Beyond Super-λ: Counting 4-Edge Cutsets in 3-Regular Graphs

by

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Abstract

A graph on n vertices and m edges is called a λ-*optimal* graph if, amongst all other (n, m)-graphs, it maximizes the minimum cutset size and minimizes the number of minimum cutsets If the only cutsets of minimum size in a λ-optimal graph are the incidence sets of vertices of minimum degree, then the graph is *super-λ* Circulant graphs are examples of super-λ graphs, and for this reason have been proposed as a network topology We compare 3-regular circulants to other $(n, \frac{3n}{2})$-graphs, and provide evidence that 3-regular circulants are perhaps the worst super-λ topology to choose A family of 3-regular graphs is proposed as an alternate choice - while not always the most reliable $(n, \frac{3n}{2})$-graphs, the members of this family tend to be near the top of the list

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Chapter 1

Introduction

A *graph* is an ordered pair $G = (V, E)$, where $V = V(G)$ is a nonempty set of elements called the *vertices* of G, and $E = E(G)$ is a set containing the *edges* of G, which are defined as follows. An edge $e = (u, v) \in E$ is associated with two vertices, u and v, called the *endpoints* of e. If an edge has the same vertex as its two endpoints, it is called a *loop*. The *order* of a graph G, denoted $n(G)$ or n if G is understood, is the number of vertices of G. The *size* of G, denoted $m(G)$ or m, is the number of edges of G. For fixed n and m, the class of (n, m)*-graphs* consists of all graphs on n vertices with m edges. Any graph theoretic terminology which is not defined in this thesis follows that of Bondy and Murty [8].

In this thesis, we examine the class of $(n, \frac{3n}{2})$-graphs (where n is even), and compare members of this class based on concepts explained below. A computer network is modelled by a graph by letting the sites be represented by the vertices and letting the links between the sites be represented by the edges. Consider a network N in which the sites are assumed to be perfectly reliable and each link l fails independently with probability $q(l)$. A *probabilistic graph* is a graph $G = (V, E)$, where there is a probability of operation associated with each vertex $v \in V$ and each edge $e \in E$. Then N is modelled by a probabilistic graph G, where for each edge e

corresponding to some link l, $q(e)$ is a real number associated with e, where $q(e)$ is equal to $q(l)$ and $0 \leq q(e) \leq 1$ If $p(e) = 1 - q(e)$, then $p(e)$ is the probability that the edge e of the graph is operational The convention used here is to assume that the edge success probabilities are equal This assumption is sensible under certain circumstances, since the links between the sites in a network are often composed of the same material

Before we can define our network reliability problem, we require a few definitions, any network reliability terminology which is not defined in this thesis follows that of Colbourn in his monograph [12], which is a survey on network reliability A *subgraph* H of a graph G is defined as a graph $H = (V'(G), E'(G))$, where $V'(G) \subseteq V(G)$ and $E'(G) \subseteq E(G)$ A subgraph $P = (V', E')$ of a graph $G = (V, E)$ is a *path of G* if the vertices of P can be arranged in a sequence, i e $v_1 v_2 v_3 \cdots v_{|V'|}$, such that $E' = \{(v_i, v_{i+1}) : 1 \leq i \leq |V'| - 1\}$ If such a subgraph P exists, we say that P is a path from v_1 to $v_{|V'|}$ A graph G is *connected* if for every pair of vertices u and v in G, there exists a path of G from u to v A graph is *disconnected* if it is not connected

We can now define the *all-terminal reliability* problem as follows given a network in which the links fail uniformly at random with probability q, the probability that the network is *operational* is defined to be the probability that the probabilistic graph which models this network is connected For any probabilistic graph G, we define a state of G to be a subset $E' \subseteq E$, we interpret this to mean that all edges in E' are operational and all edges in $E \backslash E'$ are failed Under the all-terminal reliability model, for a state of the probabilistic graph to be connected no vertices can fail, i e $p(v) = 1$ for all $v \in V(G)$ Let $Rel(G)$ denote the all-terminal reliability without node failure If we change our assumption, and let p_i be the probability that vertex v_i is operational, then the probability that the graph is operational, allowing for both vertex and edge failure, is

$$\prod_{i=0}^{n-1} p_i \cdot Rel(G).$$

From [12], we know that the problem of calculating $Rel(G)$ is #P-complete. Since it is simple to compute the product of vertex probabilities, and #P-complete to compute $Rel(G)$, we focus our attention on the latter. For a very good survey of the computational complexity of network reliability analysis, see [2] by Ball.

We now define some of the terminology which appears in the formula for $Rel(G)$. For a graph G, a subset $F \subseteq E(G)$ is a *cutset*, or *cut*, of G if $G' = (V(G), E(G) \setminus F)$ is disconnected. For a cutset F of G, the *size* of the cut is $|F|$; a cutset F of G is a *minimum cutset* if $|F|$ is minimized. We let $N_i(G)$, or N_i when G is understood, denote the number of i-edge cutsets of G. Then $Rel(G)$ can be expressed as:

$$Rel(G) = 1 - \sum_{i=0}^{m} N_i(G) \cdot (1-p)^i \cdot p^{m-i}.$$

The values N_i, for $0 \leq i \leq m$, can be used to compare two (n, m)-graphs as follows. The *cut frequency vector* of a graph G is given by $(N_0, N_1, N_2, \ldots, N_m)$. Note that N_i equals $\binom{m}{i}$ for $m - (n-2) \leq i \leq m$, since any spanning tree contains $n-1$ edges. An (n, m)-graph G_1 is *uniformly-most reliable* if $Rel(G_1)$ is greater than or equal to $Rel(G_2)$ for all (n, m)-graphs G_2. Thus for an (n, m)-graph G_1, if $N_i(G_1)$ is less than or equal to $N_i(G_2)$ for all other (n, m)-graphs G_2, for $0 \leq i \leq m$, then G_1 is the uniformly-most reliable graph for those values of n and m. However, uniformly-most reliable graphs do not exist for all values of n and m; this was first proved by Kel'mans in 1981 [18], conjectured by Boesch in 1986 [4], and then later independently rediscovered in 1991 [21]. In this thesis, we only compare the cut frequency vectors of two graphs if both graphs have the same number of vertices and edges. Furthermore, we only examine the reliability of connected graphs, since disconnected graphs are always non-operational.

When examining the cut frequency vector of a graph, we are interested in determining the first nonzero entry. The *edge-connectivity* (or *line-connectivity*) of a graph G, denoted $\lambda(G)$, or λ, is the minimum size of a cutset of G. Thus $N_i(G)$ is equal to zero for all i strictly less than the edge-connectivity of G. The *degree* of a vertex v in G, denoted by $deg(v)$, is the total number of times in which edges of G are incident to v (loops are incident twice). Note that the edge-connectivity never exceeds the minimum degree of a vertex of G. Thus if G_1 and G_2 are both (n,m)-graphs, but $\lambda(G_1)$ is greater than $\lambda(G_2)$, then the cut frequency vector of G_1 is lexicographically smaller than that of G_2.

We now consider the number of cutsets having size equal to the edge-connectivity. A graph G is *super-λ* if the only λ-edge cutsets are the incidence sets of vertices of minimum degree in G. A graph G is *r-regular* if all vertices in G are of degree r. Thus an r-regular graph G is super-λ if $\lambda(G)$ equals r, and the only λ-edge cutsets are the incidence sets of the vertices of G. Consequently, for an r-regular super-λ graph G, $\lambda(G)$ is equal to r, and $N_\lambda(G)$ is equal to n, the number of vertices of G. Thus if G_1 and G_2 are both r-regular graphs, and only G_1 is super-λ, then the cut frequency vector of G_1 is lexicographically smaller than that of G_2. A graph is *cyclically t-edge connected* if upon deletion of fewer than t edges, there is at most one component containing a cycle. Any r-regular super-λ graph G is cyclically $(\lambda+1)$-edge connected, since there are no cutsets of size less than λ, and the only λ-edge cutsets are minimum cuts, each of which is the incidence set of a vertex of G. A necessary and sufficient condition for a graph to be cyclically t-edge connected is provided in [9] by Cai, along with a method for producing such graphs for t equal to 2, 3, or 4. In [14], Fleischner and Jackson discuss some conjectures on cyclically 4-edge connected 3-regular graphs.

Cutsets are important to network reliability because of their relation to the approximation of $Rel(G)$. Kel'mans observes in [17] that determining an (n,m)-

graph G which is more reliable than any other (n, m)-graph for large edge success probabilities can be reduced to the problem of finding graphs which minimize a single term of the reliability polynomial. Specifically, if the edge success probability is large enough, then $Rel(G)$ can be approximated by the term $N_\lambda(G)\,(1-p)^\lambda\,p^{m-\lambda}$. Details justifying this reduction using an edge failure model are given by Bauer et al. in [3].

This approximation is used in the following two papers, both of which propose a particular class of super-λ graphs as reliable network topologies [4] [6]. Boesch and Wang provide results in [6] related to the problem of determining a circulant graph which has minimum diameter and maximum connectivity over all (n, m)-graphs. In [4], Boesch gives an excellent survey of the graph theoretic notions which are relevant to the problem of constructing reliable networks. In both [4] and [6], circulant graphs are proposed as a reliable network topology partly because they are super-λ graphs. However, examination of the cut frequency vectors of 3-regular graphs of order 16 or less shows that the cut frequency vectors of 3-regular circulants are the lexicographically largest amongst 3-regular super-λ graphs, and that other classes of graphs have cut frequency vectors which are consistently smaller. Among the latter group are cycle permutation graphs and generalized Petersen graphs.

