PAINLEVÉ'S CONJECTURE

by

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A-t-on tout à fait le droit d'établir une séparation entre les deux grands aspects de la vie de Painlevé, son côté scientifique et son côté humain? Ce n'est point certain et, devant nous, récement, l'homme d'État qui a peut-être été le plus près de sa pensée et de son action, faisait ressortir l'unité secrète par laquelle toutes les manifestations de cette admirable nature sont solidaires les unes des autres.

Jaques Hadamard: L'oeuvre scientifique de Paul Painlevé,
Revue de Métaphysique XLI (1934), 289-325.

This is a story about celestial mechanics and mathematics and about a question older than Bieberbach's conjecture; a question that died close to its 100 years birthday but which - like any good question - left behind it many other unanswered questions as well as a universe of intelectual achievements.

The n-body problem

The roots of the n-body problem get lost somewhere in the early history of human kind but we can easily recognize its modern birth certificate signed by Isaac Newton in his fundamental *Philosophiae Naturalis Principia Mathematica*, published for the first time in 1687. The clear formulation of the problem in terms of differential equations is based on the inverse square force law of mutual attraction between particles and can be stated in the following way: Consider n particles in the ambient space whose positions are given by the vectors $\mathbf{q}_i$, $i = \overline{1,n}$ (with respect to a fixed frame) and let $\mathbf{q} = (\mathbf{q}_1,\dots,\mathbf{q}_n)$ be the *configuration* of the system. Determine the motion of the n particles by finding the general solution $(\mathbf{q},\dot{\mathbf{q}})$ of the second order system

$$\ddot{\mathbf{q}} = M^{-1}\nabla U(\mathbf{q}),$$

where $U : \mathbf{R}^{3n} \setminus \Delta \to \mathbf{R}_+$, $U(\mathbf{q}) = \sum m_i m_j |\mathbf{q}_i - \mathbf{q}_j|^{-1}$ is called *potential function* (or *force function*) of the system of particles, $\Delta = \bigcup \{\mathbf{q} |\ \mathbf{q}_i = \mathbf{q}_j\}$ is the *collision set* and $M = \mathrm{diag}\,(m_1,m_1,m_1,\dots,m_n,m_n,m_n)$ is a 3n-dimensional diagonal matrix, $m_1,m_2,\dots,m_n$ being the masses of the n particles. The usual formulation is that of an initial value problem for a system of 6n differential equations: Solve

$$\begin{cases} \dot{\mathbf{q}} = M^{-1}\mathbf{p} \\ \dot{\mathbf{p}} = \nabla U(\mathbf{q}) \end{cases} \qquad (1)$$

subject to the initial conditions $(\mathbf{q},\mathbf{p})(0) \in (\mathbf{R}^{3n} \setminus \Delta) \times \mathbf{R}^{3n}$, where $\mathbf{p} = M\dot{\mathbf{q}}$ denotes the *momentum* of the system.

For n = 2 the problem is not difficult and its solution can be found in any celestial mechanics or astronomy text-book under the name of the *two body problem* or the *Kepler problem* (in the honour of the famous German astronomer Johannes Kepler who actually provided to Newton the inspiration of the inverse square attraction law). Depending on the initial conditions, the motion of one particle with respect to the other can be an ellipse (including possibly a circle), a parabola, a branch of hyperbola or a line. This last case, of the *rectilinear* motion, is the only one when collisions between the two particles can take place. It is interesting to know that the complete solution as described above was not given by Newton as one would expect, but by Johann Bernoulli and only in 1710 (see [W,1941]).

For $n \geq 3$ the problem is still open even after three centuries of intense efforts to find its solution. Almost all important mathematicians up to the first quarter of this century attacked some aspect of the n-body problem bringing important contributions to the understanding of the subject. In spite of this fact the global image we have today is still far of being complete.

