ELECTROMAGNETIC INDUCTION
IN A STRATIFIED CONDUCTING HALF-SPACE

by

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ABSTRACT

The mathematical analysis of electromagnetic induction in a stratified conductor is a problem with considerable geophysical application. In this thesis, induction in an N-layered conducting half-space, due to an arbitrary external time-harmonic magnetic source, is considered. The field is represented in terms of electric and magnetic Hertz potentials following the theory developed by Weaver (1971) for the case of a homogeneous conductor. It is shown that the electric Hertz potential vanishes everywhere in the conductor; thus the complete N-layer induction problem can be solved in terms of one scalar component of the magnetic Hertz potential which satisfies a second-order differential equation in each layer. The equations are solved using Fourier transforms, and general integral solutions are found for the N-layer problem in terms of the surface value of the Fourier-transformed Hertz potentials of the source. The solutions are also expressed in terms of source and image fields, together with the magnetic Hertz potential alone. The general solutions for the special cases N = 1 and N = 2 are compared with previous analytic results for the homogeneous conductor, and the two-layer conductor. To illustrate the theory, three particular inducing sources are considered: the horizontal alternating line-current; the vertically oriented magnetic dipole; and the horizontally oriented magnetic dipole. The integral solutions for the first two of these sources are evaluated numerically for various distributions of the conducting layers in the half-space.

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Cosi discesi del cerchio primaio
giu nel secondo, che men loco cinghia
e tanto piu dolor, che punge a guaio.

(Inferno,V,1-3)

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INTRODUCTION

1.1 The induction phenomenon

The phenomenon of electromagnetic induction is one of the most important properties of electromagnetic fields. Since Faraday's first experimental investigations, and Maxwell's subsequent A Treatise on Electricity and Magnetism (1873), induction has attracted the interest of researchers in many different scientific contexts. In the last thirty years, with the renewed interest in the earth's magnetic field, and in the response of the earth to external magnetic fields, electromagnetic induction theory has assumed particular importance with respect to geophysical problems.

Perhaps at this point it would be wise to define what is meant by induction. The law of induction is a fundamental relationship between the magnetic and electric fields of a time-variant source; it is simply stated by the Maxwell equation,

$$\text{curl } \underline{E} = - \partial \underline{B}/\partial t \qquad (1.1)$$

where $\underline{E}$ represents the electric field, and $\partial \underline{B}/\partial t$ the time rate of change of the magnetic flux density (which in this thesis, will be called simply the magnetic field). Since we are considering time-varying fields we can omit from our discussion the irrotational, or static, parts of the electromagnetic fields.

Equation (1.1) can be physically interpreted to mean that a time-varying magnetic field generates an associate electric field. To see how the induction phenomenon is contained in this equation, one may build up the Maxwell equation from a causality argument. If a time-varying magnetic source field $\underline{B}^{(s)}$ is in the presence of a conductor, currents will flow as a result of the associate electric field, $\underline{E}^{(1)}$. This is a consequence of Ohm's law for a conductor. $\underline{E}^{(1)}$ is the solution to

$$\text{curl } \underline{E}^{(1)} = - \partial \underline{B}^{(s)}/\partial t \qquad (1.2)$$

The currents in the conductor will themselves establish a magnetic field, $\underline{B}^{(1)}$, which is called the 'induced' magnetic field. Of course the time-derivative of $\underline{B}^{(1)}$ will also give rise to a higher order associate electric field, $\underline{E}^{(2)}$, such that

$$\text{curl } \underline{E}^{(2)} = - \partial \underline{B}^{(1)}/\partial t$$

This procedure may be continued, and if the resulting equations are summed we obtain

$$\text{curl } \underline{E} = - \partial \underline{B}/\partial t$$

where $\underline{E}$ and $\underline{B}$ are the total resultant electric and magnetic fields to arbitrary order, i.e.

$$\underline{E} = \underline{E}^{(1)} + \underline{E}^{(2)} + \underline{E}^{(3)} + \cdots$$

and

$$\underline{B} = \underline{B}^{(s)} + \underline{B}^{(1)} + \underline{B}^{(2)} + \cdots$$

The term, $\underline{B}^{(s)}$, is called the source, or primary, magnetic field. Thus $\underline{B} - \underline{B}^{(s)}$ is the induced, or secondary, magnetic field. Formally, primary and induced electric fields can also be defined, although physically the distinction has less meaning. We define the primary electric field as the electric solution to the Maxwell equation (1.1) in free-space, that is, in the absence of any conducting medium. Calling this primary field $\underline{E}^{(s)}$, then the induced electric field is defined formally as $\underline{E} - \underline{E}^{(s)}$. The solution to equation (1.1) in the presence of a conductor gives $\underline{E}$ rather than $\underline{E}^{(s)}$ since conductivity enters into the boundary conditions for $\underline{E}$ at any conductor/free-space interface.

The induced current in the conductor will flow in such a direction as to produce a magnetic field which will oppose the change in the magnetic flux represented by $\partial\underline{B}/\partial t$ -- hence the minus sign in the Maxwell equation. If equation (1.1) is rewritten in integral form, we obtain

$$\oint \underline{E} \cdot \underline{dl} = -\frac{\partial}{\partial t} \oiint \underline{B} \cdot \underline{n} \, dS \qquad (1.3)$$

where dS is an element of surface, $\underline{n}$ the unit normal vector to that surface, and $\underline{dl}$ is an element of the perimeter enclosing the surface. Equation (1.3) illustrates how the electromotive force generated in a surface (the left-hand side of the equation) is related to the time rate of change of magnetic flux through that surface (the right-hand side of the equation). One can note that the magnetic flux depends only on the normal component of $\underline{B}$ at the surface.

If $\partial\underline{B}/\partial t$ constitutes an abrupt aperiodic change in $\underline{B}$, the induced current will decay freely in the conductor. If the change is oscillatory, then an oscillatory current will be set up in the conductor.

Thus, incorporated into Maxwell's theory is the fact that a time-variant magnetic field induces electric currents in a conducting medium, and these currents in turn establish an induced magnetic field. Both the electric and magnetic fields will be everywhere separable into induced and primary parts. The magnitude of the induced fields will depend both upon the time rate of change of the source magnetic field, and upon the conductivity of the conductor.

1.2 Geophysical applications

The geomagnetic field at the surface of the earth is composed of two distinct parts: the largely static 'main' field of the earth; and a time-variant, or transient, magnetic field.

From a spherical harmonic analysis of the earth's main field, Gauss (1839) could show that, to first order, this was a static magnetic dipole field, such as would be created by a dipole situated at the earth's centre. In order to describe non-dipole parts of the static field, as well as to incorporate surface anomalies, higher order harmonics were considered, however these were found to be relatively small. From his analysis Gauss was able to deduce that the earth's main field was of internal origin.

The time-variant part of the earth's field consists of superposed variations of different frequencies. Schuster in 1889 performed a spherical harmonic analysis on the Diurnal Variation, and showed that this varying field was largely of external origin, but that there was also a significant 'internal' contribution. He showed further that it was not unreasonable to suppose that the internal part was produced by electric currents induced in the earth by the externally varying transient field. A more detailed analysis by Chapman (1919) confirmed that the earth behaved as a conducting sphere in a time-varying external magnetic field. Since these early theoretical studies, 'earth currents' have been widely detected and correlated to external magnetic sources.

The origins of these external magnetic fields are not completely understood, and are the subject of much contemporary research. In general these fields arise from the interaction of the earth's magnetic dipole field with the solar wind. The solar wind is the plasma transport of particles ejected by the sun: the velocity and particle-density of this 'wind', through a series of complicated mechanisms, influence the currents which flow in the ionosphere (which starts at a height of 70 km above the earth). This region is strongly ionized as a result of solar X-ray and ultraviolet absorption in the upper atmosphere. The transient currents in the ionosphere produce source magnetic fields which in turn induce electric currents in the earth.

Temperature gradients in the ionosphere, as well as the solar influence just mentioned, cause ionospheric tides; this transport of ionic material will contribute to the current system. Other source currents are also thought to contribute to the external magnetic fields experienced by the earth, in particular the Ring Current situated some 1500 km from the earth's surface, and the associated mechanics of the radio-active Van Allen Belt. (See Rikitaki, 1966, for a complete discussion of this subject.)

The inductive response of the earth to ionospheric fields will depend on the frequency of the time variation, and on the conductivity of the earth. It's response to sources of known frequency provides a way of inferring the conductivity distribution within the earth. Different frequencies penetrate to different depths -- the depth of penetration is inversely proportional to frequency. By observing the inductive response of a conductor, like the earth, to a number of fields comprising a spectrum of frequencies, a conductivity profile for the earth's interior can be pieced together. Information about the earth's internal conductivity can in turn help to elucidate other aspects of the earth's internal composition.

Since it is not always possible, technically, to separate the induced and primary parts of naturally occuring fields, and thus determine the response of the earth, artificial magnetic sources have also been used to determine conductivity and compositional anomalies below the surface of the earth. A branch of mining technology has grown up around the use of electromagnetic induction in geophysical prospecting. (For further discussion see Keller and Frischknecht, 1966.)

1.3 Theoretical investigations

To understand the response of a given conductivity distribution to a given inducing source, the Maxwell field equations of the system must be solved.

The advent of radio science in this century motivated some mathematical study of various sources above a plane conducting earth (see Sommerfeld, 1926). Typical fields considered were the dipole antennas, horizontal line currents, etc. This research was usually oriented toward specific engineering problems, and often concerned itself with the radiative part of the field, rather than the inductive part. The distinction between these two solutions will be made shortly.

About the same time early theoretical work was done on the problem of naturally induced geomagnetic fields (see for example, Chapman, 1919; Lahiri and Price, 1939). Since these researches confined themselves to problems associated with sources which were expanded in a spherical harmonic series, localized sources were not conveniently considered since high order harmonics are required for such problems.

In 1950 A. T. Price published probably the first general mathematical analysis of the induction problem in a plane earth. His analysis was for a completely arbitrary inducing magnetic source above a homogeneous conducting half-space. We mean by half-space the region defined by $z > 0$ in three-dimensional Cartesian coordinates; by homogeneous, we refer to the conductor's electromagnetic properties:

permittivity, conductivity, and permeability. By avoiding the complications of spherical harmonics, and eliminating the character of the source from the analysis, Price was able to determine some general mathematical properties of induced currents and fields.

His analysis is of practical importance since a plane-earth model is useful when one is considering conductivity anomalies that are sufficiently localized that that the earth's curvature may be ignored. Price's treatment can be applied to various localized source fields as well.

Price began with the four Maxwell equations:

$$\text{curl } \underline{H} = \underline{J} + \partial \underline{D}/\partial t \tag{1.4}$$

$$\text{curl } \underline{E} = - \partial \underline{B}/\partial t \tag{1.5}$$

$$\text{div } \underline{B} = 0 \tag{1.6}$$

$$\text{div } \underline{D} = \rho_c \tag{1.7}$$

In these equations $\underline{H}$ is the magnetic intensity; $\underline{B}$ is the magnetic flux density (or simply the magnetic field); $\underline{E}$ is the electric field intensity; $\underline{D}$ is the electric displacement. The vector $\underline{J}$ is the current volume density;

and ρ_c is the volume charge density. The term $\partial \underline{D}/\partial t$ is called the 'displacement current'.

In a region of conductivity σ, and without source currents, the current density is given by Ohm's Law, namely:

$$\underline{J} = \sigma \underline{E} \qquad (1.8)$$

Also we have the constitutive equations,

$$\underline{B} = \mu \underline{H}$$

and

$$\underline{D} = \varepsilon \underline{E}$$

where μ is the permeability of the region (in free space it is $4\pi \times 10^{-7}$ n/amp) and ε is the permittivity (in free space, about 8.9×10^{-12} f/m).

Price shows that if a charge distribution exists in a conductor, the distribution will decay independently of any field conditions. So without loss of generality, ρ_c may be taken to be zero in the conductor. We consider the free-space region also to be free of charges. If we introduce the vector, $\underline{F}$, which represents either $\underline{E}$ or $\underline{B}$, the four Maxwell equations in an isotropic, source-free region imply the following differential equation:

$$\nabla^2 \underline{F} - \sigma\mu(\partial \underline{F}/\partial t) - \varepsilon\mu(\partial^2 \underline{F}/\partial t^2) = 0 \qquad (1.9)$$

For the induction problem, Price makes two important approximations. In the conductor, the conduction term dominates over the radiative term in equation (1.9). That is to say

$$|\sigma \partial \underline{E}/\partial t| \gg |\varepsilon \partial^2 \underline{E}/\partial t^2| \qquad (1.10)$$

Dimensionally this implies that $T \gg \varepsilon/\sigma$, where T is the period of the changing source. Thus for non-zero conductivity, and for variations slower than 10^{12} Hz, we may expect this first approximation to be valid, remembering that ε is always close to the free-space value of permittivity. Thus in the conductor one obtains the familiar diffusion equation:

$$\nabla^2 \underline{E} - \sigma\mu(\partial \underline{E}/\partial t) = 0 \qquad (\sigma \neq 0) \qquad (1.11)$$

Secondly, Price makes the quasi-static approximation in the free-space region where $\sigma = 0$. Induction phenomena are characterized by the fact that the field changes are slow compared with the time of propogation of an electromagnetic wave in a dielectric -- and this is precisely the quasi-static condition. It is obtained by setting in equation (1.9) the relation,

$$|\nabla^2 \underline{F}| \gg |\mu\varepsilon(\partial^2 \underline{F}/\partial t^2)| \qquad (1.12)$$

Dimensionally, if we define L to be the spatial extent of a region, c to be the velocity of propagation of an electromagnetic wave, and T the period of change of a source, we see that (1.12) implies

$$L/T \ll c \qquad (1.13)$$

This condition can now be written as $L \ll \lambda$, where λ is the wavelength of the electromagnetic field, and for this reason (1.13) is also referred to as the 'near field' approximation. Making the near field approximation isolates the induction solution from the complete radiative solution of equation (1.9). So in free-space one obtains Laplace's equation:

$$\nabla^2 \underline{F} = 0 \qquad (\sigma = 0) \qquad (1.14)$$

It can be seen that equations (1.11) and (1.14) could have been effectively obtained directly from Maxwell's equations by neglecting the displacement current, $\partial \underline{D}/\partial t$, in equation (1.4). So Price, by making the induction approximations, was effectively neglecting the magnetic effects of displacement currents from the field equations. To illustrate this, one can analyse dimensionally Maxwell's equations, and determine that if $|\partial \underline{D}/\partial t| << |\text{curl } \underline{H}|$ in free space, then

$$L/T << c$$

which we obtained previously from (1.12). For ionospheric fields for example, T can be about 10^5 sec, and L (the spatial dimensions of the system) can be about 100 km. Thus $L/T \simeq 1$, which clearly satisfies condition (1.13) since c is about 10^8 m/sec.

Imposing the well-known boundary conditions for $\underline{E}$, $\underline{D}$, $\underline{H}$, and $\underline{B}$ across a discontinuity in conductivity, Price solved the differential equations (1.11) and (1.14) by the method of separation of variables. The solutions he obtained were functions of a separation constant which was real and positive, and the complete solution consisted of a summation (or continuous integration) over all possible values of that constant. His solutions were in terms of functions which had to be identified when specific sources were considered. Two types of elementary solutions were obtained from his analysis. Solutions of the first type were those associated with currents induced by an external magnetic source, and were thus of importance to induction problems. Price observed from his solution that in the conductor the vertical component of the electric field was everywhere zero. This is to say that induced currents flow everywhere parallel to the surface of the conductor. The Solutions of the second type corresponded to free decays of current distributions associated with zero external magnetic field.

The semi-infinite conductor with a plane surface considered by Price is a suitable approximation for problems in which a sufficiently small area on the surface of a finite sphere is examined. However, the fact that the plane is infinite restricts the applicability of his treatment. In particular the problem of a uniform field is indeterminate since a uniform field over an infinite plane surface implies a field infinite in extent. At any point it is then impossible to determine the contribution due to the field at infinity. Thus the problems of 'sheet current' or plane waves for example, cannot be solved with the plane-earth formulation. A uniform field surrounding a finite sphere, can of course, be solved.

It is experimentally evident that the earth is not homogeneous with respect to conductivity. Lahiri and Price (1939) considered a continuous radial conductivity distribution for the earth such that

$$\sigma \propto r^{-1}$$

where r is the radial distance from the earth's centre, and σ is the conductivity which was considered to be finite at the centre. It is possible however that the earth's conductivity is not continuous, but that there may be sharp discontinuities; the conductivity profile likely is horizontally stratified. Thus the problem of induction in a layered earth is an important one.

Since Price's 1950 paper much work has been done to apply induction theory to various source fields (see Ward, 1967, for references to this work which are too numerous to mention here). Weaver (1964) considerably modernized Price's general theory by solving the differential equations employing Fourier transforms. Price's integral solutions now appeared quite naturally as Fourier integrals. Also in this work, Weaver extended the solutions to the problem of a two-layer conductor. However the extension of Price's theory to the problem of a conductor with an arbitrary number of layers results in an algebraically intractable problem.

A number of researchers have considered the radiative solutions for the N-layer problem (examples of these are: Wait, 1962; Dosso, 1966; Mundry, 1967; Ward, 1967; Bannister, 1968; Guldberg and Brock-Nannestad, 1970). This approach involves summing an infinite number of plane waves incident on the surface of a conductor over a range of angles. If the near-field approximation in the final wave equations is made, finite inducing sources can be represented. However it is very difficult to take the near-field limit in the complicated radiative solutions which result in the N-layer treatment. At any rate the method does not seem to have been applied practically to half-spaces consisting of more than four layers.

Schmucker (1970) has derived a technique for determining the attenuation of naturally induced fields for an N-layered conducting earth, in terms of the measured field values at the surface. However he does not solve generally for the induced field in terms of the inducing field.

Mathematically it would be more elegant to try and solve directly equations (1.11) and (1.14) generally for the case of an N-layered conducting half-space, analogous to Price's solution for the homogeneous case.

1.4 The Hertz potential formalism

For an isotropic medium the solution to the electromagnetic field consists in solving Maxwell's equations for the six electromagnetic field components of $\underline{E}$ and $\underline{B}$. In the previous section, it was indicated that when the four Maxwell equations were combined, they separate naturally into two vector second-order differential equations (that is to say, six scalar second-order differential equations).

In fact, the representation of the electromagnetic field is not unique, and the task of solving for the field can be simplified by choosing an alternative representation. If we consider any scalar function ϕ , and any vector function $\underline{A}$, such that

$$\underline{E} = -\nabla\phi - \partial\underline{A}/\partial t \tag{1.15}$$

and

$$\underline{B} = \text{curl}\ \underline{A} \tag{1.16}$$

then the problem of solving for the six components of the field can be reduced to the problem of solving for three scalar components of $\underline{A}$, and the scalar function ϕ . The transformation implicit in equations (1.15) and (1.16) is such that both ϕ and $\underline{A}$ have some degree of arbitrariness. In particular the representation is unaffected by the Gauge transformation which is defined by

$$\underline{A}' = \underline{A} + \nabla f$$

$$\phi' = \phi + \partial f/\partial t$$

where f is a completely arbitrary scalar function. When Maxwell's equations are expressed in terms of $\underline{A}$ and ϕ , they cannot be combined into two naturally separated second-order differential wave equations in ϕ and $\underline{A}$. However, by imposing the additional relationship (called the Lorentz condition)

$$\text{div}\,\underline{A} + \mu\sigma\,\phi + \mu\varepsilon\,\partial\phi/\partial t = 0 \qquad (1.17)$$

this 'separation' can be effected and we obtain the equations

$$\nabla^2\underline{A} - \mu\varepsilon\,\partial^2\underline{A}/\partial t^2 - \mu\sigma\,\partial\underline{A}/\partial t = 0 \qquad (1.18)$$

and

$$\nabla^2\phi - \mu\varepsilon\,\partial^2\phi/\partial t^2 - \mu\sigma\,\partial\phi/\partial t = 0 \qquad (1.19)$$

with relevant boundary conditions. We have reduced the problem of solving for six scalar variables to one of solving for four scalar variables. Imposing the Lorentz condition (1.17) has the effect of restricting the arbitrariness of the scalar function f of the Gauge transformation, since it now must satisfy

$$\nabla^2 f - \mu\varepsilon\,\partial^2 f/\partial t^2 - \mu\sigma\,\partial f/\partial t = 0$$

Still another representation can be chosen to simplify further the problem. We make the following transformation (see Jones, 1964):

$$\phi = - \text{div } \underline{\Pi} \qquad (1.20)$$

and

$$\underline{A} = \text{curl } \underline{\Gamma} + \mu\sigma \underline{\Pi} + \mu\varepsilon\, \partial\underline{\Pi}/\partial t \qquad (1.21)$$

where $\underline{\Pi}$ is called the electric Hertz potential, and $\underline{\Gamma}$, the magnetic Hertz potential; these vectors are both directed along the z-axis, the orientation of which can be chosen at will for a particular problem. If $\hat{k}$ is the unit vector in the z-direction, we can write $\underline{\Pi} = \Pi\hat{k}$ and $\underline{\Gamma} = \Gamma\hat{k}$, and if we substitute (1.20) and (1.21) into equations (1.18) and (1.19), we find the problem of solving for the field separates itself naturally into finding the solutions to the following two scalar differential equations

$$\nabla^2\Pi - \mu\varepsilon\, \partial^2\Pi/\partial t^2 - \mu\sigma\, \partial\Pi/\partial t = 0 \qquad (1.22)$$

and

$$\nabla^2\Gamma - \mu\varepsilon\, \partial^2\Gamma/\partial t^2 - \mu\sigma\, \partial\Gamma/\partial t = 0 \qquad (1.23)$$

with their suitable boundary conditions. Our original problem of solving for six scalar variables has been reduced to a problem of solving for two scalar variables. Implicit in the transformation of (1.20) and (1.21) is the fact that the variable f is now specified uniquely, and thus is no longer in any way arbitrary.

Weaver (1971) appreciated the simplification the Hertz potential representation afforded to the induction problem, and rewrote the general induction theory over a homogeneous half-space in terms of that formalism. Of course for the induction problem the 'radiation terms' can be neglected and we need only solve (instead of (1.22) and (1.23)) the following diffusion equations:

$$\nabla^2\Pi - \mu\sigma\ \partial\Pi/\partial t = 0 \qquad (1.24)$$

and

$$\nabla^2\Gamma - \mu\sigma\ \partial\Gamma/\partial t = 0 \qquad (1.25)$$

An important motivation for considering such a simplified representation of the field, is that it may render tractable the problem of induction in a half-space of N layers. It is the object of this thesis to apply the Hertz potential formalism to this problem.

1.5 Summary of work covered in this thesis

In this thesis we develop a theory for electromagnetic induction over a plane N-layered conductor, due to an external magnetic source which is finite and time-harmonic. The diagram in Figure 1. illustrates the coordinate system we are considering; it will be noticed that the z-axis is positive in the downward direction. The source is situated at a height of $z = -h$; the bottom layer extends to positive infinity.

The electromagnetic fields are expressed in terms of the electric and magnetic Hertz potentials in equations (2.16) and (2.17). Solving the induction problem for a system of N layers consists in solving the two differential equations (2.10) and (2.21) in each of the layers, and applying boundary conditions across each of the N interfaces. In expressions (2.27) and (2.28) the Fourier transforms of Π and Γ are obtained in terms of unknown coefficient functions. These coefficients are determined by imposing the boundary conditions (which themselves must be expressed in terms of the Fourier-transformed Hertz potentials). These boundary conditions are stated in equations (2.45) through (2.48), and state the usual properties of an electromagnetic field at a boundary: viz. the tangential components of $\underline{E}$ and $\underline{H}$, and the normal component of $\underline{B}$ are continuous. They also imply that the source is finite, and that $\underline{E}$ and $\underline{B}$ vanish as either $|x|$ or $|y|$ approaches infinity.

We have across each interface four conditions; we also have the condition that the inducing field vanishes as z approaches positive infinity. So for each Hertz potential (Γ for example) we have a problem of solving a system of 2N+2 coefficients, subject to 2N+1 boundary conditions. This system can be solved for the Hertz potentials anywhere, parametrically in terms of the Hertz potential of the source.

The theory is found to be consistent with the previous results of Weaver (1964) for the case of a two-layer conducting half-space. The theory is illustrated by considering several particular source fields and by solving the problem numerically for various conductivity distributions in the N-layered half-space.

CHAPTER 2

FORMULATING THE PROBLEM

2.1 The mathematical model

The N-layered half-space occupying the region $z > 0$ in a right-handed Cartesian coordinate system (x,y,z) is illustrated in Figure 1.

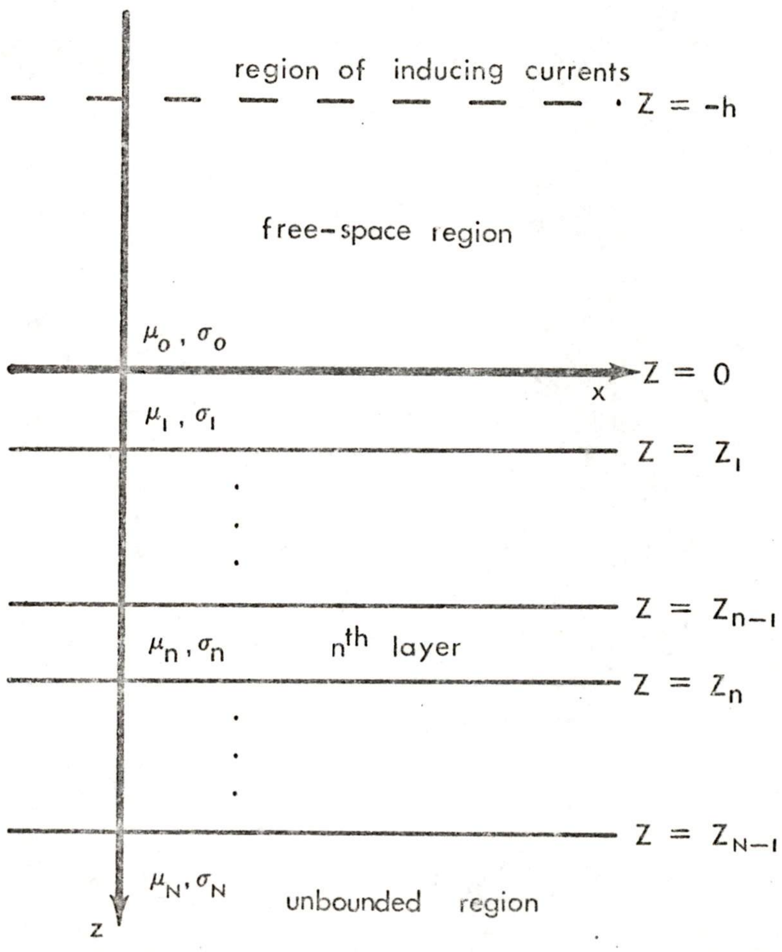

Figure 1. The N-layered half-space

We pointed out in Section 1.4 of the previous Chapter that the directions of the Hertz potentials $\underline{\Pi}$ and $\underline{\Gamma}$ are arbitrary. For our problem we choose them to be in the $\hat{k}$-direction, where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors in the x-, y-, and z-directions respectively. The region $-h < z < 0$ is assumed to be free-space, i.e. $\sigma_0 = 0$; the conductivity of the region $z > 0$ is assumed to be non-zero. We consider the source to be located in the region $z \leq -h$.

We consider the equations satisfied by a quasi-static electromagnetic field in the n^{th} layer $(n=0,1,...,N)$. Neglecting displacement currents in Maxwell's equations (1.4) and (1.5), we have

$$\text{curl } \underline{E} = -\partial\underline{B}/\partial t \tag{2.1}$$

and

$$\text{curl } \underline{B} = \mu_n\sigma_n \underline{E} \tag{2.2}$$

We introduce the magnetic vector potential $\underline{A}$, and the electric scalar potential ϕ, defined by (1.15) and (1.16). For a quasi-static field the appropriate Lorentz condition relevant to the n^{th} region, is obtained by neglecting the final term in (1.17) which arises from displacement currents. Thus in the n^{th} layer we have

$$\text{div}\ \underline{A} + \mu_n\sigma_n\phi = 0 \tag{2.3}$$

Expressing $\underline{E}$ and $\underline{B}$ in (2.1) and (2.2) in terms of $\underline{A}$ and ϕ , and imposing the condition (2.3), we obtain the following electromagnetic diffusion equations:

$$\nabla^2\phi = \mu_n\sigma_n\ \partial\phi/\partial t \tag{2.4}$$

$$\nabla^2\underline{A} = \mu_n\sigma_n\ \partial\underline{A}/\partial t \tag{2.5}$$

The quasi-static electric vector Hertz potential $\underline{\Pi}$, and magnetic Hertz potential vector $\underline{\Gamma}$, in the n^{th} region are defined by

$$\phi = -\ \text{div}\ \underline{\Pi} \tag{2.6}$$

$$\underline{A} = \mu_n\sigma_n\underline{\Pi} + \text{curl}\ \underline{\Gamma} \tag{2.7}$$

(Compare with (1.20) and (1.21).) Substituting the expressions (2.6) and (2.7) into the diffusion equations (2.4) and (2.5), a set of diffusion equations in $\underline{\Pi}$ and $\underline{\Gamma}$ results:

$$\nabla^2\underline{\Pi} = \mu_n\sigma_n\ \partial\underline{\Pi}/\partial t \tag{2.8}$$

$$\nabla^2 \underline{\Gamma} = \mu_n \sigma_n \; \partial \underline{\Gamma} / \partial t \qquad (2.9)$$

Since $\underline{\Pi}$ and $\underline{\Gamma}$ have been chosen to be in the $\hat{k}$-direction, equations (1.10) and (1.11) may be rewritten as scalar equations

$$\nabla^2 \Pi = \mu_n \sigma_n \; \partial \Pi / \partial t \qquad (2.10)$$

$$\nabla^2 \Gamma = \mu_n \sigma_n \; \partial \Gamma / \partial t \qquad (2.11)$$

where Π and Γ are the magnitudes of the Hertz vectors and will be called the electric and magnetic Hertz potentials in subsequent discussion.

The fields in any of the layered regions of the half-space are specified by the two Hertz potentials Π and Γ in those regions. Substituting the values of $\underline{A}$ and ϕ from (2.6) and (2.7) into (1.15) and (1.16), the electromagnetic fields $\underline{E}$ and $\underline{B}$ may be expressed in terms of these Hertz potentials

$$\underline{E} = \text{curl curl } (\Pi\hat{k}) - \text{curl } (\hat{k} \; \partial\Gamma/\partial t) \qquad (2.12)$$

$$\underline{B} = \text{curl curl } (\Gamma\hat{k}) + \mu_n \sigma_n \text{ curl } (\Pi\hat{k}) \qquad (2.13)$$

Expanding these vector operations explicitly, we obtain

$$\underline{E} = \left(\frac{\partial^2\Pi}{\partial z\partial x} - \frac{\partial^2\Gamma}{\partial y\partial t}\right)\hat{i} + \left(\frac{\partial^2\Pi}{\partial z\partial y} + \frac{\partial^2\Gamma}{\partial x\partial t}\right)\hat{j} - \left(\frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2}\right)\hat{k} \qquad (2.14)$$

and

$$\underline{B} = \left(\frac{\partial^2\Gamma}{\partial z\partial x} + \mu_n\sigma_n\frac{\partial\Pi}{\partial y}\right)\hat{i} + \left(\frac{\partial^2\Gamma}{\partial z\partial y} - \mu_n\sigma_n\frac{\partial\Pi}{\partial x}\right)\hat{j} - \left(\frac{\partial^2\Gamma}{\partial x^2} + \frac{\partial^2\Gamma}{\partial y^2}\right)\hat{k} \qquad (2.15)$$

These equations are valid in the n^{th} region where Π and Γ satisfy (2.10) and (2.11). It has been pointed out by Weaver (1971) that the "Π-field" has no magnetic $\hat{k}$-component, and the "Γ-field" has no electric $\hat{k}$-component; in the case where $\sigma_n=0$, $\underline{B}$ is specified entirely by Γ .

2.2 Solving the diffusion equations

The time-dependence of Π and Γ is assumed to be periodic with angular frequency ω , so these Hertz potentials may be written as

$$\Pi(\underline{r},z,t) = \Pi(\underline{r},z)e^{i\omega t} \qquad (2.16)$$

and

$$\Gamma(\underline{r},z,t) = \Gamma(\underline{r},z)e^{i\omega t} \qquad (2.17)$$

where $\underline{r} = x\hat{i} + y\hat{j}$. In this notation, $\partial/\partial t \equiv i\omega$. So equations (2.10) and (2.11) become

$$\nabla^2\Pi(\underline{r},z) - i\omega\mu_n\sigma_n\Pi(\underline{r},z) = 0 \quad , \qquad (2.18)$$

and

$$\nabla^2\Gamma(\underline{r},z) - i\omega\mu_n\sigma_n\Gamma(\underline{r},z) = 0 \quad . \qquad (2.19)$$

The time-dependence has been factored out of (2.18) and (2.19). Taking the double Fourier transform of (2.18) and (2.19), and using the fact that $\Pi \to 0$ and $\Gamma \to 0$ as $r \to \infty$, we arrive at the following relations (see Appendix A):

$$P_{33}(\underline{\rho},z) = \nu_n^2 P(\underline{\rho},z) \tag{2.20}$$

$$G_{33}(\underline{\rho},z) = \nu_n^2 G(\underline{\rho},z) \tag{2.21}$$

where $P(\underline{\rho},z)$ and $G(\underline{\rho},z)$ are the Fourier transforms of the Hertz potentials and are defined by

$$P(\underline{\rho},z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Pi(\underline{r},z)e^{i\underline{r}\cdot\underline{\rho}}\, d\underline{r} \tag{2.22}$$

and

$$G(\underline{\rho},z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(\underline{r},z)e^{i\underline{r}\cdot\underline{\rho}}\, d\underline{r} \tag{2.23}$$

ν_n is defined by

$$\nu_n \equiv (\rho^2 + i\omega\mu_n\sigma_n)^{\frac{1}{2}} \tag{2.24}$$

$\underline{\rho} = \xi\hat{i} + \eta\hat{j}$ where ξ and η are transform-space variables and the subscripted G and P indicate partial differentiation in the Landau notation (e.g. $\Gamma_3(\underline{r},z) \equiv \partial\Gamma/\partial z$, $\Gamma_{12}(\underline{r},z) \equiv \partial^2\Gamma/\partial y\partial x$, etc.).

