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Abstract. An explicit formula for the polar decomposition of an $n \times n$ nonsingular companion matrix is derived. This involves the largest and smallest singular values of the companion matrix.

1. INTRODUCTION. Every matrix $A \in \mathbb{C}^{n \times n}$ has a polar decomposition, namely $A = PU$ where $P \in \mathbb{C}^{n \times n}$ is a (unique) positive semidefinite matrix and $U \in \mathbb{C}^{n \times n}$ is a unitary matrix. If A is nonsingular, then P is positive definite and U is unique. In view of the wide range of applications, both theoretical and numerical aspects of the polar decomposition have been studied in detail; see, for example, [3], [5, Ch. 7].

It is of interest to single out special classes of matrices with polar decompositions that can be obtained in an explicit form. In this paper we show that nonsingular companion matrices are such a class.

For $n \geq 2$, let the complex companion matrix

$$C \equiv \begin{bmatrix} 0 & 1 & & & & \\ \vdots & & \ddots & & & \mathbf{0} \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ 0 & & & & 0 & 1 \\ a_0 & a_1 & & & & a_{n-1} \end{bmatrix}$$

be associated with the complex polynomial $z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0$. With

$$d^* \equiv [a_1, \cdots, a_{n-1}],$$

the matrix C can be written as

$$C = \left[\begin{array}{c|c} 0 & I_{n-1} \\ \hline a_0 & d^* \end{array}\right]. \tag{1.1}$$

It will be assumed that C is nonsingular, equivalently $a_0 \neq 0$. The aim of this paper is to derive the polar decomposition of C in terms of the data a_0 and d together with the largest and smallest singular values, σ_n and σ_1, of C. This explicit formula is stated in Theorem 1, and is proved by developing properties of these singular values in Lemmas 2 and 3.

2. POLAR DECOMPOSITION FORMULA

THEOREM 1. *Assume that the companion matrix* C *in* (1.1) *is nonsingular. Let* σ_1 *and* σ_n *be positive real numbers such that* σ_1^2 *and* σ_n^2 *are the roots of*

$$F(z) \equiv z^2 - z\left(|a_0|^2 + \cdots + |a_{n-1}|^2 + 1\right) + |a_0|^2. \tag{2.1}$$

Then $C = PU$ *is the polar decomposition with* $P = P^*$ *positive definite,* U *unitary and the factors* $P \equiv [p_1, \cdots, p_n]$, *and* $U \equiv [v, p_1, \cdots, p_{n-1}]$ *given by:*

$$P = \frac{1}{\sigma_1 + \sigma_n}\left[\begin{array}{c|c} (\sigma_1 + \sigma_n)I_{n-1} - \dfrac{1}{(\sigma_1 + 1)(\sigma_n + 1)}dd^* & d \\ \hline d^* & \sigma_1^2 + \sigma_n^2 + \sigma_1\sigma_n - 1 \end{array}\right] \tag{2.2}$$

and

$$v = \frac{a_0}{|a_0|(\sigma_1 + \sigma_n)}\begin{bmatrix} -d \\ 1 + |a_0| \end{bmatrix}. \tag{2.3}$$

To prove Theorem 1, we first consider the singular values of C (i.e. the positive square roots of the eigenvalues of CC^*). The following result is known (see, for example, [1, p. 224-5]). We include a simple proof as it clarifies the role of the quadratic polynomial $F(z)$ in (2.1).

LEMMA 2. *Let* $0 \le \sigma_1 \le \sigma_2 \le \cdots \sigma_{n-1} \le \sigma_n$ *be the singular values of* C. *Then*

$$\sigma_2 = \cdots = \sigma_{n-1} = 1, \tag{2.4}$$

and σ_1^2, σ_n^2 *are the zeros of* (2.1), *namely* $F(z) = z^2 - z\sum_{i=0}^{n} |a_i|^2 + |a_0|^2$, *with* $a_n \equiv 1$.

