Well-posedness and Blowup Results for the Swirl-free and Axisymmetric Primitive Equations in a Cylinder

by

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Abstract

This thesis is devoted to the motion of the incompressible and inviscid flow which is axisymmetric and swirl-free in a cylinder, where the hydrostatic approximation is made in the axial direction. It addresses the problem of local existence and uniqueness in the spaces of analytic functions for the Cauchy problem for the inviscid primitive equations, also called the hydrostatic incompressible Euler equations, on a cylinder, under some extra conditions. Following the method introduced by Kukavica-Temam-Vicol-Ziane in Int. J. Differ. Equ. 250 (2011) , we use the suitable extension of the Cauchy-Kowalewski theorem to construct locally in time, unique and real-analytic solution, and find the explicit rate of decay of the radius of real-analiticity. Furthermore, this thesis discusses the problem of finite-time blowup of the solution of the system of equations. Following a part of the method introduced by Wong in Proc Am Math Soc. 143 (2015), we prove that the first derivative of the radial velocity blows up in time, using primary functional analysis tools for a certain class of initial data. Taking the solution frozen at $r = 0$, we can apply an a priori estimate on the second derivative of the pressure term, to derive a Ricatti type inequality.

Table of Contents

Dedications

I would like to dedicate this dissertation to my supervisors Slim Ibrahim and David Goluskin for the guidance, direction, and pearls of wisdom. I acknowledge committee members, friends and family for all their support in completion of this program. The knowledge I have gained will follow me in my future endeavors.

Chapter 1

Introduction to the primitive equations in a bounded domain and a quick history of the problem

1.1 Derivation of the primitive equations

Geophysical fluids all exhibit a common feature: depth to horizontal width, also known as aspect ratio, is very small. This leads to an asymptotic model widely used in meteorology and oceanography: the primitive equations (PEs) which are the hydrostatic approximation of the time-dependent incompressible Navier–Stokes equations. It relies on the hypothesis that pressure increases linearly in the vertical direction. While the mathematically derivation of the primitive equations is rigorous, Physicists derive primitive equations using scaling analysis.

Rigorous justification of the derivation of the weak formulation of the 3D dimensionless primitive equations from time-dependent incompressible Navier–Stokes equations, was done by Azerad-Guillen [1]. Li-Titi [20] derived the strong formulation of primitive equations with error estimates in terms of the aspect ratio.

Here we just include a physical scaling analysis on the inviscid Boussinesq equations in order to derive inviscid primitive equations [15, 11]. Plugging in the physical estimates for each parameter, we will see that vertical conservation of momentum equation is not balanced. Navier-Stokes equation are accepted as a fundamental model in both geophysical flows and industrial purpose. The velocity vector field v and its pressure p are

$$v : D \times (0, \infty) \longrightarrow \mathbb{R}^d, \quad p : D \times (0, \infty) \longrightarrow \mathbb{R},$$

with $D = \mathbb{R}^d$, torus, finite or infinite channel, etc., such that $d = 2, 3$.

Navier-Stokes equations consists of two sets of equations

1. Conservation of momentum
2. Incompressibility.

In what follows we give a brief explanation for each of them.

Conservation of momentum equation is

$$\partial_t v - \nu \Delta v + (v \cdot \nabla) v + \frac{1}{\rho_0} \nabla p = 0,$$

showing that by Newton's law the mass time acceleration is equal to all other forces. Here ν is the viscosity, $\vec{v}.\nabla$ is advection or diffusivity. This fluid is set in d space dimension. ∇ and Δ are $(\frac{\partial}{\partial_{x_1}}, \frac{\partial}{\partial_{x_2}}, ..., \frac{\partial}{\partial_{x_d}})$ and $\frac{\partial^2}{\partial^2_{x_1}} + \frac{\partial^2}{\partial^2_{x_2}} + ... + \frac{\partial^2}{\partial^2_{x_d}}$, respectively. In the Boussinesq approximation the density assumed to be constant.

The second equation is coming from the incompressibility of the fluid. Also can be called the divergence free condition

$$\nabla \cdot v = 0.$$

For large scale oceanic circulations or for geophysical flows, there is a distinction between the horizontal scale and the vertical one. The height is 10 kilometers, while the width is about thousands of kilometers [11]. Using this huge difference one can simplify Boussinesq equations. We are going to rewrite equations for both the horizontal and vertical motions of the flow

$$\begin{aligned}
&\frac{\partial}{\partial t}v_H - (\nu_H\Delta_H + \nu_3\frac{\partial^2}{\partial z^2})v_H + (v_H.\nabla_H)v_H + w\frac{\partial}{\partial z}v_H + \frac{1}{\rho_0}\nabla_H p = 0,\\
&\frac{\partial}{\partial t}w - (\nu_H\Delta_H + \nu_3\frac{\partial^2}{\partial z^2})w + (v_H.\nabla_H)w + w\frac{\partial}{\partial z}w + \frac{1}{\rho_0}\frac{\partial}{\partial z}p = 0,\\
&\nabla_H.v_H + \frac{\partial}{\partial z}w = 0,
\end{aligned}$$

Now we can do scale analysis on the horizontal and vertical motion of the inviscid flow

$$\begin{aligned}
&\frac{\partial}{\partial t}v_H + (v_H.\nabla_H)v_H + w\frac{\partial}{\partial z}v_H + \frac{1}{\rho_0}\nabla_H p = 0,\\
&\frac{\partial}{\partial t}w + (v_H.\nabla_H)w + w\frac{\partial}{\partial z}w + \frac{1}{\rho_0}\frac{\partial}{\partial z}p = 0,\\
&\nabla_H.v_H + \frac{\partial}{\partial z}w = 0.
\end{aligned}$$

We have some physical estimates for each parameter [11]

- The horizontal distance $L \sim 10^6\ m$
- The horizontal velocity $U \sim 10^{-1}\ m/s$
- The depth $H \sim 10^3\ m.$

Using these parameters we can calculate the typical values

- The vertical velocity $W = \frac{UH}{L} \sim 10^{-4}\ m/s$
- The pressure $p = \rho_0 g H \sim 10^7\ pa$
- The time scale $T = \frac{L}{U} \sim 10^7\ s,$

and we can plug in these values in the inviscid equations. For the vertical motion of the flow we have

$$\begin{gathered}
\frac{W}{T} + \frac{UW}{L} + \frac{W^2}{H} + \frac{P}{\rho_0 H} = 0,\\
10^{-11} + 10^{-11} + 10^{-11} + 10 = 0
\end{gathered}$$

where for those values the acceleration terms are several orders smaller than the last term, then the acceleration terms are negligible and we have

$$\frac{1}{\rho_0}\frac{\partial}{\partial_z}p = 0,$$

and we call this new equation, the hydrostatic balance equation. This means that the vertical variation of the pressure is balanced. We do the same for the horizontal motion of the flow

$$\frac{U}{T}+\frac{U^2}{L^2}+\frac{UW}{H}+\frac{P}{L\rho_0}=0,$$
$$10^{-8}+10^{-8}+10^{-8}+10^{-2}=0.$$

Plugging the values in this equation we can say that the equation is balanced itself, for small aspect ratio, all the terms are small at approximately same order.

In conclusion, under the hydrostatic balance approximation, we obtain the primitive equations

$$\partial_t v + v.\nabla v + w\partial_z v + \nabla p = 0, \tag{1.1}$$ {Yek}

$$\partial_z p = 0, \tag{1.2}$$ {Do}

$$\nabla.v + \partial_z w = 0,, \tag{1.3}$$ {Seh}

Finally, we shall focus on this system following with specific boundary conditions that we introduce in the next section of this chapter.

1.2 Recent results on viscous and inviscid primitive equations

The mathematical literature on incompressible homogeneous geophysical flows in the hydrostatic limit is vast and several models have been proposed both in the viscous and the inviscid cases. In what follows first we mention some of the research in viscous cases and the rest of the thesis is devoted to the inviscid case in a cylinder.

The first work in the topic of viscous primitive equation belongs to Lions, Temam and Wang [22, 23], who constructed the global weak solution to the viscous primitive equations. The problem of uniquiness of the weak solution of the viscous primitive equations in 2D was solved by Guillen-Gonzales, Masmoudi and Rodriguez-Bellido [5], while it remained open in 3D. In the topic of strong solution, Guillen-Gonzalez, Masmoudi and Rodriguez-Bellido in [4] proved the local existence of the solution.

Global existence of strong solution to 3D viscous primitive equations first was proved by Cao and Titi in [12], and later by Kobelkov in [16], and by Kukavica and Ziane in [18] and [19] for different boundary conditions. Also Hieber and Kashiwabara in [14] proved global existence for less smooth initial data by using the semigroup method.

Global well-posedness of the 3D PEs with full viscosity and horizontal diffusivity where just the horizontal part of conservation of momentum equation was considered, was solved by Cao, Titi and Li in [9] and [10]. Global well-posedness with only horizontal viscosity and horizontal diffusivity was proved by Cao, Li and Titi in [7] and [8].

So far we referred to the studies in the viscous PEs, while we are more interested in the inviscid case. Here we consider the inviscid hydrostatic model which is classical in geophysical fluid mechanics. In [21] section 4.6 and [24] the author raises the question of existence and uniqueness of solutions. These equations are formally derived from the three-dimensional incompressible Euler equations for a fluid between two horizontal plates, in the limit of vanishing distance between the plates [21, 3, 13].

The existence and uniqueness of solutions to the hydrostatic Euler equations is an outstanding open problem (cf. [21]). In 2009, Renardy [25] proved that linearization of the hydrostatic Euler equations at specific parallel shear flows is ill-posed in the sense of Hadamard. The local existence result available for the nonlinear problem was obtained in two dimensions by Brenier [2] under the assumptions of convexity of v in the $z-$variable, constant normal derivative of v on Γ_z, and of periodicity of (v, w, p) in the $x-$variable. Using the suitable extension of Cauchy-Kowalewski theorem, Kukavica-Temam-Vicol-Zian proved local well-posedness for inviscid primitive equations in the space of real analytic functions [17].

There are some results regarding blowup for incompressible geophysical flows in the hydrostatic limit. In [6], the authors show that for a certain class of initial data the corresponding smooth solutions of the inviscid primitive equations blow up in finite time. Specifically, authors considered the three-dimensional inviscid primitive equations in a three-dimensional infinite horizontal channel, subject to periodic boundary conditions in the horizontal directions, and with no normal flow boundary conditions on the solid, top and bottom boundaries. For certain class of initial data, they reduced the system into the two-dimensional system of primitive equations in an infinite horizontal strip with the same type of boundary conditions, and then proved that, for specific sub-class of initial data, the corresponding smooth solutions of the reduced inviscid two-dimensional system develop singularities in finite time. There is another blowup result for the hydrostatic Euler equations. Wong in his work in 2012 proved that for a certain class of initial data, smooth solutions of the hydrostatic Euler equations blow up in finite time [27].

1.3 Purpose of the thesis

Taking the velocity field $u = (v_1, v_2, w) = (v, w)$, the scalar temperature function T, and the scalar pressure function p, we shall focus on invicsid primitive equations

$$\partial_t v + (v.\nabla)v + w\partial_z v + \nabla p = 0, \tag{1.4}$$ {ieq1}

$$\nabla.v + \partial_z w = 0, \tag{1.5}$$ {ieq2}

$$\partial_z p = 0, \tag{1.6}$$ {ieq3}

in $\mathcal{D} \times (0, T)$ for some $T > 0$ Here

$$\mathcal{D} = M \times (0, 1) = \{(x_1, x_2, z) = (x, z) \in \mathrm{I\!R}^3 \mid x \in M, 0 < z < 1\}, \tag{1.7}$$ {ieq5}

is a three-dimensional cylinder of height one, where $M \subset \mathrm{I\!R}^2$ is a disk with radius one and real-analytic boundary, subject to the no heat flux in the vertical direction at the top and bottom of the cylinder.

We denote by ∇ and Δ the corresponding two-dimensional operators $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, and $\frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2^2}$ acting on $x = (x_1, x_2)$, while $\partial_z = \frac{\partial}{\partial z}$. Also, we let $v^{\perp} = (v_2, -v_1)$ be the first two components of $v \times e_3$.

The boundary conditions for the top and bottom boundaries $\Gamma_z = M \times \{0, 1\}$ and on the side $\Gamma_x = \partial M \times (0, 1)$ of the cylinder D are

$$w(x, z, t) = 0 \quad \text{on} \;\; \Gamma_z \times (0, T), \tag{1.8}$$ {ieq6}

$$\int_0^1 v(x, z, t) \cdot \vec{n} \, \mathrm{d}z = 0 \quad \text{on} \;\; \Gamma_x \times (0, T), \tag{1.9}$$ {ieq7}

when $\vec{n}$ is the outward unit normal to M. The incompressibility condition and (1.8) imply that

$$w(x,z,t) = -\int_0^z \nabla \cdot v(x,\zeta,t)\,\mathrm{d}\zeta, \tag{1.10}$$

{ieq8}

for all $0 < z < h$, and $0 < t < T$, which combined again with (1.8) shows that the vertical average of $\nabla.v$ is zero, i.e.,

$$\int_0^1 \nabla.v(x,z,t)\,\mathrm{d}z = 0, \tag{1.11}$$

{ieq9}

for all $x \in M$ and $0 < t < T$. We consider a real-analytic initial datum

$$v(x,z,0) = v_0(x,z), \tag{1.12}$$

{ieq10}

in D, which satisfies the compatibility conditions arising from (1.9) and (1.11), namely

$$\int_0^1 v_0(x,z).\vec{n}\,\mathrm{d}z = 0, \tag{1.13}$$

{ieq11}

for all $x \in \partial M$, and

$$\int_0^1 \nabla.v_0(x,z)\,\mathrm{d}z = 0, \tag{1.14}$$

{ieq12}

for all $x \in M$.

The first objective of this thesis is that the solution for the incompressible inviscid primitive equations (1.4)-(1.6) under the boundary conditions (1.8) and (1.9) blows up in finite time. Specifically, we show that the first derivative of the radial velocity blows up in finite time. In order to prove the blowup result for (1.4)-(1.6), the same method was used as Wong. Furthermore, part of the method in [6] has been followed in order to find suitable self-similar solution. In this way, an explicit form of the blowup is achieved.