Our research is an attempt to determine the $(n, \frac{3n}{2})$-graph which is uniformly-most reliable. If none exists for those values of n and m, then we attempt to determine the $(n, \frac{3n}{2})$-graph with the lexicographically smallest cut frequency vector, since these graphs are guaranteed to be the most reliable when the edge success probability is sufficiently high. Our first attempt is to focus our attention on the values of N_i, for i less than or equal to four, for graphs of order ten or greater. Since super-λ graphs have cut frequency vectors which are lexicographically smaller than graphs which are not super-λ, and since 3-regular super-λ graphs have lexicographically smaller cut frequency vectors than super-λ graphs which have minimum degrees less than or equal to two, we restrict our search to $(n, \frac{3n}{2})$-graphs which are 3-regular and also super-λ. The emphasis for our search is placed on four classes of graphs: quasi-prisms, cycle permutation graphs, circulants, and generalized Petersen graphs. Circulants are examined because they have been proposed as a reliable network topology. Quasi-prisms are an easily described class of 3-regular graphs which contains models of various network topologies, including all cycle permutation graphs and generalized Petersen graphs, some of which have cut frequency vectors which are lexicographically smaller than those of circulant graphs. One particular subset of the class of cycle permutation graphs, which is also a subset of the class of generalized Petersen graphs, minimizes the first four entries of the cut frequency vector. The intersection of these classes of 3-regular graphs is examined in Chapter 4.

The cut frequency vectors of r-regular cycle permutation graphs have been examined previously. In [23], Piazza and Stueckle consider the cut frequency vector of permutation graphs, and provide lower bounds on the number of $(\lambda + i)$-edge cutsets for $0 \leq i \leq (m - \lambda)$. They show that, with some restrictions, permutation graphs are super-λ, and that some permutation graphs may minimize other entries in the cut frequency vector as well. For 3-regular permutation graphs, they do not consider bounds beyond N_λ for any coefficients.

A similar approach has been used to study the cut frequency vectors of r-regular circulants. In [7], Boesch and Wang examine r-regular circulants and determine lower bounds on the number of i-edge cutsets for $\lambda \leq i \leq 2r - 3$. They show that a special class of circulants achieves these lower bounds. However, for 3-regular circulants, they do not consider bounds for any coefficients beyond N_λ.

The cut frequency vectors of another class of graphs have been studied in the past by dividing the i-edge cutsets into two separate types. In [13], Farley and Proskurowski introduce the notion of an *isolated cutset*, which is a set of edges such that no two edges are incident to the same vertex. A cutset that isolates a single vertex must be a *nonisolated cutset*. A chordal ring is a 3-regular graph with n vertices on a cycle of order n and $\frac{n}{2}$ chords determined by a parameter h, denoted by $C(n, h)$. In [16], Hu and Hwang examine the cut frequency vectors of chordal rings by using the notion of isolated and nonisolated cutsets. They provide exact formulas for the number of minimum isolated cutsets for particular values of the parameter h. Chordal rings are mentioned in this thesis because the uniformly-most reliable graphs on 14 and 16 vertices are both chordal rings.

The results of this thesis are as follows. Computer results show the following for 3-regular graphs: for n equal to 10 and 14, the least reliable super-λ graph is a circulant, and for n equal to 10,12,14 and 16, the second least reliable super-λ graph is also a circulant. With the exception of one particular cycle permutation graph, cycle permutation graphs are more reliable than 3-regular circulants. Theoretical results show that the cut frequency vectors of 3-regular circulants are lexicographically larger than the cut frequency vectors of those cycle permutation graphs which are not isomorphic to one particular cycle permutation graph. A subclass of cycle permutation graphs, which is also a subclass of generalized Petersen graphs, is proposed as an asymptotically reliable choice of network topology. Members of this subclass minimize the first five entries in the cut frequency vector, for graphs of order ten or greater; this family is described in Chapter 7.

This thesis is laid out in the following manner. Chapter 2 contains some basic results in graph theory, network reliability, permutations, and number theory. Quasi-prisms are introduced and their properties are explored in Chapter 3. In Chapter 4, we establish various isomorphisms between quasi-prisms, cycle permutation graphs, circulants, and generalized Petersen graphs. The focus shifts back to a specific class of graphs in Chapter 5, where we determine conditions for when cycle permutation graphs are super-λ, and enumerate the number of $(\lambda + 1)$-edge cutsets for specific cycle permutation graphs. In the next chapter, we determine $N_{\lambda+1}$ for all 3-regular circulant graphs. We then establish restrictions for a generalized Petersen graph to be super-λ, and give a formula for the number of $(\lambda + 1)$-edge cutsets for certain graphs in this class. Ideas for future research appear in Chapter 8. Finally, the appendices contain tables of reliability coefficients, examples, and proofs of some of the previously stated results.

Chapter 2

Background Definitions and Theorems

In this chapter, we present some background definitions and simple results which are used throughout this thesis. We begin with some definitions and results in graph theory, then provide selected theorems in network reliability. Since we use permutations to define families of graphs, and use number theoretic results when examining these families, we finish with definitions related to permutations and number theory.

2.1 Graph Theory

This section contains graph theoretic definitions and results which are related to enumerating cutsets in graphs. A subgraph $C = (V', E')$ of a graph $G = (V, E)$ is a *cycle of* G if the vertices of C can be arranged in a sequence, i.e. $v_1 v_2 \ldots v_{|V'|}$, such that $E' = \{(v_i, v_{i+1}) : 1 \leq i \leq |V'| - 1\} \cup \{(v_1, v_{|V'|})\}$. A *segment* of a cycle C is a subgraph of C which is a path, the *order* of a segment is the number of vertices in the segment. A *bipartite graph* is one whose vertex set can be partitioned into two subsets X and Y, so that each edge in E has one endpoint in X and one endpoint in Y, such a partition (X, Y) is called a *bipartition* of the graph If X and Y are both nonempty, then we say that (X, Y) is a *nontrivial bipartition* Let V' be a nonempty subset of $V(G)$ Then the *subgraph of G induced by V'*, $G[V']$, is the subgraph of G whose vertex set is V' and whose edge set is the set of those edges in G having both endpoints in V'

For $X \subseteq V$ and $\overline{X} = V \setminus X$, the notation $(X, \overline{X})$ denotes the cut containing exactly the edges (u, v), where $u \in X$ and $v \in \overline{X}$, and both $G[X]$ and $G[\overline{X}]$ are connected A cutset F of G is *minimal* if there is no set of edges F' properly contained in F such that F' is also a cut of G Note that the cuts $(X, \overline{X})$ are exactly the minimal cuts of a graph Note also that for a graph G, with $\lambda(G)$ equal to λ, it is possible to have a $(\lambda+1)$-edge cutset that is not minimal If F is a λ-edge cut of G, then for any edge $e \in E(G) \setminus F$, $F \cup \{e\}$ is a $(\lambda+1)$-edge cut which is not minimal It is clear that $F \cup \{e\}$ is not minimal, since $F \subsetneq F \cup \{e\}$ is also a cut of G The following lemma states a well-known property of bipartite graphs

Lemma 2 1.1 [[8], p 14] *A graph is bipartite if and only if it contains no cycles of odd order*

The observation made in the following lemma is simple, but it is important for several of the proofs in this thesis We *contract* an edge $e = (u, v) \in E(G)$ by first removing e, then removing the vertices u and v and adding the new vertex w, such that every edge involving u or v, i e (u, x) or (v, y), is replaced by an edge involving w, i e (w, x) or (w, y) Multiple edges created in this process are retained

Lemma 2 1.2 *Let $(X, \overline{X})$ be a minimal cutset of a graph G For any cycle C of G, $|E(C) \cap (X, \overline{X})|$ is even*

Proof. Consider a cycle C of G, and contract all edges of C which do not belong to the cut $(X, \overline{X})$ This results in a smaller cycle C' having $|E(C) \cap (X, \overline{X})|$

edges Clearly, $(X, \overline{X})$ is a bipartition of C', therefore C' is a bipartite graph By Lemma 2 1 1, C' must be a cycle of even order Therefore $|E(C) \cap (X, \overline{X})|$ is even
□

The following result is easily obtained, but it is useful in future chapters We refer to a $(2t)$-cut as an *even cut*, and a $(2t+1)$-cut as an *odd cut*,

Lemma 2 1 3 *If $(X, \overline{X})$ is an even cut of a 3-regular graph, then $|X|$ is even, if $(X, \overline{X})$ is an odd cut of a 3-regular graph, then $|X|$ is odd*

Proof We use proof by contradiction Assume that we have a $(2t)$-edge cut $(X, \overline{X})$, $|X| = 2r + 1$ Then for the subgraph induced by X,

$$\sum_{v \in X} deg(v) = 3|X| - 2t = 3(2r+1) - 2t,$$

which is an odd number But we know that for the subgraph induced by X, $\sum_{v \in X} deg(v)$ is an even number This is a contradiction, and therefore $|X|$ must be even Now assume that we have a $(2t+1)$-edge cut $(X, \overline{X})$, $|X| = 2r$ Then for the subgraph induced by X,

$$\sum_{v \in X} deg(v) = 3|X| - (2t+1) = 3(2r) - (2t+1),$$

which is an odd number But we know that for the subgraph induced by X, $\sum_{v \in X} deg(v)$ is an even number This is a contradiction, and therefore $|X|$ must be odd □

2.2 Network Reliability

In this section, we present results which are useful when comparing the cut frequency vectors of two (n, m)-graphs Let $\lambda_{max}(n, m)$ denote the maximum value of λ over all (n, m)-graphs G An (n, m)-graph G with $\lambda(G)$ equal to $\lambda_{max}(n, m)$ is *λ-optimal* if $N_\lambda(G)$ is minimized over all (n, m)-graphs having λ equal to $\lambda_{max}(n, m)$

Lemma 2.2.1 [[4], p. 343] *If* $[\lambda(G_1) > \lambda(G_2)]$ *or* $[\lambda(G_1) = \lambda(G_2)$ *and* $N_\lambda(G_1) < N_\lambda(G_2)]$ *then* G_1 *is more reliable than* G_2 *for sufficiently reliable edges.*

Therefore by Lemma 2.2.1, to maximize $Rel(G)$ for p close to one we must choose a λ-optimal graph. But for $(n, \frac{3n}{2})$-graphs, any λ-optimal graph is a 3-regular super-λ graph. Thus we observe that any super-λ graph is more reliable than a graph which is not super-λ, for very reliable edges. The following lemma allows us to compare two graphs which are super-λ.

