FURTHER NOTES ON THE ASSOCIATED ALPHA-FUNCTIONS AND RELATED INTEGRALS ARISING IN THE THEORY OF THE LIGHT CHANGES OF ECLIPSING VARIABLES

by

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FURTHER NOTES ON THE ASSOCIATED ALPHA-FUNCTIONS AND RELATED INTEGRALS ARISING IN THE THEORY OF THE LIGHT CHANGES OF ECLIPSING VARIABLES

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Abstract. With a view to furthering the theory of the light changes of eclipsing variables, developed rather systematically by Z. Kopal, this paper presents a number of new (and computable) expressions for the associated alpha-function $\alpha^0_{n,\lambda}(r_1,r_2,\delta)$ (and also for its partial derivatives), where $\alpha^0_{n,\lambda}(r_1,r_2,\delta)$ represents the fractional loss of light suffered by an eclipse of a circular disc of fractional radius r_1 (and darkened at the limb to the nth degree) by an opaque disc of radius r_2, with their centres separated by a fractional (projected) distance δ, provided that the transparency of the occulting disc increases with the angle of foreshortening in the same manner as the limb-darkening of the eclipsed star (that is, when the transparency function $g(\rho,\zeta)$ of the second aperture is given by Equation (4) below). Many of the explicit expressions derived here are valid for any type of eclipse, occultation or transit, regardless of whether $r_1 > r_2$ or $r_1 < r_2$, and for any degree n of the adopted law of limb-darkening. It is also pointed out how some of the results obtained in this paper are related to the various representations given earlier in the literature

for the case $\lambda = 0$.

1. Introduction

The associated alpha-function $\alpha_n^0(r_1, r_2, \delta)$ of order n represents the fractional loss of light suffered by an eclipse of a circular disc of fractional radius r_1 (and darkened at the limb to the *n*th degree) by an opaque disc of radius r_2, with their centres separated by a fractional (projected) distance δ. In fact, in terms of the Bessel function $J_\mu(x)$ defined by (*cf.*, *e.g.*, Watson, 1944)

$$J_\mu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \; (\tfrac{1}{2}z)^{\mu+2m}}{m! \; \Gamma(\mu+m+1)} \qquad (|z| < \infty), \tag{1}$$

it is fairly well known that (Kopal, 1977; see also Sections I.3 and III.3 of Kopal, 1979)

$$\alpha_n^0(r_1, r_2, \delta) = 2^\nu \, \Gamma(\nu) \int_0^\infty (kx)^{-\nu} \, J_\nu(kx) \; J_1(x) \; J_0(hx) \; dx, \tag{2}$$

where, for convenience,

$$\nu = \frac{n+2}{2}, \qquad h = \frac{\delta}{r_2}, \qquad \text{and} \qquad k = \frac{r_1}{r_2}. \tag{3}$$

More generally, if the transparency of the occulting disc increases with the angle of foreshortening in the same manner as the limb-darkening of the eclipsed star, that is, if the transparency function $g(\rho, \zeta)$ of the second aperture is given by (*cf*. Kopal, 1977, p. 232, Equation (3.36))

$$g(\rho,\zeta) = g_\lambda(\rho,\zeta) = \begin{cases} [1-(\rho/r_1)^2]^\lambda & (\rho \leqq r_2), \\ 0 & (\rho > r_2), \end{cases} \tag{4}$$

in place of

$$g(\rho,\zeta) = g_0(\rho,\zeta) = \begin{cases} 1 & (\rho \leqq r_2), \\ 0 & (\rho > r_2), \end{cases} \tag{5}$$

then Equation (2) is easily replaced by (*cf*. Kopal, 1979, p. 34, Equation (3.38))

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta) = 2^{\nu+\lambda}\,\Gamma(\nu)\Gamma(\lambda+1)\,k^{-\nu} \cdot \int_0^\infty x^{-\nu-\lambda}\,J_\nu(kx)\,J_{\lambda+1}(x)\,J_0(hx)\,dx. \tag{6}$$

Clearly,

$$\alpha^0_n(r_1,r_2,\delta) = \alpha^0_{n,0}(r_1,r_2,\delta), \tag{7}$$

which follows also in view of Equation (5).

Motivated by the continuing importance of these associated alpha-functions in, for example, an interpretation of the observed light changes of eclipsing variables, Kopal (1983) gave a number of new and simple expressions for the partial derivatives[1]

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[1] A couple of misprints in Kopal (1983, p. 445, Equations (3) and (4)) have been corrected here.

$$r_2 \frac{\partial \alpha_n^0}{\partial r_2} = 2^\nu \, \Gamma(\nu) \, k^{-\nu} \int_0^\infty x^{1-\nu} \, J_\nu(kx) \, J_0(hx) \, J_0(x) \, dx \tag{8}$$

and

$$\delta \frac{\partial \alpha_n^0}{\partial \delta} = -2^\nu \, \Gamma(\nu) \, hk^{-\nu} \int_0^\infty x^{1-\nu} \, J_\nu(kx) \, J_1(hx) \, J_1(x) \, dx, \tag{9}$$

where, and throughout this investigation,

$$\alpha_n^0 \equiv \alpha_n^0(r_1, r_2, \delta). \tag{10}$$

The object of this paper is to present a systematic study of the relatively more general alpha-function defined by Equation (6). In view of the relationship (7), each of our results involving $\alpha_{n,\lambda}^0(r_1, r_2, \delta)$ would yield, upon setting $\lambda = 0$, the corresponding result for the associated alpha-function $\alpha_n^0(r_1, r_2, \delta)$.

2. Identification with the Appell Function F_4 and the Lauricella function $F_C^{(3)}$

We begin by recalling the derivative formulas (*cf*. Watson, 1944, p. 45)

$$\frac{d}{dz} \left\{ z^\mu \, J_\mu(z) \right\} = z^\mu \, J_{\mu-1}(z) \tag{11}$$

and

$$\frac{d}{dz} \left\{ z^{-\mu} \, J_\mu(z) \right\} = -z^{-\mu} \, J_{\mu+1}(z) \tag{12}$$

satisfied by the Bessel function $J_\mu(z)$ defined by Equation (1). Making use of these formulas, it is readily seen from the definitions (3) and (6) that

$$r_1 \frac{\partial \alpha^0_{n,\lambda}}{\partial r_1} = -2^{\nu+\lambda}\,\Gamma(\nu)\Gamma(\lambda+1)\,k^{1-\nu} \cdot \int_0^\infty x^{1-\nu-\lambda}\,J_{\nu+1}(kx)\,J_{\lambda+1}(x)\,J_0(hx)\,dx, \tag{13}$$

$$r_2 \frac{\partial \alpha^0_{n,\lambda}}{\partial r_2} = 2^{\nu+\lambda}\,\Gamma(\nu)\Gamma(\lambda+1)\,k^{-\nu} \cdot \left\{\int_0^\infty x^{1-\nu-\lambda}\,J_\nu(kx)\,J_\lambda(x)\,J_0(hx)\,dx - 2\lambda \int_0^\infty x^{-\nu-\lambda}\,J_\nu(kx)\,J_{\lambda+1}(x)\,J_0(hx)\,dx\right\}, \tag{14}$$

and

$$\delta \frac{\partial \alpha^0_{n,\lambda}}{\partial \delta} = -\,2^{\nu+\lambda}\,\Gamma(\nu)\Gamma(\lambda+1)\,h\,k^{1-\nu} \cdot \int_0^\infty x^{1-\nu-\lambda}\,J_\nu(kx)\,J_{\lambda+1}(x)\,J_1(hx)\,dx, \tag{15}$$

where, and in what follows,

$$\alpha^0_{n,\lambda} \equiv \alpha^0_{n,\lambda}(r_1, r_2, \delta). \tag{16}$$

Since

$$\alpha^0_{n,\lambda}(r_1\tau, r_2\tau, \delta\tau) = \tau^0\, \alpha^0_{n,\lambda}(r_1, r_2, \delta), \tag{17}$$

which results immediately from the definitions (3) and (6), by appealing to Euler's theorem for homogeneous functions, we have

$$r_1 \frac{\partial\alpha^0_{n,\lambda}}{\partial r_1} + r_2 \frac{\partial\alpha^0_{n,\lambda}}{\partial r_2} + \delta \frac{\partial\alpha^0_{n,\lambda}}{\partial\delta} = 0. \tag{18}$$

Equation (18) would enable us to express any one of the three partial derivatives occurring in (13), (14), and (15) in terms of the remaining two. Thus, in particular, Equation (8) and (9) are complemented by

$$r_1 \frac{\partial\alpha^0_n}{\partial r_1} = -r_2 \frac{\partial\alpha^0_n}{\partial r_2} - \delta \frac{\partial\alpha^0_n}{\partial\delta} \tag{19}$$

or, equivalently, by

$$r_1 \frac{\partial\alpha^0_n}{\partial r_1} = -2^\nu\, \Gamma(\nu) k^{1-\nu} \int_0^\infty x^{1-\nu}\, J_{\nu+1}(kx)\, J_0(hx)\, J_1(x)\, dx, \tag{20}$$

which follows also from (13) in the special case $\lambda = 0$.

