GENERAL RELATIVISTIC INCOMPRESSIBILITY

BY

ROBERT SINCLAIR SARRACINO

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ABSTRACT

In this thesis we advance a new definition of incompressibility in general relativity, and develop and analyse the equation of state for a uniform proper-density static, perfect fluid.

Schwarzschild published the first exact interior solution to Einstein's equations: the field inside a static, spherically symmetric perfect fluid with equation of state $\rho = \rho_o$ = constant, where ρ is the energy-density component of the energy-momentum tensor. He interpreted the ρ-constant condition to be the equation of state for a 'uniform density', hence 'incompressible', perfect fluid, an interpretation that has been generally accepted. As a consequence, the ρ-constant equation of state has been taken to be the limiting equation of state for superdense nonrotating stellar bodies. While there is no general agreement over what constitutes a 'limiting' equation of state for a physically realistic star, to date, to the best of our knowledge, there has been no argument that the Schwarzschild fluid represents the absolute limit for any static system.

In this thesis we argue that the Schwarzschild fluid is not an incompressible fluid, and advance a new definition of incompressibility. An incompressible fluid would be one for which the limit of infinite central pressure, and the static limit $2m/r_o = 1$ at the surface, where m is the mass and r_o the radius of the body, would coincide. In addition we develop the equation of state for a uniform proper-density perfect fluid, and, using the new definition of incompressibility, compare this with the Schwarzschild fluid, and with a third proposed by Eddington. We find that the constant proper-density equation of state more closely approximates incompressibility than either the Schwarzschild or Eddington equations of state, and briefly explore a few of its properties, such as a greater maximum gravitational red-shift, $z \equiv \Delta\lambda/\lambda = 2.48$, compared with $z = 2.00$ for the Schwarzschild case, and note that the constant proper-density equation of state would lead to a larger upper limit to the mass of a neutron star.

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CHAPTER ONE

INTRODUCTION

Einstein's general theory of relativity was published in 1915, and in 1916, Schwarzschild published the spherically symmetric time independent 'exterior solution' to Einstein's vacuum field equations.

$$R_{\mu\nu} = 0 \tag{1.1}$$

(Misner, Thorne and Wheeler, 1973). A short time later Schwarzschild (1916) published the exact interior solution to Einstein's equations for a static, spherically symmetric perfect fluid with equation of state $\rho = \rho_o = \text{constant}$, where ρ is the energy-density component of the energy-momentum tensor.

Schwarzschild interpreted the $\rho = \rho_o = \text{constant}$ condition to be the equation of state for a uniform density, incompressible perfect fluid, an interpretation that has been almost universally accepted. In this thesis Schwarzschild's interpretation of this equation of state is re-examined, and a new definition of incompressibility advanced. A new equation of state, representing constancy of proper energy density, is then examined and compared with the Schwarzschild equation of state, and with a third equation of state, representing uniform particle density, proposed by Eddington.

In Chapter Two a brief development of Einstein's special and general theories of relativity is presented.

In Chapter Three the Schwarzschild exterior and interior solutions are derived, and the Schwarzschild and Eddington equations of state examined. The Schwarzschild exterior solution is the line-element exterior to a spherically symmetric mass distribution,

$$ds^2 = (1 - \frac{2m}{r})(dx^o)^2 - \frac{dr^2}{(1 - \frac{2m}{r})} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad , \quad (1.2)$$

where m is the total mass of the object, (G = c = 1) and $dx^o = cdt$. The line element becomes singular at

$$\frac{2m}{r} = 1 \quad . \quad (1.3)$$

At this value g_{rr} becomes infinite, and g_{oo} vanishes. Thus, the exterior metric imposes a limiting condition on the size, or mass, of a static body. For a static body, the $r_o = 2m$ radius must fall inside the body, where the exterior metric is inapplicable. For a static body of given radius, condition (1.3) imposes an upper limit to the mass, and conversely, for a static body of given mass, this condition imposes a lower limit on its radius.

The interior equations are then derived. Schwarzschild found an exact solution to the interior equations by choosing the simple equation of state $\rho = \rho_o$ = constant, which has the solution

$$ds^2 = \left| \frac{3}{2} \left(1 - \frac{r_o^{\;2}}{\hat{R}^2}\right)^{1/2} - \frac{1}{2} \left(1 - \frac{r^2}{\hat{R}^2}\right)^{1/2} \right|^2 (dx^o)^2$$

$$- \frac{dr^2}{1 - \frac{r^2}{\hat{R}^2}} - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \; , \qquad (1.4)$$

where $$\frac{1}{\hat{R}^2} = \frac{2m}{r_o^{\;3}} \; .$$

This line element contains two singularities, the first for the value $2m/r_o = 1$, and the second for the value $2m/r_o = 8/9$ where r_o is the radius of the body. For $2m/r_o = 8/9$ the metric component g_{oo} vanishes at the center of the body, and central pressure becomes infinite. The existence of the singularity at $2m/r_o = 8/9$ imposes a more severe restriction on the mass and radius of a static body than that which arises from the singularity at $2m/r_o = 1$.

Since Schwarzschild had set ρ, the energy-density component of the energy-momentum tensor, to be constant throughout the body, he interpreted this equation of state to be the one for a 'uniform density', and therefore 'incompressible', fluid. The function ρ, however, represents proper energy-density only within the context of special relativity, as it includes energy contributions from rest-mass and from motion, and includes binding energy of all interactions excluding gravitation. It does not include gravitational binding energy, and consequently does not represent proper energy-density within the context of general relativity.

Although several authors have noted that the $\rho = \rho_o$ = constant condition does not represent constancy of proper energy density, it has been generally accepted that the Schwarzschild equation of state does represent an 'incompressible' fluid and, as such, is the limiting equation of state for any physically realistic system. It should be noted that many workers in the field have proposed more restrictive equations of state as the limit for physically realistic systems; however, as far as we are aware, all have agreed, up to the present, that no system could be contemplated which has a less restrictive equation of state than the Schwarzschild equation of state.

For the Schwarzschild system,

$$\frac{d\rho}{dr} = 0 \quad , \tag{1.5}$$

Buchdahl (1959) has proved that for any equation of state satisfying the conditions

$$\frac{d\rho}{dr} \leq 0 \quad , \quad \left| \frac{2m}{r} < 1 \right| \quad , \tag{1.6}$$

for bodies of fixed radius and mass, the maximum value of $2m/r_o$ is less than or equal to 8/9. Since it has been generally considered that the condition (1.6) must hold for <u>any</u> physically realistic system, as a matter distribution with ρ increasing outwards would be unstable, it has been concluded that 8/9 is the maximum possible value of $2m/r_o$ for any static configuration.

Eddington advanced an alternate definition of incompressibility. An incompressible fluid should have constancy, not of energy-density, but of particle density. Eddington's equation of state

$$\rho - 3p \equiv B = \text{constant} \quad , \tag{1.7}$$

where p is the pressure, represents a system in which particle-density, the closeness of packing of matter, is a constant throughout the body, and is independent of pressure. Eddington's equation of state implies a more severe restriction than that of Schwarzschild in that the maximum value of $2m/r_o$ is less than 8/9. This is seen from Buchdahl's proof:

$$\frac{d\rho}{dr} = 3\,\frac{dp}{dr} < 0 \quad . \tag{1.8}$$

In Chapter Four, we advance a new definition of incompressibility, consistent with basic principles of relativity. It is motivated by the observation that the Schwarzschild body is not a true uniform-density body, in that proper energy-density is not constant throughout the body. The equation of state for a truly incompressible fluid would be one for which the two limits, representing metric singularity at the surface and infinite pressure at the center, would coincide. That is, an incompressible body would have an equation of state such that pressure at the center of the body becomes infinite precisely when the limit $2m/r_o = 1$ is reached. This, surely, represents the ultimate stiffness of matter which can be envisaged. Incompressibility is an idealization in the realm of classical physics, and an impossibility in special relativity, since it implies the instantaneous velocity of propagation of interactions. Accordingly, it would be expected that such an equation of state would be unattainable.

The work of Bondi (1959, 1964) shows that, indeed, for a body with non-negative pressure and density, the limit $2m/r_o = 1$ cannot be reached. Thus, by the new definition, an incompressible fluid cannot exist. Moreover, the definition is very useful as a means of comparing the relative compressibility of bodies represented by different equations of state: the closer the maximum possible surface value of $2m/r_o$ to 1, for a given equation of state, the less compressible that body would be.

In Chapter Five an equation of state representing constancy of proper energy-density is developed, and this new, truly 'uniform density' body is compared with the Schwarzschild body, against the new definition of incompressibility advanced in Chapter Four.

The equations obtained from the new equation of state were integrated numerically. As a check of the integration methods used, the Schwarzschild equations were first integrated. The maximum value obtained for $2m/r_o$ at the surface agreed with the maximum value of $2m/r_o$ obtained from the exact analytic solution to five significant figures.

Although it was known from Buchdahl's proof, and from Bondi's work, that the Eddington fluid was more compressible than the Schwarzschild fluid, the Eddington equation of state was integrated as well, as a further check of the accuracy of the program, and to determine the extent to which the Eddington fluid is less compressible than the

Schwarzschild fluid. Unfortunately, severe 'bundling' of the Eddington equations, a consequence of the ρ - p coupling in the Eddington equation of state, led to cross-overs which made the results unreliable. As a further check of the integration methods, various 'extreme' equations of state were programmed, in an attempt to find the maximum possible value of $2m/r_o$ for any system with non-negative pressure and density. The results agreed with those of Bondi.

By numerical integration, the maximum possible value of $2m/r_o$ for the equation of state representing constant proper energy-density was found to be .917, larger than the .889 for the Schwarzschild case. From these results it is concluded that a uniform proper-density body, which represents the limit for a stable matter distribution, is less compressible than the uniform density body traditionally considered to represent incompressibility.

The last sections of Chapter Five contain a discussion on gravitational red-shift and the maximum mass of neutron stars. The maximum gravitational red-shift from the surface of a uniform proper-density body is 2.48, as compared with 2.00 for the Schwarzschild body. At the same time, the maximum mass of a neutron star obeying the constant proper-density equation of state would be greater than that for the Schwarzschild equation of state.

The Eddington equation of state, representing constancy of particle-density, is equivalent to the classical concept of uniform density and incompressibility since classically, particle-density and mass density are proportional. The Schwarzschild fluid would correspond to a uniform-density body in the realm of special relativity, since ρ is the special relativistic energy-density. The constant proper-density fluid is the general relativistic uniform-density model, since it includes gravitational energy density as well as all other forms of energy. Accordingly, a stable body would be one in which proper density did not increase outwards from the center. The limiting condition for stability, rather than the previously considered equation (1.6), would be the condition

$$\frac{d\,\rho_{proper}}{dr} \leq 0 \quad , \qquad (1.9)$$

which is more in accord with principles of general relativity.

CHAPTER TWO

THE FOUNDATIONS OF RELATIVITY

2.1 Special Relativity

The principle of relativity was first formulated by Galileo, suspended when its classical formulation came into conflict with Maxwell's equations of electrodynamics, and later reformulated by Einstein; a reformulation which reconciled relativity with electrodynamics. (Møller, 1972; Balazs, 1972).

The classical principle of relativity stated that the laws of mechanics were the same with respect to all inertial observers. An 'inertial' frame is one in which particles that are not acted on by any forces remain unaccelerated. Any frame moving with constant velocity relative to an inertial frame is itself an inertial frame. Galileo illustrated his principle of relativity with an example:

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in

> throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though there is no doubt that when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air...
>
> (Taylor and Wheeler, 1966)

Before Maxwell's formulation of the laws of electrodynamics it might have been natural to infer, from Galilean relativity, a more general principle: that all the laws of physics, not just the laws of mechanics, are the same with respect to all inertial observers. If the speed of light is taken to be infinite, Newton's laws would be compatible with this more general principle of relativity. Maxwell's equations, however, showed light as an electromagnetic wave that propagated with finite velocity. Thus, either Maxwell's or Newton's laws had to be modified, or the principle of relativity was not valid for all physical phenomena. To explain the wave-like propagation of light, physicists hypothesized the existence of an 'ether'--a highly tenuous medium that filled all space; the medium in which light propagated as a disturbance. If Maxwell's and Newton's laws were both correct, then one could distinguish between inertial frames, because each frame would measure a different speed of light. Thus, the 'absolute' frame of reference could be discovered: it would be the frame relative to which the speed of light was the constant in Maxwell's equations.

In 1881 Michelson attempted to measure the velocity of the earth relative to the 'ether'. In his experiment he measured the relative speeds of light beams which were sent along paths of equal lengths, parallel and perpendicular to the earth's direction of motion. From the difference in time taken by the two beams, the speed of the earth relative to the 'ether' could easily be calculated. Michelson's experiment gave a null result: the beams, emitted simultaneously, returned at the same time. The null result persisted throughout the year, even though the earth's velocity continued to change.

In 1905 Albert Einstein, although apparently unaware of the Michelson experiment, (Holton, 1960), reconciled Maxwell's laws of electrodynamics with the Galilean principle of relativity. Einstein's theory was based on two principles.

(a) All inertial frames are equivalent, with respect to all the laws of physics

(b) The speed of light in vacuum is the maximum speed of signal propagation

As a consequence of these two postulates time, and the separation between two points at rest relative to each other, could no longer be considered invariant quantities.

In 1908 Minkowski found a geometric framework for special relativity, and in doing so introduced the concept of a four-dimensional space-time, paving the way for Einstein's later formulation of a geometric theory of gravitation.

