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L. Carlitz extended certain known generating functions for Laguerre and Jacobi polynomials to the forms:

$$\sum_{n=0}^{\infty} c_n^{(\alpha+\lambda n)} \frac{t^n}{n!} \quad \text{and} \quad \sum_{n=0}^{\infty} d_n^{(\alpha+\lambda n, \beta+\mu n)} \frac{t^n}{n!},$$

respectively, where $c_n^{(\alpha)}$ and $d_n^{(\alpha,\beta)}$ are general one– and two–parameter coefficients. Subsequently, H.M. Srivastava considered a number of generalizations of Carlitz's results and showed how these general multivariable (and multiparameter) generating functions would apply to a wide variety of polynomial systems studied in the literature. The object of the present note is to deduce several new generating functions for the Hermite polynomials by applying some of these earlier results.

1. INTRODUCTION

In the usual notations for the classical orthogonal polynomials, let $L_n^{(\alpha)}(x)$ and $P_n^{(\alpha,\beta)}(x)$ denote the Laguerre and Jacobi polynomials, respectively (see, *e.g.*, Szegö [12] and Rainville [8]). The following generating function was given by Carlitz [1, p. 826, Equation (8)]:

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\lambda n)}(x)\, t^n = \frac{(1+v)^{\alpha+1}}{1-\lambda v} \exp(-xv), \tag{1}$$

where α and λ are arbitrary (real or complex) numbers, and v is a function of t defined by

$$v = t(1+v)^{\lambda+1}, \quad v(0) = 0. \tag{2}$$

We also recall a subsequent generalization of Carlitz's result (1) due to Srivastava and Singhal [11, p. 749, Equation (8)]:

$$\sum_{n=0}^{\infty} P_n^{(\alpha+\lambda n,\beta+\mu n)}(x)\, t^n \tag{3}$$

$$= (1+\zeta)^{\alpha+1}(1+\eta)^{\beta+1}\{1-\lambda\xi-\mu\eta-(1+\lambda+\mu)\xi\eta\}^{-1},$$

where ξ and η satisfy

$$(x+1)^{-1}\xi = (x-1)^{-1}\eta = \tfrac{1}{2}\, t(1+\xi)^{\lambda+1}(1+\eta)^{\mu+1}. \tag{4}$$

Motivated by (1) and (3), Carlitz [3] derived generating functions for certain general one– and two–parameter coefficients. Subsequently, Srivastava [9] presented several generalizations of Carlitz's result and applied his general multivariable (and multiparameter) generating functions to various systems of polynomials in one and more variables.

The generating functions (1) and (3), and their numerous special cases, have been recorded systematically by Hansen [6], and (more recently) by Srivastava and Manocha

[10]. As a matter of fact, Srivastava and Manocha [10, Chapter 7] have also included a systematic presentation of the aforementioned works of Carlitz [3] and Srivastava [9].

For the classical Hermite polynomials $H_n(x)$ defined by (*cf.*, *e.g.*, [12, p. 106, Equation (5.5.4)])

$$(5)\qquad H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \begin{bmatrix} n \\ 2k \end{bmatrix} \frac{(2k)!}{k!} (2x)^{n-2k},$$

Carlitz [3] showed that

$$(6)\qquad \sum_{n=0}^{\infty} H_n(x+ny) \frac{t^n}{n!} = \frac{\exp(2xw-w^2)}{1-2yw}, \quad w = t\exp(2yw),$$

and Cohen [5] and Srivastava [9] gave further generating functions analogous to (6). (See also Srivastava and Manocha [10, Chapter 7, Problems 24 and 29].)

Each of the generating functions (1), (3), and (6), and their analogues and generalizations referred to above, has tremendous potential for applications. For example, the generating function (3) was applied recently by Chen and Ismail [4] in order to determine the asymptotic behavior of the Jacobi polynomials $P_n^{(\alpha+\lambda n,\beta+\mu n)}(x)$ when $n \to \infty$ and α, β, λ, μ, and x remain fixed. With such objectives as these in view, we derive here a number of new generating functions for Hermite polynomials analogous to Carlitz's formula (6).