Lemma 2.2.2 [[20], p. 4] *If* $N_i(G_1) = N_i(G_2)$ *for* $i = 0, 1, 2, \ldots, j$ *and* $N_{j+1}(G_1) < N_{j+1}(G_2)$*, then* G_1 *is more reliable than* G_2 *for* p *in the range* $1 > p > 1 - \epsilon$ *for some* $\epsilon > 0$.

Observe that Lemma 2.2.1 is a corollary of Lemma 2.2.2. For two super-λ graphs G_1 and G_2, $N_i(G_1)$ is equal to $N_i(G_2)$ for $0 \leq i \leq \lambda$. Therefore by Lemma 2.2.2, if a strict inequality exists, say $N_{\lambda+1}(G_1) < N_{\lambda+1}(G_2)$, then G_1 is more reliable than G_2 for values of p close to one. Therefore a first step is to compare the cut frequency vectors of 3-regular super-λ graphs, and attempt to find the graph with N_4, the number of 4-edge cutsets, minimized.

2.3 Permutations and Number Theory

Permutations are relevant to this thesis for the following reason. A permutation is associated with each quasi-prism and each cycle permutation graph, and the permutation is used to describe the edge sets of these graphs. This section defines permutations and explains the notation which is used in this thesis.

The group theory notation used in this thesis follows that of Gallian in his abstract algebra text [15]. A function f from a set A to a set B is an *injection* if, for $a \in A$ and $b \in A$, $f(a) = f(b)$ implies $a = b$. The function f is a *surjection* if, for each $b \in B$, there is some $a \in A$ such that $f(a) = b$. If f is both injective and surjective, then f is a *bijection*.

A *permutation* of a set Z is a bijection from Z to Z. Letting $Z = \{0, 1, 2, \ldots, r-1\}$, the set of all permutations on Z is called the *symmetric group of degree* r, and is denoted by $Sym(r)$. See [15] for a description of the properties of $Sym(r)$ and groups in general. A permutation $\sigma \in Sym(r)$ can be represented in *matrix form*, as follows. The matrix form of σ is denoted by

$$\sigma = \begin{bmatrix} 0 & 1 & 2 & \cdots & (r-1) \\ \sigma(0) & \sigma(1) & \sigma(2) & \cdots & \sigma(r-1) \end{bmatrix}$$

In this form, $\sigma(i)$ is placed directly below i, for $0 \leq i \leq r-1$. For example, letting $\sigma \in Sym(4)$ be

$$\sigma(0) = 1,\ \sigma(1) = 3,\ \sigma(2) = 2, \text{ and } \sigma(3) = 0,$$

then the matrix form of σ is denoted by

$$\sigma = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 2 & 0 \end{bmatrix}$$

A permutation can also be written in *cycle notation*, which is defined as follows. Let π be a permutation on the set $Z = \{0, 1, \ldots, r-1\}$. To write the permutation π in cycle form, we start by choosing any member of Z, say a_0, and let $a_1 = \pi(a_0)$, $a_2 = \pi(\pi(a_0))$, and so on, until $a_0 = \pi^{(x)}(a_0)$ for some x. Such an x is guaranteed to exist since the set $\{a_0, \pi(a_0), \pi^{(2)}(a_0), \ldots\}$ must be finite. Therefore there must eventually be some repeated element, say $\pi^{(i)}(a_0) = \pi^{(j)}(a_0)$ for some i and j with $i < j$. Then $a_0 = \pi^{(x)}a_0$, where $x = j - i$. The first cycle has x elements in it, therefore this cycle is of order x. We express this relationship as $\pi = (a_0 a_1 \cdots a_{x-1}) \cdots$, where the $\cdots$ represents that we may not yet have exhausted the set Z. If the set Z has not yet been exhausted, we choose a remaining element of Z, say b_0, and repeat the above process This is repeated until the set Z is exhausted, the permutation can then be written as

$$\pi = (a_0 a_1 \cdots a_{x-1})(b_0 b_1 \cdots b_{y-1}) \cdots (c_0 c_1 \cdots c_{z-1})$$

Thus every permutation can be written as a succession of disjoint cycles, say $\pi = C_0\, C_1 \cdots C_h$, where each C_i is a cycle For example, if

$$\pi = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 0 & 2 & 6 & 1 & 4 \end{bmatrix},$$

then writing π in cycle form yields

$$\pi = (032)(15)(46)$$

Note that for a permutation $\pi = C_0\, C_1 \cdots C_h$, the cycles C_i can be reordered and still yield the same permutation For example, for π written above,

$$\begin{aligned} \pi &= (032)(15)(46) \\ &= (15)(032)(46) \\ &= (46)(032)(15) \end{aligned}$$

Throughout this thesis, we let $\mathcal{I}$ denote the identity permutation, where

$$\mathcal{I} = \begin{bmatrix} 0 & 1 & 2 & \cdots & r-1 \\ 0 & 1 & 2 & \cdots & r-1 \end{bmatrix}$$

In cycle notation, the identity permutation is written as $\mathcal{I} = (0)(1)(2) \cdots (r-1)$

Let π be a permutation expressed in cycle notation, where $\pi \in Sym(r)$ and π is a cycle of order r, i e $\pi = (a_0\, a_1 \cdots a_{r-1})$ If a_0 is not equal to zero, and instead a_i equals zero, then it is always possible to express π as π', where a'_0 is equal to zero, i e $\pi' = (a_i\, a_{i+1}\, a_{i+2} \cdots a_{i-2}\, a_{i-1})$ This corresponds to a *rotation* of the permutation π It is also possible to express π^{-1}, where $\pi^{-1} = (a_0 a_{r-1} a_{r-2} \cdots a_2 a_1)$ This corresponds to the *inverse* of the permutation π Given two permutations π

and π' in $Sym(r)$, where both π and π' are expressed in cycle notation and both permutations are cycles of order r, we say that π and π' are *cyclically equivalent* if it is possible to rotate and/or take the inverse of π' so as to obtain π.

The *greatest common divisor* of two nonzero integers a and b, denoted by $gcd(a, b)$, is the largest integer which divides both a and b. When $gcd(a, b)$ equals one, a and b are *relatively prime*. The following number theoretic result is required for several results in this thesis.

Lemma 2.3.1 [[15], pp. 62-63] *If* $gcd(a, b) = 1$, *then*

$$\{b \cdot i \,(\mathrm{mod}\, a) : 0 \leq i \leq a-1\} = \{0, 1, 2, \ldots, (a-1)\}.$$

Chapter 3

Quasi-Prisms

In this chapter, we introduce the class of graphs known as quasi-prisms. We first define this class of graphs, and show that they are always 3-regular. We then state conditions for when a quasi-prism is super-λ.

3.1 Properties of Quasi-Prisms

We begin this section with the definition of quasi-prisms. We then give a brief proof showing that all quasi-prisms are 3-regular.

Definition: Quasi-prism $Q_k[\pi]$

The *quasi-prism* of order $n = 2k$ associated with a permutation $\pi \in Sym(k)$ expressed in cycle notation, denoted $Q_k[\pi]$, has vertex set partitioned into the two parts $\mathcal{S} = \{s_0, s_1, \ldots, s_{k-1}\}$ (the *outer cycle vertices*), and $\mathcal{T} = \{t_0, t_1, \ldots, t_{k-1}\}$ (the *inner cycle vertices*), and edges as follows, for $i = 0, 1, \ldots, k-1$ (all arithmetic is modulo k):

1. (s_i, s_{i+1}), the *outer cycle edges*,
2. $(t_i, t_{\pi(i)})$, the *inner cycle edges*, and

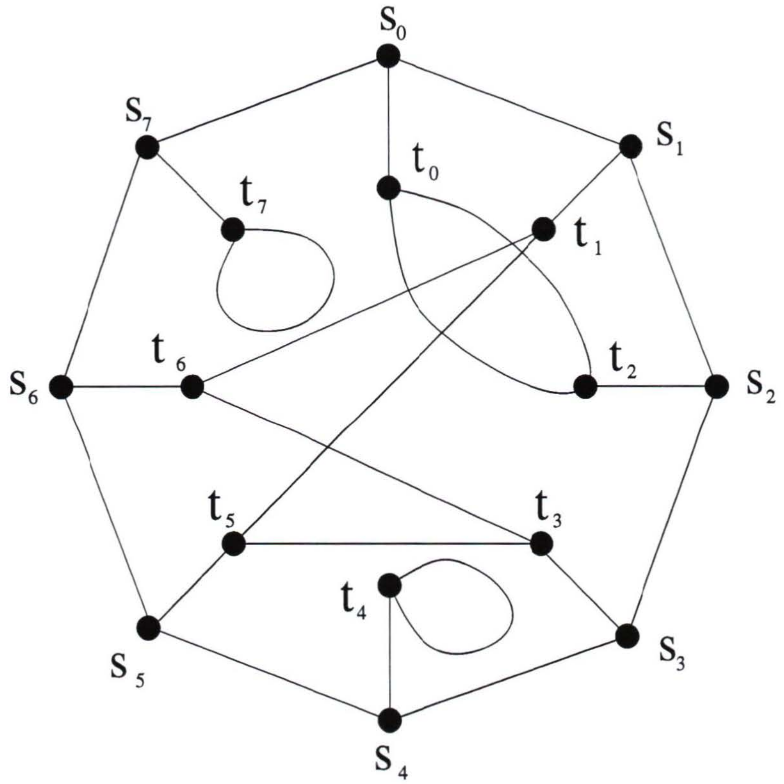

Figure 3.1 Quasi-prism $Q_8[(02)(1536)(4)(7)]$

3. (s_i, t_i), the *spoke edges*.

