ORDERED COLOURINGS OF LINE GRAPHS OF TREES

by

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Abstract. An ordered colouring of a graph G is a function c from V(G) into the positive integers such that for every pair of vertices u and v and for every (u,v)–path P, if $c(u) = c(v)$ then there exists an internal vertex x of P with $c(u) < c(x)$. An ordered colouring of G is minimal if the largest integer in the range is minimal. We give a polynomial algorithm for finding a minimal ordered colouring of a line graph of a tree. We then extend the algorithm to a larger class of graphs..

Introduction

We use the notation and terminology of Bondy and Murty [1]. We restrict ourselves to simple graphs.

We will refer to the positive integers as *colours*. An *ordered colouring* of a graph G is a function from V(G) into the positive integers such that for every path P between two vertices of the same colour there is an internal vertex of P with a larger colour. Equivalently, a colouring c of G with the positive integers is an ordered colouring if for every colour i, each component of $G - \{v | i < c(v)\}$ has at most one vertex with colour i.

If we assign a different colour to every vertex of G, then we trivially have an ordered colouring of G. The smallest k such that G has an ordered colouring using only the integers from 1 to k is called the *ordered chromatic number* of G and is denoted by $\chi_0(G)$. An ordered colouring of G using only the integers from 1 to $\chi_0(G)$ is called a *minimal ordered colouring.*

A *partial ordered colouring* of G is an ordered colouring of an induced subgraph of

G. If c and d are partial ordered colourings of G such that $c \subsetneq d$, then we say that c is a *restriction* of d and d is an *extension* of c. If c is an ordered colouring of the induced subgraph H of G, then we use c_H to denote c.

Next we define an ordering on the finite sets of colours. Let A and B be finite sets of positive integers. $A < B$ if $A \Delta B \neq \emptyset$ and the grestest integer in $A \Delta B$ is in B.

Let c be an ordered colouring of an induced subgraph H of G and let $x \in V(G) - V(H)$. We say that a colour a is *represented at* x *by* c if there exists a vertex v in V(H) and a (v, x)–path with all its internal vertices in V(H) such that $c(v) = a$ and all internal vertices have colours smaller than a. Equivalently, a is represented at x by c if $c \cup \{(x,a)\}$ is not a partial ordered colouring of G. We use $r(x,c)$ to denote the set of all colours represented at x by c. Further, we define $r_a^b(x,c) = \{a' \in r(x,c) \mid a \leq a' \leq b\}$, $r_a(x,c) = \{a' \in r(x,c) \mid a \leq a'\}$ and $r^b(x,c) = \{a' \in r(x,c) \mid a' \leq b\}$. We say that c is *minimal with respect to* x if $r(x,c) \leq r(x,d)$, for every ordered colouring d of H.

The class $\mathscr{L}$ of line graphs of trees is the class of all connected graphs G with the following properties.

α) G is the union of edge–disjoint complete graphs.

β) Any two distinct maximal cliques have at most one vertex in common.

γ) Any cycle of G is a subgraph of some maximal clique.

δ) Every vertex is in the vertex set of at most two maximal cliques.

A maximal clique K of a graph G in $\mathscr{L}$ is a *leaf clique* if at most one vertex in V(K) is in the vertex set of another maximal clique. If G is not a complete graph, then every leaf clique shares a unique vertex x with another maximal clique. In this case we say that x is the *attachment vertex* of K. It is easy to show that every noncomplete graph in $\mathscr{L}$ has at least two leaf cliques.

We will present a polynomial algorithm for finding a minimal ordered colouring of a graph in $\mathscr{L}$ The main component of this algorithm is the extension procedure discussed in the next section. We also show how the algorithm can be used to find a minimal ordered colouring of a connected graph which only satisfies conditions α, β, and γ.

Other results on ordered colourings can be found in Katchalski, McCuaig, and Seager [2].

An Extension Procedure

Consider a graph G constructed as follows. Let S be the complete graph on $\{v, u_1, \cdots, u_n\}$. Let $H_1', \cdots, H_n'$ be disjoint graphs such that $V(H_i') \cap V(S) = \{u_i\}$ and $H_i = H_i' - u_i$ is connected, $i = 1, \cdots, n$. Let $G = S \cup \bigcup_{i=1}^{n} H_i'$ and $H = \bigcup_{i=1}^{n} H_i$.