The assertion (18), as also its special case $\lambda = 0$ given by (19), can indeed be verified directly by substituting from (13), (14), and (15), or from (8), (9), and (20), and making use of the derivative formulas (11) and (12), and the recurrence relations (*cf.* Watson, 1944, p. 45):

$$J_{\mu-1}(z) + J_{\mu+1}(z) = \frac{2\mu}{z} J_\mu(z) \tag{21}$$

and

$$J_{\mu-1}(z) - J_{\mu+1}(z) = 2\,\frac{d}{dz}\left\{J_{\mu}(z)\right\}, \tag{22}$$

which incidentally are implied by (11) and (12).

The associated alpha–function $\alpha_n^0(r_1,r_2,\delta)$, and each of its partial derivatives considered here, are expressed in terms of infinite integrals[2] involving certain products of three Bessel functions of different orders. A general class of such infinite integrals, involving the product of N Bessel functions of different orders, was considered by Srivastava and Exton (1979) who gave the formula (*cf.* also Srivastava and Karlsson, 1985, p. 50, Equation 1.7 (12)):

$$\int_0^\infty t^{\rho-1} \prod_{j=1}^{N} \left\{J_{\mu_j}(x_j t)\right\} dt$$

$$= \frac{2^{\rho-1}\, x_1^{\mu_1} \cdots x_{N-1}^{\mu_{N-1}}\, x_N^{\mu_N - M}\, \Gamma(\tfrac{1}{2}M)}{\Gamma(\mu_1+1) \cdots \Gamma(\mu_{N-1}+1)\, \Gamma(\mu_N - \tfrac{1}{2}M+1)}$$

$$\cdot\, F_C^{(N-1)}\left[\frac{1}{2}M,\, \frac{1}{2}M-\mu_N;\, \mu_1+1, \cdots, \mu_{N-1}+1;\, \frac{x_1^2}{x_N^2}, \cdots, \frac{x_{N-1}^2}{x_N^2}\right], \tag{23}$$

where $x_1, \cdots, x_N$ are positive real numbers such that

$$x_N > x_1 + \cdots + x_{N-1} \qquad (N = 2,3,4,\cdots), \tag{24}$$

$$M = \rho + \mu_1 + \cdots + \mu_N\,, \tag{25}$$

---

[2] Each of these infinite integrals can easily be interpreted as the Hankel transform of the product of two Bessel functions.

$$\mathrm{Re}(1 + \mu_1 + \cdots + \mu_N) > \mathrm{Re}(1-\rho) > -\frac{1}{2} N, \tag{26}$$

and $F_C^{(N)}$ denotes one of Lauricella's hypergeometric functions of N variables, defined by (*cf.* Lauricella, 1893; see also Srivastava and Karlsson, 1985, p. 33)

$$F_C^{(N)}[a,b; c_1,\cdots,c_N; z_1,\cdots,z_N]$$

$$= \sum_{m_1,\cdots,m_N=0}^{\infty} \frac{(a)_{m_1+\cdots+m_N}\,(b)_{m_1+\cdots+m_N}}{(c_1)_{m_1}\cdots(c_N)_{m_N}} \frac{z_1^{m_1}}{m_1!}\cdots\frac{z_N^{m_N}}{m_N!} \tag{27}$$

$$(\sqrt{|z_1|} + \cdots + \sqrt{|z_N|} < 1),$$

with, as usual, the Pochhammer symbol $(\lambda)_m$ given by

$$(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } m = 0, \\ \lambda(\lambda+1)\cdots(\lambda+m-1), & \forall\ m \in \{1,2,3,\cdots\}. \end{cases} \tag{28}$$

For $N = 2$, Formula (23) evidently reduces to the familiar (discontinuous) integral of Weber and Schafheitlin (*cf.*, *e.g.*, Watson, 1944, p. 398 *et seq.*), whose analytic expression is different, according as x_1 is smaller than, equal to, or larger than x_2. Of our immediate interest here is the special case of Formula (23) when $N = 3$, which was proven *independently* by Bailey (1936, p. 45, Equation (7.1)) and Rice (1935, p. 60, Equation (2.6)), using markedly *different* methods. Since $F_C^{(2)} = F_4$, where F_4 denotes the fourth type of Appell's double hypergeometric functions, defined by (*cf.* Appell, 1880; see also Appell *et* Kampé de Fériet, 1926, p. 14)

$$F_4[a,b;\, c,\, c';\, x,\, y]$$

$$= \sum_{\ell,m=0}^{\infty} \frac{(a)_{\ell+m}\,(b)_{\ell+m}}{(c)_\ell\,(c')_m} \frac{x^\ell}{\ell!} \frac{y^m}{m!} \tag{29}$$

$$(\sqrt{|x|} + \sqrt{|y|} < 1)\ ,$$

we can rewrite this last special case of Formula (23) in the (convenient) form (see also Erdélyi *et al.*, 1953b, p. 93, Equations 7.14.2 (40) and 7.14.2 (41)):

$$\int_0^\infty x^{\rho-1}\, J_\kappa(ax)\, J_\mu(bx)\, J_\sigma(cx)\, dx$$

$$= \frac{2^{\rho-1}\, a^\kappa\, b^\mu\, \Gamma\{\tfrac{1}{2}(\rho+\sigma+\kappa+\mu)\}}{c^{\rho+\kappa+\mu}\, \Gamma(\kappa+1)\, \Gamma(\mu+1)\, \Gamma\{1-\tfrac{1}{2}(\rho-\sigma+\kappa+\mu)\}}$$

$$\cdot\, F_4\left[\frac{1}{2}(\rho+\sigma+\kappa+\mu),\, \frac{1}{2}(\rho-\sigma+\kappa+\mu);\, \kappa+1,\, \mu+1;\, \frac{a^2}{c^2},\, \frac{b^2}{c^2}\right], \tag{30}$$

where

$$\min\{a,b,c\} > 0;\quad c > a + b;\quad -\mathrm{Re}(\sigma+\kappa+\mu) < \mathrm{Re}(\rho) < \tfrac{5}{2}\ . \tag{31}$$

We shall now apply Formula (30) to find closed-form representations for the associated alpha-function $\alpha^0_{n,\lambda}(r_1,r_2,\delta)$, given by Equation (6), under the following cases.

Case 1 (Outside Eclipses). In the case of outside eclipses, the parameters r_1, r_2, and δ are constrained by

$$\delta \geqq r_1 + r_2 \tag{32}$$

or, equivalently, by [*cf.* Equation (3)]

$$h \geqq k + 1, \tag{33}$$

where the equality sign corresponds to the moment of first contact. In this case, we can set

$$\rho = 1 - \nu - \lambda, \quad \kappa = \nu = \frac{n + 2}{2}, \quad \mu = \lambda + 1, \quad \text{and} \quad \sigma = 0, \tag{34}$$

so that

$$\rho + \sigma + \kappa + \mu = 2 > 0 \tag{35}$$

and

$$\rho = 1 - \nu - \lambda = 1 - \frac{n + 2}{2} - \lambda = -\lambda - \frac{1}{2} n < \frac{5}{2}. \tag{36}$$

Letting

$$a = k = \frac{r_1}{r_2}, \quad b = 1, \quad \text{and} \quad c = h = \frac{\delta}{r_2}, \tag{37}$$

we see that the Appell series F_4 occurring on the right-hand side of Formula (30) converges under the condition (32) or (33). However, in view of the choice (34),

$$\Gamma\left\{1-\frac{1}{2}\left(\rho-\sigma+\kappa+\mu\right)\right\}=\Gamma(0)\longrightarrow\infty, \tag{38}$$

and Formula (30) immediately yields

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta)\bigg|_{\delta\geqq r_1+r_2}=0. \tag{39}$$

This is not really an unexpected result; in fact, it proves that the definition of $\alpha^0_{n,\lambda}(r_1,r_2,\delta)$ in the form (6) remains valid outside minima as well.