There were several ways to approach the problem. A modern methods to tackle systems of differential equations in 19th century mathematics was to find *first integrals* and consequently to reduce the dimension of the system. More precisely, a function

$$F: (\mathbf{R}^{3n} \setminus \Delta)\times\mathbf{R}^{3n} \to \mathbf{R}$$

is said to be a *first integral* for the equations (1) if $F(\mathbf{q},\mathbf{p}) = c$ (constant), along a solution $(\mathbf{q},\mathbf{p})$ of it. A relation like this between the components of a solution reduces the dimension of the system by one. It is known that systems of k equations have (locally) k linearly independent first integrals and it was an important goal to find as many integrals as possible. For the equations (1), ten of them were easy to obtain: three integrals of the impulse, three integrals of the center of mass, three of the angular momentum and one energy integral, namely

$$\sum \mathbf{p}_i = \mathbf{a}, \quad \sum m_i\mathbf{q}_i - \mathbf{a}t = \mathbf{b}, \quad \sum \mathbf{q}_i\times\mathbf{p}_i = \mathbf{c} \quad \text{and} \quad T(\mathbf{p}) - U(\mathbf{q}) = h,$$

where **a**, **b**, **c** are constant vectors and h is a real constant, T denoting the *kinetic energy*.

Any furhter attempt to find new ones was unsuccesful and people started to look for other methods. The decisive result which stopped completely the search for first integrals was published in 1887 by Bruns. In a long paper [B,1887] he proved the following negative statement

Theorem 1. *The only linearly independent integrals of the equations (1), algebraic with respect to* **q**, **p** *and* t, *are the ten described above.*

This was an important moment in the history of mathematics which changed the way of thinking since Gallilei. After a long period of quantitative methods, mathematicians understood that the class of problems possible to solve in this way is very small and a large window towards qualitative methods was opened. It is, of course, not the only result signaling a new era. For example Liapunov obtained, approximately at the same time, his stability criteria in a nonquantitative way, studing a question coming also from celestial mechanics.

Approximately one hundered years ago the interest in the problem reached a high level. Advised by Gustav Mittag-Leffler (at that time Editor-in-Chief of *Acta Mathematica*), King Oscar II of Sweden and Norway, a protector and supporter of science and especially of mathematics, established in 1887 an important prize for solving the 3-body problem. The formulation was very precise and asked *to obtain, for any choice of the initial data, a solution of the coordinates expressed as a power series, convergent for all real values of the time variable.* The idea of attacking the problem in this way is attributed to Dirichlet (see

[SM,1971]). Bruns's result was at that time still too fresh to change the belief in success for quantitave methods. A quick battle was expected, unfortunately nobody could provide the desired solution. In spite of this fact the prize was awarded to Henri Poincaré in 1889 for his mémoire *Le problème des trois corps et les equations de la dynamique*, published in *Acta Mathematica* one year later [P, 1890]. The motivation was the stimulating value of this paper for further research in mathematics and mechanics and indeed this choice was a good one. Poincaré's interest was aroused by this success and he continued the investigation into the mysterious n-body problem to which he further dedicated many years of work. He also wrote the even today famous *Les nouvelles methodes de la mecanique celèste*, in three volumes [P, 1890, 1891, 1899], where the idea of *chaos* appears for the first time.

It is interesting to remark not only the fact that many mathematical theories were born from the study of the n-body problem but also that the strength of several new theories is checked today by trying to find applications of them to this old problem. The time when only classical analysis, differential equations and sometimes function theory were the research tools, is history. New fields like dynamical systems, differential topology, differential geometry, Morse theory, algebraic geometry, algebraic topology, symplectic manifolds, Lie groups and algebras, ergodic theory, numerical analysis and computers, operator theory and C*-algebras and many other independent branches of mathematics have been involved in one way or another with this problem.

The conjecture of Painlevé

In 1895, at 32 years of age, Paul Painlevé was already one of the most famous mathematicians of his time and King Oscar II invited him to give a series of lectures at the University of Stockholm in September-November of that year. The event was considered of paramount importance and even the King attended the introductory lecture. The notes were published in 1897 in a hand written form under the title *Leçons sur la théorie analytique des équations différentielles* [Pa,1897] and can be found today also in Painlevé's Complete Works [Pa,1972]. The last pages contain an application of the results to the 3-body problem and the opinion of the author with respect to a question concerning the n-body case, formulated as a statement which was known afterwards as the *Conjecture of Painlevé*. Let us try to understand first its natural birth.