Let c be an ordered colouring of $G - V(S)$ such that c_{H_i} is minimal with respect to u_i, $i = 1, \cdots, n$. We first show that c can be extended to an ordered colouring d_0 of $G - v$ which is minimal with respect to v. Afterwards we give a procedure to find such an extension.

Lemma 1. There exists an extension d_0 of c to $G - v$ which is minimal with respect to v.

Proof. Let d be an ordered colouring of $G - v$ which is minimal with respect to v.

Suppose $i \in \{1, \cdots, n\}$ and let $d(u_i) = a$. Since c_{H_i} is minimal with respect to u_i, $r_a(c_{H_i}, u_i) \leq r(d_{H_i}, u_i)$. Then either $r_a(c_{H_i}, u_i) = r_a(d_{H_i}, u_i)$ or there exists colour b such that $b \in r_a(d_{H_i}, u_i) - r_a(c_{H_i}, u_i)$ and $r_{b+1}(c_{H_i}, u_i) = r_{b+1}(d_{H_i}, u_i)$. Suppose we have the latter case. Since $b \in r_a(d_{H_i}, u_i)$ and $d(u_i) = a$, $a < b$. Then we have an ordered colouring d' of $G - v$, where

$$d'(x) = \begin{cases} c(x), & \text{if } x \in V(H_i) \\ b, & \text{if } x = u_i \\ d(x), & \text{otherwise.} \end{cases}$$

But now $r(d',v) \subsetneq r(d,v) - \{a\}$ and we have contradicted the fact that d is minimal with respect to v. Therefore, $r_a(c_{H_i},u_i) = r_a(d_{H_i},u_i)$.

We can now obtain d_0 from d by replacing d_{H_i} by c_{H_i} for every i in $\{1,\cdots,n)$, that is, by letting

$$d_0(x) = \begin{cases} d(x), & \text{if } x \in V(S) - \{v\} \\ c(x), & \text{otherwise.} \end{cases}$$

■

We now recursively define a sequence, $c_0, c_1, \cdots, c_n$, of partial ordered colourings of $G - v$ and a sequence $v_1, \cdots, v_n$, of vertices in $V(S) - \{v\}$. Let $c_0 = c$. Given c_i, we define c_{i+1} as follows. Let m_{i+1} be the smallest colour m such that c_i has an extension c_i' to $G - v$ using additional colours only in $\{1,\cdots,m\}$. Let v_{i+1} be a vertex u in $V(S) - \{v,v_1,\cdots,v_i\}$ for which $r^{m_{i+1}}(c,u)$ is maximal. Let $c_{i+1} = c_i \cup \{(v_{i+1},m_{i+1})\}$.

It is easy to prove by induction that c_i is a function on $V(H) \cup \{v_1,\cdots,v_i\}$, $i = 1, \cdots, n$, and that $m_1 > m_2 > \cdots > m_n$. The significance of $c_0, c_1, \cdots, c_n$ is shown by the next theorem.

Theorem 1. For every i in $\{0,1,\cdots,n\}$, c_i is an ordered colouring and c_i can be extended to an ordered colouring d_i of $G - v$ which is minimal with respect to v.

Proof. We prove the result by induction. It is true for c_0 by Lemma 1. Suppose the result holds for c_i, where $i < n$. We will find d_{i+1} using d_i.

Let $b = \max\{d_i(v_k) \mid k = i+1, \cdots, n\}$. By definition $d_i(v_k) = c_i(v_k) = c_i'(v_k)$, $k = 1, \cdots, i$, and $b \geq d_i(v_k)$ and $b \geq m_{i+1} \geq c_i'(v_k)$, $k = i+1, \cdots, n$. Thus, $r_{b+1}(d_i,v) = r_{b+1}(c_i',v)$. Hence, if $m_{i+1} < b$, then $b \in r(d_i,v) - r(c_i',v)$. But then $r(c_i',v) < r(d_i,v)$ and we have contradicted the fact that d_i is minimal with respect to v. Therefore, $m_{i+1} = b$.

