MATCHING SOCKS

(American Mathematical Monthly Problem E3148)

by

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Let n distinct pairs of socks be put into the laundry. (It is assumed that each of the $2n$ socks has precisely one mate.) When the laundry is returned, the socks are drawn out one at a time; each is matched with its mate, if the mate has previously been drawn.

Find a formula for the expected number $E(k)$ of pairs formed after k socks have been drawn, $k = 1, 2, \ldots, 2n$.

Solution by Bruce R. Johnson, University of Victoria, Canada. First we will show that $E(k) = k(k-1)/2(2n-1)$. Then we will use this result to derive the simple formula $(j-1)/(2n-1)$ for the probability that a match occurs with the jth draw (i.e. that the mate of the jth sock drawn was drawn previously).

To derive a formula for the probability that exactly m matched pairs occur, we use the unordered sample space consisting of $\binom{2n}{k}$ equally likely possible subsets of k socks. Precisely $\binom{n}{m}\binom{n-m}{k-2m}2^{k-2m}$ of these subsets contain m matched pairs. This follows from the multiplication principle because there are $\binom{n}{m}$ different choices of pairs for inclusion of both mates, then $\binom{n-m}{k-2m}$ different choices of pairs for inclusion of one mate each, and then 2^{k-2m} different choices for the odd sock inclusions. Therefore,

(1) $P(\text{exactly } m \text{ pairs}) = \binom{n}{m}\binom{n-m}{k-2m}2^{k-2m}\Big/\binom{2n}{k}$, $\max\{0,k-n\} \le m \le [k/2]$.

Now we assume that $n \ge 2$ and $k \ge 2$ (omitting the trivial cases).

$$E(k) = \sum_{m=\max\{0,k-n\}}^{[k/2]} m\binom{n}{m}\binom{n-m}{k-2m}2^{k-2m}\Big/\binom{2n}{k}$$

$$= \frac{k(k-1)}{2(2n-1)}\sum_{m=\max\{1,k-n\}}^{[k/2]} 2(2n-1)m\binom{n}{m}\binom{n-m}{k-2m}2^{k-2m}\Big/k(k-1)\binom{2n}{k}$$

$$= \frac{k(k-1)}{2(2n-1)}\sum_{m=\max\{1,k-n\}}^{[k/2]} \binom{n-1}{m-1}\binom{(n-1)-(m-1)}{(k-2)-2(m-1)}2^{(k-2)-2(m-1)}\Big/\binom{2(n-1)}{k-2}$$

$$= \frac{k(k-1)}{2(2n-1)}\sum_{r=\max\{0,(k-2)-(n-1)\}}^{[(k-2)/2]} \binom{n-1}{r}\binom{(n-1)-r}{(k-2)-2r}2^{(k-2)-2r}\Big/\binom{2(n-1)}{k-2}$$

$= \frac{k(k-1)}{2(2n-1)} \cdot 1$, because the sum is an exhaustive sum of probabilities having the same form as (1) except that n and k are replaced by $n-1$ and $k-2$, respectively.

$$P\begin{bmatrix}\text{match on}\\ j\text{th draw}\end{bmatrix} = \sum_{m=\max\{0,(j-1)-n\}}^{[(j-1)/2]} P\begin{bmatrix}m \text{ matches from}\\ \text{first } j-1 \text{ draws}\end{bmatrix} P\left[\begin{matrix}\text{match on}\\ j\text{th draw}\end{matrix}\middle|\begin{matrix}m \text{ matches from}\\ \text{first } j-1 \text{ draws}\end{matrix}\right]$$

$$= \sum_{m=\max\{0,(j-1)-n\}}^{[(j-1)/2]} \frac{\binom{n}{m}\binom{n-m}{(j-1)-2m}2^{(j-1)-2m}}{\binom{2n}{j-1}} \cdot \frac{(j-1) - 2m}{2n - (j-1)}$$

$$= \frac{(j-1) - 2E(j-1)}{2n - (j-1)} = \frac{j - 1}{2n - 1}, \qquad j = 1, 2, \ldots, 2n.$$