Standard results of the differential equations theory ensure, for any $(\mathbf{q},\mathbf{p})(0) \in (\mathbf{R}^{3n} \setminus \Delta) \times \mathbf{R}^{3n}$ the existence and uniqueness of an analytic solution of the equations (1) defined locally on, let's say (t^-,t^+), with 0 contained in this interval. Due to the symmetry of mechanical laws with respect to past and future one can study the problem on $(t^-,0]$ or on $[0,t^+)$, without loss of generality. Since many scientists have a natural desire to predict future phenomena, let's choose the second interval. We can extend the solution annalytically such that to be defined on a maximum interval $[0,t^*)$, with $0 < t^- \leq t^* \leq \infty$. In case $t^* = \infty$ the motion is called regular while if t^* is finite we say that the solution experiences a *singularity.* What is the physical meaning of such a singularity and is it important to be studied? One obvious possibility for a solution to encounter a singularity is if a collision occurs. Indeed, the configuration vector $\mathbf{q}$ will tend to the set Δ, so at least two position vectors have the same value, consequently ∇U tends to infinity and the equations of motion (1) become meaningless. The establishment of the prize made the importance of such a study very clear. Since a series expansion of the coordinates, convergent for every real value of t was asked, solutions leading to singularities were expected to be prolonged somehow beyond the singularity.

Although very young in 1887, Painlevé was working at his doctoral thesis and knew about the famous problem. He tried therefore to understand if in the 3-body problem the only possible singularities are collisions. His worry on the occurence of other singularities was motivated by the possible appearance of large oscillations (suspected already by Poincaré). For example, one particle could oscillate between the other two without to collide but coming closer and closer to a collision by each close encounter. Under such circumstances one can find a subsequence t_n of time instants converging to a finite t^* such that $\nabla U(\mathbf{q}(t_n)) \to \infty$. This cancels again the meaning of the equations (1) and t^* is also a singularity. In a modern terminology we would say that $\mathbf{q}(t) \to \Delta$ *without asymptotic phase* (i.e. by oscillating between different elements of the set) to distinguish it from the collision case when $\mathbf{q}(t) \to \Delta$ *with asymptotic phase* (i.e. by tending to a definite element of Δ). This new type of singularity was called *pseudocollision* or *noncollision singularity.* Consequently the following natural definition was accepted

Definition 2. Consider $(\mathbf{q},\mathbf{p})$ to be a solution of the equations (1) defined on $[0,t^*)$ with t^* a singularity. Then t^* will be called a *collision singularity* if $\mathbf{q}(t)$ tends to a definite limit when $t \to t^*$, $t < t^*$. If the limit doesn't exist then the singularity will be called a *pseudocollision.*

It is clear that these singularities (especially the noncollision ones) are an important obstacle in trying to accomplish King Oscar's goal. Indeed, one could eventually try to extend a collision as an elastic bounce and possibly to obtain a globally convergent power series, but how to do that with pseudocollisions? Painlevé doubted that pseudocollisions can actually appear and succeeded to prove that for the 3-body case

Theorem 3. *For* $n = 3$, *any solution of the the equations* (1) *defined on* $[0,t^*)$ *with* t^* *finite, encounters a collision when* $t \to t^*$.

Any attempt to extend this result to the n-body problem, with $n > 3$, failed and the intuition of Painlevé was that pseudocollisions may indeed arise for more than 4 bodies. Thus, his famous Stockholm lectures end with the following

Conjecture. *For* $n \geq 4$ *the equations* (1) *admit solutions with noncollision singularities.*

Painlevé understood that this is a very hard problem and his future mathematical work contains some papers dealing with singularities, none however dedicated to prove the conjecture. After 1905 Painlevé's scientific activity becomes less intense because of his deep involvement in politics. His political career is also exceptional, Paul Pinlevé being elected several times as deputy, detaining the War then Finance, again War and finaly Air portfolios and serving as President of the Deputy Chamber of France. In 1918 he became "Président de l'Académie des Sciences" and in 1927 the University of Cambridge offered him the title of Doctor Honoris Causa. Indeed a remarkable and succesful life! His famous conjecture continued to remain open, however, and will stay unsolved for more than half a century after his death.