Case 2 (Total Eclipses). The total eclipses occur whenever

$$r_2\geqq\delta+r_1 \tag{40}$$

or, in view of Equation (3),

$$1\geqq h+k. \tag{41}$$

In this case, the infinite integral in (6) can be evaluated by means of Formula (30) with

$$a=h=\frac{\delta}{r_2}\,,\quad b=k=\frac{r_1}{r_2}\,,\quad \text{and}\quad c=1, \tag{42}$$

and

$$\rho=1-\nu-\lambda,\ \ \kappa=0,\ \ \mu=\nu,\ \ \text{and}\ \ \sigma=\lambda+1. \tag{43}$$

We thus find that

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta)\Big|_{r_2 \geqq \delta+r_1}$$

$$= \frac{2}{n+2}\, F_4\left[1, -\lambda;\ 1, \frac{n+4}{2};\ \frac{\delta^2}{r_2^2}, \frac{r_1^2}{r_2^2}\right], \tag{44}$$

which, for $\lambda = 0$, immediately yields (*cf*. Kopal, 1979, p. 38, Equation (3.73))

$$\alpha^0_n(r_1,r_2,\delta)\Big|_{r_2 \geqq \delta+r_1} = \frac{2}{n+2}, \tag{45}$$

in complete agreement with the expected results.

Case 3 (Annular Eclipses). For annular eclipses, the parameters r_1, r_2, and δ are constrained by

$$r_1 \geqq \delta + r_2 \tag{46}$$

or, equivalently, by [*cf*. Equation (3)]

$$k \geqq h + 1. \tag{47}$$

In order to evaluate the infinite integral occurring on the right-hand side of Equation (6), under the constraint (47), we apply Formula (30) with

$$a = 1, \quad b = h = \frac{\delta}{r_2}, \quad \text{and} \quad c = k = \frac{r_1}{r_2}, \tag{48}$$

and

$$\rho = 1 - \nu - \lambda, \quad \kappa = \lambda + 1, \quad \mu = 0, \quad \text{and} \quad \sigma = \nu = \frac{n+2}{2}. \tag{49}$$

We thus obtain the following explicit representation for the associated alpha-function $\alpha^0_{n,\lambda}(r_1,r_2,\delta)$ in the case of annular eclipses:

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta)\Big|_{r_1 \geqq \delta + r_2}$$

$$= \frac{r_2^2}{(\lambda+1)r_1^2} F_4\left[1, -\frac{1}{2}n;\ \lambda+2,\ 1;\ \frac{r_2^2}{r_1^2}, \frac{\delta^2}{r_1^2}\right]. \tag{50}$$

Some simple consequences of Equation (50) are worthy of mention. First of all, for uniformly bright discs, we can set $\nu = 1$ (*i.e.*, $n = 0$), and (50) reduces at once to

$$\alpha^0_{0,\lambda}(r_1,r_2,\delta)\Big|_{r_1 \geqq \delta + r_2} = \frac{r_2^2}{(\lambda+1)r_1^2}. \tag{51}$$

On the other hand, the Appell function F_4 occurring on the right-hand side of Equation (50) would reduce to a polynomial when n is an even positive integer, but will remain an infinite (double) series when n is an odd positive integer. Thirdly, corresponding to the moment of central eclipse, we have $\delta = 0$, and Equation (50) immediately yields

$$\alpha^0_{n,\lambda}(r_1,r_2,0) = \frac{r_2^2}{(\lambda+1)r_1^2}\ {}_2F_1\left[\begin{matrix}1, -\frac{1}{2}n; & \frac{r_2^2}{r_1^2} \\ \lambda+2; & \end{matrix}\right] \quad (r_1 \geqq r_2), \tag{52}$$

where ${}_2F_1$ denotes the Gaussian hypergeometric function defined, in the special case when $p = 2$ and $q = 1$, by the generalized hypergeometric series (*cf.*, *e.g.*, Bailey, 1935a; see also Erdélyi *et al.*, 1953a, and Slater, 1966):

$$
\begin{aligned}
{}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) \\
&= {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p; \\ \\ b_1, \cdots, b_q; \end{matrix} \quad z\right] \\
&= \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{z^m}{m!} \qquad (53)
\end{aligned}
$$

$$
\left(p, q = 0, 1, 2, \cdots; \;\; |z| < \infty \text{ when } p \leqq q; \;\; |z| < 1 \text{ when } p = q + 1; \right.
$$

$$
\left. |z| = 1 \text{ when } p = q + 1, \text{ provided } \mathrm{Re}\left[\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right] > 0\right),
$$

it being understood that no zeros appear in the denominator of (53). Lastly, at the moment of internal tangency (when $\delta = r_1 - r_2$), Equation (50) assumes the form:

$$
\begin{aligned}
\alpha^0_{n,\lambda}(r_1, r_2, r_1 - r_2) \\
&= \frac{r_2^2}{(\lambda+1) r_1^2} F_4\left[1, -\frac{1}{2}n; \lambda+2, 1; \frac{r_2^2}{r_1^2}, \left(1 - \frac{r_2}{r_1}\right)^2\right]. \qquad (54)
\end{aligned}
$$

Case 4 (Partial Eclipses). In the case of a partial eclipse, none of the three parameters r_1, r_2, and δ can be equal to, or larger than, the sum of the remaining two parameters. Consequently, the condition corresponding to the second one in (31) cannot be satisfied, and Formula (30) can no longer be applied to evaluate the infinite integral occurring on the right-hand side of Equation (6). With a view to evaluating the integral on the right-hand side of Equation (6) under these conditions, we shall make use of the known expansion[3] (Srivastava, 1966, p. 150, Equation (5.1))

$$(\tfrac{1}{2}z)^{\rho-\mu_1-\cdots-\mu_N} \prod_{j=1}^{N} \left\{J_{\mu_j}(a_j z)\right\}$$

$$= \frac{a_1^{\mu_1} \cdots a_N^{\mu_N}}{\Gamma(\mu_1+1)\cdots\Gamma(\mu_N+1)} \sum_{m=0}^{\infty} \frac{(\rho+2m)\Gamma(\rho+m)}{m!} J_{\rho+2m}(z)$$

$$\cdot F_C^{(N)}\left[-m, \rho+m; \mu_1+1, \cdots, \mu_N+1; a_1^2, \cdots, a_N^2\right]. \qquad (55)$$

Since $F_C^{(2)} = F_4$, as we observed above, a special case of Formula (55) when $N = 2$ may be rewritten in the form:

$$(\tfrac{1}{2}z)^{\rho-\kappa-\mu} J_\kappa(az)\, J_\mu(bz) = \frac{a^\kappa\, b^\mu}{\Gamma(\kappa+1)\Gamma(\mu+1)} \sum_{m=0}^{\infty} \frac{(\rho+2m)\Gamma(\rho+m)}{m!}$$

$$\cdot J_{\rho+2m}(z)\; F_4\left[-m,\ \rho+m;\ \kappa+1,\ \mu+1;\ a^2,\ b^2\right], \qquad (56)$$

which is due to Bailey (1935b, p. 235, Equation (3.1)).

There are two ways in which these expansion formulas can be applied to evaluate

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[3] Much more general polynomial expansions have since been given for multivariable functions defined by multiple series with essentially arbitrary terms (see, *e.g.*, Srivastava, 1987, Section 2; *cf.* especially p. 851, Equation (21)).

the infinite integral occurring on the right–hand side of Equation (6). Firstly, if we apply (55) with

$$N = 3, \quad \mu_1 = \nu = \frac{n+2}{2}, \quad \mu_2 = \lambda + 1, \quad \mu_3 = 0, \tag{57}$$

and

$$z = x, \quad a_1 = k = \frac{r_1}{r_2}, \quad a_2 = 1, \quad \text{and} \quad a_3 = h = \frac{\delta}{r_2}, \tag{58}$$

we find from (6) that

$$\alpha^0_{n,\lambda}(r_1, r_2, \delta) = \frac{2^\rho}{(n+2)(\lambda+1)} \sum_{m=0}^{\infty} \frac{(\rho+2m)\Gamma(\rho+m)}{m!}$$

$$\cdot F_C^{(3)}\left[-m, \rho+m; \frac{n+4}{2}, \lambda+2, 1; \frac{r_1^2}{r_2^2}, 1, \frac{\delta^2}{r_2^2}\right]$$

$$\cdot \int_0^\infty x^{1-\rho} J_{\rho+2m}(x)\, dx \quad \left[\mathrm{Re}(\rho) > \tfrac{1}{2}\right]. \tag{59}$$

Upon evaluating this simpler integral by appealing to the known result (Erdélyi *et al.*, 1953b, p. 49, Equation 7.7.3 (19)):

$$\int_0^\infty x^{\sigma-1} J_\mu(ax)\,dx = \frac{2^{\sigma-1}\,\Gamma\{\frac{1}{2}(\mu+\sigma)\}}{a^\sigma\,\Gamma\{1+\frac{1}{2}(\mu-\sigma)\}} \tag{60}$$

$$\left[-\mathrm{Re}(\mu) < \mathrm{Re}(\sigma) < \frac{3}{2};\ a > 0\right],$$

which incidentally is an obvious[4] special case of the Srivastava–Exton theorem (23 when $N = 1$, $\rho = \sigma$, and $\mu_1 = \mu$, we finally obtain the following class of representations for $\alpha^0_{n,\lambda}$:

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta) = \frac{2}{(n+2)(\lambda+1)} \sum_{m=0}^{\infty} (\rho+2m) \cdot F_C^{(3)}\left[-m,\ \rho+m;\ \frac{n+4}{2},\ \lambda+2,\ 1;\ \frac{r_1^2}{r_2^2},\ 1,\ \frac{\delta^2}{r_2^2}\right], \tag{61}$$

where $\mathrm{Re}(\rho) > \frac{1}{2}$.