The second objective here is to show that the system of partial differential equations supplemented with suitable boundary conditions, possesses a unique solution in certain spaces, and that the solution depends continuously on the data. Here is why we take this specific boundary condition. For hydrostatic Euler equations or our original system, $v = (v_1, v_2)$ are called prognostic variables in the language of geophysical fluid mechanics, but p and w are diagnostic variables. At each instant of time, p and w can be expressed as functions of v. Furthermore, as we mention below, p is determined by the solution of an elliptic Neumann problem. Using (1.4) and (1.5), we get

$$\begin{aligned} -\Delta p = {}& \partial_{x_1} \int_0^1 v_1.\partial_{x_1} v_1 + v_2 \partial_{x_2} v_1 + w \partial_z v_1 \,\mathrm{d}z \\ & + \partial_{x_2} \int_0^1 v_2.\partial_{x_2} v_2 + v_1 \partial_{x_1} v_2 + w \partial_z v_2 \,\mathrm{d}z, \end{aligned}$$

and then using (1.8) and (1.10)

$$-\Delta p = \partial_k \partial_j \int_0^1 v_j v_k \,\mathrm{d}z,$$

such that $1 \leq j, k \leq 2$ are the repeated indices for the summation convention. This is an elliptic boundary value problem which introduces some form of ellipticity in the original system, which is otherwise essentially hyperbolic. This is solvable using the boundary conditions (1.8) and (1.9), we later discuss it in chapter three. In this way, we can describe the existence and uniqueness result as a non-local Cauchy-Kowalewski-type of result for the hydrostatic Euler equations, same as the method in [17] in 2009 for the first time.

Finally, as the ultimate result of this thesis, the existence of the solution to the primitive equations is guaranteed by the second part of the thesis in chapter three which satisfies the assumption of the blowup theorem, guarantees the existence of a blowup solution.

Chapter 2

The Blowup result

We consider the PE's in a three dimensional cylinder $\mathcal{D} = \mathcal{M} \times (0,1) = \{(x_1, x_2, z) | x_1^2 + x_2^2 \leq 1,\ 0 \leq z \leq 1\}$

$$\partial_t v + (v.\nabla)v + w\partial_z v + \nabla p = 0, \tag{2.1} \qquad \texttt{\{beq1\}}$$

$$\nabla.v + \partial_z w = 0, \tag{2.2} \qquad \texttt{\{beq2\}}$$

$$\partial_z p = 0, \tag{2.3} \qquad \texttt{\{beq3\}}$$

which are supplemented with the initial value $v_0 = (v_1(0, x_1, x_2, z), v_2(0, x_1, x_2, z))$, where $v = (v_1, v_2)$ and $(v_1(t, x_1, x_2, z), v_2(t, x_1, x_2, z), w(t, x_1, x_2, z), p(t, x_1, x_2, z))$ are unknowns.

The boundary conditions for the top and bottom $\Gamma_z = \mathcal{M} \times \{0, 1\}$ and the side $\Gamma_{(x_1,x_2)} = \partial\mathcal{M} \times (0, 1)$ of the cylinder $\mathcal{D}$ are

$$w(t, x_1, x_2, z) = 0, \quad \mathit{on}\ \Gamma_z \times (0, T), \tag{2.4} \qquad \texttt{\{BD1\}}$$

$$\int_0^1 v(t, x_1, x_2, z)\ dz.n = 0, \quad \mathit{on}\ \Gamma_{(x_1,x_2)} \times (0, T), \tag{2.5} \qquad \texttt{\{BD2\}}$$

for some $T > 0$, where n is the outward unit normal to $\mathcal{M}$. In our case the system is swirl free and axis symmetric.

2.1 Cylindrical coordinate

In this section, we aim to derive the pde's in cylindrical coordinates:

$$\begin{aligned} x_1 &= r\cos(\theta), \\ x_2 &= r\sin(\theta), \\ z &= z, \\ 0 &< r \leq 1, \end{aligned} \tag{2.6} \qquad \texttt{\{eq1\}}$$

Applying the new coordinate we get

$$\begin{aligned} r^2 &= x_1^2 + x_2^2, \\ 2rr_{x_1} &= 2x_1, \\ r_{x_1} &= \frac{x_1}{r}. \end{aligned}$$

Differentiating both sides of (2.6) with respect to x_1:

$$\begin{aligned} r_{x_1}\sin(\theta) + r\cos(\theta)\theta_{x_1} &= 0, \\ \theta_{x_1} &= -\frac{x_2}{r^2}. \end{aligned}$$

Using the same method we can get r_{x_2} and θ_{x_2}:

$$r_{x_2} = \sin(\theta),$$
$$\theta_{x_2} = \frac{\cos(\theta)}{r}.$$

$$\begin{bmatrix} v_1 \\ v_2 \\ w \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_r \\ v_\theta \\ w \end{bmatrix}$$
$$v_1(x_1, x_2, z, t) = v_r \cos(\theta) - v_\theta \sin(\theta),$$
$$v_2(x_1, x_2, z, t) = v_r \sin(\theta) + v_\theta \cos(\theta),$$
$$p(x_1, x_2, t) = p(r, \theta, t).$$

and

$$\begin{aligned} \partial_{x_1} v_1 &= \partial_r v_1 r_{x_1} + \partial_\theta v_1 \theta_{x_1}, \\ \partial_{x_1} v_1 &= \partial_r (v_r \cos(\theta) - v_\theta \sin(\theta)) \cos(\theta) - \frac{1}{r} \partial_\theta (v_r \cos(\theta) - v_\theta \sin(\theta)) \sin(\theta), \\ \partial_{x_1} v_1 &= \partial_r v_r \cos^2(\theta) - \partial_r v_\theta \sin(\theta)\cos(\theta) + \frac{1}{r}\partial_\theta v_r \cos(\theta)\sin(\theta) - \frac{v_r}{r}\sin^2(\theta) \\ &\quad - \frac{1}{r}\partial_\theta v_\theta \sin^2(\theta) - \frac{1}{r} v_\theta \cos(\theta)\sin(\theta), \end{aligned} \tag{2.7}$$
{eq2}

$$\begin{aligned} \partial_{x_2} v_2 &= \partial_r v_2 r_{x_2} + \partial_\theta v_2 \theta_{x_2}, \\ \partial_{x_2} v_2 &= \partial_r (v_r \sin(\theta) + v_\theta \cos(\theta)) \sin(\theta) + \frac{1}{r} \partial_\theta (v_r \sin(\theta) + v_\theta \cos(\theta)) \cos(\theta), \\ \partial_{x_2} v_2 &= \partial_r v_r \sin^2(\theta) + \partial_r v_\theta \cos(\theta)\sin(\theta) + \frac{1}{r}\partial_\theta v_r \cos(\theta)\sin(\theta) + \frac{1}{r} v_r \cos^2(\theta) \\ &\quad + \frac{1}{r}\partial_\theta v_\theta \cos^2(\theta) - \frac{1}{r} v_\theta \sin(\theta)\cos(\theta). \end{aligned} \tag{2.8}$$
{eq3}

As the system is swirl free, we can drop the terms including $\partial_r v_\theta$,$\partial_\theta v_\theta$ and v_θ in both (2.7) and (2.8):

$$\begin{aligned} \partial_{x_1} v_1 &= \partial_r v_r \cos^2(\theta) - \frac{1}{r}\partial_\theta v_r \cos(\theta)\sin(\theta) + \frac{v_r}{r}\sin^2(\theta), \\ \partial_{x_2} v_2 &= \partial_r v_r \sin^2(\theta) + \frac{1}{r}\partial_\theta v_r \cos(\theta)\sin(\theta) + \frac{v_r}{r}\cos^2(\theta), \\ \partial_{x_1} p &= \partial_r p \cos(\theta) - \frac{1}{r}\partial_\theta p \sin(\theta), \\ \partial_{x_2} p &= \partial_r p \sin(\theta) + \frac{1}{r}\partial_\theta p \cos(\theta), \\ v_2 \partial_{x_2} v_1 &= v_r \partial_r v_r \cos(\theta)\sin^2(\theta) - \frac{v_r^2}{r}\cos(\theta)\sin^2(\theta), \\ v_1 \partial_{x_1} v_2 &= v_r \partial_r v_r \sin(\theta)\cos^2(\theta) - \frac{v_r^2}{r}\cos^2(\theta)\sin(\theta). \end{aligned}$$

Applying the change of coordinates to the equations (2.1),(2.2), and (2.3) we have:

$$\begin{aligned}
&\cos(\theta)\partial_t v_r + v_r\partial_r v_r\cos^3(\theta) - \frac{1}{r}v_r\partial_\theta v_r\cos^2(\theta)\sin(\theta) + \frac{v_r^2}{r}\cos(\theta)\sin^2(\theta) + w\cos(\theta)\partial_z v_r\\
&\quad \partial_r p\cos(\theta) - \frac{1}{r}\partial_\theta p\sin(\theta) + v_r\partial_r v_r\cos(\theta)\sin^2(\theta) - \frac{v_r^2}{r}\cos(\theta)\sin^2(\theta) = 0, \qquad (2.9) \quad \texttt{\{eq7\}}\\
&\sin(\theta)\partial_t v_r + \sin^3(\theta)v_r\partial_r v_r + \frac{1}{r}\sin^2(\theta)\cos(\theta)v_r\partial_\theta v_r + \frac{v_r^2}{r}\sin(\theta)\cos^2(\theta) + w\partial_z v_r\sin(\theta) +\\
&\quad \partial_r p\sin(\theta) + \frac{1}{r}\partial_\theta p\cos(\theta) + \cos^2(\theta)\sin(\theta)v_r\partial_r v_r - \frac{v_r^2}{r}\cos^2(\theta)\sin(\theta) = 0. \qquad (2.10) \quad \texttt{\{eq8\}}\\
&\partial_r v_r + \frac{v_r}{r} + \partial_z w = 0.
\end{aligned}$$

Dropping $\cos\theta$ in (2.9) and $\sin\theta$ in (2.10), and then summing we do have:

$$\partial_t v_r + v_r\partial_r v_r + w\partial_z v_r = -\partial_r p, \qquad (2.11) \quad \texttt{\{eq9\}}$$

$$\partial_r v_r + \frac{v_r}{r} + \partial_z w = 0, \quad 0 < r \leq 1. \qquad (2.12) \quad \texttt{\{eq10\}}$$

We can take $\partial_\theta p = 0$ and $\partial_\theta v_r = 0$, because we mentioned that the system is swirl free and axisymmetric.

We need to transform the boundary conditions too. On $\Gamma_{(x_1,x_2)} = \partial\mathcal{M} \times (0,1)$

$$\int_0^1 v_1(x_1,x_2,z).x_1 \,\mathrm{d}z + \int_0^1 v_2(x_1,x_2,z).x_2 \,\mathrm{d}z = 0,$$

Applying the transformation at $r = 1$, we do have

$$\int_0^1 (v_r\cos\theta - v_\theta\sin\theta).r\cos\theta \,\mathrm{d}z + \int_0^1 (v_r\sin\theta + v_\theta\cos\theta).r\sin\theta \,\mathrm{d}z = 0.$$

Finally we have

$$\int_0^1 rv_r\cos\theta^2 - rv_\theta\sin\theta\cos\theta \,\mathrm{d}z + \int_0^1 rv_r\sin\theta^2 + rv_\theta\sin\theta\cos\theta \,\mathrm{d}z = 0,$$

we get

$$\int_0^1 rv_r \,\mathrm{d}z = 0.$$

The final boundary condition is

$$\int_0^1 v_r \,\mathrm{d}z = 0, \quad r = 1.$$

Using (2.12), we have

$$w = -\int_0^z (\partial_r v_r + \frac{v_r}{r})(r,s,t) \,\mathrm{d}s = -\int_0^z \frac{1}{r}\partial_r(rv_r)(r,s,t) \,\mathrm{d}s \qquad 0 < r \leq 1.$$

The final transformed equation is

$$\partial_t v_r + v_r\partial_r v_r - \int_0^z \frac{1}{r}\partial_r(rv_r)(r,s,t)\ \mathrm{d}s\ \partial_z v_r = -\partial_r p. \tag{2.13}$$ `{eq11}`

2.2 A self-similar solution

Let us first see that, one can construct the blowup result using the existence of a self-similar solution, as this was done in [6]. The idea is to find a self-similar solution $v = \frac{U(r,z)}{1-t}$, and to construct a nontrivial initial profile $U(r,z)$.

First, we are going to use Barotropic and Baroclinic decomposition

$$\begin{aligned} \bar{v_r} &= \int_0^1 v_r(r,z)\ \mathrm{d}z := P_0 v_r, \\ \tilde{v_r} &= v_r - \bar{v_r}. \end{aligned}$$

According to the incompressibility condition we do have:

$$\partial_r \bar{v_r} + \frac{\bar{v_r}}{r} = 0,$$

whose the solution is a function $\bar{v_r} = \frac{C(t)}{r}$. We need to consider $\bar{v_r} = 0$ otherwise the L^2-norm is infinity.
Using the mentioned decomposition into (2.13) and the fact that $\bar{v_r} = 0$, we do have

$$\partial_r \tilde{v_r} + \tilde{v_r}\partial_r \tilde{v_r} - \int_0^z \frac{1}{r}\partial_r(r\tilde{v_r})(r,s,t)\ \mathrm{d}s\ \partial_z \tilde{v_r} = -\partial_r p. \tag{2.14}$$ `{eq15}`

Integrating (2.11), we have

$$\partial_t \bar{v_r} + \int_0^1 v_r\partial_r v_r\ \mathrm{d}z + \int_0^1 w\partial_z v_r\ \mathrm{d}z = -\partial_r p, \tag{2.15}$$ `{eq13}`

simplifying (2.15) and the fact that $\bar{v_r} = 0$, we have

$$\partial_r p = -\int_0^1 \tilde{v_r}\partial_r \tilde{v_r}\ \mathrm{d}z - \int_0^1 w\partial_z \tilde{v_r}\ \mathrm{d}z. \tag{2.16}$$ `{eq14}`

Substituting (2.16) in (2.14) we do have

$$\partial_t \tilde{v_r} + (I-P_0)(\tilde{v_r}\partial_r \tilde{v_r}) + (I-P_0)(w\partial_z \tilde{v_r}) = 0, \tag{2.17}$$ `{eq16}`

$$w = -\int_0^z \frac{1}{r}\partial_r(r\tilde{v_r})\ ds, \tag{2.18}$$ `{eq17}`

$$\int_0^1 \tilde{v_r}\ \mathrm{d}s = 0 \quad r \leq 1. \tag{2.19}$$ `{eq18}`

As we observe, the boundary condition at $r = 1$, follows from the fact that $\bar{v}_r = 0$ (2.19).