Euclidean geometry was the geometric framework of classical physics. In classical physics the 'interval'

$$ds^2 = \sum_{i,k} g_{ik}\, dx^i\, dx^k \quad ,$$

the square of the separation between two points at rest relative to each other, is an invariant quantity. We will adopt the Einstein summation convention, where repeated indices are to be summed. Then,

$$ds^2 = g_{ik}\, dx^i\, dx^k \quad , \qquad (2.1.1)$$

If dx^1, dx^2, dx^3 are rectangular Cartesian coordinates,

$$g_{ik} = \delta_{ik} \equiv \begin{cases} 1 & i=k \\ 0 & i \neq k \end{cases} \quad , \qquad (2.1.2)$$

In Euclidean geometry and in classical physics the distance between two points at rest relative to each other is independent of the velocity or orientation of the observer, or of the coordinate system. In special relativity this invariance, as well as the invariance of time intervals, no longer exists. An invariant quantity does exist in special relativity, however, but within a four-dimensional, rather than three-dimensional, framework. Let $dx^o = cdt$ and dx^1, dx^2, dx^3 have the same meaning they have in Euclidean geometry. The quantity

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \quad , \qquad (2.1.3)$$

where α, β take values from zero to three, is an invariant. In a Cartesian coordinate system, $g_{\alpha\beta}$ takes the values

$$g_{oo} = 1 \quad g_{oi} = 0 \quad g_{ik} = \begin{cases} -1 & i=k \\ 0 & i\neq k \end{cases} , \qquad (2.1.4)$$

where we adopt the convention that Greek indices take values from zero to three, and Latin indices from one to three. Thus, in special relativity, the interval

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \qquad (2.1.5)$$

is an invariant quantity.

2.2 The Principle of Equivalence

After the publication of his special theory in 1905, Einstein turned his attention to the problem of formulating the laws of physics so as to be covariant with respect to all reference frames, not just inertial frames. This generalization seemed even more natural when one considered that the Newtonian reference frame was experimentally undiscoverable. In order to find an inertial frame, an experimenter would have to observe the motion of a body which had been freed from the influence of all forces. This can be done for the electrical force by choosing a neutral test particle. However, there is no way to completely eliminate the influence of a gravitational field, since the gravitational force is proportional to the inertial mass of the test particle.

The equivalence between inertial mass and passive gravitational mass, the mass with which a particle responds to a gravitational field, baffled Newton. Within the framework of classical physics this equivalence is a coincidence. Einstein, however, saw a deeper relationship. Motion in a gravitational field is mass-independent, as is motion in a non-inertial reference frame.

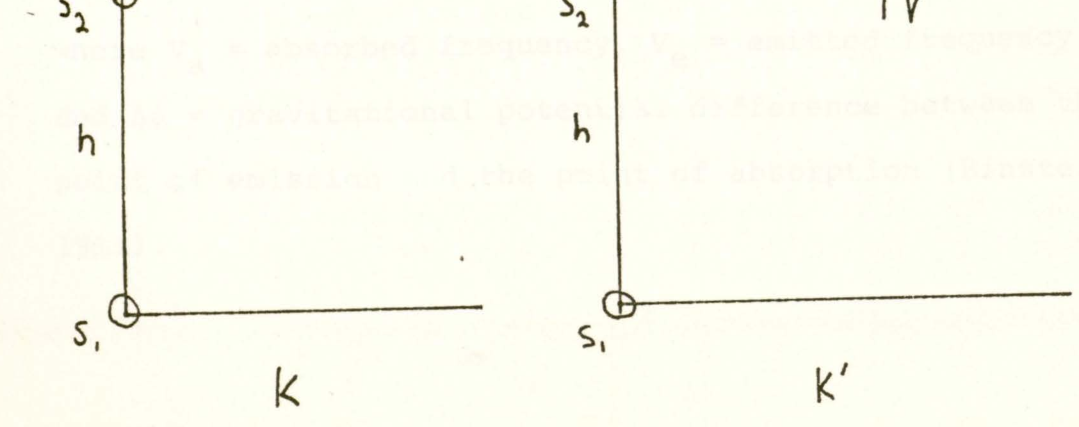
In 1907 Einstein formulated his Principle of Equivalence, the foundation stone of general relativity. The principle states that 'locally', the laws of physics in gravitational fields and non-inertial frames of reference are equivalent, not just with respect to the laws of mechanics, but with respect to all the laws of physics. Consider two laboratories, one in a uniform gravitational field, and the other far from all gravitational influences, but accelerated 'upwards' with a uniform acceleration equal to the acceleration experienced by the first laboratory, in the gravitational field. The Principle of Equivalence then states that experimenters in the two laboratories, no matter what experiments they perform, will be unable to discover in which system they are--the accelerated system, or the system stationary in the gravitational field. This same principle would apply to a system falling freely in a gravitational field. The observer would be unable to tell by local measurement whether he was in free-fall, or whether he was motionless (or, rather, unaccelerated) in a gravity-free region.

The equivalence principle does not say that a gravitational field is 'nothing but' a non-inertial reference frame. Because true gravitational fields are non-uniform it would be impossible to find a single reference frame which 'in the large'--over sufficient distances, or for sufficient times--would reproduce the effects of the field. In the above case, if the laboratory were large enough or were to fall long enough, the experimenter would observe effects which would derive from gravitation: he would see neighboring particles accelerate towards each other if separated horizontally, or away from each other if separated vertically. Furthermore, the relative acceleration would be directly proportional to separation. Uniformly accelerated frames and gravitational fields are only equivalent 'locally'.

2.3 Some Consequences of the Principle of Equivalence

The equivalence of gravitational and inertial energy follows as a direct consequence of the equivalence principle. Special relativity showed that an increase in energy ΔE in a system resulted in an increase in inertial mass $\Delta E/c^2$. From the equivalence principle, Einstein showed that this also corresponded to an increase $\Delta E/c^2$ in the gravitational mass (Einstein, 1911). Two further consequences of the equivalence principle, which also served as tests of its validity, were gravitational red-shift (or time dilation) and the deflection of light by gravitational fields (Lanczos, 1972).

Consider two systems, S_1 and S_2, separated by a distance h along the Z-axis, in a frame K. Assume that the gravitational potential is greater at S_2 than at S_1, by $\Delta\phi = \gamma h$. By the equivalence principle, this situation is the same as the one in which S_2 and S_1 are rigidly connected to a frame K', which is in a gravitation-free region, but accelerated in the z-direction with acceleration γ.



Assume that S_2 radiates a pulse of light, with frequency V_2 , which arrives at S_1 a time t = h/c later. At this time S_1, in the K' frame, will have, to a first approximation, velocity $\gamma h/c$ relative to S_2 at the time of emission. The relativistic Doppler effect predicts that, to a first approximation, the frequency of the absorbed light at S_1 will be greater than the frequency of the emitted light at S_2.

$$V_1 = V_2\left(1 + \frac{v_{S_1}}{c^2}\right) = V_2\left(1 + \frac{\gamma h}{c^2}\right) \quad . \qquad (2.3.1)$$

By the equivalence principle, the frequency of the absorbed light at S_1 in K will be

$$V_1 = V_2\left(1 + \frac{\gamma h}{c^2}\right) = V_2\left(1 + \frac{\Delta\phi}{c^2}\right) \quad . \qquad (2.3.2)$$

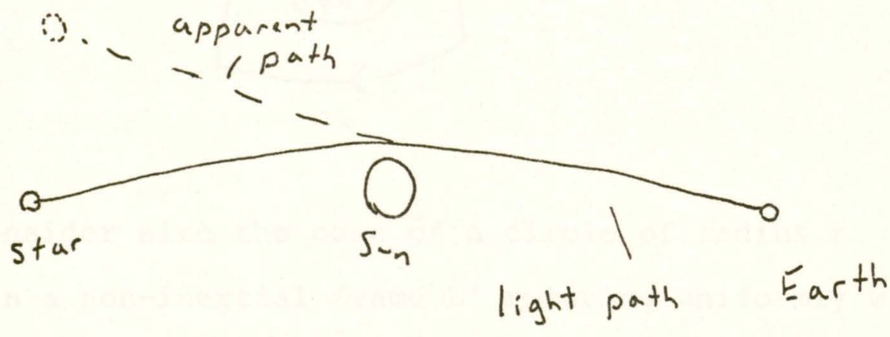
In general,

$$\frac{V_a - V_e}{V_e} = \frac{\Delta\phi}{c^2} \left(= \frac{\phi_e - \phi_a}{c^2}\right) \quad ,$$

where V_a = absorbed frequency, V_e = emitted frequency, and $\Delta\phi$ = gravitational potential difference between the point of emission and the point of absorption (Einstein, 1911).

Although the predicted red-shift of light from the sun is not too small to be observed, it is masked by other effects. It was not until 1960 that the gravitational red-shift effect was unambiguously confirmed, by Pound and Rebka, in an experiment comparing two clocks in the earth's gravitational field separated by a height of 22 meters (Lanczos, 1972).

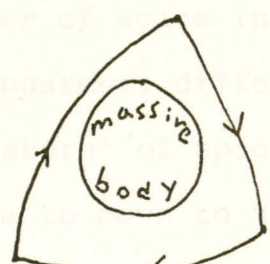
Light will deviate from a straight-line path, relative to an accelerated reference system, and hence should be deflected by a gravitational field. Einstein's prediction was confirmed in 1919, when observations made during a solar eclipse showed an apparent shift in position of stars eclipsed by the sun.



The equivalence principle showed Einstein that special relativity, based on the constancy of the velocity of light, was limited. A more general theory of relativity, one which incorporated gravitation, would have to be formulated.

2.4 The Choice of Riemannian Geometry

The equivalence principle not only showed that special relativity had to be modified, but it also showed that the Lorentzian (or pseudo-Euclidean) geometry of special relativity did not apply in gravitational fields. For instance, a triangle bound by light rays will, in the presence of gravitational fields, be 'distorted' from a triangle in Euclidean space; its interior angles will be greater than Π radians.



Consider also the case of a circle of radius r fixed in a non-inertial frame L' rotating uniformly with respect to an inertial frame L. An observer in L' will measure the same circumference-to-diameter ratio as he would measure if L' were stationary with respect to L, since the dimensions of his measuring instruments and the dimensions of the circle are equally affected by the rotation. However, an observer in L will measure a different ratio. Since the line of radius is perpendicular to its motion, he will measure the same radius as the observer in L', but the circumference, parallel to its motion, will be shortened. The figure is still a 'circle' in the sense that each point on the circumference is equidistant from the center, but the circumference-to-diameter ratio is not Π. Space itself appears to be 'curved'.

The geometry in the frame is non-Euclidean. By the equivalence principle, the geometry in a gravitational field would also be non-Euclidean. However, the non-Euclidean character of space in a gravitational field is a real, and not apparent, difference. In the above example the 'curvature' of space would vanish if the observer in L were to move to the frame L', while an observer in a gravitational field will still observe space to be curved.

In a non-inertial reference frame $g_{\mu\nu}$ is a general function, and not the Lorentzian

$$g_{\mu\nu} \equiv \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

However, by a suitable transformation, $g_{\mu\nu}$ can be reduced to the Lorentzian form. That the metric tensor describing a gravitational field can be reduced to the above diagonal form at a given point follows from the equivalence principle. However, there is no transformation which will reduce the tensor to that form everywhere.

In searching for a geometry for gravitational fields, we need a four-dimensional geometry which reduces to Lorentzian, or 'flat', space locally, but which globally remains curved. Just such a geometry had been developed many years earlier by Reimann. It was in the discovery of Riemannian geometry that Einstein found the final key to his theory of gravitation.

As a final note, it should be mentioned that attempts have been made to develop a theory of gravitation which utilizes the equivalence principle while keeping the geometric framework of special relativity, but none of these attempts have ever been successful.

2.5 The Theory of General Relativity

In gravitational fields, global inertial frames, which are impossible to obtain, lose their special significance. The laws of physics should have the same form in all coordinate systems. This principle of general covariance, from which general relativity gets its name, led Einstein to conclude that a successful theory of gravitation would have to be a tensor theory.

The field equation of Newton's theory of gravitation is

$$\nabla^2\phi = -4\pi G\rho \quad , \tag{2.5.1}$$

where ϕ is the gravitational potential ρ the mass density, and G the gravitational constant. This is a scalar theory. We would like to replace the scalars ϕ and ρ by tensors.

In special relativity mass-density is replaced by total energy-density, which becomes one component of a second-rank tensor in 4-space, the energy-momentum tensor $T_{\mu\nu}$. For this tensor,

T_{oo} represents total non-gravitational energy-density

T_{oi} represents momentum-density, or energy flux density

T_{ij} represents momentum-flux density.

Thus, in a tensor theory, the scalar ρ of Newton's theory should be replaced by the tensor $T_{\mu\nu}$ of special relativity.

The simplest equation would be

$$F_{\mu\nu} = T_{\mu\nu} \quad , \tag{2.5.2}$$

where $F_{\mu\nu}$ is some derivative of a tensor-potential.

The equation of motion for a force-free particle in an inertial frame in the Minkowski space of special relativity is

$$\frac{d^2x^\alpha}{ds^2} = 0 \quad . \tag{2.5.3}$$

In a non-inertial frame--in an arbitrary coordinate system--its equation of motion would be

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\nu}{ds} = 0 \tag{2.5.4}$$

where

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\delta}(g_{\gamma\delta,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta}) \quad ,$$

representing the inertial forces that arise in transforming to a non-inertial frame. By the inverse transformation the Γ's can be made to vanish everywhere, and equations (2.5.4) will revert to (2.5.3).

In general relativity $\Gamma^\alpha_{\beta\gamma}$ will represent both inertial and gravitational forces. The Γ-terms cannot be split into 'inertial' and 'gravitational' terms, nor can they, by any transformation, be made to vanish everywhere. Thus, (2.5.4) becomes the equation of motion for a force-free particle in the curved, Riemann space. The $g_{\alpha\beta}$'s, which compase the metric, become the gravitational potentials.