It is interesting to note that collision orbits are very improbable. Donald Saari proved that in the n-body problem they are of *Lebesgue measure zero* and of the *first Baire category*. Moreover, this is true for all singularities in the 4-body problem (see [S,1971,1972,1973a, 1975]). Some of these results were generalized and are expressed in terms of lower dimensional manifolds [S,1984]. It is also expected that, for any n, singularities are improbable. However, these results did not diminish the interest in the study of singularities.

Singularity criteria

Many of Painlevé's contemporaries tried to find examples of solutions with pseudocollisions but nobody succeeded. Their attention was therefore concentrated towards understanding theoretical aspects and especially in getting criteria for obtaining noncollision singularities.

A way of finding singularities was already known at that time but it is quite hard to decide when and who obtained this theorem first. The current literature claims the following to be a result of astronomical tradition, its formulation being stated below

Theorem 4. *Consider a solution* $(\mathbf{q},\mathbf{p})$ *of the equations* (1). *Then* t^* *is a singularity of this solution iff*

$$\liminf_{t \to t^*} \min_{i<j} q_{ij}(t) = 0, \qquad (2)$$

where $q_{ij} = |\mathbf{q}_i - \mathbf{q}_j|$.

Painlevé himself improved this result [Pa,1897] in order to use the stronger one for proving Theorem 3. He showed that condition (2) can be actually replaced by

$$\lim_{t \to t^*} \min_{i<j} q_{ij}(t) = 0.$$

The first important condition for the occurence of noncollision singularities was found and published only in 1908 by a Swedesh mathematician of German origin, Hugo von Zeipel [Z,1908]. His result has not only a nice formulation but also an unusual history and played a fundamental role in the life of Painlevé's conjecture.

Theorem 5. *If* t^* *is a collision singularity for a solution* $(\mathbf{q},\mathbf{p})$ *of the equations* (1), *then* $J(\mathbf{q}(t))$ *tends to a definite limit when* $t \to t^*$, *where* $J(\mathbf{q}) = \sum m_i|\mathbf{q}_i|^2$ *is the moment of inertia.*

This implies, of course, that a necessary condition for having a noncollision singularity is that the motion has to become unbounded in finite time, since the moment of inertia is a measure of the distribution of particles in space. This is not as obvious as a first look would suggest, because the classical differential equations result - who everybody thinks at when reading this statement - implies only that the whole $|(\mathbf{q},\mathbf{p})|$ has to become unbounded. This happens always (if it is to give a mechanical argument) because at a collision instant the velocities are infinite. Anyway it is not clear why this would be true in the configuration space (i.e. for the vector $\mathbf{q}$) and here lies von Zeipel's contribution. His paper appeared in a less famous journal (see [Z,1908]) and was therefore not well known in the literature. Personally I've tried to find it in several good university libraries in Eastern and Western Europe (not in Sweden, of course) as well as in North America, but without success. An article of Dick McGehee [Mc,1986], who spent a period in Stockholm and was interested in this subject, mollified my interest in reading the original.

At the end of the second decade of this century the French astronomer Jean Chazy announced the same statement without making any reference to von Zeipel's paper [C,1920] and Aurel Wintner wrote in 1941 that the proof of the Swedesh mathematician has some gaps and there is no complete argument for the theorem [W,1941]. Hans Sperling gave 30 years later a detailed proof [Sp,1970], closing apparently the subject of dispute. However, McGehee's paper cited above provides a translation in modern mathematical language and notations of von Zeipel's proof, showing that his performance was actually correct from the very beginning.

Today we know a beautiful generalization of this result which is due to Donald Saari from Northwestern University [S,1973]. He proves that if $J \circ q$ is a *slowly varying* function as $t \to t^*$ for a solution $(\mathbf{q},\mathbf{p})$ of the equations (1), then the singularity t^* is necessarily a collision.

Theorem 5 is a fundamental contribution to the subject of singularities in the n-body problem and the elucidation of Painlevé's conjecture would have been hard to imagine without it.