Proof. We have

$$CC^* = \left[\begin{array}{c|c} I_{n-1} & d \\ \hline d^* & s \end{array}\right]$$

where $s \equiv \sum_{i=0}^{n-1} |a_i|^2 = d^*d + |a_0|^2$. Hence, by expanding the determinant,

$$\begin{aligned} \det(zI_n - CC^*) &= (z-s)\det((z-1)I_{n-1}) - d^* \operatorname{adj}((z-1)I_{n-1})d \\ &= (z-1)^{n-2}(z^2 - (s+1)z + s - d^*d) \\ &= (z-1)^{n-2} F(z). \end{aligned}$$

hence $F(1) = 0$ is equivalent to $d = 0$. In this case $F(z) = (z-1)(z-|a_0|^2)$, and $\{\sigma_1^2, \sigma_n^2\} = \{1, |a_0|^2\}$. If $F(1) \neq 0$, then $F(1) < 0$. Thus when $d \neq 0$, $a_0 \neq 0$, from $F(0) = |a_0|^2 > 0$ and $F(\infty) = +\infty$, it follows that $\sigma_1^2 < 1 < \sigma_n^2$. When $d \neq 0$, $a_0 = 0$, then $\sigma_1^2 = 0$, and $\sigma_n^2 = \sum_{i=0}^{n} |a_i|^2 > 1$. $\square$

The subsequent observations will be useful.

LEMMA 3. *Let σ_1 be the smallest and σ_n be the largest singular values of C. Then*

$$\sigma_1\sigma_n = |a_0| \tag{2.5}$$

and

$$d^*d = -(\sigma_n^2 - 1)(\sigma_1^2 - 1). \tag{2.6}$$

In addition, assuming $a_0 \neq 0$, and defining

$$g_i \equiv \begin{bmatrix} d \\ \sigma_i^2 - 1 \end{bmatrix}, \quad i = 1, n, \tag{2.7}$$

gives

$$CC^*g_i = \sigma_i^2 g_i, \quad i = 1, n, \tag{2.8}$$

and

$$|g_i|^2 = g_i^* g_i = \left(\sigma_i^2 - 1\right)\left(\sigma_i^2 - \sigma_j^2\right), \ \{i,j\} = \{1,n\}. \tag{2.9}$$

Proof: From (2.4) it follows that

$$\prod_{i=1}^{n} \sigma_i^2 = \sigma_1^2\sigma_n^2 = \det\ CC^* = |\det C|^2 = |a_0|^2,$$

which yields (2.5). To prove (2.6), note that trace (CC^*) is equal to the sum of the eigenvalues of CC^* and (by the proof of Lemma 2) is also equal to $n - 1 + d^*d + |a_0|^2$. Thus

$$n - 2 + \sigma_1^2 + \sigma_n^2 = n - 1 + d^*d + |a_0|^2,$$

giving, by (2.5),

$$d^*d = \sigma_1^2 + \sigma_n^2 - \sigma_1^2\sigma_n^2 - 1,$$

which is (2.6).

We prove (2.8) for nonsingular C in the equivalent form

$$C^*g_i = \sigma_i^2 C^{-1} g_i, \quad i = 1, n. \tag{2.10}$$

From (2.5), $\bar{a}_0 = \sigma_1^2\sigma_n^2/a_0$, thus

$$C^*g_i = \left[\begin{array}{c|c} 0 & \bar{a}_0 \\ I_{n-1} & d \end{array}\right] \begin{bmatrix} d \\ \sigma_i^2 - 1 \end{bmatrix} = \sigma_i^2 \begin{bmatrix} \sigma_j^2\left(\sigma_i^2 - 1\right)/a_0 \\ d \end{bmatrix},$$

where $\{i,j\} = \{1,n\}$. Note that (2.6) is equivalent to $-d^*d + (\sigma_i^2 - 1) = \sigma_j^2(\sigma_i^2 - 1)$. It is well known that

$$C^{-1} = \left[\begin{array}{c|c} -d^*/a_0 & 1/a_0 \\ \hline I_{n-1} & 0 \end{array}\right], \tag{2.11}$$

hence

$$\sigma_i^2 C^{-1} g_i = \sigma_i^2 \begin{bmatrix} (-d^*d + \sigma_i^2 - 1)/a_0 \\ d \end{bmatrix},$$

which establishes (2.10). Using (2.7) and (2.6) again, it is straightforward to verify (2.9). □