The solutions to equations (2.20) and (2.21) are:

$$P(\underline{\rho},z) = a_n(\underline{\rho})e^{\nu_n z} + b_n(\underline{\rho})e^{-\nu_n z} \tag{2.25}$$

and

$$G(\underline{\rho},z) = c_n(\underline{\rho})e^{\nu_n z} + d_n(\underline{\rho})e^{-\nu_n z} \tag{2.26}$$

The coefficients $a_n(\underline{\rho})$, $b_n(\underline{\rho})$, $c_n(\underline{\rho})$, and $d_n(\underline{\rho})$ are determined by the boundary conditions associated with the field at the boundaries of the n^{th} region.

2.3 Boundary conditions

The boundary conditions at an interface $z = z_n$ can be derived in terms of electric and magnetic Hertz potentials by a procedure identical to that developed by Weaver (1971) for the homogeneous half-space. Functions $\Lambda(\underline{r})$, $T(\underline{r})$, $\Psi(\underline{r})$, and $\Theta(\underline{r})$ are defined such that

$$\Gamma(\underline{r}, z_n+0) - \Gamma(\underline{r}, z_n-0) = \Lambda(\underline{r}) \tag{2.27}$$

$$\frac{1}{\mu_{n+1}}\Gamma_3(r, z_n+0) - \frac{1}{\mu_n}\Gamma_3(\underline{r}, z_n-0) = T(\underline{r}) \tag{2.28}$$

$$\Pi_3(\underline{r}, z_n+0) - \Pi_3(\underline{r}, z_n-0) = \Psi(\underline{r}) \tag{2.29}$$

$$\sigma_{n+1}\Pi(\underline{r}, z_n+0) - \sigma_n\Pi(\underline{r}, z_n-0) = \Theta(\underline{r}) \tag{2.30}$$

The electromagnetic field boundary conditions may now be imposed across $z = z_n$. The vertical component of $\underline{B}$ and the tangential components of $\underline{H}$ are continuous across the interface, so from equations (2.14) and (2.15) we have:

$$\Lambda_{11}(\underline{r}) + \Lambda_{22}(\underline{r}) = 0 \tag{2.31}$$

$$T_1(\underline{r}) + \Theta_2(\underline{r}) = 0 \tag{2.32}$$

$$T_2(\underline{r}) - \Theta_1(\underline{r}) = 0 \tag{2.33}$$

Differentiating (2.32) with respect to x, and (2.33) with respect to y, and adding the resulting expressions, we obtain:

$$T_{11}(\underline{r}) + T_{22}(\underline{r}) = 0 \tag{2.34}$$

By differentiating (2.32) with respect to y and (2.33) with respect to x and subtracting the resulting expressions, we obtain:

$$\Theta_{11}(\underline{r}) + \Theta_{22}(\underline{r}) = 0 \tag{2.35}$$

Equating the tangential components of $\underline{E}$ across the interface we see that

$$\Psi_{1}(\underline{r}) - i\omega\Lambda_{2}(\underline{r}) = 0 \tag{2.36}$$

and

$$\Psi_{2}(\underline{r}) + i\omega\Lambda_{1}(\underline{r}) = 0 \tag{2.37}$$

By differentiating (2.36) with respect to x, and (2.37) with respect to y and adding the resulting expressions, we get

$$\Psi_{11}(\underline{r}) + \Psi_{22}(\underline{r}) = 0 \tag{2.38}$$

Therefore, from (2.31), (2.34), (2.35) and (2.38), it is seen that $\Lambda(\underline{r})$, $T(\underline{r})$, $\Theta(\underline{r})$, and $\Psi(\underline{r})$ all satisfy Laplace's equation in two dimensions; also they all tend to zero as $r \to \infty$, since we are considering finite source fields. The only solution to Laplace's equation for which this is true, is the trivial solution. Thus $\Lambda(\underline{r})$, $T(\underline{r})$, $\Theta(\underline{r})$, and $\Psi(\underline{r})$ all vanish identically. Equations (2.27) through (2.30) can now be rewritten:

$$\Gamma(\underline{r},z_n+0) = \Gamma(\underline{r},z_n-0) \tag{2.39}$$

$$\frac{1}{\mu_{n+1}}\Gamma_3(r,z_n+0) = \frac{1}{\mu_n}\Gamma_3(r,z_n-0) \tag{2.40}$$

$$\Pi_3(\underline{r},z_n+0) = \Pi_3(\underline{r},z_n-0) \tag{2.41}$$

$$\sigma_{n+1}\Pi(r,z_n+0) = \sigma_n\Pi(r,z_n-0) \tag{2.42}$$

Taking the Fourier transforms of these boundary condition equations, we obtain finally:

$$G(\rho,z_n+0) = G(\underline{\rho},z_n-0) \tag{2.43}$$

$$\frac{1}{\mu_{n+1}} G_3(\underline{\rho},z_n+0) = \frac{1}{\mu_n}G_3(\underline{\rho},z_n-0) \tag{2.44}$$

$$P_3(\underline{\rho},z_n+0) = P_3(\underline{\rho},z_n-0) \tag{2.45}$$

$$\sigma_{n+1} \quad P(\underline{\rho},z_n+0) = \sigma_n \; P(\underline{\rho},z_n-0) \tag{2.46}$$

CHAPTER 3

THE RECURSION RELATION

3.1 Developing the recursion relation

Equations (2.25) and (2.26) express $G(\underline{\rho},z)$ and $P(\underline{\rho},z)$, the Fourier transforms of the magnetic and electric Hertz potentials respectively, in terms of coefficients $a_n(\underline{\rho})$, $b_n(\underline{\rho})$, $c_n(\underline{\rho})$, and $d_n(\underline{\rho})$, which are themselves determined by the boundary conditions relevant to the n^{th} region. It is convenient at this time to eliminate these coefficients from the formalism, and express all quantities in terms of $G(\underline{\rho},z_n)$, $P(\underline{\rho},z_n)$, $G(\underline{\rho},z_{n-1})$, and $P(\underline{\rho},z_{n-1})$; that is to say, entirely in terms of the Hertz potentials evaluated at the boundaries of the n^{th} region.

It may be observed that $G(\underline{\rho},z)$ and $P(\underline{\rho},z)$ both satisfy the same diffusion equation, (2.20) and (2.21), but subject to different boundary conditions. In order to encompass both solutions within the one treatment, it is convenient to consider a general set of boundary conditions which may be made to express those satisfied by $G(\underline{\rho},z)$ or $P(\underline{\rho},z)$.

Thus, consider a general function $F(\underline{\rho},z)$ which in the layer $z_{n-1} < z < z_n$ satisfies the differential equation,

$$F_{33}(\underline{\rho},z) = \nu_n^2 F(\underline{\rho},z) \tag{3.1}$$

and the general homogeneous boundary conditions

$$a_{n+1}F(\underline{\rho},z_n+0) - a_nF(\underline{\rho},z_n-0) = 0 \tag{3.2}$$

$$b_{n+1}F_3(\underline{\rho},z_n+0) - b_nF_3(\underline{\rho},z_n-0) = 0 \tag{3.3}$$

By an appropriate choice of a_n and b_n in (3.2) and (3.3) we can express the boundary conditions (2.43) through (2.46).

The solution to (3.1) in the n^{th} layer, $1 \leq n \leq N-1$, is

$$F(\underline{\rho},z) = e_n(\underline{\rho})e^{\nu_n z} + f_n(\underline{\rho})e^{-\nu_n z} \tag{3.4}$$

The cases for $n = 0$ and $n = N$ (the free-space region, and the bottom layer) require special treatment and will be considered separately later.

For convenience the dependence of $e_n(\underline{\rho})$, $f_n(\underline{\rho})$, and $F(\underline{\rho},z)$ on $\underline{\rho}$ will be suppressed in the notation, and the z-derivatives of F will be denoted by primes in the usual way. Thus evaluating (3.4) at $z = z_n-0$ and at $z = z_{n-1}+0$, we obtain

$$F(z_n-0) = e_n e^{\nu_n z_n} + f_n e^{-\nu_n z_n} \tag{3.5}$$

and

$$F(z_{n-1}+0) = e_n e^{\nu_n z_{n-1}} + f_n e^{-\nu_n z_{n-1}} \tag{3.6}$$

The coefficients e_n and f_n may be eliminated from (3.5) and (3.6) to yield

$$F(z_n-0)e^{\nu_n z_{n-1}} - F(z_{n-1}+0)e^{\nu_n z_n} = -f_n\left\{e^{\nu_n(z_n-z_{n-1})} - e^{-\nu_n(z_n-z_{n-1})}\right\} \tag{3.7}$$

Defining $d_n \equiv z_n - z_{n-1}$, the depth of the n^{th} region, and rearranging (3.7), we obtain

$$f_n = \frac{F(z_{n-1}+0)e^{\nu_n z_n} - F(z_n-0)e^{\nu_n z_{n-1}}}{2\sinh \nu_n d_n}$$

and

$$e_n = \frac{F(z_n-0)e^{-\nu_n z_{n-1}} - F(z_{n-1}+0)e^{-\nu_n z_n}}{2\sinh \nu_n d_n} \tag{3.8}$$

After substituting the expressions (3.8) and suitably re-arranging the exponential terms, equation (3.4) becomes

$$F(z) = \frac{F(z_n-0)\sinh \nu_n(z-z_{n-1}) + F(z_{n-1}+0)\sinh \nu_n(z_n-z)}{\sinh \nu_n d_n} \tag{3.9}$$

Multiplying (3.9) by b_n, differentiating with respect to z, and evaluating the resulting expression at $z = z_n - 0$, we obtain

$$b_n F'(z_n-0) = \frac{b_n\nu_n F(z_n-0)\cosh \nu_n d_n - b_n\nu_n F(z_{n-1}+0)}{\sinh \nu_n d_n} \tag{3.10}$$

The corresponding expression applicable to the $n+1^{th}$ region, $z_n < z < z_{n+1}$, is found by replacing n by n+1 in (3.9) (provided that $1 \le n+1 \le N-1$). Differentiating the expression which results and evaluating it at $z = z_n+0$, we then deduce that for $0 \le n \le N-2$,

$$b_{n+1}F'(z_n+0) = \frac{b_{n+1}\nu_{n+1}F(z_{n+1}-0) - b_{n+1}\nu_{n+1}F(z_n+0)\cosh \nu_{n+1}d_{n+1}}{\sinh \nu_{n+1}d_{n+1}} \tag{3.11}$$

The second boundary condition (3.3) demands the equality of the expressions (3.10) and (3.11), that is

$$- b_n\nu_nF(z_{n-1}+0)\mathrm{csch}(\nu_nd_n) + b_n\nu_nF(z_n-0)\coth(\nu_nd_n)$$

$$= - b_{n+1}\nu_{n+1}F(z_n+0)\coth(\nu_{n+1}d_{n+1}) + b_{n+1}\nu_{n+1}F(z_{n+1}-0)\mathrm{csch}\nu_{n+1}d_{n+1} \tag{3.12}$$

When the first boundary condition (3.2) is multiplied by the factor $b_n\nu_n\coth(\nu_nd_n)$, it follows that

$$a_{n+1}b_n\nu_nF(z_n+0)\coth\nu_nd_n - a_nb_n\nu_nF(z_n-0)\coth \nu_nd_n = 0 \tag{3.13}$$

Multiplying (3.12) by a_n and adding to it (3.13), we obtain:

$$-a_nb_n\nu_nF(z_{n-1}+0)\mathrm{csch}(\nu_nd_n) + a_{n+1}b_n\nu_nF(z_n+0)\coth(\nu_nd_n)$$

$$+ a_nb_{n+1}\nu_{n+1}F(z_n+0)\coth(\nu_{n+1}d_{n+1})$$

$$- a_nb_{n+1}\nu_{n+1}F(z_{n+1}-0)\mathrm{csch}(\nu_{n+1}d_{n+1}) = 0 \tag{3.14}$$

Defining u_n and v_n by

$$u_n \equiv \nu_n \operatorname{csch}(\nu_n d_n)$$

$$v_n \equiv \cosh(\nu_n d_n)$$

we can write equation (3.14) as

$$a_n b_n u_n F(z_{n-1}+0) - \left[a_{n+1} b_n u_n v_n + a_n b_{n+1} u_{n+1} v_{n+1}\right] F(z_n+0) + a_n b_{n+1} u_{n+1} F(z_{n+1}-0) = 0 \tag{3.15}$$

Boundary condition (3.2) relevant to the interface $z=z_{n+1}$, and multiplied through by the factor $a_n b_{n+1} u_{n+1}$, is:

$$a_n a_{n+2} b_{n+1} u_{n+1} F(z_{n+1}+0) - a_n a_{n+1} b_{n+1} u_{n+1} F(z_{n+1}-0) = 0 \tag{3.16}$$

Multiplying (3.14) by a_{n+1} and subtracting the resulting expression from (3.16) we obtain:

$$a_n a_{n+1} b_n u_n F(z_{n-1}+0) - \left[a_{n+1}{}^2 b_n u_n v_n + a_n a_{n+1} b_{n+1} u_{n+1} v_{n+1}\right] F(z_n+0) + a_n a_{n+2} b_{n+1} u_{n+1} F(z_{n+1}+0) = 0 \tag{3.17}$$

Equation (3.17) is the recursion relation for $F(z_n)$, applicable when $1 \leq n \leq N-2$.

Let us now consider the N^{th} region, $z_{N-1} < z < +\infty$. We may assume F(z) satisfies the condition

$$\lim_{z\to\infty} F(z) = 0$$

since the general function F(z) is designed to represent either $P(\underline{\rho},z)$ or $G(\underline{\rho},z)$, both of which vanish as $z \to +\infty$, that is, as the distance from the inducing source becomes great. Moreover, since the bottom layer is unbounded, d_N is infinite and we have

$$u_N = \lim_{d_N \to \infty} \operatorname{csch}(\nu_N d_N) = 0$$

$$u_N v_N = \lim_{d_N \to \infty} \nu_N \coth(\nu_N d_N) = \nu_N$$

Thus the N^{th} recursion relation becomes:

$$a_{N-1} a_N b_{N-1} u_{N-1} F(z_{N-2}+0) - \left[a_N^2 b_{N-1} u_{N-1} v_{N-1} + a_{N-1} a_N b_N \nu_N \right] F(z_{N-1}+0) = 0 \qquad (3.18)$$

The expression for F(z) in the N^{th} region may be determined by the same procedure as that which lead to (3.9). Since $\lim_{z \to \infty} F(z) = 0$, F(z) may be written,

$$F(z) = ke^{-\nu_N z}$$

Thus, at $z = z_{N-1}+0$,

$$F(z_{N-1}+0) = ke^{-\nu_N z_{N-1}}$$

and we may write F(z) in the N^{th} region as

$$F(z) = F(z_{N-1}+0)e^{\nu_N(z_{N-1} - z)} \qquad (3.19)$$

So far we have considered the recursion relation in the n^{th} region bounded by $z = z_{n-1}$ and $z = z_n$ $(1 \leq n \leq N-1)$ and we have also considered the unbounded N^{th} region beneath the plane $z = z_{N-1}$. To complete the analysis we must now consider the surface boundary conditions, at $z = 0$. Equations (3.2) and (3.3) can be written as follows:

$$a_1 F(+0) - a_o F(-0) = 0 \qquad (3.20)$$

$$b_1 F'(+0) - b_o F'(-0) = 0 \qquad (3.21)$$

Again, since the function F(z) is designed to represent $P(\underline{\rho},z)$ and $G(\underline{\rho},z)$ it is convenient to separate the field in the free space region into two parts, one associated with the source, and the other with the induced currents.

Mathematically, this is equivalent to writing

$$F(z) = F^s(z) + F^i(z) \qquad (3.22)$$

where

$$\lim_{z \to -\infty} F^i(z) = 0 \qquad (3.23)$$

and

$$\lim_{z \to +\infty} F^s(z) = 0 \qquad (3.24)$$

The function $F^s(z)$ represents the Fourier transforms of the Hertz potentials associated with the source which is assumed to be a known function. The function $F^i(z)$ represents the Fourier transform of either Hertz potential associated with the induced field.

The equivalent form of (3.4) giving the solution in the region $-h < z < 0$, is

$$F(z) = F^i(-0)e^{\nu_o z} + F^s(-0)e^{-\nu_o z} \qquad (3.25)$$

where the coefficients may be written in this form by virtue of (3.23) and (3.24) above. At the surface $z = -0$, the following equations are true:

$$F(-0) = F^i(-0) + F^s(-0) \qquad (3.26)$$

$$F'(-0) = \nu_o\left[F^i(-0) - F^s(-0)\right] \qquad (3.27)$$

From equations (3.17), (3.21), and (3.22), it follows that

$$a_1 b_o \nu_o F(+0) = b_o a_o \nu_o\left[F^i(-0) + F^s(-0)\right] \qquad (3.28)$$

$$a_o b_1 F'(+0) = a_o b_o \nu_o\left[F^i(-0) - F^s(-0)\right] \qquad (3.29)$$

Subtracting (3.29) from (3.28) we obtain the result

$$a_1 b_o \nu_o F(+0) - a_o b_1 F'(+0) = 2a_o b_o \nu_o F^s(-0) \qquad (3.30)$$

Putting $n = 0$ in (3.11) and noting that $z_o = 0$, we deduce

$$\begin{aligned} b_1 F'(+0) &= b_1 \nu_1 F(z_1 - 0)\operatorname{csch}(\nu_1 d_1) - b_1 \nu_1 F(+0)\coth(\nu_1 d_1) \\ &= b_1 u_1 F(z_1 - 0) - b_1 u_1 v_1 F(+0) \end{aligned} \qquad (3.31)$$

If we insert this expression for $b_1F'(+0)$ into the previous equation (3.30), we obtain:

$$a_1b_o\nu_1\ F(+0)\ -\ a_ob_1\left(u_1F(z_1-0)\ -\ u_1v_1F(+0)\right)$$
$$=\ 2a_ob_o\nu_o\ F^s(-0) \qquad (3.32)$$

From (3.2) we know the following must be true:

$$a_oa_1b_1u_1\ F(z_1-0)\ -\ a_oa_2b_1u_1\ F(z_1+0)\ =\ 0 \qquad (3.33)$$

Multiplying equation (3.32) by a_1 and adding to the resulting expression equation (3.26), we obtain:

$$a_oa_2b_1u_1\ F(z_1+0)\ -\ \left(a_1{}^2b_o\nu_o\ +\ a_oa_1b_1u_1v_1\right)F(+0)$$
$$=\ -\ 2a_oa_1b_o\nu_o\ F^s(-0) \qquad (3.34)$$

Equations (3.15), (3.18), and (3.34) determine a set of N equations in the N unknown variables $F(z_n+0)$, $n = 1,2,\ldots,N-1$.

The system may be written compactly by making the following definitions:

$$
\left.
\begin{aligned}
B_1 &= -\ a_1{}^2 b_o \nu_o \ -\ a_o a_1 b_1 u_1 v_1 \\
C_1 &= a_o a_2 b_1 u_1 \\
A_N &= a_{N-1} a_N b_{N-1} u_{N-1} \\
B_N &= -\ a_N{}^2 b_{N-1} u_{N-1} v_{N-1} \ -\ a_{N-1} a_N b_{N\ N}
\end{aligned}
\right\}
$$

and for n = 2,3,...,N-1, we have

$$A_n = a_{n-1} a_n b_{n-1} u_{n-1}$$

$$B_n = -\ a_n{}^2 b_{n-1} u_{n-1} v_{n-1} \ -\ a_{n-1} a_n b_n u_n v_n \qquad (3.35)$$

$$C_n = a_{n-1} a_{n+1} b_n u_n$$

We also denote the unknown variable by w_n, i.e.

$$w_n = F(z_{n-1}+0)$$

and denote the source function by w^s, i.e.

$$w^s = -\ 2a_o a_1 b_o \nu_o \ F^s(-0)$$

The system of equations which we must solve is thus simplified to the matrix equation,

$$\underline{\underline{S}} \cdot \underline{w} = w^s\ \underline{e} \qquad (3.36)$$

where

$$\underline{\underline{S}} = \begin{bmatrix} B_1 & C_1 & & & & & \\ A_2 & B_2 & C_2 & & & & \\ & A_3 & B_3 & C_3 & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & A_{N-1} & B_{N-1} & C_{N-1} \\ & & & & & A_N & B_N \end{bmatrix} \qquad (3.37)$$

and $\underline{w} \equiv (w_1, w_2, w_3, \ldots , w_N)^T$, and $\underline{e} \equiv (1,0,0, \ldots , 0)^T$.

3.2 The electric Hertz potential

Comparing (2.45) and (2.46) with (3.2) and (3.3), we see that if we let $a_n = \sigma_n$, and $b_n = 1$, for all $n = 0,1,\ldots,N$, then F(z) represents the transformed electric Hertz potential, $P(\underline{\rho},z)$. Noting that $\sigma_o = 0$, we obtain from (3.35):

$$B_1 = -\sigma_1{}^2\nu_o$$

$$C_1 = 0 .$$

$$A_N = \sigma_{N-1}\sigma_N u_{N-1}$$

$$B_N = -\sigma_N{}^2 u_{N-1}v_{N-1} - \sigma_{N-1}\sigma_N \nu_N$$

and for $n = 2,3,\ldots,N-1$:

$$A_n = \sigma_{n-1}\sigma_n u_{n-1}$$

$$B_n = -\sigma_n{}^2 u_{n-1}v_{n-1} - \sigma_{n-1}\sigma_n u_n v_n$$

$$C_n = \sigma_{n-1}\sigma_{n+1}u_n \qquad (3.38)$$

The variable now becomes,

$$w_n = P(\underline{\rho}, z_{n-1}+0) ,$$

and the source function is

$$w^s = 0 .$$

We now have the following homogeneous system for $P(\underline{\rho},z_{n-1}+0)$:

$$\underline{\underline{S}}\cdot\underline{w} = \underline{0} \qquad (3.39)$$

where $\underline{\underline{S}}$ is formally defined by (3.37).

The matrix equation (3.39) has only the trivial solution $\underline{w} = 0$ provided det $\underline{\underline{S}} \neq 0$. This condition is satisfied since the coefficients depend on arbitrary physical parameters, and we have assumed that $\sigma_n \neq 0$ for all n, $1 \leq n \leq N$. Thus, the unique solution of the system is

$$P(\underline{\rho},+0) = P(\underline{\rho},z_1+0) = \cdots = P(\underline{\rho},z_{N-1}+0) = 0 \tag{3.40}$$

and hence from (2.46) we deduce that

$$P(\underline{\rho},z_1-0) = P(\underline{\rho},z_2-0) = \cdots = P(\underline{\rho},z_{N-1}-0) = 0. \tag{3.41}$$

Replacing F(z) by $P(\underline{\rho},z)$ in equation (3.9) for $1 \leq n \leq N-1$, and in (3.19) for n = N , we have

$$P(\underline{\rho},z) = \begin{cases} \dfrac{P(\underline{\rho},z_n-0)\sinh \nu_n(z-z_{n-1}) + P(\underline{\rho},z_{n-1}+0)\sinh \nu_n(z_n-z)}{\sinh \nu_n d_n} & (z_{n-1} < z < z_n) \\ P(z_{N-1}+0)e^{\nu_n(z_{N-1} - z)} & (z > z_{N-1}) \end{cases}$$

Substituting (3.40) and (3.41) into this expression and its derivative with respect to z, we see immediately that

$$P(\underline{\rho},z) = P_3(\underline{\rho},z) = 0 \qquad (z > 0) \tag{3.42}$$

In the case where there exist layers of zero conductivity below the surface, an alternative procedure described in Appendix B also shows $P(\underline{\rho},z)$ vanishes in $z > 0$.

In the free-space region, $-h < z < 0$, $P(\underline{\rho},z)$ need not of course be zero. To determine the electric Hertz potential there in terms of the source function, we replace $F(z)$ by $P(\underline{\rho},z)$ in equation (3.25) and write

$$P(\underline{\rho},z) = P^s(\underline{\rho},-0)e^{-\nu_o z} + P^i(\underline{\rho},-0)e^{\nu_o z} \qquad (3.43)$$

When differentiated with respect to z, and evaluated at $z = -0$, this becomes

$$P_3(\underline{\rho},-0) = \nu_o\left[P^i(\underline{\rho},-0) - P^s(\underline{\rho},-0)\right] \qquad (3.44)$$

Since it follows from (3.42) that $P_3(\underline{\rho},+0) = 0$, the boundary condition (2.45) requires $P_3(\underline{\rho},-0) = 0$. Accordingly,

$$P^i(\underline{\rho},-0) = P^s(\underline{\rho},-0) \qquad (3.45)$$

by (3.44). Thus (3.43) expresses the transformed electric Hertz potential in the free-space region in terms of the transformed source potential as follows:

$$P(\underline{\rho},z) = 2P^s(\underline{\rho},-0)\cosh(\nu_o z) \qquad (3.46)$$

Following Weaver (1971), $P(\underline{\rho},z)$ may also be written as the sum of the transformed source potential and its geometrical image. For, on replacing z by -z in

$$P^s(\underline{\rho},z) = P^s(\underline{\rho},-0)e^{-\nu_0 z}$$

and adding these two equations together, we obtain, by (3.46),

$$P(\underline{\rho},z) = P^s(\underline{\rho},z) + P^s(\underline{\rho},-z) \qquad (3.47)$$

3.3 The magnetic Hertz potential

The general system (3.36) can also be made to represent the equations satisfied by the transformed magnetic Hertz potentials evaluated at the interfaces in the conductor. From (2.43) it is seen that $G(\underline{\rho},z)$ is continuous across every interface, so there is no ambiguity in writing $G(\underline{\rho},z_n)$ rather than $G(\underline{\rho},z_n \pm 0)$. In (3.2) and (3.3) we must let $a_n = 1$, and $b_n = 1/\mu_n$ for all n. The definitions (3.35) become:

$$
\left.
\begin{aligned}
B_1 &= -\nu_o/\mu_o - u_1 v_1/\mu_1 \\
C_1 &= u_1/\mu_1 \\
A_N &= u_{N-1}/\mu_{N-1} \\
B_N &= - u_{N-1} v_{N-1}/\mu_{N-1} - \nu_N/\mu_N \\
&\text{and for } n = 2,3,\ldots,N-1 \text{, we have} \\
A_n &= u_{n-1}/\mu_{n-1} \\
B_n &= - u_n v_n/\mu_n - u_{n-1} v_{n-1}/\mu_{n-1} \\
C_n &= u_n/\mu_n = A_{n+1} \\
&\text{The variable becomes} \\
w_n &= G(\underline{\rho}, z_{n-1}) \\
&\text{and the source function, } w^s \text{, is} \\
w^s &= - 2\nu_o G^s(\underline{\rho},0)/\mu_o \;.
\end{aligned}
\right\} \qquad (3.48)
$$

Equation (3.36) now represents the inhomogeneous system of equations,

$$\underline{\underline{S}}\cdot\underline{w} = -\frac{2\nu_o}{\mu_o} G^s(\underline{\rho},0)\underline{e} \tag{3.49}$$

Since the determinant of the coefficient matrix must be non-zero (the coefficients, remember, depend on arbitrary physical quantities), the system is consistent; that is, the system possesses a solution which is expressible in terms of physical parameters, including the transformed source potential at the surface, $G^s(\underline{\rho},0)$. Replacing F(z) by $G(\underline{\rho},z)$, and remembering the continuity of $G(\underline{\rho},z)$ across each interface, we find that the equation corresponding to (3.9) is

$$G(\underline{\rho},z) = \frac{G(\underline{\rho},z_n)\sinh \nu_n(z-z_{n-1}) + G(\underline{\rho},z_{n-1})\sinh \nu_n(z_n-z)}{\sinh \nu_n d_n} \tag{3.50}$$

in the region $z_{n-1} < z < z_n$, $(1 \leq n \leq N-1)$. Equation (3.19) becomes

$$G(\underline{\rho},z) = G(\underline{\rho},z_{N-1})e^{\nu_N(z_{N-1} - z)} \tag{3.51}$$

for the region $z > z_{N-1}$, and equation (3.25) becomes

$$G(\underline{\rho},z) = G^i(\underline{\rho},0)e^{\nu_o z} + G^s(\underline{\rho},0)e^{-\nu_o z} \tag{3.52}$$

in $-h < z < 0$. The recursion relation (3.15) becomes

$$\frac{u_n}{\mu_n} G(\underline{\rho},z_{n-1}) - \left\{ \frac{u_n v_n}{\mu_n} + \frac{u_{n+1} v_{n+1}}{\mu_{n+1}} \right\} G(\underline{\rho},z_n)$$

$$+ \frac{u_{n+1}}{\mu_{n+1}} G(\underline{\rho},z_{n+1}) = 0 \tag{3.53}$$

for $n = 1,2,\ldots,N-2$, and the N^{th} recursion is

$$\frac{u_{N-1}}{\mu_{N-1}} G(\underline{\rho},z_{N-2}) - \left\{ \frac{u_{N-1} v_{N-1}}{\mu_{N-1}} + \frac{\nu_N}{\mu_N} \right\} G(\underline{\rho},z_{N-1}) = 0 \tag{3.54}$$

3.4 Summary

We are now in a position to make some important observations. Since $P(\underline{\rho},z)$ vanishes in the conductor, it follows by taking an inverse Fourier transform that

$$\Pi(\underline{r},z) = 0 \qquad (0 < z < \infty) \tag{3.55}$$

Hence, (2.14) and (2.15) simplify to the expressions

$$\underline{E} = -\frac{\partial^2\Gamma}{\partial y\partial t}\hat{i} + \frac{\partial^2\Gamma}{\partial x\partial t}\hat{j} \tag{3.56}$$

$$\underline{B} = \frac{\partial^2\Gamma}{\partial z\partial x}\hat{i} + \frac{\partial^2\Gamma}{\partial z\partial y}\hat{j} - \left(\frac{\partial^2\Gamma}{\partial x^2} + \frac{\partial^2\Gamma}{\partial y^2}\right)\hat{k} \tag{3.57}$$

in the conducting region $z > 0$. From (3.56) and (3.57) it may be seen that the normal component of the $\underline{E}$-field is zero everywhere in the conductor, and that both the $\underline{E}$ and $\underline{B}$ fields are specified by the magnetic Hertz potential, $\Gamma(\underline{r},z)$, alone.

Since $G(\underline{\rho},z)$ is known from (3.49) in terms of its source function, its inverse Fourier transform, $\Gamma(\underline{r},z)$, will be equal to a Fourier integral with $G^S(\underline{\rho},0)$ appearing in the integrand. Equations (2.14) and (2.15) in the free-space region $-h < z < 0$ become:

$$\underline{E} = \left(\frac{\partial^2\Pi}{\partial z\partial x} - \frac{\partial^2\Gamma}{\partial y\partial t}\right)\hat{i} + \left(\frac{\partial^2\Pi}{\partial z\partial y} + \frac{\partial^2\Gamma}{\partial x\partial t}\right)\hat{j} - \left(\frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2}\right)\hat{k} \tag{3.58}$$

$$\underline{B} = \frac{\partial^2\Gamma}{\partial z\partial x}\,\hat{i} + \frac{\partial^2\Gamma}{\partial z\partial y}\,\hat{j} - \left(\frac{\partial^2\Gamma}{\partial x^2} + \frac{\partial^2\Gamma}{\partial y^2}\right)\hat{k} \tag{3.59}$$

We see that the $\underline{B}$ field in this region is again specified by the magnetic Hertz potential alone. Thus the $\underline{B}$ field is known everywhere in the whole region of interest $-h < z < \infty$, in terms of $\Gamma(\underline{r},z)$ which itself is known in terms of the source term $G^S(\underline{\rho},0)$. The $\underline{E}$ field in the free-space region is specified by both $G^S(\underline{\rho},0)$ and $P^S(\underline{\rho},-0)$. Since $\Gamma(\underline{r},z)$ satisfies Laplace's equation in the free-space region, c f. equation (2.11), we can deduce from (3.59) that

$$\underline{B} = \nabla\left(\frac{\partial\Gamma}{\partial z}\right) \quad . \tag{3.60}$$

Thus the magnetic scalar potential, $\Omega^S(\underline{r},z)$ (say) , is given in terms of the Hertz potential by

$$\Omega^S(\underline{r},z) = -\frac{\partial\Gamma}{\partial z} \quad . \tag{3.61}$$

The electric Hertz potential also satisfies Laplace's equation in $-h < z < 0$. Thus from (2.12) we see that

$$\underline{E} = \nabla\left(\frac{\partial\Pi}{\partial z}\right) - \text{curl}\left(\frac{\partial\Gamma}{\partial t}\,\hat{k}\right) \quad . \tag{3.62}$$

CHAPTER 4

SOLVING THE SYSTEM

4.1 Hertz potential solutions

From (3.50) it is seen that in order to evaluate $G(\underline{\rho},z)$ in the n^{th} region we must solve the matrix equation (3.49) for $G(\underline{\rho},z_{n-1})$ and $G(\underline{\rho},z_n)$. Also, from the discussion of the preceding section we saw that the problem of solving for the electromagnetic field components in any n^{th} layer, is a problem of determining the function $G(\underline{\rho},z)$ in that layer.