An example of the quasi-prism $Q_8[(02)(1536)(4)(7)]$ is shown in Figure 3.1.

Lemma 3.1.1 *Quasi-prisms are 3-regular.*

Proof Let the vertex and edge sets of $Q_k[\pi]$ be as follows:

$$V(Q_k[\pi]) = \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and}$$
$$E(Q_k[\pi]) = \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \le i \le k-1\}.$$

For any vertex $s_i \in \mathcal{S}$, the only edges incident with s_i are $(s_{i-1}, s_i), (s_i, s_{i+1})$, and (s_i, t_i). Therefore all vertices in $\mathcal{S}$ are of degree three. For any vertex $t_i \in \mathcal{T}$, the only edges incident with t_i are $(t_i, t_{\pi(i)}), (t_j, t_i)$, and (s_i, t_i), where $i = \pi(j)$ for some $0 \le j \le k-1$. Therefore all vertices in $\mathcal{T}$ are of degree three. □

Note that if π contains a cycle of order one, say (i), then $\pi(i)$ is equal to i. Therefore the edge $(t_i, t_{\pi(i)}) \in Q_k[\pi]$ is an edge having t_i as both endpoints, which is a loop. Note also that if π contains a cycle of order two, say (ij), then $\pi(i)$ equals j and $\pi(j)$ equals i. Therefore the edge $(t_i, t_{\pi(i)})$ and the edge $(t_j, t_{\pi(j)})$ are both edges having t_i and t_j as endpoints, which means that $Q_k[\pi]$ has multiple edges.

3.2 Super-λ Properties of Quasi-prisms

This section contains the result that quasi-prisms are super-λ under certain conditions. In establishing when a quasi-prism is super-λ, we discuss how the outer cycle vertices, the inner cycle vertices, and the inner cycles can be bipartitioned. A portion of this discussion appears below, set apart from the actual proofs, since the ideas contained within are also used in other proofs in this thesis.

Let the vertex and edge sets of $Q_k[\pi]$ be as follows:

$$\begin{aligned} V(Q_k[\pi]) &= \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and} \\ E(Q_k[\pi]) &= \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \le i \le k-1\}. \end{aligned}$$

We denote a minimal cutset of a quasi-prism $Q_k[\pi]$ as $(X, \overline{X})$. This cut induces a bipartition of the outer cycle vertices $\{s_0, s_1, \ldots, s_{k-1}\}$ into $(S, \overline{S})$, where $S \subseteq X$ and $\overline{S} \subseteq \overline{X}$. Similarly, $(X, \overline{X})$ induces a bipartition of the inner cycle vertices $\{t_0, t_1, \ldots, t_{k-1}\}$ into $(T, \overline{T})$, where $T \subseteq X$ and $\overline{T} \subseteq \overline{X}$. Note that if $(X, \overline{X})$ does not contain any inner cycle edges, then $(T, \overline{T})$ induces a bipartition of the cycles of π. This observation is explored further in Chapter 7.

Let $(T, \overline{T})$ be a nontrivial bipartition of the inner cycle vertices induced by a minimal cut. Then $(T, \overline{T})$ induces the nontrivial bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$, where $I = \{i : t_i \in T\}$ and $\overline{I} = \{i : t_i \in \overline{T}\}$. The values of a set I are *contiguous* (modulo k) if they can be arranged in a contiguous sequence,

$$\text{i.e. } q, \; q+1, \; q+2, \; \ldots, \; q+|I|-1$$

For example, the values of the set $I = \{0, 1, 2, 7, 8\}$ are contiguous modulo 9, since they can be arranged in a contiguous sequence, i.e. 7 8 0 1 2. The values of a set I are *almost contiguous* (modulo k) if all but one value can be arranged in a contiguous sequence.

Let $(X, \overline{X})$ be a minimal cutset of $Q_k[\pi]$ having three or fewer edges. Then by Lemma 2.1.2, $(X, \overline{X})$ contains zero or two outer cycle edges and zero or two inner cycle edges. Therefore a minimal cutset $(X, \overline{X})$ with three or fewer edges can take one of the following forms:

(1) $(X, \overline{X})$ contains only spoke edges (one, two, or three),

(2) $(X, \overline{X})$ contains two outer cycle edges and zero or one spoke edge, or

(3) $(X, \overline{X})$ contains two inner cycle edges and zero or one spoke edge.

Our proof that $Q_k[\pi]$ is super-λ, under certain conditions, follows a case analysis based on the three cases stated above. The next three lemmas establish results which are then used in the proof of the main theorem.

Lemma 3.2.1 *Let $k \geq 1$ and $\pi \in Sym(k)$, where π is expressed in cycle notation. The quasi-prism $Q_k[\pi]$ has a minimal j-edge cutset containing only spoke edges if and only if π has a cycle of order j.*

Proof. Let the vertex and edge sets of $Q_k[\pi]$ be as follows:

$$\begin{aligned} V(Q_k[\pi]) &= \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and} \\ E(Q_k[\pi]) &= \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \leq i \leq k-1\}. \end{aligned}$$

[$\Rightarrow$] Let $Q_k[\pi]$ have a minimal j-edge cutset $(X, \overline{X})$ containing only spoke edges. Without loss of generality, we can assume that $\{s_0, s_1, \ldots, s_{k-1}\} \subseteq \overline{X}$. Then since $(X, \overline{X})$ induces a bipartition $(T, \overline{T})$ which induces a bipartition of the cycles of π, $G[X]$ contains one or more inner cycles of $Q_k[\pi]$. Since $(X, \overline{X})$ is minimal, $G[X]$ can contain exactly one inner cycle. Since $(X, \overline{X}) = \{(s_i, t_i) : t_i \in T\}$, $G[T]$ is a cycle of order $|(X,\overline{X})|$ in $Q_k[\pi]$ Therefore π contains a cycle of order j, whose values belong to the set $I = \{i : t_i \in T\}$

[$\Leftarrow$] Let π have a cycle C of order j Then $F = \{(s_i, t_i) : i \in C\}$ is a j-edge cutset of $Q_k[\pi]$ containing only spoke edges □

Lemma 3 2 2 *Let $k \geq 2$ and $\pi \in Sym(k)$, where π is expressed in cycle notation The quasi-prism $Q_k[\pi]$ has a 2-edge minimal cutset $(X,\overline{X})$ containing only outer cycle edges if and only if there exists a nontrivial bipartition of the cycles of π inducing a bipartition $(I,\overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is contiguous*

Proof Let the vertex and edge sets of $Q_k[\pi]$ be as follows

$$\begin{aligned} V(Q_k[\pi]) &= \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and} \\ E(Q_k[\pi]) &= \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \leq i \leq k-1\} \end{aligned}$$

[$\Rightarrow$] Assume that $Q_k[\pi]$ has a 2-edge cutset $(X,\overline{X})$ containing only outer cycle edges, say (s_i, s_{i+1}) and (s_j, s_{j+1}) Without loss of generality, we can assume that $\{s_i, s_{j+1}\} \subseteq \overline{S}$ and $\{s_{i+1}, s_j\} \subseteq S$ Therefore $S = \{s_{i+1}, s_{i+2}, \ldots, s_j\}$ and $\overline{S} = \{s_{j+1}, s_{j+2}, \ldots, s_i\}$, and thus $T = \{t_{i+1}, t_{i+2}, \ldots, t_j\}$ and $\overline{T} = \{t_{j+1}, t_{j+2}, \ldots, t_i\}$ Since $(X,\overline{X})$ does not contain any inner cycle edges, and $(T,\overline{T})$ is a nontrivial bipartition, $(T,\overline{T})$ induces a nontrivial bipartition of the cycles of π Thus $(T,\overline{T})$ induces the nontrivial bipartition $(I,\overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$, where $I = \{i : t_i \in T\}$ and $\overline{I} = \{i : t_i \in \overline{T}\}$ But then the values of $I = \{i+1, i+2, \ldots, j\}$ can be arranged in a sequence which is contiguous (modulo k), i e $i+1 \; i+2 \; \ldots \; j$

[$\Leftarrow$] Assume that there exists a nontrivial bipartition of the cycles of π inducing a bipartition $(I,\overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is contiguous Then the values of I can be arranged in a sequence which is contiguous (modulo k), i e $q \, q{+}1 \; \ldots \; q{+}|I|{-}1$ Therefore without loss of generality, let $I = \{q, q{+}1, q{+}2, \ldots, q{+}|I|-1\}$ Letting all subscripts be taken modulo k, $F = \{(s_{q-1}, s_q), (s_{q+|I|-1}, s_{q+|I|})\}$ is a 2-edge cutset of $Q_k[\pi]$ containing only outer cycle edges (See Figure 3 2) □

