
Faculty of Science

Faculty Publications

Continuous Wavelet Transform of Schwartz Tempered Distributions in $S'(\mathbb{R}^n)$

Jagdish Narayan Pandey, Jay Singh Maurya, Santosh Kumar Upadhyay and Hari Mohan Srivastava

February 2019

©2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).

This article was originally published at:

<http://dx.doi.org/10.3390/sym11020235>

Citation for this paper:

Pandey, J.N., Maurya, J.S., Upadhyay, S.K. & Srivastava, H.M. (2019). Continuous Wavelet Transform of Schwartz Tempered Distributions in $S'(\mathbb{R}^n)$. *Symmetry*, 11(2), 235. <https://doi.org/10.3390/sym11020235>

Article

Continuous Wavelet Transform of Schwartz Tempered Distributions in $S'(\mathbb{R}^n)$

Jagdish Narayan Pandey ¹, Jay Singh Maurya ² and Santosh Kumar Upadhyay ^{2,*}
and Hari Mohan Srivastava ^{3,4,*} 

¹ School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6, Canada; jimpandey1936@gmail.com

² Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi-221005, India; jaysinghmaurya.rs.mat17@itbhu.ac.in

³ Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada

⁴ Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

* Correspondence: sk_upadhyay2001@yahoo.com (S.K.U.); harimsri@math.uvic.ca (H.M.S.); Tel.: +91-94501-12714 (S.K.U.); +1-250-472-5313 (H.M.S.)

Received: 11 January 2019; Accepted: 12 February 2019; Published: 15 February 2019



Abstract: In this paper, we define a continuous wavelet transform of a Schwartz tempered distribution $f \in S'(\mathbb{R}^n)$ with wavelet kernel $\psi \in S(\mathbb{R}^n)$ and derive the corresponding wavelet inversion formula interpreting convergence in the weak topology of $S'(\mathbb{R}^n)$. It turns out that the wavelet transform of a constant distribution is zero and our wavelet inversion formula is not true for constant distribution, but it is true for a non-constant distribution which is not equal to the sum of a non-constant distribution with a non-zero constant distribution.

Keywords: function spaces and their duals; distributions; tempered distributions; Schwartz testing function space; generalized functions; distribution space; wavelet transform of generalized functions; Fourier transform

1. Introduction

As studied in the earlier works (see, for example, [1–12]), we define a Schwartz testing function space $S(\mathbb{R}^n)$ to consist of C^∞ functions ϕ defined on \mathbb{R}^n and satisfying the following conditions:

$$\sup_{x \in \mathbb{R}^n} \left| x_n^{m_n} \cdots x_2^{m_2} x_1^{m_1} \frac{\partial^{k_n}}{\partial x_n} \frac{\partial^{k_{n-1}}}{\partial x_{n-1}} \cdots \frac{\partial^{k_2}}{\partial x_2} \frac{\partial^{k_1}}{\partial x_1} \phi(x_1, x_2, x_3, \dots, x_n) \right| < \infty \quad (1)$$

$|m|, |k| = 0, 1, 2, \dots$

The topology over $S(\mathbb{R}^n)$ is generated by the following sequence of semi-norms:

$$\{\gamma_{m,k}\}_{|m|,|k|=0}^\infty$$

where

$$\gamma_{m,k}(\phi) = \sup_{x \in \mathbb{R}^n} \left| |x^m| \phi^{(k)}(x) \right|, \quad (2)$$

$$|m| = m_1 + m_2 + m_3 + \cdots + m_n,$$

$$|k| = k_1 + k_2 + k_3 + \cdots + k_n,$$

$$|x^m| = |x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_n^{m_n}|,$$

$$\phi^{(k)}(x) = \frac{\partial^{k_n}}{\partial x_n} \cdots \frac{\partial^{k_2}}{\partial x_2} \frac{\partial^{k_1}}{\partial x_1} \phi(x).$$