Alternatively, we can apply Bailey's expansion (56) with

$$\kappa = \nu = \frac{n+2}{2},\quad \mu = \lambda + 1,\quad z = x,\quad a = k = \frac{r_1}{r_2},\quad \text{and}\quad b = 1, \tag{62}$$

so that Equation (6) assumes the form:

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta) = \frac{2^\rho}{(n+2)(\lambda+1)} \sum_{m=0}^{\infty} \frac{(\rho+2m)\Gamma(\rho+m)}{m!} \cdot F_4\left[-m,\ \rho+m;\ \frac{n+4}{2},\ \lambda+2;\ \frac{r_1^2}{r_2^2},\ 1\right]$$

[4] Here it is understood, as usual, that $F_C^{(0)} = 1$.

$$\cdot \int_0^\infty x^{1-\rho} J_{\rho+2m}(x) J_0\left[\frac{\delta x}{r_2}\right] dx \quad (\mathrm{Re}(\rho) > 0). \tag{63}$$

Now evaluate this infinite integral by making use of the following well-known special case $N = 2$ of the Srivastava–Exton theorem (23) above (*cf.* Erdélyi *et al.* 1953b, p. 51, Equation 7.7.4 (29); 1954, p. 349, Equation 19.3 (1)):

$$\int_0^\infty x^{\sigma-1} J_\kappa(ax) J_\mu(bx)\, dx = \frac{2^{\sigma-1} a^\kappa b^{-\kappa-\sigma} \Gamma\{\frac{1}{2}(\sigma+\kappa+\mu)\}}{\Gamma(\kappa+1)\ \Gamma\{1-\frac{1}{2}(\sigma+\kappa-\mu)\}}$$

$$\cdot {}_2F_1\left[\begin{matrix}\frac{1}{2}(\sigma+\kappa+\mu),\ \frac{1}{2}(\sigma+\kappa-\mu); \\ \kappa+1;\end{matrix}\ \frac{a^2}{b^2}\right] \tag{64}$$

$$(-\mathrm{Re}(\kappa+\mu) < \mathrm{Re}(\sigma) < 2;\ \ 0 < a < b),$$

which indeed provides one form of the Weber–Schafheitlin integral (*cf.* Watson, 1944, p. 398 *et seq.*). Equation (63) thus yields the class of representations for $\alpha^0_{n,\lambda}(r_1,r_2,\delta)$:

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta) = \frac{2}{(n+2)(\lambda+1)} \sum_{m=0}^\infty (\rho+2m)$$

$$\cdot F_4\left[-m,\ \rho+m;\ \frac{n+4}{2},\ \lambda+2;\ \frac{r_1^2}{r_2^2},\ 1\right]$$

$$\cdot {}_2F_1\left[\begin{matrix}m+1,\ 1-\rho-m; \\ 1;\end{matrix}\ \frac{\delta^2}{r_2^2}\right] \quad (\mathrm{Re}(\rho) > 0;\ \ r_2 \geqq \delta). \tag{65}$$

Since, by Euler's transformation (*cf.* Erdélyi *et al.*, 1953a, p. 64, Equation 2.1.4 (23)), we have

$$ {}_2F_1\left[\begin{matrix} m+1,\ 1-\rho-m; \\ 1; \end{matrix}\ \frac{\delta^2}{r_2^2}\right] = \left(1-\frac{\delta^2}{r_2^2}\right)^{\rho-1} {}_2F_1\left[\begin{matrix} -m,\ \rho+m; \\ 1; \end{matrix}\ \frac{\delta^2}{r_2^2}\right], \tag{66} $$

Equation (65) may be rewritten at once in the form:

$$ \alpha^0_{n,\lambda}(r_1,r_2,\delta) = \frac{2}{(n+2)(\lambda+1)}\left(1-\frac{\delta^2}{r_2^2}\right)^{\rho-1} \sum_{m=0}^{\infty} (\rho+2m) \cdot F_4\left[-m,\ \rho+m;\ \frac{n+4}{2},\ \lambda+2;\ \frac{r_1^2}{r_2^2},\ 1\right] \cdot {}_2F_1\left[\begin{matrix} -m,\ \rho+m; \\ 1; \end{matrix}\ \frac{\delta^2}{r_2^2}\right] \quad (\mathrm{Re}(\rho) > 0), \tag{67} $$

in which each of the hypergeometric series is terminating (*i.e.*, finite).

This last representation (67) obviously holds true for every type of eclipse, and for any arbitrary degree n of the adopted law of limb-darkening.

3. Further Representations and Remarks

Our techniques of obtaining the various representations for the associated alpha–function $\alpha^0_{n,\lambda}(r_1,r_2,\delta)$, detailed in the preceding section, can be applied *mutatis mutandis* in order to derive the corresponding representations for the partial derivatives

$$r_1 \frac{\partial\alpha^0_{n,\lambda}}{\partial r_1}, \quad r_2 \frac{\partial\alpha^0_{n,\lambda}}{\partial r_2}, \quad \text{and} \quad \delta \frac{\partial\alpha^0_{n,\lambda}}{\partial \delta},$$

defined by Equations (13), (14), and (15), respectively. For example, in the case of total eclipses ($r_2 \geqq \delta + r_1$ or $1 \geqq h + k$), the Bailey–Rice integral (30) applies readily to (13) and (15), and we thus find that

$$r_1 \frac{\partial\alpha^0_{n,\lambda}}{\partial r_1}\bigg|_{r_2 \geqq \delta+r_1}$$

$$= -\frac{8\lambda}{(n+2)(n+4)} \left[\frac{r_1}{r_2}\right]^2 F_4\left[2,\ 1-\lambda;\ 1,\ \frac{n+6}{2}\ ;\ \frac{\delta^2}{r_2^2},\ \frac{r_1^2}{r_2^2}\right] \tag{68}$$

and

$$\delta \frac{\partial\alpha^0_{n,\lambda}}{\partial \delta}\bigg|_{r_2 \geqq \delta+r_1}$$

$$= -\frac{4\lambda r_1\ \delta^2}{(n+2)r_2^3} F_4\left[2,\ 1-\lambda;\ 2,\ \frac{n+4}{2}\ ;\ \frac{\delta^2}{r_2^2},\ \frac{r_1^2}{r_2^2}\right]. \tag{69}$$

Moreover, in view of (18), we have

$$r_2 \frac{\partial\alpha^0_{n,\lambda}}{\partial r_2}\bigg|_{r_2 \geqq \delta+r_1}$$

$$= -r_1 \left.\frac{\partial \alpha^0_{n,\lambda}}{\partial r_1}\right|_{r_2 \geqq \delta + r_1} - \delta \left.\frac{\partial \alpha^0_{n,\lambda}}{\partial \delta}\right|_{r_2 \geqq \delta + r_1}$$

$$= \frac{4\lambda\, r_1}{(n+2)(n+4) r_2^3} \left\{ 2r_1 r_2 \; F_4\left[2,\, 1-\lambda;\, 1,\, \frac{n+6}{2}\,;\, \frac{\delta^2}{r_2^2}\,,\, \frac{r_1^2}{r_2^2}\right]\right.$$

$$\left. + (n+4)\delta^2 \; F_4\left[2,\, 1-\lambda;\, 2,\, \frac{n+4}{2}\,;\, \frac{\delta^2}{r_2^2}\,,\, \frac{r_1^2}{r_2^2}\right]\right\}. \tag{70}$$

Setting $\lambda = 0$ in Equations (68), (69), and (70), we immediately verify the fact that

$$r_1 \left.\frac{\partial \alpha^0_n}{\partial r_1}\right|_{r_2 \geqq \delta + r_1} = 0\,, \tag{71}$$

$$r_2 \left.\frac{\partial \alpha^0_n}{\partial r_2}\right|_{r_2 \geqq \delta + r_1} = 0\,, \tag{72}$$

and

$$\delta \left.\frac{\partial \alpha^0_n}{\partial \delta}\right|_{r_2 \geqq \delta + r_1} = 0\,, \tag{73}$$

in the case of total eclipses.