The computer and the idea

As many times in the sciences, the idea to solve Painlevé's conjecture came by looking for something else and the electronic computers had an important role to play in this discovery. In 1893 Meissel proposed the investigation of a so called Pythagorean problem, in which three gravitationally attracting particles of masses 3, 4 and 5 are initially located at the vertices of a triangle with sides 3, 4 and 5 such that the corresponding point masses and sides are opposite. Releasing the paritcles with zero initial velocities from their positions, how will they move in the future? Burrau investigated the problem numerically in 1913 but without reaching important conclusions. Several computer investigations in 1966 and 1967 [Sz,1967] helped to go much further by showing a surprising qualitative behavior: After passing close to a triple collision, two particles will form a binary while the other one is expelled with high velocity in the opposite direction, as one can see in Figure 1 (see also [A,1988]). The formation of the binary was an interesting point for astronomers, while the high speed escape of the third particle attracted the attention of mathematicians. It provided the idea that an example of a noncollision singualrity solution might be possible to construct. The main reason for this qualitative feature is due to the triple approach of the particles and this fact was recognized in [Mc,1974,1975] and [Wa,1975,1976]. Crucial in this sense were the ideas in McGehee's 1974 paper and we will sketch them below. Dick McGehee considered the case of the rectilinear 3-body problem, i.e. when the masses m_1, m_2, m_3 move all the time on a fixed line. He was interested in understanding the behavior of the motion near a triple collision , or - in a more standard formulation - to give a description of the flow in the neighborhood of triple collision solutions. This was indeed a hard problem since previous numerical investigations suggested the idea of a chaotic behavior near a *total collapse* (i.e. a simultaneous collision of all bodies). Only qualitatively speaking, the particles behave by forming a binary and an escape, numerical endeavors showing a highly sensitive dependence with respect to initial data. For example, for some initial conditions the particles m_1 and m_2 form a binary and m_3 escapes, while perturbing the data a little bit it may be possible that m_1 and m_3 form a binary and m_2 escapes. Two such solutions look very different in a phase space picture in spite of being close to one another at some initial moment of time.

McGehee's idea was to restrict the equations of motion to an arbitrary energy level, then to blow up the singularity (by using certain transformations which bear today his name) and finally to paste instead a so called *collision manifold* which is proved to be independent of the chosen energy level. In the rectilinear 3-body problem the collision manifold happens to be a sphere without four points like in Figure 2.

Roughly speaking the McGehee coordinates introduce polar coordinates for the configuration vector and a decomposition of the velocity into a radial and a tangential component, rescaled by a suitable transformation of time, which makes the collision manifold to be reached asymptotically by the real flow, when the new (fictitious) time variable goes to infinity.

The equations of motion restricted to the collision manifold do not describe a real physical situation. However, due to the continuity property of the solutions with respect to initial data, a