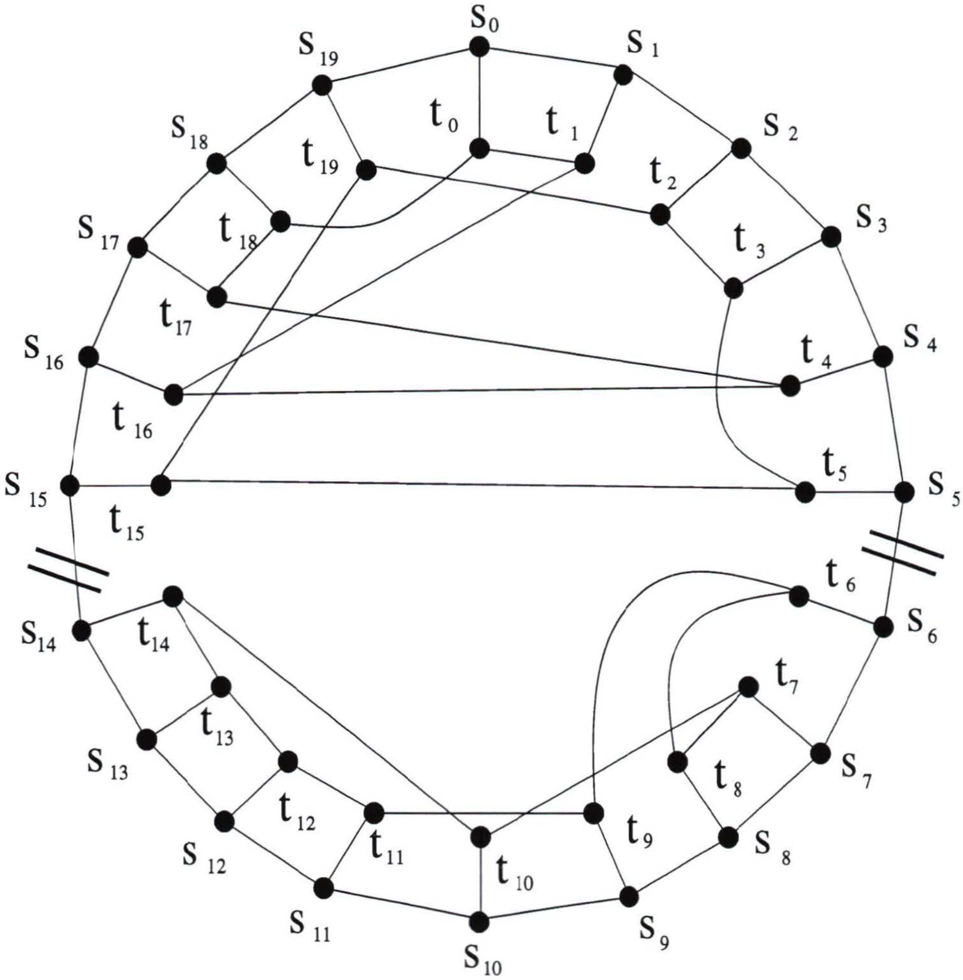

Figure 3.2: $Q_{20}[(0\ 1\ 16\ 4\ 17\ 18)(2\ 3\ 5\ 15\ 19)(6\ 8\ 7\ 10\ 14\ 13\ 12\ 11\ 9)]$.
Example of 2-edge cutset for this graph is marked.

Lemma 3 2 3 *Let $k \geq 4$ and $\pi \in Sym(k)$, where π is expressed in cycle notation The quasi-prism $Q_k[\pi]$ has a 3-edge minimal cutset $(X, \overline{X})$ containing two outer cycle edges and one spoke edge, with both $|X|$ and $|\overline{X}|$ strictly greater than one, if and only if there exists a nontrivial bipartition of the cycles of π inducing a bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is almost contiguous*

Proof Let the vertex and edge sets of $Q_k[\pi]$ be as follows

$$V(Q_k[\pi]) = \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and}$$
$$E(Q_k[\pi]) = \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \leq i \leq k-1\}$$

[⇒] Assume that $Q_k[\pi]$ has a 3-edge cutset $(X, \overline{X})$ containing two outer cycle edges and one spoke edge, with both $|X|$ and $|\overline{X}|$ strictly greater than one Let $(X, \overline{X}) = \{(s_i, s_{i+1}), (s_j, s_{j+1}), (s_b, t_b)\}$ Without loss of generality, we can assume that $S = \{s_{i+1}, s_{i+2}, \ldots, s_j\}$ and $\overline{S} = \{s_{j+1}, s_{j+2}, \ldots, s_b, \ldots, s_i\}$ Then since $(s_b, t_b) \in (X, \overline{X})$, we have $T = \{t_{i+1}, t_{i+2}, \ldots, t_j, t_b\}$ and $\overline{T} = \{t_{j+1}, t_{j+2}, \ldots, t_{b-1}, t_{b+1}, \ldots, t_i\}$ Since $(X, \overline{X})$ does not contain any inner cycle edges, and $(T, \overline{T})$ is a nontrivial bipartition of the inner cycle vertices, $(T, \overline{T})$ induces a nontrivial bipartition of the cycles of π Thus $(T, \overline{T})$ induces the nontrivial bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$, where $I = \{i : t_i \in T\}$ and $\overline{I} = \{i : t_i \in \overline{T}\}$ But then $I = \{i+1, i+2, \ldots, j, b\}$ is almost contiguous, since the values of I are such that all but one value, b, can be arranged in a contiguous sequence (modulo k), i e $i+1\ i+2 \ldots j$

[⇐] Assume that there exists a nontrivial bipartition of the cycles of π inducing a bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is almost contiguous Then all but one of the values of I can be arranged in a sequence which is contiguous (modulo k), i e $q\ q+1 \ldots q+|I|-2$ Then let $I = \{q, q+1, \ldots, q+|I|-2\} \cup \{b\}$, where b is the one value which does not belong to the contiguous sequence Then letting all subscripts be taken modulo k, $F = \{(s_{q-1}, s_q), (s_{q+|I|-2}, s_{q+|I|-1}), (s_b, t_b)\}$ is a 3-edge cutset of $Q_k[\pi]$ containing two outer cycle edges and one spoke edge, with both $|X|$ and $|\overline{X}|$ strictly greater than one (See Figure 3 3) □

Theorem 3.2.4 *Let $k \geq 4$ and $\pi \in Sym(k)$, where π is expressed in cycle notation. Then the quasi-prism $Q_k[\pi]$ is super-λ if and only if*

(i) π does not contain any cycles of order less than or equal to three, and

(ii) there is no nontrivial bipartition of the cycles of π inducing a bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is contiguous or almost contiguous.

Proof: Let the vertex and edge sets of $Q_k[\pi]$ be as follows.

$$V(Q_k[\pi]) = \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and}$$
$$E(Q_k[\pi]) = \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \leq i \leq k-1\}.$$

Since λ must be equal to three for 3-regular super-λ graphs, we need to show that $\lambda(Q_k[\pi])$ is equal to three and $N_3(Q_k[\pi])$ is equal to n. Therefore in this proof, we enumerate the number of i-edge cutsets for $1 \leq i \leq 3$.

As stated earlier, any minimal cutset $(X, \overline{X})$ of $Q_k[\pi]$ having three or fewer edges can take one of the following three forms:

(1) $(X, \overline{X})$ contains only spoke edges (one, two, or three),

(2) $(X, \overline{X})$ contains two outer cycle edges and zero or one spoke edge, or

(3) $(X, \overline{X})$ contains two inner cycle edges and zero or one spoke edge.

We analyze each of these three cases separately.

Case 1: $(X, \overline{X})$ contains only spoke edges (one, two, or three).

By Lemma 3.2.1, $Q_k[\pi]$ contains a cut $(X, \overline{X})$ of order j, containing only spoke edges, if and only if π has a cycle of order j.

Case 2: $(X, \overline{X})$ contains two outer cycle edges and zero or one spoke edge.

Subcase (i): $(X, \overline{X})$ contains two outer cycle edges and zero spoke edges.

By Lemma 3.2.2, $Q_k[\pi]$ has a 2-edge cut $(X, \overline{X})$ containing two outer cycle edges if and only if there exists a nontrivial bipartition of the

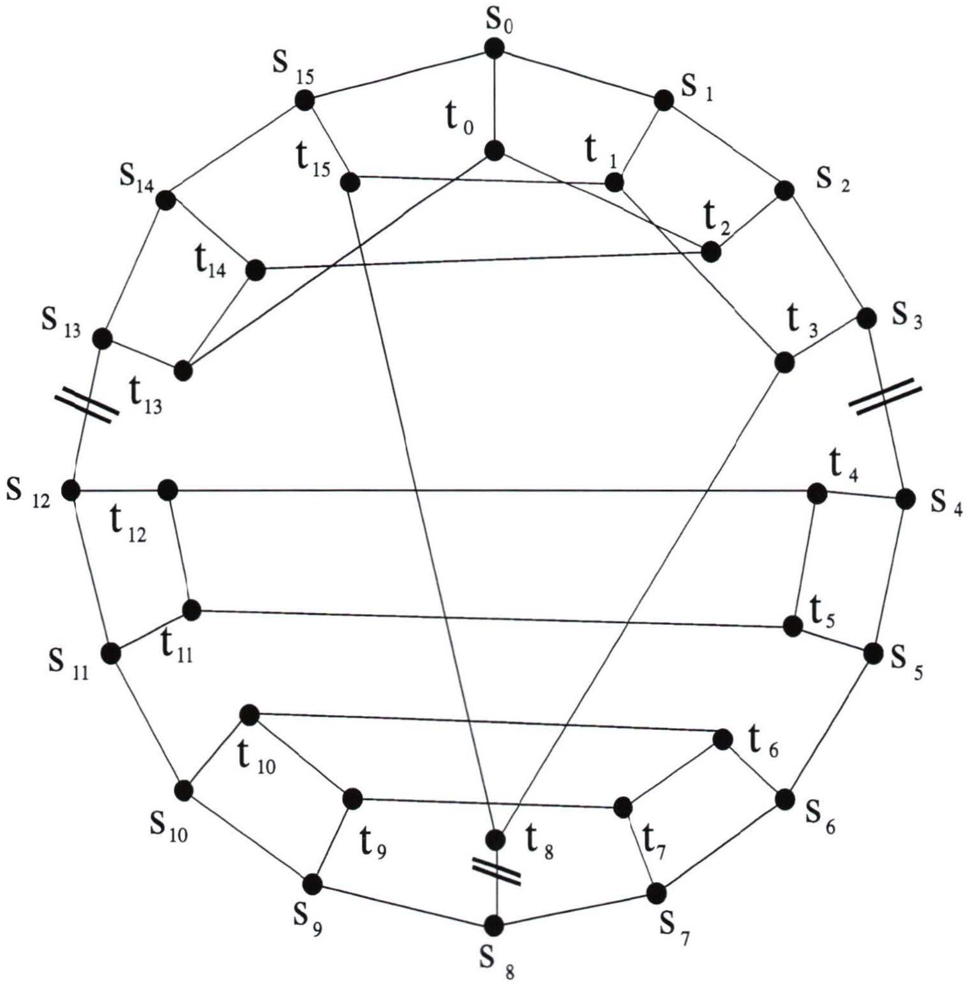

Figure 3.3: $Q_{16}[(0\ 2\ 14\ 13)(1\ 3\ 8\ 15)(4\ 5\ 11\ 12)(6\ 7\ 9\ 10)]$.
Example of 3-edge cutset for this graph is marked.

cycles of π inducing a bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is contiguous.