Let v_ℓ be the vertex in $\{v_{i+1},\ldots,v_n\}$ such that $d_i(v_\ell) = b$. If $v_\ell = v_{i+1}$, then we let $d_{i+1} = d_i$. Suppose $v_\ell \neq v_{i+1}$ and let $d_i(v_{i+1}) = a$.

By the choice of v_{i+1}, $r^b(c,v_\ell) \leq r^b(c,v_{i+1})$. Then either $r_a^b(c,v_\ell) = r_a^b(c,v_{i+1})$ or there exists colour a' such that $a' \in r_a^b(c,v_{i+1}) - r_a^b(c,v_\ell)$ and $r_{a'+1}^b(c,v_\ell) = r_{a'+1}^b(c,v_{i+1})$. Suppose we have the latter case. Since d_i is an extension of c and $d_i(v_\ell) = b > a = d_i(v_{i+1})$, we know that $a,b \notin r(c,v_{i+1})$. Then $a' \in r_a^b(c,v_{i+1})$ implies $a < a' < b$. Then we have an ordered colouring d_i' of $G - v$, where

$$d_i'(x) = \begin{cases} a' & , \text{ if } x = v_\ell \\ b & , \text{ if } x = v_{i+1} \\ d_i(x), & \text{otherwise.} \end{cases}$$

But now $r(d_i',v) \subseteq r(d_i,v) - \{a\}$ and we have contradicted the fact that d_i is minimal with respect to v. Therefore, $r_a^b(c,v_\ell) = r_a^b(c,v_{i+1})$. Therefore, we can exchange the colours of v_{i+1} and v_k in d_i to obtain d_{i+1}.

Thus, we have shown the existence of d_{i+1}. Since c_{i+1} is a restriction of the ordered colouring d_{i+1}, c_{i+1} is also an ordered colouring. ■

The recursive definition of the sequence $c_0, c_1, \cdots, c_n$ will give an algorithm for finding an extension c_n of c to $G - v$ which is minimal with respect to v. All that is needed is a method for determining m_{i+1} given c_i, $i = 0, 1, \cdots, n-1$.

Let b be the largest colour represented by c at some vertex $V(S) - \{v\}$. Since $m_1 > m_2 > \cdots > m_n$, $m_1 \in \{n,n+1,\cdots,b+n\}$ and $m_{i+1} \in \{n-i,n-i+1,\cdots,m_i-1\}$, $i = 1, \cdots, n-1$. Therefore, to find m_{i+1} for i in $\{1,\cdots,n-1\}$ (respectively, $i = 0$) we only need to determine for each m in $\{n-i,\cdots,m_i-1\}$ (respectively, $\{n,\cdots,b+n\}$) if there exists an extension $c_{i,m}$ of c_i to $G - v$ such that the vertices in $\{v_{i+1},\cdots,v_n\}$ receive colours in $\{1,\cdots,m\}$. The smallest such m is m_{i+1}.

For each m we try to find $c_{i,m}$ as follows. We recursively define a sequence $f_{m+1}, f_m, \cdots, f_1$ of extensions of c_i. Let $f_{m+1} = c_i$. Given f_j we define f_{j-1} as follows. Let $W_j = V(S) - (\{v\} \cup \operatorname{dom} f_j)$. If $j - 1 \in r(c,u)$, for some $u \in W_j$, then let $f_{j-1} = f_j$. If $j - 1 \notin r(c,u)$, for every $u \in W_j$, then choose a vertex w in W_j for which $r^{j-1}(c,w)$ is maximal and let $f_{j-1} = f_j \cup \{(w,j-1)\}$.

If f_1 is an ordered colouring of $G - v$, then it is an extension $c_{i,m}$ and m is a candidate to be m_{i+1}. But f_{m+1} could be an ordered colouring of a proper subgraph of $G - v$. In this case we would like to be able to conclude that $c_{i,m}$ does not exist and $m \neq m_{i+1}$. To do this we prove that if $c_{i,m}$ exists then f_{m+1} is an ordered colouring of $G - v$.

