Symbolic and Geometric Representations of Unimodular Pisot Substitutions

by

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Abstract

We review the construction of three Smale spaces associated to a unimodular Pisot substitution on d letters: a subshift of finite type (SFT), a substitution tiling space, and a hyperbolic toral automorphism on the Euclidean d-torus. By considering an SFT whose elements are biinfinite, rather than infinite, paths in the graph associated to the substitution, we modify a well-known map to obtain a factor map between our SFT and the hyperbolic toral automorphism on the d-torus given by the incidence matrix of the substitution. We prove that if the tiling substitution forces its border, then this factor map is the composition of an s-resolving factor map from the SFT to a one-dimensional substitution tiling space and a u-resolving factor map from the tiling space to the d-torus.

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Chapter 1

Introduction

Symbolic dynamical systems were first introduced to better understand the dynamics of geometric maps. This is done by coding the orbits of a dynamical system with respect to a well chosen finite partition indexed by an alphabet $\mathcal{A}$, and then studying the symbolic dynamical system on a subset of $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$.

Substitution sytems are an important class of symbolic dynamical systems. Although they are very interesting from a mathematical point of view, additional motivation for studying substitution dynamical system comes from other branches of science, since they provide mathematical models for systems exhibiting self-similarity. In the physics of quasicrystals, substitution sequences are used to model atomic configurations. Substitution dynamical systems are also closely related to tiling dynamical systems and adic transformations [10]. [16] and [8] are excellent references on substitution dynamical systems.

Constant length substitutions have been well understood since the late 1970s, and since then, researchers have been focusing on nonconstant length substitutions. The motivation to study unimodular Pisot substitutions is given by the key conjecture in the theory of substitutions of nonconstant length, known as the Pure

Discrete Spectrum Conjecture [4]. This conjecture states that every unimodular Pisot substitution dynamical system has pure discrete spectrum, or, equivalently, is metrically isomorphic to a translation on a compact Abelian group [10]. Under the conditions of unimodularity and Pisot type, the substitutive dynamical systems are not weakly mixing, and their spectrum is the spectrum of a toral translation. In terms of quasicrystals, pure discrete spectrum corresponds to the atomic configuration being pure point diffractive [10].

To study the question of whether unimodular Pisot substitutive dynamical systems have pure discrete spectrum, G. Rauzy developed the idea of constructing a semiconjugacy from the substitution dynamical system onto a toral translation, and then showing that it is a.e. injective.

Arnoux and Ito [2] prove the following:

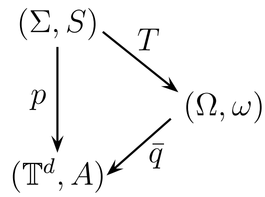
Theorem 1.1 *Let σ be a unimodular Pisot type substitution over a d-letter alphabet which satisfies the coincidence condition. Then the substitutive dynamical system (X_σ, S) associated with σ is measure-theoretically isomorphic to the exchange of d domains defined almost everywhere on the self-similar Rauzy fractal of σ. Furthermore, (X_σ, S) admits as a continuous factor an irrational translation on the torus $\mathbb{T}^{d-1}$, the fibres being finite almost everywhere.*

The work in [2] on the Rauzy fractal sparked a lot of interest from a variety of standpoints, and practically all work on unimodular Pisot substitutions now involves Rauzy fractals. Canterini and Siegel [7] give an alternate proof of the above theorem and some additional results using a "prefix-suffix automaton". Sirvent and Wang [18] prove a number of tiling properties of the Rauzy fractal using graph-directed iterated function systems. The most recent and complete study of the geometric theory of unimodular Pisot substitutions is given by Barge and Kwapisz [4]. They use the notions of a "strand space" and "tiling space" to lead to a number of necessary and sufficient conditions for unimodular Pisot substitution dynamical systems to have pure discrete spectrum, including measure and tiling properties of the Rauzy fractal of the substitution.

The foundation of almost all of the work on unimodular Pisot substitutions is the stable/unstable decompostion $\mathbb{R}^d = A^c \oplus A^e$ given by the incidence matrix A of the substitution, where $A^c \cong \mathbb{R}^{d-1}$ and $A^e \cong \mathbb{R}$. Rather than working directly with the substitution dynamical system, we take a somewhat different approach and concern ourselves with three Smale spaces associated to a unimodular Pisot substitution σ on a d-letter alphabet. These consist of a subshift of finite type (Σ, S), a substitution tiling space (Ω, ω) on A^e, and the d-torus together with the linear transformation given by A, $(\mathbb{T}^d, A)$. By considering a subshift of finite type whose elements are biinfinite, rather than infinite, paths in the graph associated to the substitution, we are able to use the same tools to study geometric representations of σ on A^e and A^c. We remark that a substitution tiling space $(\hat{\Omega}, \hat{\omega})$ can be defined on A^c such that our results involving the one-dimensional tiling space (Ω, ω) can be extended without much difficulty to this $(d-1)$-dimensional tiling space.

The goal of this thesis is twofold. We aim to summarize and unify the methods of the study of the geometric theory associated to unimodular Pisot substitutions. And in doing so, we provide one of the first concrete examples of a property of factor maps between Smale spaces proven in [14].

Assuming that (Ω, ω) satisfies the forcing the border condition, we define the following maps, prove that they are factor maps, and that the following diagram commutes:



Moreover, we prove that T is s-resolving and $\bar{q}$ is u-resolving. This gives one of the first concrete examples of what Putnam proves in [14]:

Theorem 1.2 *Let (Y, g) be an irreducible Smale space. Then there is another irreducible Smale space, (Ω, ω), an irreducible shift of finite type, (Σ, S), and two factor maps, $\phi_1 : (\Sigma, S) \to (\Omega, \omega)$ and $\phi_2 : (\Omega, \omega) \to (Y, g)$, such that ϕ_1 is s-resolving while ϕ_2 is u-resolving.*

Chapter 2 contains formal definitions and the background necessary to make this work self-contained. We introduce the concepts of (symbolic) substitutions, tiling substitutions, the geometric representation of substitutions, subshifts of finite type associated with substitutions, and finally, we say a few words on Smale spaces.

In Chapter 3, we prove our main results. We define various maps between Smale spaces associated to unimodular Pisot substitutions and prove that they satisfy a number of desirable properties.

The Appendix contains a few technical proofs omitted from Chapters 2 and 3 due to length.

Chapter 2

Definitions and Background

We begin by formally introducing the concepts we'll use to study unimodular Pisot substitutions. These include symbolic dynamical systems, tiling substitutions, subshifts of finite type, a stable/unstable decomposition of $\mathbb{R}^d$, Rauzy fractals, and Smale spaces.

2.1 Substitutions

After discussing our notation, the formal definitions of various classifications of substitutions, and a few basic results, we give a brief introduction to substitution dynamical systems.

2.1.1 Notation and Classifications

We start with an *alphabet* $\mathcal{A} = \{1, 2, \ldots, d\}$, the elements of which are called *letters*. To avoid triviality, we assume $d \geq 2$. Strings of letters of $\mathcal{A}$ are called *words* on $\mathcal{A}$. We use the notation $u = u_1 u_2 \cdots u_n$ for finite words (where n is called the length of the word, denoted $|u|$), $v = v_1 v_2 v_3 \cdots$ for infinite words, and $w =$

$\cdots w_{-2}w_{-1} \cdot w_0w_1w_2 \cdots$ for biinfinite words. The unique word on $\mathcal{A}$ containing no letters is denoted ε and called the empty word. For $n \geq 0$, the set of words on $\mathcal{A}$ of length n is denoted $\mathcal{A}^n$. The set of finite words on $\mathcal{A}$ is denoted $\mathcal{A}^* = \cup_{n=0}^{\infty}\mathcal{A}^n$, and the set of biinfinite (respectively infinte) words on $\mathcal{A}$ is denoted $\mathcal{A}^{\mathbb{Z}}$ (respectively $\mathcal{A}^{\mathbb{N}}$). In addition, we will occasionally use the notation $\mathcal{A}^{-\mathbb{N}} = \{\cdots w_{-3}w_{-2}w_{-1} \mid w_i \in \mathcal{A},\ i \leq -1\}$.

A *substitution* is a map $\sigma : \mathcal{A} \to \mathcal{A}^*\backslash\{\varepsilon\}$. We denote

$$\sigma(i) = W^{(i)} = W_1^{(i)}W_2^{(i)} \cdots W_{l(i)}^{(i)},$$

where $i \in \mathcal{A}$ and $l(i)$ is the length of $\sigma(i)$. If $l(i) = l(j)$ for all $i, j \in \mathcal{A}$, then σ is said to be of *constant length*.

The substitution σ extends naturally to an endomorphism of $\mathcal{A}^*$, also denoted σ, by the rules $\sigma(\varepsilon) = \varepsilon$ and $\sigma(u_1 \cdots u_n) = \sigma(u_1) \cdots \sigma(u_n)$ for $u = u_1 \cdots u_n \in \mathcal{A}^*$. We generalize our notation above, and write for any $u \in \mathcal{A}^*$, $\sigma(u) = W^{(u)} = W_1^{(u)}W_2^{(u)} \cdots W_{|\sigma(u)|}^{(u)}$. We further define the morphisms $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ and $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ by $\sigma(\cdots w_{-1} \cdot w_0w_1 \cdots) = \cdots \sigma(w_{-1}) \cdot \sigma(w_0)\sigma(w_1) \cdots$ and $\sigma(v_1v_2 \cdots) = \sigma(v_1)\sigma(v_2) \cdots$, respectively.

The number of occurences of the letter i in a finite word w is denoted $|w|_i$. There is a natural homomorphism $f : \mathcal{A}^* \to \mathbb{Z}^d$ given by $f(w) = (|w|_i)_{i=1}^d$ (all vectors in this work are column vectors). The mapping f is often called the *abelianization* of $\mathcal{A}^*$. The $d \times d$ *incidence matrix* $A = (A_{ij})$ associated to σ is given by the property that its jth column is $f(\sigma(j))$; that is, $A_{ij} = |\sigma(j)|_i$. Since this matrix A also represents a linear transformation on $\mathbb{Z}^d$, we have the following easily verified commuting diagram:

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\ \sigma\ } & \mathcal{A}^* \\ f\big\downarrow & & \big\downarrow f \\ \mathbb{Z}^d & \xrightarrow{\ A\ } & \mathbb{Z}^d \end{array}$$

It is clear that the incidence matrix for σ^n is A^n.

The incidence matrix of a substitution contains a lot of information about the substitution. In fact, substitutions are often classified by various properties of their incidence matrices, and these classifications have been studied separately. For example, whether or not a substitution is Pisot is determined by its incidence matrix. Although we restrict ourselves to studying only Pisot substitutions, work has also been done on non-Pisot substitutions, see for example [9].

A square matrix M is called irreducible if for any i and j, there is $n \in \mathbb{N}$ such that the ij entry of M^n is positive. It is called primitive if some power of M is positive (i.e. all entries are positive). A substitution σ is called *irreducible* if for any $a, b \in \mathcal{A}$ there is $n \in \mathbb{N}$ such that $\sigma^n(a)$ contains b. It is called *primitive* if there is $m \in \mathbb{N}$ such that $\sigma^m(a)$ contains b for all $a, b \in \mathcal{A}$. It is clear that a substitution is primitive (resp. irreducible) if and only if its incidence matrix is primitive (resp. irreducible).

A substitution is called *unimodular* if its incidence matrix has determinant 1 or -1. The inverse of a unimodular integer matrix is also a unimodular integer matrix.

An algebraic integer $\lambda > 1$ is called a *Pisot-Vijayaraghavan number*, or *Pisot number*, if all of its algebraic conjugates α other than itself satisfy $|\alpha| < 1$ (the algebraic conjugates of an algebraic integer λ are the other roots of the minimal polynomial for λ). A substitution is said to be *of Pisot type*, or simply *Pisot*, if its incidence matrix A satisfies the following property: A has a simple eigenvalue $\lambda > 1$, called the dominant eigenvalue, and for every other eigenvalue α of A, one has $0 < |\alpha| < 1$.

Theorem 2.1 *[7] Let σ be a substitution of Pisot type. Then the characteristic polynomial P of its incidence matrix A is irreducible over $\mathbb{Q}$ and the dominant eigenvalue λ is a Pisot number. Moreover, σ cannot be of constant length, the roots of P are all simple, and the matrix A is diagonalizable (over $\mathbb{C}$).*

Proof Recall that a polynomial is irreducible over $\mathbb{Q}$ if and only if it is irreducible over $\mathbb{Z}$. Suppose that P is reducible over $\mathbb{Z}$. Then there exist two non-constant polynomials Q and R with integer coefficients such that $P = QR$. Since 0, 1, and -1 are not roots of P and since the constant term of P is the product of all the roots of P, Q and R each have at least one root which is greater than 1 in modulus. Hence P is irreducible over $\mathbb{Q}$. By Theorem 6.10 of [12], it follows P has no multiple roots in $\mathbb{C}$. It follows that A is diagonalizable over $\mathbb{C}$.

If σ is of constant length, say $|\sigma(i)| = l$ for each $i \in \mathcal{A}$, then l is an eigenvalue for the eigenvector $(1, \ldots, 1)$, which implies that P is reducible over $\mathbb{Q}$. □

Theorem 2.2 *[7] Any Pisot type substitution is primitive.*

Proof We can deduce from the irreducibility of the characteristic polynomial of A that A is irreducible. The proof of this theorem is hence a direct consequence of the following classic theorem: a nonnegative matrix M is primitive if and only if M is irreducible and the spectral radius of M is greater in magnitude than any other eigenvalue [7]. □

We say that a word W occurs in the word w, or that W is a *subword* of w, if there exists s such that for every $1 \leq i \leq |W|$, $W_i = w_{s+i}$. For a finite word $u = u_1u_2 \cdots u_n$ and $0 \leq i \leq n$, the *prefix* of length i of u is the subword $u_1u_2 \cdots u_i$, sometimes denoted $u[1, i]$. Similarly, for an inifinite word $v = v_1v_2v_3 \cdots$, the prefix of length $i \geq 0$ of v is $v_1v_2 \cdots v_i$. By convention, we say that ε is the prefix of length

0 of any finite or infinite word. Analogously, the suffix of length $0 \leq j \leq n$ of u is the subword $u_{n-j+1} \cdots u_n$. And for any $j \geq 1$, the subword $v_j v_{j+1} v_{j+2} \cdots$ is a suffix of the infinite word v.

As defined in [3], a substitution σ on an alphabet $\mathcal{A}$ satisfies the *coincidence condition* if for every i, $j \in \mathcal{A}$, there are integers k and n such that

1. $\sigma^n(i)$ and $\sigma^n(j)$ have the same kth letter (i.e. $\sigma^n(i)_k = \sigma^n(j)_k$) , and
2. the prefixes of length $k-1$ of $\sigma^n(i)$ and $\sigma^n(j)$ have the same image under the abelianization map (i.e. $f(\sigma^n(i)[1, k-1]) = f(\sigma^n(j)[1, k-1])$).

In some papers, for example [2], the coincidence condition as defined above is referred to as positive strong coincidences for all letters.

No examples are known of Pisot substitutions which do not satisfy the coincidence condition [3]. The conjecture that all substitutions of Pisot type satisfy the coincidence condition is known as the *Coincidence Conjecture*.

The following is currently the most complete result on coincidence, confirming the Coincidence Conjecture for substitutions on 2 letters, and giving a partial result for substitutions on $d > 2$ letters.

Theorem 2.3 *[3] Let σ be a Pisot substitution on an alphabet $\mathcal{A} = \{1, 2, \ldots, d\}$. There are distinct letters $i, j \in \mathcal{A}$ for which there are integers k and n such that $\sigma^n(i)$ and $\sigma^n(j)$ have the same kth letter, and the prefixes of length $k-1$ of $\sigma^n(i)$ and $\sigma^n(j)$ have the same image under the abelianization map.*

Barge and Kwapisz [4] also define a stronger version of the coincidence condition, called the geometric coincidence condition, on “strands” in $\mathbb{R}^d$.

We will use the following well-studied example throughout this work.

Example 1 The Fibonacci substitution σ on $\mathcal{A} = \{1, 2\}$ is defined as follows:

$$\sigma : 1 \mapsto 12$$
$$2 \mapsto 1$$

This substitution is called the Fibonacci substitution because the lengths of the iterates of $\sigma(1)$ are the Fibonacci numbers.

The incidence matrix of this substitution is given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This matrix has determinant -1 and eigenvalues $\gamma = (1 + \sqrt{5})/2$ and $-1/\gamma$, hence σ is unimodular and of Pisot type. By Theorem 2.3, this substitution satisfies the coincidence condition. This can also be seen directly since the images under σ of both letters begin with 1.

We will only work with substitutions which are Pisot and unimodular, and for some of our results, we also assume the coincidence condition.

2.1.2 Symbolic Dynamical System

Each substitution has an associated symbolic dynamical system. Before we define this, we need a few simple results, as well as a metric and a transformation on $\mathcal{A}^{\mathbb{Z}}$.

Proposition 2.4 *Every primitive substitution σ on $\mathcal{A} = \{1, 2, \cdots, d\}$ has a periodic point in $\mathcal{A}^{\mathbb{Z}}$. That is, there is $u \in \mathcal{A}^{\mathbb{Z}}$ and $n \in \mathbb{N}$ such that $\sigma^n(u) = u$.*

Proof First, we show the result for $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$. Since $\mathcal{A}$ has d elements, there exists $i \in \mathcal{A}$ and $1 \leq k \leq d$ such that $\sigma^k(i)_1 = i$. If $|\sigma^k(i)| = 1$, then $iii\cdots$ is a fixed point of σ^k. Otherwise $|\sigma^{kn}(i)| \to \infty$ as $n \to \infty$, and the unique word having $\sigma^{kn}(i)$ as a prefix for every $n \geq 0$ is a fixed point for σ^k.

The analogous argument applied to $j \in \mathcal{A}$ such that $\sigma^K(j)_{l(j)} = j$ proves the existence of a periodic point of σ in $\mathcal{A}^{-\mathbb{N}}$. Concatenating this point with the periodic one in $\mathcal{A}^{\mathbb{N}}$ from above gives a periodic point in $\mathcal{A}^{\mathbb{Z}}$. (If the periodic point in $\mathcal{A}^{\mathbb{N}}$ has period n and the periodic point in $\mathcal{A}^{-\mathbb{N}}$ has period m, then the period of the concatenated point is nm.) □

When speaking of a σ-periodic point $u \in \mathcal{A}^{\mathbb{Z}}$, we must be careful that the point is "allowed" by the substitution. To illustrate what can go wrong, consider the following example taken from [4]. Let $\sigma : 1 \mapsto 12221, 2 \mapsto 21212212$, a unimodular Pisot substitution. The biinfinite word containing $\cdots \sigma^n(1) \cdot \sigma^n(1) \cdots$ for $n \geq 0$ is a fixed point of σ. However, we can easily see that the central subword 11 does not appear in $\sigma^m(1)$ or $\sigma^m(2)$ for any $m \in \mathbb{N}$. As a result, we say that the periodic point above is not allowed by the substitution. Barge and Kwapisz [4] prove that a primitive substitution always has an allowed periodic point in $\mathcal{A}^{\mathbb{Z}}$. From now on, by a periodic point of a substitution we will mean a periodic point that is allowed by the substitution.

As in [8], the metric on $\mathcal{A}^{\mathbb{Z}}$ is defined by

$$d(u,v) = \begin{cases} 2^{-\min\{|n| \,|\, u_n \neq v_n\}} & u \neq v \\ 0 & u = v \end{cases} .$$

Under this metric, two points u and v are close if $u_{-N} \cdots u_N = v_{-N} \cdots v_N$ for some large $N \in \mathbb{N}$. In fact, it is easy to see that $d(u,v) \leq 2^{-(n+1)}$ if and only if $u_{-n} \cdots u_n = v_{-n} \cdots v_n$.

The *left shift* map $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $(Sw)_i = w_{i+1}$, and is a homeomorphism on $\mathcal{A}^{\mathbb{Z}}$. We will also denote by $S : \mathcal{A}^* \to \mathcal{A}^*$ the left shift map on finite words, which drops the first letter of a word in $\mathcal{A}^*$. We will often refer to S as simply the shift.

Theorem 2.5 *If σ is a primitive substitution whose incidence matrix has a nonzero eigenvalue of modulus less than 1, then no periodic point of σ in $\mathcal{A}^{\mathbb{Z}}$ is shift-periodic.*

Proof See Corollary 2.7 in [11]. □

A word $w \in \mathcal{A}^{\mathbb{Z}}$ is *minimal* (or uniformly recurrent) if every finite subword of w occurs in an infinite number of positions and with bounded gaps, i.e. if for every subword W of w there exists s such that for every $n \in \mathbb{Z}$, W is a subword of $w_n \cdots w_{n+s}$.

Proposition 2.6 *If σ is primitive, then each of its periodic points is minimal.*

Proof Let n be such that $\sigma^n(j)$ contains i for all $i, j \in \mathcal{A}$, and let $w = \sigma^k(w)$ be a periodic point of σ. Then $w = \sigma^{kn}(w) = \cdots \sigma^{kn}(w_{-1}) \cdot \sigma^{kn}(w_0)\sigma^{kn}(w_1) \cdots$, and for each $m \in \mathbb{Z}$ and $i \in \mathcal{A}$, $\sigma^{kn}(w_m)$ contains i. Since for each $j \in \mathcal{A}$, the length of $\sigma^{kn}(j)$ is finite, it follows that each $i \in \mathcal{A}$ occurs in w infinitely often and with bounded gaps. Hence for $m \geq 0$, $\sigma^{km}(i)$ occurs in $w = \sigma^{km}(w)$ infinitely often and with bounded gaps, and therefore so does any word occuring in w. □

The orbit of a point $w \in \mathcal{A}^{\mathbb{Z}}$ under the shift map is the set $\{S^n w \mid n \in \mathbb{Z}\}$. It follows from Proposition 2.6 that if σ is primitive, then all of its periodic points have the same orbit closure under the shift map, where the closure is taken with respect to the above metric, d. As a result, we can associate a dynamical system to σ as follows. Let w be any biinfinite word which is periodic for σ and set

$$X_\sigma = \overline{\{S^n w \mid n \in \mathbb{Z}\}},$$

that is, X_σ is the metric closure of $\{S^n w \mid n \in \mathbb{Z}\}$. Then (X_σ, S) is the *symbolic dynamical system associated with* σ. It is clear that $S(X_\sigma) = X_\sigma$. By Proposition 2.5, Pisot substitutions do not have periodic points which are also shift-periodic, hence the shift-orbits of all σ-periodic points of Pisot substitutions are infinite. Some authors prefer to work with $\mathcal{A}^{\mathbb{N}}$ and consider the symbolic dynamical system (X_σ^+, S), where X_σ^+ is the closure of the forward orbit of a σ-periodic infinite word.

