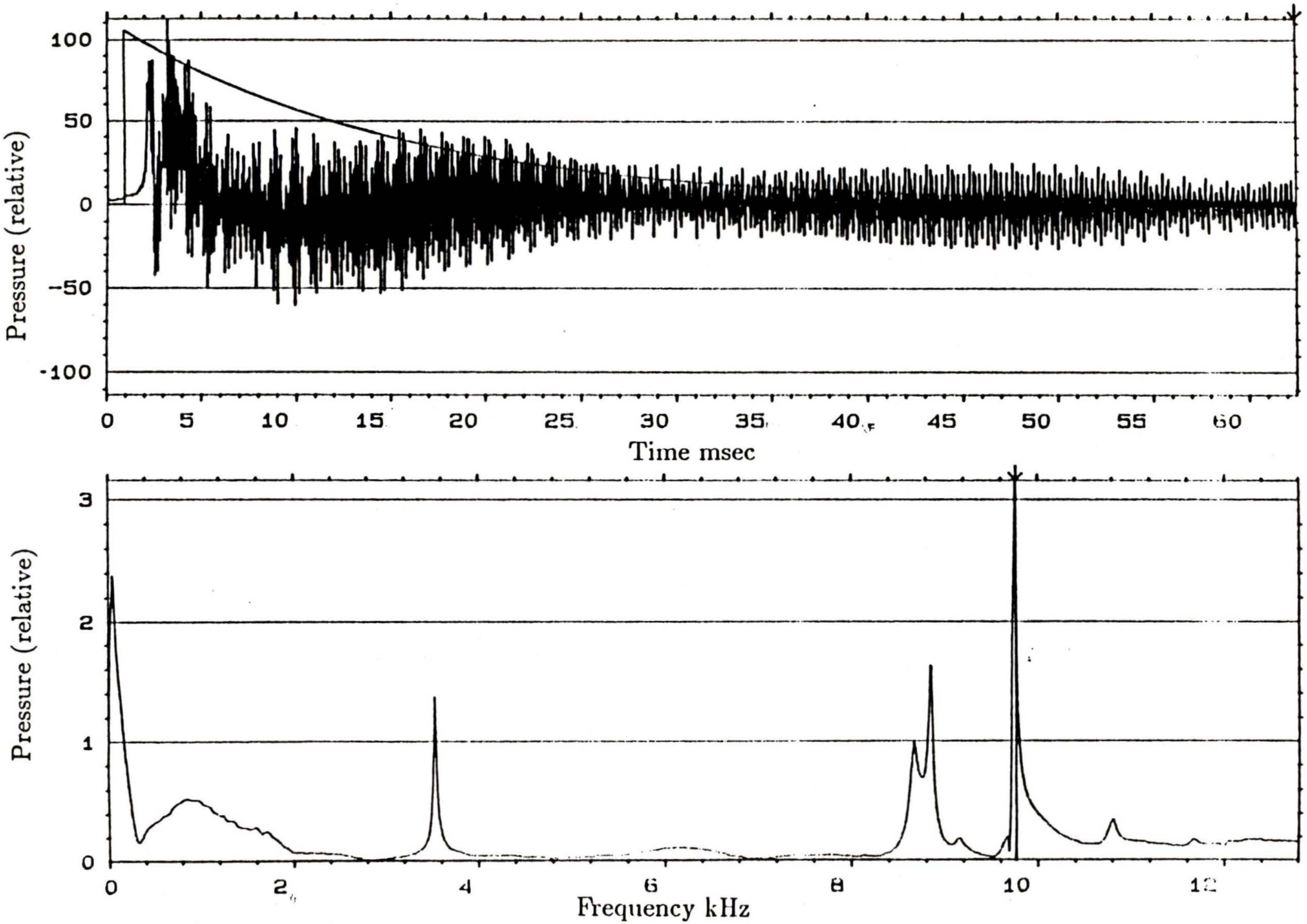

Fig. 6.5 Frequencies of Bjork-Shiley Valve Intact Inlet Strut

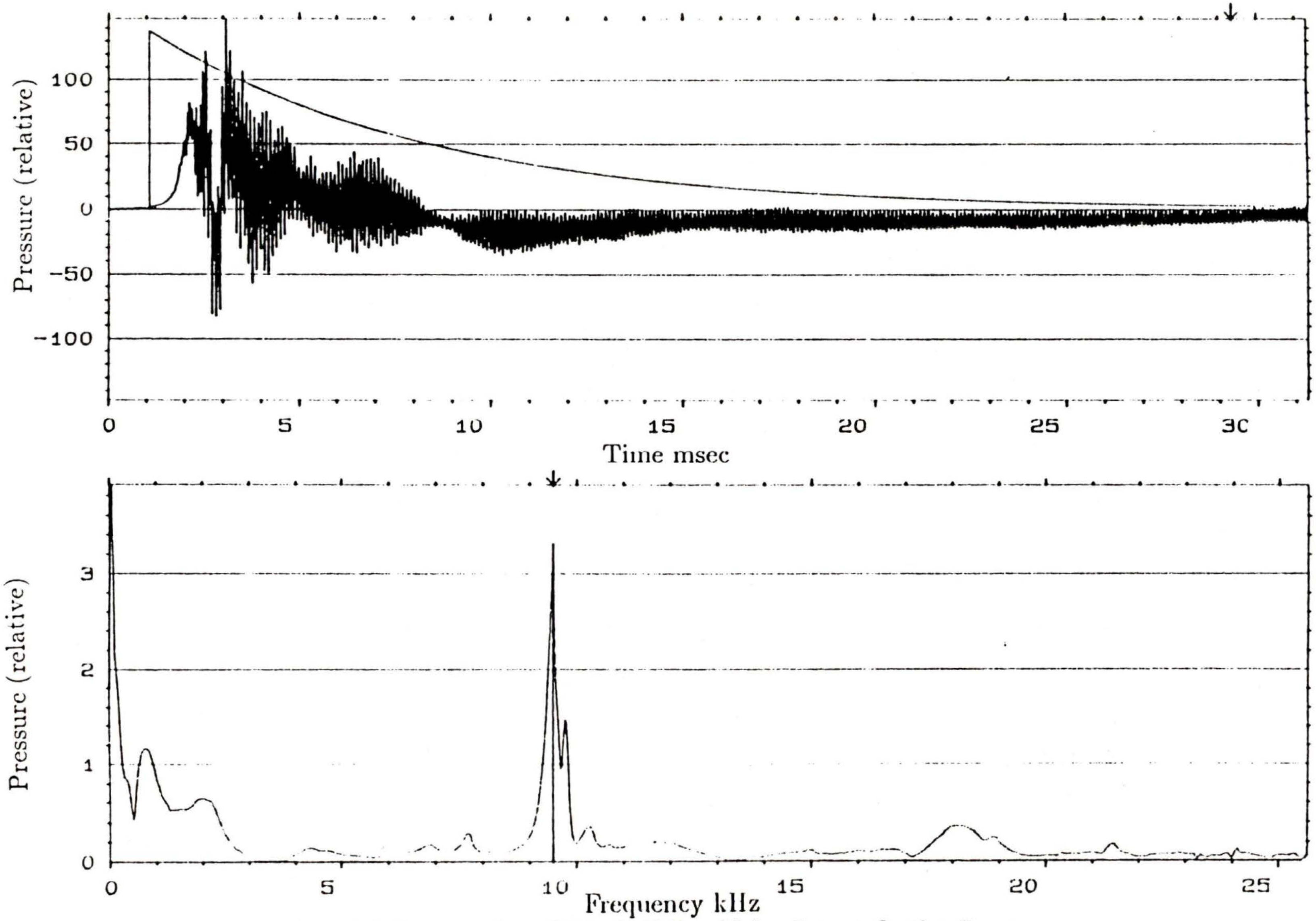

Fig. 6.6 Frequencies of Bjork-Shiley Valve Intact Outlet Strut

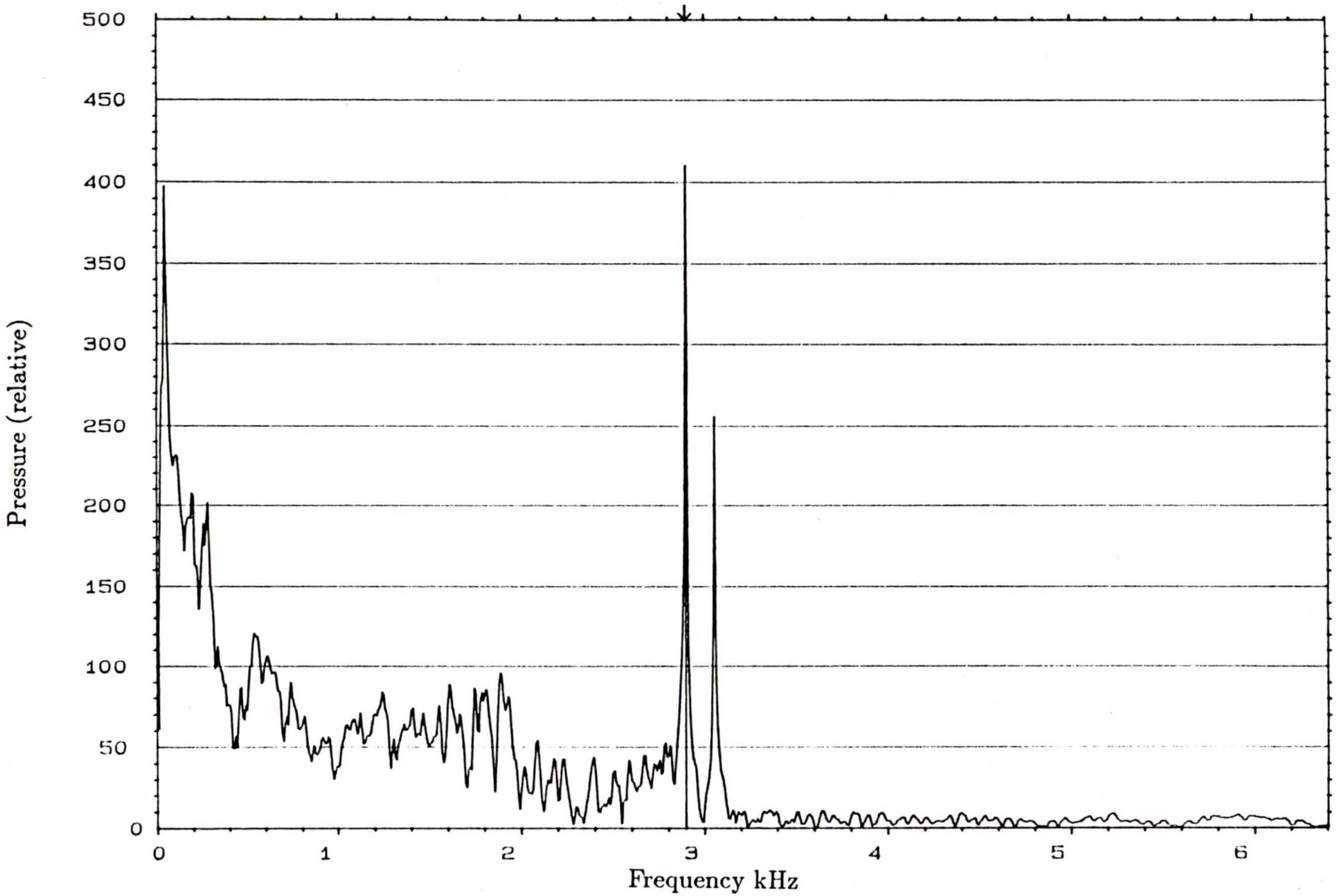

Fig. 6.7 Frequencies of Bjork-Shiley Valve Broken Outlet Strut

Even though there is a good agreement (with an accuracy to within 2.53 percent) between the two analytical methods, that is, between the ANSYS results and the helical element model results, the experimental results differ by as much as 4.5 percent in the intact model and 15.7 percent in the broken case. There are a number of sources of error the most important of which are related to material and geometric properties. For example, it was observed that probably due to poor handling the geometry of the struts had been distorted a little and also, the cross section of the struts was found to vary from region to region randomly which could not be taken into consideration in the analysis. The shear modulus for the material of the struts was not available from the manufacturer and so an empirical value was assumed for the analysis. All these may have had their impact on the accuracy of the computed frequencies.

6.3 Summary

The helical element developed in the preceding chapters was shown to provide a viable means for computing the natural frequencies of artificial heart valve struts. To gauge the differences between the natural frequencies of intact and broken strut Bjork-Shiley valve, free vibration analysis was carried out on both cases. To validate the accuracy of the results of the analysis comparison was made with experimental results and those obtained by using prismatic elements in the ANSYS package. Excellent agreement was found between ANSYS results and helical element ones. It is noted that the frequencies obtained with helical elements are lower than those with the prismatic beam elements which points to that the former are closer to the exact frequencies. The validity of these results was further supported by the closeness of the experimental results to the computed values in spite of several drawbacks of modelling due to insufficient information and/or nonconformity of the geometry of the actual structure with the drawings supplied.