$F_{\mu\nu}$, a function of $g_{\alpha\beta}$, can be obtained by demanding that in weak fields, and low velocities, the equation (2.5.2) reduce to the linear, second-order Newtonian equation (2.5.1). But the only tensors which can be constructed from the metric tensor and its derivatives, which are linear in the second derivatives, are the Riemann tensor and its contractions. (Papapetrou, 1972). The Riemann tensor is

$$R^{\rho}_{\lambda\mu\nu} = -\Gamma^{\rho}_{\lambda\mu,\nu} + \Gamma^{\rho}_{\lambda\nu,\mu} - \Gamma^{\sigma}_{\lambda\mu}\Gamma^{\rho}_{\sigma\nu} + \Gamma^{\sigma}_{\lambda\nu}\Gamma^{\rho}_{\sigma\mu} \quad . \qquad (2.5.5)$$

Contracting ρ and μ yields the double-index Ricci tensor

$$R_{\lambda\nu} \equiv R^{\rho}_{\lambda\rho\nu} \quad . \qquad (2.5.6)$$

Contracting again yields

$$R^{\lambda}_{\lambda} \equiv R \quad . \qquad (2.5.7)$$

The general form for $F_{\mu\nu}$ would be

$$F_{\mu\nu} = \alpha R_{\mu\nu} + \beta g_{\mu\nu} R + \nu g_{\mu\nu} \quad , \qquad (2.5.8)$$

and the general field equations would be

$$R_{\mu\nu} + \bar{\beta} g_{\mu\nu} R + \bar{\nu} g_{\mu\nu} = -kT_{\mu\nu} \quad . \qquad (2.5.9)$$

Einstein's original equation was the simplest:

$$R_{\mu\nu} = -kT_{\mu\nu} \quad .$$

But it was soon discovered that this would have to be modified. The law of conservation of energy requires, in special relativity, that the divergence of the energy-momentum tensor vanish;

$$T^{\mu\nu}{}_{,\nu} = 0 \quad . \qquad (2.5.10)$$

In a general coordinate system, the covariant divergence of the energy-tensor would vanish.

$$T^{\mu\nu}{}_{;\nu} = 0 \quad . \qquad (2.5.11)$$

By the equivalence principle this must hold true in the Riemann space which describes the gravitational field. If, however, this same condition is imposed on the Ricci tensor,

$$R_{\mu}{}^{\nu}{}_{;\nu} = 0 \quad ,$$

four additional equations will be obtained, and the $g_{\mu\nu}$ will be over-determined. A <u>divergenceless form</u> of the Ricci tensor must be found such that

$$(R^{\mu\nu} + \beta' g^{\mu\nu} R)_{;\nu} = 0 \quad .$$

This equation is an identity for $\beta' = -\frac{1}{2}$. The field equations in their final form are:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}R + \Lambda g^{\mu\nu} = -kT^{\mu\nu} \quad . \qquad (2.5.12)$$

The constant Λ is called the cosmological constant, and is either zero or so small that on any scale other than a cosmological scale the term $\Lambda g^{\mu\nu}$ is negligible. Neglecting the cosmological term, the field equations become

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}R = -kT^{\mu\nu} \quad . \qquad (2.5.13)$$

CHAPTER THREE

SCHWARZSCHILD AND EDDINGTON EQUATIONS OF STATE AND THEIR INTERPRETATIONS

3.1 The Schwarzschild Exterior Solution

In 1916 Schwarzschild published the first exact solution of Einstein's equations and obtained the free-space time-independent, or static, spherically symmetric line element; the line element exterior to a static, spherically symmetric mass distribution.*

By using plausibility arguments and employing symmetry conditions we can greatly simplify the metric from the general form, (2.1.3). Since the field is static we demand the metric to remain unchanged under a time-inversion, $dx^o \rightarrow -dx^o$. Hence, the metric components g_{oi} should vanish. Similarly, spherical symmetry would imply invariance under rotation. The metric should remain unchanged under the transformation $d\theta \rightarrow -d\theta$, and $d\phi \rightarrow -d\phi$. All diagonal terms would be zero, and the

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*Birkhoff (Papapetrou, 1974) later showed this to be the same as the exterior metric for non-static, spherically symmetric distributions, as well.

line element would be

$$ds^2 = A(dx^o)^2 - Bdr^2 - Cr^2d\theta^2 - Dr^2\sin^2\theta\, d\phi^2, \qquad (3.1.1)$$

where A,B,C, and D are functions of radius only. A further simplification can be made by noting that ϕ and θ, as they appear in (3.1.1), represent angular coordinates. Consider two infinitesimal displacements, one by ε along the pole, and the other by ε along the equator. The interval ds^2 should be the same for each displacement. At the pole $ds^2 = -Cr^2d\theta^2$, and at the equator $ds^2 = -Dr^2d\phi^2$. But $\varepsilon_{pole} = rd\theta$ and $\varepsilon_{equator} = rd\phi$. So, $ds^2 = -C\varepsilon^2 = -D\varepsilon^2 \Rightarrow C = D$. The line element becomes

$$ds^2 = A(dx^o)^2 - Bdr^2 - Cr^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \qquad (3.1.2)$$

One further simplification can be made, by making a suitable transformation. Define a new radial coordinate $\hat{r} = C(r)^{\frac{1}{2}}r$

$$d\hat{r} = \frac{1}{2}\, C^{-\frac{1}{2}}(\frac{dC}{dr}\, dr)r + C^{\frac{1}{2}}dr = drC^{\frac{1}{2}}(1 + \frac{r}{2C}\,\frac{dC}{dr}) \quad . \qquad (3.1.3)$$

Transforming to this new coordinate, Cr^2 becomes $\hat{r}^2$, and

$$Bdr^2 = \frac{B}{C}\,(1 + \frac{r}{2C}\,\frac{dC}{dr})^{-2} d\hat{r}^2 = \hat{B}d\hat{r}^2 \quad . \qquad (3.1.4)$$

Since A and B are positive-definite, it would be convenient to re-express them in exponential form. The final form of the metric is

$$ds^2 = \ell^{\nu}(dx^{o})^2 - \ell^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad ,$$

$$g_{\mu\nu} = \begin{pmatrix} \ell^{\nu} & 0 & 0 & 0 \\ 0 & -\ell^{\lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \quad . \qquad (3.1.5)$$

In free space

$$T_{\mu\nu} = 0 \quad .$$

Thus Einstein's free-space equations are

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}R = 0 \quad . \qquad (3.1.6)$$

Contracting (3.1.6), we obtain

$$R^{\mu}{}_{\mu} - \tfrac{1}{2}\, g^{\mu}{}_{\mu}R = -R = 0 \quad . \qquad (3.1.7)$$

Equation (3.1.6) reduces to the simpler

$$R_{\mu\nu} = 0 \quad . \qquad (3.1.8)$$

The additional conditions

$$\lim_{r\to\infty} \ell^{\nu} = \lim_{r\to\infty} \ell^{\lambda} = 1 \quad , \qquad (3.1.9)$$

are imposed on the metric. That is, infinitely far from the source, the line element approaches the special relativistic, flat-space, metric.

Employing these boundary conditions and solving the equations (3.1.8) yields

$$\ell^{\nu} = 1 - \frac{2m}{r} \quad , \quad \ell^{\lambda} = \frac{1}{1 - \frac{2m}{r}} \quad . \qquad (3.1.10)$$

Thus, the expression for the interval is

$$ds^2 = (1 - \frac{2m}{r})(dx^o)^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \qquad (3.1.11)$$

where m is a constant of integration which can be determined by demanding correspondence with Newtonian theory. In weak-field, nonrelativistic limits, $g_{oo} = 1 + \frac{2\phi}{c^2}$. In the case of a point mass,

$$\phi = - \frac{GM}{c^2}$$

where M = total mass of the source. Hence,

$$m = \frac{GM}{c^2} \quad .$$

3.2 The Schwarzschild Singularity

The exterior solution becomes singular at $r = 2m$. At this value the metric component g_{rr} becomes infinite, and g_{oo} zero. For bodies within our experience, such as the Sun and planets, this singularity poses no problems, since the 'Schwarzschild radius' (the $r = 2m$ radius) lies well within the body itself, in a region where the exterior solution does not apply.* This singularity does, however give a lower limit to the radius of a static, spherically symmetric distribution of given mass. For a static body the Schwarzschild radius would have to lie within the body itself.

3.3 The Interior Solution

Schwarzschild also found the metric inside a static spherically symmetric perfect fluid. He reduced the field equations to three equations and found an exact solution for the equation of state $\rho = \rho_o$ = constant (where ρ is the T_{oo} component of the energy-momentum tensor). His solution remains one of a handful of exact physically interesting interior solutions, and certainly the best-known and most widely utilized solution to the

*The Schwarzschild radius for the Sun is 1.5km, for the earth .44cm, and for a neutron 1.2×10^{-54}cm. (Schild, 1967) interior equations.

We can obtain the value of the constant k in equations (2.5.13) by matching with Poisson's equation for the gravitational field in the Newtonian limit. Equation (2.5.13) becomes

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}R = \frac{8\pi G}{c^4} T^{\mu\nu} \quad . \qquad (3.3.1)$$

The metric has the same form as the exterior metric (3.1.5).

In a perfect fluid characterized by a proper-density field $\rho(x)$, a scalar pressure-field $P(x)$, and a four-vector velocity field of flow $v\mu(x)$, the energy-momentum tensor will be (Adler, Bazin, and Schiffer, 1965)

$$T_{\mu\nu} = \rho v_\mu v_\nu + \frac{P}{c^2} (v_\mu v_\nu - g_{\mu\nu}) \quad . \qquad (3.3.2)$$

For a static fluid, $v_i = 0$, $v_o = (g_{oo})^{\frac{1}{2}}$. The final form of $T_{\mu\nu}$ is

$$T_{\mu\nu} = \begin{pmatrix} \rho \ell^{\nu} & 0 & 0 & 0 \\ 0 & P\ell^{\lambda} & 0 & 0 \\ 0 & 0 & Pr^2 & 0 \\ 0 & 0 & 0 & Pr^2\sin^2\theta \end{pmatrix} , \qquad (3.3.3)$$

(letting c=1).

A lengthy, but straightforward, calculation finally yields the three equations

$$C\rho = \ell^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2} \quad , \qquad (3.3.4)$$

$$CP = \frac{1}{r^2} - \ell^{-\lambda}\left(\frac{1}{r^2} + \frac{v'}{r}\right) \quad , \qquad (3.3.5)$$

$$CP = \ell^{-\lambda}\left|\frac{v'\lambda'}{4} - \frac{v'^2}{4} - \frac{v''}{2} - \frac{1}{2}\,\frac{(v'-\lambda')}{r}\right| \quad , \qquad (3.3.6)$$

from (3.1.5), (3.3.1), and (3.3.3), where the constant $C = -8\pi G$ (Adler, Bazin, and Schiffer, 1965). For convenience in performing the numerical integrations which are to follow, we will introduce a fourth equation, which can be derived from (3.3.4), (3.3.5), and (3.3.6), by differentiating with respect to r, and eliminating λ:

$$P' = -\frac{1}{2}\,v'(\rho + P) \quad . \qquad (3.3.7)$$

There are three equations in the four unknowns $\lambda(r)$, $v(r)$, $\rho(r)$, and $P(r)$. This gives the freedom to eliminate one of the variables by choosing an equation of state. Schwarzschild chose the simplest equation of state, $\rho = \rho_o$ = const., which led to an exact solution.

Equation (3.3.4) can be integrated directly. Multiplying by r^2, yields

$$C\rho r^2 = \ell^{-\lambda}(1 - \lambda' r) = (\ell^{-\lambda} r)' - 1 \quad . \qquad (3.3.8)$$

Integrating, we obtain

$$\ell^{-\lambda} = 1 + \frac{C}{r}\int_0^r \rho r^2 dr + \frac{\text{const.}}{r} \quad , \qquad (3.3.9)$$

where the integral

$$4\pi \int_0^r \rho r^2 dr$$

has been defined to be the total mass inside a sphere of radius r (Misner and Sharp, 1964; Misner, 1964; Misner, Thorne and Wheeler, 1973).