A dynamical system (Y, T) is *minimal* if the only closed sets $V \subseteq Y$ satisfying $T(V) \subseteq V$ are $\varnothing$ and Y. By Proposition 5.1.13 in [8], the word $w \in \mathcal{A}^{\mathbb{Z}}$ is minimal if and only if $(\overline{\{S^n w \mid n \in \mathbb{Z}\}}, S)$ is a minimal dynamical system. Hence (X_σ, S) is minimal.

There is a significant body of work concerning symbolic dynamical systems. See, for example, [8] for an excellent overview.

2.2 Tiling Substitutions

Most of the definitions and results in this section are taken from [1].

A *tile* is a compact subset of $\mathbb{R}^n$ satisfying the property that it is the closure of its (non-empty) interior. A *partial tiling* is a collection of tiles in $\mathbb{R}^n$ with pairwise disjoint interiors, and its *support* is the union of its tiles. A *tiling* is a partial tiling with support $\mathbb{R}^n$.

Let T be a tiling. If t is a tile in T, we will write $t \in T$. If P is a partial tiling with bounded support such that $P \subseteq T$, then we say that P is a *patch* of T. Furthermore, we say that two tilings T and T' *agree* on a set U if $\{t \in T \mid t \subseteq U\} = \{t \in T' \mid t \subseteq U\}$, and write $T = T'$ on U. For any partial tiling T, we define expansions and translations of T by

$$\alpha T = \{\alpha t \mid t \in T\}, \qquad \alpha > 0$$

$$T + \mathbf{x} = \{t + \mathbf{x} \mid t \in T\}, \qquad \mathbf{x} \in \mathbb{R}^n.$$

Clearly αT and $T + \mathbf{x}$ are also partial tilings.

We define a metric on the space of tilings of $\mathbb{R}^n$, in which two tilings are close if they almost agree on a large ball around the origin. For any tilings T and T' of $\mathbb{R}^n$, let

$$d(T, T') = \inf(\{1\} \cup \{\epsilon > 0 \mid T + \mathbf{x} = T' + \mathbf{y} \text{ on } B_{1/\epsilon}(\mathbf{0}), \text{ some } \|\mathbf{x}\|, \|\mathbf{y}\| < \epsilon\}),$$

where $\|\cdot\|$ is the usual norm on $\mathbb{R}^n$.

We now define tiling substitutions, which are very similar to symbolic substitutions. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a finite set of tiles, which we call *prototiles*. Let $\mathcal{P}^*$ be the collection of all partial tilings that only contain translations of these prototiles. We assume that there is an inflation constant $\alpha > 1$ and a substitution rule $\omega : \mathcal{P} \to \mathcal{P}^*$ that associates to each prototile P_i a partial tiling in $\mathcal{P}^*$ with support αP_i. We extend our definition of ω by setting $\omega(P_i + \mathbf{x}) = \omega(P_i) + \alpha\mathbf{x}$, for any $1 \leq i \leq m$ and $\mathbf{x} \in \mathbb{R}^n$. As for symbolic substitutions, we then define $\omega(T) = \cup_{t \in T}\omega(t)$ for any $T \in \mathcal{P}^*$. Clearly $\omega(T) \in \mathcal{P}^*$. Let Ω be the collection of tilings T of $\mathcal{P}^*$ having the property that for any patch U of T, U is contained in $\omega^m(P_i + \mathbf{x})$ for some $m, i, \mathbf{x}$; then $\omega(\Omega) \subseteq \Omega$. We are interested in studying the dynamical system (Ω, ω).

We say that the tiling substitution ω is *primitive* if there exists a positive integer N such that for each pair of prototitles P_i and P_j, the partial tiling $\omega^N(P_i)$ contains a translation of P_j. Further, Ω satisfies the *finite pattern condition*, or Ω has *finite local complexity*, if for each $r > 0$, there are only finitely many partial tilings up to translation that are subsets of tilings in Ω and whose supports have diameters less than r.

Proposition 2.7 *Ω is non-empty and $\omega(\Omega) = \Omega$.*

Proof See Propositions 2.1 and 2.2 in [1] (in Anderson and Putnam's proof of this Proposition, it served their purposes well to assume that the prototiles are homeomorphic to closed balls, but these results also hold if we simply assume that each prototile is the closure of its interior). □

Note that $\mathbb{R}^n$ acts on Ω by translation. We say that a tiling $T \in \Omega$ is translation-periodic if $T + \mathbf{x} = T$ for some $\mathbf{x} \in \mathbb{R}^n$.

Proposition 2.8

1. *ω is continuous.*
2. *Ω is compact if and only if it has finite local complexity.*
3. *$\omega : \Omega \to \Omega$ is injective if and only if Ω contains no translation-periodic tilings.*

Proof See Lemma 1.4.4 in [15]. □

It is analogous to the proof of Proposition 2.6 to prove that if ω is primitive and Ω satisfies the finite pattern condition, then for any patch P in an ω-periodic tiling $T_0 \in \Omega$, P appears infinitely often and with bounded gaps. As a result, $\Omega = \overline{\{T_0 + \mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}}$.

Remark If the substitution map $\omega : \Omega \to \Omega$ is injective, then it is hyperbolic: under iterates of the substitution, tilings that agree around the origin become exponentially closer together, while those that are close translations of each other become exponentially closer under iterations of the inverse.

In Chapter 3, we will construct a substitution tiling on an invariant subspace of the incidence matrix of a substitution.

2.3 Geometric Representation of Substitutions

The geometric theory of unimodular Pisot substitutions on $\mathcal{A} = \{1, 2, \cdots, d\}$ is based on the decomposition of $\mathbb{R}^d$ into two A-invariant subspaces, one 1-dimensional and the other $(d-1)$-dimensional. Most research on this subject involves the Rauzy fractal of a substitution, a useful compact subset of the $(d-1)$-dimensional subspace.

2.3.1 The Stable/Unstable Decomposition for A

Recall that the incidence matrix of a Pisot substitution is primitive. The following well-known theorem on primitive matrices was proved by Perron, and extended to irreducible matrices by Frobenius. The Perron-Frobenius Theorem provides a significant amount of information regarding the eigenvalues and eigenvectors of the incidence matrix of a Pisot substitution.

Theorem 2.9 *Let A be a non-negative primitive square matrix. Then there exists an eigenvalue λ of A with the following properties:*

1. $\lambda > 0$,
2. *λ is a simple root of the characteristic polynomial,*
3. *λ has a positive eigenvector $\mathbf{v}$ (i.e. all entries of $\mathbf{v}$ are positive),*
4. *if α is any other eigenvalue of A, then $|\alpha| < \lambda$,*
5. *any non-negative eigenvector of A is a positive multiple of $\mathbf{v}$.*

Proof See Theorem 3.3.1 in [6]. □

The dominant eigenvalue of an irreducible matrix is called the *Perron-Frobenius eigenvalue*.

Let A be the $d \times d$ incidence matrix of a unimodular, Pisot substitution σ. Denote by $\lambda, \lambda_2, \ldots, \lambda_d$ the eigenvalues of A, and by $\mathbf{v} = \mathbf{v}_\lambda, \mathbf{v}_{\lambda_2}, \ldots, \mathbf{v}_{\lambda_d}$ the corresponding right eigenvectors, where λ is the Perron-Frobenius eigenvalue of A. By Theorem 2.1, all of the eigenvalues of A are irrational. Since the Perron-Frobenius Theorem guarantees a positive eigenvector associated with λ, we may assume that $\mathbf{v} > 0$ and $\|\mathbf{v}\| = 1$, and we denote by $\mathbf{u} > 0$ the right eigenvector for A^T associated to λ satisfying $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \cdot \mathbf{v} = 1$. Notice that $\mathbf{u}^T$ is a left eigenvector for A associated to λ (by our convention that all vectors are column vectors, left eigenvectors aren't actually vectors, but their transposes are). Finally, denote by $\mathbf{u}_{\lambda_2}, \ldots, \mathbf{u}_{\lambda_d}$ the left eigenvectors of A corresponing to $\lambda_2, \ldots, \lambda_d$.

Define two subsets of $\mathbb{R}^d$ as follows: let $A^e = \mathbb{R}\mathbf{v}$ and $A^c = \{\mathbf{u}\}^\perp = \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{u} \rangle = 0\}$. Then clearly $A(A^e) \subseteq A^e$, and $A(A^c) \subseteq A^c$ since if $\mathbf{x} \in A^e$, then $\langle A\mathbf{x}, \mathbf{u} \rangle = (A\mathbf{x})^T \mathbf{u} = \mathbf{x}^T A^T \mathbf{u} = \mathbf{x}^T \lambda \mathbf{u} = \lambda \langle \mathbf{x}, \mathbf{u} \rangle = 0$. For $\alpha\mathbf{v} \in A^e \cap A^c$, we have $0 = \langle \mathbf{u}, \alpha\mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle = \alpha$, so that $A^e \cap A^c = \{\mathbf{0}\}$. Futhermore, $A^c + A^e = \mathbb{R}^d$ since for $\mathbf{x} \in \mathbb{R}^d$, we have $\mathbf{x} = (\mathbf{x} - \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{v}) + \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{v}$, where $\langle \mathbf{x}, \mathbf{u} \rangle \mathbf{v} \in A^e$ and $\mathbf{x} - \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{v} \in A^c$ since $\langle \mathbf{x} - \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{x}, \mathbf{u} \rangle - \langle \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{x}, \mathbf{u} \rangle - \langle \mathbf{x}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{u} \rangle = 0$. Therefore $\mathbb{R}^d = A^e \oplus A^c$ is an A-invariant decomposition of $\mathbb{R}^d$.

It is easy to see that A is expanding on A^e, since for $\mathbf{x} \in A^e$, $\|A\mathbf{x}\| = \lambda \|\mathbf{x}\| > \|\mathbf{x}\|$. Now let us consider the action of A on A^c.

For a linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ and a subspace $X \subseteq \mathbb{R}^d$ satisfying $T(X) \subseteq X$, we say that T is *contracting* on X if there are constants $C \geq 0$ and $0 < \alpha < 1$ such that $\|T^n(\mathbf{x})\| \leq C\alpha^n \|\mathbf{x}\|$ for every $\mathbf{x} \in X$ and $n \in \mathbb{N}$. In this case, $\|T^n(\mathbf{x})\| \to 0$ as $n \to \infty$, for every $\mathbf{x} \in X$.

Let us show that A is contracting on A^c. Clearly the eigenvalues of $A|_{A^c}$ are $\lambda_2, \ldots, \lambda_d$, and each of these has norm less than 1. Since each of the eigenvalues is simple, $A|_{A^c}$ is diagonalizable. That is, there exists an invertible matrix U such that $D = U^{-1}A|_{A^c}U$ is a diagonal matrix, with the eigenvalues $\lambda_2, \ldots, \lambda_d$ on the diagonal. Let $C = \max_{i,k,j}|U_{ik}U^{-1}_{kj}|$ and let $\alpha = \max_{2\le i\le d}|\lambda_i|$. If $\mathbf{x} \in A^c$, $|\mathbf{x}| = (|\mathbf{x}(i)|)_{1\le i\le d-1}$, and $\mathbf{x}_1$ is the vector with all entries 1, then

$$
\begin{aligned}
\|(A|_{A^c})^n\mathbf{x}\|^2 &= \|UD^nU^{-1}\mathbf{x}\|^2 \\
&= \sum_{i=1}^{d-1} |(UD^nU^{-1}\mathbf{x})(i)|^2 \\
&= \sum_{i=1}^{d-1} \Big| \sum_{j=1}^{d-1} \big(\sum_{k=1}^{d-1} U_{ik}D^n_{kk}U^{-1}_{kj}\mathbf{x}(j)\big)\Big|^2 \\
&\le \sum_{i=1}^{d-1} \big(\sum_{j=1}^{d-1}\sum_{k=1}^{d-1} \alpha^n C|\mathbf{x}(j)|\big)^2 \\
&= \sum_{i=1}^{d-1} ((d-1)\alpha^n C)^2 \big(\sum_{j=1}^{d-1} |\mathbf{x}(j)|\big)^2 \\
&= (d-1)^3(\alpha^n C)^2 \langle \mathbf{x}_1, |\mathbf{x}|\rangle^2 \\
&\le (d-1)^3(\alpha^n C)^2 \|\mathbf{x}_1\|^2 \||\mathbf{x}|\|^2 \qquad (*) \\
&= \alpha^{2n}(d-1)^4 C^2 \|\mathbf{x}\|^2,
\end{aligned}
$$

where in the inequality in $(*)$ follows from the Schwarz Inequality, $|\langle x, y\rangle| \le \|x\|\|y\|$. Hence $\|(A|_{A^c})^n\mathbf{x}\| \le \alpha^n(d-1)^2C\|\mathbf{x}\|$, and so A is contracting on A^c.

The decomposition $\mathbb{R}^d = A^e \oplus A^c$ is sometimes called the stable/unstable decomposition for A [4]. By Theorem 5.10.3 in [6], both A^e and A^c are dense in $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Let $\pi^e : \mathbb{R}^d \to A^e$ denote projection onto A^e, and $\pi^c : \mathbb{R}^d \to A^c$ denote projection onto A^c. More precisely, for $\mathbf{x} = \alpha\mathbf{v} + \mathbf{w} \in \mathbb{R}^d$, where $\mathbf{w} \in A^c$,

$$\pi^e(\alpha\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v}$$

and

$$\pi^c(\alpha\mathbf{v} + \mathbf{w}) = \mathbf{w}.$$

It is clear that $A\pi^e = \pi^e A$ and $A\pi^c = \pi^c A$. The two projections π^e and π^c are central to the theory of the geometric representation of Pisot substitutions.

Let m_e denote Lebesgue measure on A^e and m_c denote Lebesgue measure on A^c. Then for any Lebesgue measurable set $E \oplus C \subset \mathbb{R}^d$, $m(E \oplus C) = m_e(E)m_c(C)a$, where m is Lebesgue measure on $\mathbb{R}^d$ and $a > 0$ is a constant depending only on the angle betwen A^e and A^c. Since A is unimodular, $m(A(E \oplus C)) = m(E \oplus C)$, and hence it follows that $m_c(A(C)) = \lambda^{-1}m_c(C)$ for all measurable sets $C \subseteq A^c$.

2.3.2 Rauzy Fractals

In his analysis of unimodular Pisot substitutions, G. Rauzy developed the idea of looking for a geometric representation of substitutive dynamical systems, as a rotation on a suitable space [8]. He gave the following construction for the Tribonacci substitution, and Arnoux and Ito [2] generalized his construction to all unimodular Pisot substitutions.

Let σ be a Pisot, unimodular substitution and let $u \in X_\sigma$ be a σ-periodic word. Then the Rauzy fractal $\mathcal{R}(u) \subset A^c$ associated to u is defined as

$$\mathcal{R}(u) = \overline{\Big\{\pi^c\big(\sum_{i=0}^{n} \mathbf{e}_{u_i}\big) \mid n \geq 0\Big\}},$$

where $\mathbf{e}_i$ is the i^{th} canonical basis vector for $\mathbb{R}^d$. From Section 7.5 in [8] we have that $\mathcal{R}(u)$ is a bounded subset of A^c. There is a natural decompostion of $\mathcal{R}(u)$ into d subsets called *cylinders*. For $i \in \mathcal{A}$, the i^{th} cylinder of $\mathcal{R}(u)$ is given by

$$\mathcal{R}_i(u) = \overline{\left\{\pi^c\big(\sum_{i=0}^{n} \mathbf{e}_{u_i}\big) \mid n \geq 0,\ u_{n+1} = i\right\}}.$$

It is clear that $\bigcup_{i\in\mathcal{A}} \mathcal{R}_i(u) = \mathcal{R}(u)$.

One may well ask how the Rauzy fractals associated to distinct periodic points of a substitution relate. It is not hard to prove that every unimodular Pisot substitution has a unique Rauzy fractal.

Proposition 2.10 *Let σ be a unimodular Pisot substitution, and let u and v be two periodic points of σ. Then $\mathcal{R}(u) = \mathcal{R}(v)$.*

Proof It is clear that for every $K \geq 1$, if $\mathcal{R}_\sigma(u)$ is the Rauzy fractal associated to a σ-periodic point u, and $\mathcal{R}_{\sigma^K}(u)$ is the Rauzy fractal associated to the σ^K-periodic point u, then $\mathcal{R}_\sigma(u) = \mathcal{R}_{\sigma^K}(u)$. If u is a σ-periodic point of period m and v is σ-periodic of period n, and A^k is positive, then both u and v are fixed points of σ^{mnk}, and $\sigma^{mnk}(i)$ contains j for all $i, j \in \mathcal{A}$. Hence we may assume, w.l.o.g., that all periodic points of σ are fixed points and that $\sigma(i)$ contains j for all $i, j \in \mathcal{A}$. Let u and v be two fixed points of σ, and let $\mathcal{R}(u)$ and $\mathcal{R}(v)$ be their associated Rauzy fractals.

For a finite word w, let $\overrightarrow{w} = \{\sum_{j=1}^{n} \mathbf{e}_{w_j} \mid 1 \leq n \leq |w|\} = \{f(w_1 \cdots w_n) \mid 1 \leq n \leq |w|\}$.

Then u_0 is contained in $\sigma(v_0)$. Say $u_0 = W_k^{(v_0)}$, so that $(S^{k-1}\sigma(v_0))_1 = u_0$. Since $Af = f\sigma$, we have that $\overrightarrow{\sigma^n(u_0)} \subseteq \overrightarrow{\sigma^n(S^{k-1}\sigma(v_0))} \subseteq \overrightarrow{\sigma^{n+1}(v_0)} - A^n f(W_1^{(v_0)} \cdots W_{k-1}^{(v_0)})$ for $n \geq 0$. Let $\alpha = \pi^c(f(W_1^{(v_0)} \cdots W_{k-1}^{(v_0)}))$. Then, $\pi^c(\overrightarrow{\sigma^n(u_0)}) \subseteq \pi^c(\overrightarrow{\sigma^{n+1}(v_0)}) - A^n\alpha$ for $n \geq 0$; that is

$$\pi^c(\overrightarrow{\sigma^n(u_0)}) \subseteq B_{\|A^n\alpha\|}(\pi^c(\overrightarrow{\sigma^{n+1}(v_0)})) \subseteq B_{\|A^n\alpha\|}(\mathcal{R}(v)),$$

where $B_r(U) = \{\mathbf{x} \in A^c \mid \|\mathbf{x} - \mathbf{u}\| < r, \text{ some } \mathbf{u} \in U\}$.

Since A is contracting on A^c, $\lim_{n\to\infty} \|A^n\alpha\| = 0$. Hence

$$\mathcal{R}(u) = \overline{\bigcup_{n\geq 0} \pi^c(\overrightarrow{\sigma^n(u_0)})} \subseteq \overline{\bigcap_{n\geq 0} B_{\|A^n\alpha\|}(\mathcal{R}(v))} = \mathcal{R}(v).$$

Similarly, $\mathcal{R}(v) \subseteq \mathcal{R}(u)$. □

As a result, we can define $\mathcal{R} = \mathcal{R}(u)$, where u is any periodic point of σ.

Proposition 2.11 *[18] Let $\mathcal{R}$ be the Rauzy fractal associated to a periodic point of a unimodular Pisot substitution on $\mathcal{A}$. Then $\mathcal{R}_i$ has non-empty interior and $\overline{Int(\mathcal{R}_i)} = \mathcal{R}_i$ for each $i \in \mathcal{A}$. If, in addition, the substitution satisfies the coincidence condition, then $m_c(\mathcal{R}_i \cap \mathcal{R}_j) = 0$ for $i \neq j$.*

Proof See Theorem 4.1 and Corollary 4.5 in [18]. □

Example 2 Figure 2.1 shows the Rauzy fractal for the Tribonacci substitution.

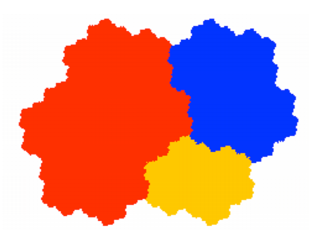


Figure 2.1: Rauzy fractal for $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

Although it is the case for the Tribonacci substitution, Rauzy fractals need not be connected. However, there is a criterion for the connectedness of the Rauzy fractal [7].

An exchange of domains can be defined a.e. by $\phi(\mathbf{x}) = \mathbf{x} + \pi^c(\mathbf{e}_i)$ if $\mathbf{x} \in \mathcal{R}_i$.