Chapter 7

Conclusions and Suggestions for Future Work

The development of a spatially curved and twisted finite element, or in other words a helical element, has been outlined. The development is based on previous works by Tabarrok *et al.* [17] and Mottershead [9] on a similar type of element. Tabarrok's formulation is not as accurate as the present formulation because of the assumption of constant strain over the element. Mottershead's treatment is as accurate as the present one because it is also based on the solutions of the governing differential equations, but it is limited to circular cross sections, whereas the helical element developed here can take care of any doubly symmetric cross section. Since the shape functions of the helical element are based on the solutions of the homogeneous differential equations for static equilibrium, this element gives exact solutions to static problems. Also, for a given number of elements, it gives better accuracy for dynamic problems than conventional prismatic elements.

This dissertation also considers the problem of describing any arbitrary spatial curve-like structure using helical elements. A procedure has been laid out which brings together all the element stiffness and mass matrices in their coordinate systems to a common reference with respect to which the problem is usually known. This procedure, called globalization in this report, has been verified by solving problems in literature. As an important application, free vibration analysis of a Bjork-Shiley tilting disc type artificial heart valve has been carried out. The natural frequencies of the struts for intact and broken conditions are required for the discriminant analysis on them for identifying them as intact or broken. The present element has been extremely useful in modelling the struts of these valves since they are spatial curves with varying geometrical properties. The results provided by the helical elements were then compared with those obtained by conventional prismatic elements and experimental results. Since, at present this element can handle a problem in a very general setting, it is hoped that this can be incorporated into a finite element package.