From the condition of elementary flatness, explored in the next section, and directly from equation (3.3.4), it is seen that $\ell^{\lambda} = 1$ at $r = 0$. It is clear that at $r = 0$ in equation (3.3.4), ρ will be infinite, which is physically impossible, unless λ approaches 0 at least like r^2 (Landau and Lifshitz, 1971). Accordingly, the constant in (3.3.9) is zero. Setting $\rho = \rho_o$ = const., (3.3.9) becomes

$$\ell^{-\lambda} = 1 + \frac{C\rho_o r^2}{3} \quad . \qquad (3.3.10)$$

Defining

$$\hat{R}^2 = -\frac{3}{C\rho_o} \quad , \qquad (3.3.11)$$

we obtain

$$\ell^{\lambda} = -g_{11} = (1 - \frac{r^2}{\hat{R}^2})^{-1} . \qquad (3.3.12)$$

Rearranging equation (3.3.7),

$$- \frac{1}{2} v''(r) = \frac{P'}{\rho_o + P} , \qquad (3.3.13)$$

and integrating, yields

$$- \frac{D}{C} \ell^{-v/2} = P + \rho_o , \qquad (3.3.14)$$

where D is a constant of integration. The right-hand term is the same as the term we would have if we were to add (3.3.5) and (3.3.6):

$$\rho_o + P = - \frac{1}{C} \frac{\ell^{-\lambda}}{r} (v'+\lambda') = - \frac{1}{Cr} \left| \ell^{-\lambda} v' - (\ell^{-\lambda})' \right| . \qquad (3.3.15)$$

Thus,

$$rD\ell^{-v/2} = \ell^{-\lambda} v' - (\ell^{-\lambda})' = (1 - \frac{r^2}{\hat{R}^2}) v'(r) + \frac{2r}{\hat{R}^2} . \qquad (3.3.16)$$

Let

$$\gamma(r) = \ell^{v/2} . \qquad (3.3.17)$$

We obtain

$$(1 - \frac{r^2}{\hat{R}^2})\gamma'(r) + \frac{r}{\hat{R}^2}\gamma = \frac{1}{2} rD \quad . \tag{3.3.18}$$

The homogeneous equation

$$(1 - \frac{r^2}{\hat{R}^2})\gamma'(r) + \frac{r}{\hat{R}^2}\gamma(r) = D \quad , \tag{3.3.19}$$

has the general solution

$$\gamma(r) = B(1 - \frac{r^2}{\hat{R}^2})^{\frac{1}{2}} \quad , \tag{3.3.20}$$

while (3.3.18) has the particular solution

$$\gamma_\rho = \frac{1}{2} D\hat{R}^2 \quad . \tag{3.3.21}$$

Thus,

$$\gamma(r) \equiv \ell^{\nu/2} = \frac{1}{2} D\hat{R}^2 - B(1 - \frac{r^2}{\hat{R}^2})^{\frac{1}{2}} \tag{3.3.22}$$

and

$$\ell^{\nu} = g_{oo} = \left| A - B(1 - \frac{r^2}{\hat{R}^2})^{\frac{1}{2}} \right|^2 \quad . \tag{3.3.23}$$

where

$$A = \frac{1}{2} D\hat{R}^2 \quad .$$

It now remains to find the constants A and B. We impose the two boundary conditions that at the surface, $r = r_o$, the pressure becomes zero, and that the metric joins smoothly at the surface with the exterior metric (3.1.9)*. From equation (3.3.13),

$$C(\rho_o + P) = \frac{-\frac{2A}{\hat{R}^2}}{A - B(1 - \frac{r^2}{\hat{R}^2})^{\frac{1}{2}}} \quad . \qquad (3.3.24)$$

At the surface $r = r_o$, $P = 0$, and

$$C\rho_o = (-\frac{3}{\rho_o \hat{R}^2})\rho_o = -\frac{2A}{\hat{R}^2}\left| \frac{1}{A - B(1 - \frac{r_o^2}{\hat{R}^2})^{\frac{1}{2}}} \right| , \qquad (3.3.25)$$

$$\Rightarrow A = 3B(1 - \frac{r_o^2}{\hat{R}^2})^{\frac{1}{2}} \quad .$$

From the second condition,

$$-(g_{11})_{int.,r=r_o} = \frac{1}{(1 - \frac{r_o^2}{\hat{R}^2})^{\frac{1}{2}}} = -(g_{11})_{ext.,r=r_o} = \frac{1}{(1 - \frac{2m}{r_o})^{\frac{1}{2}}} \quad ,$$

(3.3.26)

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*There exists an 'admissible' coordinate system such that the g_{ik} and $g_{ij,k}$ are continuous across every 3-surface. For any coordinate system obtained from admissible coordinates by a transformation continuous only in the first derivative, the g_{ik} will remain continuous, although $g_{ij,k}$ may not. The Schwarzschild coordinates are not admissible, but can be derived from an admissible coordinate system by a transformation continuous in the first derivatives. (Synge, 1966) which implies

$$(1 - \frac{r_o^2}{\hat{R}^2}) = 1 - \frac{2m}{r_o} \quad , \qquad (3.3.27)$$

and

$$(g_{oo})_{int.,r=r_o} = \left[2B(1 - \frac{r_o^2}{\hat{R}^2})^{\frac{1}{2}}\right]^2 ,$$

$$\left(= 4B^2(1 - \frac{2m}{r_o})\right) = (g_{oo})_{ext.,r=r_o} = (1 - \frac{2m}{r_o}) \quad .$$

$$(3.3.28)$$

which implies

$$B = \frac{1}{2} \quad . \qquad (3.3.29)$$

The interior Schwarzschild metric, in its final form, is

$$ds^2 = \left[\frac{3}{2}(1 - \frac{r_o^2}{\hat{R}^2})^{\frac{1}{2}} - \frac{1}{2}(1 - \frac{r^2}{\hat{R}^2})^{\frac{1}{2}}\right]^2 (dx^o)^2 - \frac{dr^2}{(1 - \frac{r^2}{\hat{R}^2})}$$

$$- r^2(d\theta^2 + \sin^2\theta \, d\phi) \quad . \qquad (3.3.30)$$

Substituting for m and R^2 in (3.3.27), we also note

$$M = \frac{4\pi}{3} r_o^3 \rho_o \quad . \qquad (3.3.31)$$

The total mass is equal to the product of the (constant) density and the coordinate volume.

3.4 Proper Quantities in General Relativity

In general relativity, as in special relativity, it is the proper time or proper distance which has physical significance rather than coordinate time or coordinate distance, since coordinate quantities are frame-dependent. The proper time would be the time interval measured between two events in a frame in which they are concurrent, and the proper distance the distance measured between two events by an observer in the frame in which they occur simultaneously.

The interval ds^2 between two events is either positive or negative. If positive, we can find a frame in which the two are concurrent, and if negative a frame in which they occur simultaneously. In the former case the proper time between the events would be

$$d\tau \equiv ds = dx^{o} \quad . \qquad (3.4.1)$$

In the latter case, the proper distance would be

$$d\sigma \equiv ds = (-g_{ik}dx^{i}dx^{k})^{\frac{1}{2}} = (dx^2 + dy^2 + dz^2)^{\frac{1}{2}} \quad . \qquad (3.4.2)$$

The generalization to general relativity is straightforward:

$$d\tau = (g_{oo})^{\frac{1}{2}}\, dx^{o} \quad , \tag{3.4.3}$$

and

$$d\sigma = \left[(-g_{ik} + \frac{g_{oi}g_{ok}}{g_{oo}})\, dx^{i} dx^{k}\right]^{\frac{1}{2}} \quad .$$

(Landau and Lifshitz, 1971).

For the body previously considered, r_o is the coordinate radius, not the proper radius. The proper radial separation of two infinitesimally separated points would be

$$d\hat{r} = (g_{rr})^{\frac{1}{2}} dr = \frac{dr}{(1 - \frac{r^2}{\hat{R}^2})^{\frac{1}{2}}} \quad ,$$

and the proper radius of the body

$$r_{proper} = \int_{r=0}^{r=r_o} (g_{rr})^{\frac{1}{2}} dr = \int_{o}^{r_o} \frac{dr}{(1 - \frac{r^2}{\hat{R}^2})^{\frac{1}{2}}} \quad . \tag{3.4.4}$$

Thus, $\frac{4\pi}{3} r_o{}^3$ is not the proper volume of the body. More will be said about this in Section 6.

Consider an infinitesimal circle of radius δr , centered at $r = 0$. The condition of elementary flatness, mentioned in the previous section, demands that, as δr approaches zero, the ratio of proper circumference to proper radius reduce to the ratio in Newtonian flat-space, 2π. (Synge, 1960). The ratio will be independent of the orientation of the circle, but for simplicity choose the circle to lie in the $\phi = \frac{\pi}{2}$ plane. The proper distance interval on the circumference is, from equation (3.4.3),

$$d\sigma = r d\theta = \delta r d\theta \quad , \qquad (3.4.5)$$

since $dr = d\phi = 0$. The proper circumference is the integral of this, or $2\pi\delta r$. From (3.3.4), the proper radius is

$$r_{proper} = \int_o^{\delta r} \ell^{\lambda/2} dr = \ell^{\lambda/2} \int_o^{\delta r} dr = \ell^{\lambda/2} \delta r \quad .$$

The function $\ell^{\lambda/2}$ can be removed from the integral since it would be approximately constant over the infinitesimal distance δr. The ratio of proper circumference to proper radius is

$$2\pi \ell^{-\lambda/2} \quad ,$$

which must equal 2π as $\delta r \to 0$, implying $\ell^{\lambda} = 1$ at the center of the body.

3.5 The Singularity in the Interior Solution

The interior solution contains a second singularity, aside from the $r = 2m$ singularity found also in the exterior solution. This imposes a second, more severe restriction on the radius of a body of given mass, or given density. Substituting for λ and ν in equations (3.3.5) or (3.3.6), we can solve for the pressure

$$P = \rho_o \left[\frac{(1 - \frac{r^2}{\hat{R}^2})^{\frac{1}{2}} - (1 - \frac{r_o^{\,2}}{\hat{R}^2})^{\frac{1}{2}}}{3(1 - \frac{r_o^{\,2}}{\hat{R}^2})^{\frac{1}{2}} - (1 - \frac{r^2}{\hat{R}^2})^{\frac{1}{2}}} \right] . \qquad (3.5.1)$$

Between the values $r_o = \frac{9m}{4}$ and $r_o = 2m$, the pressure, at some point in the body, will be infinite. The central pressure becomes infinite when $r_o = \frac{9m}{4}$. At this value of r_o the static configuration cannot be sustained, since the matter at the center could no longer 'support' the rest of the body. Hence, this larger radius would be the limit for a static fluid with the Schwarzschild equation of state.

3.6 Interpretation of the Schwarzschild Interior Form

Because the energy-density ρ_o is constant in the Schwarzschild equation of state, Schwarzschild referred to his interior solution as that of a uniform-density, incompressible fluid. However, ρ_o represents the density of all forms of energy excluding gravitational binding energy. Although it is well known that the proper energy density of the body, the density of gravitational energy as well as all other forms of energy, is not equal to ρ_o, (Adler, Bazin, and Schiffer, 1965; Møller, 1972), the Schwarzschild fluid is generally referred to as an incompressible fluid, and the condition $\rho = \rho_o$ = constant is considered to be the limiting equation of state for physically realistic, stable stars (see, for example, Hegyi, Lee, and Cohen, 1975).* The condition

$$\frac{d\rho}{dr} \leq 0 \quad . \tag{3.6.1}$$

is regarded as a necessary condition of stability.

Buchdahl (1959) and Weinberg (1971) show that, for any static spherically symmetric body with arbitrary equation of state, subject to the one limiting condition $\frac{d\rho}{dr} \leq 0$, the maximum value for $\frac{2m}{r}$ is $\frac{8}{9}$, the same as the maximum value in Schwarzschild's interior solution. The

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*Eddington advanced an alternate definition of incompressibility, which will be examined shortly.

condition $\rho' \leq 0$ is imposed since "it is difficult to imagine that a fluid sphere with a larger density near the surface than near the center could be stable" (Weinberg, 1971). From his proof, he concludes that the maximum value of $\frac{2m}{r}$ for any stable matter distribution is $\frac{8}{9}$.

The proper energy-density of the Schwarzschild fluid is not ρ_o . The average physical density of the fluid would be the total mass divided by the proper volume. The physical volume element

$$dV = ds_r ds_\theta ds_\phi \quad , \tag{3.6.2}$$

where

$$ds_r = \sqrt{g_{rr}}\, dr = (1 - \frac{r^2}{\hat{R}^2})^{-\frac{1}{2}}\, dr \quad , \tag{3.6.3}$$

$$ds_\theta = \sqrt{g_{\theta\theta}}\, d\theta = r d\theta \quad , \tag{3.6.4}$$

$$ds_\phi = \sqrt{g_{\phi\phi}}\, d\phi = r \sin\theta\, d\phi \quad . \tag{3.6.5}$$

The proper volume is

$$V = \iiint_{\theta\phi r} dV = \iiint \frac{r^2 \sin\theta\, d\theta d\phi dr}{(1 - \frac{r^2}{\hat{R}^2})^{\frac{1}{2}}} = 4\pi \int_0^{r_o} (1 - \frac{r^2}{\hat{R}^2})^{-\frac{1}{2}}\, r^2 dr \;. \tag{3.6.6}$$

Following the procedure of Adler, Bazin, and Schiffer (1965), we expand (3.6.5) in a power series, obtaining

$$V = \frac{4\pi r_o^{\,3}}{3}\left[1 + \frac{3}{10}\left(\frac{r_o}{\hat{R}}\right)^2 + 0\left(\frac{r_o}{\hat{R}}\right)^4 + \ldots\right] \quad . \tag{3.6.7}$$

We then find the average density

$$\langle\rho\rangle = \frac{M}{V} = \frac{3M}{4\pi r_o^{\,3}}\left[1 - \frac{3}{10}\left(\frac{r_o}{\hat{R}}\right)^2 + 0\left(\frac{r_o}{\hat{R}}\right)^4 + \ldots\right] \, . \tag{3.6.8}$$

By (3.3.9) this becomes

$$\langle\rho\rangle = \rho_o\left[1 - \frac{3}{10}\left(\frac{r_o}{\hat{R}}\right)^2 + 0\left(\frac{r_o}{\hat{R}}\right)^4 + \ldots\right] \quad . \tag{3.6.9}$$

The physical density is neither the same as the coordinate density ρ_o nor, as we shall see later, is it uniform throughout the body. The difference between ρ and the physical density is that ρ, the proper density locally, a function which includes both rest-energy density and energy-density due to motion, does not include energy-density due to the presence of the gravitational field. Gravitation is represented by the metric on the left side of the field equations (3.3.1), while the right-hand side of the equations, $T^{\mu\nu}$, includes, by definition, all stress-energy except gravitation. Indeed, this function, as the T_{oo} component of the energy-momentum tensor, could not include gravitational energy density since gravitational energy cannot be 'localized' in an absolute sense. At any point where a gravitational field exists we could transform to a coordinate system in which that field vanishes.