Theorem 2.12 *[2] Let σ be a unimodular Pisot type substitution over a d-letter alphabet which satisfies the coincidence condition. Then the substitutive dynamical system (X_σ, S) associated with σ is measure-theoretically isomorphic to the exchange of d domains defined almost everywhere on the self-similar Rauzy fractal of σ. Furthermore, (X_σ, S) admits as a continuous factor an irrational translation on the torus $\mathbb{T}^{d-1}$, the fibres being finite almost everywhere.*

Proof See [2]. □

2.4 Subshifts of Finite Type

Given a substitution σ on $\mathcal{A}$, the directed graph $G_\sigma = (V, E)$ associated to σ has vertex set

$$V = \mathcal{A}$$

and edge set

$$E = \{(i, j) \mid i \in \mathcal{A},\ 1 \leq j \leq l(i)\},$$

where (i, j) is an edge from vertex i to vertex $W_j^{(i)}$. This graph is commonly found in the literature on substitutions, although sometimes with the edges in the opposite direction, see for example [1] and [11].

Example 3 The graph associated to the Tribonacci substitution, $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$, is shown in Figure 3.

The *subshift of finite type* or SFT associated to σ is the set of all biinfinite paths in G_σ and is denoted Σ_σ. That is,

$$\Sigma_\sigma = \{(i_n, j_n)_{n\in\mathbb{Z}} \mid (i_n, j_n) \in E,\ W_{j_n}^{(i_n)} = i_{n+1}\ \forall\, n \in \mathbb{Z}\}.$$

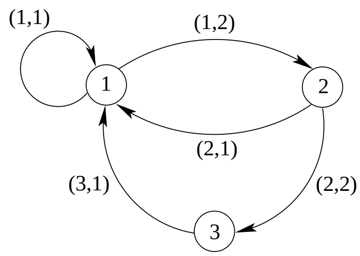


Figure 2.2: G_σ for $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$

If there is no ambiguity about which substitution we mean, we will write Σ instead of Σ_σ.

Remark The set $\{(i_n, j_n)_{n\leq 0} \mid (i_n, j_n) \in E \ \forall\ n \leq 0,\ W_{j_n}^{(i_n)} = i_{n+1} \ \forall\ n < 0\}$, is equivalent to the prefix-suffix SFT $\mathcal{D}$ found in [8] and [7].

For $x = (i_n, j_n)_{n\in\mathbb{Z}}$, we write $x_n = (i_n, j_n)$, for $n \in \mathbb{Z}$. The topology on Σ is defined by the following metric:

$$d(x,y) = \begin{cases} 2^{-\min\{|n| \mid x_n \neq y_n\}} & x \neq y \\ 0 & x = y \end{cases}.$$

For $k < l \in \mathbb{Z}$ and a path $(i_k, j_k)(i_{k+1}, j_{k+1}) \cdots (i_{l-1}, j_{l-1})$ in G_σ, we define the cylinder $U(k, l, (i_k, j_k) \cdots (i_{l-1}, j_{l-1})) = \{x \in \Sigma \,|\, x_n = (i_n, j_n),\ k \leq n \leq l-1\}$. Since we can also have a path consisting of a single vertex, i, we also define the cylinders $U(n, n, i) = \{x \in \Sigma \mid i_n = i\}$ for $n \in \mathbb{Z}$. The cylinders are clopen and form a basis for the topology on Σ [8]. It is easy to see that Σ is closed, and hence compact.

We prove in Appendix 1 that *Parry measure* on Σ can be simplified to the form

$$\mu(U(k, l, (i_k, j_k) \cdots (i_{l-1}, j_{l-1}))) = \lambda^{k-l}\mathbf{v}(i_k)\mathbf{u}(W_{j_{l-1}}^{(i_{l-1})}),$$

where λ is the Perron-Frobenius eigenvalue of the incidence matrix A of σ, and $\mathbf{v}$ and $\mathbf{u}$ are the corresponding positive right and left eigenvectors satisfying $\langle \mathbf{u}, \mathbf{v} \rangle = 1$.

Parry measure is a Borel measure and is the unique measure of maximal entropy on Σ [19]. In all that follows, we will take Parry measure to be the measure on Σ.

By an abuse of notation, we will also denote by S the left shift on Σ. That is, for $x \in \Sigma$, we define $S(x)_n = x_{n+1}$. We will always be very clear about which domain we are working with, and hence this notation should not cause a problem. Clearly $S : \Sigma \to \Sigma$ is uniformly continuous and measure-preserving.

2.5 Smale Spaces

2.5.1 Notation and Definitions

We begin with an intuitive definition and then give a precise definition. A *Smale space* is a compact metric space together with a homeomorphism, that has a certain hyperbolic structure: each point in the space is the intersection of a local stable set, on which the homeomorphism is contracting, and a local unstable set, on which the homeomorphism is expanding; moreover, the product of these sets is homeomorphic to a neighborhood of the point [17].

The following more precise definition is also taken from [17]. Let (X, d) be a compact metric space and let f be a homeomorphism of X. Then (X, d, f) is a Smale space if there exists $0 < \lambda_0 < 1$, $\epsilon_0 > 0$ and a continuous function

$$[\,,\,] : \{(x, y) \mid x, y \in X, \quad d(x, y) < \epsilon_0\} \to X$$

which satisfy the following. First we require

$$\begin{aligned}
[x, x] &= x \\
[[x, y], z] &= [x, z] \\
[x, [y, z]] &= [x, z]
\end{aligned}$$

for $x, y, z \in X$, whenever both sides of the equations are defined. We let

$$\begin{aligned}
V^S(x, \epsilon) &= \{y \in X \mid [x, y] = x \text{ and } d(x, y) < \epsilon\}, \\
V^U(x, \epsilon) &= \{y \in X \mid [y, x] = x \text{ and } d(x, y) < \epsilon\}
\end{aligned}$$

for any $0 < \epsilon \leq \epsilon_0$. These are called the local stable sets and the local unstable sets. We also require

$$[f(x), f(y)] = f([x, y]),$$

whenever both sides of the equation are defined. Finally, we assume that

$$\begin{aligned}
d(f(y), f(z)) &\leq \lambda_0 d(y, z), \qquad y, z \in V^S(x, \epsilon), \\
d(f^{-1}(y), f^{-1}(z)) &\leq \lambda_0 d(y, z), \qquad y, z \in V^U(x, \epsilon).
\end{aligned}$$

It follows from the definitions that, for any $x \in X$,

$$[\,,] : V^U(x, \epsilon_0/2) \times V^S(x, \epsilon_0/2) \to X$$

is a homeomorphism onto a neighborhood of x in X. It can also be shown that, for any $0 < \epsilon \leq \epsilon_0$,

$$V^S(x,\epsilon) = \{y \in X \mid d(f^n(x), f^n(y)) < \epsilon, \text{ for all } n = 0, 1, 2, \cdots\}$$

$$V^U(x,\epsilon) = \{y \in X \mid d(f^n(x), f^n(y)) < \epsilon, \text{ for all } n = 0, -1, -2, \cdots\}$$

and that, for x, y with $d(x, y) < \epsilon_0$,

$$V^S(x,\epsilon_0) \cap V^U(y,\epsilon_0) = \{[x, y]\}.$$

These last observations show that $[,]$, if it exists, depends only on (X, d, f).

Each of the spaces $(\mathbb{T}^d, A)$, (Ω, ω), and (Σ, S) is a Smale space, as follows.

In $(\mathbb{T}^d, A)$, the local unstable and stable sets of a point $\mathbf{x} \in \mathbb{R}^d$ are given by

$$V^U(x,\epsilon) = \{q(\mathbf{x} + \mathbf{t}) \mid \mathbf{t} \in A^c, \ \|\mathbf{t}\| \leq \epsilon\}$$

and

$$V^S(x,\epsilon) = \{q(\mathbf{x} + \mathbf{t}) \mid \mathbf{t} \in A^e, \ \|\mathbf{t}\| \leq \epsilon\}$$

where q is the quotient map from $\mathbb{R}^d$ onto $\mathbb{T}^d$. For additional details in the proof that $(\mathbb{T}^d, A)$ is a Smale space, see [15]. Since A has no eigenvalues of modulus one, $A|_{\mathbb{T}^d}$ is a hyperbolic toral automorphism, and $(\mathbb{T}^d, A)$ is one of the simplest examples of an Anosov diffeomorphism.

In the substitution tiling space (Ω, ω), the local stable set for a tiling consists of tilings that agree with it on a large ball around the origin, while the local unstable set consists of tilings that are small translations of it. For a complete proof that our substitution tiling spaces are Smale spaces, see [1].

The following proof that (Σ, S) is a Smale space closely follows Section 1.4 in

[15].

For any $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma$, we define the sets

$$V^U(x,\epsilon) = \{x' \in \Sigma \mid x'_n = x_n, \text{ for all } n \geq 0,\ d(x,x') < \epsilon\}$$

$$V^S(x,\epsilon) = \{x' \in \Sigma \mid x'_n = x_n, \text{ for all } n \leq 0,\ d(x,x') < \epsilon\}$$

Clearly these two sets intersect exactly at x. Given $x' \in V^U(x,\epsilon)$ and $x'' \in V^S(x,\epsilon)$, we form the sequence

$$y_n = \begin{cases} x'_n & n \leq 0 \\ x''_n & n \geq 0 \end{cases}$$

It is clear that this construction gives a homeomorphism between $V^U(x,\epsilon)\times V^S(x,\epsilon)$ and the set $\{y \in \Sigma \mid d(x,y) < \epsilon\}$, which is a neighborhood of x.

Let us now consider the contracting/expanding structure of S on these sets. If $x', x'' \in V^U(x,\epsilon)$, then $x'_n = x_n = x''_n$ for all $n \geq 0$ and so $d(x',x'') = 2^{-N}$, where N is the smallest positive integer such that $x'_{-N} \neq x''_{-N}$. From the definition of S, it follows that $d(Sx', Sx'') = 2^{-(N+1)}$, so that

$$d(Sx', Sx'') = \frac{1}{2}d(x',x'').$$

Similarly, for $x', x'' \in V^S(x,\epsilon)$,

$$d(S^{-1}x', S^{-1}x'') = \frac{1}{2}d(x',x'').$$

More rigorously, we can define the operation $[\,,]$ as follows. Set $\epsilon_0 = 1/2$. If x and y are in Σ and $d(x,y) \leq \epsilon_0$, it follows from the definition of the metric that we must have $x_0 = y_0$. In this case, we let

$$[x, y] = \begin{cases} y_n & n \leq 0 \\ x_n & n \geq 0 \end{cases}$$

and we observe that $[x, y] \in \Sigma$.

2.5.2 Factor Maps

Given two dynamical systems (X, f) and (Y, g), a *factor map* between them is a continuous function

$$\phi : X \to Y$$

such that

$$\phi \circ f = g \circ \phi.$$

We write this as

$$\phi : (X, f) \to (Y, g).$$

A factor map is *finite-to-one* if there is a constant M such that $|\phi^{-1}\{y\}| \leq M$, for every $y \in Y$. Here, $|B|$ denotes the cardinality of the set B.

When the dynamical systems are Smale systems, there are two special classes of factor maps. A map ϕ is *s-resolving* if $\phi|_{V^s(x,\epsilon)}$ is injective for every $x \in X$ and some $\epsilon > 0$. Similarly, it is *u-resolving* if $\phi|_{V^u(x,\epsilon)}$ is injective. Such an s-resolving map is actually a homeomorphism on the local stable sets, and s- or u-resolving maps are always finite-to-one [14].

Let (X, f) be a topological dynamical system. A point $x \in X$ is said to be *non-wandering* if for any neighborhood U of x there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \varnothing$. A Smale space is said to be *irreducible* if every point is non-wandering and there is a dense orbit.

In [14], it is shown that any finite-to-one factor map between two irreducible Smale spaces may be lifted to an s-resolving map between two others which factor onto the originals by u-resolving maps. The following is given as a corollary:

Proposition 2.13 *[14] Let (Y, g) be an irreducible Smale space. Then there is another irreducible Smale space, (Ω, ω), an irreducible shift of finite type, (Σ, S), and two factor maps*

$$\phi_1 : (\Sigma, S) \rightarrow (\Omega, \omega)$$

and

$$\phi_2 : (\Omega, \omega) \rightarrow (Y, g)$$

such that ϕ_1 is s-resolving while ϕ_2 is u-resolving.

Chapter 3

The Maps

Let σ be a unimodular Pisot substitution on the alphabet $\mathcal{A} = \{1, 2, ..., d\}$. Recall the notation

$$\sigma(i) = W_1^{(i)} W_2^{(i)} \cdots W_{l(i)}^{(i)}, \quad i \in \mathcal{A}.$$

For $i \in \mathcal{A}$ and $1 \leq j \leq l(i)$, define $\mathbf{x}_j^{(i)} = f(W_1^{(i)} \cdots W_{j-1}^{(i)}) = \sum_{k<j} \mathbf{e}_{W_k^{(i)}}$, where f is the abelianization map and $\mathbf{e}_i$ is the i^{th} canonical basis element of $\mathbb{R}^d$. Notice that $\mathbf{x}_1^{(i)} = \mathbf{0}$ and $\mathbf{x}_j^{(i)} \in \mathbb{Z}^d$ for every $i \in \mathcal{A}$ and $1 \leq j \leq l(i)$.

Further, let A be the $d \times d$ incidence matrix of σ. Denote by λ the Perron-Frobenius eigenvalue of A, and by $\mathbf{v}$ the corresponding right eigenvector such that $\mathbf{v} > 0$ and $\|\mathbf{v}\| = 1$. Denote by $\mathbf{u} > 0$ the left eigenvector associated to λ satisfying $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \cdot \mathbf{v} = 1$. Recall the A-invariant decompostion of $\mathbb{R}^d = A^e \oplus A^c$, where $A^c = \{\mathbf{u}\}^{\perp}$ is a contracting $(d-1)$-dimensional plane, and $A^e = \mathbb{R}\mathbf{v}$ is an expanding line. Let $\pi^e : \mathbb{R}^d \to A^e$ denote projection onto A^e, and $\pi^c : \mathbb{R}^d \to A^c$ denote projection onto A^c.

We begin by constructing a map from the SFT associated to σ to the d-torus. We then give two more maps, one from the SFT to a substitution tiling space, and one from the tiling space to the d-torus, and prove that our original map is the composition of these two. Along the way, we prove a number of desirable properties of these maps.

3.1 From the SFT to $\mathbb{T}^d$

Recall from Section 2.4 that for an element x of the SFT Σ associated to σ, x is of the form $(i_n, j_n)_{n\in\mathbb{Z}}$, where $i_{n+1} = W_{j_n}^{(i_n)}$ for every $n \in \mathbb{Z}$.

For $n_0 \in \mathbb{Z}$, define maps $p_{n_0}^+ : \Sigma \to A^e$ and $p_{n_0}^- : \Sigma \to A^c$ as follows:

$$p_{n_0}^+((i_n, j_n)_{n\in\mathbb{Z}}) = \sum_{n=n_0}^{\infty} A^{-n}\pi^e(\mathbf{x}_{j_n}^{(i_n)}),$$

$$p_{n_0}^-((i_n, j_n)_{n\in\mathbb{Z}}) = \sum_{n=-\infty}^{n_0-1} A^{-n}\pi^c(\mathbf{x}_{j_n}^{(i_n)}).$$

Since A^{-1} is contracting on A^e and A is contracting on A^c, it follows that both of these series converge for all $n_0 \in \mathbb{Z}$ and $(i_n, j_n)_{n\in\mathbb{Z}} \in \Sigma$.

Lemma 3.1 *For every $n_0 \in \mathbb{Z}$, $p_{n_0}^+ \circ S = A \circ p_{n_0+1}^+$ and $p_{n_0}^- \circ S = A \circ p_{n_0+1}^-$.*

Proof Let $x = (i_n, j_n)_{n\in\mathbb{Z}} \in \Sigma$ and $n_0 \in \mathbb{Z}$. Then

$$\begin{aligned} p_{n_0}^+(S(x)) &= \sum_{n=n_0}^{\infty} A^{-n}\pi^e(\mathbf{x}_{j_{n+1}}^{(i_{n+1})}) \\ &= A\sum_{n=n_0}^{\infty} A^{-(n+1)}\pi^e(\mathbf{x}_{j_{n+1}}^{(i_{n+1})}) \\ &= A\sum_{n=n_0+1}^{\infty} A^{-n}\pi^e(\mathbf{x}_{j_n}^{(i_n)}) \\ &= Ap_{n_0+1}^+(x). \end{aligned}$$

The proof that $p_{n_0}^- \circ S = A \circ p_{n_0+1}^-$ is analogous. □

We further define a map $p_{n_0} : \Sigma \to \mathbb{R}^d$ by

$$p_{n_0}(x) = p^+_{n_0}(x) - p^-_{n_0}(x).$$

Denote by $q : \mathbb{R}^d \to \mathbb{T}^d$ the standard quotient map $\mathbf{x} \mapsto \mathbf{x}(mod\, \mathbb{Z}^d)$.

Proposition 3.2 *The map $q \circ p_{n_0} : \Sigma \to \mathbb{T}^d$ is independent of n_0. That is,*

$$q \circ p_{n_0} = q \circ p_{n_0+1} \quad \forall\, n_0 \in \mathbb{Z}.$$

Proof For all $x = (i_n, j_n)_{n\in\mathbb{Z}} \in \Sigma$, we have

$$p^+_{n_0}(x) = p^+_{n_0+1}(x) + A^{-n_0}\pi^e(\mathbf{x}^{(i_{n_0})}_{j_{n_0}})$$

and

$$p^-_{n_0}(x) = p^-_{n_0+1}(x) - A^{-n_0}\pi^c(\mathbf{x}^{(i_{n_0})}_{j_{n_0}}).$$

So

$$\begin{aligned} p_{n_0}(x) &= p_{n_0+1}(x) + A^{-n_0}\big(\pi^e(\mathbf{x}^{(i_{n_0})}_{j_{n_0}}) + \pi^c(\mathbf{x}^{(i_{n_0})}_{j_{n_0}})\big) \\ &= p_{n_0+1}(x) + A^{-n_0}(\mathbf{x}^{(i_{n_0})}_{j_{n_0}}). \end{aligned}$$

Since A is a unimodular integer matrix, so is A^n for all $n \in \mathbb{Z}$. Since $\mathbf{x}^{(i_0)}_{j_0} \in \mathbb{Z}^d$, it follows that $A^{-n_0}(\mathbf{x}^{(i_{n_0})}_{j_{n_0}}) \in \mathbb{Z}^d$ for all $n_0 \in \mathbb{Z}$. $\square$

Consequently, for $p : \Sigma \to \mathbb{T}^d$ defined by

$$p = q \circ p_1,$$

we have $p = q \circ p_{n_0}$ for every $n_0 \in \mathbb{Z}$. The aim of this section is to prove that $p : (\Sigma, S) \to (\mathbb{T}^d, A)$ is a factor map.

For simplicity, we will restrict our attention to the case $n_0 = 1$. We begin our study of p_1 with an example.

Example 4 Let σ be the Fibonacci substitution $1 \mapsto 12, 2 \mapsto 1$. The two rectangles in Figure 4 correspond to the two possibilities for $i_1 = W_{j_0}^{(i_0)}$, namely 1 and 2.

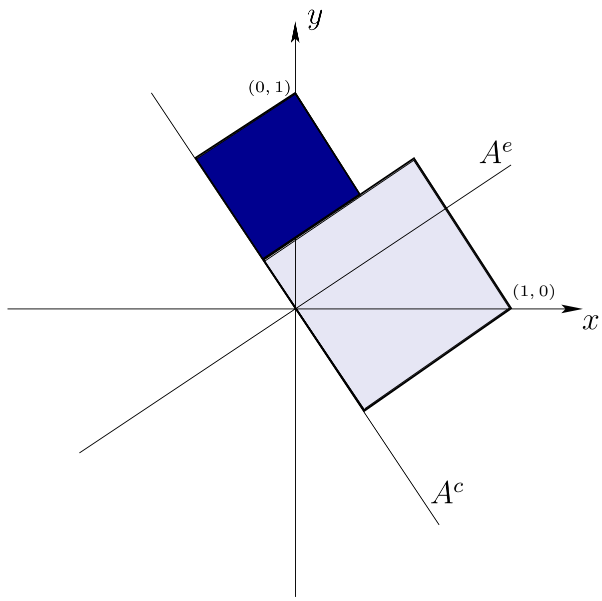


Figure 3.1: $p_1(\Sigma_\sigma)$ for $\sigma : 1 \mapsto 12, 2 \mapsto 1$.

It is easy to see that p_1 is continuous: if $x = (i_n, j_n)_{n\in\mathbb{Z}}, x' = (i'_n, j'_n)_{n\in\mathbb{Z}} \in \Sigma$ are close, then $\mathbf{x}_{j_n}^{i_n} = \mathbf{x}_{j'_n}^{i'_n}$ for all $0 \leq |n| \leq N$, where N is large. As a result, $p_1^+(x)$ and $p_1^+(x')$ (resp. $p_1^-(x)$ and $p_1^-(x')$) will be close, making p_1^+ (resp. p_1^-) continuous.

3.1.1 Properties of p_1^+

To study p_1^+, it will be useful to consider the tiling substitution on A^e induced by σ. We will use interval notation for the connected subsets of A^e as follows: $[\alpha\mathbf{v}, \beta\mathbf{v}] = \{\gamma\mathbf{v} \,|\, \alpha \leq \gamma \leq \beta\}$. We will say that a point $\alpha\mathbf{v} \in A^e$ is to the right (resp. left) of a point $\beta\mathbf{v} \in A^e$ if $\alpha > \beta$ (resp $\alpha < \beta$), and write $\alpha\mathbf{v} \preccurlyeq \beta\mathbf{v}$ iff $\alpha \leq \beta$. This terminology is quite intuitive since $\mathbf{v} > 0$.