Returning to the heart valve design one can make the following observations. The struts are rather slender as compared to the ring. Hence, the connection points of the struts to the ring represent sharp changes in geometry resulting in stress concentration sites. To mitigate their effects one should incorporate large fillets at these locations. Such fillets should be in a plane parallel to the direction of blood flow and have elliptic cross sections so as not to disturb the flow unduly.

The detection of breakages through changes of emitted frequencies is an important practical idea. The computation of natural frequencies in this report has revealed that significant changes occur in the natural frequencies when a breakage occurs. Even though the analysis carried out here was under simplified conditions these changes are expected to occur qualitatively in the actual conditions where the movements of the disc, the blood flow, and the biological environment will have their contributions. To develop an understanding of such changes in implanted valves an experimental study simulating the conditions in the body is being carried out by Walker and Scotten[21]. However, the studies in this thesis have indicated that a diagnostic detection procedure based on the frequency spectra is viable and holds good promise.

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Appendix A

Analysis Programs

A.1 Introduction

The helical element formulated in this report has been implemented in solving practical problems through the programs listed in the following pages. A brief introduction to these programs is given here. These programs could be modified and included in a large finite element package.

Both static and dynamic problems can be handled by the same programs, it being only a matter of choice. The problems dealt with may have a very general setting. That is, the structure can have any arbitrary curve-like shape, its geometrical and material properties can change over the length and finally, it can be geometrically described with respect to any convenient global reference frame. The program dicretizes the structure into helical elements and computes the stiffness and mass matrices of the entire structure with respect to the global reference frame and depending on the analysis type it determines the nodal displacements or natural frequencies.

A.2 Program Description

All the programs are written in FORTRAN [11]. The Makefile automates the compilation of the programs. The input is taken care of by the main program **helix2.f**. After a decision has been made on the type of analysis (through an option), the stiffness matrix alone or both stiffness and mass matrices are computed with respect to the element's own coordinate system making use of the subroutines in **he2.f**, **he4.f** and **bave.f**. The subroutines in **he2.f** and **he4.f** are similar in their functions. Both construct the matrix **B** which is composed of the vectors evaluated at nodal or Gauss points. The subroutine in **bave.f** delivers the element coordinte system base vectors in terms of the global reference which are used to construct the transformation matrix between the element coordinate system and global reference.

For the computation of the inverse of matrices and eigen value extraction, mathematical subroutines from IMSL (mathematical software package) are used. Any consistent system of units may be used for these programs. Every significant step in the program is commented before it appears in the program.

A.3 Sample Input File

The following is a sample input file for the inlet strut of Bjork-Shiley artificial heart valve. In addition to the nodal coordinates, components of the axial vector of the helix, radius of the helix, cross sectional properties, material properties and type of analysis are to be specified for each element. The following input file uses six elements for the inlet strut and has the input in the order outlined in the program. It provides the natural frequencies of the intact inlet strut. All units are in S.I.

```
2 6
11 12
0 0
2
1 7
1 8e-3
12.06e-3 0 0
7.503e-3 3.2544e-3 0.53082e-3
```

```
-0.000005 0.43098 -2.642333
0.1017876e-12 0.1017876e-12 0.2035752e-12 225.129e9 86e9
1.1309734e-6 1.1309734e-6 1.13097340e-6 9130
1 8e-3
7.503e-3 3.2544e-3 0.53082e-3
6.18e-3 7.97e-3 1.3e-3
-0.000008 0.309633 -1.898276
0.1017876e-12 0.1017876e-12 0.2035752e-12 225.129e9 86e9
1.1309734e-6 1.1309734e-6 1.13097340e-6 9130
1 9e-3
6.18e-3 7.97e-3 1.3e-3
0.44442e-3 9.6147e-3 3.1109e-3
-0.000019 -2.515314 2.284406
0.0858541e-12 0.0858541e-12 0.1717082e-12 225.129e9 86e9
1.0386891e-6 1.0386891e-6 1.0386891e-6 9130
1 9e-3
0.44442e-3 9.6147e-3 3.1109e-3
-6.18e-3 7.97e-3 1.3e-3
0.000019 -3.780044 3.433054
0.0858541e-12 0.0858541e-12 0.1717082e-12 225.129e9 86e9
1.0386891e-6 1.0386891e-6 1.0386891e-6 9130
1 8e-3
-6.18e-3 7.97e-3 1.3e-3
-7.503e-3 3.2544e-3 0.53082e-3
-0.000008 0.309633 -1.898276
0.1017876e-12 0.1017876e-12 0.2035752e-12 225.129e9 86e9
1.1309734e-6 1.1309734e-6 1.13097340e-6 9130
1 8e-3
-7.503e-3 3.2544e-3 0.53082e-3
-12.06e-3 0 0
-0.000005 0.43098 -2.642333
0.1017876e-12 0.1017876e-12 0.2035752e-12 225.129e9 86e9
1.1309734e-6 1.1309734e-6 1.13097340e-6 9130
```