There is yet another way of evaluating the infinite integrals representing $\alpha^0_{n,\lambda}(r_1, r_2, \delta)$ and its partial derivatives considered above. This alternative should be made available especially in situations in which the Bailey–Rice integral (30) cannot be applied *directly*. With a view to illustrating this technique, let us recall a well–known special case of Srivastava's expansion (55) when $N = 1$ in the form (*cf*. Watson, 1944, p. 140, Equation 5.21 (3)):

$$(\tfrac{1}{2}z)^{\rho-\mu} J_{\mu}(az) = \frac{a^{\mu}}{\Gamma(\mu+1)} \sum_{m=0}^{\infty} \frac{(\rho+2m)\Gamma(\rho+m)}{m!} \cdot J_{\rho+2m}(z) \; {}_2F_1(-m, \rho+m; \mu+1; a^2) \tag{74}$$

or, equivalently,

$$(\tfrac{1}{2}z)^{\kappa+1} J_{\mu}(az) = a^{\mu} \sum_{m=0}^{\infty} \frac{(\kappa+\mu+2m+1)\Gamma(\kappa+\mu+m+1)}{\Gamma(\mu+m+1)} \cdot J_{\kappa+\mu+2m+1}(z) \; P_m^{(\mu,\kappa)}(1-2a^2), \tag{75}$$

in terms of the Jacobi polynomials (*cf*. Szegö, 1975, p. 62 *et seq*.)

$$P_m^{(\xi,\eta)}(x) = \sum_{j=0}^{m} \binom{m+\xi}{m-j} \binom{m+\eta}{j} \left[\frac{x-1}{2}\right]^{j} \left[\frac{x+1}{2}\right]^{m-j}$$

$$= \binom{m+\xi}{m} \; {}_2F_1\left[-m, \xi+\eta+m+1; \xi+1; \frac{1-x}{2}\right]. \tag{76}$$

In particular, for the Gegenbauer (or ultraspherical) polynomials $C_n^{\mu}(x)$, we have (*cf*. Szegö, 1975, p. 81, Equation (4.7.1))

$$C_m^{\mu+\frac{1}{2}}(x) = \begin{bmatrix} m+\mu \\ m \end{bmatrix}^{-1} \begin{bmatrix} m+2\mu \\ m \end{bmatrix} P_m^{(\mu,\mu)}(x) \tag{77}$$

and (*cf.*, *e.g.*, Magnus *et al.*, 1966, p. 219)

$$P_m^{(\mu-\frac{1}{2},-\frac{1}{2})}(x) = \frac{(\frac{1}{2})_m}{(\mu)_m} C_{2m}^{\mu}\left[\sqrt{\frac{x+1}{2}}\right] . \tag{78}$$

In view of this last relationship (78), a further special case of the expansion (75) when $\kappa = -\frac{1}{2}$ yields (*cf.*, *e.g.*, Magnus *et al.*, 1966, p. 129)

$$J_\mu(az) = a^\mu\,\Gamma(\mu+\tfrac{1}{2}) \sqrt{\frac{2}{\pi z}} \sum_{m=0}^{\infty} \frac{(\mu+2m+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(\mu+m+1)}$$

$$\cdot\, J_{\mu+2m+\frac{1}{2}}(z)\, C_{2m}^{\mu+\frac{1}{2}}\left[\sqrt{1-a^2}\right], \tag{79}$$

which, for $\mu = 0$, reduces immediately to the elegant expansion:

$$J_0(az) = \sqrt{\frac{2}{z}} \sum_{m=0}^{\infty} (2m+\tfrac{1}{2}) \frac{\Gamma(m+\frac{1}{2})}{m!}$$

$$\cdot\, J_{2m+\frac{1}{2}}(z)\, P_{2m}\left[\sqrt{1-a^2}\right] , \tag{80}$$

in terms of the Legendre (or spherical) polynomials

$$P_m(x) = C_m^{\frac{1}{2}}(x) = P_m^{(0,0)}(x) . \tag{81}$$

Since, by the classical Vandermonde theorem (*cf.* Slater, 1966, p. 243, Equation (III.4)),

$$ {}_2F_1(-m,b;\ c;\ 1) = \frac{(c-b)_m}{(c)_m} \tag{82} $$

$$ (m = 0,1,2,\cdots;\quad c \neq 0,-1,-2,\cdots)\ , $$

a special case of Formula (74) when $a = 1$ yields (*cf.* Watson, 1944, p. 139, Equation 5.21 (1); see also Magnus *et al.*, 1966, p. 129)

$$ (\tfrac{1}{2}z)^{\rho-\mu} J_\mu(z) = \sum_{m=0}^{\infty} (-1)^m \frac{(\rho+2m)(\rho-\mu)_m\ \Gamma(\rho+m)}{\Gamma(\mu+m+1)} J_{\rho+2m}(z)\ , \tag{83} $$

where the series would terminate if

$$ \mu - \rho = 0,1,2,\cdots\ . \tag{84} $$

In order to facilitate the evaluation of the infinite integrals representing the alpha–function $\alpha^0_{n,\lambda}(r_1,r_2,\delta)$ and its three partial derivatives considered above, especially in situations in which the Bailey–Rice integral (30) ceases to be applicable *directly*, we can now replace any one of the three Bessel functions (occurring in the integrands) by an appropriate series of Bessel functions given by one (or the other) of the expansion formulas (74), (75), (79), (80), and (83). If, for example, we make use of the relatively simple expansion (80) in Equation (6) with a new variable y of integration, related with x by

$$x = \frac{r_2 y}{r_1 + r_2}, \tag{85}$$

so that

$$\alpha^0_{n,\lambda}(r_1, r_2, \delta) = 2^{\nu+\lambda}\,\Gamma(\nu)\Gamma(\lambda+1)\,\frac{(r_1+r_2)^{\nu+\lambda-1}}{r_1^{\nu}\,r_2^{\lambda-1}}$$

$$\cdot \int_0^\infty y^{-\nu-\lambda}\,J_\nu\left[\frac{r_1 y}{r_1 + r_2}\right] J_{\lambda+1}\left[\frac{r_2 y}{r_1 + r_2}\right] J_0\left[\frac{\delta y}{r_1 + r_2}\right] dy\,, \tag{86}$$

and apply the Bailey–Rice integral (30) to evaluate the resulting infinite integral, we shall obtain the representation:

$$\alpha^0_{n,\lambda}(r_1, r_2, \delta) = \frac{2}{(n+2)(\lambda+1)}\left[\frac{r_2}{r_1 + r_2}\right]^2 \sum_{m=0}^{\infty} (2m+\tfrac{1}{2})\,P_{2m}\left[\sqrt{1 - \left[\frac{\delta}{r_1+r_2}\right]^2}\,\right]$$

$$\cdot F_4\left[-m+\tfrac{1}{2},\ m+1;\ \frac{n+4}{2},\ \lambda+2;\ \left[\frac{r_1}{r_1+r_2}\right]^2,\ \left[\frac{r_2}{r_1+r_2}\right]^2\right], \tag{87}$$

which, for $\lambda = 0$, yields a known result due to Kopal (1979, p. 41, Equation (3.94)).