3.7 Eddington Incompressibility

Eddington (1924) objected to calling the Schwarzschild fluid incompressible. He advanced an alternate definition of incompressibility, based upon an alternate concept of uniform density. To understand the motivation for Eddington's definition, it is useful to derive the formula for the energy-momentum tensor (3.3.3), in the following way:

The energy-momentum tensor for a system of non-interacting particles, in Lorentz space, is

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} = \rho\gamma^2 \frac{dx^{\mu}}{dx} \frac{dx^{\nu}}{dx} \quad , \qquad (3.7.1)$$

where $u^{\mu} = \frac{dx^{\mu}}{ds}$. In Cartesian coordinates this would be

$$T_{\mu\nu} = \begin{pmatrix} \rho' & \rho' v_x & \rho' v_y & \rho' v_z \\ \rho' v_x & \rho' v_x v_x & \rho' v_x v_y & \rho' v_x v_z \\ \rho' v_y & \rho' v_x v_y & \rho' v_y v_y & \rho' v_y v_z \\ \rho' v_z & \rho' v_x v_z & \rho' v_y v_z & \rho' v_z v_z \end{pmatrix} , \quad (3.7.2)$$

where $\rho' = \rho\gamma^2$.

Consider, now, $T_{\mu\nu}$ for an element of a perfect fluid in which the individual particles interact. If (v_x, v_y, v_z) refer to the motion of the center of mass of the fluid element, and (v_x', v_y', v_z') the motion of the individual particles relative to the center of mass, each term in (3.7.2) will now be

$$\begin{aligned} T^{ik} &= \Sigma\bar{\rho}(v^i + v'^i)(v^k + v'^k) \quad , \\ &= \Sigma\bar{\rho}v^i v^k + \Sigma\bar{\rho}v'^i v'^k \quad . \end{aligned} \quad (3.7.3)$$

where $\bar{\rho}$ is the density of the individual particles. Clearly the cross terms

$$\Sigma\bar{\rho}v^i v'^k \quad (= v^i \Sigma\bar{\rho}v'^k)$$

are, by definition of center of mass, equal to zero. The terms $\Sigma\bar{\rho}v'^i v'^k$ are the internal stresses P^{ik}, i.e., the rate of transfer of the x^i-component of momentum across a plane perpendicular to the x^k-axis. To the term (3.7.2) is now added the stress term,

$$S_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P_{xx} & P_{xy} & P_{xz} \\ 0 & P_{xy} & P_{yy} & P_{yz} \\ 0 & P_{xz} & P_{yz} & P_{zz} \end{pmatrix} . \qquad (3.7.4)$$

If the perfect fluid is now referred to axes with respect to which it itself is at rest, the stress components P_{ij} become $\delta^i_j P$; $v^o = 1$, $v^i = 0$, and the energy-momentum tensor becomes

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} . \qquad (3.7.5)$$

The trace is $\eta_{\mu\nu}T^{\mu\nu} = \rho - 3P$, where $\eta_{\mu\nu}$ is the metric tensor of special relativity. Since the trace is an invariant, it will have the same value in curved space as in flat space, i.e., $T = g_{\mu\nu}T^{\mu\nu} = \rho - 3P$, from equation (3.3.3). It should be noted that, since the trace of the energy-momentum tensor for a system of non-interacting particles must be positive, and that, since the trace for the electromagnetic field is zero, the trace for a system of particles interacting electromagnetically, will be positive (see Landau and Lifshitz, 1971).

Eddington's concept of incompressibility is constancy of particle-density, not mass-density. He writes,

> "If a fluid is incompressible, i.e. if the closeness of packing of the particles is independent of p, the condition must be that... |T, the trace of the energy-momentum tensor|... is constant. Incompressibility is concerned with constancy not of mass-density but of particle-density, so that no account should be taken of increases of mass of the particles due to motion relative to the centre of mass of the matter as a whole."
>
> (Eddington, 1924)

The particle-density of a system would be proportional to the value obtained if the individual particle masses were summed, where each particle is referred to a set of axes with respect to which it itself is at rest. If we denote the sum ρ_{oo}, the energy-momentum tensor would be

$$T^{\mu\nu} = \begin{pmatrix} \rho_{oo} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \qquad (3.7.6)$$

Equating the traces of (3.7.1) and (3.7.5), gives the relationship

$$\rho_{oo} = \rho - 3P \quad . \qquad (3.7.7)$$

The equation of state proposed by Eddington for incompressible fluids is

$$\rho_{oo} = \rho - 3P \equiv B = \text{constant} \quad . \qquad (3.7.8)$$

Substituting this into (3.3.7) gives

$$P' = -\frac{v'}{2}(B + 4P) \quad . \qquad (3.7.9)$$

Integrating,

$$P = \frac{K\ell^{-2v} - B}{4} \quad , \qquad (3.7.10)$$

where K is the constant of integration. However, the coupling of the equations for density and pressure which is demanded by the Eddington equation of state leads to far more difficult equations for v and λ themselves, than the equations obtained by using the Schwarzschild equation of state. Substituting (3.7.3) into (3.3.5) and (3.3.6), and eliminating λ, gives

$$CK\ell^{-2v}\left(v'' - (v')^2 - \frac{6v'}{r} + \frac{6}{r^2}\right) + CB\left(-v'' - (v')^2 + \frac{2}{r^2}\right)$$

$$- \frac{4v'^2}{r^2} - \frac{4v''}{r} - \frac{8v'}{r^2} = 0 \quad . \qquad (3.7.11)$$

This equation was solved by numerical integration.

Eddington's condition, that particle-density, not energy-density, be independent of pressure, satisfies the classical condition for uniform density and is equivalent to the classical concept of incompressibility since, in Newtonian physics, the constancy of mass and particle density are equivalent for a homogeneous body. The Schwarzschild condition is a generalization of this classical condition to special relativity, since it takes into account changes in mass due to particle motion. Neither condition, however, would seem suitable for a general relativistic concept of uniform density or incompressibility, since neither take into account increases in mass due to the gravitational field.

CHAPTER FOUR

GENERAL RELATIVISTIC INCOMPRESSIBILITY

4.1 A New Definition of Incompressibility

The condition $\rho = \rho_o$ = constant was considered by Schwarzschild, and by many others subsequently, to be the equation of state for an incompressible fluid. The limit $\frac{2m}{r_o} = \frac{8}{9}$, at which central pressure becomes infinite for the Schwarzschild equation of state, has been generally considered to be the closest that any physically realistic body could come to the ultimate static limit, $r_o = 2m$. However, as was seen in Chapter 3, ρ_o does not represent proper energy density, since gravitational binding energy is not taken into account. The $\rho = \rho_o$ = constant condition is not the general-relativistic condition for a body which has uniform proper-density.

True incompressibility, an idealization in the realm of classical physics, is an even more remote concept in general relativity, because its realization is already precluded in special relativity. Complete rigidity implies the instantaneous velocity of propagation of interactions. Instead, we propose the following <u>definition of incompressibility</u>:

> A static spherically symmetric body is incompressible if its equation of state is such that the pressure becomes infinite at r=0 precisely when the limit $\frac{2m}{r_o} = 1$ is reached. (Cooperstock and Sarracino, 1976)

This certainly represents the ultimate stiffness of matter, a condition which could be attained only as a limit, if then.

Whereas central pressure may become infinite before the static limit $\frac{2m}{r_o} = 1$ is reached, this static limit could never be attained <u>before</u> central pressure becomes infinite. This can be seen from an examination of the Schwarzschild interior equations (3.3.4), (3.3.5), (3.3.6), and (3.3.7). The metric component g_{rr} has the value 1 at the center of the body, and increases outwards from the center. Re-expressing (3.3.5), we obtain

$$v' = \frac{\ell^{\lambda}-1}{r} - CP\ell^{\lambda}r \quad . \qquad (4.1.1)$$

Since $\ell^{\lambda} - 1 > 0$, and P, ℓ^{λ}, and $r \geq 0$, v' will be positive throughout the body. From (3.3.7), it is seen that the pressure gradient is everywhere negative, implying that the maximum value of pressure is at the center of the body.

From the continuity condition for ℓ^{λ} and ℓ^{v} , and the condition P = 0 at the boundary, it can be shown that $(\ell^{\nu})'$ is also continuous at the boundary (Schild, 1967).

From equation (3.3.5),

$$\frac{1}{r_o^{\,2}} - (\ell^{-\lambda})_{surf.}\left(\frac{1}{r_o^{\,2}} + \frac{(v')_{surf.}}{r_o}\right) = 0 \quad ,$$

$$\Rightarrow (\ell^{\nu})_{surf.} v'_{surf.} = (\ell^{\nu})'_{surf.} = \frac{1}{r_o}(\ell^{\nu}{}_{surf.}\,\ell^{\lambda}{}_{surf.} - \ell^{\nu}{}_{surf.})$$

$$= \frac{1}{r_o}(1 - \ell^{\nu}{}_{surf.}) \quad .$$

But

$$\frac{1}{r_o}(1 - \ell^{\nu}{}_{surf.}) = 2mr_o^{-2} = (\ell^{\nu})'_{surf.} \; .$$

Matching with the exterior Schwarzschild solution,

$$(\ell^{\lambda})_{surface} = \left(1 - \frac{2m}{r_o}\right)^{-1} \quad , \qquad (4.1.2)$$

$$(\ell^{\nu})_{surface} = 1 - \frac{2m}{r_o} \quad , \qquad (4.1.3)$$

$$(v')_{surface} = \frac{2m}{r_o^{\,2}}\left(1 - \frac{2m}{r_o}\right)^{-1} \quad . \qquad (4.1.4)$$

Substituting into (3.3.7) yields the pressure gradient at the surface

$$\lim_{r \to r_o} P' = \frac{1}{2r_o} \left(\frac{2m}{r_o}\right) \left(1 - \frac{2m}{r_o}\right)^{-1} \left(\lim_{r \to r_o} \rho\right) \quad . \qquad (4.1.5)$$

At $\frac{2m}{r_o} = 1$ the surface pressure gradient, and therefore the central pressure, would become infinite. Thus, the new definition of incompressibility advanced above represents a true limiting condition in that there cannot exist a body whose equation of state is such that the central pressure would be finite at $r_o = 2m$.

It should be noted that a simple substitution of $r_o = 2m$ into the pressure equation for the Schwarzschild fluid, (3.5.1), will give a negative central pressure. However, between the values $\frac{2m}{r_o} = \frac{8}{9}$ and $\frac{2m}{r_o} = 1$ the pressure will be infinite somewhere in the body. We are considering here only static bodies or bodies at the static limit, and beyond $\frac{2m}{r_o} = \frac{8}{9}$ the static configuration of a body with the Schwarzschild equation of state could not be sustained.

4.2 Incompressibility and the Work of Bondi

Since an incompressible fluid would be stiffer than a uniform-density fluid, which is generally regarded as physically unrealistic, it follows that a fluid which is incompressible by our definition would be an idealization, impossible of attainment. The definition does, moreover, provide a useful standard against which the relative compressibility or incompressibility of a fluid may be measured. The closer the surface value of $\frac{2m}{r}$ approaches to 1, the closer a given equation of state would come to representing incompressiblity. In the next chapter we shall use this standard to compare the Schwarschild and Eddington equations of state, plus a third, which represents uniform proper density.

Bondi (1959, 1964), has shown that for bodies with non-negative pressure and density, the limit $r_o = 2m$ cannot be reached. Beginning with the Schwarzschild equations (3.3.4), (3.3.5), (3.3.6), and (3.3.7), and defining

$$u(r) = \frac{\mathcal{M}(r)}{r} \quad , \qquad (4.2.1)$$

where

$$\mathcal{M}(r) = \int_o^r 4\pi r^2 \rho dr \quad , \qquad (4.2.2)$$

equation (3.3.4) can be integrated to give

$$\ell^{-\lambda} = 1 - 2u \quad . \qquad (4.2.3)$$

Defining a second function,

$$v(r) = 4\pi r^2 P \quad , \qquad (4.2.4)$$

equation (3.3.5) becomes

$$\frac{1}{2} r \frac{dv}{dr} = \frac{u+v}{1-2u} \quad , \tag{4.2.5}$$

which when substituted into (3.3.6) assumes the form

$$\frac{1}{r} \frac{dr}{dv} = \frac{(1-2u) \frac{du}{dv} + u + v}{2v-(u^2+6uv+v^2)} \quad , \tag{4.2.6}$$

the equation of hydrostatic equilibrium. Bondi then does a detailed analysis of this equation, plotting its solution curves in the u-v plane.

Imposing the condition $\rho - 3P \geq 0$, the maximum value of $\frac{2m}{r_o}$ is found to be .638.* Under the less stringent condition $\rho - P \geq 0$, the limiting equation of state for bodies in which the speed of sound, $(\frac{dP}{d\rho})^{\frac{1}{2}}$, is less than or equal to the speed of light in vacuum, the maximum value of $\frac{2m}{r_o}$ is .780. In accord with the proof of Buchdahl it found that for $\frac{d\rho}{dr} \leq 0$, the maximum value of $\frac{2m}{r_o}$ is .889. The highest value of $\frac{2m}{r_o}$, obtained by imposing no restriction except $\rho \geq 0$, $P \geq 0$, is .970. The solution yielding this extreme value corresponds to 'a thin shell of matter stuffed full of density-free

*This is the condition for a system of non-interacting particles, or particles interacting electromagnetically. (Landau and Lifshitz, 1971) pressure', and is, therefore, entirely unphysical.

As a final note, Bondi goes further still, showing that r_o approaches 2m if negative densities are allowed, and that $\frac{2m}{r_o}$ can exceed 1 if pressure is allowed to become negative. But these conditions are also completely unphysical.

CHAPTER FIVE

UNIFORM DENSITY FLUIDS

Neither the T_{oo} component of the energy-momentum tensor, nor the trace (ρ - 3P in Schwarzschild coordinates) of the energy-momentum tensor correspond to proper energy density when one wishes to take gravitational energy into account. In this chapter we find the general relativistic expression for proper energy density in Schwarzschild coordinates, and develop the equation of state for a truly 'uniform density' fluid, that is, a fluid in which the general relativistic energy density is constant. We then compare this fluid with the Schwarzschild and Eddington fluids, and determine which most closely approximates incompressibility.