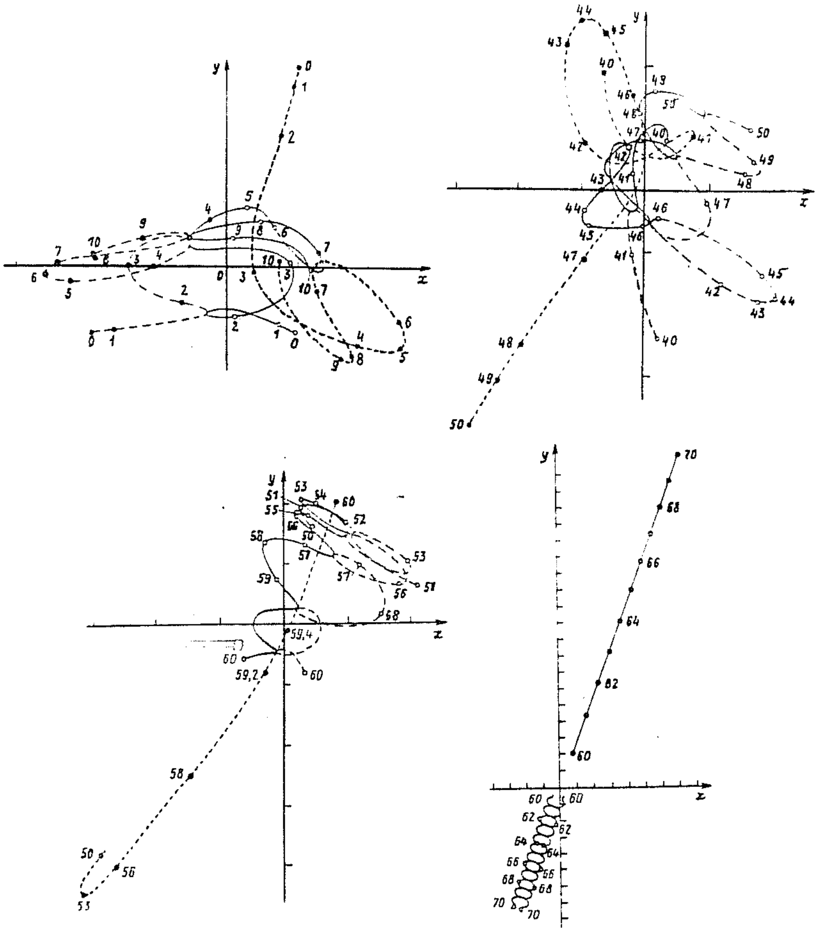

Figure 1

Numerical results in the Pythagorean problem

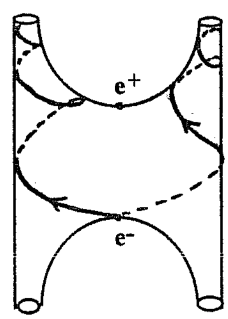

Figure 2

The flow on the collision manifold

study of the flow on this manifold provides precious information on the behavior of solutions passing close to a triple collision. Many interesting theoretical results could be proved in

McGehee's paper using these powerful instruments, including a theorem on the occurence of solutions with high velocity escapes. Studies on collision singularities are therefore hard to imagine today without McGehee's transformations.

The example of Mather and McGehee

One of the results McGehee announces (without proof) in his 1974 paper is the construction of a solution with noncollision singularities in the rectilinear 5-body problem, using the idea of a high speed escape. The trouble is not that collisions always appear in a rectilinear problem but the fact that they always arise before an eventual pseudocollision, as it was shown by Donald Saari [S,1973b]. The reader might be confused now since the solution was already defined on a maximal interval [0,t*), t* (finite) representing a singularity which can be either a collision or a noncollision one. For a specialist, however, this is not ambiguous. Binary collision solutions can be analytically extended by a mathematical procedure called *regularization.* There is a vast literature on this subject (see e.g. [D,1992]). Physically, this means that an elastic bounce, without loss or gain of energy, takes place. Now Saari's result becomes probably clear for everybody.

John Mather and Dick McGehee [MM,1975] were later able to prove completely the existence of a noncollision singularity in the rectilinear 4-body problem but only after the occurence of an infinity of (regularized) binary collisions. Let's describe the scenario proposed by these authors.

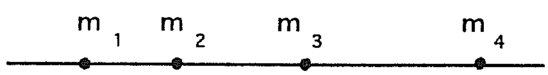

Figure 3
The example of Mather and McGehee

Consider four bodies of suitably chosen masses m_1, m_2, m_3, m_4 which lie on a straight line at some initial moment (see Figure 3). The initial data (positions and velocities) are taken such that the particles m_1 and m_2 stay close together, so we say that they form a binary system. The particle m_3 will oscillate between the binary system and the particle m_4, respectively. The motion is regularized beyond the binary collisions which take place at the instants of time t_1, $t_2,\dots,t_k,\dots$. This sequence will converge as k goes to infinity. Meanwhile the binary m_1, m_2 goes to $-\infty$, m_4 to $+\infty$ and m_3 bounces back and forth, with increased velocity after every close passage to a triple collision. This is possible also because the distance between m_1 and m_2 tends to zero, the loss of potential energy of the binary being transfered into kinetic energy for the particle m_3. It is not at all easy to prove that these facts are possible.

However, besides its mathematical beauty and interest for the dynamical systems theory, the above example is not accepted as a proof of Painlevé's conjecture because the pseudocollision appears only after (infinitely many) collisions.

Gerver's first example

In 1984 Joe Gerver from Rutgers University described a heuristic solution of a planar 5-body problem in which the particles escape to infinity in finite time [G,1984]. Although he doesn't give a complete proof, Gerver brings a lot of arguments supporting the idea for the existence of such a solution. We reproduce further his proposed scenario.