A.4 Program Listing

A.4.1 Main Program

```
C PROGRAM HELIX2.F
C THIS PROGRAM COMPUTES THE ELEMENT STIFFNESS AND
C       MASS MATRICES OF A HELICAL ELEMENT WITH RESPECT
C       TO THE GLOBAL COORDINATE
C SYSTEM AND ASSEMBLES THEM. ALSO, IT COMPUTES
C THE DISPLACEMENTS
C OF NODES FOR A STATIC ANALYSIS PROBLEM AND
C       NATURAL FREQUENCIES
C FOR A FREE VIBRATION PROBLEM

C NE: NO. OF ELEMENTS, NOEL: CURRENT ELEMENT NUMBER
C
implicit double precision (A-H,O-Z)
double precision k,Ix,Iy,Iz,Ke
        double precision AA(12,12),B(12,12),Ke(12,12),
     *                  SMe(12,12),T1(12,12),SS(12,12)
double precision XX(12,12),YY(12,12),XI(12,12),
     *                  XIT(12,12),SM(6,12)
double precision T2(12,12),SS(12,12),pt(5),co(5),con(6)
double precision ecs(3,3),RT(12,12),TRT(12,12),
     c    GMe(12,12),GKe(12,12),a1(3),b1(3)
double precision bsn(3),GM(66,66),GK(66,66),GKI(66,66),
     c  uvt(66),fvt(66),eval(66),ndc(66),ndfc(66)
C WORKSPACE ALLOCATION BY IMSL PACKAGE

   common /WORKSP/ RWKSP
   REAL RWKSP(24666)
```

```
common t,a,k,x1,x2,x3,x4,x5,x6,x7,z1,z2,z3,z4,z5,z6
open(9,file= 'out')
open(8,file= 'in')
   CALL IWKIN(24666)
C NE: NO. OF ELEMENTS, NFV: NO. OF FORCE VECTORS
C NTYPE: ANALYSIS TYPE #1: STATIC, #2: MODAL ANALYSIS
C NOC: NO. OF NODES CONSTRAINED
C NDC: # OF THE CONSTRAINED NODE

read(8,*)ntype,ne
ndf=(ne+1)*6
read(8,*)nfv1,nfv2
read(8,*)fvt(nfv1),fvt(nfv2)
read(8,*)noc
read(8,*)(ndc(i),i=1,noc)
do 152 i=1,noc
ndfc(i)=(ndc(i)-1)*6+1
152 continue

C THE FOLLOWING LOOP IS CARRIED OUT OVER ALL ELEMENTS

do 201 noel=1,ne
write (9,153)noel
153 format(5x,"current element # = ",i2,/)
read(8,*)sign1,ra
read(8,*)(a1(i),i=1,3)
read(8,*)(b1(i),i=1,3)
read(8,*)(bsn(i),i=1,3)
read(8,*) Ix,Iy,Iz,E,G
read(8,*) Ax,Ay,Az,den
```

```
C bsvect PROVIDES THE ELEMENT COORDINATE
C       SYSTEM BASE VECTORS

call bsvect(a1,b1,bsn,ra,sign1,crv,tau,s2,ecs)

C TORSION AND CURVATURE

t=tau
k=crv



C s1 IS ASSUMED TO BE 0


s=0
m=1
a=dsqrt(t*t+k*k)
al=datan(t/k)
p=s2*a/2
write (9,190) t,k,E,G,Ix,Iy,Ax,Ay,Az,s2,al,p
190 format(5x,"t=",f10.5,2x,"k=",f10.5,2x,
     c  "E=",E15.8,2x,"G=",
     c  E15.8,2x,"Ix=",E15.8,2x,"Iy=",
     c  E15.8,2x,"Ax=",E15.8,2x,
     c  "Ay=",E15.8,2x,"Az=",E15.8,2x,
     c  "s2=",E10.5,2x,"al=",
     c  E10.5,2x,"phi=",E10.5,/)

c CONSTRUCTION OF A

do 60 i=1,2

c=dcos(a*s)
```

```
sn=dsin(a*s)
AA(m,1)=a*c
AA(m,2)=a*sn
AA(m+1,1)=-t*sn
AA(m+1,2)=t*c
AA(m+1,3)=1
AA(m+2,1)=k*sn
AA(m+2,2)=-k*c
AA(m+2,3)=t/k
AA(m+3,1)=-t*s*sn
AA(m+3,2)=t*s*c
AA(m+3,4)=a*c
AA(m+3,5)=a*sn
AA(m+4,1)=-(t*t*s*c/a)-(k*k*sn/a/a)
AA(m+4,2)=-(t*t*s*sn/a)+(k*k*c/a/a)
AA(m+4,4)=-t*sn
AA(m+4,5)=t*c
AA(m+4,6)=1
AA(m+5,1)=-(k*t*sn/a/a)+(k*t*s*c/a)
AA(m+5,2)=(k*t*c/a/a)+(k*t*s*sn/a)
AA(m+5,3)=1/k
AA(m+5,4)=k*sn
AA(m+5,5)=-k*c
AA(m+5,6)=t/k
m=7
s=s2

60 continue

98 format(5x,6E17.6)
```

```
c TRANSFORMATION MATRICES T1 AND T2

c p=phi/2

sp=dsin(p)
sa=dsin(al)
cp=dcos(p)
ca=dcos(al)
do 61 i=1,10,3
  T1(i,i)=-sp
  j=i+2
  m=i+1
  T1(j,m)=sa*sp
  T1(j,j)=-ca*sp
  T2(i,m)=-sa*cp
  T2(i,j)=ca*cp
  T2(m,m)=ca
  T2(m,j)=sa
  T2(j,i)=-cp
61 continue

do 62 i=1,10,3
  if (i.ge.7) then
    sign=-1
  else
   sign=1
  endif

63   m=i+1
```

```
  j=i+2
  T1(i,m)=sign*sa*cp
  T1(i,j)=sign*(-ca*cp)
  T1(m,m)=sign*(-ca)
  T1(m,j)=sign*(-sa)
  T1(j,i)=sign*cp
  T2(i,i)=sign*sp
  T2(j,m)=sign*(-sa*sp)
  T2(j,j)=sign*ca*sp
62 continue

C COMPUTATION OF MATRIX Y

do 10 i=1,12
          do 10 j=1,12

    YY(i,j)=0
    do 11 l=1,12

11          YY(i,j)=YY(i,j)+T1(i,l)*AA(l,j)

10 continue
write(9,154)
154     format(/,10x,"matrix YY",//)
write(9,98)((YY(i,j),j=1,12),i=1,12)

c       CONSTRUCTION OF MATRIX B

        z1=E*Ix
        z2=E*Iy
        z3=G*Iz
        z4=G*Ax
```

```
      z5=G*Ay
      z6=E*Az

      x1=((z1*z3*t*t)+(z1*z2*k*k)+(z2*z3*a*a))
     c /(2.0*a*z1*z2*z3)
      x2=-t*((z1*z3*t*t)-(2.0*z1*z3*k*k)+
     c (3.0*z1*z2*k*k)-(z2*z3*a*a))
     c /(4.0*(a**3)*z1*z2*z3)
      x3=-t*((z2*x1*t)-(t*a)-(2.0*z2*x2*a*a))
     c /(4.0*(a**3)*z2)
      x41=(x1*t*t/4.0/a**4)+(t*t/2.0/z5/a)-