2. A CONSEQUENCE OF MEHLER'S FORMULA

We begin by recalling a bilinear generating function for Hermite polynomials, known as Mehler's formula [7], in its *equivalent* form:

$$(7) \qquad \sum_{n=0}^{\infty} H_n(x\sqrt{\alpha})\, H_n(y\sqrt{\alpha})\, \frac{z^n}{n!}$$

$$= (1-4z^2)^{-\frac{1}{2}} \exp\left[\alpha\left\{\frac{4xyz - 4(x^2+y^2)z^2}{1-4z^2}\right\}\right].$$

The bilinear generating function (7) is of the form considered in

THEOREM A (*cf.* Srivastava and Manocha [10, p. 376, Theorem 11]). *Let* $A(z)$ *and* $B(z)$ *be arbitrary functions which are analytic in a neighborhood of the origin, and assume that*

$$(8) \qquad A(0) = B(0) = 1.$$

Define the coefficients $\{c_n^{(\alpha)}\}$ *by means of*

$$(9) \qquad A(z)[B(z)]^{\alpha} = \sum_{n=0}^{\infty} c_n^{(\alpha)} \frac{z^n}{n!},$$

where α *is an arbitrary complex number independent of* z.

Then, for an arbitrary λ *independent of* z,

$$(10) \qquad \sum_{n=0}^{\infty} c_n^{(\alpha+\lambda n)} \frac{t^n}{n!} = \frac{A(u)[B(u)]^{\alpha+1}}{B(u) - \lambda u\, B'(u)},$$

where

$$u = t[B(u)]^{\lambda}\,. \tag{11}$$

Applying Theorem A with

$$A(z) = (1-4z^2)^{-\frac{1}{2}}, \quad B(z) = \exp\left[\frac{4xyz - 4(x^2+y^2)z^2}{1 - 4z^2}\right],$$

and

$$c_n^{(\alpha)} = H_n(x\sqrt{\alpha})\; H_n(y\sqrt{\alpha}) \qquad (n = 0,1,2,\cdots),$$

we thus find the generating function:

$$\sum_{n=0}^{\infty} H_n(x\sqrt{\alpha + \lambda n})\; H_n(y\sqrt{\alpha + \lambda n})\, \frac{t^n}{n!} \tag{12}$$

$$= (1-4u^2)^{-\frac{1}{2}} \exp\left[\alpha\left\{\frac{4xyu - 4(x^2+y^2)u^2}{1 - 4u^2}\right\}\right]$$

$$\cdot \left\{1 - \frac{\lambda u[4xy(1+4u^2) - 8(x^2+y^2)u]}{(1-4u^2)^2}\right\}^{-1},$$

where

$$u = t\exp\left[\lambda\left\{\frac{4xyu - 4(x^2+y^2)u^2}{1 - 4u^2}\right\}\right]. \tag{13}$$

The generating function (12) can also be derived by appealing to the general result contained in

THEOREM B (*cf.* Srivastava and Manocha [10, p. 378, Theorem 12]). *Let* $A(z)$, $B(z)$, *and* $z^{-1}C(z)$ *be arbitrary functions which are analytic in a neighborhood of the origin, and assume that*

$$A(0) = B(0) = C'(0) = 1. \tag{14}$$

Define the sequence of functions $\{f_n^{(\alpha)}(x)\}$ *by means of*

$$A(z)[B(z)]^{\alpha} \exp(xC(z)) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) \frac{z^n}{n!}, \tag{15}$$

where α *and* x *are arbitrary complex numbers independent of* z.

Then, for arbitrary parameters λ *and* y *independent of* z,

$$\sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny) \frac{t^n}{n!} \tag{16}$$

$$= \frac{A(\zeta)\,[\,B(\zeta)]^{\alpha} \exp(x\,C(\zeta))}{1 - \zeta\{\lambda[B'(\zeta)/B(\zeta)] + y\,C'(\zeta)\}},$$

where

$$\zeta = t[B(\zeta)]^{\lambda} \exp(yC(\zeta)). \tag{17}$$

Indeed, if in Theorem B, we set $\alpha = 0$,

$$A(z) = (1-4z^2)^{-\frac{1}{2}}, \quad B(z) = 1, \quad C(z) = \frac{4xyz - 4(x^2+y^2)z^2}{1-4z^2},$$

and

$$f_n^{(0)}(\alpha) = H_n(x\sqrt{\alpha})\, H_n(y\sqrt{\alpha}) \qquad (n = 0,1,2,\cdots),$$

we shall arrive once again at the generating function (12).