<u>Theorem 2</u>. If $c_{i,m}$ exists, then for every j in $\{m+1,m,\cdots,1\}$, f_j has an extension g_j to $G - v$ such that $g_j(u) \leq j - 1$ for every $u \in W_j$. If $c_{i,m}$ exists, then f_1 is an ordered colouring of $G - v$.

<u>Proof</u>. We prove the theorem by backwards induction on j. The result holds for $m + 1$, since $c_{i,m}$ is such an extension g_{m+1}. Suppose the result holds for f_j, where $m + 1 \geq j > 1$. We will find g_{j-1} using g_j.

Suppose $f_{j-1} = f_j$. Then g_j can not assign $j - 1$ to any vertex in W_j. In this case we let $g_{j-1} = g_j$.

Suppose $f_{j-1} = f_j \cup \{(w,j-1)\}$. If no vertex in $W_j - \{w\}$ is given colour $j - 1$ by

g_j, then define g_{j-1} by

$$g_{j-1}(x) = \begin{cases} j-1 & , \text{ if } x = w \\ g_j(x), & \text{otherwise.} \end{cases}$$

Suppose $g_j(z) = j-1$, for some $z \in W_j - \{w\}$. Then $g_j(w) = a$, for some $a \in \{1,\cdots,j-2\}$. By the definition of w, $r^{j-1}(c,z) \leq r^{j-1}(c,w)$. If $r_a^{j-1}(c,z) = r_a^{j-1}(c,w)$, then define g_{j-1} by

$$g_{j-1}(x) = \begin{cases} j-1 & , \text{ if } x = w \\ a & , \text{ if } x = z \\ g_j(x), & \text{otherwise.} \end{cases}$$

If $r_a^{j-1}(c,z) < r_a^{j-1}(c,w)$, then there exists a' in $\{a+1,\cdots,j-2\}$ such that $r_{a'+1}^{j-1}(c,z) = r_{a'+1}^{j-1}(c,w)$ and $a' \in r_a^{j-1}(c,w) - r_a^{j-1}(c,z)$. We then define g_{j-1} by

$$g_{j-1}(x) = \begin{cases} j-1 & , \text{ if } x = w \\ a' & , \text{ if } x = z \\ g_j(x), & \text{otherwise.} \end{cases}$$

In particular, we have shown that f_1 has an extension g_1 such that $g_1(w) \leq 0$, for every w in W_1, that is, $W_1 = \phi$ and g_1 is an ordered colouring of $G - v$. ■

The Algorithm

We now give an algorithm which finds a minimal ordered colouring of a graph G in $\mathcal{G}$. Suppose G is the union of m distinct maximal cliques. The ordered colouring of G

is accomplished in m stages.

Before the *(i+1)*th stage, where $0 \leq i \leq m-1$, we will have an ordered colouring f_i of an induced subgraph H_i of G and $G_i = G - V(H_i)$ will be in $\mathscr{L}$ Each component H of H_i will be adjacent to a unique vertex v in $V(G_i)$ and the restriction of f_i to H will be minimal with respect to v.

In the *(i+1)*th stage we find a leaf clique S_{i+1} of G_i and use the extension procedure to extend f_i to an ordered colouring f_{i+1} of $H_{i+1} = H_i + (S_{i+1} - v_{i+1})$, where v_{i+1} is the attachment vertex of S_{i+1} in G_i. The restriction of f_{i+1} to the component of H_{i+1} with the clique $S_{i+1} - v_{i+1}$ will be minimal with respect to v_{i+1}. Finally, in the *(i+1)*th step we determine $r(v_{i+1}, f_{i+1})$. If we let $U = V(S_{i+1}) - \{v_{i+1}\}$, then $r(v_{i+1}, f_{i+1}) = \{c(u) \mid u \in U\} \cup \bigcup_{u \in U} \{a \mid a \in r(u, f_i) \text{ and } c(u) < a\}$.

If we add a new vertex v_m to G which is adjacent to the vertices of S_m, then the final ordered colouring f_m will be minimal with respect to v_m. Hence, f_m will be a minimal colouring of G.