These collections of semi-norms in Equation (2) are separating which means that an element $\phi \in S(\mathbb{R}^n)$ is non-zero if and only if there exists at least one of the semi-norms $\gamma_{m,k}$ satisfying $\gamma_{m,k}(\phi) \neq 0$. A sequence $\{\phi_\nu\}_{\nu=1}^\infty$ in $S(\mathbb{R}^n)$ tends to ϕ in $S(\mathbb{R}^n)$ if and only if $\gamma_{m,k}(\phi_\nu - \phi) \rightarrow 0$ as ν goes to ∞ for each of the subscripts $|m|, |k| = 0, 1, 2, \dots$, are as defined above. Now, one can verify that the function $e^{-(t_1^2+t_2^2+t_3^2+\dots+t_n^2)} \in S(\mathbb{R}^n)$ and the sequence

$$\frac{\nu - 1}{\nu} e^{-(t_1^2+t_2^2+t_3^2+\dots+t_n^2)} \rightarrow e^{-(t_1^2+t_2^2+t_3^2+\dots+t_n^2)}$$

in $S(\mathbb{R}^n)$ as $\nu \rightarrow \infty$. The Dirac delta function $\delta(t)$ is defined by

$$\left. \begin{aligned} \delta(t) &= 0, \quad t \neq 0, \\ \int_{\mathbb{R}^n} \delta(t) dt &= 1, \end{aligned} \right\}$$

where $t = (t_1, t_2, t_3, \dots, t_n) \in \mathbb{R}^n$.

It is easy to check that $\delta(t_1, t_2, \dots, t_n)$ is a continuous linear functional on $S(\mathbb{R}^n)$. A regular distribution generated by a locally integrable function is an element of $S'(\mathbb{R}^n)$.

Our objective now is to find an element $\psi \in S(\mathbb{R}^n)$, which is a wavelet, so as to be able to define the wavelet transform of $f \in S'(\mathbb{R}^n)$ with respect to this kernel.

A function $\psi \in L^2(\mathbb{R}^n)$ is a window function if it satisfies the following conditions:

$$x_i \psi(x), x_i x_j \psi(x), \dots, x_1 x_2 x_3 \cdots x_n \psi(x) \tag{3}$$

belonging to $L^2(\mathbb{R}^n)$. Here, i, j, k, \dots take on all assumes values $1, 2, 3, \dots$ and all the lower suffixes in a term in Equation (3) are different. It has been proved by Pandey et al. [4,13] that a window function which is an element of $L^2(\mathbb{R}^n)$ belongs to $L^1(\mathbb{R}^n)$. It is easy to verify that every element of $S(\mathbb{R}^n)$ is a window function.

A window function ψ belonging to $L^2(\mathbb{R}^n)$ and satisfying the following condition:

$$\int_{-\infty}^{\infty} \psi(x_1, x_2, x_3, \dots, x_i, \dots, x_n) dx_i = 0 \quad (\forall i = 1, 2, 3, \dots, n) \tag{4}$$

also satisfies the admissibility condition given by

$$\int_{\mathbb{R}^n} \frac{|\hat{\psi}(\Lambda)|^2}{|\Lambda|} d\Lambda < \infty, \tag{5}$$

where

$$\begin{aligned} \hat{\psi}(\Lambda) &= \hat{\psi}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n), \\ |\Lambda| &= |\lambda_1 \lambda_2 \cdots \lambda_n| \end{aligned}$$

and $\hat{\psi}(\Lambda)$ is the Fourier transform of $\psi(x) \equiv \psi(x_1, x_2, x_3, \dots, x_n)$ (see also a recent work [14]). Clearly, ψ in Equation (4) is a wavelet [13]. As an example, one can easily verify that the function given by

$$\psi(x) = x_1 x_2 \cdots x_n e^{-(x_1^2+x_2^2+x_3^2+\dots+x_n^2)}$$

is a wavelet belonging to $S(\mathbb{R}^n)$. Let $s(\mathbb{R}^n)$ be a subspace of $S(\mathbb{R}^n)$ such that every element $\phi \in s(\mathbb{R}^n)$ satisfies Equation (4). Clearly, every element of $s(\mathbb{R}^n)$ is a wavelet [4].