We will now solve the system represented by (3.49), which may be written alternatively as a set of N equations:

$$\begin{aligned}
B_1w_1 + A_2w_2 &= -2\nu_oG^s(\underline{\rho},0)/\mu_o \\
A_2w_1 + B_2w_2 + A_3w_3 &= 0 \\
A_3w_2 + B_3w_3 + A_4w_4 &= 0 \\
\cdots\cdots & \\
A_{N-1}w_{N-2} + B_{N-1}w_{N-1} + A_Nw_N &= 0 \\
A_Nw_{N-1} + B_Nw_N &= 0
\end{aligned} \tag{4.1}$$

This system may be solved by a method of successive substitution. From the N^{th} equation of (4.1) it is seen that

$$w_N = Q_{N-1}w_{N-1} \tag{4.2}$$

where

$$Q_{N-1} = - A_N/B_N \qquad (4.3)$$

Similarly substituting (4.2) into the N-1th equation of (4.1) we have

$$w_{N-1} = Q_{N-2}w_{N-2} \qquad (4.4)$$

where

$$Q_{N-2} = \frac{- A_{N-1}}{B_{N-1} + A_N Q_{N-1}} \qquad (4.5)$$

If we continue this procedure, each unknown variable w_n is expressed in terms of the variable associated with the next higher interface, w_{n-1}, by the relation

$$w_n = Q_{n-1}w_{n-1} \qquad (2 \leq n \leq N) \qquad (4.6)$$

where Q_n satisfies the generating equation,

$$Q_{n-1} = \frac{- A_n}{B_n + A_{n+1}Q_n} \qquad (2 \leq n \leq N-1) \qquad (4.7)$$

which, together with (4.3) determines the factor Q_n at any interface. Eventually all the w_n's are expressed parametrically in terms of w_1 , which in turn may be expressed in terms of $G^s(\underline{\rho},0)$. After N-1 steps, a factor Q_1 is obtained such that

$$w_2 = Q_1 w_1 \tag{4.8}$$

where Q_1 is of the form of a continued fraction:

$$Q_1 = \cfrac{-A_2}{B_2 - \cfrac{A_3^{\,2}}{B_3 - \cfrac{A_4^{\,2}}{B_4 - \cdots}}} \tag{4.9}$$

where the A's and B's are defined explicitly in equations (3.48).

The first equation of (4.1) contains the surface boundary conditions. When (4.8) is inserted into this first equation, we obtain the result:

$$(B_1 + A_2 Q_1) w_1 = -\frac{2\nu_o}{\mu_o} G^s(\underline{\rho},0) \tag{4.10}$$

which, when written explicitly, is

$$\left\{ -\left[\frac{\nu_o}{\mu_o} + \frac{u_1 v_1}{\mu_1}\right] + \frac{u_1 Q_1}{\mu_1} \right\} G(\underline{\rho},0) = -\frac{2\nu_o}{\mu_o} G^s(\underline{\rho},0) \tag{4.11}$$

Since this equation expresses $G(\underline{\rho},0)$ as a function of $G^s(\underline{\rho},0)$, all the $G(\underline{\rho},z_n)$'s are now expressible in terms of the source function. For example, by (4.8) we have

$$G(\underline{\rho},z_1) = Q_1 G(\underline{\rho},0) \quad , \tag{4.12}$$

so we see from (4.11) that $G(\underline{\rho},z_1)$ can be expressed in terms of $G^s(\underline{\rho},0)$.

In the free space region it is necessary to eliminate $G^i(\underline{\rho},0)$ from equation (3.52) in order to know the field there solely in terms of the source function. We may write (3.52) in the form

$$G(\underline{\rho},z) = G^s(\underline{\rho},0)\left\{e^{-\nu_o z} + S(\rho)e^{\nu_o z}\right\} . \qquad (4.13)$$

where $S(\rho) = G^i(\underline{\rho},0)/G^s(\underline{\rho},0)$ is the ratio of the internal to external parts of the transformed Hertz potential at the surface. Surface ratios of the internal to external parts of the field are commonly defined in geophysical analysis, see for example Schmucker (1970).

Putting $z = 0$ in (4.13), and eliminating $G(\underline{\rho},0)$ with the aid of (4.11), we find that

$$S(\rho) = \frac{\mu_1\nu_o - \mu_o u_1(v_1 - Q_1)}{\mu_1\nu_o + \mu_o u_1(v_1 - Q_1)} . \qquad (4.14)$$

It should be remembered, of course, that u, v, ν, and Q are all functions of ρ. From (4.13) we have the relationship

$$G(\underline{\rho},0) = K(\rho)\, G^s(\underline{\rho},0) \qquad (4.15)$$

where

$$K(\rho) = 1 + S(\rho) \qquad (4.16)$$

which we can also deduce directly from (4.11). It can be seen that $K(\rho)$ represents the ratio at the surface of the total field to the source field.

Substituting (4.12) and (4.15) back into equation (3.50) with $n = 1$, we obtain for the first layer, $0 < z < z_1$,

$$G(\underline{\rho}z) = G^s(\underline{\rho},0)K(\rho)\left\{Q_1 \sinh \nu_1 z + \sinh \nu_1(z_1 - z)\right\} \operatorname{csch} \nu_1 d_1 \tag{4.17}$$

In the same manner, in the layer $z_1 < z < z_2$, we obtain

$$G(\underline{\rho},z) = G^s(\underline{\rho},0)K(\rho)Q_1\left\{Q_2 \sinh \nu_2(z-z_1) + \sinh \nu_2(z_2-z)\right\} \operatorname{csch} \nu_2 d_2$$

We can continue this procedure, and for the n^{th} region, $n = 1,2,\ldots,N-1$, we can then devise a general expression

$$G(\underline{\rho},z) = G^s(\underline{\rho},0)K(\rho)\left\{Q_n \sinh \nu_n(z-z_{n-1}) + \sinh \nu_n(z_n-z)\right\} \times \operatorname{csch}[\nu_n d_n] \prod_{i=0}^{n-1} Q_i \tag{4.18}$$

where we understand $Q_o = 1$. And finally in the unbounded N^{th} region we obtain, from (3.51),

$$G(\underline{\rho},z) = G^s(\underline{\rho},0)K(\rho)e^{\nu_N(z_{N-1}-z)} \prod_{i=0}^{N-1} Q_i \tag{4.19}$$

With (4.13), (4.17), (4.18) and (4.19) we have determined the transformed magnetic Hertz potential $G(\underline{\rho},z)$ everywhere in the region of interest, $-h < z < \infty$, in terms of known parameters, and the known source potential.

4.2 Field solutions

We are now able to determine the electromagnetic field everywhere in the region of interest: $-h < z < \infty$. In the following discussion, the Fourier transform will be denoted by the symbolic operator $\mathfrak{F}\{\ \}$ (see Appendix A); the inverse Fourier transform will be denoted by $\mathfrak{F}^{-1}\{\ \}$. Since we have shown that $\Pi = 0$ everywhere inside the conductor it follows from equation (2.15) that

$$\mathfrak{F}\{B_x\} = -\,i\xi G_3(\underline{\rho},z) \qquad (4.20)$$

$$\mathfrak{F}\{B_y\} = -\,i\eta G_3(\underline{\rho},z) \qquad (4.21)$$

$$\mathfrak{F}\{B_z\} = \rho^2 G(\underline{\rho},z) \qquad (4.22)$$

In the n^{th} region, $z_{n-1} < z < z_n$, differentiating (4.18) with respect to z, we get

$$G_3(\underline{\rho},z) = G^s(\underline{\rho},0)K(\rho)u_n\left\{Q_n\cosh \nu_n(z-z_{n-1}) - \cosh \nu_n(z_n-z)\right\} \times \prod_{i=0}^{n-1} Q_i \qquad (4.23)$$

recalling that $u_n = \nu_n \operatorname{csch} \nu_n d_n$

From (4.18), (4.20), (4,21), (4.22) and (4.23), we then deduce for the n^{th} region:

$$B_x = \mathfrak{F}^{-1}\left\{i\xi G^s(\underline{\rho},0)K(\rho)u_n\left[\cosh \nu_n(z_n-z) - Q_n\cosh \nu_n(z-z_{n-1})\right] \times \prod_{i=0}^{n-1} Q_i\right\} \qquad (4.24)$$

$$B_y = \mathcal{F}^{-1}\left\{ i\eta G^s(\underline{\rho},0)K(\rho)u_n\left[\cosh \nu_n(z_n-z) - Q_n\cosh \nu_n(z-z_{n-1})\right] \prod_{i=0}^{n-1} Q_i \right\} \quad (4.25)$$

$$B_z = \mathcal{F}^{-1}\left\{ \rho^2 G^s(\underline{\rho},0)K(\rho)\operatorname{csch}\nu_n d_n\left[\sinh \nu_n(z_n-z) + Q_n\sinh \nu_n(z-z_{n-1})\right] \prod_{i=0}^{n-1} Q_i \right\} \quad (4.26)$$

Similarly for the free-space region, the magnetic field components are given by (4.13), (4.20), (4.21), and (4.22),

$$B_x = \mathcal{F}^{-1}\left\{ i\xi\nu_o G^s(\underline{\rho},0)\left(\left[1-K(\rho)\right] e^{\nu_o z} + e^{-\nu_o z}\right)\right\} \quad (4.27)$$

$$B_y = \mathcal{F}^{-1}\left\{ i\eta\nu_o G^s(\underline{\rho},0)\left(\left[1-K(\rho)\right] e^{\nu_o z} + e^{-\nu_o z}\right)\right\} \quad (4.28)$$

$$B_z = \mathcal{F}^{-1}\left\{ -\rho^2 G^s(\underline{\rho},0)\left(\left[1-K(\rho)\right] e^{\nu_o z} - e^{-\nu_o z}\right)\right\} \quad (4.29)$$

In the unbounded N^{th} region, $z_{N-1} < z < \infty$, from (4.19) we have:

$$B_x = \mathcal{F}^{-1}\left\{ i\xi\nu_N G^s(\underline{\rho},0)K(\rho)e^{\nu_N(z_{N-1}-z)} \cdot \prod_{i=0}^{N-1} Q_i \right\} \quad (4.30)$$

$$B_y = \mathcal{F}^{-1}\left\{ i\eta\nu_N G^s(\underline{\rho},0)K(\rho)e^{\nu_N(z_{N-1}-z)} \cdot \prod_{i=0}^{N-1} Q_i \right\} \quad (4.31)$$

$$B_z = \mathcal{F}^{-1}\left\{\rho^2 G^s(\underline{\rho},0)K(\rho)e^{\nu_N(z_{N-1}-z)} \prod_{i=0}^{N-1} Q_i\right\} \quad (4.32)$$

Analogous to equations (4.20), (4.21) and (4.22), the electric field components are given by

$$\mathcal{F}\{E_x\} = -\omega\eta G(\underline{\rho},z) - i\xi P_3(\underline{\rho},z) \quad (4.33)$$

$$\mathcal{F}\{E_y\} = \omega\xi G(\underline{\rho},z) - i\eta P_3(\underline{\rho},z) \quad (4.34)$$

$$\mathcal{F}\{E_z\} = \rho^2 P(\underline{\rho},z) \quad (4.35)$$

Thus in the n^{th} layer of the conductor, $z_{n-1} < z < z_n$, we have for the components of the electric field,

$$E_x = \mathcal{F}^{-1}\left\{-\omega\eta G^s(\underline{\rho},0)K(\rho)\left[Q_n \sinh \nu_n(z-z_{n-1}) + \sinh \nu_n(z_n-z)\right] \operatorname{csch} \nu_n d_n \cdot \prod_{i=0}^{n-1} Q_i\right\} \quad (4.36)$$

$$E_y = \mathcal{F}^{-1}\left\{\omega\xi G^s(\underline{\rho},0)K(\rho)\left[Q_n \sinh \nu_n(z-z_{n-1}) + \sinh \nu_n(z_n-z)\right] \operatorname{csch} \nu_n d_n \cdot \prod_{i=0}^{n-1} Q_i\right\} \quad (4.37)$$

$$E_z = 0 \quad (4.38)$$

And in the free-space region we have,

$$E_x = \mathfrak{F}^{-1}\left\{- \omega\eta G^s(\underline{\rho},0)\left[e^{-\nu_o z} - [1-K(\rho)]\, e^{\nu_o z}\right] - 2i\xi\rho P^s(\underline{\rho},0)\sinh \nu_o z\right\} \tag{4.39}$$

$$E_y = \mathfrak{F}^{-1}\left\{\omega\xi G^s(\underline{\rho},0)\left[e^{-\nu_o z} - [1-K(\rho)]\, e^{\nu_o z}\right] - 2i\eta\rho P^s(\underline{\rho},0)\sinh \nu_o z\right\} \tag{4.40}$$

$$E_z = \mathfrak{F}^{-1}\left\{2\rho^2 P^s(\underline{\rho},0)\cosh \nu_o z\right\} \tag{4.41}$$

In the unbounded N^{th} region we have:

$$E_x = \mathfrak{F}^{-1}\left\{-\omega\eta G^s(\underline{\rho},0)K(\rho)e^{\nu_N(z_{N-1}-z)} \cdot \prod_{i=0}^{N-1} Q_i\right\} \tag{4.42}$$

$$E_y = \mathfrak{F}^{-1}\left\{\omega\xi G^s(\underline{\rho},0)K(\rho)e^{\nu_N(z_{N-1}-z)} \cdot \prod_{i=0}^{N-1} Q_i\right\} \tag{4.43}$$

$$E_z = 0 \tag{4.44}$$

4.3 Simplified field solutions

The solutions for $\underline{E}$ and $\underline{B}$ in the region $-h < z < \infty$ can be simplified considerably by expressing the Hertz potentials in terms of source Hertz potentials and their geometric images in the same manner as Weaver expresses his solutions for the case of the homogeneous conductor (1971). As we shall see, expressing the solutions in this form eliminates the necessity of determining $P(\underline{\rho},0)$ or $\Pi(\underline{r},0)$.

Fourier inverting equation (3.47) we obtain

$$\Pi(\underline{r},z) = \Pi^s(\underline{r},z) + \Pi^s(\underline{r},-z) \quad ; \qquad (4.45)$$

differentiating this with respect to z, we have

$$\Pi_3(\underline{r},z) = \Pi^s{}_3(\underline{r},z) - \Pi^s{}_3(\underline{r},-z) \quad . \qquad (4.46)$$

Equation (4.13) can be rewritten in the following form:

$$G(\underline{\rho},z) = G^s(\underline{\rho},z) + S(\rho)G^s(\underline{\rho},-z) \qquad (4.47)$$

Inserting (4.16) into (4.47) we have

$$G(\underline{\rho},z) = G^s(\underline{\rho},z) - G^s(\underline{\rho},-z) + K(\rho)G^s(\underline{\rho},-z) \qquad (4.48)$$

which, after a Fourier inversion, gives us

$$\Gamma(\underline{r},z) = \Gamma^s(\underline{r},z) - \Gamma^s(\underline{r},-z) + \Gamma^*(\underline{r},z) \qquad (4.49)$$

where we define Γ^* by the following integral

$$\Gamma^*(\underline{r},z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} K(\rho)G^s(\underline{\rho},-z)e^{-i\underline{\rho}\cdot\underline{r}}\,d\underline{\rho} \qquad (4.50)$$

Equation (4.49) corresponds to Weaver's equation (5.6) in his treatment of the homogeneous conductor; our equation (4.50) corresponds to his equation (9.6).

Inserting our expressions (4.46) and (4.49) into (2.14) and (2.15), we have for the electric field components in the free-space region:

$$E_x = \Pi^S_{31}(\underline{r},z) - i\omega\Gamma^S_2(\underline{r},z) - \Pi^S_{31}(\underline{r},-z) + i\omega\Gamma^S_2(\underline{r},-z) - i\omega\Gamma^*_2(\underline{r},z)$$

$$E_y = \Pi^S_{32}(\underline{r},z) + i\omega\Gamma^S_1(\underline{r},z) - \Pi^S_{32}(\underline{r},-z) - i\omega\Gamma^S_1(\underline{r},-z) + i\omega\Gamma^*_1(\underline{r},z)$$

$$E_z = - \Pi^S_{11}(\underline{r},z) - \Pi^S_{22}(\underline{r},z) - \Pi^S_{11}(r,-z) - \Pi^S_{22}(r,-z) ,$$

and for the magnetic field components we have

$$B_x = \Gamma^S_{31}(\underline{r},z) + \Gamma^S_{31}(\underline{r},-z) + \Gamma^*_{31}(\underline{r},z)$$

$$B_y = \Gamma^S_{32}(\underline{r},z) + \Gamma^S_{32}(\underline{r},-z) + \Gamma^*_{32}(\underline{r},z)$$

$$B_z = \Gamma^S_{11}(\underline{r},z) + \Gamma^S_{22}(\underline{r},z) - \Gamma^S_{11}(\underline{r},-z) - \Gamma^S_{22}(\underline{r},-z) + \Gamma^*_{11}(\underline{r},z) + \Gamma^*_{22}(\underline{r},z) .$$

These field equations can be written as Weaver has written them,

$$\underline{E}_{||} = \underline{E}^S_{||} - \underline{E}^{\dagger}_{||} - \mathrm{curl}(\hat{k}\partial\Gamma^*/\partial t) \qquad (4.51)$$

$$E_z = E_z^s + E_z^\dagger \tag{4.52}$$

$$\underline{B} = \underline{B}^s + \underline{B}^\dagger + \nabla(\partial\Gamma^*/\partial z) - 2B_z^\dagger\,\hat{k} \tag{4.53}$$

where $\underline{B}^\dagger$ and $\underline{E}^\dagger$ are defined as the images of the source,

$$\underline{B}^\dagger(\underline{r},z) = \underline{B}^s(\underline{r},-z)$$

$$\underline{E}^\dagger(\underline{r},z) = E^s(\underline{r},-z)$$

and $\underline{E}_{\parallel} = \underline{E} - E_z\hat{k}$ is the component of the electric field parallel to the surface of the conductor.

We can also write our solutions for the field inside the conductor, $z > 0$, in a more compact notation. We have in the n^{th} conducting region:

$$E_z = 0 \tag{4.54}$$

$$\underline{E}_{\parallel} = -\,\mathrm{curl}(\hat{k}\partial\Gamma/\partial t) \tag{4.55}$$

$$\underline{B} = \nabla(\partial\Gamma/\partial z) - \mu_n\sigma_n\hat{k}\;\partial\Gamma/\partial t \tag{4.56}$$

where $\Gamma(\underline{r},z)$ is given by Fourier inverting equation (4.18) if $n = 1,2,\ldots,N-1$; that is

$$\Gamma(\underline{r},z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G^s(\underline{\rho},0)K(\rho)\left\{Q_n \sinh\nu_n(z-z_{n-1}) + \sinh\nu_n(z_n-z)\right\} \times \operatorname{csch}\nu_n d_n\left[\prod_{i=0}^{n-1}Q_i\right]e^{-i\underline{\rho}\cdot\underline{r}}\,d\underline{\rho}\;; \tag{4.57}$$

and for $n = N$ in the unbounded N^{th} region, we obtain $\Gamma(\underline{r},z)$

by the Fourier inversion of equation (4.19):

$$\Gamma(\underline{r},z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G^{s}(\underline{\rho},0)K(\rho)e^{\nu_N(z_{N-1}-z)}\left[\prod_{i=0}^{N-1} Q_i\right]e^{-i\underline{\rho}\cdot\underline{r}}\,d\underline{\rho} \quad . \tag{4.58}$$

CHAPTER 5

CONDUCTORS WITH FEW LAYERS

5.1 The homogeneous half-space

For many important geophysical problems, it is often useful to consider the simple models of one-, two- and three-layered conducting half-spaces. We should be able to extract these simple cases from our formalism for a half-space with an arbitrary number of layers.

In problems where the depth of penetration of the inducing field is thought to be small, it is sometimes useful to consider a homogeneous conducting half-space. The general Hertz potential theory developed by Weaver (1971) for a homogeneous conductor can be shown to be a special case of our N-layer theory, if we take the limit in our expressions as $z_1 \to \infty$, and consequently as $d_1 \to \infty$.

In the free-space region, $-h < z < 0$, we take the limit as $z_1 \to \infty$ of $K(\rho)$ as defined in equations (4.16) and (4.14) to find

$$\lim_{z_1 \to \infty} K(\rho) = \frac{2\mu_1 \nu_0}{\mu_1 \nu_0 + \mu_0 \nu_1} \qquad (5.1)$$

If we observe that $\nu_0 \equiv \rho$, and define relative permeability κ , as $\kappa = \mu_1/\mu_0$, we see from (4.50) that

$$\lim_{z_1 \to \infty} \Gamma^*(\underline{r},z) = \frac{\kappa}{\pi} \int_{-\infty}^{+\infty} \frac{\rho}{\rho+\kappa\nu_1} G^S(\underline{\rho},-z) e^{-i\underline{\rho}\cdot\underline{r}} \, d\underline{\rho} \tag{5.2}$$

which is simply Weaver's expression (9.6) for Γ^* in the homogeneous case (1971).

Inside the conductor, $z > 0$, we take the limit as $z_1 \to \infty$ (and consequently as $d_1 \to \infty$) of (4.17) and find that

$$\lim_{z_1 \to \infty} Q_1 \sinh(\nu_1 z) \operatorname{csch}(\nu_1 d_1) = 0$$

and

$$\lim_{z_1 \to \infty} \operatorname{csch}(\nu_1 d_1) \sinh \nu_1 (z_1 - z) = e^{-\nu_1 z} .$$

Thus the inverse Fourier transform of (4.17) in this limit is given by

$$\lim_{z_1 \to \infty} \Gamma(\underline{r},z) = \frac{\kappa}{\pi} \int_{-\infty}^{+\infty} \frac{\rho}{\rho+\kappa\nu_1} G^S(\underline{\rho},z) e^{-\nu_1 z + \rho z} \, e^{-i\underline{\rho}\cdot\underline{r}} \, d\underline{\rho} \tag{5.3}$$

where we have observed that $G^S(\underline{\rho},z) = G^S(\underline{\rho},0)e^{-\rho z}$. Equation (5.3) is identically Weaver's expression (9.5) in his 1971 treatment of the homogeneous case.

5.2 The two-layered half-space

The two-layered conducting half-space is frequently a useful model for investigating analytically such problems as induction in sea-water over a sea-bed, or induction in the earth's crust over the earth's mantle. In such geophysical applications the variations in the permeability are usually quite unimportant (Tozer, 1959), so that we may take $\mu_1 = \mu_2 = \mu_o$. This has the added advantage of making the formulae less cumbersome.

Weaver (1964) has solved the problem of induction in a two-layered half-space directly from the field equations and the associated field boundary conditions. We shall compare his solutions with the expressions we obtain by setting N = 2 in our N-layer Hertz potential formalism.

The model we are considering is illustrated in Figure 2.

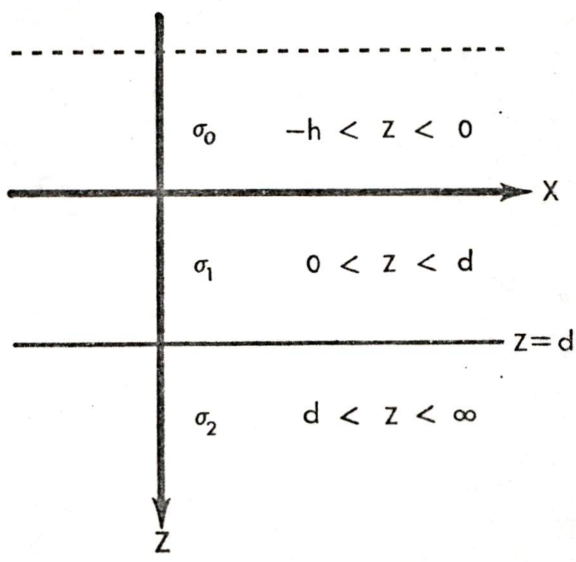

Figure 2. The two-layered half-space

In the second layer, $d < z < \infty$, setting $N = 2$ in equation (3.54), and assuming $\mu_2 = \mu_1 = \mu_o$, we obtain:

$$G(\underline{\rho},d) = \frac{u_1 G(\underline{\rho},0)}{u_1 v_1 + \nu_2} \tag{5.4}$$

Comparing (5.4) with (4.12) we see that

$$Q_1 = \frac{u_1}{u_1 v_1 + \nu_2} \tag{5.5}$$

and inserting this value of Q_1 into (4.14) we obtain

$$S(\rho) = \frac{\nu_o - u_1\left(v_1 - \dfrac{u_1}{u_1 v_1 + \nu_2}\right)}{\nu_o + u_1\left(v_1 - \dfrac{u_1}{u_1 v_1 + \nu_2}\right)} \tag{5.6}$$

If (5.6) is rationalized, we have explicitly

$$S(\rho) = \frac{\nu_o\nu_1\coth(\nu_1 d) + \nu_o\nu_2 - \nu_1{}^2 - \nu_1\nu_2\coth(\nu_1 d)}{\nu_o\nu_1\coth(\nu_1 d) + \nu_o\nu_2 + \nu_1{}^2 + \nu_1\nu_2\coth(\nu_1 d)} \tag{5.7}$$

hence from (4.16) we get

$$K(\rho) = \frac{2\nu_o\left\{\nu_1\coth(\nu_1 d) + \nu_2\right\}}{\nu_o\nu_1\coth(\nu_1 d) + \nu_o\nu_2 + \nu_1{}^2 + \nu_1\nu_2\coth(\nu_1 d)} \tag{5.8}$$

The field solutions follow by substituting (5.5) and (5.8) into the appropriate field expressions derived in the previous chapter. For example, by (4.24), (4.25) and (4.26) with $n = 1$, the magnetic field components in the first layer, $0 < z < d$, after considerable simplification are:

$$B_x = \frac{i}{\pi} \int_{-\infty}^{+\infty} \xi \nu_o \nu_1 G^s(\underline{\rho}, 0) \times \frac{\nu_1 \sinh \nu_1(d-z) + \nu_2 \cosh \nu_1(d-z)}{\nu_1(\nu_o+\nu_2)\cosh(\nu_1 d) + \nu_o \nu_2 \sinh(\nu_1 d) + \nu_1^{2} \sinh(\nu_1 d)} e^{-i\underline{\rho}\cdot\underline{r}} d\underline{\rho} \tag{5.9}$$

$$B_y = \frac{i}{\pi} \int_{-\infty}^{+\infty} \eta \nu_o \nu_1 G^s(\underline{\rho}, 0) \times \frac{\nu_1 \sinh \nu_1(d-z) + \nu_2 \cosh \nu_1(d-z)}{\nu_1(\nu_o+\nu_2)\cosh(\nu_1 d) + \nu_o \nu_2 \sinh(\nu_1 d) + \nu_1^{2} \sinh(\nu_1 d)} e^{-i\underline{\rho}\cdot\underline{r}} d\underline{\rho} \tag{5.10}$$

$$B_z = \frac{1}{\pi}\int_{-\infty}^{+\infty} \nu_o{}^3 G^s(\underline{\rho},0)$$

$$\times \frac{\nu_1 \cosh \nu_1(d-z) - \nu_2 \sinh \nu_1(d-z)}{\nu_1(\nu_o+\nu_2)\cosh(\nu_1 d) + \nu_o\nu_2 \sinh(\nu_1 d) + \nu_1{}^2 \sinh(\nu_1 d)} e^{-i\underline{\rho}\cdot\underline{r}}\, d\underline{\rho}$$

(5.11)

In order to compare these results with those obtained by Weaver (1964), we follow his notation by writing

$$\mathcal{F}\{\Omega^s(\underline{r},z)\} = f(\underline{\rho})e^{-\nu_o z}$$

and deduce from (3.61) that

$$f(\underline{\rho}) = \nu_o G^s(\underline{\rho},0) \qquad (5.12)$$

Weaver defined $\Delta = (\nu_1+\nu_2)(\nu_1+\nu_o) - (\nu_1-\nu_2)(\nu_1-\nu_o)e^{-2\nu_1 d}$ which can be shown to be the denominator of the integrand in (5.9) multiplied by $e^{-\nu_1 d}$. Thus with the use of (5.12), and expanding the hyperbolic functions, we obtain Weaver's (1964) solutions:

$$B_x = \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{\xi\nu_1 f(\underline{\rho})}{\Delta}\left\{(\nu_1+\nu_2)e^{-\nu_1 z} - (\nu_1-\nu_2)e^{\nu_1(z-2d)}\right\} e^{-i\underline{\rho}\cdot\underline{r}}\, d\underline{\rho}$$

(5.13)

$$B_y = \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{\xi\nu_1 f(\underline{\rho})}{\Delta} \left\{(\nu_1+\nu_2)e^{-\nu_1 z} - (\nu_1-\nu_2)e^{\nu_1(z-2d)}\right\} e^{-i\underline{\rho}\cdot\underline{r}}\, d\underline{\rho} \tag{5.14}$$

$$B_z = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\nu_o^{\,2} f(\underline{\rho})}{\Delta} \left\{(\nu_1+\nu_2)e^{-\nu_1 z} + (\nu_1-\nu_2)e^{\nu_1(z-2d)}\right\} e^{-i\underline{\rho}\cdot\underline{r}}\, d\underline{\rho} \tag{5.15}$$

5.3 The three-layer half-space

We can also express analytically the solutions for a three-layered half-space, but solutions for models with more than three layers become too impractical to express by explicit integrands.

We will now consider the three-layered half-space illustrated in Figure 3.

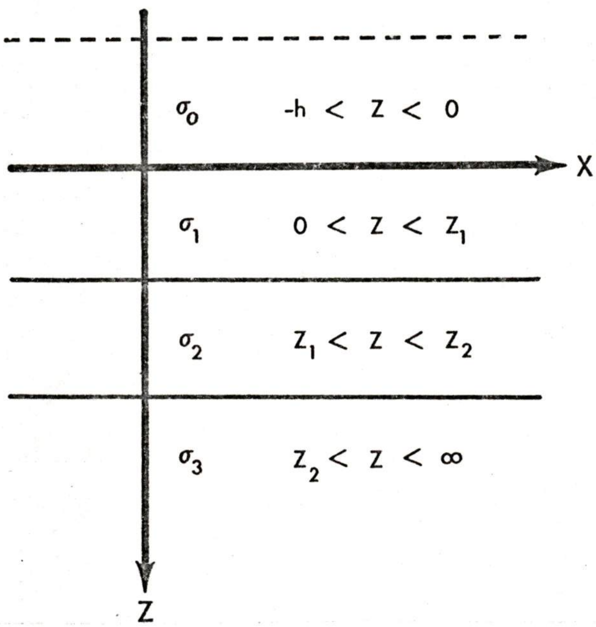

Figure 3. The three-layered half-space

If we set N = 3 in equation (3.54), and again set $\mu_3 = \mu_2 = \mu_1 = \mu_o$, we obtain for the region $z_2 < z < \infty$, the relation

$$G(\underline{\rho}, z_2) = \frac{u_2}{u_2 v_2 + \nu_3} G(\underline{\rho}, z_1) = Q_2 G(\underline{\rho}, z_1) \qquad (5.16)$$

The recursion relation associated with the region $z_1 < z < z_2$ is obtained by setting n = 1 in equation (3.53):

$$- G(\underline{\rho},0)u_1 + G(\underline{\rho},z_1)\, u_1v_1 + u_2v_2 \; - G(\underline{\rho},z_2)u_2 = 0 \tag{5.17}$$

If we insert (5.16) into (5.17) we obtain:

$$G(\underline{\rho},z_1) = \frac{u_1(u_2v_2+\nu_3)}{(u_2v_2+\nu_3)(u_1v_1+u_2v_2)-u_2^{\,2}}\; G(\underline{\rho},0) \tag{5.18}$$

which identifies Q_1 for the three-layer case as

$$Q_1 = \frac{u_1(u_2v_2+\nu_3)}{(u_2v_2+\nu_3)(u_1v_1+u_2v_2) - u_2^{\,2}} \tag{5.19}$$

From (4.14) it follows that:

$$S(\rho) =$$

$$\frac{(\nu_o-u_1v_1)(u_1v_1+u_2v_2)(u_2v_2+\nu_3)+u_1^{\,2}(u_2v_2+\nu_3)-u_2^{\,2}(\nu_o-u_1v_1)}{(\nu_o+u_1v_1)(u_1v_1+u_2v_2)(u_2v_2+\nu_3)-u_1^{\,2}(u_2v_2+\nu_3)-u_2^{\,2}(\nu_o+u_1v_1)} \tag{5.20}$$

and from (4.15) we have:

$$K(\rho) = \frac{2\nu_o(u_1v_1+u_2v_2)(u_2v_2+\nu_3) - u_2{}^2}{(\nu_o+u_1v_1)(u_1v_1+u_2v_2)(u_2v_2+\nu_3)-u_1{}^2(u_2v_2+\nu_3)-u_2{}^2(\nu_o+u_1v_1)} \tag{5.21}$$

Thus having found $K(\rho)$, we can now solve for the field components everywhere from equations (4.24) through (4.31) and equations (4.36) through (4.44). The expressions are lengthy and will not be written out in full here.

CHAPTER 6

PARTICULAR SOURCE FIELDS

6.1 Preliminary remarks

In Chapter 4 we determined the electromagnetic field components due to an arbitrary source field in terms of inverse Fourier integrals; in free space our solutions are expressed in terms of the source fields and their geometric images as well. Thus, in order to completely specify the field everywhere it remains to supply the source function $G^s(\underline{\rho},0)$ in equations (4.50), (4.57) and (4.58), and the formulae for the source fields and image fields, E^s, $E^\dagger$, B^s, and $B^\dagger$, in equations (4.51), (4.52) and (4.53).

In order to illustrate the general N-layer formalism we have developed, we shall consider three particular source fields: an infinite alternating line current parallel to the conducting surface; a quasi-static magnetic dipole vertically oriented with respect to the conducting surface; and a quasi-static magnetic dipole horizontally oriented with respect to the conducting surface.

We exclude the electric dipole from the sources we are considering. To the order of neglected displacement current in the Maxwell equations, the magnetic field of an electric dipole is zero, and thus, to this order, no induction occurs.

For a given field, the magnetic Hertz potential must be determined. If the magnetic field of the source in free space is expressible as a pure gradient (as must be the case if we neglect displacement current in Maxwell's equations), then it is a simple matter to infer the magnetic Hertz potential by equation (3.60). By taking the double Fourier transform of $\Gamma^s(\underline{r},z)$, and evaluating this at $z = 0$ we obtain $G^s(\underline{\rho},0)$.