Taking the blowup of the form $\tilde{v_r} = \frac{U(r,z)}{1-t}$, and substituting into (2.17) we have

$$\frac{U(r,z)}{(1-t)^2} + (I-P_0)(\frac{U(r,z)\partial_r U(r,z)}{(1-t)^2}) \\ - (I-P_0)(\frac{1}{(1-t)^2}\int_0^z \frac{1}{r}\partial_r(rU(r,z))\,\mathrm{d}s\partial_z U(r,z)) = 0,$$

and

$$U + (I-P_0)(U\partial_r U) - (I-P_0)(\int_0^z \frac{\partial_r(rU)}{r}\,\mathrm{d}s\ \partial_z U) = 0.$$

Taking $U(r,z) = arf(z)$ with constant a, and the function f such that

$$\int_0^1 f(z)dz = 0, \tag{2.20}$$

{eq20}

we get

$$f + (I-P_0)(af^2) - (I-P_0)(\int_0^z 2af\ \mathrm{d}s\ \partial_z f) = 0. \tag{2.21}$$

{eq19}

or equivalently, for $\phi(z) = -\int_0^z f(s)\mathrm{d}s$, one has

$$\phi' - a\phi'^2 + a\phi\phi'' + 2a\int_0^1 (\phi'(z))^2\mathrm{d}z = 0, \quad \phi(0) = \phi(1) = 0. \tag{2.22}$$

{ZI}

It has been proved in [6] that there is a nontrivial solution to this boundary value problem. Consequently, we obtained a one-parameter family of blowup solutions to (2.17)-(2.19), which blow up as $t \longrightarrow 1$. We finish this discussion with the following theorem.

Theorem 1. *$V(t,r,z) = \frac{arf(z)}{1-t}$ is a solution of (2.17)-(2.19) in the space of $L^2((0,T);H^2)$, for all $T \in [0,1)$, with initial data $V_0(r,z) = arf(z)$, where $\phi(z) = -\int_0^z f(s)\mathrm{d}s$ and $\phi(z)$ is the nontrivial solution in $C^2([0,1])$ of the boundary value problem (2.22).*

This shows that the radial velocity blows up as $t \longrightarrow 1$. consequently the there exists a family of self-similar solution to (3.1)-(3.3) subjected to the boundary conditions (2.4) and (2.5), such that each component of the horizontal velocity blows up in finite time.

Remark 1. *For $\tilde{v}_r$, we have the nice property $\frac{\tilde{v}_r}{r}|_{t=0:r=0} = 0$ implies $\frac{\tilde{v}_r}{r}|_{r=0} = 0$ for any $t \geq 0$. Indeed, letting $f = \frac{\tilde{v}}{r}$ then (2.17) becomes*

$$\partial_t f + (I-P_0)(f.(f\partial_r f)) + (I-P_0)w\partial_z f = 0$$

which, when restricted to $r = 0$ having the form of convection–diffusion equation, gives the result.

2.3 Wong's method

We aim to apply Wong's method on the transformed system in the cylindrical coordinate. From now we denote $v = \tilde{v}_r$ for simplicity. We consider

$$\partial_t v + v\partial_r v + w\partial_z v = -\partial_r p, \tag{2.23}$$ {eq21}

$$\partial_r v + \frac{v}{r} + \partial_z w = 0, \quad 0 < r \leq 1, \tag{2.24}$$ {eq22}

$$v(0, r, z) = v_0(r, z), \tag{2.25}$$ {eq24}

$$w(t, r, 0) = w(t, r, 1) = 0 \tag{2.26}$$ {eq25}

set in the two dimensional space $[0, 1] \times [0, 1] = \{(r, z) \mid 0 \leq r \leq 1, \ 0 \leq z \leq 1\}$. In what follows we include the statement of the main theorem.

Theorem 2. *Let* (v, w, p) *be a smooth solution to* (2.23) - (2.26)*. Suppose that the initial data satisfies the following properties at* $r = 0$*:*

$$\lim_{r \longrightarrow 0} \frac{v_0}{r}(0, z) \equiv 0$$

$$\int_0^1 v_0(r, z)\mathrm{d}z = 0 \quad r = 1,$$

$$v_{0rz}(0, 0) = 0, \mathit{and} \tag{2.27}$$ {eq37}

$$v_{0rzz}(0, z) < 0, \ \mathit{for\ all}\ z \in (0, 1). \tag{2.28}$$ {eq38}

Then, there exists a finite time $T > 0$ *such that*

$$\lim_{t \longrightarrow T^-} \partial_r v(t, R(t, 0, 1), 1) = -\infty, \tag{2.29}$$ {eq34}

where $R(t, 0, 1)$ *is the* $r-$*component of the characteristic starting from* $(0, 1)$*.*

2.3.1 The characteristics

Definition 1. *The functions* $R(t, r_0, z_0)$ *and* $Z(t, r_0, z_0)$ *are called the components of the characteristics starting from* (r_0, z_0) *if they satisfy*

$$\dot{R} = v, \mathit{and}\ \ R(0, r_0, z_0) = R_0,$$
$$\dot{Z} = w, \mathit{and}\ \ Z(0, r_0, z_0) = Z_0,$$

where the dot represents $\frac{d}{dt}$*. We may also write* $R(t)$*, and* $Z(t)$*.*

Following the characteristics we have the second order ODE

$$\begin{aligned}\ddot{R} &= \partial_t v + \dot{R}\partial_r v + \dot{Z}\partial_z v, \\ &= \partial_t v + v\partial_r v + w\partial_z v, \\ &= -\partial_r p\end{aligned}$$

with initial conditions

$$\begin{aligned} R(0) &= r_0 \\ \dot{R}(0) &= v_0(r_0, z_0). \end{aligned}$$

Proposition 1. *If* $v_0(r_0, z_0) = v_0(r_0, z_1)$, *then*

$$\begin{aligned} &R(t, r_0, z_0) = R(t, r_0, z_1), \text{ and} && (2.30) \quad \{\texttt{eq27}\} \\ &v(t, R(t, r_0, z_0), Z(t, r_0, z_0)) = v(t, R(t, r_0, z_1), Z(t, r_0, z_1)) && (2.31) \quad \{\texttt{eq28}\} \end{aligned}$$

Proof. Both $R(t, r_0, z_0)$ and $R(t, r_0, z_1)$ satisfy the same ODE

$$\ddot{R} = P_0(v\partial_r v) + P_0(w\partial_z v)$$

with the same initial data

$$\begin{aligned} R(0) &= r_0, \\ \dot{R}(0) &= v_0(r_0, z_0). \end{aligned}$$

According to Picard-Lindelof theorem, the equity (2.30) follows from the uniqueness of this ODE. Equality (2.31) comes from differentiation of (2.30) with respect to time t. ☐

According to the last proposition we do have the following important result:

Corollary 1. *If* $v_0(\hat{r}, z_0)$ *is independent of* z_0, *then* $R(t, \hat{r}, z_0)$ *and* $u(t, R(t, \hat{r}, z_0), z)$ *are also independent of* z_0 *and* z.

2.3.2 Proof of the main theorem

The objective of this part is to proof the main theorem (2).

Proof. As $v_0(0, z) = 0$ propagates in time due to remark (1), if we start at $\hat{r} = 0$, we stay there. It means

$$\begin{aligned} R(t, 0, 1) &= 0, && (2.32) \quad \{\texttt{eq36}\} \\ \partial_r v(t, 0, z) &= 0, && (2.33) \quad \{\texttt{AHI}\} \end{aligned}$$

and at $r = 0$ we have

$$\partial_r v + \partial_z w = 0. \qquad (2.34) \quad \{\texttt{eq23}\}$$

Then we can rewrite (2.29),

$$\lim_{t \longrightarrow T^-} \partial_r v(t, 0, 1) = -\infty.$$

We will further simplify the system as follows.
Differentiating (2.23) with respect to r, we obtain

$$\partial_r v_t + v\partial_{rr} v + \partial_r v^2 + w\partial_r v_z + \partial_r w v_z = -\partial_{rr} p. \tag{2.35}$$ {eq35}

Taking (2.35) for $r = 0$ and integrating that with respect to z over $[0, 1]$, using (2.34) , (2.26), and the fact that p is independent of z, we obtain an integral representation of the pressure

$$-\partial_{rr} p = 2 \int_0^1 v\partial_{rr} v + \partial_r v^2 \, dz \ \ for\ r = 0.$$

Now let $a(t, z) := -v_r(t, 0, z)$, $a_0(z) := -v_{0r}$ and $W(t, z) = w(t, 0, z)$. Using $v(t, 0, z) \equiv 0$, (2.33) and (2.32) we have

$$a_t + W a_z = a^2 - 2 \int_0^1 a^2 \, \mathrm{d}z. \tag{2.36}$$ {eq39}

Using the incompressibility condition we do have

$$W_z = a, \tag{2.37}$$ {eq40}

and hence, the boundary condition implies that

$$W(t, 0) = W(t, 1) = 0, \ \textit{and} \ \int_0^1 a \, \mathrm{d}z = 0. \tag{2.38}$$ {eq41}

Furthermore the initial data will be

$$a(0, y) = a_0(z), \tag{2.39}$$ {eq42}

and hence the hypotheses (2.27) and (2.28) become

$$a_{0z}(0) = 0, \tag{2.40}$$ {eq43}

$$a_{0zz} > 0. \tag{2.41}$$ {eq44}

Lastly, our aim (2.29) becomes

$$\lim_{t \longrightarrow T_-} a(t, 1) = +\infty. \tag{2.42}$$ {FINAL}

Before going through the rest of the proof, we need these two lemmas from [27],

Lemma 1. *Let $a : R^+ \times [0, 1] \longrightarrow R$ be a smooth solution to ((2.36)-(2.39)). If a_0 satisfies ((2.40)-(2.41)), then*

$$a_z(t, 0) \equiv 0 \ \textit{and} \ a_{zz} > 0.$$

Lemma 2. *Let $f : [0, 1] \longrightarrow R$ be a C^2 function with the following properties:*

- *$f'(0) = 0$ and $f'' > 0$,*

- $\int_0^1 f \mathrm{d}y = 0$.

Then $f(1) > 0$ *and*

$$\int_0^1 f^2 \mathrm{d}y \leq \frac{1}{3} f(1)^2.$$

We will give the proofs respectively.

Proof. We differentiate (2.36) with respect to y once, then we have

$$a_{zt} + V a_{zz} = a a_z,$$

and for initial conditions we have $a_{0z}(0) = 0$ and $V(t,0) \equiv 0$, then $a_z(t,0) \equiv 0$. We differentiate one more time and we obtain

$$a_{zzt} + V a_{zzz} = a_z^2 \geq 0,$$

then a_{zz} is increasing along every characteristic, and $a_{zz} > 0$ based on (2.41). ☐

Proof. We know that

$$\int_0^1 f \mathrm{d}y \leq \frac{f(1) + f(0)}{2},$$

then $f(1) \geq -f(0)$.
We define a new function which is concave,

$$g(x) = f(1) - f(x)$$

in this way $f(x) = f(1) - g(x)$. The properties of g are

$$\begin{aligned} & g'(0) = 0, \\ & g'(x) < 0, \\ & \int_0^1 g(x) \mathrm{d}x = f(1), \\ & g(x) \geq 0, \\ & g(1) = 0. \end{aligned}$$

Calculating $\int_0^1 f(y)^2 \mathrm{d}y = -f(1)^2 + \int_0^1 g^2(y) \mathrm{d}y$, we just need to prove

$$\int_0^1 g^2 \mathrm{d}y < \frac{4}{3} f^2(1).$$

As g is concave, we have $g(x) \leq x g(1) + (1-x) g(0)$, and then $g(x) < (1-x) g(0)$. Integrating the square of g,

$$\begin{aligned} \int_0^1 g^2(y) \mathrm{d}y &\leq \int_0^1 (1-y)^2 \mathrm{d}y g^2(0) \\ &\leq \frac{1}{3} (2f(1))^2 = \frac{4}{3} f^2(1). \end{aligned}$$

☐

Based on lemma (1), and (2.38), we can conclude that $a(t,.)$ satisfies the hypotheses of lemma (2), so we can use the L^2-estimate given in Lemma (2). The estimate implies that

$$a_t + Wa_z = a^2 - 2\int_0^1 a^2\,dz \geq a^2 - \frac{2}{3}a(t,1).$$

Since $W(t,1) = 0$, we obtain a Ricatti type inequality

$$a_t(t,1) \geq \frac{1}{3}a(t,1)^2,$$

and hence,

$$a(t,1) \geq \frac{3a_0(1)}{3 - a_0(1)t}. \tag{2.43}$$ {DOP}

Applying lemma (2) to a_0, we have $a_0(1) > 0$, then (2.43) imply that there exists a finite time $T > 0$ such that (2.42) holds. This completes the proof. □

Remark 2. *We have the similar blow up result for the same system with different boundary condition. Suppose that we change the boundary condition to*

$$\begin{aligned} w(t,x_1,x_2,z) &= 0, \quad \mathit{on}\ \Gamma_z \times (0,T), \\ v(t,x_1,x_2,z).n &= 0, \quad \mathit{on}\ \Gamma_{(x_1,x_2)} \times (0,T). \end{aligned} \tag{2.44}$$ {BD3}

Applying the cylindrical coordinate to (2.44), we have

$$v = 0 \quad r = 1, \tag{2.45}$$ {BD4}

where $v = v_r$, and in what follows we have the statement of the result.

Suppose (v,w,p) be a smooth solution to (2.23) - (2.26) and (2.45). Suppose the initial data satisfies the following properties at $r = 1$:

$$\begin{aligned} & v_0(1,z) \equiv 0, \\ & v_{0rz}(1,0) = 0, \mathit{and} \\ & v_{0rzz}(1,z) < 0,\ \mathit{for\ all}\ z \in (0,1), \\ & \int_0^1 v_0(r,z)\ \mathrm{d}z = 0. \end{aligned}$$

There exists a finite time $T > 0$ such that

$$\lim_{t \longrightarrow T^-} \partial_r v(t, R(t,1,1), 1) = -\infty,$$

where $R(t,1,1)$ is the $r-$component of the characteristic starting from $(1,1)$.
The proof of this theorem follows in analogy to theorem (2).