     c (t*t/4.0/z2/a**3)-(x2*t/2/a**2)
      x42=(k*k/2.0/z6/a)+(k*k/2.0/z2/a**3)+(a/2.0/z4)
      x4=x41+x42
      x5=-(t/z6/a**2)+(t/z5/a**2)+(2.0*t/z3/a**4)
      x6=(t/2.0/z2/a)-(x1*t/2.0/a**2)
      x7=(2.0*t*t/z3/a**4)-(2.0*t*t/z2/a**4)
     c +(1.0/z2/a**2)



c       COMPUTATION OF MATRIX B

call bmatrx(s2,B)
write(9,155)
155     format(/,10x,"matrix B",//)
write(9,98)((B(i,j),j=1,12),i=1,12)
c       Matrix X

do 81 i=1,12
do 81 j=1,12
```

```
  XX(i,j)=0
  do 82 l=1,12
82       XX(i,j)=XX(i,j)+T2(i,l)*B(l,j)
81 continue

write(9,156)
156      format(/,10x,"matrix XX",//)
write(9,98)((XX(i,j),j=1,12),i=1,12)
call DLINRG(12,XX,12,XI,12)
write(9,164)
164      format(/,10x,"matrix XI",//)
write(9,98)((XI(i,j),j=1,12),i=1,12)


c        ELEMENTAL STIFFNESS MATRIX Ke

do 83 i=1,12
do 83 j=1,12
     Ke(i,j)=0
          do 83 l=1,12
83         Ke(i,j)=Ke(i,j)+YY(i,l)*XI(l,j)

write(9,114)
114      format(/,10x,"matrix Ke",//)
write(9,98)((Ke(i,j),j=1,12),i=1,12)

C COMPUTATION OF ELEMENTAL MASS MATRIX, Me

195 pt(1)=0.9061798459
pt(2)=0.5384693101
pt(3)=0.0
pt(4)=-0.5384693101
```

```
pt(5)=-0.9061798459
co(1)=0.2369268850
co(2)=0.4786286705
co(3)=0.56888888889
co(4)=0.4786286705
co(5)=0.2369268850

con(1)=Az
con(2)=Az
con(3)=Az
con(4)=Ix
con(5)=Iy
con(6)=Iz

c INITIALIZING MASS MATRIX


do 527 i=1,12
do 527 j=1,12
SMe(i,j)=0.0
527 SS(i,j)=0.0


do 90 i=1,5

pt1=(pt(i)+1.0)*s2/2.0
  call Massm(pt1,SM)

    do 91 l=1,12
    do 91 m=1,12
do 92 j=1,6

92       SS(l,m)=co(i)*con(j)*SM(j,l)*SM(j,m)+SS(l,m)
91      continue
90 continue
```

```
do 525 i=1,12
do 525 j=1,12
525      SS(i,j)=SS(i,j)*s2/2

c XIT -X INVERSE TRANSPOSE

do 94 i=1,12
do 94 j=1,12
94   XIT(i,j)=XI(j,i)

c MASS MATRIX SMe

do 96 i=1,12
do 96 j=1,12
  do 95 m=1,12
  do 95 l=1,12
   SMe(i,j)=SMe(i,j)+XIT(i,m)*SS(m,l)*XI(l,j)
95 continue
96 SMe(i,j)=SMe(i,j)*den
write(9,134)
134      format(/,10x,"elem Mass Matrix",//)
write(9,98)((SMe(i,j),j=1,12),i=1,12)

C  ELEMENTAL TRANSFORMATION MATRIX

do 202 i=1,10,3
k1=0
do 203 j=i,i+2
k1=k1+1
m=0
do 204 l=i,i+2
m=m+1
204 RT(j,l)=ecs(k1,m)
```

```
203 continue
202 continue

do 205 i=1,12
do 205 j=1,12
205 TRT(i,j)=RT(j,i)
write(9,157)
157      format(/,10x,"RT matrix",//)
write(9,98)((RT(i,j),j=1,12),i=1,12)

C GLOBAL ELEMENTAL MASS AND STIFFNESS MATRICES

do 207 i=1,12
do 207 j=1,12
GMe(i,j)=0.0
GKe(i,j)=0.0
do 208 m=1,12
do 208 l=1,12
GMe(i,j)=GMe(i,j)+TRT(i,m)*SMe(m,l)*RT(l,j)
GKe(i,j)=GKe(i,j)+TRT(i,m)*Ke(m,l)*RT(l,j)
208 continue
207 continue
write(9,158)
158      format(/,10x,"GME matrix",//)
write(9,98)((GMe(i,j),j=1,12),i=1,12)
write(9,159)
159      format(/,10x,"GKematrix",//)
write(9,98)((GKe(i,j),j=1,12),i=1,12)

C ASSEMBLY OF GLOBAL ELEMENTAL MATRICES

ne1=noel+(noel-1)*5
```

```
k1=0
do 209 ne2=ne1,ne1+11
k1=k1+1
j=0
do 210 ne3=ne1,ne1+11
j=j+1
GK(ne2,ne3)=GK(ne2,ne3)+GKe(k1,j)
210 GM(ne2,ne3)=GM(ne2,ne3)+GMe(k1,j)
209 continue

201 continue
399 format(5x,6E17.6)

c  APPLY BOUNDARY CONDITIONS, ETC.

do 219 nc=1,noc
do 213 i=ndfc(nc),ndfc(nc)+5
do 213 j=1,ndf
GK(i,j)=0.0
GK(j,i)=0.0
    GK(i,i)=10e14
GM(i,j)=0.0
GM(j,i)=0.0
GM(i,i)=1.0
213 continue
219 continue

C CHOICE OF ANALYSIS TYPE

if(ntype.eq.1)then

217 write(9,160)
160     format(/,10x,"GKmatrix",//)
```

```
write(9,399)((GK(i,j),j=1,ndf),i=1,ndf)
call DLINRG(ndf,GK,ndf,GKI,ndf)
write(9,162)

162      format(/,10x,"GKI matrix",//)
write(9,399)((GKI(i,j),j=1,ndf),i=1,ndf)

C DISPLACEMENTS FOR A STATIC ANALYSIS PROBLEM

do 211 i=1,ndf
uvt(i)=0.0
do 211 j=1,ndf
211 uvt(i)=GKI(i,j)*fvt(j)+uvt(i)
220 format(5x,"Displacements of Nodes",/)
write(9,220)
write(9,215)(uvt(i),i=1,ndf)
stop
else

write(9,161)
161      format(/,10x,"GMmatrix",//)
write(9,399)((GM(i,j),j=1,ndf),i=1,ndf)

C EIGEN VALUE EXTRACTION FOR FREQUENCIES
C        FOR A DYNAMIC PROBLEM

310 call DGVLSP(ndf,GK,ndf,GM,ndf,eval)

write (9,417)
417 format(5x,"frequency in c/s",/)
do 418 i=1,ndf
418 eval(i)=sqrt(eval(i))/2/3.1416
write(9,215)(eval(i),i=1,ndf)
```

```
215 format(5x,e14.7,/)
endif
stop
end
```