Now, if $f \in S'(\mathbb{R}^n)$ and ψ is a wavelet belonging to $S(\mathbb{R}^n)$, the wavelet transform of f can be defined by

$$W_f(a, b) = \left\langle f(x), \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product and (a, b) is the argument of wavelet transform $W_f(a, b)$ of f with respect to wavelet ψ ,

$$\begin{aligned} \psi\left(\frac{x-b}{a}\right) &= \psi\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \dots, \frac{x_n-b_n}{a_n}\right) \\ &(a_i \neq 0 \quad (\forall i = 1, 2, 3, \dots, n)) \end{aligned}$$

and

$$|a| = |a_1 a_2 \cdots a_n|.$$

Our objective next is to prove the following inversion formula:

$$\left\langle \frac{1}{C_\psi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W_f(a, b) \psi\left(\frac{t-b}{a}\right) \frac{db da}{\sqrt{|a|a^2}}, \phi(t) \right\rangle \rightarrow \langle f, \phi \rangle, \quad \phi \in S(\mathbb{R}^n) \quad (6)$$

by interpreting the convergence in $S'(\mathbb{R}^n)$. Here, we have

$$C_\psi = (2\pi)^n \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\wedge)|^2}{|\wedge|} d\wedge.$$

The derivation of the inversion formula given by Formula (6) is difficult. We, therefore, make an easy approach. The work on the multidimensional wavelet transform with positive scale [$a > 0$] was done by Daubechies [15], Meyer [16], Pathak [17], and some others. Motivated by the earlier works [6,8,12], Pandey et al. [4] studied a generalization of these works and extended the multidimensional wavelet transform with real scale [$a \neq 0$]. In the year 1995, Holschneider [18] extended the multidimensional wavelet transform to Schwartz tempered distributions with positive scales [$a > 0$]. Recently, Weisz [19,20] studied the inversion formula for the continuous wavelet transform and found its convergence in L^p and Wiener amalgam spaces. Postnikov et al. [21] studied computational implementation of the inverse continuous wavelet transform without a requirement of the admissibility condition.

Our objective in this investigation is to extend the wavelet transform to Schwartz tempered distributions with real scale [$a \neq 0$]. The standard cut off of negative frequencies (which is required to apply continuous wavelet transform with $a > 0$) may result in a loss of information if the transformed functions were non-symmetric (in the Fourier space) mixture of real and imaginary frequency components. Our proposed and proven inversion formula is free from the mentioned defect. The main advantage of our work is a possible further practical utility of the proven result and the simplicity of calculation; in addition, our extension of the multidimensional wavelet inversion formula is the most general. In [4], it is proved that a window function $\psi(x) \in L^2(\mathbb{R}^n)$ is a wavelet if and only if the integral of ψ along each of the axes is zero; therefore, any $\psi(x) \in s(\mathbb{R}^n)$ is a wavelet. Hence, the wavelet transform of a constant distribution is zero.

We thus realize that two elements of $S'(\mathbb{R}^n)$ having equal wavelet transform will differ by a constant in general. Holschneider [18] uses the wavelet inversion formula for $f \in S'(\mathbb{R}^n)$, but he does not mention the wavelet inversion formula and its convergence in $S'(\mathbb{R}^n)$. Perhaps, he takes it for granted, as such an inversion formula is valid for elements of $L^2(\mathbb{R}^n)$ by interpreting convergence in $L^2(\mathbb{R}^n)$. Our objective in this paper is to fill up all these gaps. We will prove the inversion Formula (6) in Section 3.

2. Structure of Generalized Functions of Slow Growth

Elements of $S'(\mathbb{R}^n)$ are called tempered distributions or distributions of slow growth.

Definition 1. A function $f(x)$ is said to be a function of slow growth in \mathbb{R}^n if, for $m \geq 0$, we have

$$\int_{\mathbb{R}^n} |f(x)| (1 + |x|)^{-m} dx < \infty$$

and it determines a regular functional f in $S'(\mathbb{R}^n)$ by the formula given by

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx, \quad \phi \in S(\mathbb{R}^n). \quad (7)$$

It is easy to verify that the functional f defined by Equation (7) exists $\forall \phi \in S(\mathbb{R}^n)$ and that it is linear as well as continuous on $S(\mathbb{R}^n)$.