5.1 The Constant ρ_{proper} Equation of State

In section 3.4 it was shown that

$$V_{proper} = 4\pi \int_o^{r_o} \ell^{\lambda/2} r^2 \, dr \quad , \qquad (5.1.1)$$

and that for the Schwarzschild equation of state,

$$<\rho_{proper}> = \frac{\int_o^{r_o} \rho_o r^2 \, dr}{\int_o^{r_o} \ell^{\lambda/2} r^2 \, dr} \neq \rho_o \quad . \qquad (5.1.2)$$

It can be further concluded that $\langle\rho\rangle$ in the Schwarzschild fluid is a function of r, and not constant throughout the body, as is ρ_o . In general we cannot associate a definite proportion of the total gravitational energy of a body to any particular region inside it. Strictly speaking, gravitational energy cannot be localized. However, in the special case of a spherically symmetric body this can be done (Misner, Thorne, and Wheeler, 1972). The total mass, that is, mass due to all sources other than gravitation, and mass due to gravitational binding energy, contained in a sphere of radius r, of a spherically symmetric matter distribution, is

$$\mathcal{M}(r) = 4\pi \int_o^r \rho r'^2 \, dr' \quad . \tag{5.1.3}$$

From (5.1.3) and (5.1.2) we see that ρ_{proper} is not constant throughout the fluid. By a straightforward calculation the general expression for proper energy density can be derived. The mass of a spherically symmetric system is

$$M = \int (\text{proper density})(\text{proper volume element}) \quad ,$$

$$= 4\pi\int \rho r^2 \, dr = \int \left(\frac{\rho}{\sqrt{g_{rr}}}\right)(4\pi r^2 \sqrt{g_{rr}} \, dr) \quad , \tag{5.1.4}$$

But from (3.6.1) we know that

$$dV_{proper} = 4\pi r^2 \sqrt{g_{rr}}\, dr \quad . \tag{5.1.5}$$

Therefore,

$$\rho_{proper} = \frac{\rho}{\sqrt{g_{rr}}} = \rho \ell^{-\lambda/2} \tag{5.1.6}$$

in Schwarzschild coordinates.

We now consider the equation of state

$$\rho_{proper} \equiv A = \text{const.} \quad , \tag{5.1.7}$$

or

$$\rho = A\ell^{\lambda/2} \quad .$$

Substituting into equations (3.1.4), (3.1.5), (3.1.6), and (3.1.7) yields the equations

$$-8\pi G A \ell^{\lambda/2} = \ell^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2} \quad , \tag{5.1.8}$$

$$-8\pi G P = \frac{1}{r^2} - \ell^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) \quad , \tag{5.1.9}$$

$$-8\pi G P = \ell^{-\lambda}\left|\frac{\nu'\lambda'}{4} - \frac{\nu'^2}{4} - \frac{\nu''}{2} - \frac{1}{2}\left(\frac{\nu'-\lambda'}{r}\right)\right| \quad , \tag{5.1.10}$$

$$P' = -\frac{1}{2}\nu'(A\ell^{\lambda/2} + P) \quad . \tag{5.1.11}$$

These equations were integrated numerically. The program used was "Variable Order Integrator for the Numerical Solution of Ordinary Differential Equations", by F.T. Krogh, Jet Propulsion Laboratory, California Institute of Technology, May, 1969. The program was carried out on the IBM 370/145 Computer at the University of Victoria. Work was done in double precision (16 significant figures).

5.2 Preliminary Work and the Schwarzschild Equation of State

As a check, and in order to determine the accuracy, of the program, the equations for the Schwarzschild equation of state were first solved numerically. The equations (3.3.4), (3.3.5), (3.3.6), and (3.3.7) were re-written in a form more suitable for integration,

$$(\ell^{\lambda/2})' = \frac{\ell^{\lambda/2} - \ell^{3\lambda/2}}{2r} - \frac{C\rho}{2}\ell^{3\lambda/2}r \quad , \qquad (5.2.1)$$

$$v' = \frac{\ell^{\lambda} - 1}{r} - CP\ell^{\lambda}r \quad , \qquad (5.2.2)$$

$$v'' = -\frac{v'^2}{2} + v'\left[\frac{(\ell^{\lambda/2})'}{\ell^{\lambda/2}} - \frac{1}{r}\right] + \frac{2(\ell^{\lambda/2})'}{\ell^{\lambda/2}r} - 2CP\ell^{\lambda} \quad , (5.2.3)$$

$$P' = \frac{-v'\rho}{2} - \frac{v'P}{2} \quad . \qquad (5.2.4)$$

In solving differential equations numerically precautions must be taken to assure that the solution does not cross onto a curve which diverges radically from the true curve represented by the given initial conditions. To guard against this, a 'vector plot' of the solution curves is made, to find the areas of 'bundling', that is, the areas where the solution curves concentrate. If bundling is severe, but occurs at one of the extremes of the curves, the boundary conditions must be set, and integration performed, from the other extreme. If bundling occurs in the middle, very little can be done short of using a more accurate and sophisticated program.

Figures 5.1 and 5.2 show solution curves for two hypothetical equations. If integration begins at $t = 0$ for the equation represented by Figure 5.1, small errors made at the start will be greatly magnified as integration proceeds. If, however, integration is performed from t_o to 0, small errors will be reduced as integration proceeds, and the correct curve will be obtained. In Figure 5.2, severe bundling occurs approximately mid-way between $t = 0$ and $t = t_o$. Regardless of which end of the curve the integration is begun, crossing to the wrong curve will result.

A vector plot was made of equation (5.2.1) by calculating $(\ell^{\lambda/2})'$ for different values of $\ell^{\lambda/2}$ and r.

Figure 5.3 shows the results of the plot.

The curves become singular at r = 1 and, with the exception of the curve with boundary value $\ell^{\lambda/2}(0)=1$, at r = 0. Bundling occurs between these two values of r, but fortunately it did not prove severe, except for the Eddington equation of state. From the vector plot, it is clear that integration should be carried outwards from r = 0. As expected, a few attempts to integrate from the surface towards the center did result in the curve approaching negative infinity as r approached zero.

The equations (5.2.1), (5.2.2), (5.2.3), and (5.2.4) were re-written in the form

$$f' = \frac{f-f^3}{2\hat{r}} + \frac{kAr_m^{\,2}}{2} f^3\hat{r} \quad , \tag{5.2.5}$$

$$g = \frac{f^2-1}{\hat{r}} + KP_m r_m^{\,2}\hat{P}f^2\hat{r} \quad , \tag{5.2.6}$$

$$g' = -\frac{g^2}{2} + g\left(\frac{f'}{f} - \frac{1}{\hat{r}}\right) + \frac{2f'}{f\hat{r}} + 2KP_m r_m^{\,2}\hat{P}f^2 \quad , \tag{5.2.7}$$

$$\hat{P} = -\frac{g}{2}\frac{A}{P_m} - \frac{g}{2}\hat{P} \quad , \tag{5.2.8}$$

where $K = -C = 8\pi G$, $f = \ell^{\lambda/2}$, $\rho = A =$ const., $\hat{r} = \frac{r}{r_m}$ $\hat{P} = \frac{P}{P_m}$, $g = \frac{dv}{dr}$, and " ' " denotes differentiation with respect to $\hat{r}$.

In evaluating the constants, r_m was taken to be the radius which gives the singularity in the exterior Schwarzschild metric, and the ratio A/P_m was chosen to be within an order of magnitude of the density-to-pressure ratio at the center of a neutron star obeying the Schwarzschild equation of state, with the mass and radius of a 'maximum mass' neutron star (Weinberg, 1972).*

For a 'uniform density' body (in the Schwarzschild sense)

$$r_o = 2m = 2G\,\frac{4\pi}{3}\,Ar^2 = \frac{KAr_o^{\,3}}{3} \quad .$$

Accordingly, we choose

$$r_m^{\,2} = \frac{3}{KA} \quad .$$

The constant $KAr_m^{\,2}$ in (5.2.5) is 3. The constant in (5.2.6) and (5.2.7) becomes

$$KP_m r_m^{\,2} = \frac{3P_m}{A} \quad ,$$

*Recent work has resulted in much larger theoretical values for the maximum mass of neutron stars (see, for instance, Hegyi, Lee, and Cohen, 1975; Brecher and Caporaso, 1975). However, the constants were chosen simply to be in the neighborhood of values found in the interior of neutron stars.

the inverse of the constant in (5.2.8).

A 'maximum mass' neutron star (Weinberg, 1972), has mass $.7M_\odot$, and radius 9.6km. This gives a value of $\frac{2m}{r_o} = .215$. From (3.5.1),

$$\frac{P_c}{\rho_o} = \frac{1 - (1 - \frac{2m}{r_o})^{\frac{1}{2}}}{3(1 - \frac{2m}{r_o})^{\frac{1}{2}} - 1} \approx .07 \quad ,$$

where P_c = central pressure, and r = radius at the surface. We set $\frac{P_o}{A} = 5$, or about seven times the above value. If $A = 10^{14}$g/cm^2 then P_o represents $4.5 \cdot 10^{34}$ dynes/cm^2 .

Making the appropriate substitutions we obtain the equations in their final form, suitable for integration.

$$f' = \frac{f-f^3}{2r} + \frac{3}{2} f^3 r \quad , \tag{5.2.9}$$

$$g = \frac{f^2-1}{r} + \frac{3}{2} Pf^2 r \quad , \tag{5.2.10}$$

$$g' = -\frac{g^2}{2} + g(\frac{f'}{f} - \frac{1}{r}) + \frac{2f'}{fr} + 3Pf^2 \quad , \tag{5.2.11}$$

$$P' = -g - \frac{gP}{2} \quad . \tag{5.2.12}$$

We have the freedom of choosing P at the origin. The values of f', g, g', and P' can be found at the origin by expanding f, P, and g in a power series about r = 0. We find

$$f'(0) = 0 \ ,$$

$$P'(0) = 0 \ ,$$

$$g(0) = 0 \ ,$$

$$g'(0) = 1 + \frac{3P}{2} \ .$$

The central pressure was set at the maximum value allowed by the computer, and the program stopped when P became negative. A simple interpolation between the final positive and first negative value of pressure was used to give the value of r and f at the surface. From f, $\frac{2m}{r_o}$ can be readily found by matching with the exterior solution:

$$f \equiv \ell^{\lambda/2} = \frac{1}{(1 - \frac{2m}{r_o})^{\frac{1}{2}}} \ . \qquad (5.2.13)$$

As a preliminary check of the accuracy of the program we integrated equation (5.2.9). The function f went off the computer register (maximum setting, ~ 10^{74}) at the expected value of r = 1. The computer value agreed with the analytic value to 10 significant figures.

Then the two equations (5.2.9) and

$$P' = -\frac{3}{4} f^2 r P^2 + P\left(\frac{1}{2r} - \frac{f^2}{2r} - \frac{3}{2} f^2 r\right) - \frac{f^2}{r} + \frac{1}{r} \quad , \qquad (5.2.14)$$

obtained by eliminating g between (5.2.10) and (5.2.12), were integrated. With central pressure set at 10^{36}, f at the surface was found to agree with the analytic value for infinite central pressure to 5 significant figures. Then a number of programs were run in which all four equations (5.2.9) to (5.2.12) were solved. For a number of values of central pressure, to values as low as 10^5, the computer solutions agreed with the infinite-central-pressure analytic solution to five significant figures. These checks were performed to assure the accuracy of the program and the consistency of the procedures.

Table 5.1 lists the most significant of the programs run, and their results.

5.3 Numerical Solutions for the Constant ρ_{proper} Equation of State

Equations (3.1.4) to (3.1.7) were integrated using the constant-ρ_{proper} equation of state,

$$\rho = A\ell^{\lambda/2} \equiv Af \quad . \qquad (5.3.1)$$

As with the Schwarzschild equation of state, two sets of equations were integrated: the first set corresponding to (5.2.5) and (5.2.15), and the second set corresponding to (5.2.9), (5.2.10), (5.2.11), and (5.2.12). The first set of equations were

$$f' = \frac{f-f^3}{2r} + \frac{3}{2} f^4 r \quad , \tag{5.3.2}$$

and

$$P' = -\frac{3}{4} f^2 r P^2 + P\left(\frac{1}{2r} - \frac{f^2}{2r} - \frac{3}{2} f^3 r\right) - \frac{f^3}{r} + \frac{f}{r} \; . \tag{5.3.3}$$

The second set of equations were (5.3.2), (5.2.10), (5.2.11), and

$$P' = -gf - \frac{gP}{2} \quad . \tag{5.3.4}$$

These equations have the same initial values for f, f', g, g', and P' as those for the Schwarzschild equation of state.

The maximum value obtained for $(\ell^{\lambda/2})_{surface}$ corresponded to a maximum value for $2m/r_o$ of .917, as compared to .889 for the Schwarzschild fluid. The constant-ρ_{proper} equation does lead to a stiffer fluid, more closely approximating an incompressible fluid, than the Schwarzschild constant-ρ equation.

5.4 Numerical Solutions for the Eddington Equation of State

Substituting the Eddington equation of state

$$\rho = A + 3P \tag{5.4.1}$$

into equations (5.2.5) and (5.2.8), we obtain

$$f' = \frac{f-f^3}{2r} + \frac{3}{2} f^3 r + \frac{9}{4} P f^3 r \ , \tag{5.4.2}$$

$$P' = -g - 2gP \ . \tag{5.4.3}$$

These equations, together with (5.2.10) and (5.2.11), were integrated in the same manner as in the previous two cases. Expanding f, P, and g in power series about r = 0, we find the initial values

$$f'(0) = 0 \ ,$$

$$g(0) = 0 \ ,$$

$$P'(0) = 0 \ ,$$

$$g'(0) = 1 + 3P(0) \ .$$

For low values of central pressure, $(\frac{2m}{r_o})_{surface}$ increases as central pressure is increased, as is expected. However, for values of central pressure between 10 and 100 the value of $\frac{2m}{r_o}$ at the surface levels off, and then decreases, as central pressure is increased further still.