A.4.2 Stiffness Matrix

```
c        SUBROUTINES START HERE
c VECTORS USED TO CONSTRUCT MATRIX B ARE COMPUTED HERE

subroutine bmatrx(s2,B)
common t,a,k,x1,x2,x3,x4,x5,x6,x7,z1,z2,z3,z4,z5,z6
implicit double precision (A-H,O-Z)
double precision uv(12),up(12),vv(12),k,iQv(12),iMv(12)
double precision iup(12),tup(12),wv(12),tuv(12),tvv(12)
double precision twv(12), itup(12),Qu(12),Qv(12),Qw(12)
double precision Mu(12),Mv(12),Mw(12),dup(12),dtup(12)
double precision k,B(12,12)

      write(9,120) t,a,k,x1,x2,x3,x4,x5,x6,x7,z1,z2,z3,z4,z5,z6
120     format(5x,"check",5x,8f9.3,/,5x,8f12.5)
  write (9,190) t,k,E,G,Ix,Iy,Ax,Ay,Az,s2,al,p
190     format(5x,6f16.10,/,5x,6f16.10)

C THE FOLLOWING LOOP IS CARRIED OUT AT THE TWO NODES
C OF THE ELEMENT

do 20 i=1,2

  if(i.eq.1)then
    s=0.0
    m=0
  else
    s=s2
    m=6
  endif

c=dcos(a*s)
```

```
sn=dsin(a*s)
uv(10)=a*c
uv(11)=a*sn
up(1)=(-t*t*x1/6.0/a**2)*c*s**3-(x3*sn*s*s)+(x4*s*c)
up(2)=(-t*t*x1*sn*s**3/6.0/a**2)+(x3*c*s*s)+(x4*s*sn)
up(3)=x5
up(4)=(-t*x1*sn*s*s/2.0/a)+(x6*s*c)
up(5)=(t*x1*c*s*s/2.0/a)+(x6*sn*s)
up(6)=x7
up(7)=-t*s*sn
up(8)=t*s*c


vv(7)=-sn
vv(8)=c
vv(10)=-t*sn
vv(11)=t*c
vv(12)=1.0


iQv(1)=c*t/a
iQv(2)=sn*t/a
iQv(3)=s

iMv(1)=-((a*s*sn+c)*t**2/a**3)+(c*k**2)/a**3
iMv(2)=-((sn-a*c*s)*t**2/a**3)+(sn*k**2)/a**3
iMv(4)=c*t/a
iMv(5)=sn*t/a
iMv(6)=s

iup(1)=-(x1*(((a**3)*(s**3)-6.0*a*s)*sn+
     c  c*(3*a*a*s*s-6.0))*t*t)/6.0/(a**6)
```

```
     c  -x3*(2.0*a*s*sn+c*(2.0-a*a*s*s))/(a**3)+
     c  (x4*(a*s*sn+c)/a/a)
iup(2)=-(x1*((3.0*a*a*s*s-6.0)*sn+c*(6.0*a*s-
     c  (a*s)**3))*t*t)/(6.0*a**6)+
     c  (x3*((a*a*s*s-2.0)*sn+2.0*a*c*s)/(a**3))+
     c  (x4*(sn-a*c*s)/a/a)
iup(3)=x5*s
iup(4)=(x6*(a*s*sn+c)/a/a)-
     c  (x1*(2.0*a*s*sn+c*(2.0-a*a*s*s))*t/(2.0*a**4))
iup(5)=(x1*((a*a*s*s-2.0)*sn+2.0*a*c*s)*t/
     c  (2.0*a**4))+(x6*(sn-a*c*s)/a/a)
iup(6)=x7*s
iup(7)=-(sn-a*c*s)*t/a/a
iup(8)=(a*s*sn+c)*t/a/a

tup(1)=(-t*x1*s*s*sn/2.0/a)+x2*s*c
tup(2)=(t*x1*s*s*c/2.0/a)+x2*s*sn
tup(3)=-1.0/(a*a*z3)
tup(4)=x1*s*c
tup(5)=x1*s*sn
tup(6)=t*(z3-z2)/(a*a*z2*z3)

wv(7)=-t*sn/k
wv(8)=t*c/k
wv(10)=k*sn
wv(11)=-k*c
wv(12)=t/k


tuv(7)=a*c
tuv(8)=a*sn
```

```
tvv(7)=-t*sn
tvv(8)=t*c
tvv(9)=1.0

twv(7)=k*sn
twv(8)=-k*c
twv(9)=t/k

itup(1)=(x2*(a*s*sn+c)/a/a)-
     c  (x1*(2.0*a*s*sn+c*(2.0-a*a*s*s))*t/2.0/a**4)
itup(2)=(x1*((a*a*s*s-2.0)*sn+2.0*a*c*s)*t/(2.0*a**4))+
     c  (x2*(sn-a*c*s)/a/a)
itup(3)=-s/(a*a*z3)
itup(4)=x1*(a*s*sn+c)/a/a
itup(5)=x1*(sn-a*c*s)/a/a
itup(6)=(z3-z2)*s*t/(a*a*z2*z3)

Qu(1)=a*c
Qu(2)=a*sn

Qv(1)=-t*sn
        Qv(2)=t*c
Qv(3)=1.0

Qw(1)=k*sn
Qw(2)=-k*c
Qw(3)=t/k

Mu(1)=-t*s*sn
Mu(2)=t*s*c
Mu(4)=a*c
Mu(5)=a*sn

Mv(1)=-(t*t*s*c/a)-(k*k*sn/a/a)
```

```
Mv(2)=-(t*t*s*sn/a)+(k*k*c/a/a)
Mv(4)=-t*sn
Mv(5)=t*c
Mv(6)=1.0

Mw(1)=-(k*t*sn/a/a)+(k*t*c*s/a)
Mw(2)=(k*t*c/a/a)+(k*t*s*sn/a)
Mw(3)=1.0/k
Mw(4)=k*sn
Mw(5)=-k*c
Mw(6)=t/k

dup(1)=(x1*(s**3)*sn*t*t/6.0/a)-
     c  (x1*c*s*s*t*t/2.0/a/a)-(a*x4*s*sn)-
     c  (2.0*x3*s*sn)-(a*x3*c*s*s)+(x4*c)
dup(2)=-(x1*s*s*sn*t*t/2.0/a/a)-
     c  (x1*c*s**3.0*t*t/6.0/a)-(a*x3*s*s*sn)+
     c  (x4*sn)+(a*c*x4*s)+(2.0*x3*c*s)
dup(3)=0.0
dup(4)=-(x1*s*sn*t/a)-(x1*c*s*s*t/2.0)-
     c  (a*x6*s*sn)+(x6*c)
dup(5)=-(x1*s*s*sn*t/2.0)+(x1*c*s*t/a)+
     c  (x6*sn)+(a*x6*c*s)
dup(7)=-sn*t-a*c*s*t
dup(8)=c*t-(a*s*sn*t)

dtup(1)=-(x1*s*sn*t/a)-(x1*c*s*s*t/2.0)-
     c  (a*x2*s*sn)+(x2*c)
dtup(2)=-(x1*s*s*sn*t/2.0)+(x1*c*s*t/a)+
     c  (x2*sn)+(a*x2*c*s)
dtup(4)=x1*c-(a*x1*s*sn)
dtup(5)=x1*sn+(a*x1*c*s)
```

```
C MATRIX B IS CONSTRUCTED BELOW

do 70  j=1,12

  B(m+1,j)=uv(j)+up(j)
 B(m+2,j)=vv(j)+(iQv(j)/z5)-(t*iup(j))-itup(j)
B(m+3,j)=(tvv(j)/k)+(iMv(j)/z2/k)-(t*t*iup(j)/k)+
     c  (t*iQv(j)/z5/k)-(2.0*t*itup(j)/k)+
     c  (Qu(j)/z4/k)-(dup(j)/k)+wv(j)
  B(m+4,j)=tuv(j)+tup(j)
 B(m+5,j)=tvv(j)-(t*itup(j))+(iMv(j)/z2)
 B(m+6,j)=twv(j)+(Mu(j)/z1/k)+(iMv(j)*t/z2/k)-
     c  (dtup(j)/k)-
     c  (t*t*itup(j)/k)

70 continue

20 continue
return
end
```