We now quote a theorem of Vladimirov proved in his book [8].

Theorem 1. If $f \in S'(\mathbb{R}^n)$, then there exists a continuous function g of slow growth in \mathbb{R}^n and an integer $m \geq 0$ such that

$$f(x) = D_1^m D_2^m D_3^m \cdots D_n^m g(x), \quad \frac{\partial}{\partial x_i} \equiv D_i \quad (8)$$

or, equivalently,

$$f(x) = D^m g(x) \quad (D := D_1 D_2 D_3 \cdots D_n). \quad (9)$$

The n -dimensional wavelet inversion formula for tempered distributions will now be proved very simply by using the structure Formula (9). This structure formula enables us to reduce the wavelet analysis problem relating to tempered distributions to the classical wavelet analysis problem of $L^2(\mathbb{R}^n)$ functions. Our wavelet inversion formula of $L^2(\mathbb{R}^n)$ functions will be used quite successfully in order to derive the wavelet inversion formula for the wavelet transform of tempered distributions.

3. Wavelet Transform of Tempered Distributions in \mathbb{R}^n and Its Inversion

Henceforth, we assume that $a \neq 0$ implies each of the components $a_i \neq 0$ for all $i = 1, 2, 3, \dots, n$ and that $a > 0$ means each of the component a_i of a is greater than zero. Moreover, $|a| > \epsilon$ will mean that $|a_i| > \epsilon$ for all $i = 1, 2, 3, \dots, n$.

Let $\psi(x) = \psi(x_1, x_2, \dots, x_n) \in S(\mathbb{R}^n)$. Then $\psi(x)$ is a window function and is a wavelet if and only if

$$\int_{-\infty}^{\infty} \psi(x_1, x_2, \dots, x_i, \dots, x_n) dx_i = 0 \quad (\forall i = 1, 2, \dots, n). \quad (10)$$

We define $\psi\left(\frac{x-b}{a}\right) \equiv \psi\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \dots, \frac{x_n-b_n}{a_n}\right)$, where a_i, b_i are real numbers and none of the a_i is zero. The wavelet transform $W_f(a, b)$ of f with respect to the kernel $\frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right)$ is defined by

$$W_f(a, b) = \left\langle f(x), \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right) \right\rangle. \quad (11)$$

Here, we assume that

$$|a| = |a_1 a_2 a_3 \cdots a_n| \quad (a_i \neq 0 \quad (i = 1, 2, 3, \dots, n)).$$

We now prove the following lemmas which will be used to prove the main inversion formula.

Lemma 1. (see [13]) Let $\phi \in S(\mathbb{R}^n)$ and ψ be a wavelet belonging to $S(\mathbb{R}^n)$.

$$\begin{aligned} \frac{1}{C_\psi} \int_{a \in \mathbb{R}^n} \int_{b \in \mathbb{R}^n} \int_{t \in \mathbb{R}^n} (-D_t)^m \phi(t) \bar{\psi}\left(\frac{t-b}{a}\right) \psi\left(\frac{x_0-b}{a}\right) \frac{dt db da}{a^2|a|} \\ = (-D_x)^m \phi(x)|_{x=x_0} \quad (\forall x_0 \in \mathbb{R}^n). \end{aligned}$$

This is called pointwise convergence of the wavelet inversion formula.

Lemma 2. Let $\phi \in S(\mathbb{R}^n)$ and let ψ be a wavelet belonging to $S(\mathbb{R}^n)$. Then

$$\frac{1}{C_\psi} \int_{a \in \mathbb{R}^n} \int_{b \in \mathbb{R}^n} \int_{t \in \mathbb{R}^n} (-D_t)^m \phi(t) \bar{\psi}\left(\frac{t-b}{a}\right) \psi\left(\frac{x-b}{a}\right) \frac{dt db da}{a^2|a|}$$

converges to $(-D_x^m) \phi(x)$ uniformly for all $x \in \mathbb{R}^n$.