3. FURTHER GENERATING FUNCTIONS

The method used in the derivation of the generating function (12) applies to the known result [2, p. 117, Equation (1.2)]

$$(18) \qquad \sum_{\ell,m=0}^{\infty} H_{\ell+m}(x)\, H_\ell(y)\, H_m(z) \frac{u^\ell}{\ell!} \frac{v^m}{m!}$$

$$= (1-4u^2-4v^2)^{-\frac{1}{2}} \exp\left[\frac{-4x^2(u^2+v^2) + 4x(yu+zv) - 4(yu+zv)^2}{1-4u^2-4v^2}\right].$$

Rewriting (18) in the form:

$$(19) \qquad \sum_{\ell,m=0}^{\infty} H_{\ell+m}(x\sqrt{\alpha})\, H_\ell(y\sqrt{\alpha})\, H_m(z\sqrt{\alpha})\, t^{\ell+m} \frac{u^\ell}{\ell!} \frac{v^m}{m!}$$

$$= (1-4u^2t^2-4v^2t^2)^{-\frac{1}{2}} \exp\left[\alpha\left\{\frac{-4x^2(u^2+v^2)t^2 + 4x(yu+zv)t - 4(yu+zv)^2t^2}{1-4u^2t^2-4v^2t^2}\right\}\right],$$

it is clearly seen that Theorem A and Theorem B would apply with

$$c_n^{\alpha} = f_n^{(0)}(\alpha) = H_n(x\sqrt{\alpha}) \sum_{\ell+m=n} \begin{bmatrix} n \\ \ell \end{bmatrix} H_{\ell}(y\sqrt{\alpha})\, H_m(z\sqrt{\alpha})\, u^{\ell} v^{m} .$$

We thus obtain the generating function:

$$(20) \qquad \sum_{n=0}^{\infty} H_n(x\sqrt{\alpha + \lambda n}) \frac{t^n}{n!} \sum_{\ell+m=n} \begin{bmatrix} n \\ \ell \end{bmatrix} H_{\ell}(y\sqrt{\alpha + \lambda n})\, H_m(z\sqrt{\alpha + \lambda n})\, u^{\ell} v^{m}$$

$$= (1-4u^2\zeta^2 - 4v^2\zeta^2)^{-\frac{1}{2}} \exp\left[\alpha\left\{\frac{-4x^2(u^2+v^2)\zeta^2 + 4x(yu+zv)\zeta - 4(yu+zv)^2\zeta^2}{1 - 4u^2\zeta^2 - 4v^2\zeta^2}\right\}\right]$$

$$\cdot \left\{1 - \frac{\lambda\zeta[4x(yu+zv) - 8\{x^2(u^2+v^2) + (yu+zv)^2\}\zeta + 16x(yu+zv)(u^2+v^2)\zeta^2]}{(1 - 4u^2\zeta^2 - 4v^2\zeta^2)^2}\right\}^{-1},$$

where

$$(21) \qquad \zeta = t \exp\left[\lambda\left\{\frac{-4x^2(u^2+v^2)\zeta^2 + 4x(yu+zv)\zeta - 4(yu+zv)^2\zeta^2}{1 - 4u^2\zeta^2 - 4v^2\zeta^2}\right\}\right].$$

More generally, the case $m = 0$ of Problem 11 of Srivastava and Manocha [10, p. 496] (see also [2, p. 120, Equation (2.3)]), *viz*

$$(22)\qquad \sum_{n_1,\cdots,n_k=0}^{\infty} H_{m+n_1+\cdots+n_k}(x)\, H_{n_1}(y_1)\cdots H_{n_k}(y_k)\, \frac{u_1^{n_1}}{n_1!}\cdots\frac{u_k^{n_k}}{n_k!}$$

$$= (1-4\,\Sigma u_i^2)^{-m-\frac{1}{2}} \exp\left[x^2 - \frac{(x-2\,\Sigma u_i y_i)^2}{1-4\,\Sigma u_i^2}\right]$$

$$\cdot\, H_m\left[\frac{x-2\,\Sigma u_i y_i}{\sqrt{(1-4\,\Sigma u_i^2)}}\right],$$

similarly yields the generating function:

$$(23)\qquad \sum_{n=0}^{\infty} H_n(x\sqrt{\alpha+\lambda n})\,\frac{t^n}{n!}$$

$$\cdot \sum_{m_1+\cdots+m_k=n} \begin{bmatrix} n \\ m_1,\cdots,m_k \end{bmatrix} H_{m_1}(y_1\sqrt{\alpha+\lambda n})\cdots H_{m_k}(y_k\sqrt{\alpha+\lambda n})\, u_1^{m_1}\cdots u_k^{m_k}$$

$$= (1-4\omega^2\,\Sigma u_i^2)^{-\frac{1}{2}} \exp\left[\alpha\left\{x^2 - \frac{(x-2\omega\,\Sigma u_i y_i)^2}{1-4\omega^2\,\Sigma u_i^2}\right\}\right]$$