Subcase (ii): $(X, \overline{X})$ contains two outer cycle edges and one spoke edge.

Assume that both $|X|$ and $|\overline{X}|$ are strictly greater than one. By Lemma 3.2.3, $Q_k[\pi]$ has a 3-edge cut $(X, \overline{X})$ containing two outer cycle edges and one spoke edge, with both $|X|$ and $|\overline{X}|$ strictly greater than one, if and only if there exists a nontrivial bipartition of the cycles of π inducing a bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is almost contiguous. Now assume that, without loss of generality, $|X|$ is equal to one, and thus $|S|$ is equal to one. Then for each outer cycle vertex $s_i \in \mathcal{S}$, $F = \{(s_{i-1}, s_i), (s_i, s_{i+1}), (s_i, t_i)\}$ is a 3-edge cutset of $Q_k[\pi]$ containing two outer cycle edges and one spoke edge; there are k such 3-edge cutsets.

Case 3: $(X, \overline{X})$ contains two inner cycle edges and zero or one spoke edge.

Subcase (i): $(X, \overline{X})$ contains two inner cycle edges and zero spoke edges.

Since the outer cycle is not disconnected and no spoke edges belong to $(X, \overline{X})$, there can be no cutset of this type.

Subcase (ii): $(X, \overline{X})$ contains two inner cycle edges and one spoke edge.

Since $(X, \overline{X})$ does not contain any outer cycle edges, without loss of generality, $(S, \overline{S})$ consists of $S = \{s_0, s_1, \ldots, s_{k-1}\}$ and $\overline{S} = \emptyset$, and thus is a trivial bipartition. Then for any edge $(s_i, t_i) \in (X, \overline{X})$, $t_i \in \overline{T}$. Since only one spoke edge belongs to $(X, \overline{X})$, only one inner cycle vertex can belong to $\overline{T}$, therefore $|\overline{T}|$ equals one. For each inner cycle vertex $t_i \in \mathcal{T}$, letting $i = \pi(j)$, $F = \{(t_j, t_i), (t_i, t_{\pi(i)}), (s_i, t_i)\}$ is a 3-edge cutset of $Q_k[\pi]$ containing two inner cycle edges and one spoke edge; there are k such 3-edge cutsets.

Therefore since $\lambda(Q_k[\pi])$ is equal to three and $N_3(Q_k[\pi])$ is equal to n, $Q_k[\pi]$ is super-λ (with π satisfying the conditions as stated in the theorem). $\square$

Chapter 4

Isomorphisms

In this chapter, we define cycle permutation graphs, circulants, and generalized Petersen graphs. We also establish isomorphisms between these classes of graphs. These are not the major results of this thesis, but are included to clarify the interrelationships between the classes and simplify certain proofs. Since the isomorphism results in this chapter are easily established, their proofs are omitted from the body of this chapter. They can be found in Appendix F.

We first define what it means for two graphs to be isomorphic. Two graphs G_1 and G_2 are *isomorphic*, denoted by $G_1 \cong G_2$, if there exist bijections $f : V(G_1) \rightarrow V(G_2)$ and $g : E(G_1) \rightarrow E(G_2)$, such that (u, v) is an edge of G_1 if and only if $g(u, v) = (f(u), f(v))$ is an edge of G_2.

4.1 Cycle Permutation Graphs

This section includes a definition of cycle permutation graphs and a result which establishes that any cycle permutation graph is also a quasi-prism. Let G_1 be some graph on k vertices with vertex set $V(G_1) = \{x_0, x_1, \ldots, x_{k-1}\}$. Let G_2 be a copy of G_1, with the vertices relabelled from x_i to y_i, $0 \leq i \leq k-1$, resulting in the vertex set $V(G_2) = \{y_0, y_1, \ldots, y_{k-1}\}$. Let $\sigma \in Sym(k)$. Then $\sigma(G)$ is a *permutation graph*, or *σ-permutation graph*, defined by $\sigma(G) = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup \{(x_i, y_{\sigma(i)}) : 0 \leq i \leq k-1\})$. We use the matrix notation to represent the permutation associated with the permutation graphs, rather than the cycle notation. Since the top row of the matrix is always equal to $[0\ 1\ \ldots\ (k-1)]$, it is omitted in the notation we use in this thesis. For example, using σ as defined below,

$$\sigma = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 2 & 0 \end{bmatrix},$$

we write

$$\sigma = [1320]$$

Permutation graphs were first considered in 1967 by Chartrand and Harary [11]. Permutation graphs have also been referred to as *generalized prisms* in the literature (see for example [22]). In [22], Piazza and Ringeisen examine the connectivity of permutation graphs using various topologies for G, and provide upper and lower bounds for the connectivity of G. When G is a k-cycle, the graph is called a *cycle permutation graph*, redefined below.

Definition Cycle Permutation Graph (C_k, σ)
The *cycle permutation graph* of order $n = 2k$, associated with a permutation $\sigma \in Sym(k)$ expressed in matrix notation, denoted by (C_k, σ), has vertex set partitioned into the two parts $\mathcal{A} = \{a_0, a_1, \ldots, a_{k-1}\}$ (the *outer cycle vertices*), and $\mathcal{B} = \{b_0, b_1, \ldots, b_{k-1}\}$ (the *inner cycle vertices*), and edges as follows, for $i = 0, 1, \ldots, k-1$ (all arithmetic is modulo k):

1. (a_i, a_{i+1}), the *outer cycle edges*,
2. (b_i, b_{i+1}), the *inner cycle edges*, and
3. $(a_i, b_{\sigma(i)})$, the *permutation edges*.

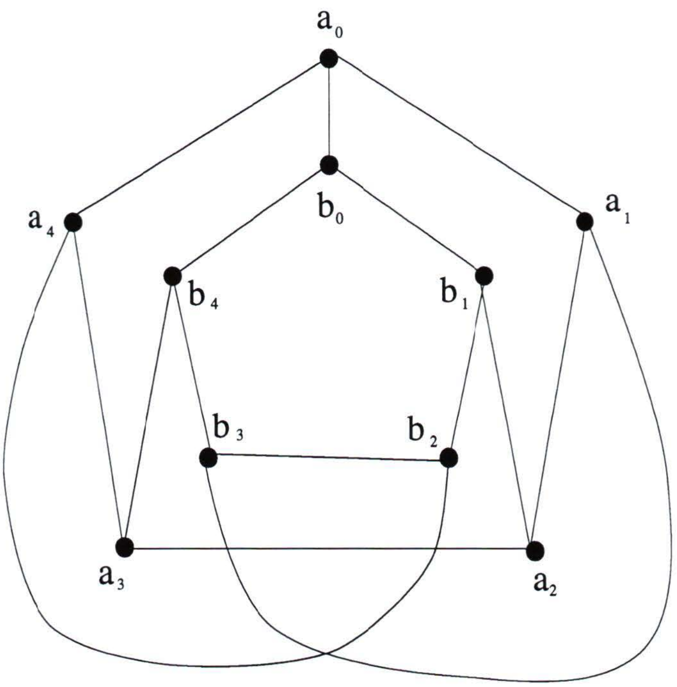

Figure 4 1 Petersen graph ($(C_5, [03142])$)

The famous Petersen graph is an example of a cycle permutation graph In this notation, it can be represented by $(C_5, [03142])$, as shown in Figure 4 1

For $n = 2, 4, 6$, and 8, the cycle permutation graphs are easily enumerated They are included with their cut frequency vectors in Tables B 1 through B 4 in Appendix B It is easy to show that cycle permutation graphs are not super-λ for $n < 8$, so these cases are not considered in detail here Since we are only interested in super-λ graphs in this thesis, cycle permutation graphs are only considered for $n \geq 8$ A proof that all cycle permutation graphs are super-λ for $n \geq 8$ can be found in Chapter 5, Lemma 5 1 2

Note that (C_k, σ_1) can be isomorphic to (C_k, σ_2) when σ_1 is not equal to σ_2 For example, $(C_5, [04132])$ and $(C_5, [02431])$ are isomorphic These two graphs are shown in Figure 4 2, and the isomorphism is as indicated