Proof of Theorem 1: The case with $\sigma_1 = 1$ or $\sigma_n = 1$, equivalently $d = 0$ (see the proof of Lemma 2), is trivial. In this case, $C = PU$ with $P = \mathrm{diag}(1, ..., 1, |a_o|)$ and

$$U \equiv \begin{bmatrix} 0 & 1 & & & \mathbf{0} \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ \frac{a_0}{|a_0|} & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

agreeing with (2.2) and (2.3).

Assume now $d \neq 0$, that is $\sigma_1 < 1 < \sigma_n$. Let $y_1, \cdots y_n$ be an orthonormal set of eigenvectors of CC^* such that $CC^* y_i = \sigma_i^2 y_i$, $i = 1, \cdots, n$ (cf. (2.8)). Then $P^2 = CC^*$ and the fact that P is the unique positive definite square root of CC^* imply

$$Py_i = \sigma_i y_i, \quad i = 1, \cdots, n.$$

Thus P has the spectral decomposition

$$P = \sum_{i=1}^{n} \sigma_i y_i y_i^* = \sum_{i \in \{1,n\}} \sigma_i y_i y_i^* + \sum_{i=2}^{n-1} y_i y_i^* = \sum_{i \in \{1,n\}} (\sigma_i - 1) y_i y_i^* + I_n. \tag{2.12}$$

Taking $y_i = g_i / |g_i|$, $i = 1, n$, with g_1, g_n given by (2.7) and using (2.9), we calculate the explicit representation for P. From (2.12)

$$\begin{aligned} P - I_n &= \frac{(\sigma_1 - 1)}{|g_1|^2} g_1 g_1^* + \frac{(\sigma_n - 1)}{|g_n|^2} g_n g_n^* \\ &= \frac{(\sigma_1 - 1)}{(\sigma_1^2 - 1)(\sigma_1^2 - \sigma_n^2)} \begin{bmatrix} d \\ \sigma_1^2 - 1 \end{bmatrix} \begin{bmatrix} d^* & \sigma_1^2 - 1 \end{bmatrix} + \frac{(\sigma_n - 1)}{(\sigma_n^2 - 1)(\sigma_n^2 - \sigma_1^2)} \begin{bmatrix} d \\ \sigma_n^2 - 1 \end{bmatrix} \begin{bmatrix} d^* & \sigma_n^2 - 1 \end{bmatrix} \\ &= \left[\begin{array}{c|c} \alpha d d^* & \beta d \\ \beta d^* & \gamma \end{array}\right], \end{aligned}$$

where $\alpha = -\dfrac{1}{(\sigma_1 + 1)(\sigma_n + 1)(\sigma_1 + \sigma_n)}$,

$$\beta = \frac{1}{\sigma_1 + \sigma_n},$$

and $\gamma = \dfrac{\sigma_1^2 + \sigma_n^2 + \sigma_1 \sigma_n - 1}{\sigma_1 + \sigma_n} - 1.$

Hence P is of the form (2.2).

For U we have $U^{-1} = C^{-1}P$, and since U is unitary $U = P(C^{-1})^*$. Hence, by (2.11),

$$U = [p_1, \cdots, p_{n-1}, p_n]\left[\begin{array}{c|c} -d/\bar{a}_0 & I_{n-1} \\ \hline 1/\bar{a}_0 & 0 \end{array}\right] = [v, p_1, \cdots, p_{n-1}].$$

