MAXIMUM EXPECTED LENGTH OF A WORLD SERIES

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E3386. *Proposed by Eugene F. Schuster, University of Texas, El Paso.*

Let L be the length of a (2N–1)-game World Series, modeled as a sequence of independent identically distributed Bernoulli trials which terminates as soon as one team wins N games. (The length is the number of games actually played.) Prove the seeming obvious observation that the expected length E(L) of the series is maximized when the two teams are evenly matched.

Solution by Bruce R. Johnson, University of Victoria, Canada. In addition to giving a proof of the proposition, we derive a simple formula for the maximal expected length of the series.

Let L_α be the length of a series in which the favorite has probability $\frac{1}{2}+\alpha$ of winning any game, $0<\alpha<\frac{1}{2}$; and let L_0 be the length of an evenly matched series. For $k \in \{N,N+1,\cdots,2N-1\}$,

$P(L_\alpha=k)$ = P(favorite wins N–1 of the first k–1 games and favorite wins *k*th game)

+ P(underdog wins N–1 of the first k–1 games and underdog wins *k*th game)

$$= \left[\begin{matrix}k-1\\N-1\end{matrix}\right](\tfrac{1}{2}+\alpha)^N(\tfrac{1}{2}-\alpha)^{k-N} + \left[\begin{matrix}k-1\\N-1\end{matrix}\right](\tfrac{1}{2}-\alpha)^N(\tfrac{1}{2}+\alpha)^{k-N}.$$

Letting r_k denote the ratio $P(L_\alpha=k)/P(L_0=k)$, we discover the strict ordering $r_N > r_{N+1} > \cdots > r_{2N-1}$, because

$$r_k = \frac{P(L_a=k)}{P(L_0=k)} = \frac{(\frac{1}{2}+a)^N(\frac{1}{2}-a)^{k-N} + (\frac{1}{2}-a)^N(\frac{1}{2}+a)^{k-N}}{(\frac{1}{2})^k + (\frac{1}{2})^k}$$

$$= \tfrac{1}{2}\{(1+2a)^N(1-2a)^{k-N} + (1-2a)^N(1+2a)^{k-N}\},$$

and the difference

$$r_k - r_{k+1} = \tfrac{1}{2}\{(1+2a)^N(1-2a)^{k-N}(1-(1-2a)) + (1-2a)^N(1+2a)^{k-N}(1-(1+2a))\}$$

$$= a(1+2a)^{k-N}(1-2a)^{k-N}\{(1+2a)^{2N-k} - (1-2a)^{2N-k}\}$$

is seen to be positive for $k \in \{N,N+1,\cdots,2N-2\}$. The strict ordering of these ratios leads to the inequalities

$$P(L_a \geq k) < P(L_0 \geq k) \quad \text{for} \quad k \in \{N+1,N+2,\cdots,2N-1\},$$

as seen by fixing k and considering two possible cases. If $r_k \geq 1$, then

$$P(L_a \geq k) = 1 - P(L_a < k) = 1 - \sum_{i=N}^{k-1} r_i\, P(L_0=i)$$

$$< 1 - \sum_{i=N}^{k-1} P(L_0=i) = P(L_0 \geq k);$$

on the other hand if $r_k < 1$, then

$$P(L_a \geq k) = \sum_{i=k}^{2N-1} r_i\, P(L_0=i) < \sum_{i=k}^{2N-1} P(L_0=i) = P(L_0 \geq k).$$

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The desired result follows by expressing the expectations as the sum of probabilities of right-tailed events,

$$E(L_\alpha) = \sum_{k=N}^{2N-1} kP(L_\alpha=k) = \sum_{k=1}^{\infty} kP(L_\alpha=k) = \sum_{k=1}^{\infty}\sum_{i=1}^{k} P(L_\alpha=k)$$

$$= \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} P(L_\alpha=k) = \sum_{i=1}^{\infty} P(L_\alpha\geq i) < \sum_{i=1}^{\infty} P(L_0\geq i) = E(L_0).$$

By manipulating the defining expression for $E(L_0)$ to involve the sum over all probability weights for the length of an evenly matched $(2(N+1)-1)$-game world series, we obtain

$$E(L_0) = \sum_{k=N}^{2N-1} k\begin{bmatrix}k-1\\N-1\end{bmatrix}\left[\frac{1}{2^k}+\frac{1}{2^k}\right] = \sum_{k=N}^{2N-1} N\begin{bmatrix}k\\N\end{bmatrix}\frac{1}{2^{k-1}}$$

$$= N\sum_{m=N+1}^{2N}\begin{bmatrix}m-1\\(N+1)-1\end{bmatrix}\frac{1}{2^{m-2}} = 2N\left\{\sum_{m=N+1}^{2(N+1)-1}\begin{bmatrix}m-1\\(N+1)-1\end{bmatrix}\frac{1}{2^{m-1}} - \begin{bmatrix}2N\\N\end{bmatrix}\frac{1}{2^{2N}}\right\}$$

$$= 2N\left\{1 - \begin{bmatrix}2N\\N\end{bmatrix}\frac{1}{2^{2N}}\right\}.$$

Some common special cases of $E(L_0)$ are given in the following table:

	N	$E(L_0)$
Best 2 out of 3	2	2.5
Best 3 out of 5	3	4.125
Best 4 out of 7	4	5.8125
Best 5 out of 9	5	7.5390625