6.2 Horizontal line current

Consider the case of an infinite alternating line current flowing parallel to the conducting surface. We choose a current amplitude, I , and take the current to be positive in the y-direction along the line x = 0, z = - h. The source fields are given by Weaver (1971):

$$\underline{B}^s(\underline{r},z) = \frac{\mu_o I}{2\pi} \nabla\left(\arctan\left(\frac{x}{h+z}\right)\right) \tag{6.1}$$

and

$$\underline{E}^s(\underline{r},z) - \underline{E}^s(\underline{0},0) = \frac{i\omega\mu_o I}{2\pi} \ln(S/h)\ \hat{j} \tag{6.2}$$

where $S = \{x^2 + (h+z)^2\}^{\frac{1}{2}}$ and the time factor $e^{i\omega t}$ is, of course, understood. From (3.60) we can deduce

$$\Gamma^s_{\ 3}(\underline{r},z) = \frac{\mu_o I}{2\pi} \arctan\left(\frac{x}{h+z}\right) \tag{6.3}$$

By taking the double Fourier transform of (6.3) and then integrating over z we obtain (Weaver,1971) the expression:

$$G^s(\underline{\rho},z) = -\left(\frac{i\mu_o I}{2}\right)\delta(\eta)\xi^{-1}|\xi|^{-1}e^{-(h+z)|\xi|} \tag{6.4}$$

where $\delta(\eta)$ is the Dirac delta-function. Hence from (6.4) we have:

$$G^s(\underline{\rho},0) = -\frac{i\mu_o I}{2}\ \delta(\eta)\xi^{-1}|\xi|^{-1}e^{-h|\xi|}\ . \tag{6.5}$$

Substituting (6.5) into (4.50), (4.57), and (4.58), as well as the source fields and images as determined from (6.1) and (6.2) into equations (4.51), (4.52), and (4.53), we determine everywhere the total electromagnetic field. Note that the factor $\delta(\eta)$ in equation (6.5) will reduce the double Fourier integrals of the general solutions to single integrals.

6.3 Vertical magnetic dipole

We consider the case of a quasi-static time-harmonic magnetic dipole, with moment M oriented vertically at z = - h above the semi-infinite conductor. The source fields are given by:

$$\underline{E}^s = \frac{i\omega M(y\hat{i} - x\hat{j})}{4\pi R^3} \tag{6.6}$$

and

$$\underline{B}^s = -\frac{M}{4\pi}\nabla\left\{\frac{h+z}{R^3}\right\} \tag{6.7}$$

where $R^2 = r^2 + (h+z)^2$, and again $e^{i\omega t}$ is suppressed.

Since we can see from (6.7) that

$$\Omega^s(\underline{r},z) = \frac{M}{4\pi}\left\{\frac{h+z}{R^3}\right\}$$

we can infer from equation (3.60) that

$$\Gamma^s(\underline{r},z) = M/4\pi R^2 \tag{6.8}$$

Weaver (1971) shows this has the Fourier transform

$$G^s(\underline{\rho},z) = \frac{M}{4\pi}\rho^{-1}e^{-\rho(h+z)} \tag{6.9}$$

whence

$$G^s(\underline{\rho},0) = Me^{-\rho h}/4\pi\rho \tag{6.10}$$

When (6.6), (6.7) and (6.10) are substituted into equations

(4.50), (4.51), (4.52), (4.53), (4.57) and (4.58), we obtain the solutions for the electromagnetic field everywhere. It will be noted that these solutions include double Fourier integrals; in fact, by transforming these integrals into cylindrical coordinates, the symmetry of the source can be used to reduce these double integrals to single integrals as shown in Appendix C.

6.4 Horizontal magnetic dipole

If we now consider the same magnetic dipole at $z = -h$ as in section 6.3 , but now oriented parallel to the x-axis, we have the source fields given by

$$\underline{B}^s = -\frac{M}{4\pi}\nabla\left(\frac{x}{R^3}\right) \tag{6.11}$$

and

$$\underline{E}^s = \frac{i\omega M}{4\pi}\{(h+z)\,\hat{j} - y\,\hat{k}\}/R^3 \quad . \tag{6.12}$$

It follows (Weaver, 1971) that

$$\Gamma^s(\underline{r},z) = Mx/4\pi(R+h+z)R \tag{6.13}$$

from which we can deduce

$$G^s(\underline{\rho},z) = iM\xi e^{-\rho(h+z)}/4\pi\rho^2 \tag{6.14}$$

and

$$G^s(\underline{\rho},0) = iM\xi e^{-\rho h}/4\pi\rho^2 \quad . \tag{6.15}$$

Thus from (6.11), (6.12), and (6.15) we can determine the electromagnetic field everywhere. Once again the double Fourier transform integrals which we obtain can be reduced to single integrals by a transformation to cylindrical coordinates. Again, this transformation is made in Appendix C.

CHAPTER 7

NUMERICAL EVALUATION OF THE FIELDS

7.1 Introductory comments

To solve numerically for the fields in the general case (with arbitrary source) we must evaluate the double inverse Fourier integrals appearing in the solutions of Chapter 4. We indicated that for the particular sources we considered in Chapter 6, the double integrals in the solutions could be reduced to single integrals; but in general a numerical technique is needed with which we can compute a two-dimensional infinite integral transform.

We shall use an integration technique applicable to any arbitrary magnetic source, in keeping with the general treatment with which the Hertz potential theory has been developed. This is to say we shall compute numerically the Fourier transforms as they appear in Chapter 4, rather than use a variety of numerical techniques to compute a variety of specific single integrals. The numerical method described in this Chapter is one which can be used independently of the geometrical properties of the source. Thus even inducing sources not expressible by a single analytic function could conceivably be considered, as long as the appropriate magnetic Hertz potential is specified numerically. However, in this thesis our discussion will be confined to the numerical solution to the problem of the horizontal line current source, and the vertical magnetic dipole source.

7.2 The discrete Fourier transform

Of course, in order to determine numerically the value of infinite integrals, the integrals must be made into finite discrete series. Cooley et al. (1967) provide a concise derivation of the one-dimensional discrete Fourier transform from the continuous integral. The two-dimensional Fourier transform can be made finite and discrete in a completely analogous manner; we shall follow here only the one-dimensional derivation.

Cooley et al. use an alternative definiton of the Fourier transform from ξ-space to s-space:

$$\gamma(s) = \int_{-\infty}^{+\infty} g(\xi)\, e^{-2\pi i\xi s}\, d\xi \tag{7.1}$$

with the inverse

$$g(\xi) = \int_{-\infty}^{+\infty} \gamma(s)\, e^{2\pi i\xi s}\, ds \tag{7.2}$$

If we set $x = 2\pi s$, $\sqrt{2\pi}\Phi(x) = \gamma(x/2\pi)$, and $g(\xi) = F(\xi)$ in (7.1) we arrive at our definition of the one-dimensional inverse Fourier transform of Appendix A. We recall that it is the inverse Fourier transform which must be computed to determine numerically our fields.

Evaluating $g(\xi)$ at the equally spaced discrete points $m\Delta\xi$ $(m=0,\pm1,\pm2,\ldots)$ we have from (7.2)

$$g(m\Delta\xi) = \int_{-\infty}^{+\infty} \gamma(s)\, e^{2\pi i s m\Delta\xi}\, ds \quad . \tag{7.3}$$

If we define $\lambda = 1/\Delta\xi$, then (7.3) can be written

$$g(m\Delta\xi) = \int_{-\infty}^{+\infty} \gamma(s) e^{2\pi ims/\lambda}\, ds = \sum_{k=-\infty}^{\infty} \int_{k\lambda}^{(k+1)\lambda} \gamma(s) e^{2\pi ims/\lambda}\, ds \qquad (7.4)$$

Making the substitution $s = u + k\lambda$, and noting that $\exp\{-2\pi imk\} = 1$, we obtain

$$g(m\Delta\xi) = \int_0^{\lambda} \gamma_p(u) e^{2\pi imu/\lambda}\, du \qquad (7.5)$$

where

$$\gamma_p(u) = \sum_{k=-\infty}^{\infty} \gamma(u + k\lambda) \qquad (7.6)$$

The periodic function γ_p is called the aliased function. Thus $g(m\Delta\xi)$ can be thought of as a coefficient in a Fourier series expansion of $\gamma_p(u)$ which has periodicity λ, so that writing the dummy variable u as s again, we obtain

$$\gamma_p(s) = \frac{1}{\lambda} \sum_{m=-\infty}^{\infty} g(m\Delta\xi)\; e^{-2\pi ims/\lambda} \qquad (7.7)$$

We shall sample γ_p at the N points $n\Delta s$ $(n=0,1,\ldots,N-1)$ between 0 and λ, i.e. $\Delta s = \lambda/N = 1/N\Delta\xi$. By (7.7) it follows that

$$\gamma_p(n\Delta s) = \frac{1}{\lambda} \sum_{m=-\infty}^{\infty} g(m\Delta\xi)\; e^{-2\pi imn/N} \qquad (7.8)$$

We note that the summation over m can be rewritten as a double summation by the following identity

$$\sum_{m=-\infty}^{\infty} a_m \equiv \sum_{m=0}^{N-1} \sum_{\ell=-\infty}^{\infty} a_{m+\ell N}$$

Observing also that

$$\exp\{-2\pi in(m+\ell N)/N\} = \exp\{-2\pi imn/N\}$$

and recalling the definition of λ, we can rewrite (7.8) in the form

$$\gamma_p(n\Delta s) = \Delta\xi \sum_{m=0}^{N-1} \sum_{\ell=-\infty}^{\infty} g(m\Delta\xi + \ell N\Delta\xi)\, e^{-2\pi imn/N} \tag{7.9}$$

By analogy with (7.6), let us define the aliased transform function as

$$g_p(u) = \sum_{\ell=-\infty}^{\infty} g(u + \ell\tau) \tag{7.10}$$

Then, putting $\tau = N\Delta\xi$, by (7.9) we have

$$\gamma_p(n\Delta s) = \Delta\xi \sum_{m=0}^{N-1} g_p(m\Delta\xi)\, e^{-2\pi imn/N} \tag{7.11}$$

An analogous analysis applied to (7.1) rather than (7.2) yields the corresponding result:

$$g_p(m\Delta\xi) = \Delta s \sum_{n=0}^{N-1} \gamma_p(n\Delta s)\, e^{2\pi imn/N} \tag{7.12}$$

Equations (7.11) and (7.12) constitute the discrete Fourier transform and discrete inverse Fourier transform respectively (if we call equation (7.1) the Fourier transform as Cooley et al. have done).

So far, we have made no approximations. We are free, of course, to select the period τ judiciously, and we choose it such that

$$g(\xi) \simeq 0 \quad \text{for} \quad |\xi| > \tfrac{1}{2}\tau \qquad (7.13)$$

This ensures that there will be a negligible contribution to the integral (7.1) outside the finite range $-\tfrac{1}{2}\tau < \xi < \tfrac{1}{2}\tau$. If we view $g_p(m\Delta\xi)$ on the domain $(0,\tfrac{1}{2}\tau)$, then the only non-vanishing term in the sum (7.10) is for $\ell = 0$, so that

$$g_p(m\Delta\xi) \simeq g(m\Delta\xi) \quad , \; 0 < m\Delta\xi < \tfrac{1}{2}\tau \qquad (7.14)$$

Similarly, on the domain $(\tfrac{1}{2}\tau,\tau)$, the only non-vanishing term is for $\ell = -1$, i.e.

$$g_p(m\Delta\xi) \simeq g(m\Delta\xi - \tau) \; , \; \tfrac{1}{2}\tau < m\Delta\xi < \tau \qquad (7.15)$$

The expressions (7.14) and (7.15) show that the function g, non-vanishing on the domain $(-\tfrac{1}{2}\tau,\tfrac{1}{2}\tau)$ is approximated by the restriction of the aliased function g_p on the domain $(0,\tau)$ with that part of g defined in the negative axis corresponding to g_p defined on $(\tfrac{1}{2}\tau,\tau)$.

Following a similar procedure for γ , we select λ such that

$$\gamma(s) \simeq 0 \quad \text{for} \; |s| > \tfrac{1}{2}\lambda$$

Hence if we view the function on the domain $(0,\lambda)$, then

$$\left.\begin{aligned} \gamma_p(n\Delta s) &\simeq \gamma(n\Delta s) \quad , \quad 0 < n\Delta s < \tfrac{1}{2}\lambda \\ \gamma_p(n\Delta s) &\simeq \gamma(n\Delta s - \lambda) \quad , \quad \tfrac{1}{2}\lambda < n\Delta s < \lambda \end{aligned}\right\} \quad (7.16)$$

We define the 'aliasing error of g_p to be

$$\left.\begin{aligned} &|g_p(u) - g(u)| \; , \quad 0 < u < \tfrac{1}{2}\tau \\ &|g_p(u) - g(u-\tau)| \; , \quad \tfrac{1}{2}\tau < u < \tau \end{aligned}\right\} \quad (7.17)$$

and of γ_p, to be

$$\left.\begin{aligned} &|\gamma_p(u) - \gamma(u)| \; , \quad 0 < u < \tfrac{1}{2}\lambda \\ &|\gamma_p(u) - \gamma(u-\lambda)| \; , \quad \tfrac{1}{2}\lambda < u < \lambda \end{aligned}\right\} \quad (7.18)$$

These errors may be made arbitrarily small by making τ and λ sufficiently large to encompass the domains in which the functions g and γ differ substantially from zero. Since $\tau = N\Delta\xi$ and $\lambda = N\Delta s$, this can be satisfied by choosing the appropriate N.

The relationship between the sampling intervals Δs and $\Delta\xi$ is given above as

$$\Delta s = 1/N\Delta\xi \quad (7.19)$$

It can be seen that a consideration in choosing $\Delta\xi$ is that it must be sufficiently small to define well the integrand. Of course, we wish the transformed function to be defined clearly as well, so Δs must also be suitably small. We see from (7.19) that these considerations imply that again a judicious choice of N (and thus of τ and λ) must be made.

The two-dimensional discrete transform analogous to equation (7.11) is given by

$$\gamma_p(n_1\Delta s, n_2\Delta s) = (\Delta\xi)^2 \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} g_p(m_1\Delta\xi, m_2\Delta\xi) \times \exp\left[-\frac{2\pi i}{N}(m_1 n_1 + m_2 n_2)\right] \tag{7.20}$$

which has the inverse

$$g_p(m_1\Delta\xi, m_2\Delta\xi) = (\Delta s)^2 \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} \gamma_p(n_1\Delta s, n_2\Delta s) \times \exp\left[\frac{2\pi i}{N}(m_1 n_1 + m_2 n_2)\right] \tag{7.21}$$

where, in both (7.20) and (7.21), we have assumed the range of discrete variables constitutes a square array of sampling points, i.e. $\Delta\xi = \Delta\eta$, $\Delta s = \Delta t$, if (ξ,η) and (s,t) represent the transform space and configuration space respectively for the case of the continuous two-dimensional transform.

7.3 The Fast Fourier Transform algorithm

The series appearing in equations (7.11), (7.12), (7.20), and (7.21) can be computed directly by conventional techniques. However, a machine algorithm has been developed (Cooley and Tukey, 1965) which greatly reduces the computation time; this algorithm is called the Fast Fourier Transform (or FFT).

The essential feature of the algorithm is that it divides N into its multiplicative factors (e.g. $N = A \cdot B$). If then the dummy indices of summation are rewritten in the series, it is found that the series separates into a double series, each part of which is a sum over a smaller range.

If there is an integer n such that $N = 2^n$, then the FFT is particularly useful since it will compute the discrete Fourier transform purely by addition and subtraction, thus avoiding complex exponentiation arithmetic. The number of operations in this case will be $2N\log_2 N$; this is to be compared with N^2 operations using direct conventional techniques. Thus, according to published reports, for N = 8192 , the FFT computations require about 5 seconds on an I.B.M. 7094 computer, whereas conventional procedures might take about 1800 seconds (Cochran et al., 1967). A Fortran listing of the (inverse) Fourier transform program used in the computations of this Chapter is found in Appendix D.

7.4 Dimensionless form of solutions

Before investigating the particular source fields we wish to consider numerically, it is convenient to make the variables non-dimensional by scaling all distances in skin-depths. We shall denote the variables transformed into this dimensionless form by placing a tilde on the variable symbol. By making such a transformation, the range of integration is more manageable since the new variables will be of order unity.

In Fourier transform space the vector $\underline{\rho}$, having the dimension of reciprocal length, can be made dimensionless by defining

$$\underline{\tilde{\rho}} = \underline{\rho}\,\delta \tag{7.22}$$

where δ is the skin-depth of the first layer, i.e.

$$\delta \equiv (\tfrac{1}{2}\omega\mu_o\sigma_1)^{-\frac{1}{2}} \tag{7.23}$$

(We have assumed again a uniform magnetic permeability, μ_o, in the conductor.) The variable $\underline{r}$, on the other hand, has the dimension of length and is therefore transformed into dimensionless form by the definition:

$$\underline{\tilde{r}} = \underline{r}/\delta \quad . \tag{7.24}$$

Similarly we can transform the variables z and d_n by writing

$\tilde{z} = z/\delta$, and $\tilde{d}_n = d_n/\delta$.

Substituting (7.22) and (7.23) in equation (2.9) we see that

$$\nu_n = (\delta^2\rho^2 + i2\delta^2\sigma_n/\sigma_1)^{\frac{1}{2}}$$

Thus we define the dimensionless variable

$$\tilde{\nu}_n = \nu_n/\delta = (\tilde{\rho}^2 + i2\sigma_n/\sigma_1)^{\frac{1}{2}}$$

We can see now that $\nu_n d_n = \tilde{\nu}_n \tilde{d}_n$ and likewise other dimensionless factors appearing in the field expressions, e.g. $K(\rho)$ and Q_i , remain unchanged when the variable transformations are made. We also note that under this transformation the differential $d\underline{\rho} \equiv d\xi d\eta$ becomes $d\underline{\tilde{\rho}}/\delta^2$ in the double integrals.

We shall make this transformation in the field integrals in Chapter 4. Also, in order to make the fields themselves dimensionless, we must divide our solutions by any constant dimensional factors that appear in the term $G^s(\underline{\rho},0)$ in the integrand. When we come to evaluate the integrals numerically we shall concern ourselves exclusively with dimensionless field components.

7.5 Field integrals

In Chapter 4 we expressed explicitly the field solutions in terms of the electric and magnetic Hertz potentials (see Section 4.2) . Also, we showed that the solutions could be expressed more concisely in terms of the source fields, the image fields, and the magnetic Hertz potential alone (see Section 4.3) . For the case of the horizontally oriented line current described in Section 6.2, we see from equation (6.2) that $E_z^s = 0$. It follows from equation (3.62) that in free space Π^s vanishes, and hence by (4.45), the total electric Hertz potential, Π, vanishes everywhere. Similarly, for the vertical magnetic dipole of Section 6.3, we find that $E_z^s = 0$, so that we can again deduce that $\Pi = 0$ everywhere for this source. Thus for both our line current source, and our vertical magnetic dipole source, we can substitute $G^s(\underline{\rho},0)$ from (6.5) and (6.10) directly into the field expressions of Section 4.2*.

After the transformation to dimensionless variables described in Section 7.4 , the total field at the surface of the conductor of a line current horizontally oriented at $z = -h$, will be given by

$$B_x = \frac{I\mu_o}{4\pi\delta} \int_{-\infty}^{+\infty} [2 - K(\xi)]\; e^{-\tilde{h}|\tilde{\xi}|}\; e^{-i\tilde{\xi}\tilde{x}}\; d\tilde{\xi} \qquad (7.25)$$

* Note that throughout this Chapter we shall consider the conducting half-space to be of uniform permeability.

$$B_z = -\frac{iI\mu_o}{4\pi\delta}\int_{-\infty}^{+\infty} K(\tilde{\xi})\ \mathrm{sgn}(\tilde{\xi})\ e^{-\tilde{h}|\tilde{\xi}|}\ e^{-i\tilde{\xi}\tilde{x}}\ d\tilde{\xi} \qquad (7.26)$$

$$E_y = -\frac{iI\omega\mu_o}{4\pi}\int_{-\infty}^{+\infty} K(\tilde{\xi})\,|\tilde{\xi}|^{-1}\ e^{-\tilde{h}|\tilde{\xi}|}\ e^{-i\tilde{\xi}\tilde{x}}\ d\tilde{\xi} \qquad (7.27)$$

and $B_y = E_x = E_z = 0$. Note that the integrands themselves are now dimensionless since all the dimensions are contained in the factors outside the integrals. Advantage can be taken of the oddness or evenness of the integrands when computing these fields.

For the vertical magnetic dipole at $z = -h$, the fields at the surface of the conductor are

$$B_x = \frac{iM}{8\pi^2\delta^3}\int_{-\infty}^{+\infty} [2 - K(\tilde{\rho})]\ \tilde{\xi}\ e^{-\tilde{\rho}\tilde{h}}\ e^{-i\underline{\tilde{\rho}}\cdot\underline{\tilde{r}}}\ d\underline{\tilde{\rho}} \qquad (7.28)$$

$$B_y = \frac{iM}{8\pi^2\delta^3}\int_{-\infty}^{+\infty} [2 - K(\tilde{\rho})]\ \tilde{\eta}\ e^{-\tilde{\rho}\tilde{h}}\ e^{-i\underline{\tilde{\rho}}\cdot\underline{\tilde{r}}}\ d\underline{\tilde{\rho}} \qquad (7.29)$$

$$B_z = \frac{M}{8\pi^2\delta^3}\int_{-\infty}^{+\infty} K(\tilde{\rho})\ \tilde{\rho}\ e^{-\tilde{\rho}\tilde{h}}\ e^{-i\underline{\tilde{\rho}}\cdot\underline{\tilde{r}}}\ d\underline{\tilde{\rho}} \qquad (7.30)$$

$$E_x = -\frac{M\omega}{8\pi^2\delta^2}\int_{-\infty}^{+\infty} K(\tilde{\rho})\ \frac{\tilde{\eta}}{\tilde{\rho}}\ e^{-\tilde{\rho}\tilde{h}}\ e^{-i\underline{\tilde{\rho}}\cdot\underline{\tilde{r}}}\ d\underline{\tilde{\rho}} \qquad (7.31)$$

$$E_y = \frac{M\omega}{8\pi^2\delta^2}\int_{-\infty}^{+\infty} K(\tilde{\rho})\ \frac{\tilde{\xi}}{\tilde{\rho}}\ e^{-\tilde{\rho}\tilde{h}}\ e^{-i\underline{\tilde{\rho}}\cdot\underline{\tilde{r}}}\ d\underline{\tilde{\rho}} \qquad (7.32)$$

As we indicated previously, $E_z = 0$.

For the horizontal magnetic dipole, the field integrals are most easily obtained by substituting E^s, $E^\dagger$, B^s, $B^\dagger$, and $G^s(\underline{\rho},0)$ into the simplified solutions of Section 4.3 . Although we do not compute numerically these integrals, for completeness we state them here; the fields at the surface of the conductor due to a horizontal magnetic dipole at $z = -h$ are given by:

$$B_x = \frac{M(3\tilde{x}^2-\tilde{R}^2)}{2\pi\tilde{R}^5\delta^3} + \frac{M}{8\pi^2\delta^3}\int_{-\infty}^{+\infty} K(\tilde{\rho})\,\frac{\tilde{\xi}^2}{\tilde{\rho}}\,e^{-\tilde{\rho}\tilde{h}}\,e^{-i\underline{\tilde{\rho}}\cdot\underline{\tilde{r}}}\,d\underline{\tilde{\rho}} \tag{7.33}$$

$$B_y = \frac{3M\tilde{x}\tilde{y}}{2\pi\tilde{R}^5\delta^3} + \frac{M}{8\pi^2\delta^3}\int_{-\infty}^{+\infty} K(\tilde{\rho})\,\frac{\tilde{\xi}\tilde{\eta}}{\tilde{\rho}}\,e^{-\tilde{\rho}\tilde{h}}\,e^{-i\underline{\tilde{\rho}}\cdot\underline{\tilde{r}}}\,d\underline{\tilde{\rho}} \tag{7.34}$$

$$B_z = \frac{iM}{8\pi^2\delta^3}\int_{-\infty}^{+\infty} K(\tilde{\rho})\,\tilde{\xi}\,e^{-\tilde{\rho}\tilde{h}}\,e^{-i\underline{\tilde{\rho}}\cdot\underline{\tilde{r}}}\,d\underline{\tilde{\rho}} \tag{7.35}$$

$$E_x = -\frac{iM\omega}{8\pi^2\delta^2}\int_{-\infty}^{+\infty} K(\tilde{\rho})\,\frac{\tilde{\xi}\tilde{\eta}}{\tilde{\rho}^2}\,e^{-\tilde{\rho}\tilde{h}}\,e^{-i\underline{\tilde{\rho}}\cdot\underline{\tilde{r}}}\,d\underline{\tilde{\rho}} \tag{7.36}$$

$$E_y = \frac{iM\omega}{8\pi^2\delta^2}\int_{-\infty}^{+\infty} K(\tilde{\rho})\,\frac{\tilde{\xi}^2}{\tilde{\rho}^2}\,e^{-\tilde{\rho}\tilde{h}}\,e^{-i\underline{\tilde{\rho}}\cdot\underline{\tilde{r}}}\,d\underline{\tilde{\rho}} \tag{7.37}$$

$$E_z = -\frac{iM\omega\tilde{y}}{2\pi\tilde{R}^3\delta^2} \tag{7.38}$$

7.6 Numerical comparisons

To check our numerical procedures we shall first use a 10-layer model in which all the layer conductivities are set equal in order to generate surface field solutions for a homogeneous conductor. The numerical results obtained in this way are then compared with those found by numerically evaluating the exact integral solutions for this simple case. The variance of our 10-layer solutions with the homogeneous solutions will give some indication of the accuracy with which the factor $K(\rho)$ can be generated numerically. In a similar manner, the 10-layer model can be used to simulate a two-layer conductor. Numerical evaluations of the exact integral solutions for certain sources over a two-layer conductor are available for comparison.

We consider first a line-current source at a height of one skin-depth over a homogeneous conducting half-space. The Fast Fourier Transform is applied in equations (7.25), (7.26), and (7.27). We consider the 10-layer model illustrated in Figure 4a. Recalling that σ_n depends upon the ratio of layer conductivities to the conductivity of the first layer, σ_1 , we set $\sigma_n/\sigma_1 = 1$ $(n = 1,2,...,10)$; for the depth of each layer we set $d_n = 0.1$ skin-depths $(n = 1,2,...,9)$. We compute, and plot, the amplitudes of the complex field components, and compare these to the components found by evaluating Weaver's (1971) line-current solutions for a homogeneous earth using a Gauss-Laguerre quadrature technique. Figure 5 illustrates the comparison of these two computations of the amplitudes of the field components. (Note we plot only the dimensionless part of the field, i.e. we evaluate $\underline{\tilde{B}} = (2\delta^3/I\mu_o)\ \underline{B}$ and $\underline{\tilde{E}} = (2\delta^2/I\omega\mu_o)\ \underline{E}$. In fact we have labelled our plots in terms of components of $\underline{\tilde{H}}$; by the definition of dimensionless fields we can write $\underline{\tilde{B}} \equiv \underline{\tilde{H}}$.)

Likewise, we follow the same procedure for a vertical magnetic dipole one skin-depth above a homogeneous conductor, applying the Fast Fourier Transform to equations (7.28) through (7.32). To simulate this conductor, we again use the model in Figure 4a. In Figure 6 we compare our results for the field amplitudes to those obtained by Thomson (1970) using conventional integration techniques applied to Weaver's (1964) solutions for a vertical dipole.

To simulate a two-layer conductor, we consider the 10-layer model illustrated by Figure 4b, where we set the depth of each layer to be d_n = 0.05 skin-depths, and we set $\sigma_n/\sigma_1 = 1$ $(n = 1,2,\ldots,5)$ and $\sigma_n/\sigma_1 = 10$ $(n = 6,\ldots,10)$. This is effectively a two-layer conductor with an upper layer of 0.25 skin-depths, and the ratio of the second layer conductivity to that of the first is 10. We again compare our evaluations of the amplitudes of the field components due to a vertical dipole source with those obtained by Thomson (1970) using conventional techniques; this comparison is found in Figure 7.

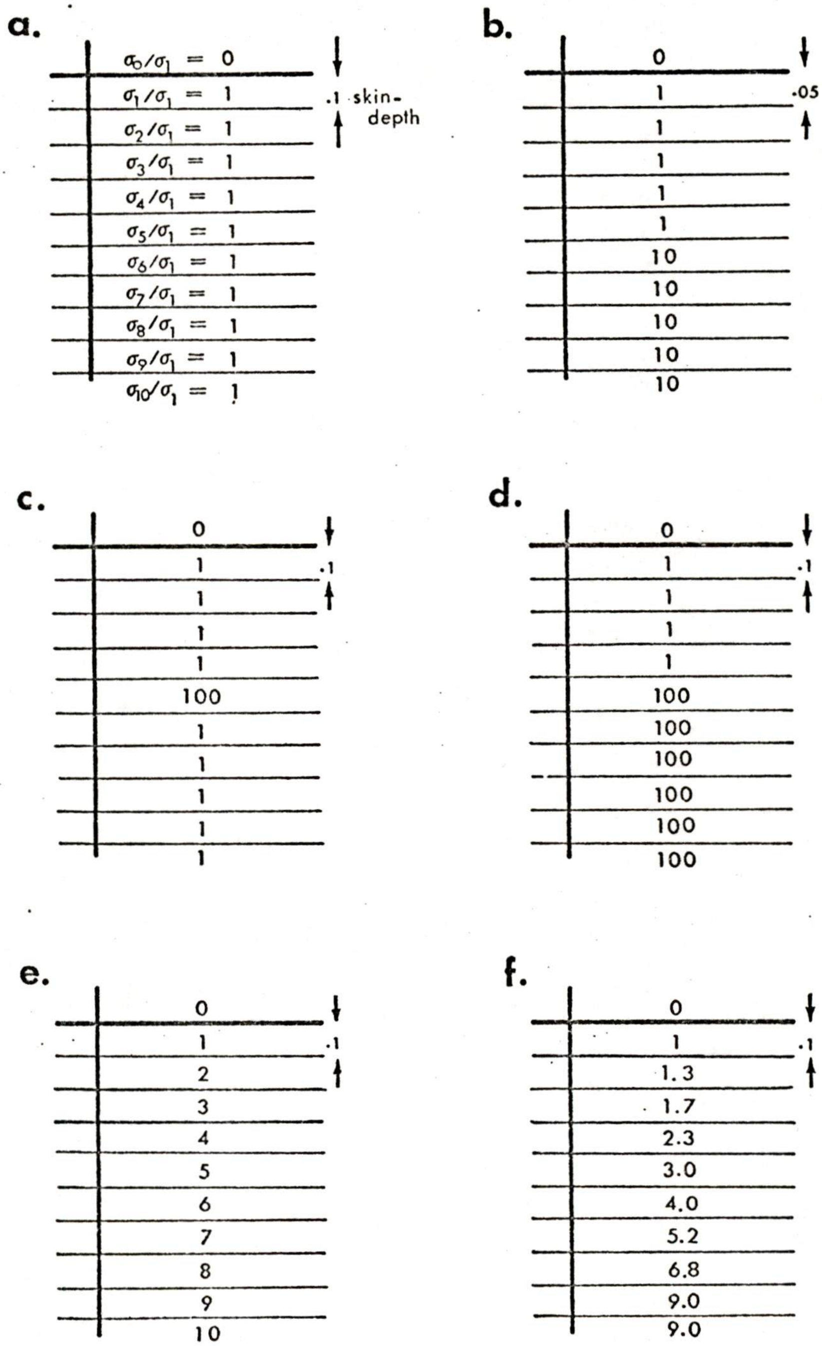

FIGURE 4.

The various conductivity distributions considered in Sections 7.6 and 7.7 .

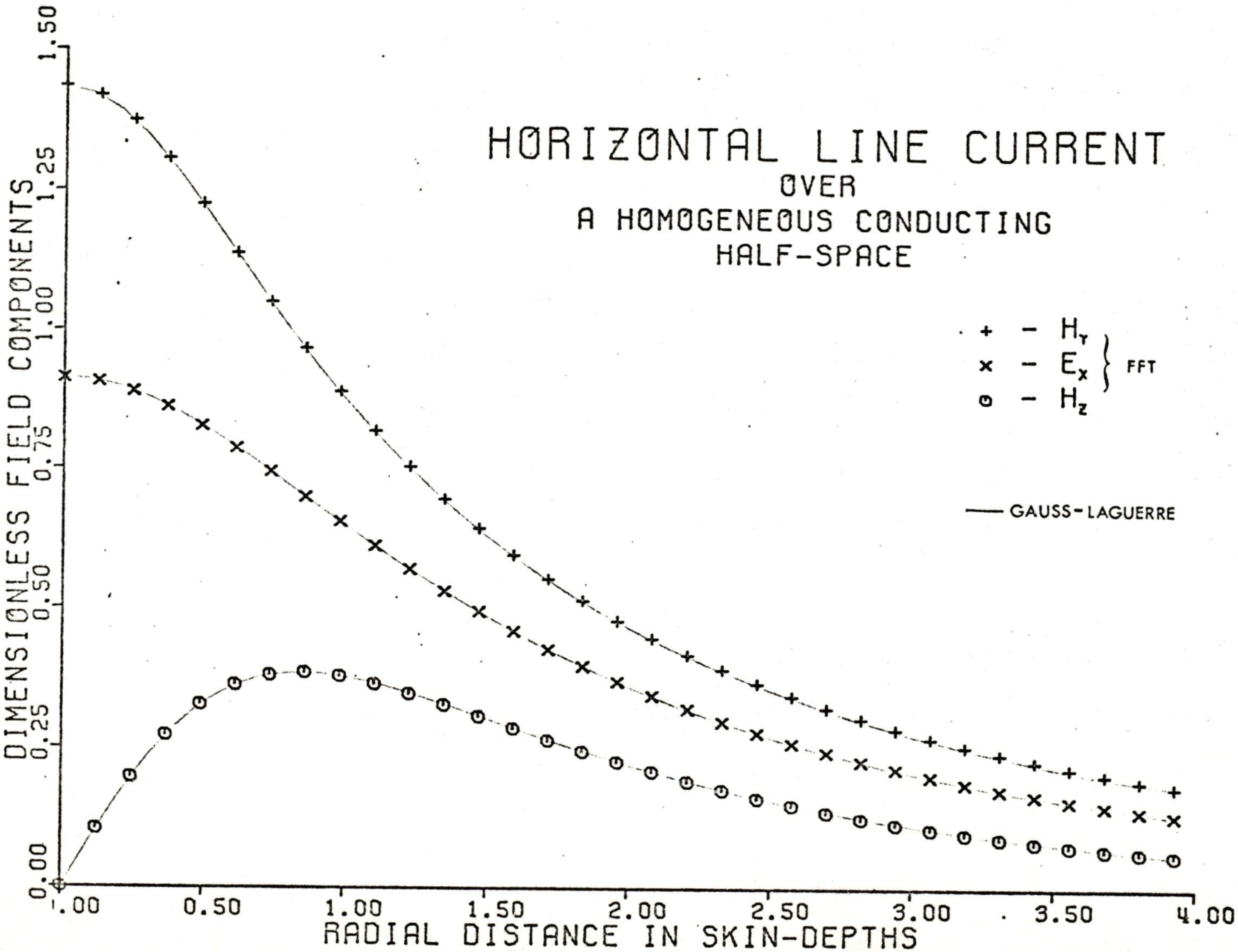

Figure 5. Amplitude comparisons for a line-current over a homogeneous half-space, and the ten-layer system illustrated in Fig. 4a .