Taking (2.5) and (2.6) into account, the first column v can be determined as follows

$$\begin{aligned} v &= \frac{1}{\bar{a}_0}\begin{bmatrix} I_{n-1} + \alpha dd^* & \beta d \\ \beta d^* & \gamma + 1 \end{bmatrix}\begin{bmatrix} -d \\ 1 \end{bmatrix} \\ &= \frac{1}{\bar{a}_0}\begin{bmatrix} (-1 + \alpha d^* d + \beta)d \\ -\beta d^* d + \gamma + 1 \end{bmatrix} \\ &= \frac{1}{\bar{a}_0}\,\frac{1}{(\sigma_1 + \sigma_n)}\begin{bmatrix} -\sigma_1\sigma_n d \\ \sigma_1\sigma_n(1 + \sigma_1\sigma_n) \end{bmatrix} \\ &= \frac{|a_0|}{\bar{a}_0(\sigma_1 + \sigma_n)}\begin{bmatrix} -d \\ 1 + |a_0| \end{bmatrix} \end{aligned}$$

which yields (2.3) and completes the proof. □

3. ADDITIONAL REMARKS

3.1 For a nonsingular 2×2 matrix A, the explicit polar decomposition $A = VP$ with V unitary and P positive definite is given for A real in [7, Th 1] and for A complex in

[6, Th 1]. Specializing to a 2×2 complex companion matrix $C = PU$, gives

$$P = |\det(C^* + |\det C|\, C^{-1})|^{-1/2} \left(CC^* + |\det C| I_2\right),$$

and

$$U = |\det(C^* + |\det C|\, C^{-1})|^{-1/2} \left(C + |\det C|\, (C^*)^{-1}\right),$$

with

$$|\det(C^* + |\det C|\, C^{-1})| = |a_1|^2 + (1 + |a_0|)^2 = (\sigma_1 + \sigma_2)^2;$$

in agreement with our Theorem 1 above. For an $n \times n$ companion matrix C, we have

$$|\det(C^* + |\det C|\, C^{-1})| = (1 + |a_0|)^{n-2} \left\{ \sum_{i=1}^{n-1} |a_i|^2 + (1 + |a_0|)^2 \right\},$$

but this does not appear in our polar decomposition formula for $n > 2$.

3.2 Closely related to the polar decomposition of a matrix is the singular value decomposition, see, for example, [3], [5, Ch. 7]. For a nonsingular companion matrix C, this decomposition is $C = V \Lambda W^*$, where V and W are unitary and $\Lambda = diag(\sigma_n, 1, \cdots, 1, \sigma_1)$. The columns of V are the (ordered) orthonormal eigenvectors of CC^*. The first and last columns are given by (2.7), (2.8) normalized by (2.9). Each remaining column $(v_1, \cdots, v_n)^T$ has $v_n = 0$ and is a normalized linearly independent solution of $\sum_{i=1}^{n-1} a_i v_i = 0$. Matrix decompositions are important in best approximation properties. With the Frobenius norm, the nearest unitary matrix to a given matrix is the unitary factor in its polar decomposition [3]. Thus for a companion matrix C, the nearest unitary matrix is given by U in Theorem 1.

3.3 Some spectral properties of polar decompositions of general matrices have been considered in the literature, see, for example [2], [4]. Restricting our attention to real nonsingular companion matrices, we are interested in the relation of the spectrums of C and U, where $C = PU$, and have some preliminary results. For 2×2 matrices, explicit calculation of eigenvalues shows that C and U have the same inertia. However, this is not in general true as the following 3×3 example shows. Let

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 48 & -44 & 12 \end{bmatrix}$$

with spectrum $\{2, 4, 6\}$. The polar factorization PU has the spectrum U equal to $[1, -.0477 \pm .9989i\}$, thus inertia $(U) \neq$ inertia (C). The field of values $F(C)$ (numerical range) of C, always contains 0, as $z^T C z = 0$ for $z = (1, 0, \cdots, 0)^T$. In some cases $0 \in F(U)$ and in others $0 \notin F(U)$. For $n = 2$, we know precisely: $0 \notin F(U)$ if and only if $a_0 > 0$ and $a_1 \neq 0$.

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