It was suspected that the ρ-P coupling in the equation of state led to severe bundling, which caused crossing from one solution curve to another in the critical region. Graphs of the pressure, g, and f curves for values of central pressure varying from 1 to 100 confirmed this suspicion (see graphs 5.5).

However, from the proof of Buchdahl, it is clear that the Eddington equation of state will lead to a more compressible body than that corresponding to the Schwarzschild equation of state. This follows from the Eddington equation of state (5.4.1):

$$\frac{d\rho}{dr} = 3\,\frac{dP}{dr} < 0 \quad , \qquad (5.4.5)$$

which satisfies the restriction on ρ necessary for the proof. In addition, the work of Bondi (1959, 1964) shows that for any fluid obeying the equation of state $\rho - 3P \geq 0$, the maximum value of $\frac{2m}{r_o}$ is .68, well below the maximum values for the constant-ρ and constant-ρ_{proper} fluids. The condition $\rho - 3P \geq 0$ is less restrictive than the Eddington equation of state, hence $\frac{2m}{r_o}$ in the Eddington case would be even lower than .638. The Eddington equation of state was analyzed numerically in order to determine just how much lower the maximum value of $\frac{2m}{r}$ would be, and also in order to gain more familiarity with the program.

5.5 Other Numerical Solutions

We programmed several other equations of state, including two which approximated the extreme configuration of 'a thin shell of matter containing a space stuffed full with density-free pressure' (Bondi, 1964). The equations

$$\rho = A\ell^{45\lambda} \tag{5.5.1}$$

and

$$\rho = Ar^{98} \tag{5.5.2}$$

produced maximum surface values of .970 and .969, respectively, in close agreement with Bondi's maximum .970. This computer analysis served as an additional check of the accuracy of the integration methods used.

5.6 Gravitational Red-Shift

The gravitational red-shift from the surface of a stellar object is

$$z = \frac{\Delta\lambda}{\lambda} = (1 - \frac{2m}{r_o})^{-\frac{1}{2}} - 1 = f - 1 \quad . \tag{5.6.1}$$

(Weinberg, 1972).

The gravitational red-shift for most stellar objects is extremely small, 4×10^{-4} for a 'maximum mass' white dwarf, for instance (Weinberg, 1972). In the early 1960's, however, a new class of objects (the quasi-stellar objects, or QSO's) were discovered, with starlike optical images, and with red-shifts varying from .1 to values in excess of 3 (Weinberg, 1972; De Veny, Osborn, and Janes, 1971). These extraordinarily large red-shifts could arise from a Doppler effect, in which case the QSO's would have an absolute luminosity far exceeding that of the largest galaxies, or, they could be caused by powerful gravitational fields. The gravitational red-shift from the surface of a 'maximum mass' neutron star (Weinberg, 1972) would be .13. The maximum red-shift from the surface of a star with Schwarzschild equation of state would be 2.00, and for awhile, when red-shifts were piling up to 1.95, it was tempting to ascribe them to stars that had approached the 'limit' of 'compressibility'. Since then, however, red-shifts have been found to exceed 2.00, and in fact, red-shifts as high as 3.2 have been observed (De Veny, Osborn, and Janes, 1971). The maximum gravitational red-shift from the surface of a star with constant ρ_{proper} equation of state is 2.48, higher than the red-shifts from all but a very few QSO's. This new value represents an improvement over the Schwarzschild limit, but would not alone suffice to provide an alternative hypothesis to the cosmological interpretation of QSO's.

It should be noted, as well, that some ascribe the large QSO red-shifts to light coming from the interior of stars (Weinberg, 1972). The red-shift from the center of a star that had reached its static limit, would be infinite.

5.7 The Maximum Mass of Non-rotating Neutron Stars

The question of the maximum possible mass of a neutron star is a topical one, since it has been claimed that the x-ray source Cygnus X-1 is too massive to be a neutron star, and, therefore, must be a black hole.

Many different limiting equations of state for neutron stars have been advanced over the years, leading to different values for the maximum possible mass of non-rotating neutron stars (Hegyi, Lee, and Cohen, 1975; Brecher and Caporaso, 1976). All equations of state proposed so far have been more restrictive than the Schwarzschild condition, which, considered to represent an 'incompressible' fluid, has been taken as the absolute limiting equation of state for any stable matter distribution (see, for example, Hegyi, Lee, and Cohen, 1975; Weinberg, 1972).

Oppenheimer and Volkoff (1939), using an independent particle model, originally calculated the maximum mass of a neutron star to be $.7M_\odot$. Recent work has yielded theoretical neutron star masses up to $5M_\odot$ using a simple pressure-dependent density function in the equation of state (Hegyi, Lee, and Cohen, 1975; Brecher and Caporaso, 1976). The equation of state

$$\rho = P \qquad (5.7.1)$$

has been taken by some to be the limiting equation of state for a physically realistic system, since the speed of sound is equal to the speed of light in vacuum. However, Bludman and Ruderman (1968 , 1970) argue that larger sound velocities can exist which do not violate causality. Hegyi, Lee, and Cohen (1975) have calculated the maximum possible mass for the Schwarzschild equation of state, and have found the value to be $8M_\odot$. A neutron star with a constant ρ_{proper} equation of state would have a maximum mass larger than $8M_\odot$, although how much larger has yet to be determined.

5.8 Physically Realistic Stars

The limiting condition for a stable mass distribution would be

$$\frac{d\rho_{proper}}{dr} \leq 0 \quad , \qquad (5.8.1)$$

not the condition

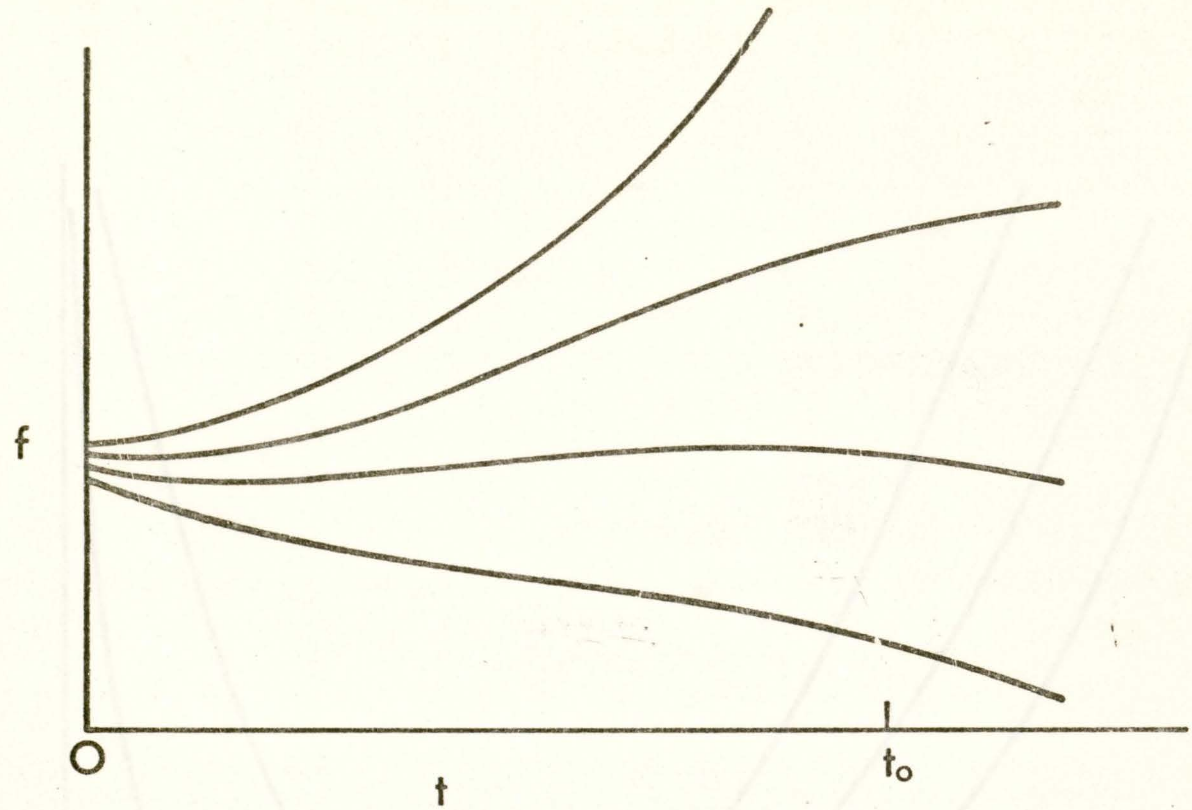
$$\frac{d\rho}{dr} \leq 0 \quad , \qquad (5.8.2)$$

which previously, and traditionally, has been considered (Weinberg, 1971; Wasserman and Brecher, 1977). For our model,

$$\frac{d\rho}{dr} = \frac{d}{dr}(A\ell^{\lambda/2}) \geq 0 \quad . \qquad (5.8.3)$$

Wasserman and Brecher (1977) have observed that our model violates the condition (5.8.2), and therefore should be unstable. This argument, we feel, suffers from the mistaken association of ρ, and not ρ_{proper}, with the physically meaningful energy-density, an association too closely wedded to classical concepts. All energy, including gravitational energy, contributes to the total mass of a body, and hence it is ρ_{proper} which is the physically relevant quantity in general relativity.

Stability, then, is determined by the proper-density distribution, not the density distribution represented by the function ρ. "...we would find it hard to imagine that a physically realistic star with ρ_{proper} increasing outward can be made. Clearly our equation of state was chosen as the limiting case which fulfills this requirement" (Cooperstock and Sarracino, 1977).