For $i \in \mathcal{A} = \{1, 2, \ldots, d\}$, let $t_i = [\mathbf{0}, \lambda\pi^e(\mathbf{e}_i)] \subset A^e$ be the prototile corresponding to i. (The scaling factor λ will simplify our calculations later.) Denote by ω the substitution rule on the set of prototiles $\{t_i | i \in \mathcal{A}\}$ corresponding to the substitution σ on the alphabet $\mathcal{A}$. That is,

$$\begin{aligned}\omega(t_i) &= \{t_{W_1^{(i)}},\ t_{W_2^{(i)}} + \lambda\pi^e(\mathbf{e}_{W_1^{(i)}}),\ \cdots,\ t_{W_{l(i)}^{(i)}} + \sum_{j=1}^{l(i)-1} \lambda\pi^e(\mathbf{e}_{W_j^{(i)}})\} \\ &= \{t_{W_k^{(i)}} + \sum_{j=1}^{k-1} \lambda\pi^e(\mathbf{e}_{W_j^{(i)}}) \mid 1 \le k \le l(i)\} = \{t_{W_k^{(i)}} + \lambda\pi^e(\mathbf{x}_k^{(i)}) \mid 1 \le k \le l(i)\}.\end{aligned}$$

It is clear that the intersection of any two of the tiles in this union contains exactly one point if the tiles are adjacent and is empty otherwise.

We remark that ω is a tiling substitution in the sense of Section 2.2: ω scales t_i by λ and then replaces it by adjacent translations of $t_{W_1^{(i)}}, t_{W_2^{(i)}}, \ldots, t_{W_{l(i)}^{(i)}}$, in that order. To see this, we observe that $[\pi^e(\mathbf{e}_1)|\pi^e(\mathbf{e}_2)|\cdots|\pi^e(\mathbf{e}_d)]\cdot A = \lambda[\pi^e(\mathbf{e}_1)|\pi^e(\mathbf{e}_2)|\cdots|\pi^e(\mathbf{e}_d)]$ since for all $j \in \mathcal{A}$,

$$\lambda\pi^e(\mathbf{e}_j) = A\pi^e(\mathbf{e}_j) = \pi^e A(\mathbf{e}_j) = \pi^e(A_{ij})_{1\le i\le d} = [\pi^e(\mathbf{e}_1)|\pi^e(\mathbf{e}_2)|\cdots|\pi^e(\mathbf{e}_d)](A_{ij})_{1\le i\le d}.$$

As a result,

$$\sum_{j=1}^{l(i)} \lambda\pi^e(\mathbf{e}_{W_j^{(i)}}) = \sum_{k=1}^{d} A_{ki}\lambda\pi^e(\mathbf{e}_k) = \lambda(\lambda\pi^e(\mathbf{e}_i)).$$

That is, $supp(\omega(t_i)) = [\mathbf{0}, \sum_{j=1}^{l(i)} \lambda\pi^e(\mathbf{e}_{W_j^{(i)}})] = \lambda t_i$. This gives a useful representation of t_i as

$$t_i = supp(\lambda^{-1}\omega(t_i)) = \bigcup_{k=1}^{l(i)} \Big(\lambda^{-1} t_{W_k^{(i)}} + \pi^e(\mathbf{x}_k^{(i)})\Big), \tag{3.1}$$

where the unions are measure-wise disjoint. We will study the substitution tiling space associated with ω in Section 3.2, but for now we only need the substitution.

Proposition 3.3 *The image of Σ under p_1^+ is given by*

$$p_1^+(\Sigma) = \bigcup_{i=1}^{d} \lambda^{-1} t_i = \bigcup_{i=1}^{d} [0, \pi^e(\mathbf{e}_i)].$$

Proof We begin with proving that $p_1^+(\Sigma) \supseteq \bigcup_{i=1}^{d} \lambda^{-1} t_i$.

Let $\alpha \in \bigcup_{i=1}^{d} \lambda^{-1} t_i$. Then $\alpha \in \lambda^{-1} t_{i_1}$ for some $1 \leqslant i_1 \leqslant d$. From (3.1) we have

$$t_{i_1} = \lambda^{-1} \bigcup_{k=1}^{l(i_1)} \Big(t_{W_k^{(i_1)}} + \lambda \pi^e(\mathbf{x}_k^{(i_1)}) \Big).$$

So $\alpha \in \lambda^{-2}(t_{W_{j_1}^{(i_1)}} + \lambda \pi^e(\mathbf{x}_{j_1}^{(i_1)})$ for some $1 \leqslant j_1 \leqslant l(i_1)$. Let $i_2 = W_{j_1}^{i_1}$. Again we have

$$t_{i_2} = \lambda^{-1} \bigcup_{k=1}^{l(i_2)} \Big(t_{W_k^{(i_2)}} + \lambda \pi^e(\mathbf{x}_k^{(i_2)}) \Big).$$

So $\alpha \in \lambda^{-3}(t_{W_{j_2}^{(i_2)}} + \lambda \pi^e(\mathbf{x}_{j_2}^{(i_2)})) + \lambda^{-1}\pi^e(\mathbf{x}_{j_1}^{(i_1)})$ for some $1 \leqslant j_2 \leqslant l(i_2)$.

For $n \geqslant 3$, repeat the above procedure to get $i_n = W_{j_{n-1}}^{i_{n-1}}$ and $1 \leqslant j_n \leqslant l(i_n)$ such that $\alpha \in \lambda^{-(n+1)} t_{W_{j_n}^{(i_n)}} + \sum_{k=1}^{n} \lambda^{-k} \pi^e(\mathbf{x}_{j_k}^{(i_k)})$.

Since $\cap_{n\geq 1} \lambda^{-(n+1)} t_{W_{j_n}^{(i_n)}} = \cap_{n \geq 1}[0, \lambda^{-n}\pi^e(\mathbf{e}_{W_{j_n}^{(i_n)}})] = \{0\}$, we can see that

$$\alpha = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda^{-k} \pi^e(\mathbf{x}_{j_k}^{(i_k)}).$$

Since we had $i_{n+1} = W_{j_n}^{i_n}$ for all $n \geqslant 1$, there exists $x \in \Sigma$ such that i_n and j_n are as in the construction above, for $n \geqslant 1$. Then $\alpha = p_1^+(x)$.

Now we'll show that $p_1^+(\Sigma) \subseteq \bigcup_{i=1}^{d} \lambda^{-1} t_i$.

Let $x = (i_n, j_n)_{n\in\mathbb{Z}} \in \Sigma$. It is easy to see that

$$\lambda^{-1} t_{i_1} \supseteq \lambda^{-(n+1)} t_{W_{j_n}^{(i_n)}} + \sum_{k=1}^{n} \lambda^{-k} \pi^e(\mathbf{x}_{j_k}^{(i_k)}) \quad \forall\, n \in \mathbb{N}.$$

It follows that

$$\begin{aligned}\lambda^{-1}t_{i_1} &\supseteq \bigcap_{n=1}^{\infty}\left(\lambda^{-(n+1)}t_{W_{j_n}^{(i_n)}} + \sum_{k=1}^{n}\lambda^{-k}\pi^e(\mathbf{x}_{j_k}^{(i_k)})\right)\\ &= \{\lim_{n\to\infty}\sum_{k=1}^{n}\lambda^{-k}\pi^e(\mathbf{x}_{j_k}^{(i_k)})\}\\ &= \{p_1^+(x)\}\end{aligned}$$

□

It is clear from the preceding proof that p_1^+ is not injective. However, a study of the injectivity of p_1^+ on cylinders of Σ yields a useful result. Before we continue in this direction, consider the following easy consequences of the preceding proof.

Corollary 3.4 *For $i \in \mathcal{A}$,*

$$p_1^+(U(1,1,i)) = \lambda^{-1}t_i,$$

and for $k \in \mathbb{N}$,

$$p_1^+(U(1,k+1,(i_1,j_1)\cdots(i_k,j_k))) = \lambda^{-(k+1)}t_{W_{j_k}^{(i_k)}} + \sum_{n=1}^{k}\lambda^{-n}\pi^e(\mathbf{x}_{j_n}^{(i_n)}).$$

In more intuitive terms, given $x \in \Sigma$, $(i_1,j_1)\cdots(i_k,j_k)$ tells us in which subinterval of $\lambda^{-1}t_{i_1}$ $p_1^+(x)$ lies. As k increases, the length of the subinterval decreases.

Corollary 3.5 *For any $n \in \mathbb{Z}$, $p_n^+(U(n,n,i)) = \lambda^{-n}t_i$. In particular,*

$$\mathbf{0} \preccurlyeq p_n^+(x) \preccurlyeq \lambda^{-(n-1)}\pi^e(\mathbf{e}_{i_n}),$$

with $p_n^+(x) = \mathbf{0}$ if and only if $j_m = 1$ for every $m \geq n$, and $p_n^+(x) = \lambda^{-(n-1)}\pi^e(\mathbf{e}_{i_n})$ if and only if $j_m = l(i_m)$ for every $m \geq n$.

Recall that our measure on Σ is defined by

$$\mu(U(k,l,(i_k,j_k)\cdots(i_{l-1},j_{l-1}))) = \lambda^{k-l}\mathbf{v}(i_k)\mathbf{u}(W_{j_{l-1}}^{(i_{l-1})}).$$

Lemma 3.6 *There exists a nullset $\mathcal{N} \in \Sigma$ such that if $x, x' \in U(1,1,i)$ for some $i \in \mathcal{A}$ and $p_1^+(x) = p_1^+(x')$, then either $(i_n, j_n)_{n\geq 1} = (i'_n, j'_n)_{n\geq 1}$ or $x, x' \in \mathcal{N}$.*

Proof Let $x \neq x'$ such that $(i_n, j_n)_{n\geq 1} \neq (i'_n, j'_n)_{n\geq 1}$ and $i_1 = i'_1$. If $p_1^+(x) = p_1^+(x')$, then

$$p_1^+(x), p_1^+(x') \in \big(\lambda^{-(n+1)}t_{W_{j_n}^{(i_n)}} + \sum_{k=1}^{n} \lambda^{-k}\pi^e(\mathbf{x}_{j_k}^{(i_k)})\big) \cap \big(\lambda^{-(n+1)}t_{W_{j'_n}^{(i'_n)}} + \sum_{k=1}^{n} \lambda^{-k}\pi^e(\mathbf{x}_{j'_k}^{(i'_k)})\big)$$

for $n \geq 1$. Let s be minimal such that $j_s \neq j'_s$. Then $i_s = i'_s$, and the above non-empty intersection for $n = s$ implies $|j_s - j'_s| = 1$. Say $j'_s = j_s + 1$. Since $(i_n, j_n) = (i'_n, j'_n)$ for $1 \leq n < s$ and $p_1^+(x) = p_1^+(x')$, it follows that $p_s^+(x) = p_s^+(x')$. However,

$$p_s^+(x) = \lambda^{-s}\pi^e(\mathbf{x}_{j_s}^{(i_s)}) + p_{s+1}^+(x) \preccurlyeq \lambda^{-s}\pi^e(\mathbf{x}_{j_s}^{(i_s)}) + \lambda^{-s}\pi^e(\mathbf{e}_{W_{j_s}^{(i_s)}}) = \lambda^{-s}\pi^e(\mathbf{x}_{j_s+1}^{(i_s)}) \preccurlyeq p_s^+(x'),$$

with equality holding iff $p_{s+1}^+(x) = \lambda^{-s}\pi^e(\mathbf{e}_{W_{j_s}^{(i_s)}})$ and $p_{s+1}^+(x') = \mathbf{0}$. By Corollary 3.5, this happens iff $j_n = l(i_n)$ and $j'_n = 1$ for all $n > s$.

For $s \in \mathbb{N}$, let

$$\mathcal{N}_s = \bigcup_{i\in\mathcal{A}} U(s,\infty,(i,1)(W_1^{(i)},1)\cdots) \ \cup \ \bigcup_{i\in\mathcal{A}} U(s,\infty,(i,l(i))(W_1^{(i)},l(W_1^{(i)}))\cdots).$$

Clearly each $\mathcal{N}_s$ is a nullset, and hence so is $\mathcal{N} = \bigcup_{s\in\mathbb{N}} \mathcal{N}_s$. □

3.1.2 Properties of p_1^-

Maps from Σ to A^c, differing mostly in notation, are common in the literature on SFT's associated to substitutions of Pisot type, see for example [7], [2], and [8]. In Appendix 2, we prove that p_1^- is identical to a map defined in [7], which is also found in [8].

In Lemma 3.7, we prove that for any $x \in \Sigma$ for which there exists a largest integer $n \geq 0$ such that $j_{-n} \neq 1$, $p_1^-(x)$ is equal to the projection of the abelianization of the prefix of $\sigma^{n+1}(i_{-n})$ given by

$$\sigma^n(W_1^{(i_{-n})} \cdots W_{j_{-n}-1}^{(i_{-n})})\sigma^{n-1}(W_1^{(i_{-(n-1)})} \cdots W_{j_{-(n-1)}-1}^{(i_{-(n-1)})}) \cdots (W_1^{(i_0)} \cdots W_{j_0-1}^{(i_0)}).$$

A slightly different version of this lemma appears in [5].

Our previous notation of $\mathbf{x}_j^{(i)} = \sum_{k<j} \mathbf{e}_{W_k^{(i)}}$ extends easily to finite words $u \in \mathcal{A}^*$, so that $\mathbf{x}_j^{(u)} = \sum_{k<j} \mathbf{e}_{W_k^{(u)}}$, where $\sigma(u) = W^{(u)} = W_1^{(u)} \cdots W_{|u|}^{(u)}$.

Lemma 3.7 *Let $n \geqslant 0$ and let $((i_{-n}, j_{-n}) \cdots (i_0, j_0))$ be a path in G_σ. Then*

$$\pi^c\left(\mathbf{x}_k^{(\sigma^n(i_{-n}))}\right) = \sum_{m=0}^{n} A^m \pi^c\left(\mathbf{x}_{j_{-m}}^{(i_{-m})}\right), \text{ where } k = 1 + \sum_{m=0}^{n} \sum_{j<j_{-m}} |\sigma^m(W_j^{(i_{-m})})|.$$

Proof We begin by representing $\sigma^{n+1}(i_{-n})$ as follows:

$$\begin{aligned}
\sigma^{n+1}(i_{-n}) &= \sigma^n\left(W_1^{(i_{-n})} \cdots W_{l(i_{-n})}^{(i_{-n})}\right) \\
&= \sigma^n\left(W_1^{(i_{-n})} \cdots W_{j_{-n}-1}^{(i_{-n})}\right)\sigma^n\left(i_{-(n-1)}\right)\sigma^n\left(W_{j_{-n}+1}^{(i_{-n})} \cdots W_{l(i_{-n})}^{(i_{-n})}\right) \\
&= \sigma^n\left(W_1^{(i_{-n})} \cdots W_{j_{-n}-1}^{(i_{-n})}\right)\sigma^{n-1}\left(W_1^{(i_{-(n-1)})} \cdots W_{j_{-(n-1)}-1}^{(i_{-(n-1)})}\right) \cdots \\
&\quad \cdots \sigma\left(W_1^{(i_{-1})} \cdots W_{j_{-1}-1}^{(i_{-1})}\right) W_1^{(i_0)} \cdots W_{j_0-1}^{(i_0)} W_{j_0}^{(i_0)} W_{j_0+1}^{(i_0)} \cdots W_{l(i_0)}^{(i_0)} \\
&\quad \sigma\left(W_{j_{-1}+1}^{(i_{-1})} \cdots W_{l(i_{-1})}^{(i_{-1})}\right) \cdots \sigma^n\left(W_{j_{-n}+1}^{(i_{-n})} \cdots W_{l(i_{-n})}^{(i_{-n})}\right).
\end{aligned}$$

Since $A\pi^c = \pi^c A$ and $Af = f\sigma$, we have $A^n\pi^c f = \pi^c f\sigma^n$ for all $n \geqslant 0$. As a result,

$$\begin{aligned}
A^n\pi^c(\mathbf{x}_{j-n}^{(i-n)}) &= \pi^c\left(\sum_{j=1}^{k_n}\mathbf{e}_{W_j^{(\sigma^n(i_{-n}))}}\right)\\
A^{n-1}\pi^c(\mathbf{x}_{j-(n-1)}^{(i-(n-1))}) &= \pi^c\left(f\left(\sigma^{n-1}\left(W_1^{(i-(n-1))}\cdots W_{j-(n-1)-1}^{(i-(n-1))}\right)\right)\right)\\
&= \pi^c\left(\sum_{j=k_n+1}^{k_n+k_{n-1}}\mathbf{e}_{W_j^{(\sigma^n(i_{-n}))}}\right)\\
&\vdots\\
\pi^c(\mathbf{x}_{j_0}^{(i_0)}) &= \pi^c\left(\sum_{j=k_n+\cdots k_1+1}^{k_n+\cdots+k_0}\mathbf{e}_{W_j^{(\sigma^n(i_{-n}))}}\right),
\end{aligned}$$

where $k_s = |\sigma^s(W_1^{(i-s)}\cdots W_{j-s-1}^{(i-s)})|$, $0 \leqslant s \leqslant n$. The result follows directly. □

By a slight abuse of notation, we'll also write $p_1^- : \Sigma_{\sigma^N} \to (A^N)^c = A^c$ for $N \in \mathbb{N}$. To prevent ambiguity, we will take the domain of p_1^- to be $\Sigma = \Sigma_\sigma$ unless specified otherwise. Let us clarify what we mean when the domain is Σ_{σ^N}. Denote $\sigma^N(i) = U^{(i)} = U_1^{(i)}U_2^{(i)}\cdots U_{h(i)}^{(i)}$ and let $\mathbf{y}_j^{(i)} = \sum_{k<j}\mathbf{e}_{U_k^{(i)}}$. Note that $\mathbf{y}_j^{(i)} = \mathbf{x}_j^{(\sigma^{N-1}(i))}$. Since A^N is the incidence matrix for σ^N, if $X = (I_n, J_n)_{n\in\mathbb{Z}} \in \Sigma_{\sigma^N}$ then $p_1^-(X) = \sum_{n\leq 0} A^{-nN}\pi^c(\mathbf{y}_{J_n}^{(I_n)})$.

Due to the length and technical nature of the proof of the following lemma, it has been moved to Appendix 3.

Lemma 3.8 *For any $N \in \mathbb{N}$,*

$$p_1^-(\Sigma_\sigma) = p_1^-(\Sigma_{\sigma^N}).$$

Proposition 3.9 *The image of Σ under p_1^- is the Rauzy fractal associated to σ. That is,*

$$p_1^-(\Sigma) = \mathcal{R}.$$

Proof By Proposition 2.10, the Rauzy fractals associated to a periodic point u of σ and to a periodic point w of σ^n coincide for any $n \in \mathbb{N}$. Since we also have from Lemma 3.8 that $p_1^-(\Sigma_\sigma) = p_1^-(\Sigma_{\sigma^n})$, we may assume w.l.o.g. that every periodic point of σ is a fixed point.

We begin by showing that $p_1^-(\Sigma) \supseteq \mathcal{R}$.

Let $u = \cdots u_{-1} \cdot u_0 u_1 u_2 \cdots$ be a fixed point of σ. Then $W_1^{(u_0)} = u_0$ and $\sigma^k(u_0) = u_0 u_1 \cdots u_{|\sigma^k(u_0)|-1}$ for all $k \geqslant 1$.

Recall that $\mathcal{R} = \{\pi^c(\sum_{i=0}^n \mathbf{e}_{u_i}) \mid n \geq 0\}$. Let $r = \pi^c(\sum_{i=0}^n \mathbf{e}_{u_i}) \in \mathcal{R}$, and let $m \in \mathbb{N}$ be minimal such that $n+1 \leqslant |\sigma^{m+1}(u_0)|$. Then $u_0 u_1 \ldots u_n$ is a prefix of $\sigma^{m+1}(u_0)$.

Let $i_{-m} = u_0$ and let $1 \leq j_{-m} \leq l(i_{-m})$ be the largest integer satisfying

$$|\sigma^m(W_1^{(i_{-m})} \cdots W_{j_{-m}-1}^{(i_{-m})})| < n+1.$$

Then let $i_{-(m-1)} = W_{j_{-m}}^{(i_{-m})}$ and let $1 \leq j_{-(m-1)} \leq l(i_{-(m-1)})$ be the largest integer satisfying

$$|\sigma^m(W_1^{(i_{-m})} \cdots W_{j_{-m}-1}^{(i_{-m})})\sigma^{m-1}(W_1^{(i_{-(m-1)})} \cdots W_{j_{-(m-1)}-1}^{(i_{-(m-1)})})| < n+1.$$

Again, let $i_{-(m-2)} = W_{j_{-(m-1)}}^{(i_{-(m-1)})}$, and continue as above until we reach $i_0 = W_{j_{-1}}^{(i_{-1})}$. From our choice of m and j_k for $-m \leq k \leq -1$, there exists $1 \leq j_0 \leq l(i_0)$ such that

$$\sigma^m\left(W_1^{(i_{-m})} \cdots W_{j_{-m}-1}^{(i_{-m})}\right) \cdots \sigma\left(W_1^{(i_{-1})} \cdots W_{j_{-1}-1}^{(i_{-1})}\right) W_1^{(i_0)} \cdots W_{j_0-1}^{(i_0)} W_{j_0}^{(i_0)} = u_0 \ldots u_n$$

and hence

$$|\sigma^m\left(W_1^{(i_{-m})} \cdots W_{j_{-m}-1}^{(i_{-m})}\right) \cdots W_1^{(i_0)} \cdots W_{j_0}^{(i_0)}| = n+1.$$

By Lemma 3.7, we have $r = \pi^c(\mathbf{x}_{n+1}^{(\sigma^m(u_0))}) = \sum_{k=0}^{m} A^k \pi^c(\mathbf{x}_{j_{-k}}^{(i_{-k})})$. So let us define $x \in \Sigma$ as follows:

$$(i_s, j_s) = \begin{cases} \text{as above} & -m \leqslant s \leqslant 0 \\ (u_0, 1) & s < -m \\ (W_{j_{s-1}}^{(i_{s-1})}, 1) & s > 0 \end{cases}$$

Then $r = p_1^-(x)$, since $\mathbf{x}_{j_s}^{(i_s)} = \mathbf{0}$ for $s < -m$.