For $\mu = 0$, Equation (75) reduces at once to the form:

$$J_0(az) = \left[\frac{2}{z}\right]^{\kappa+1} \sum_{m=0}^{\infty} \frac{(\kappa+2m+1)\Gamma(\kappa+m+1)}{m!}$$

$$\cdot\, J_{\kappa+2m+1}(z)\ P_m^{(0,\kappa)}(1-2a^2) \tag{88}$$

in terms of a certain special class of Jacobi polynomials defined by (76). Now insert this expansion with

$$a = \frac{\delta}{r_1 + r_2} \quad \text{and} \quad z = y \tag{89}$$

into the right-hand side of the representation (86), and evaluate the resulting infinite integral by means of the Bailey-Rice integral (30). We are thus led to the following class of explicit expressions for $\alpha^0_{n,\lambda}(r_1,r_2,\delta)$:

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta) = \frac{2}{(n+2)(\lambda+1)}\left[\frac{r_2}{r_1+r_2}\right]^2 \sum_{m=0}^{\infty} (\kappa+2m+1)\ P_m^{(0,\kappa)}\left[1-\frac{2\delta^2}{(r_1+r_2)^2}\right]$$

$$\cdot\, F_4\left[-\kappa-m,\ 1+m;\ \frac{n+4}{2},\ \lambda+2;\ \left[\frac{r_1}{r_1+r_2}\right]^2,\ \left[\frac{r_2}{r_1+r_2}\right]^2\right], \tag{90}$$

where, for the convergence of the integral evaluated,

$$\mathrm{Re}(\kappa) > -\frac{5}{2} - \lambda - \frac{n+2}{2}\,. \tag{91}$$

Obviously, this last condition (91) is satisfied when $\kappa = -1$ in which case the series in Equation (90) begins actually from $m = 1$. Thus, if $\kappa = -1$, Equation (90) yields the simple representation:

$$\alpha^{0}_{n,\lambda}(r_1,r_2,\delta) = \frac{4}{(n+2)(\lambda+1)}\left[\frac{r_2}{r_1+r_2}\right]^2 \sum_{m=1}^{\infty} m\, P^{(0,-1)}_{m}\left[1-\frac{2\delta^2}{(r_1+r_2)^2}\right]$$

$$\cdot\, F_4\left[1-m,\ 1+m;\ \frac{n+4}{2},\ \lambda+2;\ \left[\frac{r_1}{r_1+r_2}\right]^2,\ \left[\frac{r_2}{r_1+r_2}\right]^2\right], \tag{92}$$

where the Appell series involved is terminating (and the Jacobi polynomial occurring on the right-hand side is independent of n).

It may be remarked in passing that, since both functions occurring in the summand of (92) are polynomials, the representation (92) will greatly facilitate the computations of the alpha-function $\alpha^{0}_{n,\lambda}(r_1,r_2,\delta)$. Moreover, the above expansion converges under all (admissible) circumstances, and will thus remain valid for any type of eclipse, occultation or transit, regardless of whether

$$r_1 > r_2 \quad \text{or} \quad r_1 < r_2\,, \tag{93}$$

and for any degree n of the adopted law of limb-darkening, which (as we observed already) does not occur in the Jacobi polynomials involved in (92).

Since [*cf.* Equation (76)]

$$P^{(0,-1)}_{m}(1-2x) = {}_2F_1(-m,m;\ 1;\ x), \tag{94}$$

a special case of (92) when $\lambda = 0$ will lead us immediately to a remarkably simple representation of $\alpha^{0}_{n}(r_1,r_2,\delta)$ given earlier by Demircan (1978, p. 321, Equation (2.32)). Furthermore, since [*cf.* Equation (76)]

$$P_m^{(\xi,\eta)}(1-2x) = \begin{bmatrix} m+\xi \\ m \end{bmatrix} {}_2F_1(-m,\ \xi+\eta+m+1;\ \xi+1;\ x)$$

$$= \begin{bmatrix} m+\xi \\ m \end{bmatrix} G_m(\xi+\eta+1,\ \xi+1;\ x) \tag{95}$$

or, inversely,

$$G_m(\xi,\eta;\ x) = {}_2F_1(-m,\ m+\xi;\ \eta;\ x)$$

$$= \begin{bmatrix} m+\eta-1 \\ m \end{bmatrix}^{-1} P_m^{(\eta-1,\xi-\eta)}(1-2x), \tag{96}$$

we can easily rewrite Equation (92) in its *equivalent* form:

$$\alpha_{n,\lambda}^0(r_1,r_2,\delta) = \frac{4}{(n+2)(\lambda+1)} \left[\frac{r_2}{r_1+r_2}\right]^2 \sum_{m=1}^{\infty} m\, G_m\left[0,\ 1;\ \frac{\delta^2}{(r_1+r_2)^2}\right]$$

$$\cdot F_4\left[1-m,\ 1+m;\ \frac{n+4}{2},\ \lambda+2;\ \left[\frac{r_1}{r_1+r_2}\right]^2,\ \left[\frac{r_2}{r_1+r_2}\right]^2\right], \tag{97}$$

which, for $\lambda = 0$, would provide the *corrected* version of Demircan's result as reproduced by Kopal[5] (1979, p. 42, Equation (3.99)).

There are a number of situations in which many of the representations (given in this paper) will simplify considerably. With a view to facilitating the

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[5] The factor $G_n(0,1;c^2)$ in Kopal (1979, p. 42, Equation (3.99)) should be replaced by

$$n\, G_n(0,1;c^2).$$

interested reader, we shall now discuss some of these situations rather briefly.

First of all, Lauricella's triple hypergeometric function $F_C^{(3)}$ (defined by Equation (27) with $N = 3$) can be reduced to an Appell function F_4 by means of the known formula (*cf.*, *e.g.*, Srivastava and Karlsson, 1985, p. 329, Equation 9.4 (218)):

$$F_C^{(3)}[\xi+\eta+1,\ \eta+1;\ \xi+1,\ \eta+1,\ \eta+1;\ x,y,z]$$

$$= (1+x-y-z)^{-\lambda-\mu-1}$$

$$\cdot\ F_4\left[\tfrac{1}{2}(\xi+\eta+1),\ \tfrac{1}{2}(\xi+\eta+2);\ \xi+1,\ \eta+1;\ X,\ Y\right], \tag{98}$$

where, for convenience,

$$X = \frac{4x}{(1+x-y-z)^2}, \qquad Y = \frac{4yz}{(1+x-y-z)^2}. \tag{99}$$

On the other hand, the Lauricella function $F_C^{(3)}$ can be expressed as a bilinear generating function for the Jacobi polynomials defined by Equation (76) in the form (*cf.* Srivastava and Manocha, 1984, p. 115, Equation 2.3 (45)):

$$\sum_{n=0}^{\infty} \frac{(m+n)!\,(\alpha+\beta+m+1)_n}{(\gamma+1)_n(\delta+1)_n}\ P_{m+n}^{(\alpha,\beta)}(x)\ P_n^{(\gamma,\delta)}(y)\ t^n$$

$$= (\alpha+1)_m \left[\frac{x+1}{2}\right]^{-\alpha-\beta-m-1}$$

$$\cdot\ F_C^{(3)}\left[\alpha+\beta+m+1,\ \alpha+m+1;\ \alpha+1,\ \gamma+1,\ \delta+1;\ \frac{x-1}{x+1},\ \frac{(y-1)t}{x+1},\ \frac{(y+1)t}{x+1}\right], \tag{100}$$

where, for convergence,

$$\sqrt{|t|} < \frac{\sqrt{|x+1|} - \sqrt{|x-1|}}{\sqrt{|y+1|} + \sqrt{|y-1|}} . \tag{101}$$

The Appell function F_4 reduces, in turn, to simpler functions when its parameters and variables are constrained appropriately. For example, we have Bailey's formula (Bailey, 1933, p. 306, Equation (2.1); 1935a, p. 81, Equation 9.6 (1))

$$F_4[\alpha,\beta;\ \gamma,\ \alpha+\beta-\gamma+1;\ z(1-\zeta),\ \zeta(1-z)]$$

$$= {}_2F_1(\alpha,\beta;\ \gamma;\ z)\ \ {}_2F_1(\alpha,\beta;\ \alpha+\beta-\gamma+1;\ \zeta), \tag{102}$$

which holds true inside simply-connected regions surrounding $z = 0$ and $\zeta = 0$ for which

$$\sqrt{|z(1-\zeta)|} + \sqrt{|\zeta(1-z)|} < 1. \tag{103}$$

Bailey's formula (102) was indeed motivated, at least partially, by an interesting observation by Appell (1884, pp. 418–421) that the functions

$$F_4\left[\alpha,\beta;\ \gamma,\ \alpha+\beta-\gamma+1;\ x^2,\ (1-x)^2\right] \tag{104}$$

and

$$\left\{{}_2F_1(\alpha,\beta;\ \gamma;\ x)\right\}^2 \tag{105}$$

satisfy the same differential equation of the third order. More importantly, if

$$\alpha = -m \quad (m = 0,1,2,\cdots), \tag{106}$$

Bailey's formula (102) can be rewritten as

$$F_4[-m,\ \beta+m;\ \gamma,\ \beta-\gamma+1;\ z\zeta,\ (1-z)(1-\zeta)]$$

$$= {}_2F_1(-m,\ \beta+m;\ \gamma;\ z)\ {}_2F_1(-m,\ \beta+m;\ \beta-\gamma+1;\ 1-\zeta), \tag{107}$$

which, upon rearranging the second Gaussian hypergeometric series in powers of ζ, yields

$$F_4[-m,\ \beta+m;\ \gamma,\ \beta-\gamma+1;\ z\zeta,\ (1-z)(1-\zeta)]$$

$$= \frac{(-1)^m(\gamma)_m}{(\beta-\gamma+1)_m}\ {}_2F_1(-m,\ \beta+m;\ \gamma;\ z)\ \ {}_2F_1(-m,\ \beta+m;\ \gamma;\ \zeta), \tag{108}$$

where each of the Gaussian hypergeometric functions can be expressed as a Jacobi polynomial.