A *natural isomorphism* between two cycle permutation graphs is an isomorphism

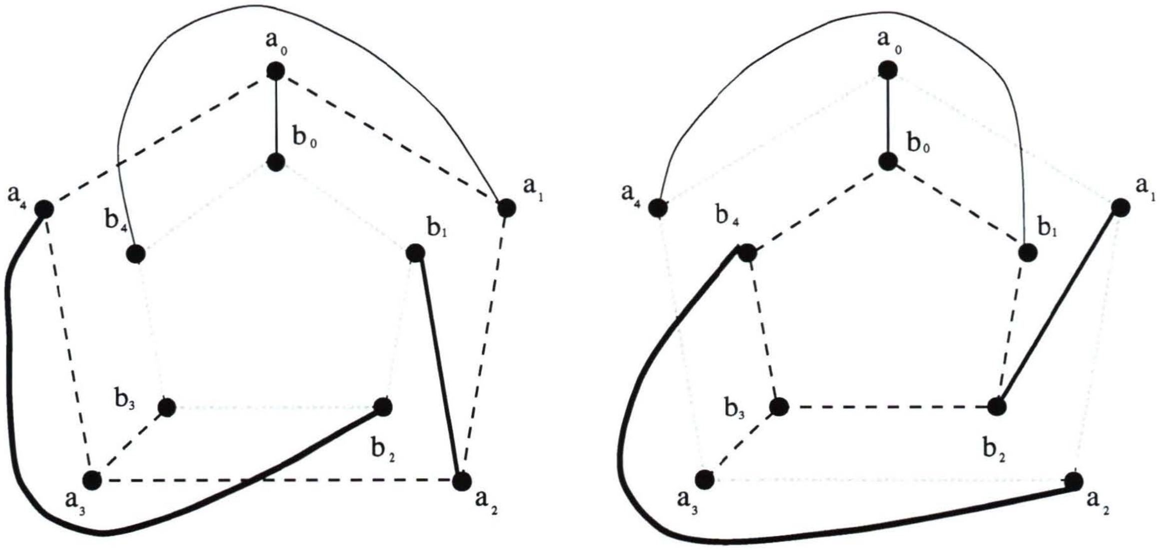

Figure 4 2 Isomorphic graphs $(C_5, [04132])$ and $(C_5, [02431])$

in which the outer and inner cycles of one cycle permutation graph are mapped to the outer and inner cycles of another cycle permutation graph Stueckle, in [24], describes when two cycle permutation graphs are naturally isomorphic It is possible for two cycle permutation graphs to be isomorphic when the isomorphism is not natural An example of this is the isomorphism between the two cycle permutation graphs $(C_{13}, [0\ 11\ 2\ 1\ 4\ 3\ 9\ 7\ 5\ 8\ 6\ 12\ 10])$ and $(C_{13}, [0\ 11\ 2\ 1\ 4\ 3\ 8\ 6\ 9\ 7\ 5\ 12\ 10])$, described in Appendix E The next result establishes that any cycle permutation graph is isomorphic to some quasi-prism, and therefore the class of quasi-prisms contains the class of cycle permutation graphs

Lemma 4 1 1 *For $\sigma \in Sym(k)$, the cycle permutation graph (C_k, σ) is isomorphic to the quasi-prism $Q_k[\pi]$ if $\pi = (\sigma(0)\ \sigma(1)\ \cdots\ \sigma(k-1))$*

Proof. We map the outer cycle of (C_k, σ) to the inner cycle of $Q_k[\pi]$, the inner cycle of (C_k, σ) to the outer cycle of $Q_k[\pi]$, and the permutation edges of (C_k, σ) to the spoke edges of $Q_k[\pi]$ A formal proof of this isomorphism can be found in Appendix F □

4.2 Circulant Graphs

This section contains a definition of circulant graphs, and results establishing isomorphisms between circulants and other classes of graphs Since we are examining the cut frequency vectors of circulants, and comparing the cut frequency vectors of circulants and cycle permutation graphs, we want to establish when circulants and cycle permutation graphs are isomorphic This isomorphism is established by first showing an isomorphism between quasi-prisms and circulant graphs

The *circulant graph* $C_n\langle a_1, a_2, \ldots, a_j\rangle$, where $0 < a_1 < a_2 < \ldots < a_j < \frac{n+1}{2}$, has vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and vertices $v_{i \pm a_h}$ adjacent to each vertex v_i, for $1 \leq h \leq j$, with all values taken modulo n The sequence $\langle a_1, a_2, \ldots, a_j\rangle$ is called the *jump sequence* and the a_i are called the *jumps* A *diagonal jump* is a jump of size $\frac{n}{2}$, such a jump is only possible if n is even If a circulant has a diagonal jump, it is regular of degree $2j - 1$, otherwise it is regular of degree $2j$ Thus all 3-regular circulants are of the form $C_n\langle a, \frac{n}{2}\rangle$ where $a < \frac{n}{2}$, they are redefined below

Definition Circulant $C_n\langle a, \frac{n}{2}\rangle$
The 3-regular *circulant* of order n, with $0 < a < \frac{n}{2}$, denoted $C_n\langle a, \frac{n}{2}\rangle$, has vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$, and edges as follows (all arithmetic is modulo n)

1 (v_i, v_{i+a}), for $0 \leq i \leq n-1$ (the *cycle edges*), and
2 $(v_j, v_{j+\frac{n}{2}})$, for $0 \leq j \leq \frac{n}{2} - 1$ (the *chord edges*)

Two examples of circulant graphs are shown in Figure 4 3 Conditions for a circulant to be connected are known and are summarized in Chapter 6, Lemma 6 1 1

The *reflexive modular reduction* of a sequence $\langle a_1, a_2, \ldots, a_j\rangle$, which is denoted by $\langle a_1, a_2, \ldots, a_j\rangle_x^*$, is the sequence obtained by reducing each a_i modulo x to yield a_i', and then reducing all resulting terms a_i' which are larger than $\frac{x}{2}$ by $x - a$ Thus $\langle 3, 5, 6\rangle_7^*$ is $\langle 3, 2, 1\rangle$ It was first noted by Ádám in [1] that if r and x are relatively

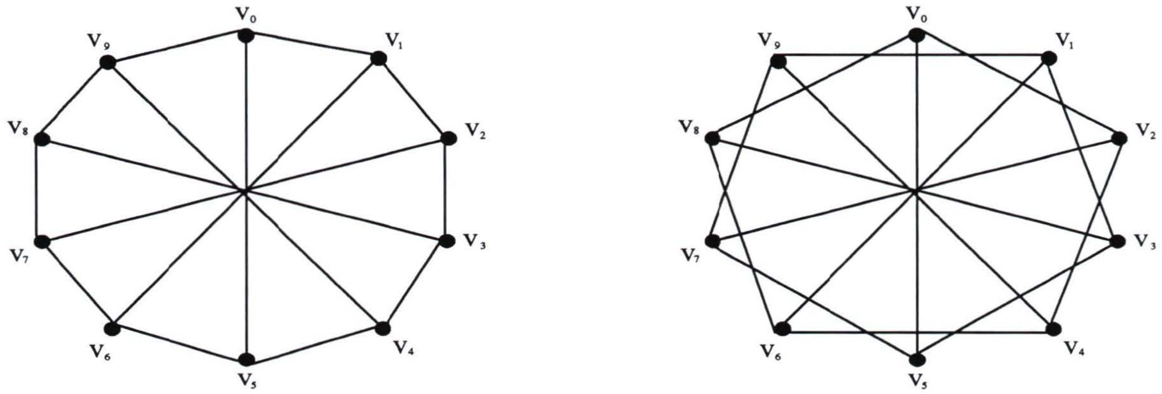

Figure 4.3: Circulants $C_{10}\langle 1, 5\rangle$ and $C_{10}\langle 2, 5\rangle$

prime, then $C_x\langle a_1, a_2, \ldots, a_j\rangle$ and $C_x\langle ra_1, ra_2, \ldots, ra_j\rangle_x^*$ are isomorphic circulants. We say two circulants $C_x\langle a_1, a_2, \ldots, a_j\rangle$ and $C_x\langle b_1, b_2, \ldots, b_j\rangle$ are *Ádám isomorphic* if there exists an r relatively prime to x with $\langle b_1, b_2, \ldots, b_j\rangle = \langle ra_1, ra_2, \ldots, ra_j\rangle_x^*$. Boesch and Tindell made the following conjecture in [5], it was proven by Zhou in [26]. It should be noted that we only read an English review of [26], and that the original paper is in Chinese.

Lemma 4.2.1 [[5], p. 492 and [26]] *All circulants $C_n\langle a_1, a_2\rangle$ which are isomorphic are Ádám isomorphic.*

The following consequence of Lemma 4.2.1, noted by Boesch and Tindell in [5], concerns those two jump circulants which are 3-regular.

Lemma 4.2.2 [[5], p. 492] *$C_n\langle a, \frac{n}{2}\rangle$ and $C_n\langle b, \frac{n}{2}\rangle$ are isomorphic if and only if $gcd(a, n) = gcd(b, n)$.*

Lemma 4.2.3 *For $n \geq 4$, any 3-regular connected circulant $C_n\langle a, \frac{n}{2}\rangle$ is isomorphic to either $C_n\langle 1, \frac{n}{2}\rangle$ or $C_n\langle 2, \frac{n}{2}\rangle$.*

Proof: See Appendix F. □

The above lemma establishes that there are at most two connected 3-regular circulants up to isomorphism. For this reason, it is sufficient to show that $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to some quasi-prism, and then show that $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to any quasi-prism, to completely characterize the class interrelationships between quasi-prisms and 3-regular connected circulants.

Lemma 4.2.4 *For $n \geq 6$ and $\frac{n}{2}$ odd, the circulant $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to the quasi-prism $Q_k[\pi]$ if $\pi = (0\ 1\ 2\ \ldots\ (k-1))$.*

Proof. The cycle C_1 of order $\frac{n}{2}$ in $C_n\langle 2, \frac{n}{2}\rangle$, with the vertices in $V(C_1)$ having even subscripts, is mapped to the outer cycle of $Q_k[\pi]$. The cycle C_2 of order $\frac{n}{2}$ in $C_n\langle 2, \frac{n}{2}\rangle$, with the vertices in $V(C_2)$ having odd subscripts, is mapped to the inner cycle of $Q_k[\pi]$. The chord edges of $C_n\langle 2, \frac{n}{2}\rangle$ are mapped to the spoke edges of $Q_k[\pi]$. A formal proof of this isomorphism is included in Appendix F. □

Corollary 4.2.5 *For $n \geq 6$ and $\frac{n}{2}$ odd, the circulant $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to the cycle permutation graph $(C_k, \mathcal{I})$.*

Proof. Since (C_k, σ) is isomorphic to $Q_k[\pi]$ if $\pi = (\sigma(0)\ \sigma(1)\ \ldots\ \sigma(k-1))$ [Lemma 4.1.1], and $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to $Q_k[\pi]$ for $n \geq 6$ and $\frac{n}{2}$ odd if $\pi = (0\ 1\ 2\ \ldots\ k-1)$ [Lemma 4.2.4], $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to $(C_k, \mathcal{I})$ for $n \geq 6$ and $\frac{n}{2}$ odd. □