Chapter 3

Well-posedness

In this section of the thesis, we aim to answer the question of the local existence of the solution to (1.4)-(1.6) with the boundary conditions (1.8)-(1.9). Here we just follow the method in [17] by Kukavica-Temam-Vicol-Ziane.

3.1 Preliminaries and the main theorem

Through this chapter, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ and $\beta = (\beta_1, \beta_2, \beta) \in \mathbb{N}^3$ denote multi-indices, where $\mathbb{N} = \{0, 1, 2, ...\}$ is the set of all non-negative integers. We use $\partial^\alpha = \partial_x^{\alpha'}\partial_z^{\alpha_z} = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_z^{\alpha_z}$, where $\alpha' = (\alpha_1, \alpha_2)$ throughout the text. We introduced space D and all the other preliminaries in section (1.3).

Here we take the homogeneous Sobolev space $\dot{H}^m(\mathbb{R}^3)$ which is the completion of $C_c^\infty(\mathbb{R}^3)$, under the norm

$$||v||_{\dot{H}^m} := |||\xi|^m \hat{v}(\xi)||_{L^2(\mathbb{R}^3)}$$

with the semi-norm $|.|_m$, for $m \in \mathbb{N}$

$$|v|_m = \sum_{|\alpha|=m} ||\partial^\alpha v||_{L^2(D)}, \tag{3.1}$$ {peq1}

where $||\partial^\alpha v||_{L^2}^2 = ||\partial^\alpha v_1||_{L^2}^2 + ||\partial^\alpha v_2||_{L^2}^2$.

In what follows, for $r \geq 0$ and $\tau > 0$ fixed, we define the spaces of real-analytic functions

$$X_\tau = \{v \in C^\infty(D) : \int_0^1 v|_{\Gamma_x}\ \mathrm{d}z.n = 0,\ \int_0^1 \mathrm{div}(v)\ \mathrm{d}z = 0,\ ||v||_{X_\tau} < \infty\}$$

where

$$||v||_{X_\tau} = \sum_{m=0}^{\infty} |v|_m \frac{(m+1)^r \tau^m}{m!}. \tag{3.2}$$ {peq2}

Similarly, we define

$$Y_\tau = \{v \in X_\tau,\ ||v||_{Y_\tau} < \infty\},$$

where the semi-norm $||.||_{Y_\tau}$ is given by

$$||v||_{Y_\tau} = \sum_{m=1}^{\infty} |v|_m \frac{(m+1)^r \tau^{m-1}}{(m-1)!}. \tag{3.3}$$ {peq3}

The reason behind this specific subspace of X_τ will be clarified when we want to find the a priori estimate for the velocity.

A function $v(x, z)$ is real-analytic in $x = (x_1, x_2)$ and z, with radius of analyticity τ if there exists $M > 0$ such that

$$|\partial^\alpha v(x,z)| \le M\frac{|\alpha|!}{\tau^{|\alpha|}},$$

for all $(x,z) \in D$ and $\alpha \in \mathbb{N}^3$, where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_z$.

Proposition 2. *If $v \in X_\tau$ then v is real-analytic with radius of analyticity τ. Conversely, if v is real-analytic with radius of analyticity τ and satisfies the boundary conditions , then $v \in X_\tau$.*

Proof. Suppose that $v \in X_\tau$. According to the definition we have

$$||v||_{X_\tau} = \sum_{m=0}^{\infty} |v|_m \frac{(m+1)^r \tau^m}{m!} < \infty.$$

For every $m \in \mathbb{N}$, there exists K such that we have

$$\sum_{|\alpha|=m} ||\partial^\alpha v||_{L^2(D)} \le M\frac{m!}{(m+1)^r \tau^m}.$$

Using Sobolev embedding theorem, we can directly conclude that v is real-analytic with radius of analyticity τ.

Conversely, if v is real-analytic with radius of analyticity τ, then there exists $M > 0$ such that

$$|\partial^\alpha v(x,z)| \le M\frac{|\alpha|!}{\tau^{|\alpha|}}.$$

Since we get

$$|v|_m \le M m^2 \frac{m!}{\tau^m},$$

with suitable adjustment of M, then for each $\tau' < \tau$ we have

$$||v||_{X_{\tau'}} = \sum_{m=0}^{\infty} |v|_m \frac{(m+1)^r \tau'^m}{m!} \le \sum_{m=0}^{\infty} M m^{r+2} (\frac{\tau'}{\tau})^r.$$

We can conclude that for each $\tau' < \tau$, $v \in X_{\tau'}$. It remains to show that $v \in X_\tau$. As we have

$$||v||_{X_\tau} \le ||v||_{L^2(D)} + \tau ||v||_{Y_\tau},$$

we just need to show that for any $\epsilon > 0$ we have $X_{\tau+\epsilon} \subset Y_\tau$. As in section 2 of [17] we have

$$||v||_{Y_\tau} \le (e\tau \ln{(1+\frac{\epsilon}{\tau})})^{-1} ||v||_{X_{\tau+\epsilon}}$$

we conclude that $X_{\tau+\epsilon} \subset Y_\tau$, and we are done with the proof. □

Now we are allowed to work with $||.||_{X_\tau}$ norm, and we can state the main theorem.

Theorem 3. *Let D be a cylinder in $\mathbb{R}^3$, and $r \geq 3$. Assume that v_0 is real-analytic with radius of analyticity strictly larger than τ_0, and suppose that it satisfies* (1.13) *and* (1.14). *Then there exists $T_* = T_*(r, \tau_0, ||v_0||_{X_{\tau_0}}) > 0$, and a unique real-analytic solution $v(t)$ of the initial value problem associated with* (1.4)-(1.9) *with radius of analyticity τ, such that*

$$||v(t)||_{X_{\tau(t)}} + C||v_0||_{X_{\tau_0}} \int_0^t (1 + \tau^{-\frac{5}{2}}(s))||v(s)||_{Y_{\tau(s)}} \, \mathrm{d}s \leq ||v_0||_{X_{\tau_0}},$$

for all $t \in [0, T_*)$, where $C = C(D)$ is a fixed positive constant.

The rest of this chapter is devoted to the proof of this theorem. First we find the a priori estimates on the velocity and the pressure, then we construct the solution and prove the uniqueness.

3.2 Finding the a priori estimates

In this part we aim to find the formal a priori estimate needed to prove theorem 3. Assume that (u, p) is a smooth solution to (1.4)-(1.6) under the boundary conditions (1.8) and (1.9). Since the pressure is defined up to a function of time, we may assume that $\int_M p \, \mathrm{d}x = 0$. Let $v_0 \in X_{\tau_0}$ for some $\tau_0 > 0$, and fixed $r \geq 3$. From the definition (3.1), and the time derivative of (3.2), and (3.3) that we had in the previous section it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}||v(t)||_{X_{\tau(t)}} = \dot{\tau}(t)||v(t)||_{Y_{\tau(t)}} + \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha v(t)||_{L^2} \frac{(m+1)^r \tau(t)^m}{m!}, \tag{3.4}$$ {pe}

and now we can find why we need the definition of the semi-norm $||.||_{Y_\tau}$.

We define the $L^2(D)$-inner product $\langle \cdot,\cdot \rangle$ with

$$\langle \phi_1, \phi_2 \rangle = \int_M \int_0^1 \phi_1(x, z)\phi_2(x, z) \, \mathrm{d}z\mathrm{d}x,$$

for any pair of smooth real functions ϕ_1 and ϕ_2.

Now given a multi-index $\alpha \in \mathbb{N}^3$, we can estimate $\frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha v(t)||_{L^2}$ by applying ∂^α to (1.4) and taking the $L^2(D)$-inner product $\langle \cdot,\cdot \rangle$ with $\partial^\alpha v$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha v(t)||_{L^2}^2 + \langle \partial^\alpha(v.\nabla v + w\partial_z v), \partial^\alpha v \rangle + \langle \partial^\alpha \nabla p, \partial^\alpha v \rangle = 0. \tag{3.5}$$ {pe1}

Using the general Leibniz rule for the multi-variable integrals, we treat the second term on the left side of (3.5)

$$\langle \partial^\alpha(v.\nabla v + w\partial_z v), \partial^\alpha v \rangle = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \langle \partial^\beta v.\nabla \partial^{\alpha-\beta} v, \partial^\alpha v \rangle + \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \langle \partial^\beta w \partial_z \partial^{\alpha-\beta} v, \partial^\alpha v \rangle.$$

Finally using Cauchy-Schwarz inequality we get,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha v(t)||_{L^2}^2 \leq \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^\beta v.\nabla\partial^{\alpha-\beta}v||_{L^2}||\partial^\alpha v(t)||_{L^2}$$
$$+\sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^\beta w\partial_z\partial^{\alpha-\beta}v||_{L^2}||\partial^\alpha v(t)||_{L^2}+||\nabla\partial^\alpha p||_{L^2}||\partial^\alpha v(t)||_{L^2},$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha v(t)||_{L^2} \leq \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^\beta v.\nabla\partial^{\alpha-\beta}v||_{L^2}$$
$$+\sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^\beta w\partial_z\partial^{\alpha-\beta}v||_{L^2}+||\nabla\partial^\alpha p||_{L^2}. \tag{3.6}$$
{pe2}

Substituting (3.6) into (3.4), we have the a priori estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}||v(t)||_{X_{\tau(t)}} \leq \dot{\tau(t)}||v(t)||_{Y_{\tau(t)}}+\mathcal{U}(v,v)+\mathcal{V}(w,v)+\mathcal{P}, \tag{3.7}$$
{pe3}

where $\mathcal{U}(v,v)=\sum_{i,j=1}^2\mathcal{U}(v_i,v_j)$ and $\mathcal{V}(w,v)=\sum_{j=1}^2\mathcal{V}(w,v_j)$, and we are going to define $\mathcal{U}(v_i,v_j)$, $\mathcal{V}(w,v_j)$, and $\mathcal{P}$.

Using the fact that p is z-independent, and the fact that due to the boundary condition (1.9), we have $\langle\nabla p, v\rangle = \langle p, \nabla.v\rangle = -\langle p, \partial_z w\rangle = 0$, hence we can drop the first term in the summation for the pressure term. We get the third term on the right side of (3.7) as the upper bound for the pressure term in (3.4)

$$\mathcal{P} = \sum_{m=1}^\infty\sum_{|\alpha|=m}||\nabla\partial^\alpha p||_{L^2(D)}\frac{(m+1)^r\tau^m}{m!}$$
$$= \sum_{m=1}^\infty\sum_{|\alpha|=m,\alpha_3=0}||\nabla\partial^\alpha p||_{L^2(M)}\frac{(m+1)^r\tau^m}{m!}.$$

In follow, according to (3.6), we define

$$\mathcal{U}(v_i,\tilde{v}) = \sum_{m=0}^\infty\sum_{j=0}^m\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^\beta v_i\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}\frac{(m+1)^r\tau^m}{m!}, \tag{3.8}$$
{VE1}

and

$$\mathcal{V}(w,\tilde{v}) = \sum_{m=0}^\infty\sum_{j=0}^m\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^\beta w\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}\frac{(m+1)^r\tau^m}{m!}, \tag{3.9}$$
{VE2}

such that $v\in X_\tau$ is a vector function and $\tilde{v}\in C^\infty(\mathcal{D})$ is a scalar function. In the following steps, we are looking for estimates on $\mathcal{U}(v,v)$, $\mathcal{V}(w,v)$, and $\mathcal{P}$.

3.2.1 Estimate on the pressure term

To estimate $\mathcal{P}$, we use the fact that the pressure may be computed explicitly from the velocity. Note that $\int_0^1 \text{div } v \, \mathrm{d}z = 0$, and therefore, by integrating (1.4) in the z-variable, and then applying the divergence operator in the x-variable, we obtain

$$\begin{aligned} -\Delta p = \partial_1 \int_0^1 v_1.\partial_x v_1 + v_2 \partial_y v_1 + w \partial_z v_1 \, \mathrm{d}z \\ + \partial_2 \int_0^1 v_2.\partial_y v_2 + v_1 \partial_x v_2 + w \partial_z v_2 \, \mathrm{d}z, \end{aligned} \tag{3.10}$$ {PRE1}

we can use the summation convention over repeated indices.

Integrating by parts in the z-variable, it follows from (1.5) and (1.8) that

$$\begin{aligned} \int_0^1 w \partial_z v_1 \, \mathrm{d}z &= -\int_0^1 v_1 \partial_z w \, \mathrm{d}z = \int_0^1 v_1 \partial_x v_1 + v_1 \partial_y v_2 \, \mathrm{d}z, \\ \int_0^1 w \partial_z v_2 \, \mathrm{d}z &= -\int_0^1 v_2 \partial_z w \, \mathrm{d}z = \int_0^1 v_2 \partial_x v_1 + v_2 \partial_y v_2 \, \mathrm{d}z \end{aligned}$$

and therefore, by (3.10) we have

$$-\Delta p = \partial_k \partial_j \int_0^1 v_j v_k \, \mathrm{d}z, \tag{3.11}$$ {PRE2}

such that we used summation convention over repeated indices $1 \leq j, k \leq 2$. We take $F = \int_0^1 v_j v_k \, \mathrm{d}z$. Using Fourier transform on the left side of the equation we have

$$\begin{aligned} \mathcal{F}(\Delta p) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_M \Delta p \exp -i(x \cdot \xi) \, \mathrm{d}x, \\ &= -\frac{1}{(2\pi)^{\frac{3}{2}}} \int_M \sum_j \partial_j p (-i\xi_j) \exp -i(x \cdot \xi) \, \mathrm{d}x, \\ &= -\frac{1}{(2\pi)^{\frac{3}{2}}} \sum_j \xi_j^2 \int_M p \exp -i(x \cdot \xi) \, \mathrm{d}x, \\ &= -(\xi \cdot \xi) \frac{1}{(2\pi)^{\frac{3}{2}}} \int_M p \exp -i(x \cdot \xi) \, \mathrm{d}x, \\ &= -(\xi \cdot \xi) \mathcal{F} p, \end{aligned} \tag{3.12}$$

hence

$$\mathcal{F}(-\Delta p) = (\xi \cdot \xi) \mathcal{F}(p).$$

Similarly for the right side of the equation we do have

$$\mathcal{F}(\partial_k \partial_j F) = -\xi_k \xi_j \mathcal{F}(F),$$

Hence we have

$$\mathcal{F}(p) = \frac{-\xi_k \xi_j \mathcal{F}(F)}{(\xi \cdot \xi)} = -i \frac{\xi_k}{|\xi|} \times -i \frac{\xi_j}{|\xi|} \mathcal{F}(F).$$

Taking the Inverse Fourier transform of the both sides we do have

$$p = R_k R_j \int_0^1 v_j v_k \, \mathrm{d}z, \tag{3.13}$$

{PPP}

where R_j is the jth Riesz transform, classically defined by its Fourier symbol $-i\frac{\xi_j}{|\xi|}$. As we define the Riesz transform over a bounded closed domain, we need to have the cancellation property of F over the boundary. Due to the boundary condition (2.5), F is going to be vanishing on the boundary of the domain.