Now let $r = \lim_{k\to\infty} r_k$, where $r_k = \pi^c(\sum_{i=0}^{n_k} \mathbf{e}_{u_i})$. From the above step, we know that there exist $x_k \in \Sigma$ such that $p_1^-(x_k) = r_k$. Since Σ is compact, there exists an increasing sequence $k_s \to \infty$ such that x_{k_s} converges to some $x \in \Sigma$. So $r = \lim_{k\to\infty} p_1^-(x_k) = \lim_{s\to\infty} p_1^-(x_{k_s}) = p_1^-(\lim_{s\to\infty} x_{k_s}) = p_1^-(x)$, by the continuity of p_1^-.

It remains to be shown that $p_1^-(\Sigma) \subseteq \mathcal{R}$.

Let us start with the case where $x = (i_n, j_n)_{n=-\infty}^{\infty}$ has the property that $j_{-s} \neq 1$ and $j_n = 1$ for $n < -s$, some $s \geq 0$. Since we've assumed that every substitution-periodic point is fixed, the sequence $i, W_1^{(i)}, W_1^{(\sigma(i))}, W_1^{(\sigma^2(i))}, \cdots$ is eventually constant for any $i \in \mathcal{A}$. It follows that $(i_n, 1)_{n=-\infty}^{-(s+1)}$ is constant and $W_1^{(i_{-s})} = i_{-s}$, so that $\sigma^{s+1}(i_{-s}) = w_0 w_1 \ldots w_{|\sigma^{s+1}(i_{-s})|-1}$, where w is some fixed point of σ.

Let $v = \sigma^s(W_1^{(i_{-s})} \cdots W_{j_{-s}-1}^{(i_{-s})}) \cdots \sigma(W_1^{(i_{-1})} \cdots W_{j_{-1}-1}^{(i_{-1})}) W_1^{(i_0)} \cdots W_{j_0-1}^{(i_0)}$. By Lemma 3.7,

$$p_1^-(x) = \pi^c(f(v)).$$

Since v is a prefix of $\sigma^{s+1}(i_{-s})$, it follows that $v = w_0 w_1 \ldots w_{|v|-1}$. Since it does not matter which periodic point of σ is used in the construction of $\mathcal{R}$, $p_1^-(x) = \pi^c(f(v)) \in \mathcal{R}$.

Let $\Sigma' = \{(i_n, j_n) \in \Sigma \mid \exists\, s \in \mathbb{Z} \text{ such that } j_s \neq 1 \text{ and } j_n = 1 \text{ for } n < s\}$. Then for any $x \in \Sigma$, we can find a sequence (x_m) in Σ' that converges to x. So

$p_1^-(x) = p_1^-(\lim_{m\to\infty} x_m) = \lim_{m\to\infty} p_1^-(x_m) \in \mathcal{R}$, since the $p_1^-(x_m) \in \mathcal{R}$ and $\mathcal{R}$ is closed. □

Recall that the cylinders of $\mathcal{R}$ are defined as $\mathcal{R}_i = \{\sum_{k=0}^{n} \mathbf{e}_{u_k} \mid u_{n+1} = i\}$, $i \in \mathcal{A}$, where u is a periodic point of σ. From the preceding proof, we have the following consequence.

Corollary 3.10 *$\{p_1^-(x) \mid W_{j_0}^{(i_0)} = i_1 = i\} = p_1^-(U(1,1,i)) = \mathcal{R}_i$. In general, for $n \in \mathbb{Z}$, $p_n^-(U(n,n,i)) = A^{-(n-1)}\mathcal{R}_i$.*

The generalization follows from Lemma 3.1.

As a result of Corollary 3.10, we may write

$$\begin{aligned} \mathcal{R}_i &= \{p_1^-(x) \mid i_1 = i\} \\ &= \{p_1^-(Sx) \mid i_2 = i\} \\ &= \{Ap_2^-(x) \mid i_2 = i\} \\ &= \{A(p_1^-(x) + A^{-1}\pi^c(\mathbf{x}_{j_1}^{(i_1)})) \mid i_2 = i\} \\ &= \bigcup_{(k,j) \mid W_j^{(k)} = i} \left(A\mathcal{R}_k + \pi^c(\mathbf{x}_j^{(k)})\right). \end{aligned}$$

The last equality stems from the fact that $\{p_1^-(x) | (i_1, j_1) = (k, j)\} = \{p_1^-(x) | i_1 = k\}$. By [18], the unions $\left(A\mathcal{R}_k + \pi^c(\mathbf{x}_j^{(k)})\right) \cup \left(A\mathcal{R}_{k'} + \pi^c(\mathbf{x}_{j'}^{(k')})\right)$ are measure-wise disjoint for $(k, j) \neq (k', j')$ such that $W_j^{(k)} = W_{j'}^{(k')} = i$. This representation of the cylinders of the Rauzy fractal leads to the following generalization of Corollary 3.10.

Corollary 3.11 *For $n, m \geq 0$,*

$$\begin{aligned} p_1^-(U(-n, m+1, (i_{-n}, j_{-n}) \cdots (i_m, j_m))) &= p_1^-(U(-n, 1, (i_{-n}, j_{-n}) \cdots (i_0, j_0))) \\ &= A^{n+1}\mathcal{R}_{i_{-n}} + \sum_{k=-n}^{0} A^{-k}\pi^c(\mathbf{x}_{j_k}^{(i_k)}). \end{aligned}$$

Proof The first equality is trivial.

Now, since

$$\mathcal{R}_i = \bigcup_{(k,j)|W_j^{(k)}=i} \left(A\mathcal{R}_k + \pi^c(\mathbf{x}_j^{(k)}) \right),$$

it follows from the preceding Corollary that

$$\begin{aligned}
A^{n+1}\mathcal{R}_{i_{-n}} &= A^{n+1} \bigcup_{(k,j)|W_j^{(k)}=i_{-n}} \left(\{Ap_1^-(x) \mid i_1 = k\} + \pi^c(\mathbf{x}_j^{(k)}) \right) \\
&= \bigcup_{(k,j)|W_j^{(k)}=i_{-n}} \left(\{A^{n+2}p_1^-(x) \mid i_1 = k\} + A^{n+1}\pi^c(\mathbf{x}_j^{(k)}) \right) \\
&= \bigcup_{(k,j)|W_j^{(k)}=i_{-n}} \left(\{p_{-(n+1)}^-(S^{n+2}x) \mid i_1 = k\} + A^{n+1}\pi^c(\mathbf{x}_j^{(k)}) \right) \\
&= \bigcup_{(k,j)|W_j^{(k)}=i_{-n}} \left(\{p_{-(n+1)}^-(x) \mid i_{-(n+1)} = k\} + A^{n+1}\pi^c(\mathbf{x}_j^{(k)}) \right). \qquad \square
\end{aligned}$$

Hence for a path $(i_s, j_s)_{s=-n}^0$ in G_σ,

$$A^{n+1}\mathcal{R}_{i_{-n}} + \sum_{k=-n}^{0} A^{-k}\pi^c(\mathbf{x}_{j_k}^{(i_k)}) = p_1^-(U(-n, 1, (i_{-n}, j_{-n}) \cdots (i_0, j_0))).$$

The two following lemmas closely follow [7]. Recall that $m_c(AB) = \lambda^{-1}m_c(B)$ for all measurable sets $B \subseteq A^c$ and that $\mathbf{v}$ is a positive eigenvector of A associated to λ.

Lemma 3.12 *There exists a constant $C > 0$ such that for all $i \in \mathcal{A}$,*

$$m_c(\mathcal{R}_i) = C\mathbf{v}(i).$$

Proof We have from [18] that the unions on the right side of

$$\mathcal{R}_i = \bigcup_{(k,j)|W_j^{(k)}=i} \left(A\mathcal{R}_k + \pi^c(\mathbf{x}_j^{(k)})\right)$$

are measure-wise disjoint and that $m_c(\mathcal{R}) > 0$.

Hence

$$\begin{aligned} m_c(\mathcal{R}_i) &= \sum_{(k,j)|W_j^{(k)}=i} m_c(A\mathcal{R}_k + \pi^c(\mathbf{x}_j^{(k)})) \\ &= \sum_{(k,j)|W_j^{(k)}=i} m_c(A\mathcal{R}_k) \\ &= \lambda^{-1} \sum_{(k,j)|W_j^{(k)}=i} m_c(\mathcal{R}_k) \\ &= \lambda^{-1} \sum_{k=1}^{d} A_{ik} m_c(\mathcal{R}_k) \\ &= \lambda^{-1}(A \cdot (m_c(\mathcal{R}_k))_{1\le k\le d})(i). \end{aligned}$$

That is,

$$\lambda(m_c(\mathcal{R}_i))_{1\le i\le d} = A \cdot (m_c(\mathcal{R}_i))_{1\le i\le d}.$$

Since λ is a simple eigenvalue of A, this implies that there exists a constant C such that $(m_c(\mathcal{R}_i))_{1\le i\le d} = C\mathbf{v}$. Moreover, since $\mathcal{R}$ is the union of the $\mathcal{R}_i$ and $m_c(\mathcal{R}) > 0$, we must have $C > 0$. □

Recall that

$$\mathcal{R} = \bigcup_{i\in\mathcal{A}} \mathcal{R}_i$$

$$\mathcal{R}_i = \bigcup_{(i_0,j_0)\ |\ W_{j_0}^{(i_0)}=i} A\mathcal{R}_{i_0} + \pi^c(\mathbf{x}_{j_0}^{(i_0)})$$

$$A\mathcal{R}_i = \bigcup_{(i_{-1},j_{-1})(i_0,j_0)\,|\,W_{j_{-1}}^{(i_{-1})}=i} A^2\mathcal{R}_{i_{-1}} + A\pi^c(\mathbf{x}_{j_{-1}}^{(i_{-1})}) + \pi^c(\mathbf{x}_{j_0}^{(i_0)})$$

$$\vdots$$

and that these unions intersect on a set of measure 0 if σ satisfies the coincidence condition. Furthermore, $\overline{Int(\mathcal{R}_i)} = \mathcal{R}_i$, so that $m_c(Int(\mathcal{R}_i)) = m_c(\mathcal{R}_i)$. Let us call the sets $\mathcal{R}_i$ level-0 cylinders, the $A\mathcal{R}_{i_0} + \pi^c(\mathbf{x}_{j_0}^{(i_0)})$ level-1 cylinders, and so on.

Lemma 3.13 *Suppose that σ satisfies the coincidence condition and let*

$$\mathcal{N} = \bigcup_{i\neq j} \mathcal{R}_i \cap \mathcal{R}_j.$$

Then $\mu((p_1^-)^{-1}(\mathcal{N})) = 0$.

Proof Let $\delta_n \to 0$, and let us define the sets $B_{\delta_n}(\mathcal{N}) = \cup_{\mathbf{x}\in\mathcal{N}} B_{\delta_n}(\mathbf{x})$. Then $\lim_{n\to\infty} m_c(B_{\delta_n}(\mathcal{N})) = m_c(\cap_1^\infty B_{\delta_n}(\mathcal{N})) = m_c(\mathcal{N}) = 0$, since σ satisfies the coincidence condition and $\overline{\mathcal{N}} = \mathcal{N}$. As a result, for every $\epsilon > 0$, there exists $\delta > 0$ such that $m_c(B_\delta(\mathcal{N})) < \epsilon$.

Fix $\epsilon > 0$, and let $\delta > 0$ be such that $m_c(B_\delta(\mathcal{N})) < \epsilon$. We can choose $n \in \mathbb{N}$ such that all of the level-n cylinders have diameter less than $\delta/2$. There are finitely many level-n cylinders adjacent to $\mathcal{N}$, say r of them (sets are "adjacent" if they have non-emtpy intersection). Label these r cylinders by T_m, where each T_m is associated to a path $(i_{-n+1}^m, j_{-n+1}^m), \cdots (i_0^m, j_0^m)$ in the graph G associated to σ, that is

$$T_m = A^n\mathcal{R}_{i_{-n+1}^m} + \sum_{k=-n+1}^{0} A^{-k}\pi^c(\mathbf{x}_{j_k^m}^{(i_k^m)}).$$

Then

$$(p_1^-)^{-1}(\mathcal{N}) \subseteq \bigcup_{1\leq m\leq r} U(-n+1,1,(i_{-n+1}^m, j_{-n+1}^m),\cdots(i_0^m, j_0^m)),$$

since all of the other cylinders $U(-n+1,1,(i_{-n+1}, j_{-n+1}),\cdots(i_0, j_0))$ of Σ have images in level-n cylinders that are not adjacent to $\mathcal{N}$. Since $m_c(A^n\mathcal{R}_i) = C\lambda^{-n}\mathbf{v}(i)$ and since the T_m intersect on sets of measure 0, we get that

$$\begin{aligned}
\mu((p_1^-)^{-1}(\mathcal{N})) &\leq \sum_{m=1}^{r} \lambda^{-n}\mathbf{v}(i_{-n+1}^m)\mathbf{u}(W_{j_0^m}^{(i_0^m)}) \\
&= \sum_{m=1}^{r} m_c(T_m)\frac{1}{C}\mathbf{u}(W_{j_0^m}^{(i_0^m)}) \\
&\leq K\sum_{m=1}^{r} m_c(T_m) \\
&\leq K m_c(B_\delta(\mathcal{N})) \\
&< K\epsilon,
\end{aligned}$$

where $K = max\{\frac{1}{C}\mathbf{u}(i) \mid i \in \mathcal{A}\}$ ($K > 0$ since $C, \mathbf{u} > 0$).

Since ϵ was arbitrary, $\mu((p_1^-)^{-1}(\mathcal{N})) = 0$. □

Corollary 3.14 *If σ satisfies the coincidence condition, then the set containing all of the pairwise intersections of level-n cylinders for every $n \geq 0$ has preimage under p_1^- of measure* 0.

Lemma 3.15 *If σ satisfies the coincidence condition, then there exists a nullset $\mathcal{M} \subset \Sigma$ such that for $x \neq x' \in \Sigma \setminus \mathcal{M}$, either $p_1^-(x) \neq p_1^-(x')$ or $(i_n, j_n)_{n\leq 0} = (i_n', j_n')_{n\leq 0}$.*

Proof Recall that $\mathcal{R}_i = \{p_1^-(x) \mid i_1 = i\}$. We have from [18] that if σ satisfies the coincidence condition then the unions

$$\mathcal{R}_i = \bigcup_{(k,j) | W_j^{(k)} = i} \left(A\mathcal{R}_k + \pi^c(\mathbf{x}_j^{(k)}) \right)$$

and

$$\mathcal{R} = \bigcup_{i \in \mathcal{A}} \mathcal{R}_i$$

are measure-wise disjoint. That is, $m_c(\mathcal{R}_i \cap \mathcal{R}_j) = 0$ if $i \neq j$. By Corollary 3.14, this implies that for

$$\mathcal{H} = \bigcup_{i \neq j} (p_1^-)^{-1}(\mathcal{R}_i \cap \mathcal{R}_j),$$

we have $\mu(\mathcal{H}) = 0$.

Letting

$$\mathcal{N}_i = \bigcup_{\substack{(k,j) \neq (k',j') \\ W_j^{(k)} = W_{j'}^{(k')} = i}} \left(\left(A\mathcal{R}_k + \pi^c(\mathbf{x}_j^{(k)}) \right) \cap \left(A\mathcal{R}_{k'} + \pi^c(\mathbf{x}_{j'}^{(k')}) \right) \right)$$

be the set of pairwise intersections of level-1 cylinders, we apply Corollary 3.14 to get $\mu((p_1^-)^{-1}(\mathcal{N}_i)) = 0$. Furthermore, the $A\mathcal{R}_k + \pi^c(\mathbf{x}_j^{(k)})$ can be measurably partitioned infinitely many times. So for $m \leq -1$ and paths $P_{m+1,0}(i) = (k_{m+1}, j_{m+1}) \cdots (k_0, j_0)$ in G_σ satisfying $W_{j_0}^{(k_0)} = i$, let

$$\mathcal{N}_{P_{m+1,0}(i)} = \bigcup_{\substack{(k,j) \neq (k',j') \\ W_j^{(k)} = W_{j'}^{(k')} = k_{m+1}}} \Bigg(\left(A^{-m+1}\mathcal{R}_k + A^{-m}\pi^c(\mathbf{x}_j^{(k)}) \right) \cap \left(A^{-m+1}\mathcal{R}_{k'} + A^{-m}\pi^c(\mathbf{x}_{j'}^{(k')}) \right) + \sum_{n=m+1}^{0} A^{-n}\pi^c(\mathbf{x}_{j_n}^{(k_n)}) \Bigg).$$

Again, $\mu((p_1^-)^{-1}(\mathcal{N}_{P_{m+1,0}(i)})) = 0$ for every path $P_{m+1,0}(i)$.

Let

$$\mathcal{M} = \bigcup_{i\in\mathcal{A}} \left((p_1^-)^{-1}(\mathcal{N}_i) \;\cup\; \bigcup_{m\le -1} \bigcup_{P_{m+1,0}(i)} (p_1^-)^{-1}(\mathcal{N}_{P_{m+1,0}(i)}) \right) \cup \mathcal{H}.$$

Then $\mathcal{M}$ is the union of countably many nullsets, and hence $\mu(\mathcal{M}) = 0$.

Let $x, x' \in \Sigma$ such that $p_1^-(x) = p_1^-(x')$. If $i_1 \neq i'_1$ then $p_1^-(x) = p_1^-(x') \in \mathcal{R}_{i_1} \cap \mathcal{R}_{i'_1}$ and so $x, x' \in \mathcal{H} \subset \mathcal{M}$.

Otherwise $i_1 = i'_1$ and $p_1^-(x) = p_1^-(x') \in \mathcal{R}_{i_1}$. If $(i_0, j_0) \neq (i'_0, j'_0)$ then x and x' are contained in $(p_1^-)^{-1}(\mathcal{N}_{i_1}) \subset \mathcal{M}$.

Otherwise $(i_0, j_0) = (i'_0, j'_0)$ and $p_1^-(x) = p_1^-(x') \in A\mathcal{R}_{i_0} + \pi^c(\mathbf{x}_{j_0}^{(i_0)})$.

Repeating the above step infinitely many times, we obtain that either $x, x' \in \mathcal{M}$ or $(i_n, j_n)_{n\le 0} = (i'_n, j'_n)_{n\le 0}$. □

3.1.3 Properties of p_1 and p

We define the *Markov Rauzy fractal*, $\mathcal{R}'$, as the union of d cylinders in $\mathbb{R}^d$ with base components consisting of a reflection of the d cylinders $\mathcal{R}_i$ of the Rauzy fractal and transverse components along A^e of heights equal to $\pi^e(\mathbf{e}_i)$. That is,

$$\mathcal{R}' = \bigcup_{i=1}^{d} \left([0, \pi^e(\mathbf{e}_i)] - \mathcal{R}_i\right).$$

This set is similar to the set $\tilde{\mathcal{R}} = \bigcup_{i=1}^{d} \left([0, \pi^e(\mathbf{e}_i)] + \mathcal{R}_i\right)$, defined in Section 5 of [5] as the Markov Rauzy fractal. The reason for the terminology is that $q(\mathcal{R}')$ is a Markov partition of $\mathbb{T}^d$ if σ satisfies the coincidence condition [4].

By Propositions 3.3 and 3.9, we have the following.

Proposition 3.16 *The image of* Σ *under* p_1 *is given by*

$$p_1(\Sigma) = \mathcal{R}'.$$

Proposition 3.17 *The mapping* $p_1 : \Sigma \to \mathcal{R}'$ *is a.e. injective if* σ *satisfies the coincidence condition.*

Proof Suppose that $p_1(x) = p_1(x')$. Since A^e and A^c intersect only at the origin, this implies that $p_1^+(x) = p_1^+(x')$ and $p_1^-(x) = p_1^-(x')$. If $i_1 = i_1'$, then let $\mathcal{N}$ be the nullset from Lemma 3.6, and let $\mathcal{M}$ be the nullset from Lemma 3.15. Then either $x = x'$, $x, x' \in \mathcal{N} \cup \mathcal{M}$, or $i_1 \neq i_1'$. In the last case, $p_1^-(x), p_1^-(x') \in \mathcal{R}_{i_1} \cap \mathcal{R}_{i_1'}$, and hence $x, x' \in \mathcal{M}$. □

For our next proposition, we need to assume the geometric coincidence condition defined in [4]. This hypothesis is needed for the quotient map q to map the Rauzy fractal a.e. injectively onto the d-torus. The geometric coincidence condition implies the usual coincidence condition.

Proposition 3.18 *The map* $p : \Sigma \to \mathbb{T}^d$ *is continuous and onto, and if* σ *satisfies the geometric coincidence condition, then* p *is also a.e. one-to-one.*

Proof Since p_1 and the quotient map $q : \mathbb{R}^d \to \mathbb{T}^d$ are continuous, $p = q \circ p_1$ is continuous.

From Proposition 3.2, $p = q \circ p_{n_0}$ for every $n_0 \in \mathbb{Z}$. Furthermore, since $A \circ p_{n_0-1}^- = p_{n_0}^- \circ S$ and $p_1^-(\Sigma) = \mathcal{R}$, it follows that $p_{n_0}^-(\Sigma) = A^{-n_0+1}\mathcal{R}$. Hence for every $x \in \Sigma$,

$$\begin{aligned} p(x) &= \lim_{n_0 \to -\infty} q(p_{n_0}(x)) \\ &= \lim_{n_0 \to -\infty} q(p_{n_0}^+(x) - p_{n_0}^-(x)) \\ &= \lim_{n_0 \to -\infty} q(p_{n_0}^+(x)). \end{aligned}$$

From Lemma 3.5, $p^+_{n_0}(\Sigma) = \bigcup_{i=1}^{d} \lambda^{-n_0} t_i$ for every $n_0 \in \mathbb{Z}$. Therefore $p^+_{n_0}(\Sigma) \subseteq p^+_{n_0-1}(\Sigma) \subseteq p^+_{n_0-2}(\Sigma) \subseteq \cdots$ for every $n_0 \in \mathbb{Z}$. Hence

$$p(\Sigma) = q\left(\bigcup_{n_0 \le 0} p^+_{n_0}(\Sigma)\right).$$

Since

$$\bigcup_{n_0 \le 0} p^+_{n_0}(\Sigma) = \{\alpha \mathbf{v} \mid \alpha \ge 0\},$$

it follows that $p(\Sigma)$ is dense in $\mathbb{T}^d$. Since p is continuous and Σ is compact, $p(\Sigma)$ is compact, and so $p(\Sigma) = \mathbb{T}^d$.