$$\cdot \left\{1 - \frac{\lambda\omega(x-2\omega\,\Sigma u_i y_i)(4\,\Sigma u_i y_i - 8x\omega\,\Sigma u_i^2)}{(1-4\omega^2\,\Sigma u_i^2)^2}\right\}^{-1},$$

where, for convenience,

$$\begin{bmatrix} n \\ m_1, \cdots, m_k \end{bmatrix} = \frac{n!}{m_1! \cdots m_k!}$$

denotes the multinomial coefficient, the range of each i–summation is from $i = 1$ to $i = k$ $(k = 1,2,3,\cdots)$, and

$$\omega = t \exp\left[\lambda\left\{x^2 - \frac{(x - 2\omega\ \Sigma u_i y_i)^2}{1 - 4\omega^2\ \Sigma u_i^2}\right\}\right]. \tag{24}$$

For $k = 2$, (23) evidently yields the generating function (20). Formula (20), in turn, would reduce to the generating function (12) when we set $u = 1$ and $v = 0$.

Finally, we remark that the method illustrated above would apply to some other results of Carlitz [2, p. 118, Equations (1.4) and (1.5)]. The first of these results may be recalled here in its equivalent form:

$$\sum_{p,q,r=0}^{\infty} H_{q+r}(x\sqrt{\alpha})\ H_{r+p}(y\sqrt{\alpha})\ H_{p+q}(z\sqrt{\alpha})\ t^{p+q+r} \frac{u^p}{p!} \frac{v^q}{q!} \frac{w^r}{r!} \tag{25}$$

$$= (1 - 4t^2\ \Sigma u^2 + 16uvw\ t^3)^{-\frac{1}{2}}$$

$$\cdot \exp\left[\alpha\left\{\Sigma x^2 - \frac{\Sigma x^2 - 4t\ \Sigma xyw - 4t^2 (\Sigma x^2 u^2 - 2\Sigma xyuv)}{1 - 4t^2\ \Sigma u^2 + 16uvw\ t^3}\right\}\right],$$

so that

$$c_n^{\alpha} = f_n^{(0)}(\alpha) = \sum_{p+q+r=n} \begin{bmatrix} n \\ p,q,r \end{bmatrix} H_{n-p}(x\sqrt{\alpha})\ H_{n-q}(y\sqrt{\alpha})\ H_{n-r}(z\sqrt{\alpha})\ u^p v^q w^r,$$

and (by appealing to Theorem A or Theorem B) we arrive at the generating function:

$$(26) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{p+q+r=n} \begin{bmatrix} n \\ p,q,r \end{bmatrix} H_{n-p}(x\sqrt{\alpha+\lambda n})\, H_{n-q}(y\sqrt{\alpha+\lambda n})$$

$$\cdot\, H_{n-r}(z\sqrt{\alpha+\lambda n})\, u^p v^q w^r$$

$$= (1 - 4\Omega^2\, \Sigma u^2 + 16uvw\, \Omega^3)^{-\frac{1}{2}}$$

$$\cdot\, \exp\left[\alpha\left\{\Sigma x^2 - \frac{\Sigma x^2 - 4\Omega\, \Sigma xyw - 4\Omega^2(\Sigma x^2 u^2 - 2\,\Sigma xyuv)}{1 - 4\Omega^2\, \Sigma u^2 + 16uvw\, \Omega^3}\right\}\right]$$

$$\cdot\, \left\{1 - \frac{\lambda\Omega(a + b\Omega + c\Omega^2 + d\Omega^3 + e\Omega^4)}{(1 - 4\Omega^2\, \Sigma u^2 + 16uvw\, \Omega^3)^2}\right\}^{-1},$$

where

$$(27) \qquad \Omega = t\exp\left[\lambda\left\{\Sigma x^2 - \frac{\Sigma x^2 - 4\Omega\, \Sigma xyw - 4\Omega^2(\Sigma x^2 u^2 - 2\,\Sigma xyuv)}{1 - 4\Omega^2\, \Sigma u^2 + 16uvw\, \Omega^3}\right\}\right],$$

$$\begin{cases} a = 4\,\Sigma xyw, \qquad b = 8\{\Sigma x^2 u^2 - 2\Sigma xyuv - (\Sigma x^2)(\Sigma u^2)\}, \\ c = 16\{3uvw\,\Sigma x^2 + (\Sigma xyw)(\Sigma u^2)\}, \\ d = -128uvw\,\Sigma xyw, \qquad e = -64uvw(\Sigma x^2 u^2 - 2\Sigma xyuv), \end{cases}$$

and Σx^2, $\Sigma x^2 u^2$, Σxyw, etc. are symmetric functions in the indicated variables which form the triples (x,y,z) and (u,v,w).

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