Proof. Let

$$F(\wedge) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (-D_t)^m \phi(t) e^{-i \wedge \cdot t} dt$$

be the Fourier transform of $(-D_t)^m \phi(t)$. It follows that, in the sense of $L^2(\mathbb{R}^n)$ convergence [17],

$$\begin{aligned} \frac{1}{C_\psi} \int_{a \in \mathbb{R}^n} \int_{b \in \mathbb{R}^n} \int_{c \in \mathbb{R}^n} (-D_t)^m \phi(t) \bar{\psi}\left(\frac{t-b}{a}\right) \psi\left(\frac{x-b}{a}\right) \frac{dt db da}{a^2|a|} \\ = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} F(\wedge) e^{i \wedge \cdot x} d\wedge = (-D_x)^m \phi(x). \end{aligned}$$

This convergence is also uniform by a Weierstrass M -test because

$$\left| \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} F(\wedge) e^{i \wedge \cdot x} d\wedge \right| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |F(\wedge)| d\wedge < \infty$$

and

$$F(\wedge) \in S(\mathbb{R}^n).$$

□

Theorem 2. Let $f \in S'(\mathbb{R}^n)$ and $W_f(a, b)$ be its wavelet transform defined by

$$W_f(a, b) = \left\langle f(x), \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \right\rangle.$$

Then the inversion formula of the wavelet transform $W_f(a, b)$ is given by

$$\begin{aligned} \left\langle \frac{1}{C_\psi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W_f(a, b) \psi\left(\frac{t-b}{a}\right) \frac{db da}{\sqrt{|a|}a^2}, \phi(t) \right\rangle = \langle f, \phi \rangle \tag{12} \\ (\forall \phi \in S(\mathbb{R}^n)), \end{aligned}$$

where the equality holds true almost everywhere.

Proof. Using the structure formula (9) for f , we find by distributional differentiation that

$$\begin{aligned} W_f(a, b) &= \left\langle D_x^m g(x), \frac{1}{\sqrt{|a|}} \psi \left(\frac{x-b}{a} \right) \right\rangle \\ &= \left\langle g(x), (-D_x)^m \frac{1}{\sqrt{|a|}} \psi \left(\frac{x-b}{a} \right) \right\rangle. \end{aligned}$$

Here, we have

$$(-D_x) = (-D_{x_1}) (-D_{x_2}) \cdots (-D_{x_n}) D_{x_i} \equiv \frac{\partial}{\partial x_i} \quad (i = 1, 2, \dots, n).$$

We thus obtain

$$\begin{aligned} W_f(a, b) &= \left\langle g(x), (D_b)^m \frac{1}{\sqrt{|a|}} \psi \left(\frac{x-b}{a} \right) \right\rangle \\ D_b &= \left(\frac{\partial}{\partial b_1} \frac{\partial}{\partial b_2} \cdots \frac{\partial}{\partial b_n} \right). \end{aligned}$$

□

The expression on the left-hand side in (12) can be written as follows:

$$\begin{aligned} \Omega &:= \frac{1}{C_\psi} \int_{t \in \mathbb{R}^n} \int_{a \in \mathbb{R}^n} \int_{b \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} g(x) D_b^m \frac{1}{\sqrt{|a|}} \bar{\psi} \left(\frac{x-b}{a} \right) \psi \left(\frac{t-b}{a} \right) \bar{\phi}(t) dx db da dt \\ &= \frac{1}{C_\psi} \int_{t \in \mathbb{R}^n} \int_{a \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} g(x) \left[\int_{b \in \mathbb{R}^n} \left\{ D_b^m \bar{\psi} \left(\frac{x-b}{a} \right) \right\} \psi \left(\frac{t-b}{a} \right) db \right] \frac{\bar{\phi}(t) dx da dt}{a^2 |a|}. \end{aligned} \tag{13}$$