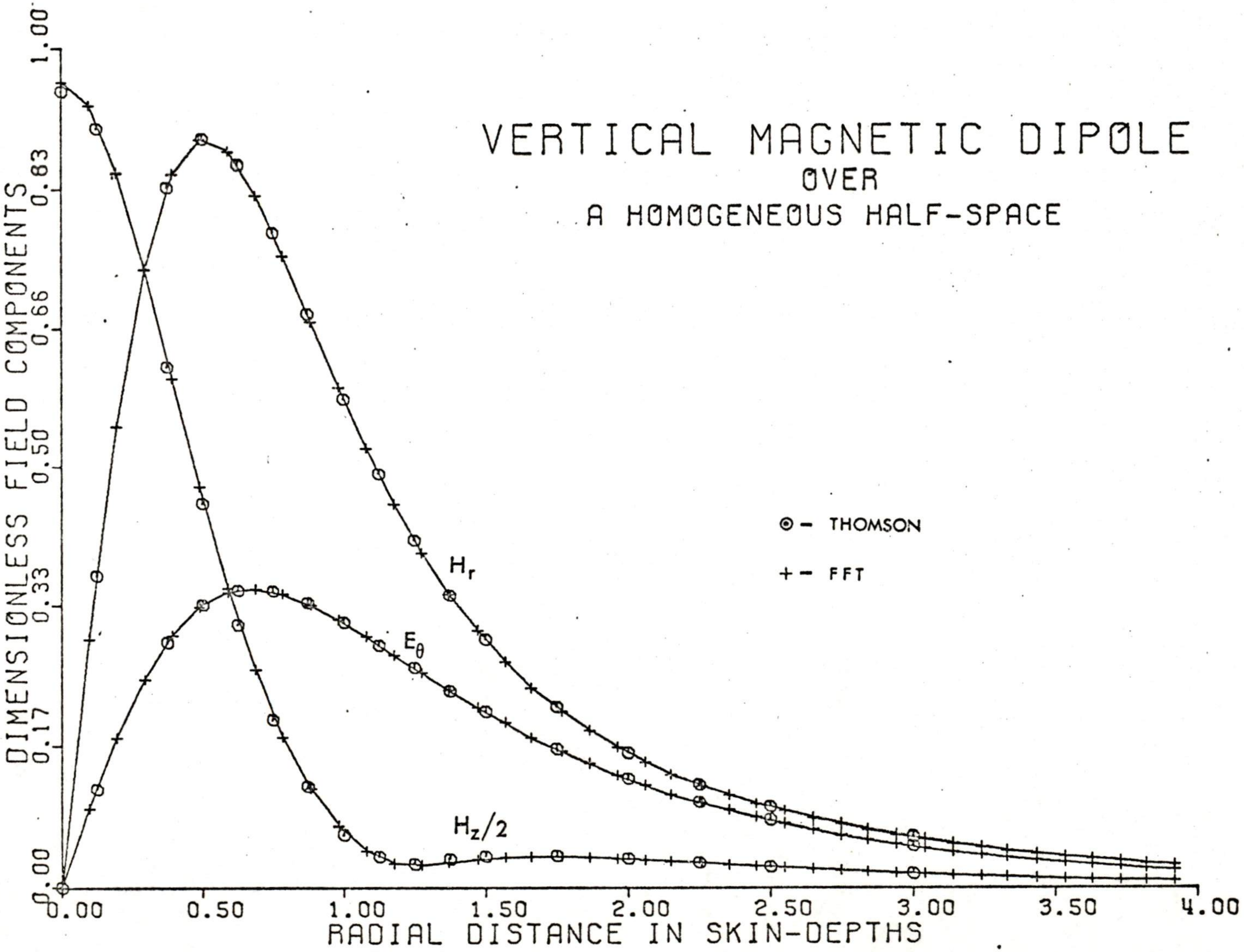

Figure 6. Amplitude comparisons for a vertical magnetic dipole over a homogeneous half-space and the model illustrated in Fig. 4a .

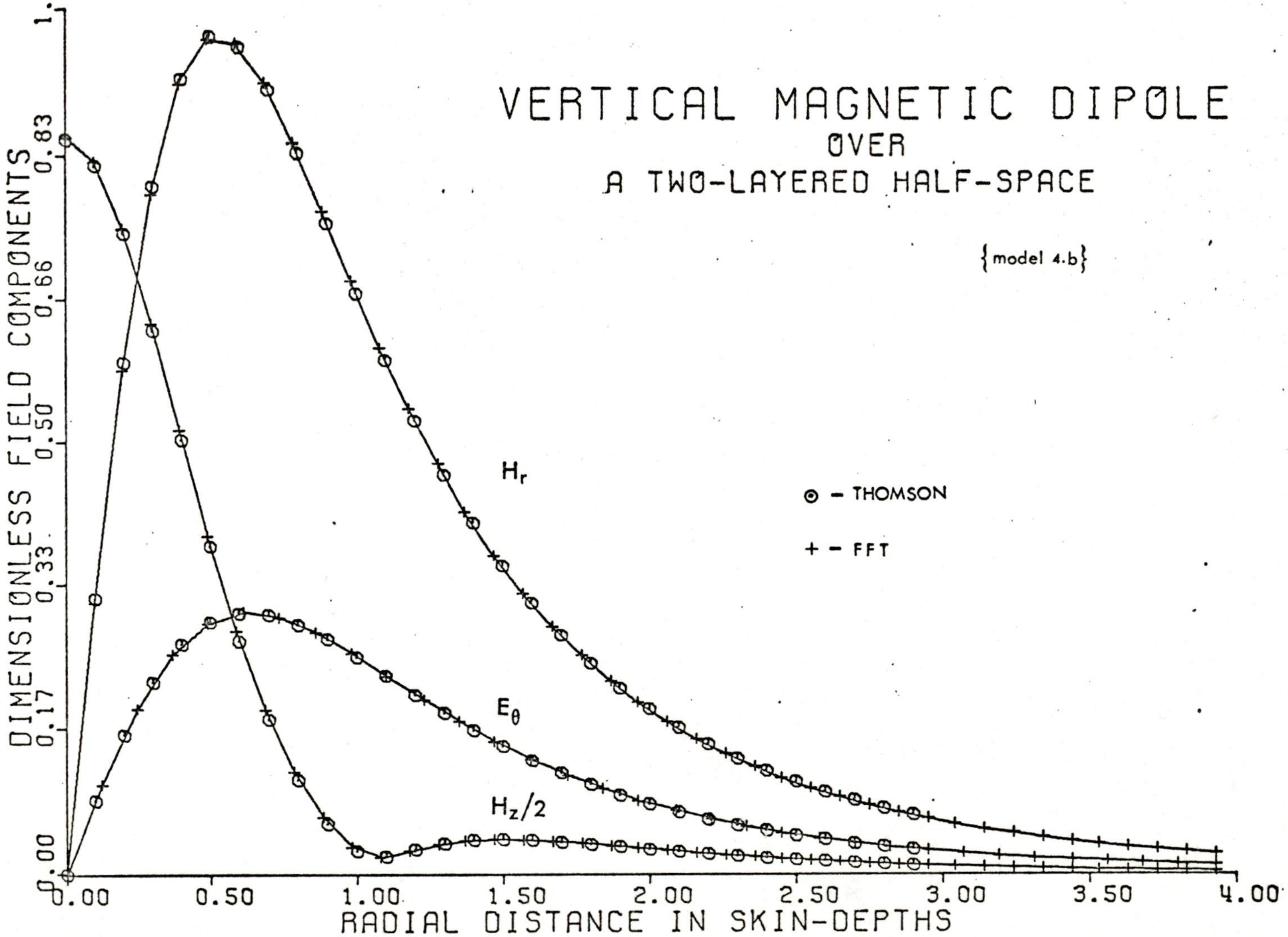

Figure 7. Amplitude comparisons for a vertical magnetic dipole over a two-layer half-space, and that illustrated in Fig. 4b .

7.7 Various conductivity distributions

To further illustrate the N-layer solutions, we now compare the surface fields due to an inducing line-current source, one skin-depth above a conducting half-space with various distributions of layer conductivities. The distributions we consider are illustrated in Figures 4c, 4d, 4e, and 4f. For comparison purposes, we consider also the homogeneous model 4a. Again we arbitrarily choose N = 10. We plot the real and imaginary parts of the field components, as opposed to the amplitude, since all possible induction information is contained in such a graph. The real part of the complex field is often called the in-phase component, and the imaginary part is called the out-of-phase component. The absolute value of the complex field is, of course, the amplitude; the inverse tangent of the ratio of imaginary to real parts, is the phase of the field.

In Figures 8, 9, and 10 we compare the field results for the model of Figure 4a (a homogeneous half-space), the model of Figure 4c (a slab 0.1 skin-depth thick, situated 0.4 skin-depths below the surface, such that the ratio of the slab conductivity to the surrounding conductivity is 100), and the model of Figure 4d (a two-layer conductor with upper layer 0.4 skin-depths thick, and the ratio of the second layer conductivity to that of the first is 100).

We see from the numerical evaluation that an inducing field over the models of Figure 4c and Figure 4d give rise to approximately the same induced fields, indicating that the inducing field is effectively screened by the conducting slab in model 4c from the region below it.

In Figures 11, 12, and 13 we compare the field results for the model in Figure 4a (a homogeneous conductor), the model of Figure 4e (a conductivity increasing linearly with depth over the upper 9 layers), and the model of Figure 4f (a conductivity increasing exponentially with depth over the upper 9 layers).

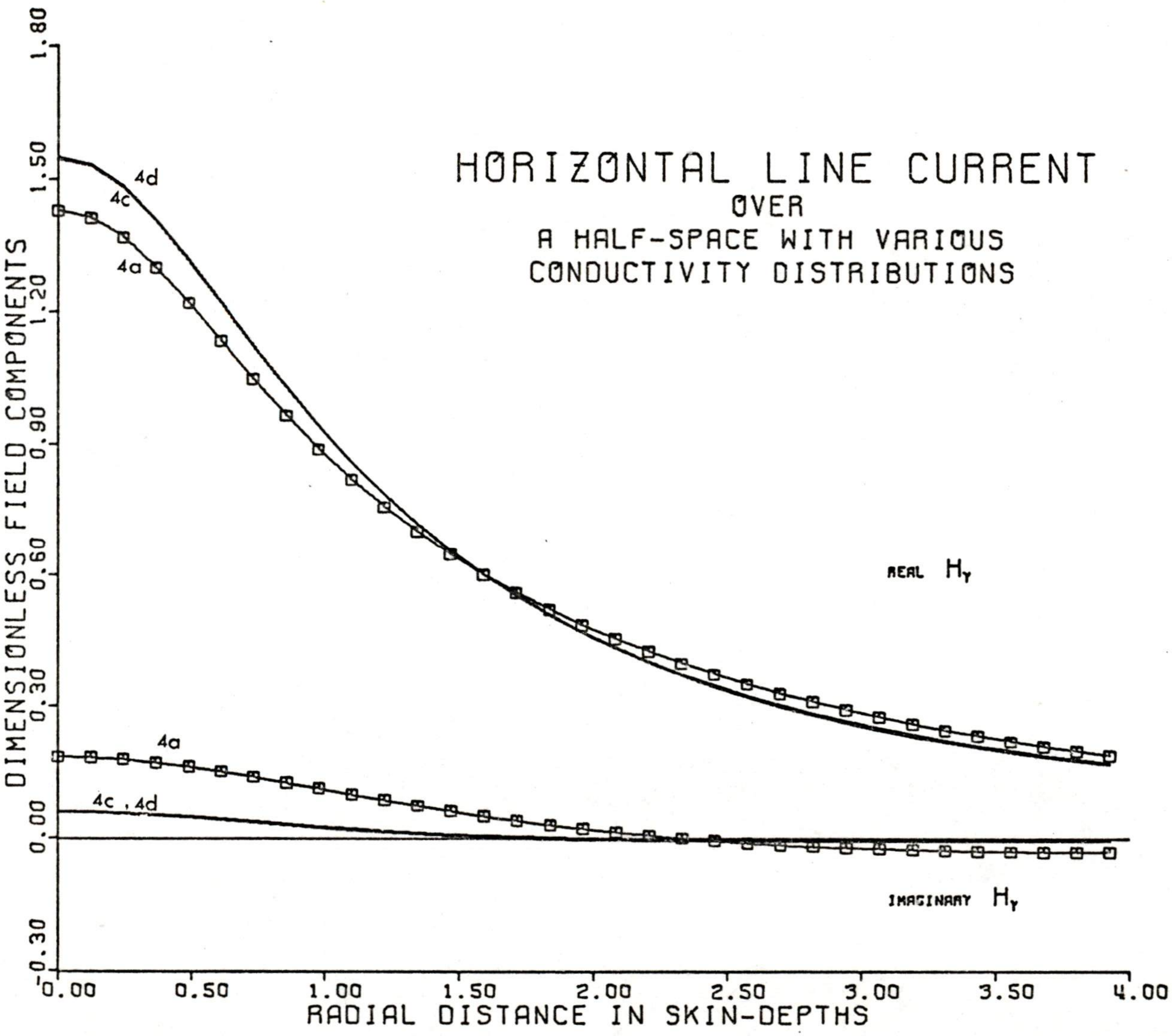

Figure 8. Real and imaginary parts of H_y for line-current over the half-space models illustrated in Figures 4a, 4c, and 4d .

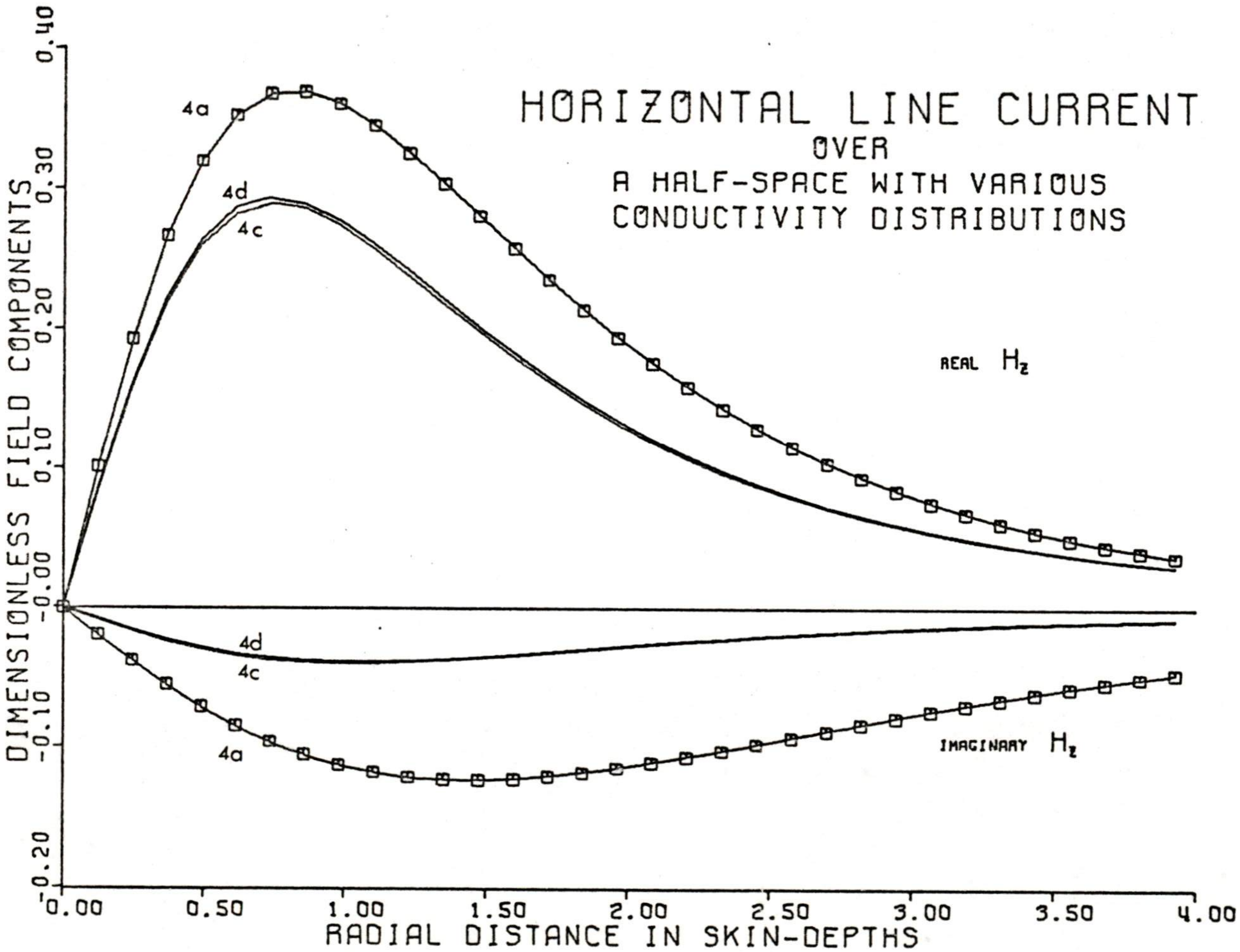

Figure 9. Real and imaginary parts of H_z for line-current over the half-space models illustrated in Figures 4a, 4c, and 4d.

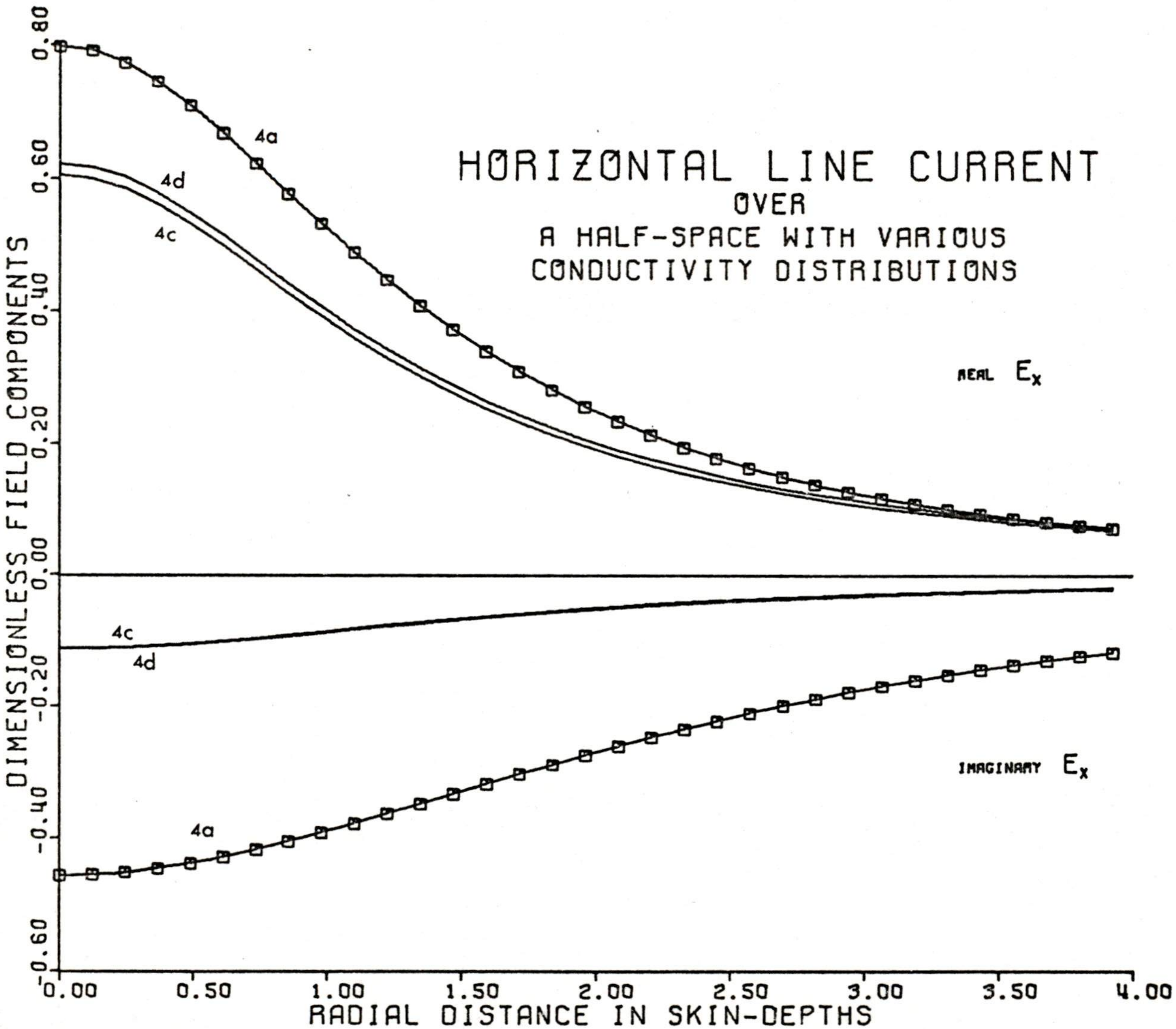

Figure 10. Real and imaginary parts of E_x for line-current over the half-space models illustrated in Figures 4a, 4c, and 4d.

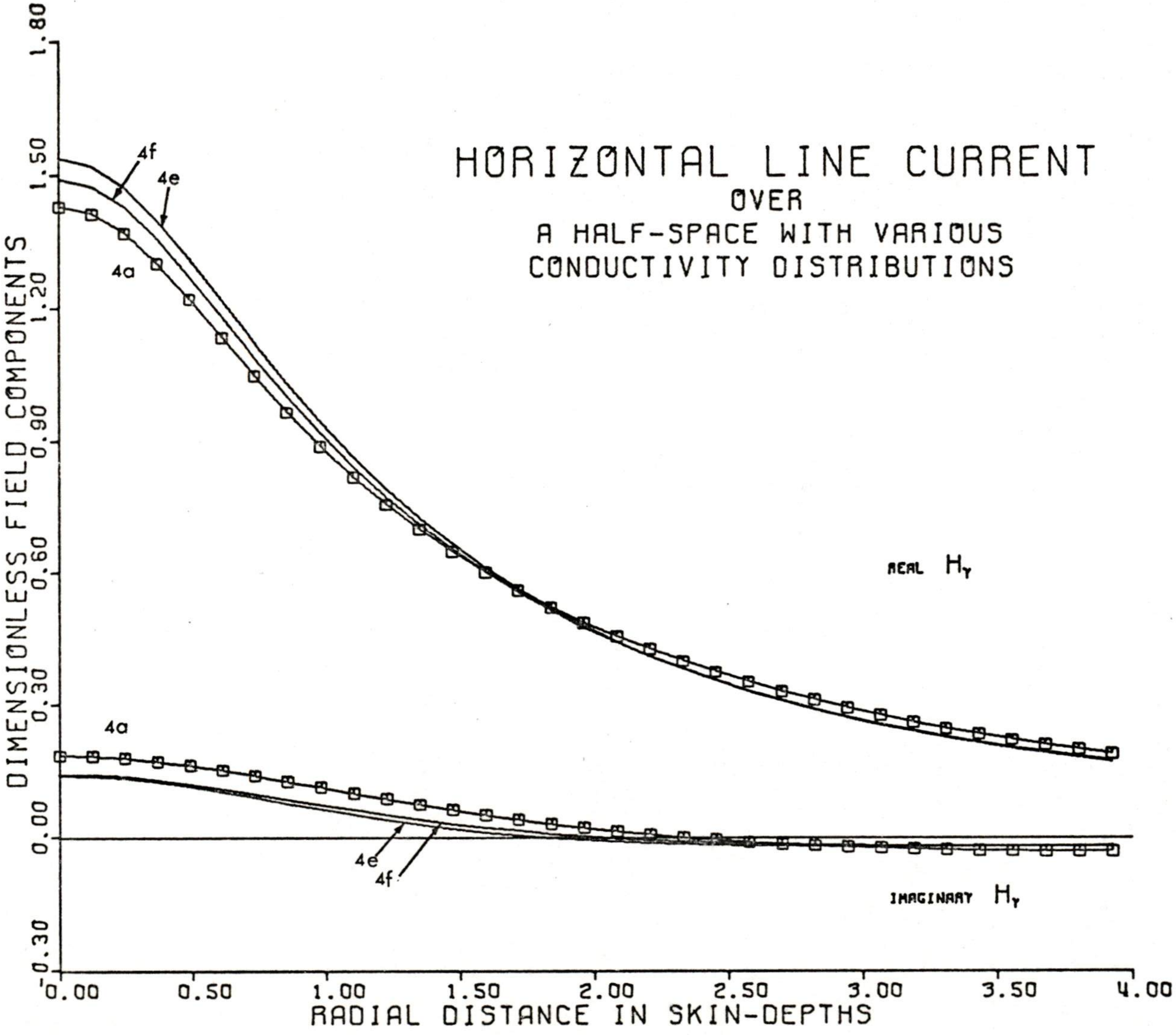

Figure 11. Real and imaginary parts of H_y for line-current over the half-space models illustrated in Figures 4a, 4e, and 4f.

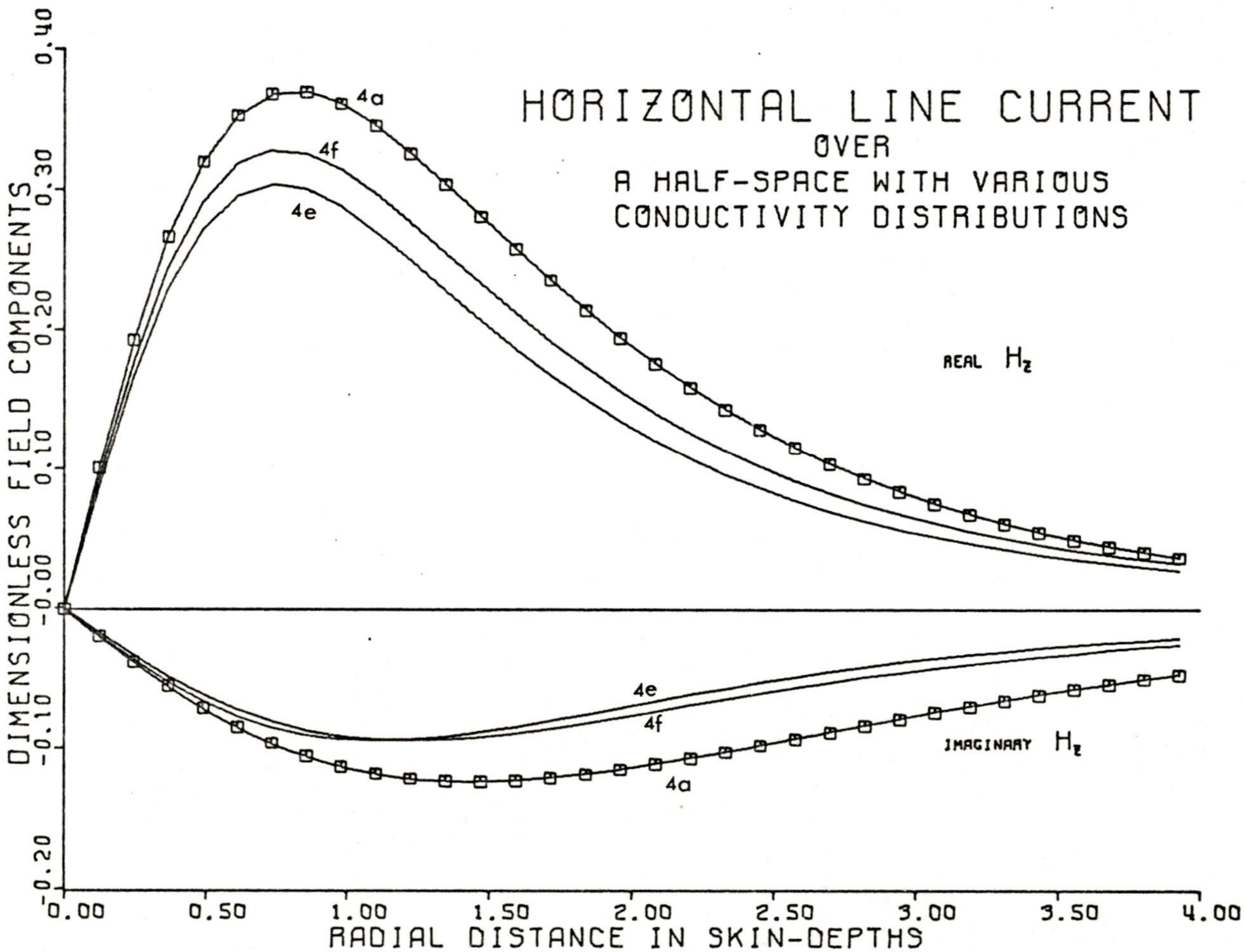

Figure 12. Real and imaginary parts of H_z for line-current over the half-space models illustrated in Figures 4a, 4e, and 4f.

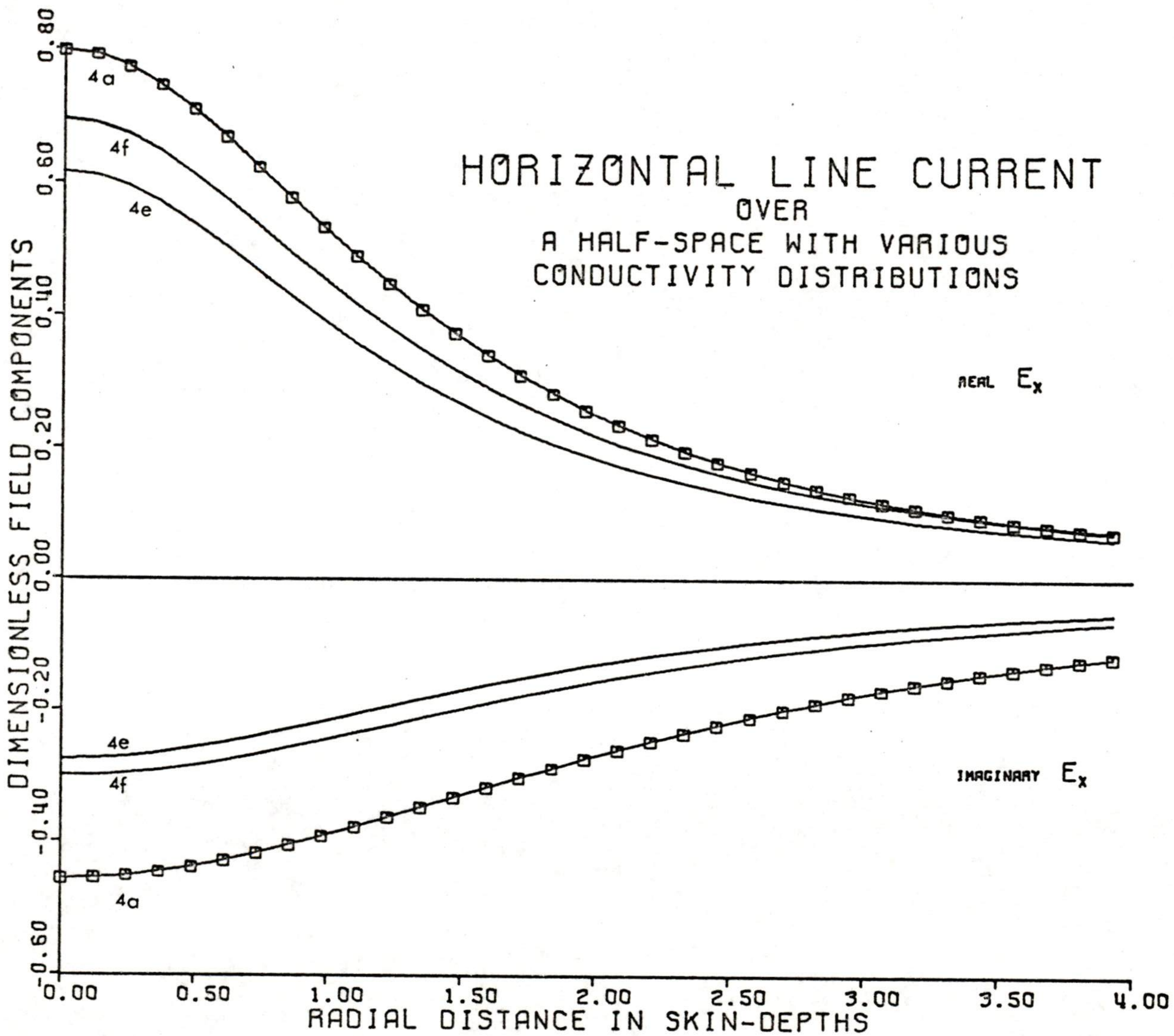

Figure 13. Real and imaginary parts of E_x for line-current over the half-space models illustrated in Figures 4a, 4e, and 4f.

7.8 The field inside the conductor

We can use the field expressions of Section 4.2 to solve numerically for the field inside a conductor due to a line current source one skin-depth above its surface. We evaluate the field at successive depths, each separated by 0.1 skin depth .

For comparison we choose three distributions of layer conductivity: the model of Figure 4a , the model illustrated in Figure 14a , and the model of Figure 14b .