The boundedness of the Riesz transforms on $L^2(M)$, the Holder inequality, and the Leibniz rule give the bound

$$\begin{aligned}
\mathcal{P} &\leq C \sum_{m=1}^{\infty} \sum_{|\alpha|=m, \alpha_3=0} \sum_{1\leq i,j,k\leq 2} \left|\left| \partial_i \partial^\alpha \left(\int_0^1 v_j v_k \mathrm{d}z \right) \right|\right|_{L^2(M)} \frac{(m+1)^r \tau^m}{m!} \\
&\leq C \sum_{m=1}^{\infty} \sum_{|\alpha|=m, \alpha_3=0} \sum_{1\leq i,j,k\leq 2} ||\partial^\alpha (v_j \partial_i v_k)||_{L^2(D)} \frac{(m+1)^r \tau^m}{m!} \\
&\leq C \sum_{m=1}^{\infty} \sum_{|\alpha|=m, \alpha_3=0} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{1\leq i,j,k\leq 2} ||\partial^\beta v_j \partial^{\alpha-\beta} \partial_i v_k||_{L^2(D)} \frac{(m+1)^r \tau^m}{m!}.
\end{aligned} \tag{3.14}$$

{PRE3}

The right hand side of (3.14) is estimated similarly to the other terms. We are looking for the limit for each terms in the summation (3.14), hence we are discussing over

$$\tilde{\mathcal{P}} = \sum_{m=1}^{\infty} \sum_{|\alpha|=m, \alpha_3=0} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} ||\partial^\beta v_j \partial^{\alpha-\beta} \partial_i v_k||_{L^2(D)} \frac{(m+1)^r \tau^m}{m!},$$

for every $1 \leq i, j, k \leq 2$.

In order to find the estimate on the pressure term, we need to estimate the terms $||\partial^\beta v_j \partial^{\alpha-\beta} \partial_i v_k||_{L^2(D)}$, for all $\alpha, \beta \in \mathbb{N}^3$. For this purpose it is convenient to distinguish between two cases: $0 \leq |\beta| \leq |\alpha - \beta|$ and $|\alpha - \beta| < |\beta| \leq |\alpha|$. When $0 \leq |\beta| \leq |\alpha - \beta|$, by the holder inequality, and by the two- dimensional Agmon inequality we have

$$\begin{aligned}
||\partial^\beta v_j \partial^{\alpha-\beta} \partial_i v_k||_{L^2(D)} &\leq C ||\partial^\beta v_j||_{L^\infty} ||\partial_i \partial^{\alpha-\beta} v_k||_{L^2} \\
&\leq C ||\partial^\beta v_j||_{L^2}^{\frac{1}{4}} ||(\Delta + \partial_{zz}) \partial^\beta v_j||_{L^2}^{\frac{3}{4}} ||\partial_{x_i} \partial^{\alpha-\beta} v_k||_{L^2} \\
&+ C ||\partial^\beta v_j||_{L^2} ||\partial_i \partial^{\alpha-\beta} v_k||_{L^2}.
\end{aligned} \tag{3.15}$$

{papa1}

As in the above estimate, for multi-indices such that $|\alpha - \beta| < |\beta| \leq |\alpha|$, we have

$$\begin{aligned}
||\partial^\beta v_j \partial^{\alpha-\beta} \partial_i v_k||_{L^2(D)} &\leq C ||\partial^\beta v_j||_{L^2} ||\partial_i \partial^{\alpha-\beta} v_k||_{L^\infty} \\
&\leq C ||\partial^\beta v_j||_{L^2} ||\partial_i \partial^{\alpha-\beta} v_j||_{L^2}^{\frac{1}{4}} ||\partial_i (\Delta + \partial_{zz}) \partial^{\alpha+\beta} v_k||_{L^2}^{\frac{3}{4}} \\
&+ C ||\partial^\beta v_j||_{L^2} ||\partial_i \partial^{\alpha-\beta} v_k||_{L^2}.
\end{aligned} \tag{3.16}$$

{papa2}

Through the rest of the proof we use the inequality $\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}$, which holds for all $\alpha, \beta \in \mathbb{N}^3$ with $\beta \leq \alpha$. Moreover, since $|\alpha-\beta| = |\alpha| - |\beta|$ for $\beta \leq \alpha$, the identity

$$\sum_{|\alpha|=m} \sum_{|\beta|=j, \beta\leq\alpha} a_\beta b_{\alpha-\beta} = \left(\sum_{|\beta|=j} a_\beta \right) \left(\sum_{|\gamma|=m-j} b_\gamma \right), \tag{3.17}$$

{UUX}

holds for all sequences a_β and b_γ, and for $j \leq m$; this identity is useful when estimating $\mathcal{U} = \mathcal{U}(v_i, \tilde{v})$, $\mathcal{V} = \mathcal{V}(w, \tilde{v})$ and $\mathcal{P}$.

With (3.15) and (3.16) in mind, we can split

$$\tilde{\mathcal{P}} = \tilde{\mathcal{P}}_1 + \tilde{\mathcal{P}}_2,$$

according to $1 \leq n \leq [\frac{m}{2}]$ and $[\frac{m}{2}] + 1 \leq n \leq m$ respectively. Using (3.15), we have

$$\begin{aligned}
\tilde{\mathcal{P}}_1 &= \sum_{m=1}^{\infty} \sum_{n=1}^{[\frac{m}{2}]} \sum_{|\alpha|=m, \alpha_3=0} \sum_{|\beta|=n, \beta\leq\alpha} \binom{\alpha}{\beta} ||\partial^\beta v_j \partial^{\alpha-\beta} \partial_i v_k||_{L^2(D)} \frac{(m+1)^r \tau^m}{m!} \\
&\leq C \sum_{m=1}^{\infty} \sum_{n=1}^{[\frac{m}{2}]} \sum_{|\alpha|=m, \alpha_3=0} \sum_{|\beta|=n, \beta\leq\alpha} \binom{\alpha}{\beta} ||\partial^\beta v_j||_{L^2}^{\frac{1}{4}} ||(\Delta + \partial_{zz}) \partial^\beta v_j||_{L^2}^{\frac{3}{4}} ||\partial_{x_i} \partial^{\alpha-\beta} v_k||_{L^2} \frac{(m+1)^r \tau^m}{m!} \\
&+ C \sum_{m=1}^{\infty} \sum_{n=1}^{[\frac{m}{2}]} \sum_{|\alpha|=m, \alpha_3=0} \sum_{|\beta|=n, \beta\leq\alpha} \binom{\alpha}{\beta} ||\partial^\beta v_j||_{L^2} ||\partial_i \partial^{\alpha-\beta} v_k||_{L^2} \frac{(m+1)^r \tau^m}{m!}.
\end{aligned} \tag{3.18}$$

{papa3}

Using discrete Holder inequality for the first summation we have

$$\begin{aligned}
&\sum_{|\alpha|=m, \alpha_3=0} \sum_{|\beta|=n, \beta\leq\alpha} \binom{\alpha}{\beta} ||\partial^\beta v_j||_{L^2}^{\frac{1}{4}} ||(\Delta + \partial_{zz}) \partial^\beta v_j||_{L^2}^{\frac{3}{4}} ||\partial_{x_i} \partial^{\alpha-\beta} v_k||_{L^2} \\
&\leq C \binom{m}{n} \left(\sum_{|\beta|=n} ||\partial^\beta v_j||_{L^2}^{\frac{1}{4}} ||(\Delta + \partial_{zz}) \partial^\beta v_j||_{L^2}^{\frac{3}{4}} \right) \left(\sum_{|\gamma|=m-j} ||\partial_i \partial^\gamma v_k||_{L^2} \right) \\
&\leq C \binom{m}{n} \left(\sum_{|\beta|=n} ||\partial^\beta v_j||_{L^2} \right)^{\frac{1}{4}} \left(\sum_{|\beta|=n} ||(\Delta + \partial_{zz}) \partial^\beta v_j||_{L^2} \right)^{\frac{3}{4}} \left(\sum_{|\gamma|=m-n} ||\partial_i \partial^\gamma v_k||_{L^2} \right).
\end{aligned} \tag{3.19}$$

{papa4}

From (3.18) and using (3.19) and

$$\begin{aligned}
&\sum_{|\gamma|=m-n} ||\partial_{x_j} \partial^\gamma v_k||_{L^2} \leq C |\tilde{v}|_{m-n-1}, \\
&\sum_{|\beta|=n} ||(\Delta + \partial_{zz}) \partial^\beta v_j||_{L^2} \leq C |v|_{j+2},
\end{aligned}$$

we have that $\tilde{\mathcal{P}}_1$ from above by

$$C\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}|v_j|_n^{\frac{1}{4}}|v_j|_{n+2}^{\frac{3}{4}}|v_k|_{m-n+1}\binom{m}{n}\frac{(m+1)^r\tau^m}{m!}+C\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}|v_j|_n|v_k|_{m-n+1}\binom{m}{n}\frac{(m+1)^r\tau^m}{m!}$$

$$\leq C\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}\left(|v_j|_n\frac{(n+1)^r\tau^n}{n!}\right)^{\frac{1}{4}}\left(|v_j|_{n+2}\frac{(n+3)^r\tau^{n+2}}{(n+2)!}\right)^{\frac{3}{4}}\left(|\tilde{v}|_{m-n+1}\frac{(m-n+2)^r\tau^{m-n}}{(m-n)!}\right)\tau^{-\frac{3}{2}}$$

$$+C\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}\left(|v_j|_n\frac{(n+1)^r\tau^n}{n!}\right)\left(|v_k|_{m-n+1}\frac{(m-n+2)^r\tau^{m-n}}{(m-n)!}\right),$$

where we used

$$\binom{m}{n}\frac{(m+1)^r}{m!}\leq C\frac{(m-n+2)^r}{(m-n)!}\frac{(n+1)^{\frac{r}{4}}(n+3)^{\frac{3r}{4}}}{n!^{\frac{1}{4}}(n+2)!^{\frac{3}{4}}},$$

as we have

$$\binom{m}{n}\frac{(m+1)^r}{m!}\frac{(m-n)!}{(m-n+2)^r}\frac{n!^{\frac{1}{4}}(n+2)!^{\frac{3}{4}}}{(n+1)^{\frac{r}{4}}(n+3)^{\frac{3r}{4}}}=\frac{(m+1)^r}{(m-n+2)^r}\frac{(n+1)^{\frac{3}{4}}(n+2)^{\frac{3}{4}}}{(n+1)^{\frac{r}{4}}(n+3)^{\frac{3r}{4}}}\leq C,$$

which is satisfied for all $m\geq 0$, $0\leq n\leq[\frac{m}{2}]$, $r\geq 3$, and sufficiently large constant C, depending only on r.

Finally, using discrete Holder inequality on n and Young inequality for products, we have

$$\tilde{\mathcal{P}}_1\leq C\left(\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}|v_j|_n\frac{(n+1)^r\tau^n}{n!}\right)^{\frac{1}{4}}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}|v_j|_{n+2}\frac{(n+3)^r\tau^{n+2}}{(n+2)!}\right)^{\frac{3}{4}}$$

$$\left(\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}|v_k|_{m-n+1}\frac{(m-n+2)^r\tau^{m-n}}{(m-n)!}\right)\tau^{-\frac{3}{2}}$$

$$+C\left(\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}|v_j|_n\frac{(n+1)^r\tau^n}{n!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}|v_k|_{m-n+1}\frac{(m-n+2)^r\tau^{m-n}}{(m-n)!}\right)$$

hence,

$$\tilde{\mathcal{P}}_1\leq C(1+\tau^{-\frac{3}{2}})||v||_{X_\tau}||\tilde{v}||_{Y_\tau}.$$

Next search for an estimate on $\tilde{\mathcal{P}}_2$ using the same procedure. By symmetry we obtain an estimate for the high values of n. We have

$$\tilde{\mathcal{P}}_2\leq C\sum_{m=0}^{\infty}\sum_{n=[\frac{m}{2}]+1}^{m}\sum_{|\alpha|=m}\sum_{|\beta|=n,\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^\beta v_j||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}v_k||_{L^2}^{\frac{1}{4}}||\partial_{x_i}(\Delta+\partial_{zz})\partial^{\alpha-\beta}v_k||_{L^2}^{\frac{3}{4}}\frac{(m+1)^r\tau^m}{m!}$$

$$+C\sum_{m=0}^{\infty}\sum_{n=[\frac{m}{2}]+1}^{m}\sum_{|\alpha|=m}\sum_{|\beta|=n,\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^\beta v_j||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}v_k||_{L^2}\frac{(m+1)^r\tau^m}{m!}.$$

Because we have

$$\sum_{|\alpha|=m}\sum_{|\beta|=n,\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^{\beta}v_j||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}v_k||_{L^2}^{\frac{1}{4}}||\partial_{x_i}(\Delta+\partial_{zz})\partial^{\alpha-\beta}v_k||_{L^2}^{\frac{3}{4}}$$

$$\leq C\binom{m}{n}\left(\sum_{|\beta|=n}||\partial^{\beta}v_j||_{L^2}\right)\left(\sum_{|\gamma=m-n|}||\partial_{x_i}\partial^{\alpha-\beta}v_k||_{L^2}\right)^{\frac{1}{4}}\left(\sum_{|\gamma|=m-n}||\partial_{x_i}(\Delta+\partial_{zz})\partial^{\alpha-\beta}v_k||_{L^2}\right)^{\frac{3}{4}},$$

finally we get

$$\begin{aligned}\tilde{\mathcal{P}}_2 &\leq C\sum_{m=1}^{\infty}\sum_{n=[\frac{m}{2}]+1}^{m}|v_j|_n|\partial_{x_i}v_k|_{m-n}^{\frac{1}{4}}|\partial_{x_i}(\Delta+\partial_{zz})v_k|_{m-j}^{\frac{3}{4}}\binom{m}{n}\frac{(m+1)^r\tau^m}{m!}\\ &+C\sum_{m=1}^{\infty}\sum_{n=[\frac{m}{2}]+1}^{m}|v_j|_n|\partial_{x_i}v_k|_{m-n}\binom{m}{n}\frac{(m+1)^r\tau^m}{m!}\\ &\leq C(1+\tau^{-\frac{3}{2}})||v||_{X_\tau}||v_k||_{Y_\tau},\end{aligned}$$

for some positive constant C depending only on r and $\mathcal{D}$. This completes the proof of

$$\mathcal{P}\leq C_0(1+\tau^{-\frac{3}{2}})||v||_{X_\tau}||v||_{Y_\tau},\tag{3.20}$$ {KK1}

such that $C_0=C_0(r,D)$.