Time Bound

We now prove that the algorithm is polynomial. We first prove that the extension procedure takes $O(\nu^3 n)$ steps.

Suppose we are determining if $c_{i,m}$ exists. This involves finding the sequence $f_{m+1}, f_m, \cdots, f_1$. Given f_j, we can determine f_{j-1} in at most $O([j-1][n-i])$ steps. Hence, finding f_1 can be done in $O\left[\sum_{j=1}^{m} j[n-i]\right] \leq O(m^2 n)$ steps.

In determining m_{i+1} we will check if $c_{i,m}$ exists for m in $\{m_i - 2, m_i - 3, \cdots, m_{i+1} - 1\}$. (We need not check if $c_{i,m_i - 1}$ exists since c_{i-1,m_i} is such a colouring.) Thus, m_{i+1} can be determined in $O\left[\left[(m_{i+1}-1)^2 + \cdots + (m_i - 2)^2\right] n\right]$ steps. Therefore,

c_n can be found in $O\left[\sum_{\ell=1}^{m_1} \ell^2 n\right] = O(m_1^3 n) \le O(\nu^3 n)$ steps.

After the extension procedure, we compute $r(v,c_n)$. This can be done in $O(\nu n)$ steps, which is dominated by the number of steps in the extension procedure.

Summing over all the extension procedures, we see that the algorithm can be done in $O(\nu^4)$ steps.

A Larger Class of Graphs

Let $\mathcal{G}$ be the class of all connected graphs satisfying conditions α, β, and γ of the introduction. In such graphs a vertex can be in the vertex sets of more than two maximal cliques. We will show how to modify the algorithm to find minimal ordered colourings of graphs in $\mathcal{G}$. First we need an extension procedure.

Consider a graph G constructed as follows. Let S be the complete graph on $\{v,u_1,\cdots,u_n\}$. Let $H_1, \cdots, H_n$ be disjoint graphs such that $V(H_i) \cap V(S) = \{u_i\}$, $i = 1, \cdots, n$. Let $G = S \cup \bigcup_{i=1}^{n} H_i$. Let $H_{i_1}, \cdots, H_{i_{\omega_i}}$ be the components of $H_i - u_i$, $i = 1, \cdots, n$.

Let c be an ordered colouring of $G - V(S)$ such that the restriction of c to H_{i_j} is minimal with respect to u_i, $j = 1, \cdots, \omega_i$, $i = 1, \cdots, n$. As in the proof of Lemma 1, we can show that c has an extension to $G - v$ which is minimal with respect to v. But we can not use the extension procedure directly to find d. This is due to the following reason: if $a \notin r(c,u_i)$, for some i in $\{1,\cdots,n\}$, then we can not necessarily assign a to u_i in any extension of c. Indeed, if $a < a'$ and $a' \in r(c_{H_{i_j}},u_i) \cap (c_{H_{i_k}},u_i)$, where j and k are distinct integers in $\{1,\cdots,\omega_i\}$, then u_i can not be coloured a in any extension of c. We must first modify the problem in order to use the extension procedure to find d.

Define $R(c,u_i)$ to be the set of colours that cannot be assigned to u_i in any extension of c, $i = 1, \cdots, n$. Equivalently, $R(c,u_i)$ is the union of $r(c,u_i)$ and the set of all colours a for which there exists a colour a' and there exist distinct j and k in $\{1,\cdots,\omega_i\}$ such that $a < a'$ and $a' \in r(c_{H_{ij}},u_i) \cap r(c_{H_{ik}},u_i)$.

Let G' be a graph constructed as follows. Let $H_1', \cdots, H_n'$ be disjoint graphs such that $V(H_i') \cap V(S) = \{u_i\}$, $H_i' - u_i$ is connected, and $H_i' - u_i$ has an ordered colouring f_i for which $r(f_i,u_i) = R(c,u_i)$, $i = 1, \cdots, n$. Let $G' = S \in \cup \bigcup_{i=1}^{n} H_i'$ and $c' = \bigcup_{i=1}^{n} f_i$. We will show how to use G', c', and the extension procedure to find an extension d of c to $G - v$ which is minimal with respect to v. First we need a lemma.