Formula (108) is due, in fact, to Watson (1922, p. 190). It is this formula that was used by Watson (1944, p. 370, Section 11.6) in proving Bateman's expansion (*cf.* Bateman, 1905):

$$\frac{1}{2} z\ \ J_\kappa(z \cos\varphi \cos\Phi)\ J_\mu(z \sin\varphi \sin\Phi)$$

$$= (\cos\varphi \cos\Phi)^\kappa\ (\sin\varphi \sin\Phi)^\mu \sum_{m=0}^{\infty} (-1)^n(\kappa+\mu+2m+1)$$

$$\cdot \frac{\Gamma(\kappa+\mu+m+1)\ \Gamma(\mu+m+1)}{m!\ \Gamma(\kappa+m+1)\ \{\Gamma(\mu+1)\}^2}\ J_{\kappa+\mu+2m+1}(z)$$

$$\cdot\ {}_2F_1(-m,\ \kappa+\mu+m+1;\ \mu+1;\ \sin^2\varphi)$$

$$\cdot\ {}_2F_1(-m,\ \kappa+\mu+m+1;\ \mu+1;\ \sin^2\Phi), \tag{109}$$

which holds true for all values of κ and μ with the exception of negative integer values.

In view of Watson's formula (108), Bateman's expansion (109) can easily be deduced as a special case of Bailey's expansion (56) when

$$\rho = \kappa + \mu + 1, \quad a = \cos\varphi\cos\Phi, \quad \text{and} \quad b = \sin\varphi\sin\Phi. \tag{110}$$

As an example of application of the reduction formulas (102) and (108), we turn to the integral representation (86) for $\alpha^0_{n,\lambda}(r_1,r_2,\delta)$ and make use of Bailey's expansion (56) with

$$\kappa = \nu = \frac{n+2}{2}, \quad \mu = \lambda + 1, \quad z = y, \quad a = \frac{r_1}{r_1+r_2}, \quad \text{and} \quad b = \frac{r_2}{r_1+r_2}, \tag{111}$$

and of the Weber–Schafheitlin integral (64). We thus arrive at the following alternative (but equivalent) form of (65):

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta) = \frac{2}{(n+2)(\lambda+1)}\left[\frac{r_2}{r_1+r_2}\right]^2 \sum_{m=0}^{\infty} (\rho+2m)$$

$$\cdot\, F_4\left[-m,\ \rho+m;\ \frac{n+4}{2},\ \lambda+2;\ \left[\frac{r_1}{r_1+r_2}\right]^2 \left[\frac{r_2}{r_1+r_2}\right]^2\right]$$

$$\cdot\, {}_2F_1\left[\begin{matrix} m+1,\ 1-\rho-m; \\ 1; \end{matrix}\ \frac{\delta^2}{(r_1+r_2)^2}\right] \quad (\mathrm{Re}(\rho) > 0;\ \ r_1 + r_2 \geqq \delta). \tag{112}$$

Since [*cf*. Equation (66)]

$${}_2F_1\left[\begin{matrix} m+1,\ 1-\rho-m; \\ 1; \end{matrix}\ \frac{\delta^2}{(r_1+r_2)^2}\right] = \left\{1 - \frac{\delta^2}{(r_1+r_2)^2}\right\}^{\rho-1}$$

$$\cdot\, {}_2F_1\left[\begin{matrix} -m,\ \rho+m; \\ 1; \end{matrix}\ \frac{\delta^2}{(r_1+r_2)^2}\right], \tag{113}$$

Equation (112) can be rewritten at once in the form:

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta) = \frac{2}{(n+2)(\lambda+1)} \left[\frac{r_2}{r_1+r_2}\right]^2 \left\{1 - \frac{\delta^2}{(r_1+r_2)^2}\right\}^{\rho-1}$$

$$\cdot \sum_{m=0}^{\infty} (\rho+2m)\ F_4\left[-m,\ \rho+m;\ \frac{n+4}{2},\ \lambda+2;\ \left[\frac{r_1}{r_1+r_2}\right]^2,\ \left[\frac{r_2}{r_1+r_2}\right]^2\right]$$

$$\cdot\, {}_2F_1\left[\begin{matrix} -m,\ \rho+m; \\ 1; \end{matrix}\ \frac{\delta^2}{(r_1+r_2)^2}\right] \quad (\mathrm{Re}(\rho) > 0), \tag{114}$$

in which, as also in the class of representations (67), each of the hypergeometric series is terminating.

The Appell series F_4 occurring in Equation (114) is reducible to that in Watson's formula (108) with, of course,

$$\beta = \rho = \frac{n+2}{2} + \lambda + 2, \quad \gamma = \frac{n+4}{2}, \quad z = \zeta = \frac{r_1}{r_1+r_2}. \tag{115}$$

The representation (114) reduces, in this special case, to the remarkably simple form:

$$\alpha^{0}_{n,\lambda}(r_1,r_2,\delta) = \frac{2\Gamma(\lambda+1)}{(n+2)\Gamma(\frac{1}{2}n+2)} \left[\frac{r_2}{r_1+r_2}\right]^2 \left\{1 - \frac{\delta^2}{(r_1+r_2)^2}\right\}^{\lambda+\frac{1}{2}n+2}$$

$$\cdot \sum_{m=0}^{\infty} (-1)^m \, (\lambda+2m+\tfrac{1}{2}n+3) \, \frac{\Gamma(m+\frac{1}{2}n+2)}{\Gamma(\lambda+m+2)}$$

$$\cdot \left\{ {}_2F_1\left[-m,\ \lambda+m+\tfrac{1}{2}n+3;\ \tfrac{1}{2}n+2;\ \frac{r_1}{r_2+r_2}\right]\right\}^2$$

$$\cdot \ {}_2F_1\left[\begin{matrix} -m,\ \lambda+m+\frac{1}{2}n+3; \\ 1; \end{matrix} \ \frac{\delta^2}{(r_1+r_2)^2}\right]. \tag{116}$$

Since

$${}_2F_1\left[-m,\ \lambda+m+\tfrac{1}{2}n+3;\ \tfrac{1}{2}n+2;\ \frac{r_1}{r_1+r_2}\right]$$

$$= \left[\frac{r_2}{r_1+r_2}\right]^{-\lambda-1} {}_2F_1\left[-\lambda-m-1,\ m+\tfrac{1}{2}n+2;\ \tfrac{1}{2}n+2;\ \frac{r_1}{r_1+r_2}\right], \tag{117}$$

again by Euler's transformation (*cf.* Erdélyi *et al.*, 1953a, p. 64, Equation 2.1.4 (23)), Equation (116) can be rewritten as

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta) = \frac{2\Gamma(\lambda+1)}{(n+2)\Gamma(\frac{1}{2}n+2)} \left[\frac{r_2}{r_1+r_2}\right]^{-2\lambda} \left\{1 - \frac{\delta^2}{(r_1+r_2)^2}\right\}^{\lambda+\frac{1}{2}n+2}$$

$$\cdot \sum_{m=0}^{\infty} (-1)^m (\lambda+2m+\tfrac{1}{2}n+3) \frac{\Gamma(m+\frac{1}{2}n+2)}{\Gamma(\lambda+m+2)}$$

$$\cdot \left\{{}_2F_1\left[-\lambda-m-1,\ m+\tfrac{1}{2}n+2;\ \tfrac{1}{2}n+2;\ \frac{r_1}{r_1+r_2}\right]\right\}^2$$

$$\cdot\ {}_2F_1\left[\begin{matrix}-m,\ \lambda+m+\frac{1}{2}n+3; \\ 1;\end{matrix}\ \frac{\delta^2}{(r_1+r_2)^2}\right]. \tag{118}$$

For nonnegative integer values of the parameter λ, each of the Gaussian hypergeometric series involved in the representation (118) can be expressed as a Jacobi polynomial. If, for the sake of convenience, we make use of the G–notation for Jacobi polynomials introduced in Equation (96), we find from (118) that