3.2.2 Estimate on the velocity term

The goal of this section is to prove the next lemma. For the matter of convenience, we suppress the time dependence of v, w, $\tilde{v}$, and τ.

Lemma 3. *Let $v\in Y_\tau$ for some $\tau>0$, and $\tilde{v}\in C^{\infty}(D)\cap L^2(D)$ with $||\tilde{v}||_{Y_\tau}<\infty$. Fix $r\geq 3$. Let w be determined from v,*

$$w(x,z)=-\int_0^z \operatorname{div}\, v(x,\zeta)\,\mathrm{d}\zeta.\tag{3.21}$$ {UX}

if $\mathcal{U}(v_i,\tilde{v})$, for $i\in\{1,2\}$, and $\mathcal{V}(w,\tilde{v})$ are as in (3.8) *and* (3.9) *respectively, then we have the estimates*

$$\mathcal{U}(v_i,\tilde{v})\leq C_0(1+\tau^{-\frac{3}{2}})||v||_{X_\tau}||\tilde{v}||_{Y_\tau},\tag{3.22}$$ {UUU}

and

$$\mathcal{V}(w,\tilde{v})\leq C_0(1+\tau^{-\frac{5}{2}})||v||_{X_\tau}||\tilde{v}||_{Y_\tau}+C_0(\tau^{-1}+\tau^{-\frac{5}{2}})||v||_{Y_\tau}||\tilde{v}||_{X_\tau},\tag{3.23}$$ {KKK}

for some positive constant $C_0=C(r,D)$.

Proof. Lets take the case when $0\leq|\beta|\leq|\alpha-\beta|$. We need to estimate the terms $||\partial^{\beta}v_i\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}$ and $||\partial^{\beta}w\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}$. In order to estimate both of them, first we use generalization of Holder inequality and then three-dimensional Agmon's inequality,

$$\begin{aligned}||\partial^{\beta}v_i\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}&\leq C||\partial^{\beta}v_i||_{L^\infty}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}\\ &\leq C||\partial^{\beta}v_i||_{L^2}^{\frac{1}{4}}||(\Delta+\partial_{zz})\partial^{\beta}v_i||_{L^2}^{\frac{3}{4}}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}\\ &+C||\partial^{\beta}v_i||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}\end{aligned}\tag{3.24}$$ {UU1}

and

$$\begin{aligned}||\partial^\beta w\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2} &\le C||\partial^\beta w||_{L^\infty}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}\\ &\le C||\partial^\beta w||_{L^2}^{\frac14}||(\Delta+\partial_{zz})\partial^\beta w||_{L^2}^{\frac34}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}\\ &+C||\partial^\beta w||_{L^2}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}.\end{aligned} \tag{3.25}$$

{UU4}

As in the above estimates, for multi-indices such that $|\alpha-\beta|<|\beta|\le|\alpha|$, we have

$$\begin{aligned}||\partial^\beta v_i\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2} &\le C||\partial^\beta v_i||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^\infty}\\ &\le C||\partial^\beta v_i||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}^{\frac14}||\partial_{x_i}(\Delta+\partial_{zz})\partial^{\alpha+\beta}\tilde{v}||_{L^2}^{\frac34}\\ &+C||\partial^\beta v_i||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2},\end{aligned} \tag{3.26}$$

{UU10}

and

$$\begin{aligned}||\partial^\beta w\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2} &\le C||\partial^\beta w||_{L^2}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^\infty}\\ &\le C||\partial^\beta w||_{L^2}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}^{\frac14}||\partial_z(\Delta+\partial_{zz})\partial^{\alpha+\beta}\tilde{v}||_{L^2}^{\frac34}\\ &+C||\partial^\beta w||_{L^2}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}.\end{aligned} \tag{3.27}$$

{UU5}

According to $0\le j\le[\frac{m}{2}]$ and $[\frac{m}{2}]+1\le j\le m$ respectively, we split $\mathcal{U}=\mathcal{U}_1+\mathcal{U}_2$. Using (3.24), we have

$$\begin{aligned}\mathcal{U}_1 &\le C\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\le\alpha}\binom{\alpha}{\beta}||\partial^\beta v_i||_{L^2}^{\frac14}||(\Delta+\partial_{zz})\partial^\beta v_i||_{L^2}^{\frac34}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}\frac{(m+1)^r\tau^m}{m!}\\ &+\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\le\alpha}\binom{\alpha}{\beta}||\partial^\beta v_i||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}\frac{(m+1)^r\tau^m}{m!}.\end{aligned} \tag{3.28}$$

{UU2}

By (3.17) and discrete Holder inequality, it follows that

$$\begin{aligned}&\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\le\alpha}\binom{\alpha}{\beta}||\partial^\beta v_i||_{L^2}^{\frac14}||(\Delta+\partial_{zz})\partial^\beta v_i||_{L^2}^{\frac34}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}\\ &\le C\binom{m}{j}\left(\sum_{|\beta|=j}||\partial^\beta v_i||_{L^2}^{\frac14}||(\Delta+\partial_{zz})\partial^\beta v_i||_{L^2}^{\frac34}\right)\left(\sum_{|\gamma|=m-j}||\partial_{x_i}\partial^\gamma\tilde{v}||_{L^2}\right)\\ &\le C\binom{m}{j}\left(\sum_{|\beta|=j}||\partial^\beta v_i||_{L^2}\right)^{\frac14}\left(\sum_{|\beta|=j}||(\Delta+\partial_{zz})\partial^\beta v_i||_{L^2}\right)^{\frac34}\left(\sum_{|\gamma|=m-j}||\partial_{x_i}\partial^\gamma\tilde{v}||_{L^2}\right).\end{aligned} \tag{3.29}$$

{UU3}

Hence, from (3.28) it follows by (3.29) and

$$\sum_{|\gamma|=m-j}||\partial_{x_i}\partial^\gamma\tilde{v}||_{L^2}\le C|\tilde{v}|_{m-j-1},$$

$$\sum_{|\beta|=j}||(\Delta+\partial_{zz})\partial^\beta v_i||_{L^2}\le C|v|_{j+2},$$

that $\mathcal{U}_1$ is bounded from above by

$$C\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}|v_i|_j^{\frac{1}{4}}|v_i|_{j+2}^{\frac{3}{4}}|\tilde{v}|_{m-j+1}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}+C\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}|v_i|_j|\tilde{v}|_{m-j+1}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}$$

$$\le C\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}\left(|v_i|_j\frac{(j+1)^r\tau^j}{j!}\right)^{\frac{1}{4}}\left(|v_i|_{j+2}\frac{(j+3)^r\tau^{j+2}}{(j+2)!}\right)^{\frac{3}{4}}\left(|\tilde{v}|_{m-j+1}\frac{(m-j+2)^r\tau^{m-j}}{(m-j)!}\right)\tau^{-\frac{3}{2}}$$

$$+C\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}\left(|v_i|_j\frac{(j+1)^r\tau^j}{j!}\right)\left(|\tilde{v}|_{m-j+1}\frac{(m-j+2)^r\tau^{m-j}}{(m-j)!}\right)$$

where we used the fact that

$$\binom{m}{j}\frac{(m+1)^r}{m!}\le C\frac{(m-j+2)^r}{(m-j)!}\frac{(j+1)^{\frac{r}{4}}(j+3)^{\frac{3r}{4}}}{j!^{\frac{1}{4}}(j+2)!^{\frac{3}{4}}},$$

because

$$\binom{m}{j}\frac{(m+1)^r}{m!}\frac{(m-j)!}{(m-j+2)^r}\frac{j!^{\frac{1}{4}}(j+2)!^{\frac{3}{4}}}{(j+1)^{\frac{r}{4}}(j+3)^{\frac{3r}{4}}}=\frac{(m+1)^r}{(m-j+2)^r}\frac{(j+1)^{\frac{3}{4}}(j+2)^{\frac{3}{4}}}{(j+1)^{\frac{r}{4}}(j+3)^{\frac{3r}{4}}}\le C,$$

holds for all $m\ge 0$, $0\le j\le[\frac{m}{2}]$, $r\ge 3$, and sufficiently large constant C, depending only on r.

Finally, using discrete Holder inequality on j component and Young inequality for products, we have

$$\mathcal{U}_1\le C\left(\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}|v_i|_j\frac{(j+1)^r\tau^j}{j!}\right)^{\frac{1}{4}}\left(\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}|v_i|_{j+2}\frac{(j+3)^r\tau^{j+2}}{(j+2)!}\right)^{\frac{3}{4}}$$
$$\left(\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}|\tilde{v}|_{m-j+1}\frac{(m-j+2)^r\tau^{m-j}}{(m-j)!}\right)\tau^{-\frac{3}{2}}$$
$$+C\left(\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}|v_i|_j\frac{(j+1)^r\tau^j}{j!}\right)\left(\sum_{m=0}^{\infty}\sum_{j=0}^{[\frac{m}{2}]}|\tilde{v}|_{m-j+1}\frac{(m-j+2)^r\tau^{m-j}}{(m-j)!}\right)$$

hence,

$$\mathcal{U}_1\le C(1+\tau^{-\frac{3}{2}})||v||_{X_\tau}||\tilde{v}||_{Y_\tau}.$$

Next we are looking for an estimate on $\mathcal{U}_2$ using the same procedure. By symmetry, from (3.29) and (3.17), we obtain an estimate for the high values of j. We have

$$\mathcal{U}_2\le C\sum_{m=0}^{\infty}\sum_{j=[\frac{m}{2}]+1}^{m}\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\le\alpha}\binom{\alpha}{\beta}||\partial^\beta v_i||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}^{\frac{1}{4}}||\partial_{x_i}(\Delta+\partial_{zz})\partial^{\alpha-\beta}\tilde{v}||_{L^2}^{\frac{3}{4}}\frac{(m+1)^r\tau^m}{m!}$$

$$+C\sum_{m=0}^{\infty}\sum_{j=[\frac{m}{2}]+1}^{m}\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\le\alpha}\binom{\alpha}{\beta}||\partial^\beta v_i||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}\frac{(m+1)^r\tau^m}{m!}.$$

As we have

$$\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^{\beta}v_i||_{L^2}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}^{\frac{1}{4}}||\partial_{x_i}(\Delta+\partial_{zz})\partial^{\alpha-\beta}\tilde{v}||_{L^2}^{\frac{3}{4}}$$

$$\leq C\binom{m}{j}\left(\sum_{|\beta|=j}||\partial^{\beta}v_i||_{L^2}\right)\left(\sum_{|\gamma=m-j|}||\partial_{x_i}\partial^{\alpha-\beta}\tilde{v}||_{L^2}\right)^{\frac{1}{4}}\left(\sum_{|\gamma|=m-j}||\partial_{x_i}(\Delta+\partial_{zz})\partial^{\alpha-\beta}\tilde{v}||_{L^2}\right)^{\frac{3}{4}},$$

finally we get

$$\begin{aligned}\mathcal{U}_2 &\leq C\sum_{m=1}^{\infty}\sum_{j=[\frac{m}{2}]+1}^{m}|v_i|_j|\partial_{x_i}\tilde{v}|_{m-j}^{\frac{1}{4}}|\partial_{x_i}(\Delta+\partial_{zz})\tilde{v}|_{m-j}^{\frac{3}{4}}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\ &+C\sum_{m=1}^{\infty}\sum_{j=[\frac{m}{2}]+1}^{m}|v_i|_j|\partial_{x_i}\tilde{v}|_{m-j}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\ &\leq C(1+\tau^{-\frac{3}{2}})||v||_{X_\tau}||\tilde{v}||_{Y_\tau},\end{aligned}$$

for some positive constant C depending only on r and $\mathcal{D}$. This completes the proof of (3.22).

We continue with the estimate on $\mathcal{V}$. Using the definition (3.21) of w and that of the semi-norms $|.|_j$ imply

$$|w|_j \leq \sum_{|\beta|=j,\beta_3\geq 1}||\partial_x^{\beta'}\partial_z^{\beta_3-1}\mathrm{div}v||_{L^2}+\sum_{|\beta|=j,\beta_3=0}\left|\left|\int_0^z\partial^{\beta}\mathrm{div}v(x,\zeta)\mathrm{d}\zeta\right|\right|_{L^2}\leq |v|j+C|v|_{j+1},$$

for some constant C. We recall that div is the divergence operator acting on $x=(x_1,x_2)$.