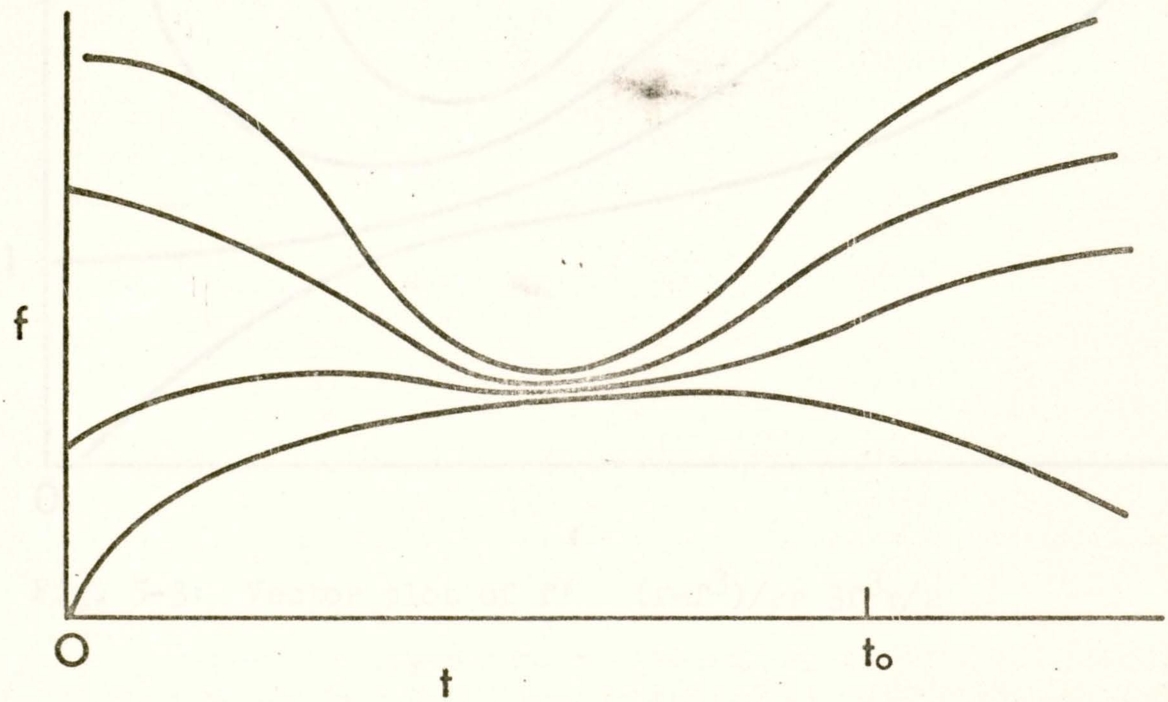


Fig. 5-1: A family of curves illustrating bundling on the t-axis

Fig. 5-2: A family of curves illustrating bundling between t 0 and t t_0

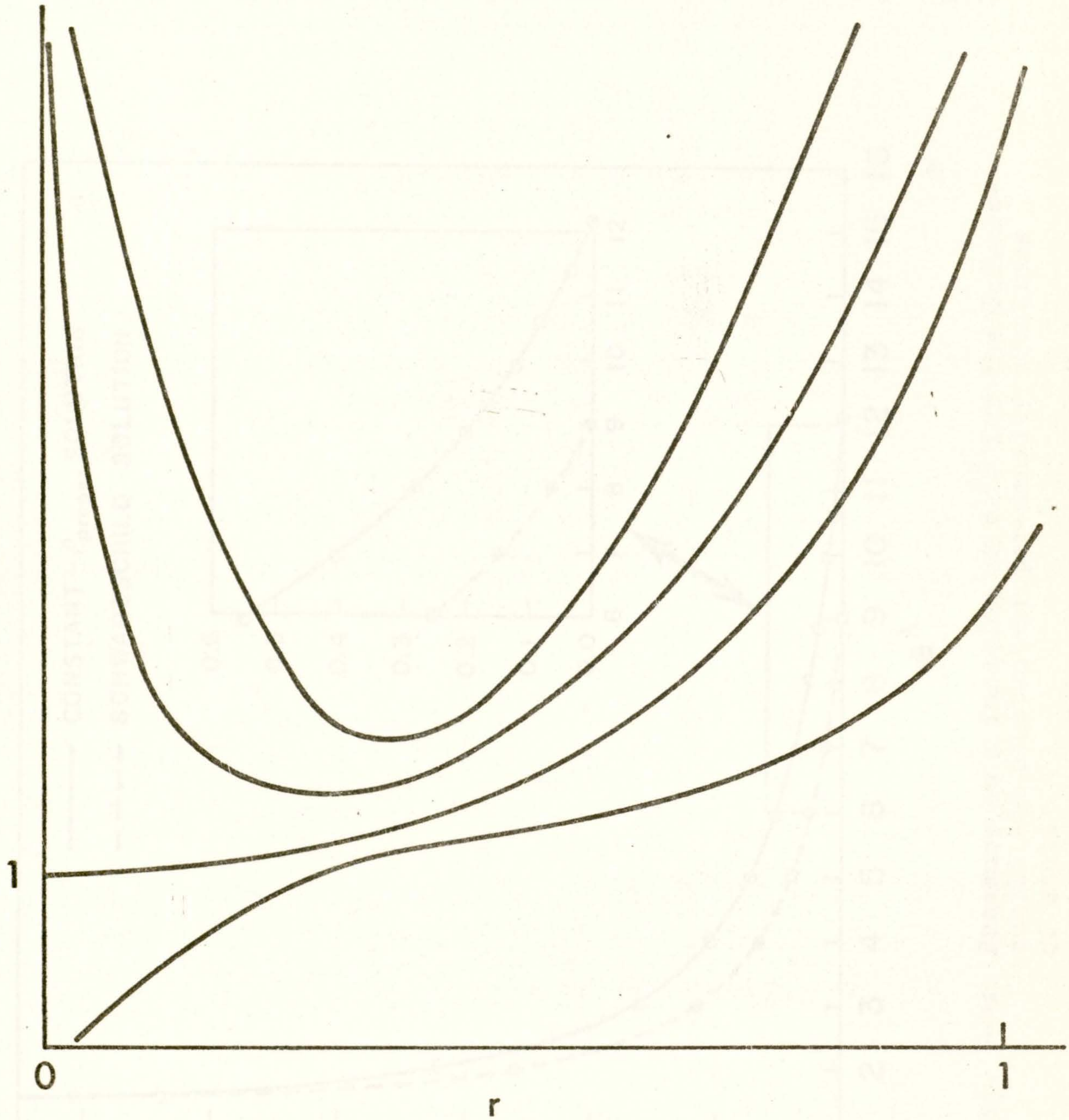


Fig. 5-3: Vector plot of f' $(f-f^3)/2r$ $3f^3r/2$

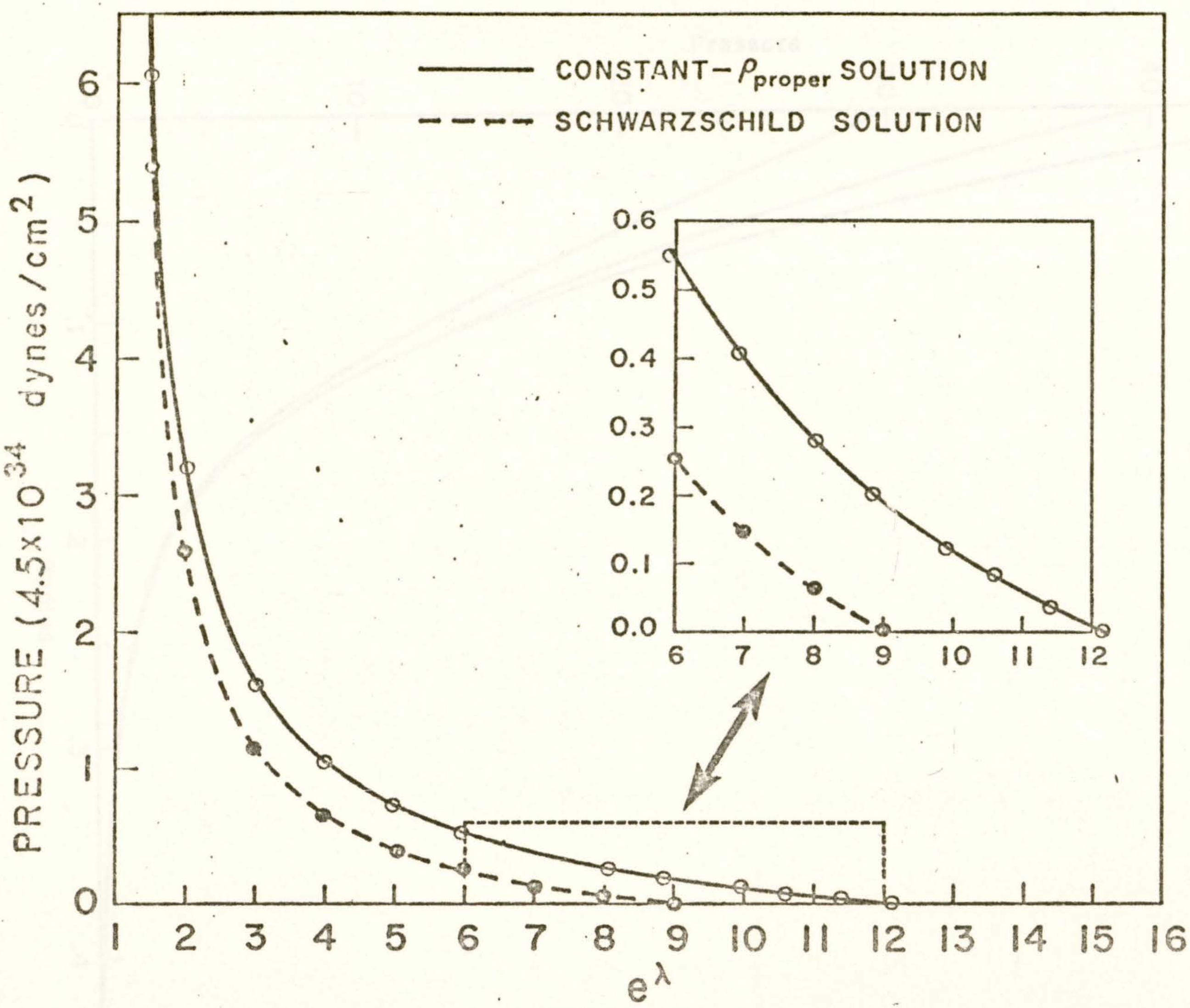


Fig. 5-4: Pressure as a function of e $^{/2}$ for the Schwarzschild and constant-proper-density equations of state

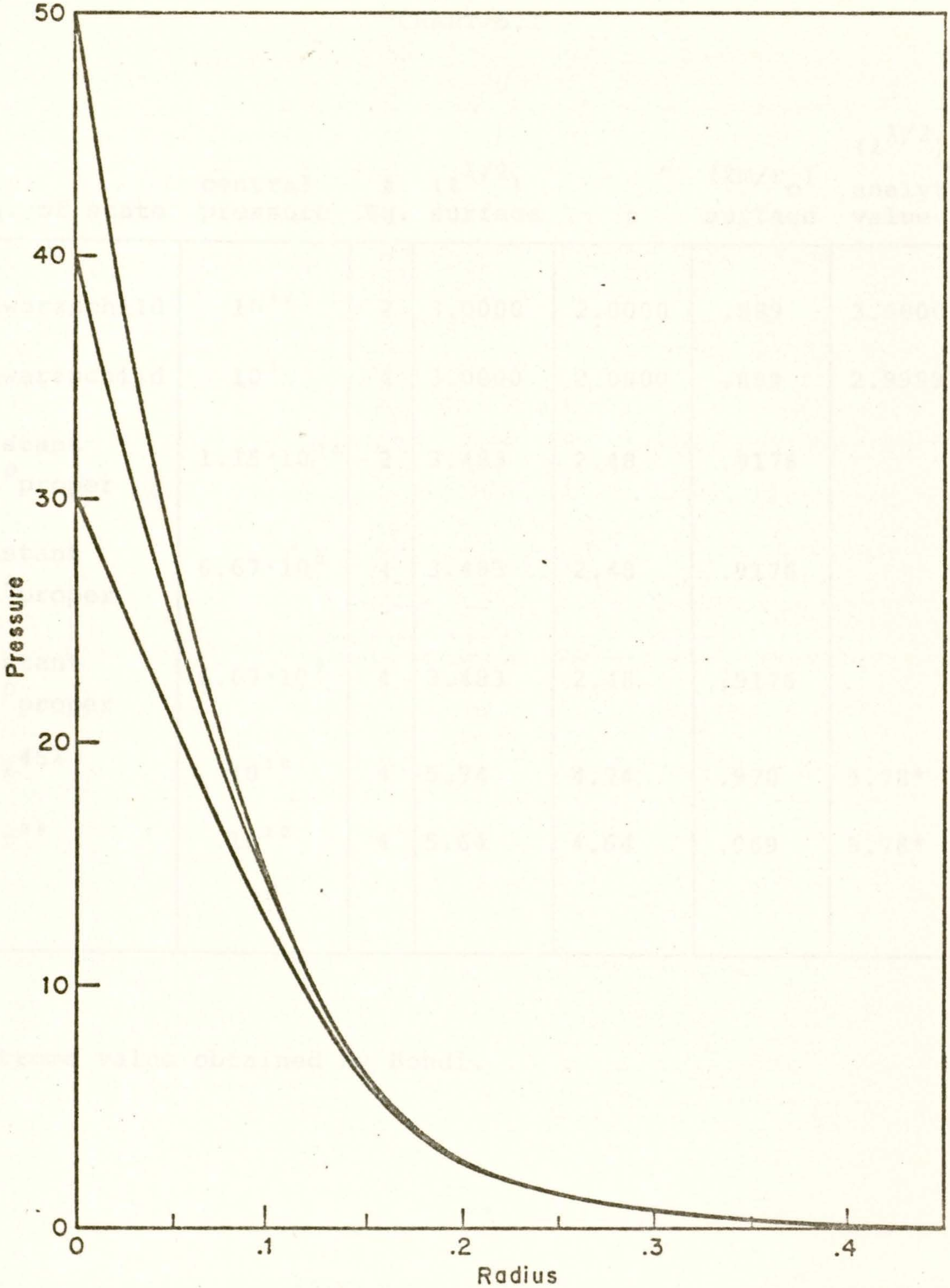


Fig. 5-5: Computer plots of pressure as a function of radius, for the Eddington equation of state.

CHART 5.1

Eq. of state	central pressure	# Eq.	$(\ell^{\lambda/2})$ surface	z	$(2m/r_o)$ surface	$(\ell^{\lambda/2})_s$ analytic value
Schwarzschild	10^{36}	2	3.0000	2.0000	.889	3.0000
Schwarzschild	10^{5}	4	3.0000	2.0000	.889	2.99997
Constant ρ_{proper}	$1.15 \cdot 10^{36}$	2	3.483	2.48	.9176	
Constant ρ_{proper}	$6.67 \cdot 10^{8}$	4	3.483	2.48	.9176	
Constant ρ_{proper}	$6.67 \cdot 10^{7}$	4	3.483	2.48	.9176	
$\rho=2\ell^{45\lambda}$	10^{10}	4	5.74	4.74	.970	5.78*
$\rho=2r^{98}$	10^{10}	4	5.64	4.64	.969	5.78*

*extreme value obtained by Bondi.

SUMMARY AND CONCLUSIONS

The two limits to the size and mass of static bodies, infinite central pressure and metric singularity at the surface, which appear from the general relativistic interior and exterior field equations, arise in general relativity. The condition that central pressure become infinite precisely when the surface value of $\frac{2m}{r}$ reaches 1, would, then, appear the natural generalization of incompressibility to general relativity. As complete rigidity is precluded by special relativity, it would be expected that an incompressible fluid, according to our new definition, could not exist, and indeed, it cannot. Moreover, the new definition provides a useful quantitative standard against which the compressibility of fluids obeying different equations of state can be measured. The larger the maximum possible surface value of $\frac{2m}{r}$ is, for a given equation of state, the closer that body approaches incompressibility.

The Eddington condition of incompressiblity is equivalent to the classical concept of a uniform-density, incompressible fluid because classically, for a homogeneous body, mass and particle density are proportional. The Schwarzschild condition is a generalization of these concepts to special relativity, since the density function takes into account changes in mass due to motion. The constant proper-density condition is a further generalization of the concept of uniform density to general relativity, since the density function includes gravitational energy as well as all other forms of energy. It is not surprising, therefore, that of the three, the Schwarzschild fluid should more closely approximate incompressiblity than the Eddington fluid, and that the constant proper-density fluid should be the least compressible.

The constant proper-density fluid has a higher maximum value of gravitational red-shift from the surface, 2.48, compared to 2.00 for the Schwarzschild fluid. At the same time, the constant proper-density equation of state would give a greater upper limit to the mass of a neutron star.

A star with uniform proper density would be stable, since it is proper density which is the physically relevant quantity in general relativity. The limiting condition for stability, traditionally considered to be the condition that ρ does not increase outwards, must, we feel, be replaced by the new stipulation that proper density does not increase outwards from the center of a body.

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VITA

Surname: SARRACINO Given Names: Robert Sinclair

Place of Birth: Ann Arbor, Mich., U.S.A. Date of Birth: 22 May, 1947

Educational Institutions Attended, with Dates of Entering and Leaving:

University of Chicago, Chicago, Ill.	1965 to 1967
Oregon State University, Corvallis, Ore.	1967 to 1969
University of Victoria, B.C., Canada.	1974 to 1977

Degrees, Diplomas, Etc., Awarded with Dates and Names of Institutions:

BS	1969	Oregon State University, Corvallis, Ore.

Publications:

Cooperstock, F.I. and Sarracino, R.S. 1976. General Relativistic Incompressibility, Nature, 264, 5586, pp. 529-531.

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