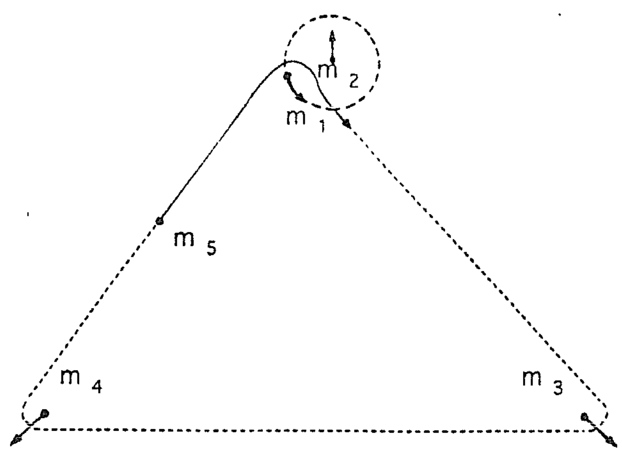

Figure 4

Gerver's heuristic example

Consider the planar motion of five particles $m_1, \dots, m_5$, with $m_3 = m_4$, m_2 somewhat greater but of the same order of magnitude as m_3, m_1 much greater than m_2 and m_5 much smaller than m_1 (see Figure 4). Initially m_1 is in a roughly circular orbit around m_2 while m_3 and m_4 are much further away. The bodies m_2, m_3, m_4 are approximately at the vertices of an obtuse triangle. Choosing suitable initial velocities the triangle is slowly expanding whilst maintaining its shape. Meanwhile m_5 moves rapidly around the triangle, coming close to each of the other four bodies, the velocity of m_5 being much greater than that of m_1. Each time m_5 passes close to m_1, it picks up a small amount of kinetic energy. This causes m_1 to fall into a lower orbit around m_2 such that the mean kinetic energy of m_1 in its orbit actually increases by about the same factor as for m_5. A small fraction of the kinetic energy of m_5 is transfered to m_2, m_3 and m_4, causing a faster rate of the triangle expanding. The time required for one trip of m_5 around the triangle decreases each time (in spite of the expansion) by a factor slightly less than one. After a finite time the geometric progression of the time instants $t_1, t_2, ..., t_k, ...$ measuring a round trip will converge and m_5 would have traveled an infinite number of times around the triangle. In the mean time the triangle has become infinitely large.

Xia's example

In his Ph.D thesis written under the supervision of Donald Saari at Northwestern University, Jeff Xia proved in 1988 that a certain type of solution in the spatial 5-body problem, leads to a noncollision singularity without to be implied by an infinite number of binary collisions, as it was the case in the example of Mather and McGehee. Painlevé's conjecture was finally proved.

The author considers two pairs of bodies, the particles in the same pair having equal masses, plus a fifth particle of small mass. The bodies in a pair move on highly eccentric orbits parallel with the (x,y) plane (see Figure 5). The binaries are on opposite sides with respect to the (x,y)-plane and have an opposite rotation such that the total angular momentum is zero. This is possible since the motion of the small particle is restricted to the z-axis. The small particle will oscillate between the two binaries determining the occurence of an unbounded motion in finite time. More precisely, suppose the particle m_5 intersects the line connecting m_3 with m_4 from above, at a moment when these particles come near to their closest approach, the motion of m_3, m_4 and m_5 being thus close to a triple collision. The body m_5 goes a little bit under the line m_3m_4 while the particles m_3 and m_4 are at their closest approach. This makes m_5 to be strongly attracted backwards and it intersects the line m_3m_4 again when these point masses start to separate. This separation reduces the retaining force on the small particle which consequently moves very fast towards the other binary system. The action-reaction effect forces the binary m_3, m_4 to move further away from the plane (x,y). The same situation described above is now repeated (in the mirror) for the binary m_1, m_2. Iterating this procedure with higher and higher accelerations for m_5, the two binaries will be forced to tend to infinity in finite time. However, the fact that this apparently simple scenario can take place, is indeed very hard to prove. There are a lot of difficulties to overcome. For example, since the motion tends to be unbounded in finite time, the acceleration effects on the small particle have to become infinitely large. This implies that the point masses in each binary will come infinitesimally close together, justifying the worry that uncontrolable collisions may appear.