We now evaluate the integral in the big bracket by parts to find from Equation (13) that

$$\begin{aligned} \Omega &= \frac{1}{C_\psi} \int_{t \in \mathbb{R}^n} \int_{a \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} g(x) \left[\int_{b \in \mathbb{R}^n} \bar{\psi} \left(\frac{x-b}{a} \right) (-D_b)^m \psi \left(\frac{t-b}{a} \right) db \right] \frac{\bar{\phi}(t) dx da dt}{a^2 |a|} \\ &= \frac{1}{C_\psi} \int_{t \in \mathbb{R}^n} \int_{a \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} g(x) \left[\int_{b \in \mathbb{R}^n} \bar{\psi} \left(\frac{x-b}{a} \right) (+D_t)^m \psi \left(\frac{t-b}{a} \right) db \right] \frac{\bar{\phi}(t) dx da dt}{a^2 |a|}, \end{aligned}$$

which, upon inverting the order of integration with respect to a and t , yields

$$\begin{aligned} \Omega &= \frac{1}{C_\psi} \int_{a \in \mathbb{R}^n} \int_{t \in \mathbb{R}^n} \int_{b \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} g(x) \bar{\psi} \left(\frac{x-b}{a} \right) dx D_t^m \psi \left(\frac{t-b}{a} \right) db \bar{\phi}(t) \frac{dt da}{|a|^2 |a|} \\ &= \frac{1}{C_\psi} \int_{a \in \mathbb{R}^n} \int_{b \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} g(x) \bar{\psi} \left(\frac{x-b}{a} \right) dx \int_{t \in \mathbb{R}^n} \psi \left(\frac{t-b}{a} \right) db (-D_t)^m \bar{\phi}(t) \frac{dt da}{|a|^2 |a|}. \end{aligned} \tag{14}$$

In order to justify the inversion of the order of integration with respect to a and t , we first perform the integration in the region $[(a, t) : |a| > \epsilon, a, t \in \mathbb{R}^n]$, invert the order of integration and then let $\epsilon \rightarrow 0$. This existence of the triple integral in terms of b, a and t in Equation (14) is proved by using the Plancherel theorem with respect to the variable b . Thus, by using

$$C_\psi = (2\pi)^n \int_{\mathbb{R}^n} \frac{|\hat{\psi}(\wedge)|^2}{|\wedge|} d\wedge,$$

we notice that the variable a disappears from the denominator and every calculation goes on smoothly. Since the functions ϕ and ψ are elements of $S(\mathbb{R}^n)$, the Fubini's theorem can be applied in order to justify the above interchanges of the order of integration.

Now, Equation (14) can be written as follows:

$$\left\langle g(x), \frac{1}{C_\psi} \int_{a \in \mathbb{R}^n} \int_{b \in \mathbb{R}^n} \int_{t \in \mathbb{R}^n} (-D_t)^m \phi(t) \bar{\psi} \left(\frac{t-b}{a} \right) dt \psi \left(\frac{x-b}{a} \right) \frac{db da}{|a|^2 |a|} \right\rangle \quad (15)$$

$$= \langle g(x), (-D_x)^m \phi(x) \rangle, \quad (16)$$

by means of the wavelet inversion formula in \mathbb{R}^n [4] and Lemma 2.

We note that the triple integral in Equation (15) converges uniformly to $(-D_x)^m \phi(x) \forall x \in \mathbb{R}^n$. Thus, Equation (15) becomes Equation (16):

$$\begin{aligned} \langle g(x), (-D_x)^m \phi(x) \rangle &= \langle (D_x)^m g(x), \phi(x) \rangle \\ &= \langle f(x), \phi(x) \rangle. \end{aligned}$$

4. Conclusions

In our present investigation, we have introduced and studied a continuous wavelet transform of a Schwartz tempered distribution $f \in S'(\mathbb{R}^n)$ with the wavelet kernel $\psi \in S(\mathbb{R}^n)$. We have successfully derived the corresponding wavelet inversion formula by interpreting convergence in the weak topology of $S'(\mathbb{R}^n)$.

We have found that the wavelet transform of a constant distribution is zero and also that our wavelet inversion formula is not true for constant distribution, but it is true for a non-constant distribution which is not equal to the sum of a non-constant distribution with a non-zero constant distribution. Our results and findings are stated and proved as Lemmas and Theorems.