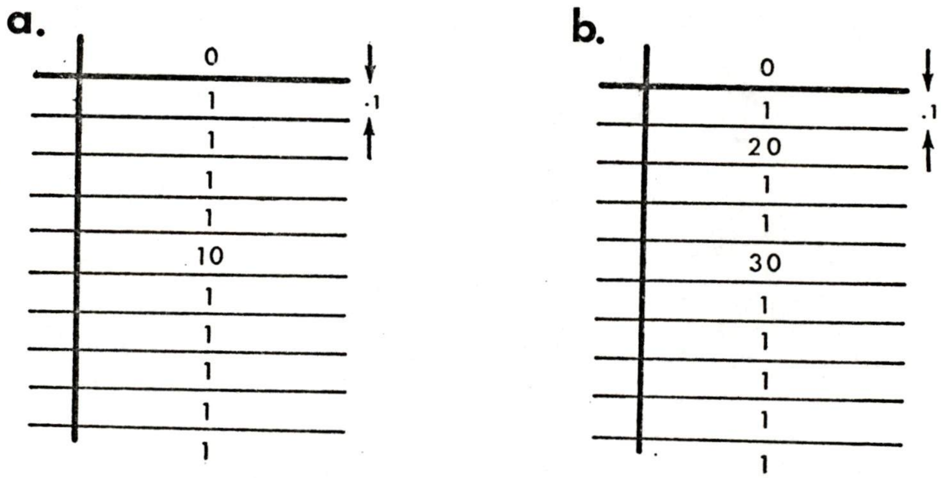

Figure 14.

Figures 15, 16, and 17 illustrate the attenuation of the amplitudes of the field components in a homogeneous conductor; Figures 18, 19, and 20 illustrate the fields inside the model of Figure 14a., and Figures 21, 22, and 23 illustrate the fields at increasing depth in the half-space of Figure 14b .

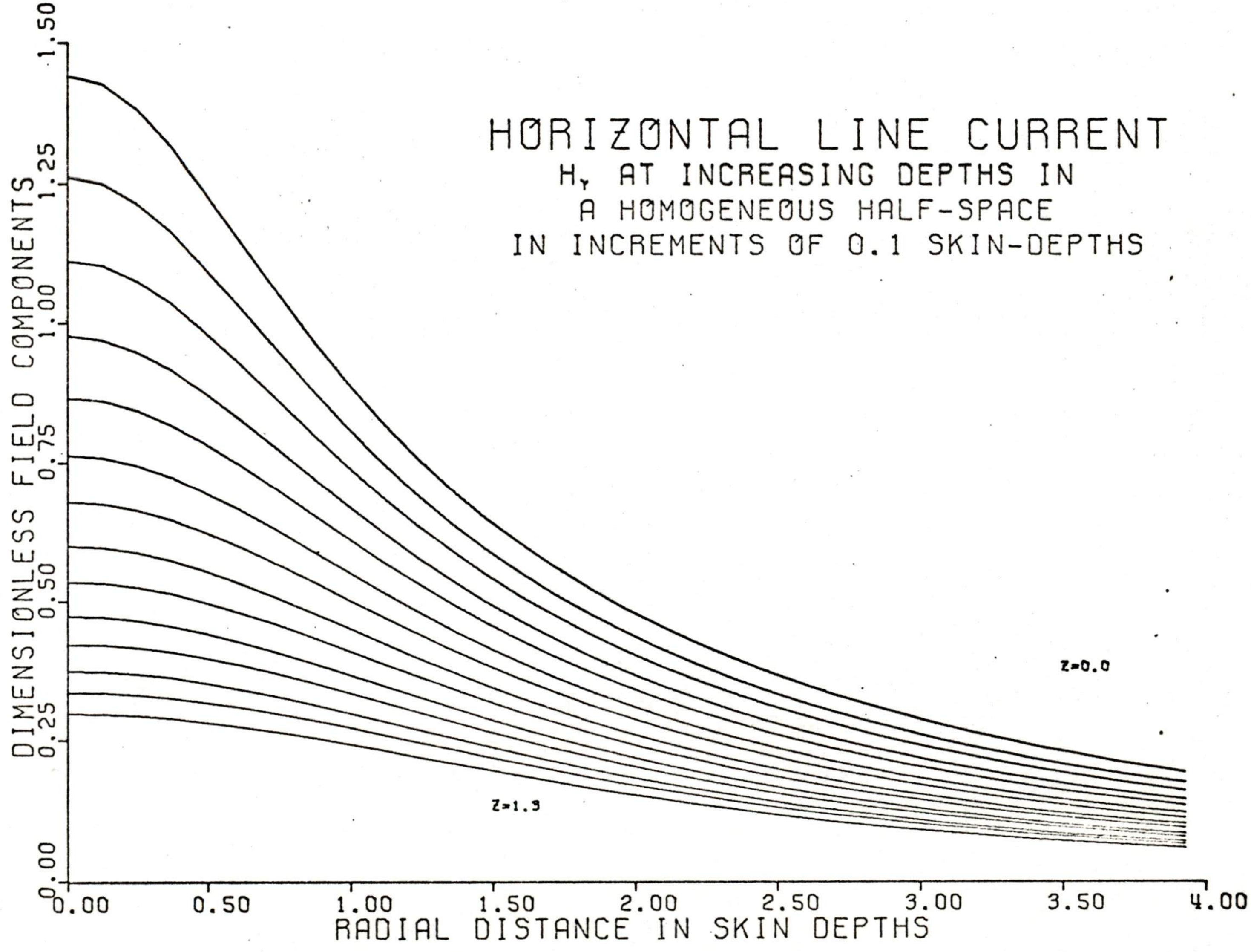

Figure 15. Illustration of amplitude attenuation of H_y inside a homogeneous conductor for a line-current source.

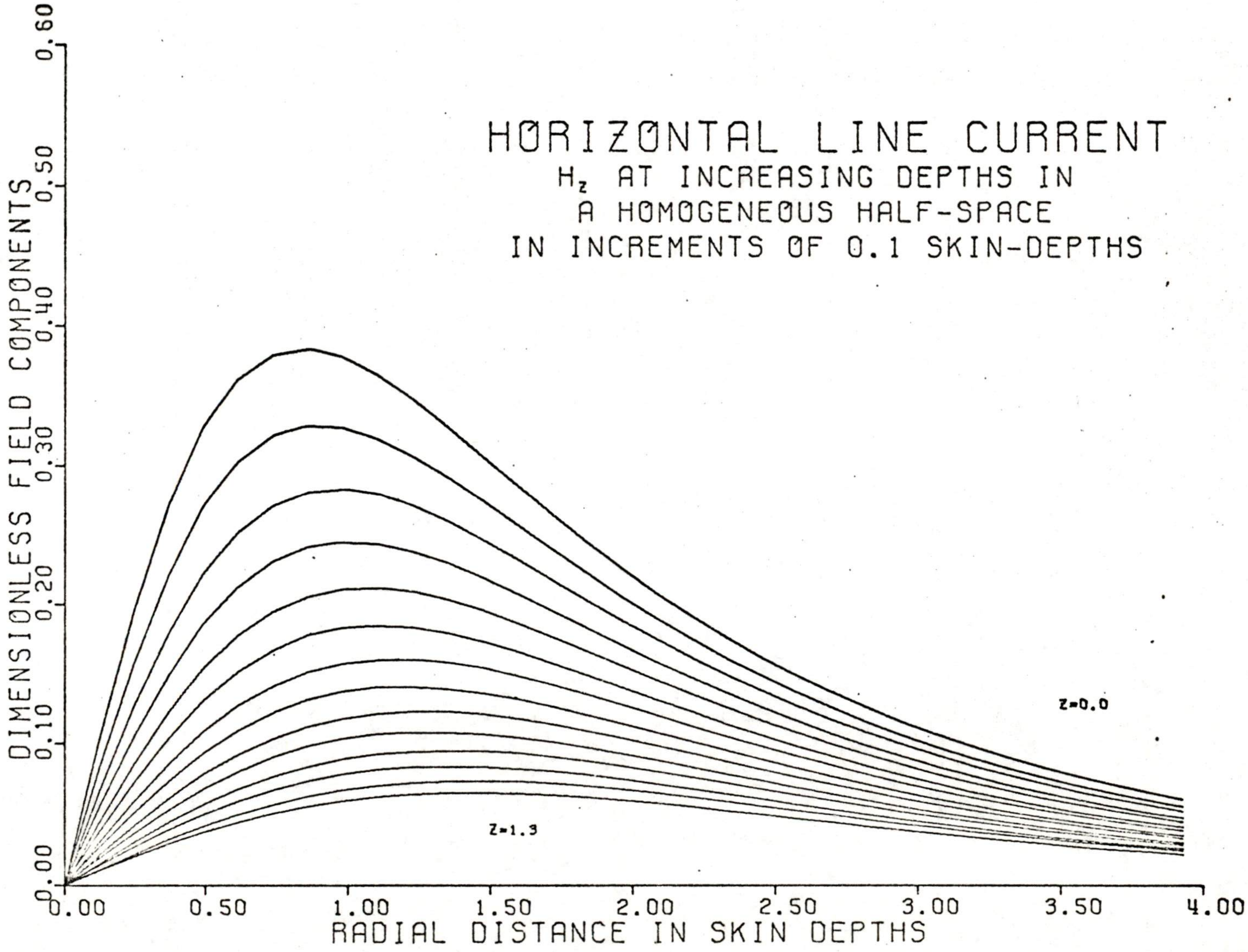

Figure 16. Illustration of amplitude attentuation of H_z inside a homogeneous conductor for a line-current source.

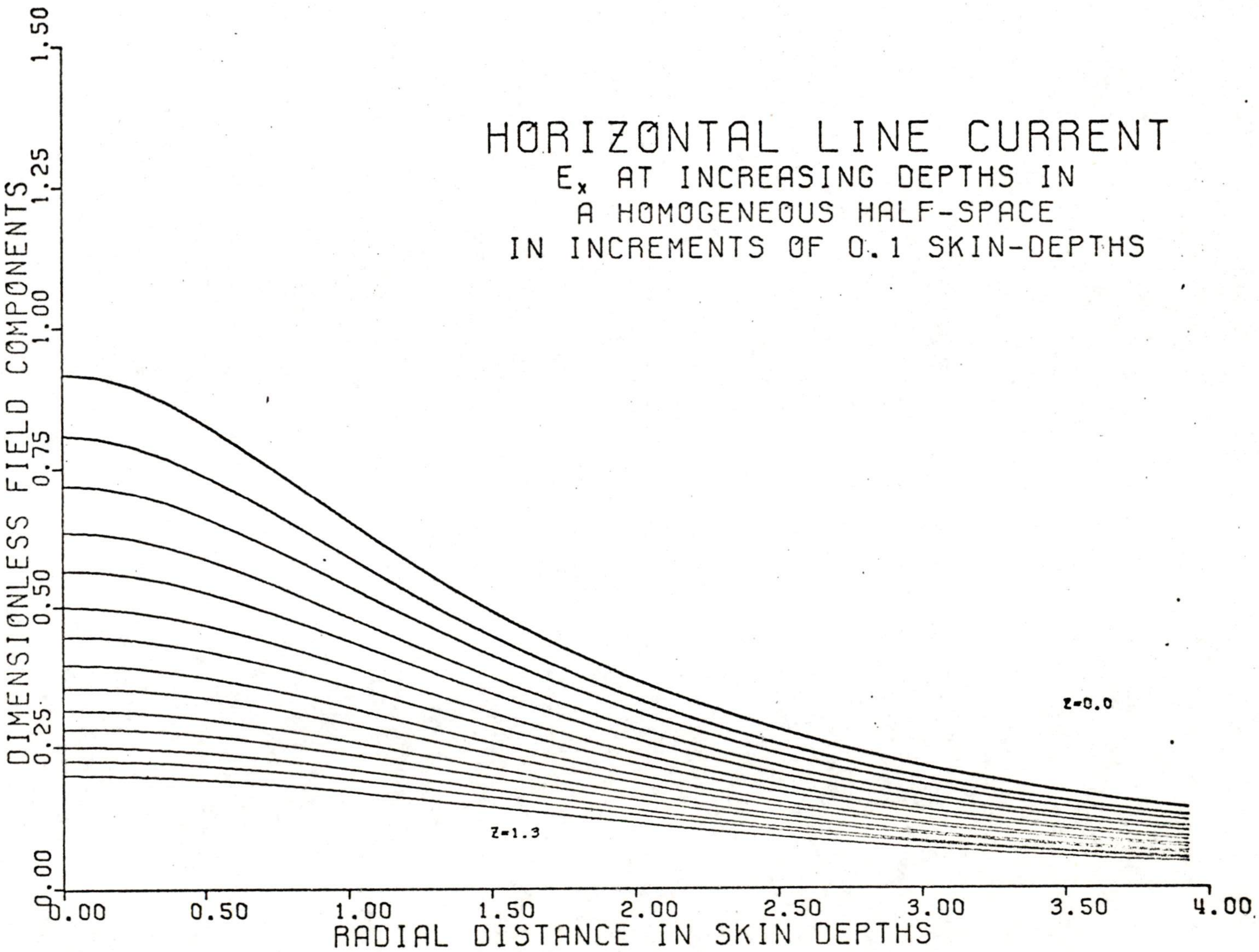

Figure 17. Illustration of amplitude attenauation of E_X inside a homogeneous conductor for a line-current source.

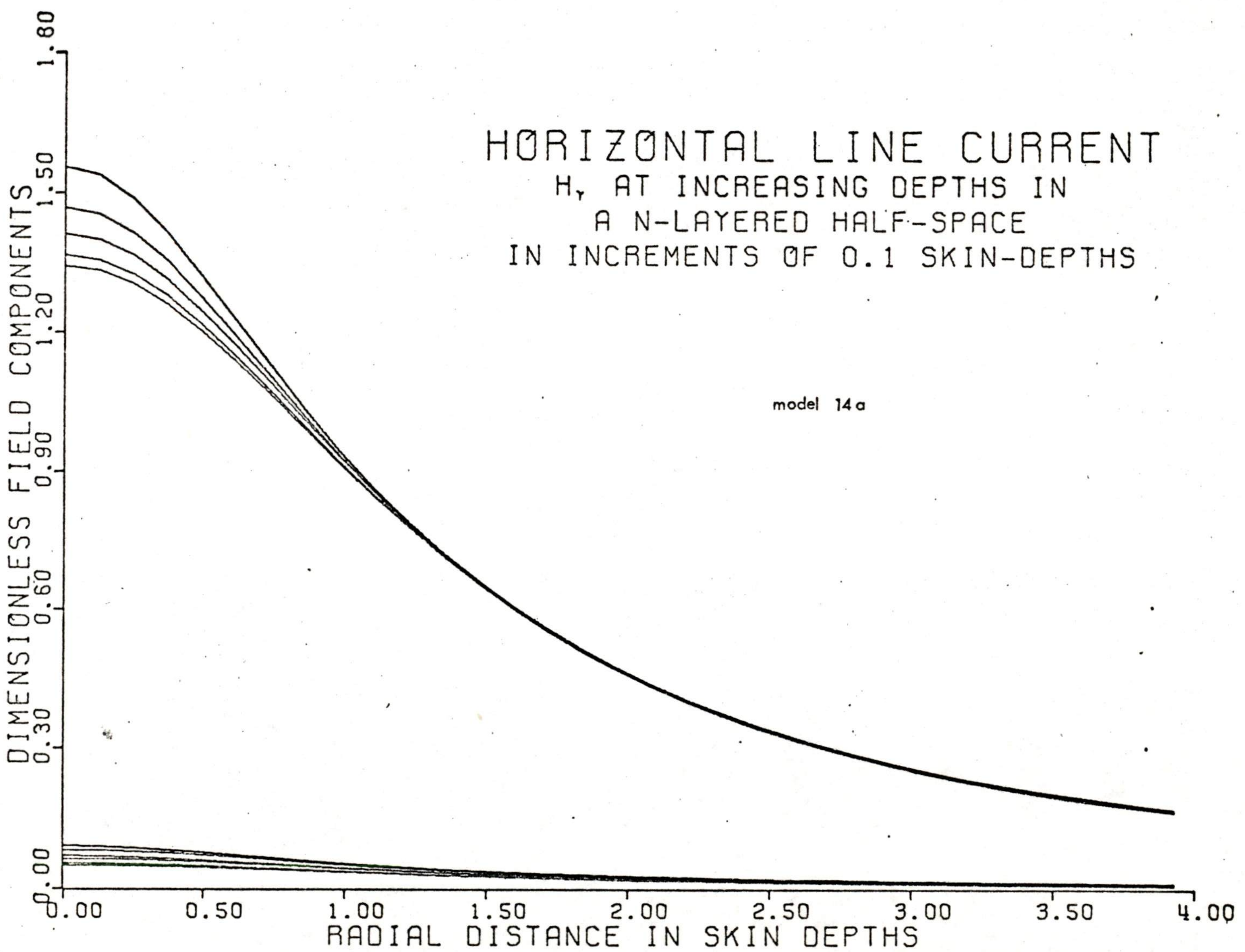

Figure 18. Amplitude attenuation of H_y inside the conductor illustrated in Fig. 14a for a line-current source.

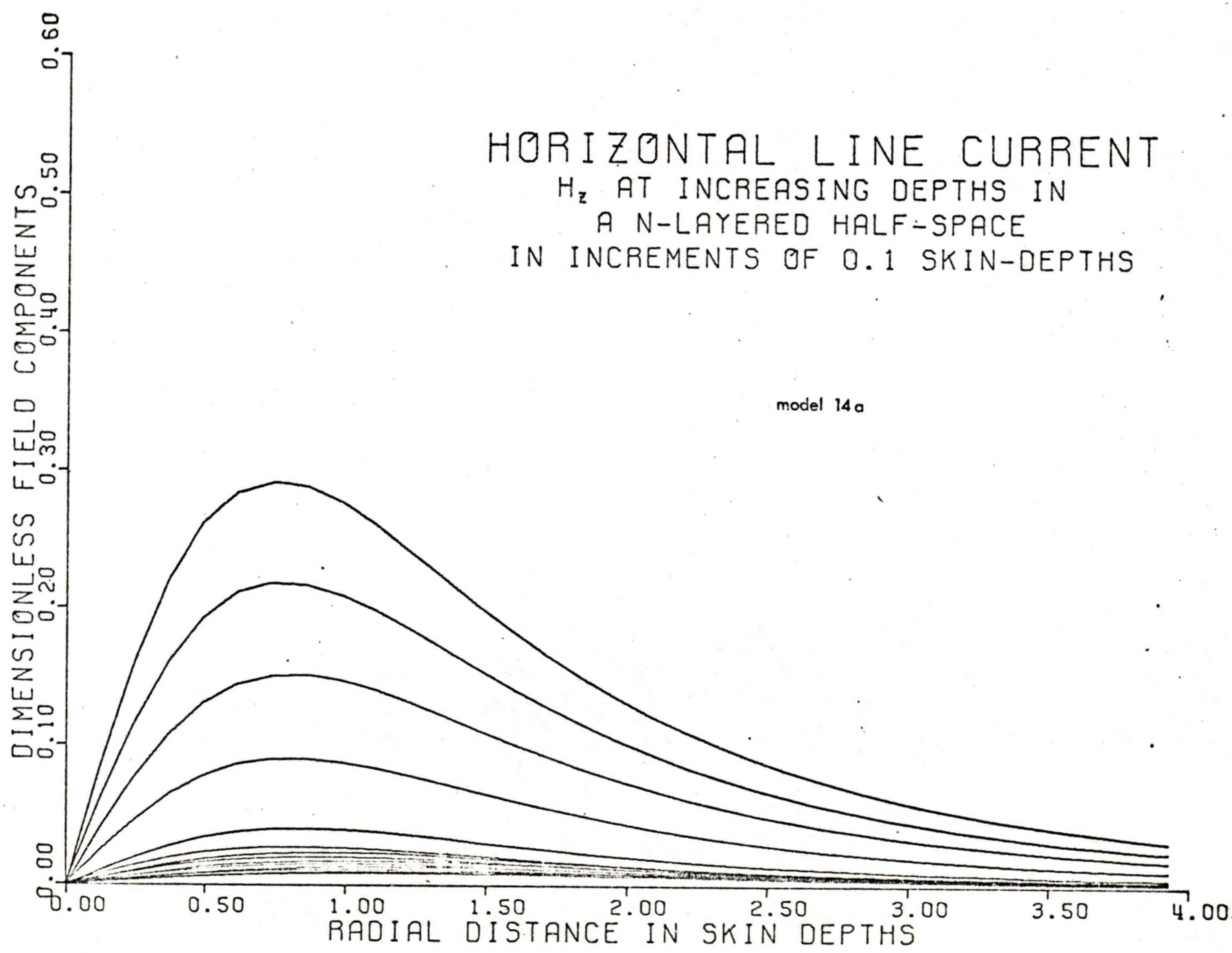

Figure 19. Amplitude attenuation of H_z inside the conductor illustrated in Fig. 14a for a line-current source.

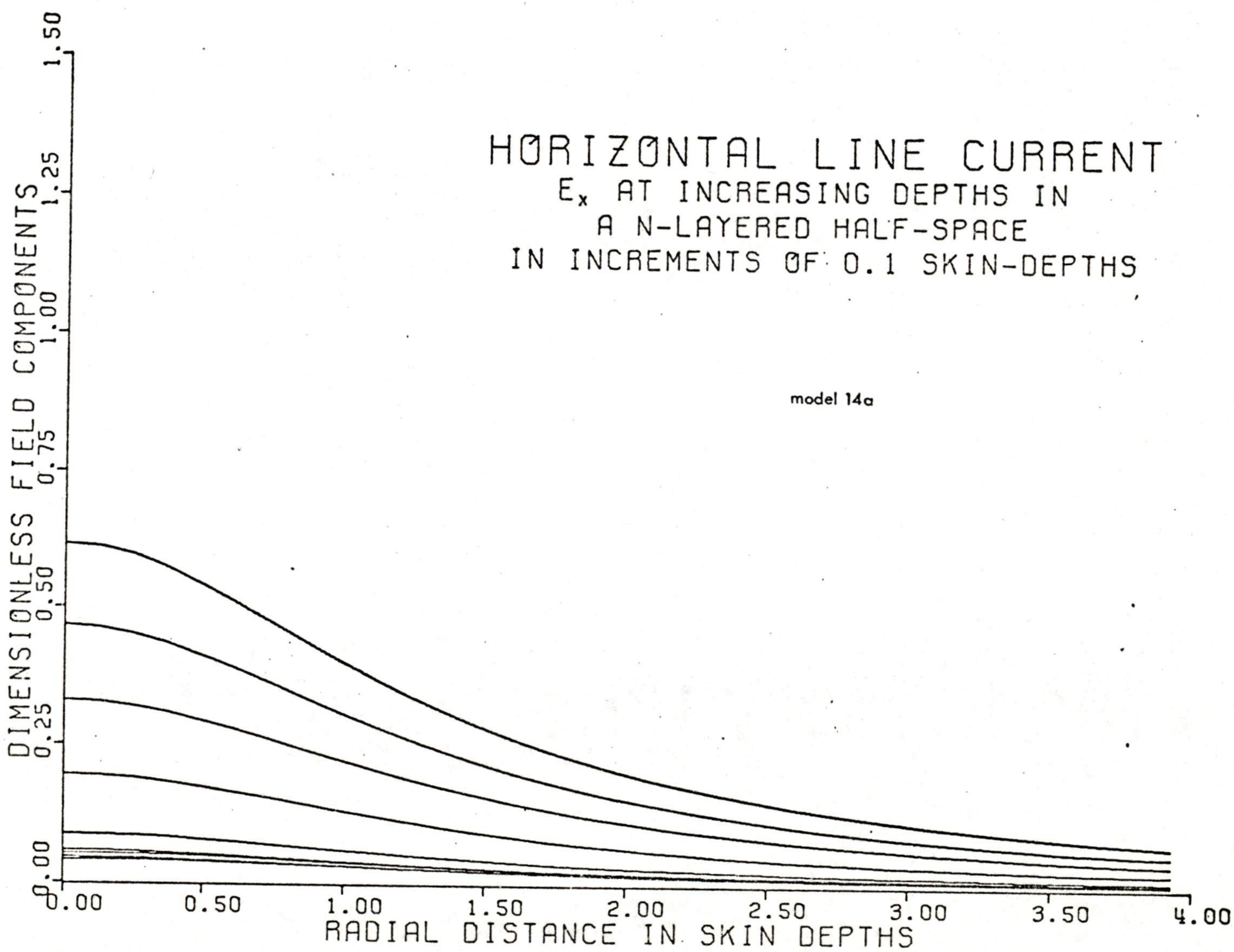

Figure 20. Amplitude attenuation of E_x inside the conductor illustrated in Fig. 14a for a line-current source.

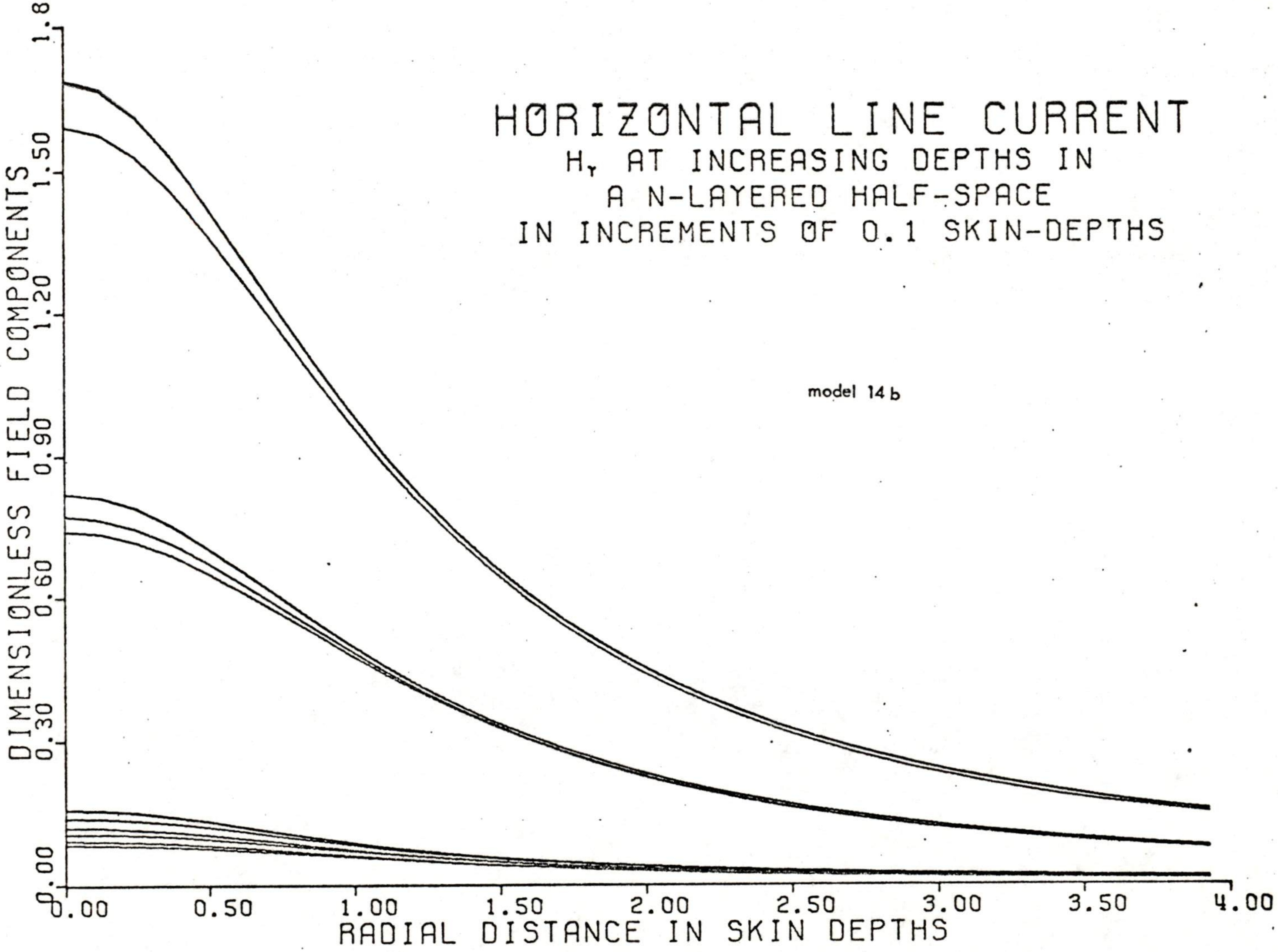

Figure 21. Amplitude attenuation of H_y inside the conductor illustrated in Fig. 14b for a line-current source.

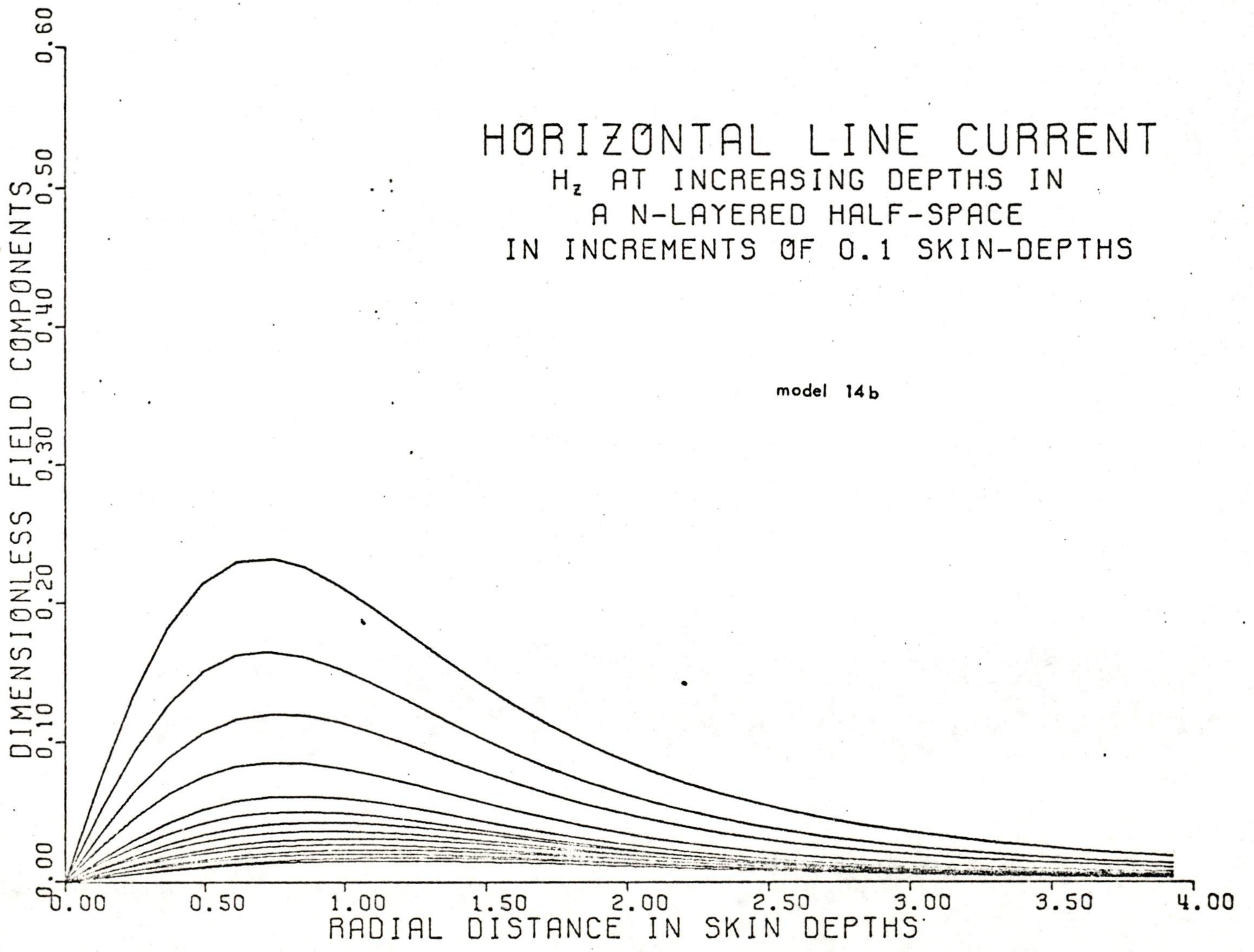

Figure 22. Amplitude attenuation of H_z inside the conductor illustrated in Fig. 14b for a line-current source.

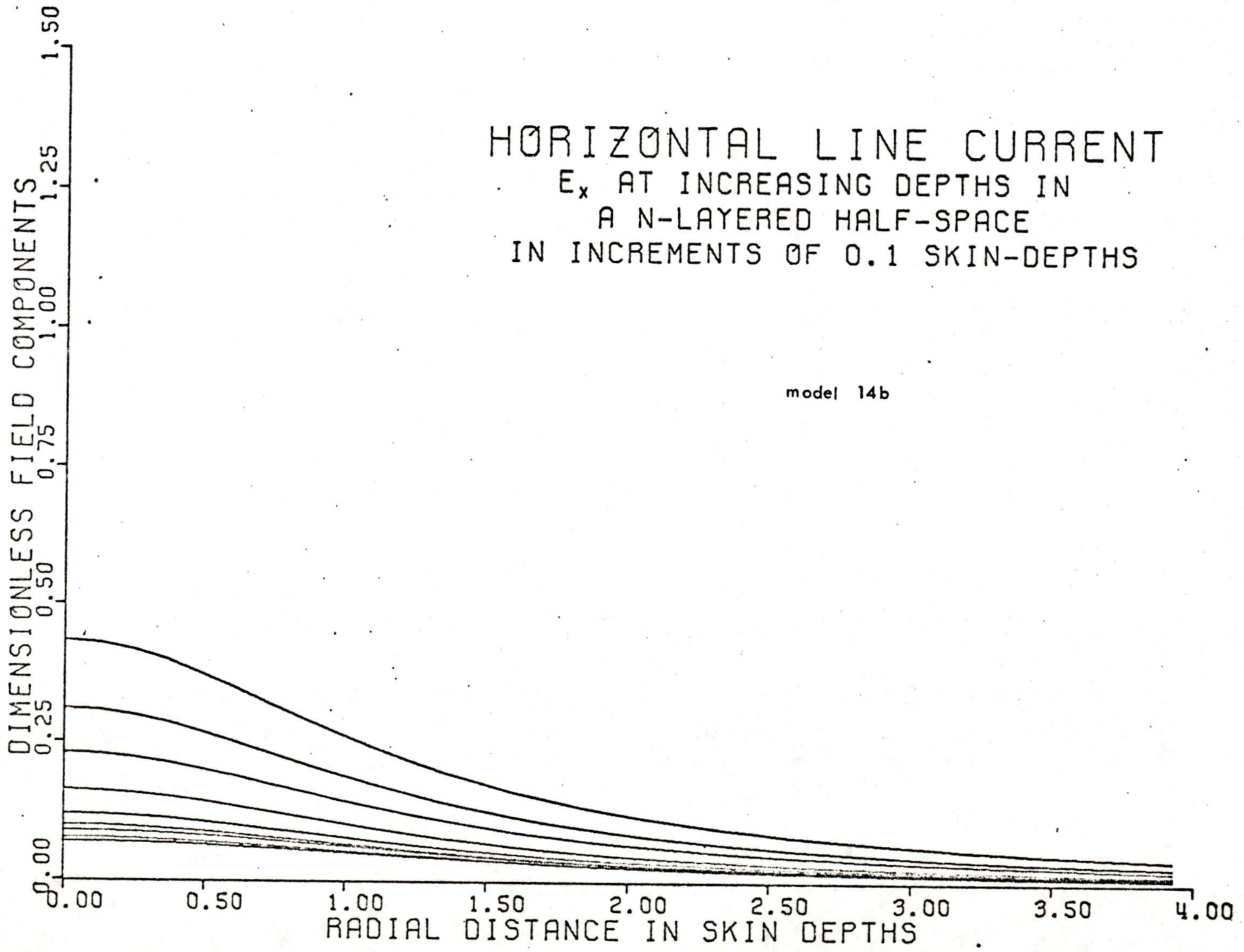

Figure 23. Amplitude attenuation of E_x inside the conductor illustrated in Fig. 14b for a line-current inducing source.