Similarly we have that

$$|\Delta w|_j \leq \sum_{|\beta|=j,\beta_3\geq 1}||\partial_x^{\beta'}\partial_z^{\beta_3-1}\mathrm{div}\Delta v||_{L^2}+\sum_{|\beta|=j,\beta_3=0}\left|\left|\int_0^z\partial^{\beta}\mathrm{div}\Delta v(x,\zeta)\mathrm{d}\zeta\right|\right|_{L^2},$$

hence

$$|\Delta w|_j \leq |v|_{j+2}+C|v|_{j+3}.$$

Also we have

$$|\partial_{zz}w|_j = |\partial_z(\mathrm{div}v)| \leq |v|_{j+2}.$$

Next we split $\mathcal{V}=\mathcal{V}_1+\mathcal{V}_2$, according to $0\leq j\leq[\frac{m}{2}]$ and $[\frac{m}{2}]\leq j\leq m$. For the estimate on $\mathcal{V}_1$, according to (3.25) and using (3.17), we have

$$\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^{\beta}w||_{L^2}^{\frac{1}{4}}||(\Delta+\partial_{zz})\partial^{\beta}w||_{L^2}^{\frac{3}{4}}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2},$$

$$\leq C\binom{m}{j}\left(\sum_{|\beta|=j}||\partial^{\beta}w||_{L^2}\right)^{\frac{1}{4}}\left(\sum_{|\beta|=j}||(\Delta+\partial_{zz})\partial^{\beta}w||_{L^2}\right)^{\frac{3}{4}}\left(\sum_{\gamma=m-j}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}\right),$$

and

$$\sum_{|\alpha|=m} \sum_{|\beta|=j, \beta\leq\alpha} \binom{\alpha}{\beta} ||\partial^\beta w||_{L^2} ||\partial_z \partial^{\alpha-\beta}\tilde{v}||_{L^2}$$

$$\leq C \binom{m}{j} \left(\sum_{|\beta|=j} ||\partial^\beta||_{L^2} \right) \left(\sum_{\gamma=m-j} ||\partial_z \partial^\gamma \tilde{v}|| \right).$$

Using the discrete Holder inequality, the estimate is going to be

$$\begin{aligned}
\mathcal{V}_1 \leq C \sum_{m=0}^{\infty} \sum_{j=0}^{[\frac{m}{2}]} (|v|_j^{\frac{1}{4}} + |v|_{j+1}^{\frac{1}{4}})(|v|_{j+2}^{\frac{3}{4}} + |v|_{j+3}^{\frac{3}{4}}) |\tilde{v}|_{m-j+1} \binom{m}{j} \frac{(m+1)^r \tau^m}{m!} \\
+ C \sum_{m=0}^{\infty} \sum_{j=0}^{[\frac{m}{2}]} (|v|_j + |v|_{j+1}) |\tilde{v}|_{m-j+1} \binom{m}{j} \frac{(m+1)^r \tau^m}{m!}, \\
\leq C \sum_{m=0}^{\infty} \sum_{j=0}^{[\frac{m}{2}]} |v|_{j+1}^{\frac{1}{4}} |v|_{j+3}^{\frac{3}{4}} \binom{m}{j} \frac{(m+1)^r \tau^m}{m!} \\
+ C \sum_{m=0}^{\infty} \sum_{j=0}^{[\frac{m}{2}]} |v|_j |\tilde{v}|_{m-j+1} \binom{m}{j} \frac{(m+1)^r \tau^m}{m!},
\end{aligned}$$

hence

$$\begin{aligned}
\mathcal{V}_1 \leq C \sum_{m=0}^{\infty} \sum_{j=0}^{[\frac{m}{2}]} \left(|v|_{j+1} \frac{(j+2)^r \tau^{j+1}}{(j+1)!} \right)^{\frac{1}{4}} \left(|v|_{j+3} \frac{(j+4)^r \tau^{j+3}}{(j+3)!} \right)^{\frac{3}{4}} \left(|\tilde{v}|_{m-j+1} \frac{(m-j+2)^r \tau^{m-j}}{(m-j)!} \right) \tau^{-\frac{5}{2}} \\
C \sum_{m=0}^{\infty} \sum_{j=0}^{[\frac{m}{2}]} \left(|v|_j \frac{(j+1)^r \tau^j}{j!} \right) \left(|\tilde{v}|_{m-j+1} \frac{(m-j+2)^r \tau^{m-j}}{(m-j)!} \right),
\end{aligned}$$

for some constant $C = C(r, D)$. Using the fact that

$$\binom{m}{j} \frac{(m+1)^r}{m!} \leq C \frac{(m-j+2)^r (j+2)^{\frac{r}{4}} (j+4)^{\frac{3r}{4}}}{(m-j)!(j+1)!^{\frac{1}{4}} (j+3)!^{\frac{3}{4}}}$$

because

$$\frac{1}{j!} \frac{(m+1)^r (j+1)}{(m-j+2)^r} \frac{(j+2^{\frac{3}{4}})(j+3)^{\frac{3}{4}}}{(j+2)^{\frac{r}{4}} (j+4)^{\frac{3r}{4}}} \leq C$$

holds for all $m \geq 0$ and $0 \leq j \leq [\frac{m}{2}]$, for some sufficiently large positive constant depending only on $r \geq 2$, we obtain

$$\mathcal{V}_1 \leq C(1 + \tau^{-\frac{5}{2}}) ||v||_{X_\tau} ||\tilde{v}||_{Y_\tau}.$$

Lastly, to estimate $\mathcal{V}_2$, we note that by (3.27) and (3.17), we have

$$\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^\beta w||_{L^2}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}^{\frac{1}{4}}||\partial_z(\Delta+\partial_{zz})\partial^{\alpha-\beta}\tilde{v}||_{L^2}^{\frac{3}{4}}$$

$$\leq C\binom{m}{j}\left(\sum_{|\beta|=j}||\partial^\beta w||_{L^2}\right)\left(\sum_{|\gamma|=m-j}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}\right)^{\frac{1}{4}}\left(\sum_{|\gamma|=m-j}||\partial_z(\Delta+\partial_{zz})\partial^\gamma\tilde{v}||_{L^2}\right)^{\frac{3}{4}},$$

and

$$\sum_{|\alpha|=m}\sum_{|\beta|=j,\beta\leq\alpha}\binom{\alpha}{\beta}||\partial^\beta w||_{L^2}||\partial_z\partial^{\alpha-\beta}\tilde{v}||_{L^2}$$

$$\leq C\binom{m}{j}\left(\sum_{|\beta|=j}||\partial^\beta w||_{L^2}\right)\left(\sum_{|\gamma|=m-j}||\partial_z\partial^\gamma\tilde{v}||_{L^2}\right).$$

The estimate on $\mathcal{V}_2$ is

$$\begin{aligned}\mathcal{V}_2 &\leq C\sum_{m=1}^{\infty}\sum_{j=[\frac{m}{2}]+1}^{m}|w|_j|\partial_z\tilde{v}|_{m-j}^{\frac{1}{4}}|\partial_z(\Delta+\partial_{zz})\tilde{v}|_{m-j}^{\frac{3}{4}}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\ &+C\sum_{m=1}^{\infty}\sum_{j=[\frac{m}{2}]+1}^{m}|w|_j|\partial_z\tilde{v}|_{m-j}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\ &\leq C\sum_{m=1}^{\infty}\sum_{j=[\frac{m}{2}]+1}^{m}(|v|_j+|v|_{j+1})|\tilde{v}|_{m-j+1}^{\frac{1}{4}}|\tilde{v}|_{m-j+3}^{\frac{3}{4}}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!}\\ &+C\sum_{m=1}^{\infty}\sum_{j=[\frac{m}{2}]+1}^{m}(|v|_j+|v|_{j+1})|\tilde{v}|_{m-j+1}\binom{m}{j}\frac{(m+1)^r\tau^m}{m!},\end{aligned}$$

and using the exact same procedure, we have

$$\mathcal{V}_2\leq C(1+\tau^{-\frac{5}{2}})||v||_{X_\tau}||\tilde{v}||_{Y_\tau}.$$

The proof of the lemma now is complete. □

3.2.3 The a priori estimate on the solution

In this section we give the formal a priori estimate in theorem (3). Following from (3.7)and using (3.20), (3.22), and (3.23), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}||v(t)||_{X_{\tau(t)}}\leq \dot{\tau}(t)||v||_{Y_{\tau(t)}}&+C_0(1+\tau^{-\frac{3}{2}})||v||_{X_\tau}||v||_{Y_\tau}+C_0(1+\tau^{-\frac{3}{2}})||v||_{X_\tau}||v||_{Y_\tau}\\ &+C_0(\tau^{-1}+\tau^{-\frac{5}{2}})||v||_{X_\tau}||v||_{Y_\tau}+C_0(1+\tau^{-\frac{5}{2}})||v||_{X_\tau}||v||_{Y_\tau}\end{aligned}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}||v(t)||_{X_{\tau(t)}}\leq(\dot{\tau}(t)+4C_0(1+\tau(t)^{-\frac{5}{2}})||v(t)||_{X_{\tau(t)}})||v(t)||_{Y_{\tau(t)}}. \tag{3.30}$$

{2}

Next we define the decreasing function $\tau(t)$ by

$$\dot{\tau}(t) + 20C_0(1+\tau^{-\frac{5}{2}})||v_0||_{X_{\tau_0}} = 0 \tag{3.31}$$ {1}

and $\tau(0) = \tau_0$; this uniquely determine τ in terms of the initial data. Let T_* be the maximal time such that $\tau \geq 0$. It is clear that $\tau(t)$ and T_* may be computed from (3.31) in terms of the initial data. By construction, we have at $t = 0$

$$\dot{\tau}(t) + 4C_0(1+\tau(t)^{-\frac{5}{2}})||v(t)||_{X_{\tau(t)}} < 0,$$

and by (3.30) we then for the short time, we have

$$||v(t)||_{X_{\tau(t)}} \leq ||v_0(t_0)||_{X_{\tau_0}}.$$

for $t < T_*$. Finally integrating both sides of (3.30), we obtain that the solution is a priori bounded in $L^\infty(0, T_*; X_\tau) \cap L^1(0, T_*; (1+\tau^{-\frac{5}{2}}))$ in the sense

$$||v(t)||_{X_{\tau(t)}} + C||v_0||_{X_{\tau_0}} \int_0^t (1+\tau^{-\frac{5}{2}}(s))||v(s)||_{Y_{\tau(s)}} \, \mathrm{d}s \leq ||v_0||_{X_{\tau_0}},$$

for all $t < T_*$.

3.3 Construction of the solution

The formal construction of the solutions is via the Picard iteration. Let $v^{(0)} = v_0$ be given, with v_0 satisfying the compatibility conditions (1.13) and (1.14). For $n \in \mathbb{N}$, let

$$w^{(n)}(x,z,t) = -\int_0^z \mathrm{div} v^{(n)}(x,\zeta,t) \, \mathrm{d}\zeta. \tag{3.32}$$ {Seq1}

Motivated by (3.13), we define the pressure

$$p^{n+1}(x,t) = -R_j R_k \int_0^1 v_j^{(n)} v_k^{(n)} \, \mathrm{d}z. \tag{3.33}$$ {Seq2}

Lastly, the velocity iterate is constructed as

$$v^{(n+1)}(t) = v_0 - \int_0^t (v^{(n)}(s).\nabla + w^{(n)}(s)\partial_z)v^{(n)}(s)\mathrm{d}s - \int_0^t \nabla p^{(n+1)}(s) \, \mathrm{d}s, \tag{3.34}$$ {Seq3}

for all $n \in \mathbb{N}$.

Taking the time derivative of (3.34), integrating in z, and using the fact that $-\Delta p^{(n+1)} = \partial_k \partial_j \int_0^1 v_j^{(n)} v_k^{(n)} \, \mathrm{d}z$, we obtain that

$$\partial_t \int_0^1 v^{(n+1)}.n \, \mathrm{d}z = 0.$$

Since the initial data v_0, has zero normal average, and satisfies (1.14), we obtain

$$\int_0^1 v^{(n)}(x,z).n \, \mathrm{d}z = 0 \quad x \in \partial M, \ n \geq 0,$$
$$\int_0^1 \nabla . v^{(n)}(x,z) \mathrm{d}z = 0 \quad x \in M, \ n \geq 0.$$

Assume that $v_0 \in X_{\tau_0+\epsilon}$, for some $0 < \epsilon < \tau_0$. According to the definition of the $||.||_{X_\tau}$ norm, we have $v_0 \in Y_{\tau_0}$. We define $\tau(t)$ by $\tau(0) = \tau_0$ and

$$\dot{\tau}(t) + 20C_0(1 + \tau^{-\frac{5}{2}}(t))||v_0||_{X_{\tau_0}} = 0, \tag{3.35}$$ {Tau}

where the constant $C_0 = C_0(r, D)$ is fixed. In the next lemma, we show that the sequence of approximations $v^{(n)}$ is bounded in $L^\infty(0, T_*; X_\tau) \cap L^1(0, T_*; (1 + \tau^{-\frac{5}{2}}))$ for some sufficiently small $T > 0$, depending on the initial data.

Lemma 4. *Let* $v_0 \in X_{\tau_0+\epsilon}$ *and* $\tau(t)$ *be defined by* (3.35). *The approximating sequence* $\{v^{(n)}\}_{n\geq 0}$*, constructed via* (3.32)-(3.34)*, satisfies*

$$\sup_{t\in[0,T]} ||v^{(n)}(t)||_{X_{\tau(t)}} + C||v_0||_{X_{\tau_0}} \int_0^T (1 + \tau^{-\frac{5}{2}}(s))||v^{(n)}(s)||_{Y_{\tau(s)}} \, \mathrm{d}s \leq ||v_0||_{X_{\tau_0}}, \tag{3.36}$$ {InE}

for all $n \geq 0$*, where* $T = T(v_0) > 0$ *is sufficiently small.*

Proof. First for $n = 0$, since τ is decreasing and since $\epsilon < \tau_0$, for all $t \geq 0$, using the definition of the $||.||_{X_\tau}$ and $||.||_{Y_\tau}$ norms, and expanding the corresponding series, we have

$$||v_0||_{Y_\tau} \leq \frac{C'||v_0||_{X_{\tau_0+\epsilon}}}{\epsilon},$$

for some constant C'.

Sufficient condition for the bound (3.36) to hold in the case $n = 0$ are that T is chosen such that

$$CC'||v_0||_{X_{\tau_0+\epsilon}} \int_0^T (1 + \tau^{-\frac{5}{2}}(s)) \, \mathrm{d}s \leq \epsilon. \tag{3.37}$$ {InE2}

According to (3.35), we have

$$20C(1 + \tau^{-\frac{5}{2}}(t)) = -\frac{\dot{\tau}}{||v_0||_{\tau_0}},$$

so that the condition (3.37) is satisfied if we choose T so that

$$\tau_0 - \tau(T) \leq \frac{\epsilon||v_0||_{X_{\tau_0}}}{C'||v_0||_{X_{\tau_0+\epsilon}}} \tag{3.38}$$ {InE3}

which is satisfied if $T \leq T_*$, where $T_*(\epsilon, \tau_0, C, C', ||v_0||_{X_{\tau_0}}, ||v_0||_{X_{\tau_0+\epsilon}}) > 0$ may be computed explicitly from (3.35) and (3.38).