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta) = \frac{\lambda!}{(\frac{1}{2}n+1)\Gamma(\frac{1}{2}n+2)} \left[\frac{r_2}{r_1+r_2}\right]^{-2\lambda} \left\{1 - \frac{\delta^2}{(r_1+r_2)^2}\right\}^{\lambda+\frac{1}{2}n+2}$$

$$\cdot \sum_{m=0}^{\infty} (-1)^m (\lambda+2m+\tfrac{1}{2}n+3) \frac{\Gamma(m+\frac{1}{2}n+2)}{(\lambda+m+1)!}$$

$$\cdot \left\{ G_{\lambda+m+1}\left[\tfrac{1}{2}n-\lambda+1,\ \tfrac{1}{2}n+2;\ \frac{r_1}{r_1+r_2}\right]\right\}^2$$

$$\cdot\ G_m\left[\lambda+\tfrac{1}{2}n+3,\ 1;\ \frac{\delta^2}{(r_1+r_2)^2}\right] \qquad (\lambda = 0,1,2,\cdots). \tag{119}$$

In a similar manner, the representation (116) yields

$$\alpha^0_{n,\lambda}(r_1,r_2,\delta) = \frac{2\Gamma(\lambda+1)}{(n+2)\Gamma(\frac{1}{2}n+2)}\left[\frac{r_2}{r_1+r_2}\right]^2\left\{1-\frac{\delta^2}{(r_1+r_2)^2}\right\}^{\lambda+\frac{1}{2}n+2}$$

$$\cdot \sum_{m=0}^{\infty} (-1)^m\ (\lambda+2m+\tfrac{1}{2}n+3)\ \frac{\Gamma(m+\frac{1}{2}n+2)}{\Gamma(\lambda+m+2)}$$

$$\cdot \left\{ G_m\left[\lambda+\tfrac{1}{2}n+3,\ \tfrac{1}{2}n+2;\ \frac{r_1}{r_1+r_2}\right]\right\}^2$$

$$\cdot\ G_m\left[\lambda+\tfrac{1}{2}n+3,\ 1;\ \frac{\delta^2}{(r_1+r_2)^2}\right], \tag{120}$$

in which each of the Jacobi polynomials involved is of the *same* degree.

The representation (119) with $\lambda = 0$ was given by Kopal[6] (1979, p. 40, Equation (3.91)).

[6] The factor $(\nu+2n+1)$ in Kopal (1979, p. 40, Equation (3.91)) should be *corrected* to read $(\nu+2n+2)$. On the other hand, in Kopal (1979, p. 40, Equation (3.87)), one should read $\Gamma(\nu+1)$ for $\Gamma(n+1)$.

Just as the Lauricella function $F_C^{(3)}$, the Appell series F_4 can be represented as a bilinear generating function for the Jacobi polynomials. Indeed, if in the known result (98), we set $m = 0$, $\gamma = \alpha$, and $\delta = \beta$, and make use of the reduction formula (98), we shall obtain Bailey's bilinear generating function (*cf*. Bailey, 1938, p. 9, Equation (2.1); see also Srivastava and Manocha, 1984, p. 116, Equation 2.3 (47)):

$$\sum_{n=0}^{\infty} \frac{n!\ (\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n}\ P_n^{(\alpha,\beta)}(x)\ P_n^{(\alpha,\beta)}(y)t^n = (1+t)^{-\alpha-\beta-1}$$

$$\cdot\ F_4\left[\tfrac{1}{2}(\alpha+\beta+1),\ \tfrac{1}{2}(\alpha+\beta+2);\ \alpha+1,\ \beta+1;\ \frac{(1-x)(1-y)t}{(1+t)^2},\ \frac{(1+x)(1+y)t}{(1+t)^2}\right], \qquad (121)$$

which, for $x = \cos 2\varphi$ and $y = \cos 2\Phi$, may be rewritten in the *equivalent* form:

$$\sum_{n=0}^{\infty} \frac{n!\ (\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n}\ P_n^{(\alpha,\beta)}(\cos 2\varphi)\ P_n^{(\alpha,\beta)}(\cos 2\Phi)\ t^n = (1+t)^{-\alpha-\beta-1}$$

$$\cdot\ F_4\left[\tfrac{1}{2}(\alpha+\beta+1),\ \tfrac{1}{2}(\alpha+\beta+2);\ \alpha+1,\ \beta+1;\ \frac{a^2}{\kappa^2},\ \frac{b^2}{\kappa^2}\right], \qquad (122)$$

where, for convenience,

$$a = \sin\varphi \sin\Phi, \quad b = \cos\varphi\cos\Phi, \quad \text{and} \quad \kappa = \frac{1}{2}\left[\sqrt{t} + \frac{1}{\sqrt{t}}\right]. \qquad (123)$$

The Appell series F_4 can also be expressed as a *linear* generating function for Jacobi polynomials (*cf*., *e.g*., Srivastava and Manocha, 1984, p. 111, Equation

2.3 (31)):

$$\sum_{n=0}^{\infty} \begin{bmatrix} m+n \\ n \end{bmatrix} \frac{(\alpha+\beta+m+1)_n}{(\gamma+1)_n} P_{m+n}^{(\alpha,\beta)}(x)t^n = \begin{bmatrix} m+\alpha \\ m \end{bmatrix} \left[\frac{x+1}{2}\right]^{-\alpha-\beta-m-1}$$

$$\cdot F_4\left[\alpha+m+1,\ \alpha+\beta+m+1;\ \alpha+1,\ \gamma+1;\ \frac{x-1}{x+1},\ \frac{2t}{x+1}\right]. \tag{124}$$

Finally, from among the various cases of reducibility of the Appell series F_4, in addition (of course) to Bailey's formula (102) and its consequence (108) due to Watson, we recall here the following reduction formulas of possible use in further investigations concerning the associated alpha-function $\alpha_{n,\lambda}^{0}(r_1,r_2,\delta)$ and its partial derivatives considered in this paper:

$$F_4[\alpha,\ \alpha+\tfrac{1}{2};\ \gamma,\ \tfrac{1}{2};\ x,\ y]$$

$$= \frac{1}{2}(1+\sqrt{y})^{-2\alpha}\ {}_2F_1\left[\begin{matrix}\alpha,\ \alpha+\tfrac{1}{2}; \\ \gamma;\end{matrix}\ \frac{x}{(1+\sqrt{y})^2}\right]$$

$$+ \frac{1}{2}(1-\sqrt{y})^{-2\alpha}\ {}_2F_1\left[\begin{matrix}\alpha,\ \alpha+\tfrac{1}{2}; \\ \gamma;\end{matrix}\ \frac{x}{(1-\sqrt{y})^2}\right], \tag{125}$$

given by Appell *et* Kampé de Fériet (1926, p. 23);

$$F_4\left[\alpha,\beta;\ \alpha,\beta;\ -\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right]$$

$$= (1-xy)^{-1}(1-x)^{\beta}(1-y)^{\alpha}, \tag{126}$$

$$F_4\left[\alpha,\beta;\ \beta,\beta;\ -\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right]$$

$$= (1-x)^{\alpha}\ (1-y)^{\alpha}\ {}_2F_1(\alpha,\ \alpha-\beta+1;\ \beta;\ xy), \tag{127}$$

and

$$F_4\left[\alpha,\beta;\ \alpha-\beta+1,\ \beta;\ -\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right]$$

$$= (1-y)^{\alpha}\ {}_2F_1\left[\begin{matrix}\alpha,\ \beta; \\ \alpha-\beta+1;\end{matrix}\ \frac{x(1-y)}{x-1}\right], \tag{128}$$

which are due to Bailey (1936, p. 42, Equations (4.2), (4.3), and (4.4); 1935a, p. 102, Example 20), and

$$F_4[\alpha,\ \beta;\ \gamma,\ \gamma';\ x,\ x]$$

$$= {}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ \frac{1}{2}(\gamma+\gamma'),\ \frac{1}{2}(\gamma+\gamma'-1); \\ \gamma,\ \gamma',\ \gamma+\gamma'-1;\end{matrix}\ 4x\right], \tag{129}$$

which was proven by Burchnall (1942, p. 101, Equation (37)) while investigating solutions of equivalent systems of hypergeometric differential equations. As already pointed out by Srivastava and Chang (1987, p. 156), Bailey's result (127) is essentially equivalent to the reduction formula (*cf*. Srivastava and Manocha, 1984, p. 112, Equation 2.3 (33)):

$$F_4[\alpha,\ \beta;\ \beta,\ \beta;\ x,\ y]$$

$$= (1-x-y)^{-\alpha}\ {}_2F_1\left[\begin{matrix}\tfrac{1}{2}\alpha,\ \tfrac{1}{2}\alpha+\tfrac{1}{2}; \\ \alpha+1;\end{matrix}\ \frac{4xy}{(1-x-y)^2}\right], \tag{130}$$

which incidentally follows immediately from a special case of (98) when $x = 0$.

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