CHAPTER 8

SUMMARY

The value of an induction theory for a multi-layered conductor is apparent -- the response of any conductivity distribution to a given inducing field can be theoretically determined. In Chapter 7 of this thesis we solved numerically for the fields induced in various distributions of conductivity; these were largely chosen to illustrate the application of the theory, and not because they represented conductivity distributions of particular geophysical interest. A detailed numerical investigation of the N-layer formalism might indicate some more detailed properties of the response of conductors to inducing fields. In particular, the problem of surface screening could be theoretically investigated. Surface screening is the damping effect that highly conducting surface layers could have on fields due to high frequency sources. These surface layers could mask the effect of the structure of the deeper layers in the earth (see Price 1970).

For a given conductivity distribution and source we can determine the profile of the total field at the surface of the half-space. A problem currently of interest to some mathematicians is that of the uniqueness with which a conductivity distribution can be inferred, given the surface fields due to sources comprising a complete range of frequencies. Bailey (1970) in particular has derived a uniqueness theorem for the case of spherical conductors. Perhaps the formalism developed in this thesis may contribute to further understanding in the matter of inferring such distributions. With this in mind, it might be interesting to see if the N-layer Hertz potential formalism could be rederived in terms of spherical coordinates.

Further research which might be relevant to the theory developed here, could consist of attempting to see if the radiative solutions of Wait (1962) do in fact reduce to our expressions when the near-field limit is taken. The time-harmonic electric dipole has been omitted from the present thesis, since it arises from the magnetic effects of displacement currents in the free-space region. With some modification of the present theory, perhaps we could include this particular source. It would be interesting to see if numerically constructed source fields could be inserted into our Hertz potential formalism for an N-layered half-space.

APPENDIX A

FOURIER TRANSFORM NOTATION

For the purposes of this thesis we will define the one-dimensional Fourier transform of the function $\Phi(x)$ as

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(x)\, e^{i\xi x}\, dx \qquad (A.1)$$

The Fourier inversion theorem states, in the notation of (A.1), that

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\xi)\, e^{-i\xi x}\, d\xi \qquad (A.2)$$

In this thesis the two-dimensional Fourier transform frequently appears. We define this as

$$F(\xi,\eta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi(x,y)\, e^{i(\xi x+\eta y)}\, dxdy \qquad (A.3)$$

and its inverse to be

$$\Phi(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\xi,\eta)\, e^{-i(\xi x+\eta y)}\, d\xi d\eta \qquad (A.4)$$

For convenience, we choose a more compact notation for (A.3) and (A.4) . We denote the functional dependence of Φ on x and y by writing $\phi(\underline{r})$ where $\underline{r} = x\hat{i} + y\hat{j}$; likewise we write for the transform function, $F(\underline{\rho})$, where $\underline{\rho} = \xi\hat{i} + \eta\hat{j}$. Thus we can write (A.3) in the following form:

$$F(\underline{\rho}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi(\underline{r})\ e^{i\underline{\rho}\cdot\underline{r}}\ d\underline{r} \qquad (A.5)$$

where $d\underline{r} \equiv dxdy$. We can write equation (A.4) as

$$\Phi(\underline{r}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\underline{\rho})\ e^{-i\underline{\rho}\cdot\underline{r}}\ d\underline{\rho} \qquad (A.6)$$

where $d\underline{\rho} \equiv d\xi d\eta$.

In order to simplify the notation in Section 4.2 (where the Fourier transforms of lengthy expressions were taken), we introduced the symbol $\mathfrak{F}$ to denote the double Fourier transform operator. In a similar manner $\mathfrak{F}^{-1}$ was introduced to denote the inverse double Fourier transform operator.

In Section 2.2 we made use of a well-known result of Fourier transform theory:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Phi'(x)\ e^{i\xi x}\ dx = i\xi\ F(\xi) \qquad (A.7)$$

This can be proven generally for an n^{th}-order derivitive (see Sneddon, 1951, Theorem 13). For two-dimensional transforms we have the following result:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\Phi_{11}(\underline{r})+\Phi_{22}(\underline{r})\right] e^{i\underline{\rho}\cdot\underline{r}}\ d\underline{r} = -\ (\xi^2+\eta^2)F(\underline{\rho})$$

$$\equiv -\rho^2 F(\underline{\rho}) \qquad (A.8)$$

and thus

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \nabla^2\Phi(r)\ e^{i\underline{\rho}\cdot\underline{r}}\ d\underline{r} = F_{33}(\underline{\rho}) - \rho^2 F(\underline{\rho}) \qquad (A.9)$$

APPENDIX B

SPECIAL CASE OF A NON-CONDUCTING LAYER

We omitted from our analysis in Section 3.2 the case of $\sigma_n = 0$, for some n (n = 2,3,...,N); that is to say, the case of layers of zero conductivity beneath the conductor surface. The determinant of the matrix $\underline{\underline{S}}$ in equation (3.39) vanishes in such a case. If we re-express the system of equations so that we have $w_n = P(\underline{\rho},z_n-0)$ instead of $w_n = P(\underline{\rho},z_{n-1}+0)$, we again find the coefficient determinant vanishes. This follows because the surface equation will be

$$(\sigma_1\nu_1 + u_1v_1\sigma_o)\sigma_oP(\underline{\rho},-0) - \sigma_ou_1P(\underline{\rho},z_1-0) = 2\sigma_o\nu_oP^s(\underline{\rho},-0) \tag{B.1}$$

each term of which vanishes since $\sigma_o = 0$. The problem of the vanishing determinant arises since boundary-value information is lost be the application of the boundary condition (2.46), i.e.

$$\sigma_nP(\underline{\rho},z_n-0) = \sigma_{n+1}P(\underline{\rho},z_n+0)$$

for the case where $\sigma_n = 0$, for some n (n=2,3,...,N)

As an alternative approach to the problem, consider the boundary condition (2.46) applied across each interface in the conductor; we have

$$\begin{aligned}
\sigma_o P(\underline{\rho},-0) &= \sigma_1 P(\underline{\rho},+0) \\
\sigma_1 P(\underline{\rho},z_1-0) &= \sigma_2 P(\underline{\rho},z_1+0) \\
&\vdots \\
\sigma_{N-1} P(\underline{\rho},z_{N-1}-0) &= \sigma_N P(\underline{\rho},z_{N-1}+0)
\end{aligned}$$

Let us write these as a matrix equation:

$$\begin{bmatrix} \sigma_o & & & \\ & \sigma_1 & & \\ & & \ddots & \\ & & & \sigma_{N-1} \end{bmatrix} \begin{bmatrix} P(\underline{\rho},-0) \\ P(\underline{\rho},z_1-0) \\ \vdots \\ P(\underline{\rho},z_{N-1}-0) \end{bmatrix} = \underline{Y}$$

$$= \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_N \end{bmatrix} \begin{bmatrix} P(\underline{\rho},+0) \\ P(\underline{\rho},z_1-0) \\ \vdots \\ P(\underline{\rho},z_{N-1}+0) \end{bmatrix} \qquad (B.3)$$

where $\underline{Y}$ is some vector.

Concentrating first on the first equality in (B.3), we see that the determinant of the coefficient matrix must vanish since $\sigma_o = 0$. Thus if $\underline{Y} \neq 0$, the system is inconsistent and does not possess a solution. Thus $\underline{Y}$ may be taken to be zero. Again from the second equality in (B.3) we see that if all of $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_N$ are non-zero, then the solution is

$$\begin{bmatrix} P(\underline{\rho},+0) \\ P(\underline{\rho},z_1+0) \\ \cdot \\ \cdot \\ \cdot \\ P(\underline{\rho},z_{N-1}+0) \end{bmatrix} = \underline{0} \qquad \text{(B.4)}$$

We saw this result previously in Chapter 3. However, if one of σ_n $(n=2,3,\ldots,N)$ is zero, the second equality of (B.3) may still entertain non-trivial solutions. In particular, if $\sigma_n = 0$, then we must determine whether $P(\underline{\rho},z_{n-1}+0)$ vanishes. To do this we must appeal to the other boundary condition on P, equation (2.45).

We show first, that in any layer of non-vanishing conductivity, the electric Hertz potential vanishes identically. For, when $\sigma_n \neq 0$, we have from (B.4) and the first equality in (B.3)

$$P(\underline{\rho},z_{n-1}+0) = 0$$

and

$$P(\underline{\rho},z_n-0) = 0$$

Thus, since $P(\underline{\rho},z)$ is expressed by (3.42) in terms of its boundary values in the layer, we deduce that $P(\underline{\rho},z) = 0$ in $z_{n-1} < z < z_n$. It follows also that $P_3(\underline{\rho},z) = 0$ in the region $z_{n-1} < z < z_n$.

If, in the k^{th} layer $\sigma_k = 0$, while $\sigma_{k\pm1} \neq 0$, the solution for P is

$$P(\underline{\rho},z) = a\, e^{\nu_o z} + b\, e^{-\nu_o z} \quad .$$

Applying the boundary condition (2.45) that P_3 is continuous across the interfaces $z = z_{k-1}$ and $z = z_k$, we obtain

$$0 = \nu_o(a\, e^{\nu_o z_{k-1}} - b\, e^{-\nu_o z_{k-1}})$$

and

$$0 = \nu_o(a\, e^{\nu_o z_k} - b\, e^{-\nu_o z_k})$$

from which we immediately deduce that $a = b = 0$. Thus $P(\underline{\rho},z) = 0$ in a non-conducting layer also. (This result clearly applies to the unbounded N^{th} region as well, since $P \to 0$ as $z \to \infty$.)

APPENDIX C

TRANSFORMATION TO POLAR COORDINATES

C.1 Bessel function properties

The electromagnetic field solutions for inducing sources possessing cylindrical symmetry can be simplified by the following transformation to polar coordinates:

$$x = r\cos\theta\ ,\ y = r\sin\theta\ ;\quad \xi = \rho\cos\psi,\ \eta = \rho\sin\psi \tag{C.1}$$

In this Appendix we will indicate how the integral field solutions of Section 7.5 for the vertical magnetic dipole, and the horizontal magnetic dipole can be transformed by the equations (C.1) .

For future reference we develop here some Bessel function properties. We start with the Bessel integral of order zero:

$$2\pi J_o(\rho r) = \int_{-\pi}^{+\pi} \exp\{-i\rho r\cos(\psi-\theta)\}\, d\psi \tag{C.2}$$

Differentiating (C.2) with respect to ρr and substituting $J_o{}'(\rho r) = -J_1(\rho r)$, we obtain

$$-2\pi i J_1(\rho r) = \int_{-\pi}^{+\pi} \cos(\psi-\theta)\exp\{-i\rho r\cos(\psi-\theta)\}\, d\psi \tag{C.3}$$

We note also that

$$0 = \int_{-\pi}^{+\pi} \sin(\psi-\theta)\ \exp\{-i\rho r\ \cos(\psi-\theta)\}\ d\psi \qquad (C.4)$$

because the integrand is an odd function on the range $-\pi < \psi-\theta < \pi$, and this is equivalent to the range of integration since the function is periodic. The identities (C.2), (C.3) and (C.4) can be used to obtain further useful results. For example, if we write

$$\cos\theta\ \cos(\psi-\theta) = \cos\psi + \sin\theta\ \sin(\psi-\theta)$$

we obtain from (C.3) and (C.4)

$$2\pi i\ J_1(\rho r)\ \cos\theta = \int_{-\pi}^{+\pi} \cos\psi\ \exp\{-i\rho r\ \cos(\psi-\theta)\}\ d\psi \qquad (C.5)$$

Differentiating again with respect to ρr, and using the recurrance relation for Bessel functions,

$$J_1'(\rho r) = J_o(\rho r) - J_1(\rho r)/\rho r \quad ,$$

we now have from (C.5)

$$2\pi\cos\theta \left\{ \frac{J_1(\rho r)}{\rho r} - J_o(\rho r) \right\}$$

$$= \int_{-\pi}^{+\pi} \cos\psi\ \cos(\psi-\theta)\ \exp\{-i\rho r\ \cos(\psi-\theta)\}\ d\psi \qquad (C.6)$$

We obtain another useful result by multiplying (C.6) by

$\sin\theta$, and inserting the following trigonometric identity,

$$\sin\theta \cos\psi \cos(\psi-\theta)$$

$$= -\tfrac{1}{2}\sin 2(\psi-\theta) + \cos\theta \cos\psi \sin(\psi-\theta) + \tfrac{1}{2}\sin 2\theta \quad ,$$

into the resulting expression; we obtain

$$2\pi \sin\theta \cos\theta \left\{ \frac{J_1(\rho r)}{\rho r} - J_o(\rho r) \right\}$$

$$= \cos\theta \sin\theta \int_{-\pi}^{+\pi} \exp\{-i\rho r \cos(\psi-\theta)\} \, d\psi$$

$$+ \cos\theta \int_{-\pi}^{+\pi} \cos\psi \sin(\psi-\theta) \exp\{-i\rho r \cos(\psi-\theta)\} \, d\psi$$

(C.7)

where we have observed that

$$\int_{-\pi}^{+\pi} \sin 2(\psi-\theta) \exp\{-i\rho r \cos(\psi-\theta) \, d\psi = 0$$

because of the nature of the integrand. For $\theta \neq \pm \tfrac{1}{2}\pi$, we may cancel the factor $\cos\theta$ appearing in each term of equation (C.7). From (C.2) we see directly the following property:

$$2\pi \sin\theta \left\{ \frac{J_1(\rho r)}{\rho r} - 2\, J_o(\rho r) \right\}$$

$$= \int_{-\pi}^{+\pi} \cos\psi \, \sin(\psi-\theta) \, \exp\{-i\rho r \cos(\psi-\theta)\} \, d\psi \qquad (C.8)$$

This relationship can be easily verified for the cases where $\theta = \pm \frac{1}{2}\pi$ as well.

C.2 The vertical dipole

If we transform equation (7.30) into polar coordinates, we obtain for the vertical magnetic dipole,

$$B_z = \frac{M}{8\pi^2\delta^3} \int_0^{+\infty}\int_{-\pi}^{+\pi} \tilde{\rho}^2 \; K(\tilde{\rho}) \; e^{-\tilde{\rho}\tilde{h}} \; \exp\{-i\tilde{\rho}\tilde{r}\cos(\psi-\theta)\} \, d\psi d\tilde{\rho}$$

which, by applying equation (C.3), becomes

$$B_z = \frac{M}{4\pi\delta^3} \int_0^{\infty} \tilde{\rho}^2 \; K(\tilde{\rho}) \; J_o(\tilde{\rho}\tilde{r}) \; e^{-\tilde{\rho}\tilde{h}} \, d\tilde{\rho} \qquad (C.9)$$

For the B_x component, we transform (7.28) to obtain

$$B_x = \frac{iM}{8\pi^2\delta^3} \int_0^{\infty}\int_{-\pi}^{+\pi} \tilde{\rho}^2\cos\psi \; [2-K(\tilde{\rho})] \; e^{-\rho\tilde{h}} \; \exp\{-i\tilde{\rho}\tilde{r}\cos(\psi-\theta)\} d\psi d\tilde{\rho}$$

and from (7.29) we similarly obtain

$$B_y = \frac{iM}{8\pi^2\delta^3} \int_0^{\infty}\int_{-\pi}^{+\pi} \tilde{\rho}^2\sin\psi[2-K(\tilde{\rho})] \; e^{-\tilde{\rho}\tilde{h}} \; \exp\{-i\tilde{\rho}\tilde{r}\cos(\psi-\theta)\} d\psi d\tilde{\rho}$$

Forming the radial component defined by

$$B_r = B_x \cos\theta + B_y \sin\theta \quad , \qquad (C.10)$$

and employing the trigonometric identity,

$$\cos\psi \cos\theta + \sin\psi \sin\theta \equiv \cos(\psi-\theta) \quad , \qquad (C.11)$$

and the Bessel relationship (C.3), we obtain

$$B_r = \frac{M}{4\pi\delta^3} \int_0^\infty \tilde{\rho}^2[2-K(\tilde{\rho})]J_1(\tilde{\rho}\tilde{r})\ e^{-\tilde{\rho}\tilde{h}}\ d\tilde{\rho} \qquad (C.12)$$

It follows from (C.4) that the azimuthal component,

$$B_\theta = - B_x \sin\theta + B_y \cos\theta \qquad (C.13)$$

vanishes for the vertical dipole, a result to be expected from the symmetry of the problem.

Making the polar coordinate transformation in equations (7.31) and (7.32), we obtain for the electric components,

$$E_x = - \frac{M\omega}{8\pi^2\delta^2} \int_0^\infty \int_{-\pi}^{+\pi} \tilde{\rho}\ \sin\psi\ K(\tilde{\rho})\ e^{-\tilde{\rho}\tilde{h}}\ \exp\{-i\tilde{\rho}\tilde{r}\ \cos(\psi-\theta)\}\ d\psi d\tilde{\rho}$$

and

$$E_y = \frac{M\omega}{8\pi^2\delta^2} \int_0^\infty \int_{-\pi}^{+\pi} \tilde{\rho}\ \cos\psi\ K(\tilde{\rho})\ e^{-\tilde{\rho}\tilde{h}}\ \exp\{-i\tilde{\rho}\tilde{r}\ \cos(\psi-\theta)\}\ d\psi d\tilde{\rho}$$

Forming the azimuthal component of the electric field (see the definition (C.13) for the azimuthal component of the magnetic field) and again using equations (C.11) and (C.3), we have

$$E_\theta = - \frac{iM\omega}{4\pi\delta^2} \int_0^\infty \tilde{\rho}\ K(\tilde{\rho})\ J_1(\tilde{\rho}\tilde{r})\ e^{-\tilde{\rho}\tilde{h}}\ d\tilde{\rho} \qquad (C.14)$$

The radial component of $\underline{E}$ vanishes of course.

C.3 The horizontal dipole

When we make the transformation into polar co-ordinates in equation (7.35), we obtain for the vertical component of the surface magnetic field of the horizontal magnetic dipole

$$B_z = \frac{iM}{8\pi^2\delta^3} \int_0^\infty \int_{-\pi}^{+\pi} \tilde{\rho}^2 \cos\psi \, K(\tilde{\rho}) \, e^{-\tilde{\rho}\tilde{h}} \exp\{-i\tilde{\rho}\tilde{r} \cos(\psi-\theta)\} \, d\psi d\tilde{\rho}$$

which, from the Bessel relationship of equation (C.5), becomes

$$B_z = -\frac{M\cos\theta}{4\pi\delta^3} \int_0^\infty \tilde{\rho}^2 \, K(\tilde{\rho}) \, J_1(\tilde{\rho}\tilde{r}) \, e^{-\tilde{\rho}\tilde{h}} \, d\tilde{\rho} \qquad (C.15)$$

Transforming equation (7.33), we obtain

$$B_x = \frac{M(3\tilde{r}^2\cos^2\theta - \tilde{R}^2)}{2\pi\delta^3\tilde{R}^5}$$

$$+ \frac{M}{8\pi^2\delta^3} \int_0^\infty \int_{-\pi}^{+\pi} \tilde{\rho}^2\cos^2\psi \, K(\tilde{\rho}) \, e^{-\tilde{\rho}\tilde{h}} \exp\{-i\tilde{\rho}\tilde{r} \cos(\psi-\theta)\} \, d\psi d\tilde{\rho}$$

Similarly, for B_y we have from (7.34)

$$B_y = \frac{3M\tilde{r}^2\cos\theta\sin\theta}{2\pi\delta^3\tilde{R}^5}$$

$$+ \frac{M}{\pi^2\delta^3}\int_0^\infty\int_{-\pi}^{+\pi}\tilde{\rho}^2\cos\psi\sin\psi K(\tilde{\rho})e^{-\tilde{\rho}\tilde{h}}\exp\{-i\tilde{\rho}\tilde{r}\cos(\psi-\theta)\}d\psi d\tilde{\rho}$$

If we now form the radial component, we have

$$B_r = \frac{M}{2\pi\delta^3}\left\{\frac{(3\tilde{r}^2-\tilde{R}^2)\cos\theta}{\tilde{R}^5}\right\}$$

$$+ \frac{M}{8\pi^2\delta^3}\int_0^\infty\int_{-\pi}^{+\pi}\tilde{\rho}^2\cos(\psi-\theta)\cos\psi\; K(\tilde{\rho})e^{-\tilde{\rho}\tilde{h}}\;\exp\{-i\tilde{\rho}\tilde{r}\cos(\psi-\theta)\}d\psi d\tilde{\rho}$$

which from the Bessel relationship of (C.5) together with (C.6) , becomes

$$B_r = \frac{M\cos\theta}{4\pi\delta^3}\left\{\frac{2(3\tilde{r}^2-\tilde{R}^2)}{\tilde{R}^5} + \int_0^\infty\tilde{\rho}^2\;K(\tilde{\rho})\left\{\frac{J_1(\tilde{\rho}\tilde{r})}{\tilde{\rho}\tilde{r}} - J_o(\tilde{\rho}\tilde{r})\right\}e^{-\tilde{\rho}\tilde{h}}\;d\tilde{\rho}\right\} \qquad (C.16)$$

Also, from equations (7.33) and (7.34), we have for the azimuthal component of the magnetic field

$$B_\theta = \frac{M \sin\theta}{2\pi\delta^3\tilde{R}^3}$$

$$+ \frac{M}{8\pi^2\delta^3}\int_0^\infty\int_{-\pi}^{+\pi} \tilde{\rho}^2 \sin(\psi-\theta)\cos\psi \; K(\tilde{\rho}) e^{-\tilde{\rho}\tilde{h}} \exp\{-i\tilde{\rho}\tilde{r}\cos(\psi-\theta)\} d\psi d\tilde{\rho}$$

which becomes as a result of (C.8)

$$B_\theta = \frac{M \sin\theta}{4\pi\delta^3} \left\{ \frac{2}{\tilde{R}^3} + \int_0^\infty \tilde{\rho}^2 K(\tilde{\rho}) \left\{ \frac{J_1(\tilde{\rho}\tilde{r})}{\tilde{\rho}\tilde{r}} - 2 J_o(\tilde{\rho}\tilde{r}) \right\} e^{-\tilde{\rho}\tilde{h}} \, d\tilde{\rho} \right\} \qquad (C.17)$$

From equation (7.36), we have the x-component of the electric field, which in polar variables becomes

$$E_x = -\frac{iM\omega}{8\pi^2\delta^2}\int_0^\infty\int_{-\pi}^{+\pi} \tilde{\rho} \cos\psi \sin\psi \; K(\tilde{\rho}) e^{-\tilde{\rho}\tilde{h}} \exp\{-i\tilde{\rho}\tilde{r}\cos(\psi-\theta)\} d\psi d\tilde{\rho}$$

The y-component from (7.37) is

$$E_y = \frac{iM\omega}{8\pi^2\delta^2}\int_0^\infty\int_{-\pi}^{+\pi} \tilde{\rho} \cos^2\psi \; K(\tilde{\rho}) e^{-\tilde{\rho}\tilde{h}} \exp\{-i\tilde{\rho}\tilde{r}\cos(\psi-\theta)\} \, d\psi d\tilde{\rho}$$

We can thus deduce that the radial component of the electric field is given by

$$E_r = -\frac{iM\omega}{8\pi^2\delta^2}\int_0^\infty\int_{-\pi}^{+\pi} \tilde{\rho} \cos\psi \sin(\theta-\psi) K(\tilde{\rho}) e^{-\tilde{\rho}\tilde{h}} \exp\{-i\tilde{\rho}\tilde{r}\cos(\psi-\theta)\} d\psi d\tilde{\rho}$$

By (C.8) this can be written

$$E_r = -\frac{iM\omega \sin\theta}{4\pi\delta^2}\int_0^\infty \tilde{\rho}\ K(\tilde{\rho})\left\{\frac{J_1(\tilde{\rho}\tilde{r})}{\tilde{\rho}\tilde{r}} - 2\ J_o(\tilde{\rho}\tilde{r})\right\} e^{-\tilde{\rho}\tilde{h}}\ d\tilde{\rho} \tag{C.18}$$

Similarly we can deduce the azimuthal component of the electric field to be

$$E_\theta = \frac{iM\omega}{8\pi^2\delta^2}\int_0^\infty\int_{-\pi}^{+\pi} \tilde{\rho}\cos\psi\cos(\psi-\theta)K(\tilde{\rho})e^{-\tilde{\rho}\tilde{h}}\ \exp\{-i\tilde{\rho}\tilde{r}\ \cos(\psi-\theta)\}d\psi d\tilde{\rho}$$

which from (C.5) becomes

$$E_\theta = \frac{iM\omega \cos\theta}{4\pi\delta^2}\int_0^\infty \tilde{\rho}K(\tilde{\rho})\left\{\frac{J_1(\tilde{\rho}\tilde{r})}{\tilde{\rho}\tilde{r}} - J_o(\tilde{\rho}\tilde{r})\right\} e^{-\tilde{\rho}\tilde{h}}\ d\tilde{\rho} \tag{C.19}$$

The z- component of the electric field from (7.38) is simply,

$$E_z = -\frac{iM\omega\tilde{r}\ \sin\theta}{2\pi\tilde{R}^3\delta^2} \tag{C.20}$$

APPENDIX D

COMPUTER PROGRAM LISTING

Program I constitutes the Fast Fourier Transform program with which the Fourier integrals of this thesis are evaluated. This particular listing is of FOUR2 which requires an input array whose dimensions are expressible as powers of two.

Program's II, III, and IV are the programs which evaluate the various components of the field due to a vertical magnetic dipole inducing source. These programs are required in the computations of the field components illustrated in Figures 6 and 7 .

Program V evaluates the various components of the field due to a line-current inducing source. This program is used in the calculations leading to Figure 4, and Figures 8, 9, 10, 11, 12, and 13.