A.4.3 Mass Matrix

```
C THIS CONSTRUCTS MATRIX B AT THE GAUSS POINTS
C       TO FORM THE MASS MATRIX

subroutine Massm(s2,B)
common t,a,k,x1,x2,x3,x4,x5,x6,x7,z1,z2,z3,z4,z5,z6
implicit double precision (A-H,O-Z)
double precision uv(12),up(12),vv(12),k,iQv(12),iMv(12)
double precision iup(12),tup(12),wv(12),tuv(12),tvv(12)
double precision twv(12), itup(12),Qu(12),Qv(12),Qw(12)
double precision Mu(12),Mv(12),Mw(12),dup(12),dtup(12)
double precision k,B(6,12)

write(9,120) t,a,k,x1,x2,x3,x4,x5,x6,x7
     c  ,z1,z2,z3,z4,z5,z6
120     format(5x,"check",5x,8f9.3,/,5x,8f12.5)
  write (9,190) t,k,E,G,Ix,Iy,Ax,Ay,Az,s2,al,p
190     format(5x,6f16.10,/,5x,6f16.10)
s=s2

c=dcos(a*s)
sn=dsin(a*s)
uv(10)=a*c
uv(11)=a*sn
up(1)=(-t*t*x1/6.0/a**2)*c*s**3-(x3*sn*s*s)+(x4*s*c)
up(2)=(-t*t*x1*sn*s**3/6.0/a**2)+(x3*c*s*s)+(x4*s*sn)
up(3)=x5
up(4)=(-t*x1*sn*s*s/2.0/a)+(x6*s*c)
up(5)=(t*x1*c*s*s/2.0/a)+(x6*sn*s)
```

```
up(6)=x7
up(7)=-t*s*sn
up(8)=t*s*c


vv(7)=-sn
vv(8)=c
vv(10)=-t*sn
vv(11)=t*c
vv(12)=1.0


iQv(1)=c*t/a
iQv(2)=sn*t/a
iQv(3)=s

iMv(1)=-((a*s*sn+c)*t**2/a**3)+(c*k**2)/a**3
iMv(2)=-((sn-a*c*s)*t**2/a**3)+(sn*k**2)/a**3
iMv(4)=c*t/a
iMv(5)=sn*t/a
iMv(6)=s

iup(1)=-(x1*(((a**3)*(s**3)-6.0*a*s)*sn+
     c  c*(3*a*a*s*s-6.0))*t*t)/6.0/(a**6)
     c  -x3*(2.0*a*s*sn+c*(2.0-a*a*s*s))/(a**3)+
     c  (x4*(a*s*sn+c)/a/a)
iup(2)=-(x1*((3.0*a*a*s*s-6.0)*sn+c*(6.0*a*s-
     c  (a*s)**3))*t*t)/(6.0*a**6)+
     c  (x3*((a*a*s*s-2.0)*sn+2.0*a*c*s)/(a**3))+
     c  (x4*(sn-a*c*s)/a/a)
iup(3)=x5*s
iup(4)=(x6*(a*s*sn+c)/a/a)-
     c  (x1*(2.0*a*s*sn+c*(2.0-a*a*s*s))*t/(2.0*a**4))
```

```
iup(5)=(x1*((a*a*s*s-2.0)*sn+2.0*a*c*s)*t/
     c  (2.0*a**4))+(x6*(sn-a*c*s)/a/a)
iup(6)=x7*s
iup(7)=-(sn-a*c*s)*t/a/a
iup(8)=(a*s*sn+c)*t/a/a

tup(1)=(-t*x1*s*s*sn/2.0/a)+x2*s*c
tup(2)=(t*x1*s*s*c/2.0/a)+x2*s*sn
tup(3)=-1.0/(a*a*z3)
tup(4)=x1*s*c
tup(5)=x1*s*sn
tup(6)=t*(z3-z2)/(a*a*z2*z3)

wv(7)=-t*sn/k
wv(8)=t*c/k
wv(10)=k*sn
wv(11)=-k*c
wv(12)=t/k


tuv(7)=a*c
tuv(8)=a*sn

tvv(7)=-t*sn
tvv(8)=t*c
tvv(9)=1.0

twv(7)=k*sn
twv(8)=-k*c
twv(9)=t/k

itup(1)=(x2*(a*s*sn+c)/a/a)-
     c  (x1*(2.0*a*s*sn+c*(2.0-a*a*s*s))*t/2.0/a**4)
```

```
itup(2)=(x1*((a*a*s*s-2.0)*sn+2.0*a*c*s)*t/(2.0*a**4))+
     c  (x2*(sn-a*c*s)/a/a)
itup(3)=-s/(a*a*z3)
itup(4)=x1*(a*s*sn+c)/a/a
itup(5)=x1*(sn-a*c*s)/a/a
itup(6)=(z3-z2)*s*t/(a*a*z2*z3)

Qu(1)=a*c
Qu(2)=a*sn

Qv(1)=-t*sn
        Qv(2)=t*c
Qv(3)=1.0

Qw(1)=k*sn
Qw(2)=-k*c
Qw(3)=t/k

Mu(1)=-t*s*sn
Mu(2)=t*s*c
Mu(4)=a*c
Mu(5)=a*sn

Mv(1)=-(t*t*s*c/a)-(k*k*sn/a/a)
Mv(2)=-(t*t*s*sn/a)+(k*k*c/a/a)
Mv(4)=-t*sn
Mv(5)=t*c
Mv(6)=1.0

Mw(1)=-(k*t*sn/a/a)+(k*t*c*s/a)
Mw(2)=(k*t*c/a/a)+(k*t*s*sn/a)
Mw(3)=1.0/k
Mw(4)=k*sn
Mw(5)=-k*c
```

```
Mw(6)=t/k

dup(1)=(x1*(s**3)*sn*t*t/6.0/a)-
     c  (x1*c*s*s*t*t/2.0/a/a)-(a*x4*s*sn)-
     c  (2.0*x3*s*sn)-(a*x3*c*s*s)+(x4*c)
dup(2)=-(x1*s*s*sn*t*t/2.0/a/a)-
     c  (x1*c*s**3.0*t*t/6.0/a)-(a*x3*s*s*sn)+
     c  (x4*sn)+(a*c*x4*s)+(2.0*x3*c*s)
dup(3)=0.0
dup(4)=-(x1*s*sn*t/a)-(x1*c*s*s*t/2.0)-
     c  (a*x6*s*sn)+(x6*c)
dup(5)=-(x1*s*s*sn*t/2.0)+(x1*c*s*t/a)+
     c  (x6*sn)+(a*x6*c*s)
dup(7)=-sn*t-a*c*s*t
dup(8)=c*t-(a*s*sn*t)

dtup(1)=-(x1*s*sn*t/a)-(x1*c*s*s*t/2.0)-
     c  (a*x2*s*sn)+(x2*c)
dtup(2)=-(x1*s*s*sn*t/2.0)+(x1*c*s*t/a)+
     c  (x2*sn)+(a*x2*c*s)
dtup(4)=x1*c-(a*x1*s*sn)
dtup(5)=x1*sn+(a*x1*c*s)



do 70  j=1,12

  B(1,j)=uv(j)+up(j)
 B(2,j)=vv(j)+(iQv(j)/z5)-(t*iup(j))-itup(j)
B(3,j)=(tvv(j)/k)+(iMv(j)/z2/k)-(t*t*iup(j)/k)+
     c  (t*iQv(j)/z5/k)-(2.0*t*itup(j)/k)+
     c  (Qu(j)/z4/k)-(dup(j)/k)+wv(j)
```

```
  B(4,j)=tuv(j)+tup(j)
 B(5,j)=tvv(j)-(t*itup(j))+(iMv(j)/z2)
 B(6,j)=twv(j)+(Mu(j)/z1/k)+(iMv(j)*t/z2/k)-
     c  (dtup(j)/k)-
     c  (t*t*itup(j)/k)

70 continue

return
end
```