<u>Lemma 2</u>. Let $H_0, H_1, \cdots, H_\omega$ be graphs such that $V(H_j) \cap V(H_k) = \{u\}$, $1 \leq j < k \leq \omega$. Let $G = \bigcup_{j=0}^{\omega} H_j$ and $H = \bigcup_{j=1}^{\omega} H_j - u$. Let c be an ordered colouring of H. Let H' be a graph such that $V(H') \cap V(H_0) = \{u\}$, $H' - u$ is connected, and $H' - u$ has an ordered colouring d such that $r(d,u) = R(c,x)$. Let $G' = H' \cup H_0$. Let $v \in V(H_0) - \{x\}$ and let g be an ordered colouring of $H_0 - v$. Then $c \cup g$ is an ordered colouring of $G - v$ if and only if $d \cup g$ is an ordered colouring of $G' - v$. If $c \cup g$ and $d \cup g$ are ordered colourings, then $r(c \cup g, v) = r(d \cup g, v)$.

<u>Proof</u>. Suppose $d \cup g$ is not an ordered colouring. Then there exists x in $V(H)$ and y in $V(H_0) - \{v\}$, and an (x,y)–path P such that $d(x) = a = g(y)$ and all internal vertices have colours less than a. Then $a \in r(d,x) = R(c,x)$. If $a \in r(c,x)$, then $c \cup g$ is not an ordered colouring. If $a \in R(c,x) - r(c,x)$, then there exists a colour a' and there exist distinct j and k in $\{1,\cdots,\omega\}$ such that $a < a'$ and $a' \in r(c_{H_j},u) \cap r(c_{H_k},u)$. Since $u \in V(P)$, $g(u) < a < a'$. Hence, $c \cup g$ is not an ordered colouring.

It is even easier to show that if $c \cup g$ is not an ordered colouring, then $d \cup g$ is not an ordered colouring.

Suppose $c \cup g$ and $d \cup g$ are ordered colourings. Since $r(c,u) \subseteq R(c,u) = r(d,u)$ it follows easily that $r(c \cup g, v) \subseteq r(d \cup g, v)$. Suppose $a \in r(d \cup g, v) - r(c \cup g, v)$. Then $a \notin r(g,v)$ and $a \in r(d,u) - r(c,u) = R(c,u) - r(c,u)$. Then there exists a colour a' and there exist distinct j and k in $\{1,\cdots,\omega\}$ such that $a < a'$ and $a' \in r(c_{H_j},u) \cap r(c_{H_k},u)$. Thus, $a < a' < g(u)$ because $c \cup g$ is an ordered colouring. But now $a \notin r(d \cup g, v)$ and we have a contradiction. Therefore, $r(c \cup g, v) = r(d \cup g, v)$. ∎

Let g be an ordered colouring of $S - v$. Using Lemma 2 and induction we can show that $c \cup g$ is an ordered colouring of $G - v$ if and only if $c' \cup g$ is an ordered colouring of $G' - v$. Furthermore, we can show that if $c \cup g$ and $c' \cup g$ are ordered colourings, then $r_G(c \cup g, v) = r_{G'}(c' \cup g, v)$. Thus, $c' \cup g$ is minimal (among extensions of c') with respect to v if and only if $c \cup g$ is minimal (among extensions of c) with respect to v. But we know that c has an extension to $G - v$ which is minimal (among all ordered colourings of $G - v$) with respect to v. So we can use the extension procedure to find an extension d' of c' to $G' - v$ which is minimal with respect to v, and then $(d' - c') \cup c$ is an ordered colouring of $G - v$ which is minimal with respect to v. (Note that we do not need to construct $H_1', \cdots, H_n'$; they are only used for exposition.)

We now have an extension procedure for graphs in $\mathscr{G}$. It is easy to show that every graph in $\mathscr{G}$ has a leaf clique. It is now clear that we can design an algorithm for finding minimal ordered colourings of graphs in $\mathscr{G}$ which is analogous to the algorithm for graphs in $\mathscr{L}$.

References

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