For a line-current source, Program VI evaluates the field components at increasing depth in the conductor. This program is used in calculations required for the models illustrated in Figures 15, 16, 17, 18, 19, 20, 21, 22, and 23 .

```
C     PROGRAM I
C
      SUBROUTINE FOUR2 (DATA,N,NDIM,ISIGN,IFORM)
C     (A DOUBLE-PRECISION FFT, DFOUR2, CAN ALSO BE CONSTRUCTED
C     FROM THE FOLLOWING PROGRAM)
C     COOLEY-TUKEY FAST FOURIER TRANSFORM IN USASI BASIC FORTRAN.
C     MULTI-DIMENSIONAL TRANSFORM, EACH DIMENSION A POWER OF TWO.
C     COMPLEX OR REAL DATA.
C     TRANSFORM(K1,K2,...) = SUM(DATA(J1,J2,...)*EXP(ISIGN*2*PI*SQRT(-1)
C     *((J1-1)*(K1-1)/N(1)+(J2-1)*(K2-1)/N(2)+...))), SUMMED FOR ALL
C     J1 AND K1 FROM 1 TO N(1), J2 AND K2 FROM 1 TO N(2), ETC. FOR ALL
C     NDIM SUBSCRIPTS.  NDIM MUST BE POSITIVE AND EACH N(IDIM) MUST BE
C     A POWER OF TWO.  ISIGN IS +1 OR -1.  LET NTOT = N(1)*N(2)...
C     ...*N(NDIM).  THEN A -1 TRANSFORM FOLLOWED BY A +1 ONE
C     (OR VICE VERSA) RETURNS NTOT (NTOT/2 IF IFORM = 0 OR
C     -1) TIMES THE ORIGINAL DATA.  IFORM = 1, 0 OR -1, AS DATA IS
C     COMPLEX, REAL OR THE FIRST HALF OF A COMPLEX ARRAY.  TRANSFORM
C     VALUES ARE RETURNED TO ARRAY DATA.  THEY ARE COMPLEX, REAL OR
C     THE FIRST HALF OF A COMPLEX ARRAY, AS IFORM = 1, -1 OR 0.
C     THE TRANSFORM OF A REAL ARRAY (IFORM = 0) DIMENSIONED N(1) BY N(2)
C     BY ... WILL BE RETURNED IN THE SAME ARRAY, NOW CONSIDERED TO
C     BE COMPLEX OF DIMENSIONS N(1)/2+1 BY N(2) BY ....  NOTE THAT IF
C     IFORM = 0 OR -1, N(1) MUST BE EVEN, AND ENOUGH ROOM MUST BE
C     RESERVED.  THE MISSING VALUES MAY BE OBTAINED BY COMPLEX CONJU-
C     GATION.  THE REVERSE TRANSFORMATION, OF A HALF COMPLEX ARRAY
C     DIMENSIONED N(1)/2+1 BY N(2) BY ...., IS ACCOMPLISHED SETTING IFORM
C     TO -1.  IN THE N ARRAY, N(1) MUST BE THE TRUE N(1), NOT N(1)/2+1.
C     THE TRANSFORM WILL BE REAL AND RETURNED TO THE INPUT ARRAY.
C     RUNNING TIME IS PROPORTIONAL TO NTOT*LOG2(NTOT), RATHER THAN
C     THE NAIVE NTOT**2.
C     WRITTEN BY NORMAN BRENNER OF MIT LINCOLN LABORATORY, JUNE 1968.
C     SEE-- IEEE AUDIO TRANSACTIONS (JUNE 1967), SPECIAL ISSUE ON FFT.
      DIMENSION DATA(1), N(1)
      NTOT=1
      DO 10 IDIM=1,NDIM
   10 NTOT=NTOT*N(IDIM)
      IF (IFORM) 70,20,20
   20 NREM=NTOT
      DO 60 IDIM=1,NDIM
      NREM=NREM/N(IDIM)
      NPREV=NTOT/(N(IDIM)*NREM)
      NCURR=N(IDIM)
      IF (IDIM-1+IFORM) 30,30,40
   30 NCURR=NCURR/2
   40 CALL BITRV (DATA,NPREV,NCURR,NREM)
      CALL COOL2 (DATA,NPREV,NCURR,NREM,ISIGN)
      IF (IDIM-1+IFORM) 50,50,60
   50 CALL FIXRL (DATA,N(1),NREM,ISIGN,IFORM)
      NTOT=(NTOT/N(1))*(N(1)/2+1)
   60 CONTINUE
      RETURN
   70 NTOT=(NTOT/N(1))*(N(1)/2+1)
      NREM=1
      DO 100 JDIM=1,NDIM
      IDIM=NDIM+1-JDIM
      NCURR=N(IDIM)
      IF (IDIM-1) 80,80,90
   80 NCURR=NCURR/2
      CALL FIXRL (DATA,N(1),NREM,ISIGN,IFORM)
      NTOT=NTOT/(N(1)/2+1)*N(1)
```

```
 90   NPREV=NTOT/(N(IDIM)*NREM)
      CALL BITRV (DATA,NPREV,NCURR,NREM)
      CALL COOL2 (DATA,NPREV,NCURR,NREM,ISIGN)
100   NREM=NREM*N(IDIM)
      RETURN
      END
      SUBROUTINE BITRV (DATA,NPREV,N,NREM)
C     SHUFFLE THE DATA BY 'BIT REVERSAL'.
C     DIMENSION DATA(NPREV,N,NREM)
C     DATA(I1,I2REV,I3) = DATA(I1,I2,I3), FOR ALL I1 FROM 1 TO NPREV,
C     ALL I2 FROM 1 TO N (WHICH MUST BE A POWER OF TWO), AND ALL I3
C     FROM 1 TO NREM, WHERE I2REV-1 IS THE BITWISE REVERSAL OF I2-1.
C     FOR EXAMPLE, N = 32, I2-1 = 10011 AND I2REV-1 = 11001.
      DIMENSION DATA(1)
      IP0=2
      IP1=IP0*NPREV
      IP4=IP1*N
      IP5=IP4*NREM
      I4REV=1
      DO 60 I4=1,IP4,IP1
      IF (I4-I4REV) 10,30,30
 10   I1MAX=I4+IP1-IP0
      DO 20 I1=I4,I1MAX,IP0
      DO 20 I5=I1,IP5,IP4
      I5REV=I4REV+I5-I4
      TEMPR=DATA(I5)
      TEMPI=DATA(I5+1)
      DATA(I5)=DATA(I5REV)
      DATA(I5+1)=DATA(I5REV+1)
      DATA(I5REV)=TEMPR
 20   DATA(I5REV+1)=TEMPI
 30   IP2=IP4/2
 40   IF (I4REV-IP2) 60,60,50
 50   I4REV=I4REV-IP2
      IP2=IP2/2
      IF (IP2-IP1) 60,40,40
 60   I4REV=I4REV+IP2
      RETURN
      END
      SUBROUTINE COOL2 (DATA,NPREV,N,NREM,ISIGN)
C     FOURIER TRANSFORM OF LENGTH N BY THE COOLEY-TUKEY ALGORITHM.
C     BIT-REVERSED TO NORMAL ORDER.
C     DIMENSION DATA(NPREV,N,NREM)
C     COMPLEX DATA
C     DATA(I1,J2,I3) = SUM(DATA(I1,I2,I3)*EXP(ISIGN*2*PI*I*((I2-1)*
C     (J2-1)/N))), SUMMED OVER I2 = 1 TO N FOR ALL I1 FROM 1 TO NPREV,
C     J2 FROM 1 TO N AND I3 FROM 1 TO NREM.  N MUST BE A POWER OF TWO.
C     FACTORING N BY FOURS SAVES ABOUT TWENTY FIVE PERCENT OVER FACTOR-
C     ING BY TWOS.
C     NOTE--IT IS NOT NECESSARY TO REWRITE THIS SUBROUTINE INTO COMPLEX
C     NOTATION SO LONG AS THE FORTRAN COMPILER USED STORES REAL AND
C     IMAGINARY PARTS IN ADJACENT STORAGE LOCATIONS.  IT MUST ALSO
C     STORE ARRAYS WITH THE FIRST SUBSCRIPT INCREASING FASTEST.
      DIMENSION DATA(1)
      TWOPI=6.2831853072*FLOAT(ISIGN)
      IP0=2
      IP1=IP0*NPREV
      IP4=IP1*N
      IP5=IP4*NREM
      IP2=IP1
```

```
      NPART=N
   10 IF (NPART-2) 50,30,20
   20 NPART=NPART/4
      GO TO 10
C     DO A FOURIER TRANSFORM OF LENGTH TWO
   30 IP3=IP2*2
      DO 40 I1=1,IP1,IP0
      DO 40 I5=I1,IP5,IP3
      J0=I5
      J1=J0+IP2
      TEMPR=DATA(J1)
      TEMPI=DATA(J1+1)
      DATA(J1)=DATA(J0)-TEMPR
      DATA(J1+1)=DATA(J0+1)-TEMPI
      DATA(J0)=DATA(J0)+TEMPR
   40 DATA(J0+1)=DATA(J0+1)+TEMPI
      GO TO 140
C     DO A FOURIER TRANSFORM OF LENGTH FOUR (FROM BIT REVERSED ORDER)
   50 IP3=IP2*4
      THETA=TWOPI/FLOAT(IP3/IP1)
      SINTH=SIN(THETA/2.)
      WSTPR=-2.*SINTH*SINTH
C     COS(THETA)-1, FOR ACCURACY.
      WSTPI=SIN(THETA)
      WR=1.
      WI=0.
      DO 130 I2=1,IP2,IP1
      IF (I2-1) 70,70,60
   60 W2R=WR*WR-WI*WI
      W2I=2.*WR*WI
      W3R=W2R*WR-W2I*WI
      W3I=W2R*WI+W2I*WR
   70 I1MAX=I2+IP1-IP0
      DO 120 I1=I2,I1MAX,IP0
      DO 120 I5=I1,IP5,IP3
      J0=I5
      J1=J0+IP2
      J2=J1+IP2
      J3=J2+IP2
      IF (I2-1) 90,90,80
C     APPLY THE PHASE SHIFT FACTORS
   80 TEMPR=DATA(J1)
      DATA(J1)=W2R*TEMPR-W2I*DATA(J1+1)
      DATA(J1+1)=W2R*DATA(J1+1)+W2I*TEMPR
      TEMPR=DATA(J2)
      DATA(J2)=WR*TEMPR-WI*DATA(J2+1)
      DATA(J2+1)=WR*DATA(J2+1)+WI*TEMPR
      TEMPR=DATA(J3)
      DATA(J3)=W3R*TEMPR-W3I*DATA(J3+1)
      DATA(J3+1)=W3R*DATA(J3+1)+W3I*TEMPR
   90 T0R=DATA(J0)+DATA(J1)
      T0I=DATA(J0+1)+DATA(J1+1)
      T1R=DATA(J0)-DATA(J1)
      T1I=DATA(J0+1)-DATA(J1+1)
      T2R=DATA(J2)+DATA(J3)
      T2I=DATA(J2+1)+DATA(J3+1)
      T3R=DATA(J2)-DATA(J3)
      T3I=DATA(J2+1)-DATA(J3+1)
      DATA(J0)=T0R+T2R
      DATA(J0+1)=T0I+T2I
```

```
      DATA(J2)=T0R-T2R
      DATA(J2+1)=T0I-T2I
      IF (ISIGN) 100,100,110
  100 T3R=-T3R
      T3I=-T3I
  110 DATA(J1)=T1R-T3I
      DATA(J1+1)=T1I+T3R
      DATA(J3)=T1R+T3I
  120 DATA(J3+1)=T1I-T3R
      TEMPR=WR
      WR=WSTPR*TEMPR-WSTPI*WI+TEMPR
  130 WI=WSTPR*WI+WSTPI*TEMPR+WI
  140 IP2=IP3
      IF (IP3-IP4) 50,150,150
  150 RETURN
      END
      SUBROUTINE FIXRL (DATA,N,NREM,ISIGN,IFORM)
C     FOR IFORM = 0, CONVERT THE TRANSFORM OF A DOUBLED-UP REAL ARRAY,
C     CONSIDERED COMPLEX, INTO ITS TRUE TRANSFORM.  SUPPLY ONLY THE
C     FIRST HALF OF THE COMPLEX TRANSFORM, AS THE SECOND HALF HAS
C     CONJUGATE SYMMETRY.  FOR IFORM = -1, CONVERT THE FIRST HALF
C     OF THE TRUE TRANSFORM INTO THE TRANSFORM OF A DOUBLED-UP REAL
C     ARRAY.  N MUST BE EVEN.
C     USING COMPLEX NOTATION AND SUBSCRIPTS STARTING AT ZERO, THE
C     TRANSFORMATION IS--
C     DIMENSION DATA(N,NREM)
C     ZSTP = EXP(ISIGN*2*PI*I/N)
C     DO 10 I2=0,NREM-1
C     DATA(0,I2) = CONJ(DATA(0,I2))*(1+I)/(1-IFORM)
C     DO 10 I1=1,N/4
C     Z = (1+(2*IFORM+1)*I*ZSTP**I1)/2
C     I1CNJ = N/2-I1
C     DIF = DATA(I1,I2)-CONJ(DATA(I1CNJ,I2))
C     TEMP = Z*DIF
C     DATA(I1,I2) = DATA(I1,I2)-TEMP
C     10  DATA(I1CNJ,I2) = DATA(I1CNJ,I2)+CONJ(TEMP)
C     IF I1=I1CNJ, THE CALCULATION FOR THAT VALUE COLLAPSES INTO
C     A SIMPLE CONJUGATION OF DATA(I1,I2).
      DIMENSION DATA(1)
      TWOPI=6.283185307*FLOAT(ISIGN)
      IP0=2
      IP1=IP0*(N/2)
      IP2=IP1*NREM
      IF (IFORM) 10,60,60
   10 J1=IP1+1
      DATA(2)=DATA(J1)
      J1=J1+IP0
      I2MIN=IP1+1
      DO 50 I2=I2MIN,IP2,IP1
      DATA(I2)=DATA(J1)
      J1=J1+IP0
      IF (N-2) 40,40,20
   20 I1MIN=I2+IP0
      I1MAX=I2+IP1-IP0
      DO 30 I1=I1MIN,I1MAX,IP0
      DATA(I1)=DATA(J1)
      DATA(I1+1)=DATA(J1+1)
   30 J1=J1+IP0
   40 DATA(I2+1)=DATA(J1)
   50 J1=J1+IP0
```

```
60    DO 80 I2=1,IP2,IP1
      TEMPR=DATA(I2)
      DATA(I2)=DATA(I2)+DATA(I2+1)
      DATA(I2+1)=TEMPR-DATA(I2+1)
      IF (IFORM) 70,80,80
70    DATA(I2)=DATA(I2)/2.
      DATA(I2+1)=DATA(I2+1)/2.
80    CONTINUE
      IF (N-2) 170,170,90
90    THETA=TWOPI/FLOAT(N)
      SINTH=SIN(THETA/2.)
      ZSTPR=-2.*SINTH*SINTH
      ZSTPI=SIN(THETA)
      ZR=(1.-ZSTPI)/2.
      ZI=(1.+ZSTPR)/2.
      IF (IFORM) 100,110,110
100   ZR=1.-ZR
      ZI=-ZI
110   I1MIN=IP0+1
      I1MAX=IP0*(N/4)+1
      DO 160 I1=I1MIN,I1MAX,IP0
      DO 150 I2=I1,IP2,IP1
      I2CNJ=N+IP0-2*I1+I2
      IF (I2-I2CNJ) 140,120,120
120   IF (ISIGN*(2*IFORM+1)) 130,150,150
130   DATA(I2+1)=-DATA(I2+1)
      GO TO 150
140   DIFR=DATA(I2)-DATA(I2CNJ)
      DIFI=DATA(I2+1)+DATA(I2CNJ+1)
      TEMPR=DIFR*ZR-DIFI*ZI
      TEMPI=DIFR*ZI+DIFI*ZR
      DATA(I2)=DATA(I2)-TEMPR
      DATA(I2+1)=DATA(I2+1)-TEMPI
      DATA(I2CNJ)=DATA(I2CNJ)+TEMPR
      DATA(I2CNJ+1)=DATA(I2CNJ+1)-TEMPI
150   CONTINUE
      TEMPR=ZR-.5
      ZR=ZSTPR*TEMPR-ZSTPI*ZI+ZR
160   ZI=ZSTPR*ZI+ZSTPI*TEMPR+ZI
170   IF (IFORM) 240,180,180
180   I2=IP2+1
      I1=I2
      J1=IP0*(N/2+1)*NREM+1
      GO TO 220
190   DATA(J1)=DATA(I1)
      DATA(J1+1)=DATA(I1+1)
      I1=I1-IP0
      J1=J1-IP0
200   IF (I2-I1) 190,210,210
210   DATA(J1)=DATA(I1)
      DATA(J1+1)=0.
220   I2=I2-IP1
      J1=J1-IP0
      DATA(J1)=DATA(I2+1)
      DATA(J1+1)=0.
      I1=I1-IP0
      J1=J1-IP0
      IF (I2-1) 230,230,200
230   DATA(2)=0.
240   RETURN
```

```
      END                                                                  FIX 105-

C        PROGRAM II
C
C       MAIN:  THE DIMENSIONLESS VERTICAL MAGNETIC FIELD AT THE
C       SURFACE OF A CONDUCTING HALF-SPACE WITH NTOT LAYERS (NTOT=10
C       IN CALCULATIONS OF THIS THESIS), DUE TO A VERTICALLY ORIENTED
C       MAGNETIC DIPOLE SITUATED AT A HEIGHT HT ABOVE THE SURFACE
C       (IN THIS THESIS, HT=1 SKIN-DEPTH).    INTO THE TWO-DIMENSIONAL
C       ARRAY DATA(I,J) IS STORED THE VALUES OF THE FOURIER TRANSFORMED
C       MAGNETIC HERTZ POTENTIAL, G,  AS EVALUATED BY THE SUBROUTINE
C       'DIPOLE'.    DATA IS INVERTED BY A FAST (INVERSE) FOURIER
C       TRANSFORM, AND THE RESULTING ARRAY IS STORED IN 'DATA'.
C       THE AMPLITUDE, AB , OF THE VERTICAL COMPONENT OF THE MAGNETIC
C       FIELD IS COMPUTED ALONG THE LINE (X,0) IN CARTESIAN SPACE.
C
      COMPLEX Z, DATA(128,128)
      REAL AB(65), J1, J2, UINC, SIG, D,T(65)
      INTEGER N(2), I, J
      COMMON/PARAM/SIG(10), D(10), I, J
      NTOT=10
      DO 75 I1=1,NTOT
      READ(5,40) SIG(I1), D(I1)
 40   FORMAT(6X, E11.4, 5X, E11.4)
 75   CONTINUE
      UINC=.4
      DO 12 I=1,128
      IF(I-64) 30,30,31
 30   J1=UINC*(I-1)
      GO TO 32
 31   J1=UINC*(129-I)
 32   DO 13 J=1,128
      IF(J-64) 33,33,34
 33   J2=UINC*(J-1)
      GO TO 35
 34   J2=UINC*(129-J)
 35   CALL DIPOLE(J1,J2,DATA(I,J))
 13   CONTINUE
 12   CONTINUE
      N(1)=128
      N(2)=128
      CALL FOUR2(DATA,N,2,+1,+1)
      TIK=(UINC**2)/(2.*3.1415)
      DO 19 M=1,65
      AB(M)=TIK*CABS(DATA(M,1))
      T(M)=2.*3.1415*(M-1)/(UINC*N(1))
      WRITE(6,18) AB(M), T(M)
 18   FORMAT(5X,E11.4,5X,E11.4)
 19   CONTINUE
      CALL EXIT
      END
C
C       SUBROUTINE DIPOLE:     THE INPUT ARRAY IS DEFINED HERE.
C       THE DO-LOOP COMPLETED BY STATEMENT 160 GENERATES THE FACTOR Q
C       APPEARING IN THE FIELD SOLUTIONS OF CHAPTER 4.    E REPRESENTS
C       THE FUNCTION K APPEARING IN CHAPTER 4.
C
```

```
      SUBRCUTINE DIPOLE(J1, J2, DATA)
      COMPLEX NU(10), NUM, DEN, A(10), V(10), U(10), B(10), P(10),
     C Z, E, DATA, CCOSH, CSINH, IM, E1
      REAL H(10), D, J1, J2, SIG, HT
      INTEGER I1, I2, K1, K2
      COMMON/PARAM/SIG(10), D(10), I, J
      CCOSH(Z)=(CEXP(Z)+CEXP(-Z))/2.
      CSINH(Z)=(CEXP(Z)-CEXP(-Z))/2.
      NTOT=10
      MTOT=NTOT-1
      LTOT=NTOT-2
      HT=1.
      G=J1**2+J2**2
      IF(G.EQ.0.) GO TO 400
      DO 110 I2=1,NTOT
      H(I2)=2.*SIG(I2)/SIG(MTOT)
      A(I2)=CMPLX(G,H(I2))
      NU(I2)=CSQRT(A(I2))
      V(I2)=CCOSH(NU(I2)*D(I2))
      U(I2)=NU(I2)/(CSINH(NU(I2)*D(I2)))
  110 CONTINUE
      B(2)=U(2)*V(2)+NU(1)
      DO 150 K1=3,MTOT
      B(K1)=U(K1)*V(K1)+U(K1-1)*V(K1-1)
  150 CONTINUE
      DO 160 K2=2,LTOT
      P(2)=+U(2)/B(2)
      P(K2+1)=U(K2+1)/(B(K2+1)-U(K2)*P(K2))
  160 CONTINUE
      NUM=NU(NTOT)-U(MTOT)*(V(MTOT)-P(MTOT))
      DEN=NU(NTOT)+U(MTOT)*(V(MTOT)-P(MTOT))
      E=1.+(NUM/DEN)
      E1=1.-(NUM/DEN)
      IM=CMPLX(0.,1.)
      DATA=NU(NTOT)*E*CEXP(-NU(NTOT)*HT)
      GO TO 281
  400 DATA=0.
  281 RETURN
      END


C     PROGRAM  III
C
C     MAIN:   IN A SIMILAR MANNER TO PROGRAM II, THE DIMENSIONLESS
C     RADIAL MAGNETIC FIELD DUE TO A VERTICAL MAGNETIC DIPOLE IS
C     EVALUATED.
C
      COMPLEX Z, DATA(128,128)
      REAL AB(65), J1, J2, UINC, SIG, D,T(65)
      INTEGER N(2), I, J
      COMMON/PARAM/SIG(10), D(10), I, J
      NTOT=10
      DO 75 I1=1,NTOT
      READ(5,40) SIG(I1), D(I1)
   40 FORMAT(6X, E11.4, 5X, E11.4)
   75 CONTINUE
      UINC=.4
      DO 12 I=1,128
```

```
      IF(I-64) 30,30,31
30    J1=UINC*(I-1)
      GO TC 32
31    J1=UINC*(129-I)
32    DO 13 J=1,128
      IF(J-64) 33,33,34
33    J2=UINC*(J-1)
      GO TC 35
34    J2=UINC*(129-J)
35    CALL DIPOLE(J1,J2,DATA(I,J))
13    CONTINUE
12    CONTINUE
      N(1)=128
      N(2)=128
      CALL FOUR2(DATA,N,2,+1,+1)
      TIK=(UINC**2)/(2.*3.1415)
      DC 19 M=1,65
      AB(M)=TIK*CABS(DATA(M,1))
      T(M)=2.*3.1415*(M-1)/(UINC*N(1))
      WRITE(6,18) AB(M), T(M)
18    FORMAT(5X,E11.4,5X,E11.4)
19    CONTINUE
      CALL EXIT
      END

      SUBRCUTINE DIPOLE(J1, J2, DATA)
      COMPLEX NU(10), NUM, DEN, A(10), V(10), U(10), B(10), P(10),
     C Z, E, DATA, CCOSH, CSINH, IM, E1
      REAL H(10), D, J1, J2, SIG, HT
      INTEGER I1, I2, K1, K2
      COMMCN/PARAM/SIG(10), D(10), I, J
      CCOSH(Z)=(CEXP(Z)+CEXP(-Z))/2.
      CSINH(Z)=(CEXP(Z)-CEXP(-Z))/2.
      NTOT=10
      MTOT=NTOT-1
      LTOT=NTOT-2
      HT=1.
      G=J1**2+J2**2
      IF(G.EQ.0.) GO TO 400
      DO 110 I2=1,NTOT
      H(I2)=2.*SIG(I2)/SIG(MTOT)
      A(I2)=CMPLX(G,H(I2))
      NU(I2)=CSQRT(A(I2))
      V(I2)=CCOSH(NU(I2)*D(I2))
      U(I2)=NU(I2)/(CSINH(NU(I2)*D(I2)))
110   CONTINUE
      B(2)=U(2)*V(2)+NU(1)
      DO 150 K1=3,MTOT
      B(K1)=U(K1)*V(K1)+U(K1-1)*V(K1-1)
150   CONTINUE
      DO 160 K2=2,LTOT
      P(2)=+U(2)/B(2)
      P(K2+1)=U(K2+1)/(B(K2+1)-U(K2)*P(K2))
160   CONTINUE
      NUM=NU(NTOT)-U(MTOT)*(V(MTOT)-P(MTOT))
      DEN=NU(NTOT)+U(MTOT)*(V(MTOT)-P(MTOT))
      E=1.+(NUM/DEN)
      E1=1.-(NUM/DEN)
      IM=CMPLX(0.,1.)
      IF(I.GT.64) GO TO 280
```

```
      DATA=IM*J1*E1*CEXP(-NU(NTOT)*HT)
      GO TC 281
280   DATA=-J1*IM*E1*CEXP(-NU(NTOT)*HT)
      GC TC 281
400   DATA=C.
281   RETURN
      END


C     PROGRAM IV
C
C     IN A SIMILAR MANNER TO PROGRAM II, THE DIMENSIONLESS AZIMUTHAL
C     COMPCNENT OF THE ELECTRIC FIELD DUE TO A VERTICAL MAGNETIC DIPOLE
C     IS EVALUATED.
C
      COMPLEX Z, DATA(128,128)
      REAL AB(65), J1, J2, UINC, SIG, D,T(65)
      INTEGER N(2), I, J
      COMMCN/PARAM/SIG(10), D(10), I, J
      NTOT=10
      DO 75 I1=1,NTOT
      READ(5,40) SIG(I1), D(I1)
40    FORMAT(6X, E11.4, 5X, E11.4)
75    CONTINUE
      UINC=.4
      DO 12 I=1,128
      IF(I-64) 30,30,31
30    J1=UINC*(I-1)
      GO TO 32
31    J1=UINC*(129-I)
32    DO 13 J=1,128
      IF(J-64) 33,33,34
33    J2=UINC*(J-1)
      GO TO 35
34    J2=UINC*(129-J)
35    CALL DIPOLE(J1,J2,DATA(I,J))
13    CONTINUE
12    CONTINUE
      N(1)=128
      N(2)=128
      CALL FOUR2(DATA,N,2,+1,+1)
      TIK=(UINC**2)/(2.*3.1415)
      DO 19 M=1,65
      AB(M)=TIK*CABS(DATA(1,M))
      T(M)=2.*3.1415*(M-1)/(UINC*N(1))
      WRITE(6,18) AB(M), T(M)
18    FORMAT(5X,E11.4,5X,E11.4)
19    CONTINUE
      CALL EXIT
      END


      SUBROUTINE DIPOLE(J1, J2, DATA)
      COMPLEX NU(10), NUM, DEN, A(10), V(10), U(10), B(10), P(10),
     C Z, E, DATA, CCOSH, CSINH, IM, E1
      REAL H(10), D, J1, J2, SIG, HT
      INTEGER I1, I2, K1, K2
      COMMCN/PARAM/SIG(10), D(10), I, J
```

```
      CCOSH(Z)=(CEXP(Z)+CEXP(-Z))/2.
      CSINH(Z)=(CEXP(Z)-CEXP(-Z))/2.
      NTOT=10
      MTOT=NTOT-1
      LTOT=NTOT-2
      HT=1.
      G=J1**2+J2**2
      IF(G.EQ.0.) GO TO 400
      DO 110 I2=1,NTOT
      H(I2)=2.*SIG(I2)/SIG(MTOT)
      A(I2)=CMPLX(G,H(I2))
      NU(I2)=CSQRT(A(I2))
      V(I2)=CCOSH(NU(I2)*D(I2))
      U(I2)=NU(I2)/(CSINH(NU( I2)*D(I2)))
  110 CONTINUE
      B(2)=U(2)*V(2)+NU(1)
      DO 150 K1=3,MTOT
      B(K1)=U(K1)*V(K1)+U(K1-1)*V(K1-1)
  150 CONTINUE
      DO 160 K2=2,LTOT
      P(2)=+U(2)/B(2)
      P(K2+1)=U(K2+1)/(B(K2+1)-U(K2)*P(K2))
  160 CONTINUE
      NUM=NU(NTOT)-U(MTOT)*(V(MTOT)-P(MTOT))
      DEN=NU(NTOT)+U(MTOT)*(V(MTOT)-P(MTOT))
      E=1.+(NUM/DEN)
      E1=1.-(NUM/DEN)
      IM=CMPLX(0.,1.)
      IF(J.GT.64) GO TO 280
      DATA=J2*E*CEXP(-NU(NTOT)*HT)/SQRT(G)
      GO TO 281
  280 DATA=-J2*E*CEXP(-NU(NTOT)*HT)/SQRT(G)
      GO TO 281
  400 DATA=0.
  281 RETURN
      END


C     PROGRAM V
C
C     MAIN:   THE DIMENSIONLESS COMPONENTS OF THE ELECTRIC AND
C     MAGNETIC FIELDS DUE TO A HORIZONTAL LINE CURRENT AT HEIGHT HT
C     ABOVE THE SURFACE OF A CONDUCTING HALF-SPACE WITH NTOT LAYERS.
C     THE ONE-DIMENSIONAL ARRAYS DATA1, DATA2, DATA3, CONTAIN THE
C     TRANSFORM-SPACE FUNCTIONS APPEARING IN THE INTEGRANDS OF THE
C     INTEGRAL SOLUTIONS FOR THE VERTICAL MAGNETIC FIELD, THE TANGENTIAL
C     MAGNETIC FIELD, AND TANGENTIAL ELECTRIC FIELD RESPECTIVELY.
C     THE ARRAYS ARE TRANSFORMED BY A DOUBLE-PRECISION FFT.   AB1,
C     AB2, AND AB3 CONTAIN THE ABSOLUTE VALUES OF THE FIELD COMPONENTS
C     HZ, HY, AND EX RESPECTIVELY.
C
      COMPLEX DATA1(514),DATA2(514),DATA3(514)
      REAL AB1(256),AB2(256),AB3(256),T(256)
      REAL*8 J1,UINC,SIG,D
      INTEGER N(1), I, J, NTOT
      DIMENSION IBUF(1000)
      COMMON/PARAM/SIG(10), D(10), J
      NTOT=512
```

```
      NT=10
      DO 75 I1=1,NT
      READ(5,40) SIG(I1), D(I1)
 40   FORMAT(6X, D20.13, 5X, D20.13)
 75   CONTINUE
      UINC=0.1D0
      DO 10 J=1,NTOT
      MTOT=NTOT/2
      LTOT=NTOT+1
      IF(J-MTOT) 30,30,31
 30   J1=UINC*(J-1)
      GO TO 32
 31   J1=UINC*(LTOT-J)
 32   CALL LINCRR(J1,DATA1(J),DATA2(J),DATA3(J))
 10   CONTINUE
      N(1)=512
      CALL FOUR2(DATA3,N,1,+1,+1)
      CALL FOUR2(DATA2,N,1,+1,+1)
      CALL FOUR2(DATA1,N,1,+1,+1)
      TIK=UINC*.5D0
      DO 19 M=1,100
      AB1(M)=TIK*CABS(DATA1(M))
      AB2(M)=TIK*CABS(DATA2(M))
      AB3(M)=TIK*CABS(DATA3(M))
      T(M)=2.D0*3.1415D0*(M-1)/(UINC*NTOT)
      WRITE(6,13) AB1(M),AB2(M),AB3(M),T(M)
 13   FORMAT(3X,E14.7,3X,E14.7,3X,E14.7,3X,E14.7)
 19   CONTINUE
 190  STOP
      END

      SUBROUTINE LINCRR(J1,DATA1,DATA2,DATA3)
C
C     SUBROUTINE LINCRR:      THE ARRAYS DATA1, DATA2, AND DATA3 ARE
C     DEFINED;   THE FACTORS Q AND K (APPEARING IN CHAPTER 4 OF
C     THIS THESIS) ARE GENERATED BY A DO-LOOP.
      COMPLEX*16 NU(10), NUM, DEN, A(10), V(10), U(10), B(10), P(10),
     C Z, DATA1, DATA2, E1, E2, CCOSH, CSINH, P1, IM, DATA3
      REAL*8 H(10), D, J1, SIG, HT, G, Q
      INTEGER K, K1, K2, NTOT, LTOT, MTOT, J
      COMMON/PARAM/SIG(10), D(10), J
      CCOSH(Z)=(CDEXP(Z)+CDEXP(-Z))/2.D0
      CSINH(Z)=(CDEXP(Z)-CDEXP(-Z))/2.D0
      NTOT=10
      MTOT=NTOT-1
      LTOT=NTOT-2
      HT=1.D0
      G=J1**2
      DO 11 K=1,NTOT
      H(K)=2.D0*SIG(K)/SIG(MTOT)
      A(K)=DCMPLX(G, H(K))
      NU(K)=CDSQRT(A(K))
      V(K)=CCOSH(NU(K)*D(K))
      Q=CDABS(NU(K))
      IF(Q.EQ.0.0) GO TO 11
      U(K)=NU(K)/(CSINH(NU(K)*D(K)))
 11   CONTINUE
      B(2)=U(2)*V(2)+NU(1)
      DO 150 K1=3,MTOT
      B(K1)=U(K1)*V(K1)+U(K1-1)*V(K1-1)
```

```
150   CONTINUE
      P(2)=U(2)/B(2)
      DO 160 K2=2,LTOT
      P(K2+1)=U(K2+1)/(B(K2+1)-U(K2)*P(K2))
160   CONTINUE
      NUM=NU(NTOT)-U(MTOT)*(V(MTOT)-P(MTOT))
      DEN=NU(NTOT)+U(MTOT)*(V(MTOT)-P(MTOT))
      E1=1.D0+(NUM/DEN)
      E2=1.D0-(NUM/DEN)
      IM=DCMPLX(0.D0, 1.D0)
      IF(J1.EQ.0.0) GO TO 171
      IF(J.GT.256) GO TO 114
      DATA1=-IM*E1*CDEXP(-NU(NTOT)*HT)
      GO TO 115
114   DATA1=+IM*E1*CDEXP(-NU(NTOT)*HT)
      GO TO 115
171   DATA1=DCMPLX(0.0D0,0.0D0)
115   DATA2=E2*CDEXP(-NU(NTOT)*HT)
202   IF(J1.EQ.0.0) GO TO 116
      DATA3=E1*CDEXP(-NU(NTOT)*HT)/J1
      GO TO 117
116   P1=DCMPLX(0.D0,-2.D0)
      DATA3=CDSQRT(P1)
117   RETURN
      END


C     PROGRAM VI
C
C     MAIN:    THE DIMENSIONLESS FIELD COMPONENTS OF PROGRAM V
C     (THE HORIZONTAL LINE-CURRENT) ARE EVALUATED AT SUCCESSIVELY
C     DEEP INTERFACES INSIDE THE LAYERED CONDUCTOR.  THESE VALUES ARE
C     STORED ON SCRATCH DISC (OUTPUT DEVICE '8') AND ARE THEN READ IN
C     A SUBSEQUENT PLOTTING ROUTINE.
C
      COMPLEX DATA1(256), DATA2(256),DATA3(256)
      REAL AB1(128),AB2(128),AB3(128),TIK,T(64),Z
      REAL*8 J1,SIG,D,UINC
      INTEGER N(1), I, J, NTOT
      COMMON/PARAM/SIG(20), D(20), J, Z, L
      NTOT=256
      NT=20
      DO 75 I1=1,NT
      READ(5,40) SIG(I1), D(I1)
40    FORMAT(6X,D20.13,5X,D20.13)
75    CONTINUE
      DO 21 L=1,7
      Z=.1*(L-1)
      UINC=0.2D0
      DO 10 J=1,NTOT
      MTOT=NTOT/2
      LTOT=NTOT+1
      IF(J-MTOT) 30,30,31
30    J1=UINC*(J-1)
      GO TO 32
31    J1=UINC*(LTOT-J)
32    CALL ATTENF(J1,DATA1(J),DATA2(J),DATA3(J))
10    CONTINUE
```

```
      N(1)=256
      CALL FOUR2(DATA1,N,1,+1,+1)
      CALL FOUR2(DATA2,N,1,+1,+1)
      CALL FOUR2(DATA3,N,1,+1,+1)
      TIK=UINC*.5
      DO 19 M=1,33
      AB1(M)=TIK*CABS(DATA1(M))
      AB2(M)=TIK*CABS(DATA2(M))
      AB3(M)=TIK*CABS(DATA3(M))
      T(M)=2.*3.1415*(M-1)/(UINC*NTOT)
      WRITE(8)     AB1(M),AB2(M),AB3(M),T(M)
19    CONTINUE
      AB1(34)=0.0
      AB2(34)=0.0
      AB3(34)=0.0
      T(34)=0.0
      AB1(35)=0.1
      AB2(35)=0.3
      AB3(35)=0.25
      T(35)=0.5
      WRITE(8) AB1(34),AB2(34),AB3(34),T(34)
      WRITE(8) AB1(35),AB2(35),AB3(35),T(35)
21    CONTINUE
      CALL EXIT
      END
      SUBROUTINE ATTENF(J1,DATA1,DATA2,DATA3)
      COMPLEX*16 NU(20),A(20),V(20),U(20),B(20),P(20),Q(20),IM,CCOSH,
     C CSINH, Z1
      COMPLEX NUM,DEN, DATA1,DATA2,E1,E2,C1,A1,F1
      COMPLEX DATA3
      REAL HT,Z,Q1
      REAL*8 H(20),D,J1,SIG,G
      INTEGER K, K1, K2, J, L, LTOT, NTOT, MTOT
      COMMON/PARAM/SIG(20), D(20), J, Z, L
      CCOSH(Z1)=(CDEXP(Z1)+CDEXP(-Z1))/2.D0
      CSINH(Z1)=(CDEXP(Z1)-CDEXP(-Z1))/2.D0
      NTOT=20
      MTOT=NTOT-1
      LTOT=NTOT-2
      HT=1.
      G=J1**2
      DO 11 K=1,NTOT
      H(K)=2.*SIG(K)/SIG(MTOT)
      A(K)=DCMPLX(G,H(K))
      NU(K)=CDSQRT(A(K))
      V(K)=CCOSH(NU(K)*D(K))
      Q1=CDABS(NU(K))
      IF(Q1.EQ.0.0) GO TO 11
      U(K)=NU(K)/CSINH(NU(K)*D(K))
11    CONTINUE
      B(2)=U(2)*V(2)+NU(1)
      DO 150 K1=3,MTOT
      B(K1)=U(K1)*V(K1)+U(K1-1)*V(K1-1)
150   CONTINUE
      P(2)=U(2)/B(2)
      DO 160 K2=2,LTOT
      P(K2+1)=U(K2+1)/(B(K2+1)-U(K2)*P(K2))
160   CONTINUE
      NUM=NU(NTOT)-U(MTOT)*(V(MTOT)-P(MTOT))
      DEN=NU(NTOT)+U(MTOT)*(V(MTOT)-P(MTOT))
```

```
      E1=1.+(NUM/DEN)
      E2=1.-(NUM/DEN)
      IM=DCMPLX(0.D0,1.D0)
      P(NTOT)=1.D0
      Q(1)=P(NTOT)
      DO 271 M1=1,14
      Q(1+M1)=Q(M1)*P(NTOT-M1)
271   CONTINUE
      A1=CDEXP(-NU(NTOT)*HT)
      C1=CSINH(NU(NTOT-L)*D(NTOT-L))
      F1=CCOSH(NU(NTOT-L)*D(NTOT-L))-P(NTOT-L)
      IF(J1.EQ.0.0) GO TO 133
      IF(J.GT.128) GO TO 114
      DATA1=-E1*A1*Q(L)*IM
      GO TO 115
114   DATA1=E1*A1*Q(L)*IM
      GO TO 115
133   DATA1=CMPLX(0.0,0.0)
115   IF(J1.EQ.0.) GO TO 112
      IF(L.EQ.18) GO TO 111
      DATA2=E1*A1*F1*Q(L)*U(NTOT-L)/J1
      GO TO 109
111   DATA2=E1*A1*Q(L)*NU(NTOT-L)/J1
      GO TO 109
112   DATA2=2.*Q(L)*U(NTOT-L)*(V(NTOT-L)-P(NTOT-L))/(U(MTOT)*(V(MTOT)
     C -P(MTOT)))
109   IF(J1.EQ.0.) GO TO 116
      DATA3=E1*A1*Q(L)/J1
      GO TO 118
116   DATA3=2.0*Q(L)/DEN
118   RETURN
      END
```

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