Author Contributions: Conceptualization, S.K.U.; Methodology, S.K.U. and J.S.M.; Software, J.S.M.; Validation, H.M.S.; Formal analysis, H.M.S.; Writing—original draft preparation, S.K.U. and J.S.M.; Writing—review and editing, S.K.U. and H.M.S.; Supervision, J.N.P. and H.M.S.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bremmermann, H. *Distributions, Complex Variables and Fourier Transforms*; Addison-Wesley Publishing Company Inc.: Boston, MA, USA, 1965.
2. Kantorovich, L.V.; Akilov, G.P. *Functional Analysis in Normed Spaces*; Macmillan: New York, NY, USA, 1963.
3. Pandey, J.N. *The Hilbert Transform of Schwartz Distributions and Applications*; John Wiley and Sons Inc.: Hoboken, NJ, USA, 1996.
4. Pandey, J.N.; Upadhyay, S.K. Continuous Wavelet transform and window functions. *Proc. Am. Math. Soc.* **2015**, *143*, 4750–4773. [[CrossRef](#)]
5. Schwartz, L. *Theorie des Distributions*; Hermann: Paris, France, 1966.
6. Shilov, G.E. *Generalized Functions and Partial Differential Equations*; Gordon and Breach Publishing Co.: Chelmsford, MA, USA, 2005.
7. Trèves, F. *Topological Vector Spaces, Distributions and Kernels*; Academic Press: New York, NY, USA, 1997.
8. Vladimirov, V.S. *Generalized Functions in Mathematical Physics*; Yankovasky, G., Eds.; Central Books Ltd: London, UK, 1976, ISBN 9780714715452.
9. Walter, G.G.; Shen, X. *Wavelets and Other Orthogonal Systems*, 2nd ed.; Chapman and Hall (CRC Press): Boca Raton, FL, USA, 2009.
10. Yosida, K. *Functional Analysis*; Springer: Berlin, Germany, 1995.
11. Zemanian, A.H. *Distribution Theory and Transform Analysis*; McGraw-Hill Book Company: New York, NY, USA, 1965.
12. Zemanian, A.H. *Generalized Integral Transformation*; Inter Science Publishers, John Wiley and Sons Inc.: Hoboken, NJ, USA, 1968.

13. Pandey, J.N.; Jha, N.K.; Singh, O.P. The continuous wavelet transform in n-dimensions. *Internat. J. Wavelets Multiresolut. Inf. Process.* **2016**, *14*, 13. [[CrossRef](#)]
14. Srivastava, H.M.; Upadhyay, S.K.; Khatterwani, K. A family of pseudo-differential operators on the Schwartz space associated with the fractional Fourier transform, *Russian J. Math. Phys.* **24**, 2017, 534–543. [[CrossRef](#)]
15. Daubechies, I. *Ten Lectures on Wavelets*; University of Lowell: Lowell, MA, USA, 1990.
16. Meyer, Y. *Wavelets and Operators*; Cambridge Studies in Advanced Mathematics (Series 37); Cambridge University Press: Cambridge, UK, 1992.
17. Pathak, R.S. *The Wavelet Transform*; Atlantic Press/World Scientific: Paris, France, 2009.
18. Holschneider, M. *Wavelets an Analysis Tool*; Oxford Science Publications, Clarendon (Oxford University) Press: Oxford, UK, 1995.
19. Weisz, F. Convergence of the inverse continuous wavelet transform in Wiener amalgam spaces. *Analysis* **2015**, *35*, 33–46. [[CrossRef](#)]
20. Weisz, F. Inversion Formulas for the Continuous Wavelet Transform. *Acta Math. Hungar.* **2013**, *138*, 237–258. [[CrossRef](#)]
21. Postnikov, E.B.; Lebedeva, E.A.; Lavrova, A.I. Computational implementation of the inverse continuous wavelet transform without a requirement of the admissibility condition. *Appl. Math. Comput.* **2016**, *282*, 128–136. [[CrossRef](#)]



© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).