We proved in Proposition 3.17 that p_1 is a.e. injective if σ satisfies the coincidence condition. And Barge and Kwapisz [4] prove that if σ satisfies the geometric coincidence condition then $q : \mathbb{R}^d|_{\mathcal{R}'} \to \mathbb{T}^d$ is one-to-one except on the boundary of $\mathcal{R}'$. It is analogous to our proofs above to show that the inverse image under p_1 of the boundary has measure 0. Hence p is a.e. injective if σ satisfies the geometric coincidence condition. $\square$

Recall that S denotes the shift map on Σ. By a slight abuse of notation, we will write $q \circ A = A : \mathbb{T}^d \to \mathbb{T}^d$.

Lemma 3.19 *The following diagram commutes:*

$$\begin{array}{ccc} \Sigma & \xrightarrow{S} & \Sigma \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{T}^d & \xrightarrow[A]{} & \mathbb{T}^d \end{array}$$

Proof It is easy to see that

$$Ap(x) = q\Big(\sum_{n \ge 1} A^{-(n-1)} \pi^e(\mathbf{x}^{(i_n)}_{j_n}) - \sum_{n \le 0} A^{-(n-1)} \pi^c(\mathbf{x}^{(i_n)}_{j_n})\Big)$$

and

$$pS(x) = q\Big(\sum_{n\geq 1} A^{-n}\pi^e(\mathbf{x}_{j_{n+1}}^{(i_{n+1})}) - \sum_{n\leq 0} A^{-n}\pi^c(\mathbf{x}_{j_{n+1}}^{(i_{n+1})})\Big).$$

Hence

$$\begin{aligned} q\left(Ap(x) - pS(x)\right) &= q\left(\pi^e(\mathbf{x}_{j_1}^{(i_1)}) + \pi^c(\mathbf{x}_{j_1}^{(i_1)})\right) \\ &= q\left(\mathbf{x}_{j_1}^{(i_1)}\right) = 0. \end{aligned}$$

□

We say that two dynamical systems (X, f) and (Y, g) are *semi-topologically conjugate* if there exists a continuous, surjective and a.e. injective map $h : X \to Y$ such that $h \circ f = g \circ h$. We have shown the following:

Proposition 3.20 *The dynamical systems* (Σ, S) *and* $(\mathbb{T}^d, A)$ *are semi-topologically conjugate via* p *if* σ *satisfies the geometric coincidence condition.*

In terms of Smale spaces, $p : (\Sigma, S) \to (\mathbb{T}^d, A)$ is a factor map.

3.2 From the SFT to a Tiling Space to $\mathbb{T}^d$

3.2.1 From the SFT to the Tiling Space

Recall our tiling substitution $\omega(t_i) = \{t_{W_k^{(i)}} + \lambda\pi^e(\mathbf{x}_k^{(i)}) \mid 1 \leq k \leq l(i)\}$, where $t_i = [\mathbf{0}, \lambda\pi^e(\mathbf{e}_i)]$ for $i \in \mathcal{A}$. As in Section 2.2, we will denote by $\{t_1, \cdots, t_d\}^*$ the collection of all partial tilings of A^e containing only translations of the prototiles t_i. Define maps $T_k : \Sigma \to \{t_1, \cdots, t_d\}^*$ for $k \geqslant 0$ as follows,

$$T_k(x) = \omega^k(t_{i_{-k}}) - p^+_{-k}(x).$$

Example 5 Let $\sigma : 1 \mapsto 12, 2 \mapsto 1$ as above. Then

$$x = (\cdots\ (2,1)\ \cdot (1,2)\ (2,1)\ (1,2)\ \cdots)$$

and

$$y = (\cdots\ (1,1)\ \cdot (1,1)\ (1,1)\ (1,1)\ \cdots)$$

are elements of Σ_σ, and we consider T_k for $k = 0, 1, 2$ (see Figure 5, where the longer tiles are translates of t_1 and the shorter tiles are translates of t_2).

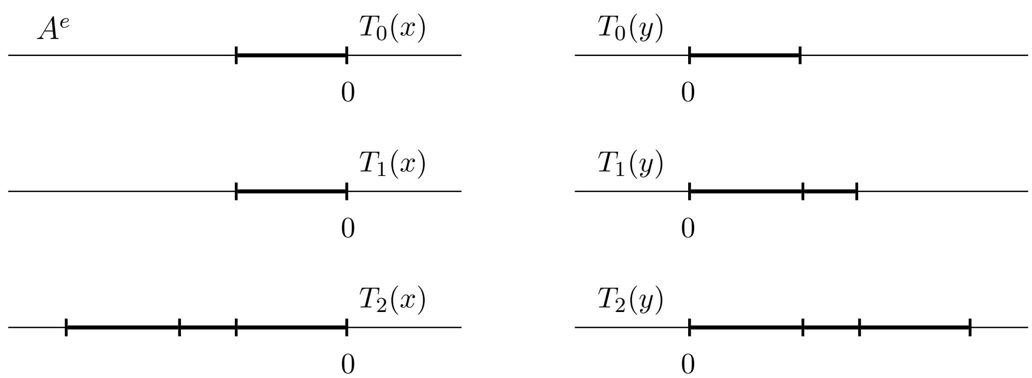


Figure 3.2: T_k for $k = 0, 1, 2$

Remark By Corollary 3.5, given $\alpha \in t_i$, there exists $x \in \Sigma$ with $i_0 = i$ such that $p_0^+(x) = \alpha$. Hence $T_0(x) = t_i - \alpha$. Since p_0^+ is not injective, neither is T_0. However, by Lemma 3.6, for almost every tile in $T_0(\Sigma)$, its inverse image under T_0 contains a unique infinite cylinder $U(1, \infty, (i_1, j_1) \cdots)$. Otherwise, its inverse image under T_0 contains two infinite cylinders.

Proposition 3.21 *$T_k(x) \subseteq T_{k+1}(x)$ for all $k \geqslant 0$ and $x \in \Sigma$.*

Proof By definition, we have

$$\omega(t_i) = \{t_{W_1^{(i)}},\ t_{W_2^{(i)}} + \lambda\pi^e(\mathbf{x}_2^{(i)}), \cdots,\ t_{W_{l(i)}^{(i)}} + \lambda\pi^e(\mathbf{x}_{l(i)}^{(i)})\}. \tag{3.2}$$

Let us begin with $k = 0$ for concreteness.

$$\begin{aligned} T_0(x) &= t_{i_0} - p_0^+(x) \\ T_1(x) &= \omega(t_{i_{-1}}) - p_{-1}^+(x) \\ &= \omega(t_{i_{-1}}) - (p_0^+(x) + \lambda\pi^e(\mathbf{x}_{j_{-1}}^{(i_{-1})})) \end{aligned}$$

The last equality follows directly from the definition of $p_{n_0}^+$.

Since $i_0 = W_{j_{-1}}^{i_{-1}}$, we have from (3.2) that $t_{i_0} + \lambda\pi^e(\mathbf{x}_{j_{-1}}^{(i_{-1})}) \in \omega(t_{i_{-1}})$. Hence

$$\begin{aligned} T_1(x) &\supseteq (t_{i_0} + \lambda\pi^e(\mathbf{x}_{j_{-1}}^{(i_{-1})})) - (p_0^+(x) + \lambda\pi^e(\mathbf{x}_{j_{-1}}^{(i_{-1})})) \\ &= t_{i_0} - p_0^+(x) \\ &= T_0(x). \end{aligned}$$

To see that $T_n(x) \subseteq T_{n+1}(x)$, we consider

$$\begin{aligned} \omega^{n+1}(t_i) &= \omega^n\left(\omega(t_i)\right) \\ &= \omega^n\Big(\{t_{W_1^{(i)}},\ t_{W_2^{(i)}} + \lambda\pi^e(\mathbf{x}_2^{(i)}), \cdots,\ t_{W_{l(i)}^{(i)}} + \lambda\pi^e(\mathbf{x}_{l(i)}^{(i)}\}\Big) \\ &= \omega^n(t_{W_1^{(i)}}) \cup \Big(\omega^n(t_{W_2^{(i)}}) + \lambda^{n+1}\pi^e(\mathbf{x}_2^{(i)})\Big) \cup \cdots \cup \Big(\omega^n(t_{W_{l(i)}^{(i)}}) + \lambda^{n+1}\pi^e(\mathbf{x}_{l(i)}^{(i)})\Big). \end{aligned}$$

As before, we have $i_{-n} = W_{j_{-(n+1)}}^{i_{-(n+1)}}$, hence

$$\omega^n(t_{i_{-n}}) + \lambda^{n+1}\pi^e(\mathbf{x}_{j_{-(n+1)}}^{i_{-(n+1)}}) \subseteq \omega^{n+1}(t_{i_{-(n+1)}}). \tag{3.3}$$

Since

$$T_{n+1}(x) = \omega^{n+1}(t_{i_{-(n+1)}}) - (p_{-n}^+(x) + \lambda^{n+1}\pi^e(\mathbf{x}_{j_{-(n+1)}}^{i_{-(n+1)}})),$$

it follows from (3.3) that

$$T_n(x) = \omega^n(t_{i_{-n}}) - p^+_{-n}(x) \subseteq T_{n+1}(x).$$

□

We will need the following lemma to construct tilings of A^e containing the partial tilings $T_k(x)$.

Lemma 3.22 *For $x \in \Sigma$, $k \geqslant 1$ and $n \in \mathbb{Z}$,*

$$(t_{i_n} - \lambda^n p^+_n(x)) \in (\omega^k(t_{i_{n-k}}) - \lambda^n p^+_{n-k}(x)).$$

Proof Let $x \in \Sigma$ and $n \in \mathbb{Z}$. From the definition of ω we have

$$\begin{aligned} t_{i_n} &\in \omega(t_{i_{n-1}}) - \lambda\pi^e(\mathbf{x}^{i_{(n-1)}}_{j_{n-1}}) \\ &\in \omega^2(t_{i_{n-2}}) - \lambda^2\pi^e(\mathbf{x}^{i_{(n-2)}}_{j_{n-2}}) - \lambda\pi^e(\mathbf{x}^{i_{(n-1)}}_{j_{n-1}}) \\ &\vdots \\ &\in \omega^k(t_{i_{n-k}}) - \sum_{m=1}^{k} \lambda^m \pi^e(\mathbf{x}^{i_{(n-m)}}_{j_{n-m}}). \end{aligned}$$

It follows that

$$\begin{aligned} \lambda^{-n} t_{i_n} - p^+_n(x) &\in \lambda^{-n}\omega^k(t_{i_{n-k}}) - p^+_n(x) - \sum_{m=1}^{k} \lambda^{-(n-m)} \pi^e(\mathbf{x}^{i_{(n-m)}}_{j_{n-m}}) \\ &= \lambda^{-n}\omega^k(t_{i_{n-k}}) - p^+_{n-k}(x). \end{aligned}$$

The result follows. □

Let us consider the substitution tiling space associated to ω. As in Section 2.2, we define Ω to be the collection of all tilings T in $\{t_1, \cdots, t_d\}^*$ having the property that for any patch $U \subseteq T$, $U \subseteq \omega^m(P_i + \mathbf{x})$ for some $m \geq 0$,$1 \leq i \leq d$, and $\mathbf{x} \in A^e$. Since σ is primitive, so is ω, and it is clear that Ω has finite local complexity. Also, since X_σ contains no shift-periodic points, Ω contains no translation-periodic points. It follows from Proposition 2.8 that $\omega : \Omega \to \Omega$ is injective. We also have that $\omega : \Omega \to \Omega$ is onto and continuous, and that Ω is compact. Hence ω^{-1} is also continuous.

Define

$$\Delta = \overline{\{\mathcal{T}_0 - \mathbf{a} \mid \mathbf{a} \in A^e\}},$$

where $\mathcal{T}_0$ is a substitution-periodic tiling of A^e. From Section 2.2, we have $\Delta = \Omega$. Notice that if $w \in X_\sigma$ is a σ-periodic word, then the tiling

$$T_w = \{\cdots,\ t_{w_{-1}} - \lambda\pi^e(\mathbf{e}_{w_{-1}}),\ t_{w_0},\ t_{w_1} + \lambda\pi^e(\mathbf{e}_{w_0}), \cdots\}$$

is $\omega - \textit{periodic}$ and clearly $T_w \in \Omega$.

In [13], a tiling substitution ω is said to satisfy the forcing the border condition if there exists $k \geqslant 1$ such that, if T, $T' \in \Omega$ are two tilings containing a tile t, then the patches in $\omega^k(T)$ and $\omega^k(T')$ consisting of all tiles which meet $\omega^k(t)$ are identical. Let us call the integer k a *forcing number* for ω.

Proposition 3.23 *If ω satisfies the forcing the border condition, then for every $x \in \Sigma$ there exists a unique tiling $\mathcal{T}_x \in \Omega$ containing $\bigcup_{n \geqslant 0} T_n(x)$.*

Proof If $\bigcup_{n \geqslant 0} T_n(x) = A^e$, we are done. Otherwise, $\bigcup_{n \geqslant 0} T_n(x)$ contains a rightmost or leftmost tile, but not both, since the support of $T_n(x)$ is a translate of the interval $\lambda^n t_{i_{-n}}$. Since the two cases are analogous, we'll consider only the case of a rightmost tile.

Let k be a forcing number for ω as defined above, and let $\mathcal{S} \in \Omega$ be any tiling containing $t_{i_{-k}} - \lambda^{-k} p^{+}_{-k}(x)$. Then there is a unique tile t which may be right adjacent to $\omega^k(t_{i_{-k}}) - p^{+}_{-k}(x) = T_k(x)$ in $\omega^k(\mathcal{S})$.

Further, let $\mathcal{S}' \in \Omega$ be any tiling containing $t_{i_{-2k}} - \lambda^{-2k} p^{+}_{-2k}(x)$. Then there is a unique tile t' which may be right adjacent to $\omega^k(t_{i_{-2k}}) - \lambda^{-k} p^{+}_{-2k}(x)$ in $\omega^k(\mathcal{S}')$. By Lemma 3.22, we have $t_{i_{-k}} - \lambda^{-k} p^{+}_{-k}(x) \subseteq \omega^k(t_{i_{-2k}}) - \lambda^{-k} p^{+}_{-2k}(x)$. Hence $\omega^{2k}(\mathcal{S}')$ contains $T_k(x) \cup t$, and $T_{2k}(x) \cup \omega^k(t')$ gives us some tiles right adjacent to $T_k(x)$.

More generally, for $l \in \mathbb{N}$ let $\Omega_l = \{\mathcal{S} \in \Omega : \mathcal{S} \text{ contains } t_{i_{-kl}} - \lambda^{-kl} p^{+}_{-kl}(x)\}$. Then, by Lemma 3.22, every tiling in $\omega^k(\Omega_l)$ contains $t_{i_{-k(l-1)}} - \lambda^{-k(l-1)} p^{+}_{-k(l-1)}(x)$ and any two have the same unique tile, s_l, right adjacent to $t_{i_{-k(l-1)}} - \lambda^{-k(l-1)} p^{+}_{-k(l-1)}(x)$ (since we know what tile is right adjacent to $\omega^k(t_{i_{-kl}} - \lambda^{-kl} p^{+}_{-kl}(x))$). It follows that every tiling in $\omega^{kl}(\Omega_l)$ contains $T_{k(l-1)}(x)$ and the same patch of tiles, $\omega^{k(l-1)}(s_l)$, right adjacent to $T_{k(l-1)}(x)$. Clearly $diam(\omega^{k(l-1)}(s_l)) \to \infty$ as $l \to \infty$.

In other words, $\omega^k(\Omega_l) \subseteq \Omega_{l-1}$, and $\omega^{kl}(\Omega_l)$ gives us growing patches of tiles right adjacent to $T_{k(l-1)}(x)$. Since we assumed that $\bigcup_{n \geqslant 0} T_n(x)$ has a rightmost tile, this rightmost tile is contained in $T_n(x)$ for all $n \geqslant N$, some $N \in \mathbb{N}$. It is clear that the patches of unique tiles right adjacent to the $T_{kl}(x)$ grow to cover all of A^e to the right of the rightmost tile. By taking the union of these patches and $\bigcup_{n \geqslant 0} T_n(x)$, we obtain a unique tiling of A^e. This tiling is clearly contained in Ω. □

Of course, many tiling substitutions, including the Fibonacci substitution, do not satisfy the forcing the border condition. However, Kellendonk and Putnam [13] describe a method of replacing any substitution tiling system by one that forces its border and has exactly the same collection of tilings. One may ask whether this new substitution preserves properties of the orginal substitution, such as being Pisot and unimodular, but we do not address this question here.

For the rest of this chapter, we assume that ω satisfies the forcing the border condition. So we can define a map $T : \Sigma \to \Omega$ by $x \mapsto \mathcal{T}_x$, where $\mathcal{T}_x$ is the tiling in

Ω containing $\bigcup_{n\geqslant 0} T_n(x)$.

The following terms are defined in [1]. Two points $x, x' \in \Sigma$ are said to be *right asymptotic* if there is an integer N such that $(i_n, j_n) = (i'_n, j'_n)$ for all $n \geq N$ (sometimes this is also referred to as being *right tail-equivalent*). A *left closing map* is a continous shift-commuting map that never identifies two distinct right asymptotic points.

Proposition 3.24 *[1] If ω satisfies the forcing the border condition, then the map $T : \Sigma \to \Omega$ is continuous, onto, left closing, and $TS = \omega T$.*

Proof See Theorem 5.1 in [1]. □

In other words, $T : (\Sigma, S) \to (\Omega, \omega)$ is an s-resolving factor map.

3.2.2 From the Tiling Space to $\mathbb{T}^d$

Since $\mathcal{A}$ is finite, there exist $a \in \mathcal{A}$ and $K \geq 1$ such that $\sigma^K(a)_1 = a$. Let $e_0 = (i_n, j_n)_{n\in\mathbb{Z}}$ where $(i_{nK}, j_{nK}) = (a, 1)$ and $j_n = 1$ for all $n \in \mathbb{Z}$. Then define

$$E = \{e \in \Sigma \mid \exists\, N \,\text{such that}\; i_{(N-n)K} = a \,\forall\, n \geq 0,\; j_m = 1 \,\forall\, m < NK\}$$

and let $\mathcal{T}_0 = T(e_0)$. Notice that $\omega^{nK}(t_a) \subseteq \mathcal{T}_0$ for $n \geq 0$.

Lemma 3.25 *For $e \in E$, $T(e) = \mathcal{T}_0 - p^+_{NK}(e)$. Furthermore,*

$$T(E) = \mathcal{T}_0 - A^e_{\geq 0} = \{\mathcal{T}_0 - \alpha\mathbf{v} \mid \alpha \geq 0\}.$$

Proof To show $T(E) \subseteq \mathcal{T}_0 - A^e_{\geq 0}$, pick $e = (i_n, j_n)_{n\in\mathbb{Z}} \in E$. There there exists N such that $i_{(N-n)K} = a$ for $n \geq 0$ and $j_m = 1$ for $m < NK$. Then clearly

$p^+_{NK-l}(e) = p^+_{NK}(e)$ for $l \geq 0$. As a result, for $n \geq max\{0,\ N\}$ we have

$$T_{-(N-n)K}(e) = \omega^{-(N-n)K}(t_a) - p^+_{NK}(e).$$

Hence

$$\bigcup_{n\geq 0} T_n(e) = \bigcup_{n\geq N} T_{-(N-n)K}(e) \subseteq \mathcal{T}_0 - p^+_{NK}(e).$$

By Proposition 3.23, $T(e) = \mathcal{T}_0 - p^+_{NK}(e)$ and clearly $p^+_{NK}(e) \in A^e_{\geq 0}$.

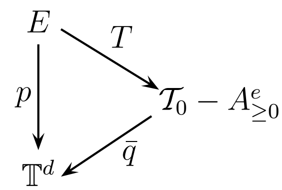
For the reverse inclusion, let $\mathbf{x} \in A^e_{\geq 0}$. Then there exists $N \leq 0$ such that $\mathbf{x} \in [\mathbf{0}, \lambda^{-NK} t_a]$. Since $p^+_{NK}(U(NK, NK, a)) = [\mathbf{0}, \lambda^{-NK} t_a]$ by Corollary 3.5, there exists $x = (i_n, j_n)_{n\in\mathbb{Z}} \in \Sigma$ with $i_{NK} = a$ such that $p^+_{NK}(x) = \mathbf{x}$. Let $e = (i'_n, j'_n)_{n\in\mathbb{Z}}$, where $(i'_n, j'_n)_{n\geq NK} = (i_n, j_n)_{n\geq NK}$ and $i_{(N-n)K} = a$ for $n \geq 0$ and $j_m = 1$ for $m < NK$. Then $e \in E$ and

$$\bigcup_{n\geq 0} T_n(e) = \bigcup_{n\geq N} T_{-(N-n)K}(e) = \bigcup_{n\geq N} \omega^{-(N-n)K}(t_a) - \mathbf{x}.$$

Since $\mathcal{T}_0$ is the unique tiling in Ω containing $\bigcup_{n\geq 0} \omega^{nK}(t_a)$, it follows that $T(e) = \mathcal{T}_0 - \mathbf{x}$. $\square$

Recall $q : \mathbb{R}^d \to \mathbb{T}^d$ by $q : \mathbf{x} \mapsto \mathbf{x}(mod\, \mathbb{Z}^d)$. Further define $\bar{q} : \mathcal{T}_0 - A^e_{\geq 0} \to \mathbb{T}^d$ by $\bar{q}(\mathcal{T}_0 - \mathbf{x}) = q(\mathbf{x})$. This map is well-defined since Ω contains no translation-periodic points.