For the rest of the proof we proceed by induction. For this purpose, similarly to the thing we did for the priori estimate on the velocity, according to the definition of the norms $||.||_{X_\tau}$ and $||.||_{Y_\tau}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}||v^{(n+1)}(t)||_{X_{\tau(t)}} = \dot{\tau}(t)||v^{(n+1)}(t)||_{Y_{\tau(t)}} + \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha v^{(n+1)}(t)||_{L^2} \frac{(m+1)^r \tau(t)^m}{m!}. \tag{3.39}$$ {Seq7}

Next, given a multi-index $\alpha \in \mathbb{N}^3$, we need to estimate $\frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha v^{(n+1)}(t)||_{L^2}$. Taking the time derivative of (3.34) and applying ∂^α next, we can take the $L^2(D)$-inner product $< .,. >$ with $\partial^\alpha v^n$:

$$\frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha v^{(n+1)}(t)||_{L^2}^2 + \left\langle \partial^\alpha (v.\nabla v + w^{(n)}\partial_z v^{(n)}), \partial^\alpha v^{(n)} \right\rangle + \left\langle \partial^\alpha \nabla p^{n+1}, \partial^\alpha v^n \right\rangle = 0. \qquad (3.40)$$

{Seq5}

Likewise what we have for the a priori estimate on the velocity term, we can find the bounds for the velocity term and the pressure term in (3.40).

Using the Schwarz inequality we have

$$\frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha v^{(n+1)}||_{L^2} \leq \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} ||\partial^\beta v^{(n)}.\nabla \partial^{\alpha-\beta} v^{(n)}||_{L^2} + \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} ||\partial^\beta w^{(n)} \partial_z \partial^{\alpha-\beta} v^{(n)}||_{L^2}. \qquad (3.41)$$

{Seq6}

Substituting estimate (3.41) above into (3.39), we have the a priori estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}||v^{(n+1)}(t)||_{X_{\tau(t)}} = \dot{\tau}(t)||v^{(n+1)}(t)||_{Y_{\tau(t)}} + \mathcal{U}(v^{(n)}, v^{(n)}) + \mathcal{V}(w^{(n)}, v^{(n)}) + \mathcal{P}$$

where $\mathcal{U}(v^{(n)}, v^{(n)}) = \sum_{i,j=1}^2 \mathcal{U}(v_i^{(n)}, v_j^{(n)})$ and $\mathcal{V}(w^{(n)}, v^{(n)}) = \sum_{j=1}^2 \mathcal{V}(w^{(n)}, v_j^{(n)})$, and

$$\begin{aligned} \mathcal{P} &= \sum_{m=1}^{\infty} \sum_{|\alpha|=m} ||\nabla \partial^\alpha p^{(n+1)}||_{L^2(D)} \frac{(m+1)^r \tau^m}{m!} \\ &= \sum_{m=1}^{\infty} \sum_{|\alpha|=m, \alpha_3=0} ||\nabla \partial^\alpha p^{(n+1)}||_{L^2(M)} \frac{(m+1)^r \tau^m}{m!}. \end{aligned}$$

Finally using the same techniques we get

$$\frac{\mathrm{d}}{\mathrm{d}t}||v^{(n+1)}(t)||_{X_{\tau(t)}} \leq \dot{\tau}||v^{(n+1)}||_{Y_\tau} + C(1+\tau^{-\frac{5}{2}})||v^{(n)}||_{X_\tau}||v^{(n)}||_{Y_\tau},$$

and using the induction and (3.37) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}||v^{(n+1)}(t)||_{X_{\tau(t)}} \leq \dot{\tau}||v^{(n+1)}||_{Y_\tau} + C(1+\tau^{-\frac{5}{2}})||v_0||_{X_{\tau_0}}||v^{(n)}||_{Y_\tau}.$$

Using the fact that τ is the solution of the (3.35), we can choose T small enough for $v^{(n)}$ to satisfies the inequality (3.36). ☐

Finally, we can conclude the construction of the solution by showing that the map $v^{(n)} \mapsto v^{(n+1)}$ is contraction in $L^\infty(0,T;X_\tau) \cap L^1(0,T;(1+\tau^{-\frac{5}{2}})Y_\tau)$, for some sufficiently small T depending on the initial data.

Lemma 5. *Let* $\tilde{v}^{(n)} = v^{(n+1)} - v^{(n)}$*, for all* $n \geq 0$*. Let* $v_0 \in X_{\tau_0+\epsilon}$*, and* $\tau(t)$ *be defined by* (3.35)*, and* T *be as in lemma* (4)*. If for all* $n \geq 0$ *we let*

$$a_n = \sup_{t\in[0,T]} ||\tilde{v}^{(n)}(t)||_{X_\tau} + C||v_0||_{X_{\tau_0}} \int_0^T (1+\tau^{-\frac{5}{2}}(s))||\tilde{v}^{(n)}(s)||_{Y_{\tau(s)}} \,\mathrm{d}s,$$

then there is a $0 < k \leq 1$ *such that* $a_n \leq k a_{n-1}$ *for all* $n \geq 1$*.*

Proof. Denote $\tilde{w}^{(n)} = w^{(n+1)} - w^{(n)}$, and $\tilde{p}^{(n)} = p^{(n+1)} - p^{(n)}$. According to the definition of the sequence (3.34), we have

$$\begin{aligned}\tilde{v}^{(n)} = &-\int_0^t (v^{(n)}(s).\nabla + w^{(n)}(s)\partial_z)v^{(n)}(s)\,\mathrm{d}s - \int_0^t \nabla p^{(n+1)}(s)\,\mathrm{d}s\\ &+\int_0^t (v^{(n-1)}(s).\nabla + w^{(n-1)}(s)\partial_z)v^{(n-1)}(s)\mathrm{d}s + \int_0^t \nabla p^{(n)}(s)\,\mathrm{d}s,\end{aligned}$$

hence by adding and deducting $w^{(n)}(s)v^{(n-1)}(s)$ and $v^{(n)}(s).\nabla v^{(n-1)}$ following with taking the time derivative, we have

$$\partial_t\tilde{v}^{(n)} + (v^{(n)}.\nabla + w^{(n)}\partial_z)\tilde{v}^{(n-1)} + (\tilde{v}^{(n-1)}.\nabla + \tilde{w}^{(n-1)}\partial_z)v^{(n-1)} + \nabla\tilde{p}^{(n)} = 0, \tag{3.42}$$

{CK}

with initial condition $\tilde{v}^{(n)}(0) = 0$, for all $n \geq 0$. Similarly to what we had in chapter (3.2), we can have the estimate over approximate solutions $v^{(n)}$.

First we have

$$\frac{\mathrm{d}}{\mathrm{d}t}||\tilde{v}^{(n)}(t)||_{X_{\tau(t)}} = \dot{\tau}(t)||\tilde{v}^{(n)}(t)||_{Y_{\tau(t)}} + \sum_{m=0}^{\infty}\sum_{|\alpha|=m} \frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha\tilde{v}^{(n)}(t)||_{L^2}\frac{(m+1)^r\tau(t)^m}{m!},$$

where we can find an estimate on $\frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha\tilde{v}^{(n)}(t)||_{L^2}$ using (3.42). Taking the ∂^α derivative and taking the inner product with $\partial^\alpha\tilde{v}^{(n)}$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}||\partial^\alpha\tilde{v}^{(n)}(t)||_{L^2}^2 + \left\langle\partial^\alpha(v^{(n)}.\nabla\tilde{v}^{(n-1)} + w^{(n)}\partial_z\tilde{v}^{(n-1)}), \partial^\alpha\tilde{v}^{(n)}\right\rangle + \langle\partial^\alpha\nabla\tilde{p}^n, \partial^\alpha\tilde{v}^n\rangle\\ &+\left\langle\partial^\alpha(\tilde{v}^{(n-1)}.\nabla v^{(n-1)} + \tilde{w}^{(n-1)}\partial_z v^{(n-1)}), \partial^\alpha\tilde{v}^{(n)}\right\rangle = 0.\end{aligned}$$

Again, using the Leibniz rule following with Schwarz inequality and using the same procedure as in lemma (3), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}||\tilde{v}^{(n)}||_{X_\tau} \leq\; &\dot{\tau}||\tilde{v}^{(n)}||_{Y_\tau}\\ &+ C(1+\tau^{-\frac{5}{2}})(||v^{(n)}||_{X_\tau} + ||v^{(n-1)}||_{X_\tau})||\tilde{v}^{(}n-1)||_{Y_\tau}\\ &+ C(1+\tau^{-\frac{5}{2}})(||v^{(n)}||_{Y_\tau} + ||v^{(n-1)}||_{Y_\tau})||\tilde{v}^{(n-1)}||_{X_\tau}.\end{aligned} \tag{3.43}$$

{Con1}

Using the definition of $\dot{\tau}$ in (3.35), and the estimate in lemma (4) for the approximate solution, from (3.43) and taking the supremum for $t \in [0,T]$ we obtain

$$
\begin{aligned}
a_n \leq C||v_0||_{X_{\tau_0}} & \int_0^T (1+\tau^{-\frac{5}{2}})||\tilde{v}^{(n-1)}||_{Y_\tau} \mathrm{d}t \\
& + (\sup_{t\in[0,T]} ||\tilde{v}^{(n-1)}||_{X_\tau}) \int_0^T C(1+\tau^{-\frac{5}{2}})(||v^{(n)}||_{Y_\tau} + ||v^{(n-1)}||_{Y_\tau})\mathrm{d}t.
\end{aligned}
$$

If T is taken such that

$$
(\sup_{t\in[0,T]} ||\tilde{v}^{(n-1)}||_{X_\tau}) \int_0^T C(1+\tau^{-\frac{5}{2}})(||v^{(n)}||_{Y_\tau} + ||v^{(n-1)}||_{Y_\tau})\mathrm{d}t < \sup_{t\in[0,T]} ||\tilde{v}^{(n-1)}||_{X_\tau}.
$$

This concludes the proof of the lemma, showing that the map $v^{(n)} \mapsto v^{(n+1)}$ is strict contraction.Now, For the existence of the solution we can apply the classical fixed point theorem. $\square$

3.4 Uniqueness

Fix the initial data $v_0 \in X_{\tau_0+\epsilon}$, real-analytic on D with radius of analyticity strictly larger than τ_0, for some positive $\epsilon < \tau_0$. Let $\tau(t)$ be defined by $\tau(0) = \tau_0$ and $\dot{\tau} + 20C(1+\tau^{-\frac{5}{2}})||v_0||_{X_{\tau_0}} = 0$, where $C = C(D,r) > 0$ is the fixed constant defined in lemma (3). Let T_* be the maximal time such that $\tau(t) \geq 0$.

For the problem of uniqueness, assume that there exist two solutions $v^{(1)}$ and $v^{(2)}$ to (1.4)-(1.9) evolving from initial data v_0, such that for $i = 1,2$, we have

$$
||v^{(i)}(t)||_{X_{\tau(t)}} + C||v_0||_{X_{\tau_0}} \int_0^t (1+\tau^{-\frac{5}{2}})||v^{(i)}||_{Y_{\tau(s)}} \,\mathrm{d}s < \infty,
$$

for all $0 \leq t < T_*$. Similarly to we have that

$$
||v^{(i)}(t)||_{X_{\tau(t)}} \leq ||v_0(t_0)||_{X_{\tau_0}}, \quad i = 1,2,
$$

for all $0 \leq t < T_*$. Let $w^{(i)}$ and $p^{(i)}$ be the vertical velocity and the pressure associated to $v^{(i)}$. We denote the difference of the solutions $v^{(1)} - v^{(2)} = v$, and similarly define w, and p. Then (v,w,p) satisfy the equations

$$
\begin{aligned}
&\partial_t v + (v^{(1)} + w^{(1)}\partial_z)v + (v.\nabla + w\partial_z)v^{(2)} + \nabla p = 0, \qquad (3.44)\\
&\nabla.v + \partial_z w = 0,\\
&\partial_z p = 0,
\end{aligned}
$$

$$(3.45) \quad \{UN1\}$$

in $\mathcal{D} \times (0,T)$, with the corresponding boundary and initial value conditions

$$
\begin{aligned}
& w(x,z,t) = 0 \quad \text{on } \Gamma_z \times (0,T),\\
& \int_0^1 v(x,z,t)\,\mathrm{d}z.n = 0, \quad \text{on } \Gamma_x \times (0,T),\\
& v(x,z,0) = 0, \quad \text{in } \mathcal{D}. \qquad (3.46) \quad \{UN2\}
\end{aligned}
$$

Similarly to the a priori estimates of the solution, by (3.45)-(3.46) and lemma (3), we obtain

$$\frac{d}{dt}||v||_{X_\tau} \leq (\dot{\tau} + 4C_0(1+\tau^{-\frac{5}{2}})(||v^{(1)}||_{X_\tau} + ||v^{(2)}||_{X_\tau}))||v||_{Y_\tau},$$

where $C_0 > 0$ is the constant from lemma (3). Using the assumption

$$||v^{(1)}||_{X_\tau} + ||v^{(2)}||_{X_\tau} \leq 2||v_0||_{X_{\tau_0}},$$

and by the construction of τ

$$\dot{\tau} + 20C_0(1+\tau^{-\frac{5}{2}})||v_0||_{X_{\tau_0}} = 0,$$

we obtain that for all $t \in [0, T_*)$, we have $||v||_{X_\tau} = 0$.

Remark 3. *Taking another form of boundary condition*

$$\begin{aligned} w(t, x_1, x_2, z) &= 0, \quad \textit{on } \Gamma_z \times (0, T), \\ v(t, x_1, x_2, z).n &= 0, \quad \textit{on } \Gamma_{(x_1,x_2)} \times (0, T), \end{aligned}$$

for inviscid primitive equations in a cylinder, we can prove wellposedness in analogy to this proof. We need to define the space of real-analytic functions as bellow

$$X_\tau = \{v \in C^\infty(D) : v|_{\Gamma_x}.n = 0, \ \int_0^1 \nabla.v \ \mathrm{d}z = 0, \ ||v||_{X_\tau} < \infty\}.$$

for fixed $r \geq 0$ *and* $\tau > 0$*. We have the same a priori estimates for both pressure and velocity terms, and we can define the pressure and the velocity iterate same as the previous case.*

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