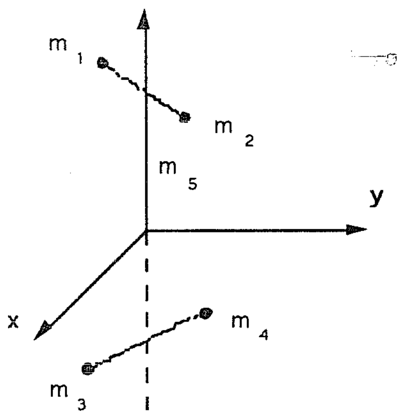

Figure 5

The example of Xia

Some mistakes occured in the first attempt of Jeff Xia but he succeeded to correct them and to show that his proof solves the problem. The paper will appear in the Annals of Mathematics.

His example can be extended to similar symmetric problems for any N.

In spite of his youth (not even 30 years old in 1992), today associate professor at Georgia Tech, Jeff Xia already brought a tremendous contribution to the field. He recently proved a new magnificent result, namely that the very rare (and hard to put into the evidence) phenomenon called *Arnold diffusion* (some kind of chaos), takes place in a very natural system, the elliptical restricted 3-body problem. Arnold himself constructed in the 60's a very sophisticated and artificial system in order to show, for the first time, that such a phenomenon exists. It is expected that Jeff Xia will bring many other important contributions in the next years.

Gerver's second example

The idea of using a radial symmetry, combined with the experience obtained by trying to prove his previous heuristic example, led Joe Gerver to the following solution for the planar case. Consider 3n bodies (n sufficiently large) in a plane like in Figure 5. 2n of the particles are arranged in n nearly circular orbiting pairs and have all the same mass. The center of mass of each binary lies at one of the vertices of a regular polygon. The other n bodies have small equal masses and move rapidly from one pair to the other like in Figure 6. When a small particle comes close to the binary it takes some kinetic energy from the pair and transfers some momentum to it, forcing on one side the binary to move into a tighter orbit and on the other side to increase its distance from the center of the polygon. Iterating this process for a suitably chosen n, suitable values of the masses and of the initial velocities, the size of the configuration will increase by each close encounter of a small particle with a binary. The sequence of times from one encounter to the next will converge to a finite value, while the system becomes unbounded in finite time. The complete proof contains very many computations and is therefore quite hard to follow (see [G,1991]).

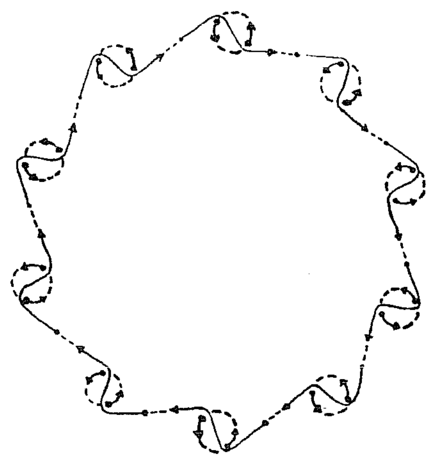

Figure 5
Gerver's planar example

Joe Gerver found out about Painlevé's conjecture 19 years before he gave the solution. Jeff

Xia succeeded to prove his example about six months before Gerver. However the above one is the first confirmation of the conjecture for the case of planar solutions and is also very elementary, using mainly 19th century mathematics. Seeing the proof one would merely say that the conjecture would have been possible to complete by Painlevé's contemporaries, but nobody did it. It was not the first time when Joe Gerver attacked a famous problem. As a graduate student at Columbia University he proved Riemann's conjecture on the ζ-function without too much effort. But this happend much before his work on Painlevé's conjecture started.

A comparison between the two solutions is hard to make. Each is interesting and valuable in its own way. Xia opened a new direction of work bringing fresh air into the field while Gerver used the old methods showing that they can be succesful too. However one has to agree that both obtained a most remarcable performance in an old and hard field where good new results are not at all easy to obtain.

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