Lemma 3.26 *The following diagram commutes:*



Proof Let $e = (i_n, j_n)_{n\in\mathbb{Z}} \in E$ such that $i_{(N-n)K} = a$ for $n \geq 0$ and $j_m = 1$ for $m < NK$. By Lemma 3.25, $T(e) = \mathcal{T}_0 - p^+_{NK}(e)$.

Since $p^-_{NK}(e) = \mathbf{0}$, we have $p^+_{NK}(e) = p_{NK}(e)$, and from Proposition 3.2, $q(p_{NK}(e)) = p(e)$.

Hence we have shown that $p(e) = \bar{q}(T(e))$. □

Although the following proof is long, the difficulty lies only in the notation.

Proposition 3.27 $\bar{q} : \mathcal{T}_0 - A^e_{\geq 0} \to \mathbb{T}^d$ *is uniformly continuous.*

Proof Let $0 < \epsilon < min\{\frac{1}{4}, \text{(largest tile length)}^{-1}\}$. We want to prove that there exists $\delta > 0$ such that for $T_1, T_2 \in \mathcal{T}_0 - A^e_{\geq 0}$ with $d(T_1, T_2) < \delta$, we have $d(\bar{q}(T_1), \bar{q}(T_2)) < \epsilon$.

By the continuity of p and the nature of the metric on Σ, there exists $K \geq 1$ such that for $x, x' \in \Sigma$ with $(i_n, j_n) = (i'_n, j'_n)$ for $-K \leq n \leq K$, $d(p(x), p(x')) < \epsilon$. By the continuity of ω^{-K}, there exists $\delta > 0$ such that for $T_1, T_2 \in \Omega$ with $d(T_1, T_2) < \delta$, we have $d(\omega^{-K}(T_1), \omega^{-K}(T_2)) < \epsilon\lambda^{-K}$.

So let $T_1, T_2 \in \mathcal{T}_0 - A^e_{\geq 0}$ with $d(T_1, T_2) < \delta$. Then $d(\omega^{-K}(T_1), \omega^{-K}(T_2)) < \epsilon\lambda^{-K}$. That is, there exist $\mathbf{x}_1, \mathbf{x}_2 \in A^e$ such that $|\mathbf{x}_1|, |\mathbf{x}_2| < \epsilon\lambda^{-K}$ and $\omega^{-K}(T_1) + \mathbf{x}_1 = \omega^{-K}(T_2) + \mathbf{x}_2$ on $B_{\epsilon^{-1}\lambda^K}(\mathbf{0})$. It is not hard to see that we may assume w.l.o.g. that $\mathbf{x}_1, \mathbf{x}_2 \in A^e_{\leq 0}$.

Notice that $\epsilon^{-1}\lambda^K > \epsilon^{-1} >$ (largest tile length). Therefore $\omega^{-K}(T_1) + \mathbf{x}_1$ and $\omega^{-K}(T_2) + \mathbf{x}_2$ have the same tile(s) containing the origin, as do $\omega^n(\omega^{-K}(T_1) + \mathbf{x}_1)$ and $\omega^n(\omega^{-K}(T_2) + \mathbf{x}_2)$ for $n \geq 1$.

By the triangle inequality,

$$\begin{aligned} d(\bar{q}(T_1), \bar{q}(T_2)) \leq\ & d(\bar{q}(T_1), \bar{q}(T_1 + \lambda^K \mathbf{x}_1)) + d(\bar{q}(T_1 + \lambda^K \mathbf{x}_1), \bar{q}(T_2 + \lambda^K \mathbf{x}_2)) \\ & + d(\bar{q}(T_2 + \lambda^K \mathbf{x}_2), \bar{q}(T_2)). \end{aligned} \tag{3.4}$$

Consider the first term on the RHS of (3.4). Since $T_1 = \mathcal{T}_0 - \mathbf{y}_1$ for some $\mathbf{y}_1 \in A^e_{\geq 0}$, we have $T_1 + \lambda^K \mathbf{x}_1 = \mathcal{T}_0 + \lambda^K \mathbf{x}_1 - \mathbf{y}_1$. Hence $\bar{q}(T_1) = q(\mathbf{y}_1)$ and $\bar{q}(T_1 + \lambda^K \mathbf{x}_1) = q(\mathbf{y}_1 - \lambda^K \mathbf{x}_1)$. However, $|\lambda^K \mathbf{x}_1| < \lambda^K \lambda^{-K} \epsilon = \epsilon < \frac{1}{4}$. Therefore,

$$d(\bar{q}(T_1),\, \bar{q}(T_1 + \lambda^K \mathbf{x}_1)) = |\lambda^K \mathbf{x}_1| < \epsilon.$$

The same argument works for the third term on the RHS of (3.4).

For the middle term on the RHS of (3.4), we need the following:

Claim: There exist $e_1 = (i_n, j_n)_{n \in \mathbb{Z}}$ and $e_2 = (i'_n, j'_n)_{n \in \mathbb{Z}}$ in E such that $T(e_1) = T_1 + \lambda^K \mathbf{x}_1$ and $T(e_2) = T_2 + \lambda^K \mathbf{x}_2$, and $(i_n, j_n) = (i'_n, j'_n)$ for $-K \leq n \leq K$.

Therefore $d(p(e_1), p(e_2)) < \epsilon$. However, $p(e_1) = \bar{q}T(e_1) = \bar{q}(T_1 + \lambda^K \mathbf{x}_1)$ and $p(e_2) = \bar{q}(T_2 + \lambda^K \mathbf{x}_2)$, and as a result the middle term in (3.4) is also less than ϵ. So to complete the proof, it suffices to prove the *Claim.*

Proof of Claim: Since $T_1 = \mathcal{T}_0 - \mathbf{y}_1$, we have

$$T_1 + \lambda^K \mathbf{x}_1 = \mathcal{T}_0 - (\mathbf{y}_1 - \lambda^K \mathbf{x}_1) \supseteq \bigcup_{n \geq 0} \omega^{nK}(t_a) - (\mathbf{y}_1 - \lambda^K \mathbf{x}_1).$$

Recall that $\mathbf{y}_1 \in A^e_{\geq 0}$ and $\mathbf{x}_1 \in A^e_{\leq 0}$. Hence $\mathbf{y}_1 - \lambda^K \mathbf{x}_1 \in A^e_{\geq 0}$. Similarly $T_2 = \mathcal{T}_0 - \mathbf{y}_2$, where $\mathbf{y}_2 \in A^e_{\geq 0}$ and $\mathbf{x}_2 \in A^e_{\leq 0}$, so $\mathbf{y}_2 - \lambda^K \mathbf{x}_2 \in A^e_{\geq 0}$. Let $N \leq -1$ be such that $\mathbf{y}_1 - \lambda^K \mathbf{x}_1,\, \mathbf{y}_2 - \lambda^K \mathbf{x}_2 \in Int(\lambda^{-NK} t_a)$.

First let us construct $e_1 \in E$. Let $i_{NK} = a$. Since

$$t_{i_{NK}} = supp\Big(\bigcup_{j=1}^{l(i_{NK})} \lambda^{-1} t_{W_j^{(i_{NK})}} + \pi^e(\mathbf{x}_j^{(i_{NK})})\Big),$$

there is $1 \leq j_{NK} \leq l(i_{NK})$ such that

$$\mathbf{y}_1 - \lambda^K \mathbf{x}_1 \in \lambda^{-NK}(\lambda^{-1} t_{W^{(i_{NK})}_{j_{NK}}} + \pi^e(\mathbf{x}^{(i_{NK})}_{j_{NK}})). \tag{3.5}$$

If (3.5) also holds if we replace j_{NK} by $j_{NK} \pm 1$, choose the greater one to be j_{NK}. Similarly, for $n \geq NK+1$, let $i_n = W^{(i_{n-1})}_{j_{n-1}}$ and let $1 \leq j_n \leq l(i_n)$ be maximal such that

$$\mathbf{y}_1 - \lambda^K \mathbf{x}_1 \in \lambda^{-(n+1)} t_{W^{(i_n)}_{j_n}} + \sum_{m=NK}^{n} \lambda^{-m} \pi^e(\mathbf{x}^{(i_m)}_{j_m}).$$

Then clearly $\mathbf{y}_1 - \lambda^K \mathbf{x}_1 = \sum_{m \geq NK} \lambda^{-m} \pi^e(\mathbf{x}^{(i_m)}_{j_m})$. Let $(i_n, j_n)_{n<NK}$ be the path in G_σ with $i_{(N-n)K} = a$ for $n \geq 1$ and $j_m = 1$ for $m < NK$, and let $e_1 = (i_n, j_n)_{n\in\mathbb{Z}}$. Then $e_1 \in E$ and $p^+_{NK}(e_1) = \mathbf{y}_1 - \lambda^K \mathbf{x}_1$. Furthermore,

$$T(e_1) \supseteq \bigcup_{n \leq N} \omega^{-nK}(t_a) - p^+_{NK}(e_1).$$

Therefore, $T(e_1) = T_1 + \lambda^K \mathbf{x}_1$.

Moreover, $(i_m, j_m)_{n \geq NK}$ satisfies the property that for every $n \geq NK$, the origin is either the left endpoint or in the interior of

$$\lambda^{-n} t_{i_n} + \sum_{m=NK}^{n-1} \lambda^{-m} \pi^e(\mathbf{x}^{(i_m)}_{j_m}) - (\mathbf{y}_1 - \lambda^K \mathbf{x}_1) = \lambda^{-n} t_{i_n} - p^+_n(e).$$

Since $TS = \omega T$ and $TS^{-1} = \omega^{-1} T$, we have

$$T_0(S^n e_1) = t_{i_n} - \lambda^n p^+_n(e_1) \subset T(S^n e_1) = \omega^n(T_1 + \lambda^K \mathbf{x}_1)$$

for $n \geq NK$. So $t_{i_n} - \lambda^n p^+_n(e_1)$ is the rightmost tile in $\omega^n(T_1 + \lambda^K \mathbf{x}_1)$ containing the origin, for $n \geq NK$.

Apply the same process to $T_2 + \lambda^K \mathbf{x}_2$ to obtain $e_2 = (i'_n, j'_n)_{n\in\mathbb{Z}} \in E$ such that $T(e_2) = T_2 + \lambda^K \mathbf{x}_2$ and such that $t_{i'_n} - \lambda^n p^+_n(e_2)$ is the rightmost tile in $\omega^n(T_2 + \lambda^K \mathbf{x}_2)$ containing the origin, for $n \geq NK$.

Since for every $n \geq 0$, $\omega^n(\omega^{-K}(T_1) + \mathbf{x}_1)$ and $\omega^n(\omega^{-K}(T_2) + \mathbf{x}_2)$ have the same rightmost tile containing the origin, $t_{i_n} - \lambda^n p_n^+(e_1) = t_{i'_n} - \lambda^n p_n^+(e_2)$ for $n \geq -K$. Hence $i_n = i'_n$ and $p_n^+(e_1) = p_n^+(e_2)$ for every $n \geq -K$. It follows that $j_n = j'_n$ for $n \geq -K$. $\square$

By Proposition 2.6, since σ is primitive, every finite word occuring in any σ-periodic point occurs infinitely often and with bounded gaps. It is clear that this extends to ω: every ω-periodic tiling has the property that if P is any patch in the tiling, then P occurs infinitely often and with bounded gaps. Therefore, it is easy to see that for any $\mathbf{y} \in A^e_{\leq 0}$, the tiling $\mathcal{T}_0 - \mathbf{y}$ has a sequence in $\mathcal{T}_0 - A^e_{\geq 0}$ converging to it. That is, $\mathcal{T}_0 - A^e_{\geq 0}$ is dense in $\mathcal{T}_0 - A^e$, and hence $\overline{\mathcal{T}_0 - A^e_{\geq 0}} = \overline{\mathcal{T}_0 - A^e}$. Since $\overline{\mathcal{T}_0 - A^e}$ is complete, it is the completion of $\mathcal{T}_0 - A^e_{\geq 0}$. Now, $\mathcal{T}_0 = T(e_0)$ is a periodic point for ω, so $\overline{\mathcal{T}_0 - A^e} = \Omega$. Therefore $\bar{q}$ extends to Ω as follows: for any $\mathcal{S} \in \Omega$, there exists a Cauchy sequence $(\mathcal{T}_n)$ in $\mathcal{T}_0 - A^e_{\geq 0}$ converging to $\mathcal{S}$. By the continuity of $\bar{q}$ on $\mathcal{T}_0 - A^e_{\geq 0}$, $\lim_{n\to\infty} \bar{q}(\mathcal{T}_n)$ exists. And so we define

$$\bar{q}(\mathcal{S}) = \lim_{n\to\infty} \bar{q}(\mathcal{T}_n), \qquad \mathcal{T}_0 - A^e_{\geq 0} \ni (\mathcal{T}_n) \to \mathcal{S}.$$

It is clear that the extension $\bar{q} : \Omega \to \mathbb{T}^d$ is continuous, and that $p = \bar{q}T$. We have from Section 2.3.1 that $q(A^e)$ is dense in $\mathbb{T}^d$. Since Ω is compact, so is $\bar{q}(\Omega)$, and therefore $\bar{q}(\Omega) = \mathbb{T}^d$. Thus we have shown the following.

Proposition 3.28 *$\bar{q} : \Omega \to \mathbb{T}^d$ is continuous, onto, and $p = \bar{q}T$.*

It remains to quantify the injectivity of $\bar{q}$ and check that $A\bar{q} = \bar{q}\omega$.

Proposition 3.29 *$\bar{q}$ is injective on the local unstable sets of Ω.*

Proof Clearly $\bar{q}$ is injective on $\mathcal{T}_0 - A^e$. Recall that for a tiling $T \in \Omega$, the local unstable sets for T consist of small translations of T. Notice that if $\mathbf{x} \in A^e$ such that $\|\mathbf{x}\| <$ (shortest tile length), then $d(T, T+\mathbf{x}) = \|\mathbf{x}\|$. Also, for $\mathbf{y}_1, \mathbf{y}_2 \in A^e$, if $d(\mathbf{y}_1, \mathbf{y}_2) \leq 1/4$ then $d(q(\mathbf{y}_1), q(\mathbf{y}_2)) = d(\mathbf{y}_1, \mathbf{y}_2)$. Let $\mathbf{y}_1, \mathbf{y}_2 \in A^e$ such that $d(\mathbf{y}_1, \mathbf{y}_2) < \min(1/4$, (shortest tile length)$/2)$. We want to show that $\bar{q}(T+\mathbf{y}_1) \neq \bar{q}(T+\mathbf{y}_2)$.

Let $0 < \epsilon < d(\mathbf{y}_1, \mathbf{y}_2)/2$. By the continuity of $\bar{q}$, there exists $\delta > 0$ such that for any tilings T_1, T_2 satisfying $d(T_1, T_2) < \delta$, we have $d(\bar{q}(T_1), \bar{q}(T_2)) < \epsilon$. By the density of $\mathcal{T}_0 - A^e$ in Ω, there exists $\mathbf{x} \in A^e$ such that $d(T+\mathbf{y}_i, \mathcal{T}_0 - \mathbf{x} + \mathbf{y}_i) < \delta$ for $i = 1, 2$. Hence $d(\bar{q}(T+\mathbf{y}_i), \bar{q}(\mathcal{T}_0 - \mathbf{x} + \mathbf{y}_i)) < \epsilon$. That is, $d(\bar{q}(T+\mathbf{y}_i), q(\mathbf{x}-\mathbf{y}_i)) < d(\mathbf{y}_1, \mathbf{y}_2)/2 = d(q(\mathbf{x}-\mathbf{y}_1), q(\mathbf{x}-\mathbf{y}_2))/2$ for $i = 1, 2$. As a result, $\bar{q}(T+\mathbf{y}_1) \neq \bar{q}(T+\mathbf{y}_2)$. □

Since T is surjective and $p = \bar{q}T$, $TS = \omega T$, and $pS = Ap$, it follows from $A\bar{q}T = \bar{q}\omega T$ that $A\bar{q} = \bar{q}\omega$. Hence we have proven that $\bar{q} : (\Omega, \omega) \to (\mathbb{T}^d, A)$ is a u-resolving factor map.

3.3 The $(d-1)$-dimensional Analogy

It is also possible to construct maps analogous to $T : \Sigma \to \Omega$ and $\bar{q} : \Omega \to \mathbb{T}^d$ by replacing (Ω, ω) with a suitable substitution tiling space on A^c. We outline the main ideas here, but do not give any proofs.

Since for every $i \in \mathcal{A}$,

$$\mathcal{R}_i = \bigcup_{(k,j) | W_j^{(k)} = i} \left(A\mathcal{R}_k + \pi^c(\mathbf{x}_j^{(k)}) \right),$$

we can define the following tiling substitution on the set of prototiles $\{\mathcal{R}_i \mid i \in \mathcal{A}\}$:

$$\hat{\omega}(\mathcal{R}_i) = \{\mathcal{R}_k + A^{-1}\pi^c(\mathbf{x}_j^{(k)}) \mid W_j^{(k)} = i\}.$$

By [18], distinct tiles in $\hat{\omega}(\mathcal{R}_i)$ have non-overlapping interiors and each of the prototiles $\mathcal{R}_i$ is the closure of its (non-empty) interior. Notice that the "incidence matrix" for $\hat{\omega}$ is A^T. That is, since A_{ij} is the number of times i occurs in $\sigma(j)$, A_{ij} is the number of translates of $\mathcal{R}_j$ in $\hat{\omega}(\mathcal{R}_i)$. Since σ is primitive, it follows that so is $\hat{\omega}$.

As in Section 2.2, let $\hat{\Omega}$ be the collection of tilings T of A^c satisfying the property that for any patch $U \subseteq T$, $U \subseteq \hat{\omega}^m(P_i + \mathbf{x})$ for some $m, i, \mathbf{x}$. By [1], $\hat{\Omega}$ contains an $\hat{\omega}$-periodic tiling. Since the $\mathcal{R}_i$ have non-empty interiors, it is clear that $\hat{\Omega}$ has finite local complexity. Hence, if $\mathcal{U}$ is an $\hat{\omega}$-periodic tiling, it follows from the primitivity of $\hat{\omega}$ that $\hat{\Omega} = \overline{\{\mathcal{U} - \mathbf{x} \mid \mathbf{x} \in A^c\}}$. It was easy to see that $\omega : \Omega \to \Omega$ was injective, but this is not the case for $\hat{\omega}$, and hence we must assume injectivity.

A mapping analogous to $T : \Sigma \to \Omega$ can be defined as follows. For $k \geq 0$, define $R_k : \Sigma \to \{\mathcal{R}_1, \cdots, \mathcal{R}_d\}^*$ by

$$R_k(x) = \hat{\omega}^k(\mathcal{R}_{i_k}) - A^{-1}p_k^-(x).$$

It can be shown that for every $n \geq 0$ and $x \in \Sigma$, $R_n(x) \subseteq R_{n+1}(x)$. Furthermore, if $\hat{\omega}$ satisfies the forcing the border condition, then for every $x \in \Sigma$ there exists a unique tiling $\mathcal{U}_x \in \hat{\Omega}$ containing $\bigcup_{n \geqslant 0} R_n(x)$. Since the Rauzy fractals for many substitutions are not homeomorphic to closed balls, checking that $\hat{\omega}$ satisfies the forcing the border condition can be a much more subtle question than checking that ω does. However, if we assume that $\hat{\omega}$ satisfies the forcing the border condition, then we can define a map $R : \Sigma \to \hat{\Omega}$ by $x \mapsto \mathcal{U}_x$, where $\mathcal{U}_x$ is the unique tiling in $\hat{\Omega}$ containing $\bigcup_{n \geqslant 0} R_n(x)$. From [1], we have that R is continuous, onto, right closing and $RS = \hat{\omega}R$. In terms of Smale spaces, this means that $R : (\Sigma, S) \to (\hat{\Omega}, \hat{\omega})$ is u-resolving.

We proceed as with the one-dimensional tiling space to define a map from $\hat{\Omega}$ to

$\mathbb{T}^d$. Recall that $e_0 = (i_n, j_n)_{n\in\mathbb{Z}}$ is the point with $(i_{nK}, j_{nK}) = (a, 1)$ and $j_n = 1$ for all $n \in \mathbb{Z}$. Define

$$F = \{f \in \Sigma \mid \exists\, N \text{ such that } i_{(N+n)K} = a\; \forall\, n \geq 0,\; j_m = 1\; \forall\, m \geq NK\}$$

and let $\mathcal{U}_0 = R(e_0)$. Since $p_1^-(U(1,1,i)) = \mathcal{R}_i$, it is easy to see that $p_n^-(U(n,n,i)) = A^{-(n-1)}\mathcal{R}_i$ for every $n \in \mathbb{Z}$. Hence $\bigcup_{n\in\mathbb{Z}} A^{nK}(\mathcal{R}_a)$ will replace the role that $A^e_{\geq 0}$ played in the preceding section. That is, $R(F) = \mathcal{U}_0 - \bigcup_{n\in\mathbb{Z}} A^{nK}(\mathcal{R}_a)$. So we define a map $\bar{r} : \mathcal{U}_0 - \bigcup_{n\in\mathbb{Z}} A^{nK}\mathcal{R}_a \to \mathbb{T}^d$ by

$$\bar{r}(\mathcal{U}_0 - \mathbf{x}) = q(-A\mathbf{x}).$$

This map is well-defined since no $\hat{\omega}$-periodic tiling is translation-periodic [18], and $p|_F = \bar{r}R|_F$.