A.4.4 Base Vectors

```
c PROGRAM : bave.f
c THIS COMPUTES ELEMENT COORDINATE SYSTEM BASE VECTORS
c WITH RESPECT TO GLOBAL COORDINATE SYSTEM
c ECS: ELEMENT COORDINATE SYSTEM
c LCS: LOCAL COORDINATE SYSTEM
c GCS: GLOBAL COORDINATE SYSTEM
subroutine bsvect(a1,b1,bsn,r,sign1,crv,tau,s2,xe)
implicit double precision (a-h,o-z)
double precision xe(3,3),xn(3,3),bpg(3),bpl(3),bl(3)
double precision a1(3),b1(3),o1(3),bsn(3),cg(3),cl(3)

C POINTS a1(i), b1(i) ARE INITIAL AND FINAL NODES RESP.


do 27 i=1,3
27 xn(3,i)=bsn(i)

C NORMALIZE AXIAL VECTOR

xn3=dsqrt(xn(3,1)**2+xn(3,2)**2+xn(3,3)**2)
xn(3,1)=xn(3,1)/xn3
xn(3,2)=xn(3,2)/xn3
xn(3,3)=xn(3,3)/xn3

if (sign1.lt.0.0) then
d=xn(3,1)*a1(1)+xn(3,2)*a1(2)+xn(3,3)*a1(3)
else
d=-(xn(3,1)*a1(1)+xn(3,2)*a1(2)+xn(3,3)*a1(3))
endif

d1=(xn(3,1)*b1(1)+xn(3,2)*b1(2)+xn(3,3)*b1(3)+d)**2
```

```
try=1
a123=a1(1)**2+a1(2)**2+a1(3)**2
b123=b1(1)**2+b1(2)**2+b1(3)**2
19  nm=1
        do 100 i=1,3
                if(nm.eq.2)go to 101
  if (i.eq.3)then
                        j=1
                        else
                        j=i+1
                endif
                if (j.eq.3)then
                        l=1
                        else
                        l=j+1
                endif
h4=(a1(j)-b1(j))*xn(3,i)-xn(3,j)*(a1(i)-b1(i))
h2=(a1(i)-b1(i))*xn(3,j)-xn(3,i)*(a1(j)-b1(j))
                if(abs(h2).le.0.0001)then
                 go to 100
                 else
                if(abs(h4).le.0.0001)then
                 go to 100
                 else
                h3=(a1(l)-b1(l))*xn(3,i)
      c         -xn(3,l)*(a1(i)-b1(i))
                h1=(a1(l)-b1(l))*xn(3,j)
      c         -xn(3,l)*(a1(j)-b1(j))
c=(2.0*d*(a1(i)-b1(i))+(d1+a123-b123)*xn(3,i))/2.0/h4
```

```
a=(2.0*d*(a1(j)-b1(j))+(d1+a123-b123)*xn(3,j))/2.0/h2
e=-h3/h4
b=-h1/h2
aa=b*b+e*e+1.0
bb=2.0*(a*b+c*e-a1(i)*b-a1(j)*e-a1(l))
cc=a*a+c*c-2.0*a*a1(i)-2.0*c*a1(j)-r**2+a123
b4ac=bb*bb-4.0*aa*cc

if(abs(b4ac).lt.0.000001) then
b4ac=0.0
else
endif

if (b4ac.lt.0.0)then
write(9,*)"b*b-4ac is negative"
stop
else
endif

if(try.eq.2) then
o1(l)=(-bb+dsqrt(b4ac))/2.0/aa
else
o1(l)=(-bb-dsqrt(b4ac))/2.0/aa
endif

o1(i)=a+b*o1(l)
o1(j)=c+e*o1(l)

                go to 101
                endif
                endif
101             nm=2
```

```
100     continue
write(9,*)"O'"
write(9,*)o1
25 format(5x,f10.5,/)

xn1=dsqrt((o1(1)-a1(1))**2+(o1(2)-a1(2))**2
     c  +(o1(3)-a1(3))**2)
xn(1,1)=(a1(1)-o1(1))/xn1
xn(1,2)=(a1(2)-o1(2))/xn1
xn(1,3)=(a1(3)-o1(3))/xn1

xn(2,1)=(xn(3,2)*xn(1,3))-(xn(3,3)*xn(1,2))
xn(2,2)=(xn(3,3)*xn(1,1))-(xn(3,1)*xn(1,3))
xn(2,3)=(xn(3,1)*xn(1,2))-(xn(3,2)*xn(1,1))

c TRANFORMATION OF POINT b TO LCS


do 20 i= 1,3
bl(i)=0.0
do 20 j=1,3
bl(i)=bl(i)+xn(i,j)*(b1(j)-o1(j))
20 continue


C CHECK FOR O'


if (bl(2).lt.0.0) then
if(try.eq.2) then
write(9,*) 'FINAL NODE IS NOT WITHIN PI RAD OF
C        FIRST NODE FOR ATLEAST ONE ELEMENT'
stop
else
try=2
go to 19
```

```
endif
else
endif

C FIND PITCH (p), ALPHA (al), TORSION (tau),
C       AND CURVATURE (crv)

abp=(bl(1)-r)**2+bl(2)**2
t=2.0*dasin(dsqrt(abp)/2.0/r)
p=dsqrt(d1)/t

crv=(1.0/(1.0+(p/r)**2))/r
tau=crv*p/r
c length of curve
s2=t/dsqrt(tau**2+crv**2)

C TRANSFORMATION OF POINT b' TO GCS


bpl(1)=bl(1)
bpl(2)=bl(2)
bpl(3)=0.0

do 21 i=1,3
bpg(i)=0.0
do 22 j=1,3
bpg(i)=xn(j,i)*bpl(j)+bpg(i)
22 continue
21 bpg(i)=bpg(i)+o1(i)
write(9,*)"B'"
write(9,25)bpg
```

```
c ELEMENT COORDINATE SYSTEM BASE VECTORS XE(i,j)

xe1=dsqrt((bpg(1)-a1(1))**2+(bpg(2)-a1(2))**2+
     c  (bpg(3)-a1(3))**2)
xe(1,1)=(bpg(1)-a1(1))/xe1
xe(1,2)=(bpg(2)-a1(2))/xe1
xe(1,3)=(bpg(3)-a1(3))/xe1

xe(2,1)=xn(3,1)
xe(2,2)=xn(3,2)
xe(2,3)=xn(3,3)

xe(3,1)=(xe(1,2)*xe(2,3))-(xe(1,3)*xe(2,2))
xe(3,2)=(xe(1,3)*xe(2,1))-(xe(1,1)*xe(2,3))
xe(3,3)=(xe(1,1)*xe(2,2))-(xe(1,2)*xe(2,1))

write(9,*)"local coordinate vectors"
write(9,24)((xn(i,j),j=1,3),i=1,3)
write(9,*)"element coordinate vectors"
write(9,24)((xe(i,j),j=1,3),i=1,3)
24 format(/,5x,3E15.4,/)

C       PRODUCTION OF POINTS ON THE HELIX

        write(9,*)"coordinates"
        th=0.0
        tinc=t/30.0
        do 28 i=1,31
        cl(1)=r*dcos(th)
        cl(2)=r*dsin(th)
        cl(3)=p*th
```

```
c         TRANSFORMATION TO GCS

          do 29 l=1,3
          cg(l)=0.0
          do 30 j=1,3
                  cg(l)=xn(j,l)*cl(j)+cg(l)
30        continue
29        cg(l)=cg(l)+o1(l)
          th=th+tinc
          write(9,31)(cg(l),l=1,3)
31        format(5x,f14.8,5x,f14.8,5x,f14.8)
28        continue

return
end
```

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Publications:

Journal Publications

1. Ramamani, S., B. Tabarrok, and W.S. Lu, "Optimization of a Discretized Brachistochrone Problem", *The International Journal of Mechanical Engineering Education*, Ellis Horwood Limited, West Sussex, England.

Title of Thesis: A Helical Finite Element Applied to the Vibration Analysis of a Mechanical Heart Valve

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