As we did for $\bar{q} : \mathcal{T}_0 - A^e_{\geq 0} \to \mathbb{T}^d$, we can show that if for every $\mathbf{x} \in \mathcal{R}_a$ and $\epsilon > 0$ there exists $\mathbf{x}' \in Int(\mathcal{R}_a)$ with $d(\mathbf{x}, \mathbf{x}') < \epsilon$, then $\bar{r} : \mathcal{U}_0 - \bigcup_{n\in\mathbb{Z}} A^{nK}(\mathcal{R}_a) \to \mathbb{T}^d$ is uniformly continuous. Since $\hat{\Omega}$ is the completion of $\mathcal{U}_0 - \bigcup_{n\in\mathbb{Z}} A^{nK}(\mathcal{R}_a)$, this means that $\bar{r}$ extends uniquely to $\hat{\Omega}$. Moreover, $\bar{r} : \hat{\Omega} \to \mathbb{T}^d$ is continuous, onto, and $p = \bar{q}T$.

Now, in terms of Smale spaces, the local stable set of a tiling $\mathcal{T} \in \hat{\Omega}$ contains small translations of $\mathcal{T}$ since on $\hat{\Omega}$, the transformation A is contracting on A^c. So it is analogous to Proposition 3.29 to show that for any $\mathcal{T} \in \hat{\Omega}$, $\bar{r}$ is injective on the local stable sets of $\mathcal{T}$. Furthermore, $A\bar{r} = \bar{r}\hat{\omega}$, and so $\bar{r} : (\hat{\Omega}, \hat{\omega}) \to (\mathbb{T}^d, A)$ is an s-resolving factor map.

That is, assuming that $\hat{\omega}$ is injective and that for every $\mathbf{x} \in \mathcal{R}_a$ and $\epsilon > 0$ there exists $\mathbf{x}' \in Int(\mathcal{R}_a)$ with $d(\mathbf{x}, \mathbf{x}') < \epsilon$, we have the following commutative diagram:

$$\begin{array}{ccc} (\Sigma, S) & \xrightarrow{R} & (\hat{\Omega}, \hat{\omega}) \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \bar{r}} & \\ (\mathbb{T}^d, A) & & \end{array}$$

where R is a u-resolving factor map, and $\bar{r}$ is an s-resolving factor map.

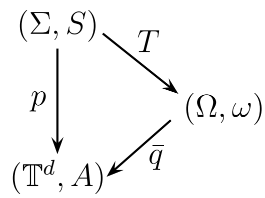
Chapter 4

Conclusion

The incidence matrix A of a unimodular Pisot substitution σ on d letters gives rise to a stable/unstable decomposition of d-dimensional Euclidean space, $\mathbb{R}^d = A^e \oplus A^c$. The one-dimensional unstable subspace A^e is inherently a simple object, and admits a substitution tiling space (Ω, ω) that is very similar to the symbolic dynamical system associated to the substitution. The complementary $(d-1)$-dimensional stable subspace A^c contains the Rauzy fractal, a useful and well-studied compact set.

By considering the set $\mathcal{D}$ of infinite paths in the graph G_σ associated to σ, Canterini and Siegel [7] prove that every dynamical system generated by σ admits as a topological factor a minimal translation on $\mathbb{T}^{d-1}$. We consider biinfinite paths in G_σ, and prove that the subshift of finite type (Σ, S) containing these paths is semi-conjugate to $(\mathbb{T}^d, A)$ via a map $p : \Sigma \to \mathbb{T}^d$. One could say that p is an extension of Canterini and Siegel's map from $\mathcal{D}$ to $\mathbb{T}^{d-1}$.

In terms of Smale spaces, the map $p : \Sigma \to \mathbb{T}^d$ turns out to have a number of very desirable properties if we assume that σ satisfies the coincidence condition and that ω satisfies the forcing the border condition. Besides being a factor map, p is the composition of s-resolving factor map $T : (\Sigma, S) \to (\Omega, \omega)$ and u-resolving factor map $\bar{q} : (\Omega, \omega) \to (\mathbb{T}^d, A)$. That is, the following diagram commutes:



Many open questions are left to be explored. For example, since every substitution is equivalent to a substitution that satisfies the forcing the border condition, it is conceivable that we should be able to eliminate the forcing the border condition altogether. It seems plausible that σ satisfying the coincidence condition is somehow related to ω satisfying the forcing the border condition. If this turned out to be the case, it might provide a new approach to proving the Coincidence Conjecture.

Many researchers, especially those working on the applications of tiling dynamical systems to quasicrystals, prefer to construct tiling spaces using the *cut and project method*. Although our method of constructing tiling spaces is much more useful for our purposes, it might be of interest to explore our results in terms of cut and project tilings.

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Appendix

Appendix 1: Parry Measure on Σ

Recall that for a given substitution σ on $\mathcal{A}$, the directed graph $G_\sigma = (V, E)$ associated to σ has vertex set

$$V = \mathcal{A}$$

and edge set

$$E = \{(i, j) \mid i \in \mathcal{A},\, 1 \leq j \leq l(i)\},$$

where (i, j) is an edge from vertex i to vertex $W_j^{(i)}$.

Notice that the incidence matrix A of σ has the property that A_{ij} is the number of edges of G_σ from j to i.

Define A^1 to be the *edge matrix* of σ indexed by the edges of G_σ and given by

$$A^1_{ef} = \begin{cases} 1 & \text{if } t(e) = i(f) \\ 0 & \text{otherwise} \end{cases},$$

where $t(e)$ denotes the terminal vertex of the edge e, and $i(f)$ denotes the initial vertex of the edge f.

Further, let I be the matrix whose rows are indexed by the vertices of G_σ and whose columns are indexed by the edges of G_σ, given by

$$I_{ve} = \begin{cases} 1 & \text{if } v = i(e) \\ 0 & \text{otherwise} \end{cases}.$$

And let T be the matrix whose rows are indexed by the edges in G_σ and whose columns are indexed by the vertices of G_σ, given by

$$T_{ev} = \begin{cases} 1 & \text{if } t(e) = v \\ 0 & \text{otherwise} \end{cases}.$$

Then $TI = A^1$ since

$$\begin{aligned} (TI)_{ef} &= \sum_{v \in V} T_{ev} I_{vf} \\ &= \begin{cases} 1 & \text{if } t(e) = i(f) \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Similarly, $IT = A^T$ since

$$\begin{aligned} (IT)_{uv} &= \sum_{e \in E} I_{ue} T_{ev} \\ &= \sum_{e \in E \mid i(e)=u, t(e)=v} 1 \\ &= A_{vu}. \end{aligned}$$

Since we had $A^T\mathbf{u} = \lambda\mathbf{u}$,

$$\begin{aligned} A^1 T\mathbf{u} &= TIT\mathbf{u} \\ &= TA^T\mathbf{u} \\ &= T\lambda\mathbf{u} \\ &= \lambda T\mathbf{u}. \end{aligned}$$

That is, λ is an eigenvalue for A^1 and $T\mathbf{u}$ is an associated right eigenvector. Similarly, $I^T\mathbf{v}$ is a left eigenvector of A^1 associated to λ. Recall that we denoted the other $d-1$ eigenvalues by $\lambda_2, \cdots, \lambda_d$ of A, and their associated left eigenvectors by $\mathbf{u}_{\lambda_2}, \cdots, \mathbf{u}_{\lambda_d}$ and right eigenvectors by $\mathbf{v}_{\lambda_2}, \cdots, \mathbf{v}_{\lambda_d}$. It is clear that $\lambda_2, \cdots, \lambda_d$ are also eigenvalues of A^1 and $T\mathbf{u}_{\lambda_2}, \cdots, T\mathbf{u}_{\lambda_d}$ are respective right eigenvectors and $I^T\mathbf{v}_{\lambda_2}, \cdots, I^T\mathbf{v}_{\lambda_d}$ are respective left eigenvectors.

Similarly, if α is an eigenvalue for A^1 and $\mathbf{x}$ is an associated right eigenvector, then α is an eigenvalue for A^T (and therefore A) and $I\mathbf{x}$ is the associated right eigenvector. Since $\lambda, \lambda_2, \cdots, \lambda_d$ are the only eigenvalues for A^T, the additional eigenvalues of A^1 must be 0. Hence λ is the Perron-Frobenius eigenvalue of A^1.

Notice that $\langle I^T\mathbf{v}, T\mathbf{u}\rangle = \mathbf{v}^T IT\mathbf{u} = \mathbf{v}^T A^T\mathbf{u} = \mathbf{v}^T\lambda\mathbf{u} = \lambda$. Hence

$$\langle \lambda^{-1} I^T\mathbf{v}, T\mathbf{u}\rangle = 1,$$

and clearly $\lambda^{-1}I^T\mathbf{v}$ and $T\mathbf{u}$ are positive.

Let s be the number of edges of G, or equivalently, the size of A^1. Parry measure is defined to be the Markov measure associated to the probability vector

$\mathbf{p} = (\lambda^{-1}(I^T\mathbf{v})(i)(T\mathbf{u})(i))_{1\le i\le s}$ and the stochastic matrix

$$P_{ij} = A^1_{ij}(T\mathbf{u})(j)/\lambda(T\mathbf{u})(i).$$

That is, Parry measure is given by the following measure on cylinders of Σ:

$$\mu(U(k, l+1, e_k \cdots e_l)) = \mathbf{p}(e_k)P_{e_k e_{k+1}} P_{e_{k+1}e_{k+2}} \cdots P_{e_{l-1}e_l}.$$

Since $U(k, l, e_k \cdots e_l)$ is a cylinder of Σ, it follows that $t(e_k) = i(e_{k+1})$, $t(e_{k+1}) = i(e_{k+2})$, $\cdots$, $t(e_{l-1}) = i(e_l)$. That is, $A^1_{e_k e_{k+1}} = A^1_{e_{k+1}e_{k+2}} = \cdots = A^1_{e_{l-1}e_l} = 1$. As a result, we easily simplify the above expression to get

$$\mu(U(k, l+1, e_k \cdots e_l)) = \lambda^{k-l-1}(I^T\mathbf{v})(e_k)(T\mathbf{u})(e_l).$$

However, $(I^T\mathbf{v})(e) = \mathbf{v}(i(e))$ and $(T\mathbf{u})(e) = \mathbf{u}(t(e))$ for every $e \in E$. Hence

$$\mu(U(k, l+1, (i_k, j_k) \cdots (i_l, j_l))) = \lambda^{k-(l+1)}\mathbf{v}(i_k)\mathbf{u}(W^{(i_l)}_{j_l}).$$

Let $\mathcal{B}$ denote the Borel subsets of Σ. By [19], the above definition of μ on cylinders of Σ extends uniquely to a probability measure on $(\Sigma, \mathcal{B})$.

Appendix 2: Proof that p_1^- is the map in [7]

Claim *The map p_1^- defined in Chapter 3 is the same map as $p_{\mathcal{H}}\varphi$ found in [7].*

Proof The SFT $\mathcal{D}$ from [7] differs from $\Sigma_1^- = \{(i_n, j_n)_{n\le 0} | (i_n, j_n)_{n\in\mathbb{Z}} \in \Sigma\}$ only in notation. In the graph $G_{\mathcal{D}}$, called the prefix-suffix automaton of σ, the set of vertices is $\mathcal{A}$ and there is an edge from $W_j^{(i)}$ to i labeled by

$$(p, a, s) = (W_1^{(i)} \cdots W_{j-1}^{(i)}, W_j^{(i)}, W_{j+1}^{(i)} \cdots W_{l(i)}^{(i)}),$$

for $i \in \mathcal{A}$ and $1 \leq j \leq l(i)$. So an edge (i, j) from i to $W_j^{(i)}$ in the graph G_Σ corresponds to the edge $(W_1^{(i)} \cdots W_{j-1}^{(i)}, W_j^{(i)}, W_{j+1}^{(i)} \cdots W_{l(i)}^{(i)})$ from $W_j^{(i)}$ to i in $G_{\mathcal{D}}$. Since Σ_1^- is indexed by $n \leq 0$ and $\mathcal{D}$ is indexed by $n \geq 0$, the seemingly opposite directions of the edges are not opposite after all. It is easy to see that for the point $(i_n, j_n)_{n\leq 0} \in \Sigma_1^-$ also described by $(p_n, a_n, s_n)_{n\geq 0} \in \mathcal{D}$, we have

$$\mathbf{x}_{j-n}^{(i-n)} = f(p_n) \; \forall \, n \geq 0.$$

The $d = r + 2s$ eigenvalues of the incidence matrix A are labelled as $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$, and $\alpha_{r+1}, \overline{\alpha_{r+1}}, \ldots, \alpha_{r+s}, \overline{\alpha_{r+s}} \in \mathbb{C}$, where α_1 is the Perron-Frobenius eigenvalue. A set of right eigenvectors $\mathbf{u}_1, \ldots \mathbf{u}_d$, where $A\mathbf{u}_k = \alpha_k \mathbf{u}_k$, and a set of left eigenvectors $\mathbf{v}_1, \ldots \mathbf{v}_d$, where $\mathbf{v}_k^T A = \alpha_k \mathbf{v}_k^T$, are constructed to satisfy certain properties. A map $\delta : \mathcal{A}^* \to \mathbb{R}^{r-1} \times \mathbb{C}^s$ is defined by $w \mapsto V \cdot f(w)$, where

$$V = \begin{pmatrix} \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_{r+s}^T \end{pmatrix},$$

and f is the abelianization map, as above.

Further, a linear bijection $p_{\mathcal{H}} : \mathbb{R}^{r-1} \times \mathbb{C}^s \to A^c$ is given and shown to satisfy $\pi^c f = p_{\mathcal{H}} \delta = p_{\mathcal{H}} V f$. That is,

$$\pi^c = p_{\mathcal{H}} V$$

on $\mathbb{Z}_{\geq 0}^d$. In addition, the definition of $p_{\mathcal{H}}$ easily leads to the equality $A p_{\mathcal{H}} = p_{\mathcal{H}} A_{diag}$, where

$$A_{diag} = \begin{pmatrix} \alpha_2 & & 0 \\ & \ddots & \\ 0 & & \alpha_{r+s} \end{pmatrix}.$$

Recall that (X_σ, S) is the symbolic dynamical system associated to σ. A map $\Gamma : X_\sigma \to \mathcal{D}$ is defined and shown to be continuous, onto and one-to-one except on the orbits of periodic points of σ. Although the following map is defined on X_σ, no information is lost if we define it on $\mathcal{D}$.

The map $\varphi : \mathcal{D} \to \mathbb{R}^{r-1} \times \mathbb{C}^s$ is defined by

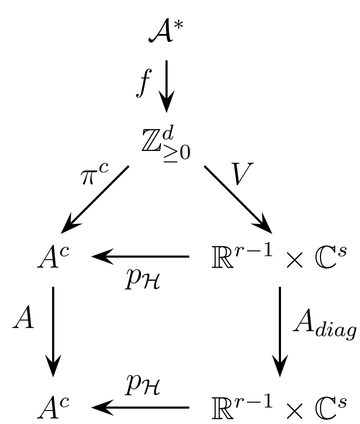
$$\varphi((p_n, a_n, s_n)_{n\geq 0}) = \sum_{n\geq 0} A^n_{diag}\delta(p_n).$$

Since $\mathbf{v}_k^T A = \alpha_k \mathbf{v}_k^T$, we have $A_{diag}\delta(w) = A_{diag}Vf(p_n) = VAf(w)$. It follows that

$$p_{\mathcal{H}} A_{diag}\delta(w) = \pi^c Af(w) = A\pi^c f(w).$$

Equivalently, we could have deduced this from $Ap_{\mathcal{H}} = p_{\mathcal{H}} A_{diag}$.

This gives the following commutative diagram, and the result follows.



□

Appendix 3: Proof of Lemma 3.8

Lemma *For any* $N \in \mathbb{N}$,

$$p_1^-(\Sigma_\sigma) = p_1^-(\Sigma_{\sigma^N}).$$

Proof This proof uses many of the elements used in the proof of Lemma 3.7, and could be seen almost as a corollary to that lemma.

First, let $x = (i_n, j_n)_{n\in\mathbb{Z}} \in \Sigma_\sigma$ and construct $X \in \Sigma_{\sigma^N}$ as follows. Let $I_0 = i_{-(N-1)}$ and $J_0 = 1 + \sum_{m=0}^{N-1} \sum_{j<j_{-m}} |\sigma^m(W_j^{(i_{-m})})|$. Then, as in the proof of Lemma 3.7,

$$U_1^{(I_0)} \cdots U_{J_0}^{(I_0)} = \sigma^{N-1}(W_1^{(i_{-(N-1)})} \cdots W_{j_{-(N-1)}-1}^{(i_{-(N-1)})})\sigma^{N-2}(W_1^{(i_{-(N-2)})} \cdots W_{j_{-(N-2)}-1}^{(i_{-(N-2)})}) \cdots$$
$$\cdots \sigma \left(W_1^{(i_{-1})} \cdots W_{j_{-1}-1}^{(i_{-1})}\right) W_1^{i_0} \cdots W_{j_0}^{i_0}.$$

Similary, let $I_{-1} = i_{-(2N-1)}$ and $J_{-1} = 1 + \sum_{m=0}^{N-1} \sum_{j<j_{-(m+N)}} |\sigma^m(W_j^{(i_{-(m+N)})})|$. Then, as before,

$$U_1^{(I_{-1})} \cdots U_{J_{-1}}^{(I_{-1})} = \sigma^{N-1}(W_1^{(i_{-(2N-1)})} \cdots W_{j_{-(2N-1)}-1}^{(i_{-(2N-1)})}) \cdots W_1^{i_{-N}} \cdots W_{j_{-N}}^{i_{-N}}.$$

Notice that $U_{J_{-1}}^{(I_{-1})} = W_{j_{-N}}^{i_{-N}} = i_{-(N-1)} = I_0$. Continuing in this manner, we define $(I_n, J_n)_{n\leq 0}$ where $U_{J_n}^{(I_n)} = I_{n+1}$ for $n < 0$. By Lemma 3.7,

$$\pi^c(\mathbf{y}_{J_0}^{(I_0)}) = \pi^c(\mathbf{x}_{J_0}^{(\sigma^{N-1}(i_{-(N-1)}))}) = \sum_{m=-(N-1)}^{0} A^{-m}\pi^c\left(\mathbf{x}_{j_m}^{(i_m)}\right),$$

$$A^N\pi^c(\mathbf{y}_{J_{-1}}^{(I_{-1})}) = A^N\pi^c(\mathbf{x}_{J_{-1}}^{(\sigma^{N-1}(i_{-(2N-1)}))}) = A^N \sum_{m=-(N-1)}^{0} A^{-m}\pi^c\left(\mathbf{x}_{j_{m-N}}^{(i_{m-N})}\right),$$

$$\vdots$$

Choosing an arbitrary path $(I_n, J_n)_{n\geq 1}$ in G_{σ^N} satisfying $I_1 = U_{J_0}^{I_0)}$, we define $X = (I_n, J_n)_{n\in\mathbb{Z}} \in \Sigma_{\sigma^N}$, and $p_1^-(X) = p_1^-(x)$. Hence $p_1^-(\Sigma_\sigma) \subseteq p_1^-(\Sigma_{\sigma^N})$.

Now, let $X = (I_n, J_n)_{n\in\mathbb{Z}} \in \Sigma_{\sigma^N}$ and we'll construct $x \in \Sigma_\sigma$ as follows. Let $i_{-(N-1)} = I_0$ and let $1 \leq j_{-(N-1)} \leq l(i_{-(N-1)})$ be maximal such that

$$|\sigma^{N-1}(W_1^{(i_{-(N-1)})} \cdots W_{j_{-(N-1)}-1}^{(i_{-(N-1)})})| < J_0.$$

Then let $i_{-(N-2)} = W_{j_{-(N-1)}}^{(i_{-(N-1)})}$ and let $1 \leq j_{-(N-2)} \leq l(i_{-(N-2)})$ be maximal such that $|\sigma^{N-1}(W_1^{(i_{-(N-1)})} \cdots W_{j_{-(N-1)}-1}^{(i_{-(N-1)})})\sigma^{N-2}(W_1^{(i_{-(N-2)})} \cdots W_{j_{-(N-2)}-1}^{(i_{-(N-2)})})| < J_0$. We continue in this manner until we arrive at $i_0 = W_{j_{-1}}^{(i_{-1})}$. From the proof of Lemma 3.7, we see that there exists $1 \leq j_0 \leq l(i_0)$ such that

$$|\sigma^{N-1}(W_1^{(i_{-(N-1)})} \cdots W_{j_{-(N-1)}-1}^{(i_{-(N-1)})})\sigma^{N-2}(W_1^{(i_{-(N-2)})} \cdots W_{j_{-(N-2)}-1}^{(i_{-(N-2)})}) \cdots W_1^{i_0} \cdots W_{j_0}^{i_0}| = J_0.$$

Hence we have obtained $(i_n, j_n)_{n=-(N-1)}^0$ satisfying

$$\sigma^{N-1}(W_1^{(i_{-(N-1)})} \cdots W_{j_{-(N-1)}-1}^{(i_{-(N-1)})}) \cdots W_1^{i_0} \cdots W_{j_0}^{i_0} = U_1^{(I_0)} \cdots U_{J_0}^{(I_0)}.$$

Next, we repeat this process starting at $i_{-(2N-1)} = I_{-1}$ to obtain $(i_n, j_n)_{n=-(2N-1)}^{-N}$. Then $W_{j_{-N}}^{(i_{-N})} = U_{J_{-1}}^{(I_{-1})} = I_0 = i_{-(N-1)}$. Continuing in this manner, we obtain $(i_n, j_n)_{n\leq 0}$ satisfying $W_{j_n}^{(i_n)} = i_{n+1}$ for $n < 0$. Define $x = (i_n, j_n)_{n\in\mathbb{Z}}$, where $(i_n, j_n)_{n\geq 1}$ is an arbitrary path in G_σ satisfying $i_1 = W_{j_0}^{i_0}$. Clearly the same equations as above hold, and we get $p_1^-(x) = p_1^-(X)$. □