Lemma 4.2.6 *For $n \geq 10$, $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to $Q_k[\pi]$ for any permutation π, $\pi \in Sym(k)$.*

Proof. See Appendix F. □

Corollary 4.2.7 *For $n \geq 10$, $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to (C_k, σ) for any permutation $\sigma \in Sym(k)$.*

Proof Since (C_k, σ) is isomorphic to $Q_k[\pi]$ if $\pi = (\sigma(0)\ \sigma(1)\ \cdots\ \sigma(k-1))$ [Lemma 4 1 1], and $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to $Q_k[\pi]$ for any $\pi \in Sym(k)$ for $n \geq 10$ [Lemma 4 2 6], $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to (C_k, σ) for any $\sigma \in Sym(k)$ for $n \geq 10$ □

Corollary 4.2 8 *For $n \geq 10$, $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to $C_n\langle 2, \frac{n}{2}\rangle$*

Proof Since $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to $(C_k, \mathcal{I})$ for $n \geq 6$ and $\frac{n}{2}$ odd [Corollary 4 2 5], and $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to (C_k, σ) for any $\sigma \in Sym(k)$ for $n \geq 10$ [Corollary 4 2 7], $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to $C_n\langle 2, \frac{n}{2}\rangle$ for $n \geq 10$ □

Corollary 4.2.9 *For $n \geq 10$, there are at most two 3-regular connected circulants up to isomorphism $C_n\langle 1, \frac{n}{2}\rangle$ for $\frac{n}{2}$ even, and $C_n\langle 1, \frac{n}{2}\rangle$ and $C_n\langle 2, \frac{n}{2}\rangle$ for $\frac{n}{2}$ odd*

Proof Since $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to $C_n\langle 2, \frac{n}{2}\rangle$ for $n \geq 10$ [Corollary 4 2 8], and any 3-regular connected circulant $C_n\langle a, \frac{n}{2}\rangle$ is isomorphic to either $C_n\langle 1, \frac{n}{2}\rangle$ or $C_n\langle 2, \frac{n}{2}\rangle$ for $n \geq 4$ [Lemma 4 2 3], there are at most two 3-regular connected circulants up to isomorphism for $n \geq 10$ If $\frac{n}{2}$ is even, $C_n\langle 2, \frac{n}{2}\rangle$ is disconnected, this result is a consequence of Lemma 6 1 1, from Chapter 6 □

4.3 Generalized Petersen Graphs

We begin this section with a definition of generalized Petersen graphs We then establish the intersection between generalized Petersen graphs and each of the following quasi-prisms, cycle permutation graphs, and circulants

Definition Generalized Petersen Graph $GP(k, j)$
The *generalized Petersen graph* of order $n = 2k$ associated with a *jump* of size j, $1 \leq j \leq k-1$ and j not equal to $\frac{k}{2}$, denoted $GP(k, j)$, has vertex set partitioned into the

Proof Since (C_k, σ) is isomorphic to $Q_k[\pi]$ if $\pi = (\sigma(0)\ \sigma(1) \ldots \sigma(k-1))$ [Lemma 4 1 1], and $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to $Q_k[\pi]$ for any $\pi \in Sym(k)$ for $n \geq 10$ [Lemma 4 2 6], $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to (C_k, σ) for any $\sigma \in Sym(k)$ for $n \geq 10$ □

Corollary 4 2 8 *For $n \geq 10$, $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to $C_n\langle 2, \frac{n}{2}\rangle$*

Proof Since $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to $(C_k, \mathcal{I})$ for $n \geq 6$ and $\frac{n}{2}$ odd [Corollary 4 2 5], and $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to (C_k, σ) for any $\sigma \in Sym(k)$ for $n \geq 10$ [Corollary 4 2 7], $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to $C_n\langle 2, \frac{n}{2}\rangle$ for $n \geq 10$ □

Corollary 4 2.9 *For $n \geq 10$, there are at most two 3-regular connected circulants up to isomorphism $C_n\langle 1, \frac{n}{2}\rangle$ for $\frac{n}{2}$ even, and $C_n\langle 1, \frac{n}{2}\rangle$ and $C_n\langle 2, \frac{n}{2}\rangle$ for $\frac{n}{2}$ odd*

Proof. Since $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to $C_n\langle 2, \frac{n}{2}\rangle$ for $n \geq 10$ [Corollary 4 2 8], and any 3-regular connected circulant $C_n\langle a, \frac{n}{2}\rangle$ is isomorphic to either $C_n\langle 1, \frac{n}{2}\rangle$ or $C_n\langle 2, \frac{n}{2}\rangle$ for $n \geq 4$ [Lemma 4 2 3], there are at most two 3-regular connected circulants up to isomorphism for $n \geq 10$ If $\frac{n}{2}$ is even, $C_n\langle 2, \frac{n}{2}\rangle$ is disconnected, this result is a consequence of Lemma 6 1 1, from Chapter 6 □

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The *generalized Petersen graph* of order $n = 2k$ associated with a *jump* of size j, $1 \leq j \leq k-1$ and j not equal to $\frac{k}{2}$, denoted $GP(k, j)$, has vertex set partitioned into the

two parts $\mathcal{U} = \{u_0, u_1, \ldots, u_{k-1}\}$ (the *outer rim vertices*), and $\mathcal{V} = \{v_0, v_1, \ldots, v_{k-1}\}$ (the *inner rim vertices*), and edges as follows, for $i = 0, 1, \ldots, k-1$ (all arithmetic is modulo k):

1. (u_i, u_{i+1}), the *outer rim edges*,
2. (v_i, v_{i+j}), the *inner rim edges*, and
3. (u_i, v_i), the *spoke edges*.

Note that $GP(k, j)$ is isomorphic to the graph $GP(k, k-j)$.

Lemma 4.3.1 *The quasi-prism $Q_k[\pi]$ is isomorphic to the generalized Petersen graph $GP(k, j)$ if the permutation π has $\pi(i) = i + j$.*

Proof. We map the outer cycle of $Q_k[\pi]$ to the outer rim of $GP(k, j)$, the inner cycles of $Q_k[\pi]$ to the inner rim cycles of $GP(k, j)$, and the spoke edges of $Q_k[\pi]$ to the spoke edges of $GP(k, j)$. A formal proof of this isomorphism appears in Appendix F. □

Corollary 4.3.2 *For $n \geq 6$ and $\frac{n}{2}$ odd, the circulant $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to the generalized Petersen graph $GP(k, 1)$.*

Proof. Since $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to $Q_k[\pi]$ if $\pi = (0\ 1\ 2\ \ldots\ (k-1))$ for $n \geq 6$ and $\frac{n}{2}$ odd [Lemma 4.2.4], and $Q_k[\pi]$ is isomorphic to $GP(k, j)$ if $\pi(i) = i + j$ [Lemma 4.3.1], $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to $GP(k, 1)$. □

By Lemma 4.3.1, any generalized Petersen graph is isomorphic to some quasi-prism, and by Lemma 4.1.1, any cycle permutation graph is isomorphic to some quasi-prism. However, these lemmas do not fully characterize the intersection between generalized Petersen graphs and cycle permutation graphs. In [25], Stueckle and Ringeisen provide a characterization of the intersection between cycle permutation graphs and generalized Petersen graphs. The following lemma, proven by Stueckle and Ringeisen in [25], provides necessary and sufficient conditions for such an isomorphism.

Lemma 4.3.3 [[25], pp. 148-149] *A generalized Petersen graph $GP(k, j)$, is a cycle permutation graph if and only if one of the following is true.*

1. $gcd(k, j) = 1$
2. $k \equiv 0 \pmod 4$, $j \equiv \pm 1 \pmod 4$, *and* $gcd(\frac{k}{4}, \frac{j \pm 1}{4}) = 1$
3. $k \equiv 0 \pmod 5$, $j \equiv \pm 2 \pmod 5$, $gcd(\frac{k}{5}, \frac{j \pm 2}{5}) = 1$, *and* $gcd(\frac{k}{5}, \frac{2j \pm 1}{5}) = 1$

The next three figures illustrate each of the cases in Lemma 4.3.3 as follows. In Figure 4.4, the first condition is satisfied by $k = 8$ and $j = 3$, and $GP(8, 3)$ is isomorphic to $(C_8, [0\ 3\ 6\ 1\ 4\ 7\ 2\ 5])$. The second condition is satisfied by $k = 12$ and $j = 3$, and Figure 4.5 shows that $GP(12, 3)$ is isomorphic to $(C_{12}, [0\ 3\ 10\ 9\ 4\ 7\ 2\ 1\ 8\ 11\ 6\ 5])$. Finally, we show an isomorphism between $(C_{15}, [0\ 12\ 9\ 1\ 3\ 10\ 7\ 4\ 11\ 13\ 5\ 2\ 14\ 6\ 8])$ and $GP(15, 3)$ in Figure 4.6, where $k = 15$ and $j = 3$ satisfy the third condition.

Corollary 4.3.4 *If $gcd(k, j)$ is equal to one, then $GP(k, j)$ is isomorphic to (C_k, σ) for $\sigma = [0\ j\ 2j\ \ldots\ (k-1)j]$.*

Proof. Since (C_k, σ) is isomorphic to $Q_k[\pi]$ if $\pi = (\sigma(0)\ \sigma(1)\ \ldots\ \sigma(k-1))$ [Lemma 4.1.1], $Q_k[\pi]$ is isomorphic to $GP(k, j)$ for $\pi(i) = i + j$ [Lemma 4.3.1], and $GP(k, j)$ is isomorphic to (C_k, σ) if $gcd(k, j)$ equals one [Lemma 4.3.3], $GP(k, j)$ is isomorphic to (C_k, σ) if $gcd(k, j)$ equals one, for $\sigma = [0\ j\ 2j\ \ldots\ (k-1)j]$. □

We have shown that the intersection of the class of cycle permutation graphs and the class of generalized Petersen graphs is nonempty, but we have yet to show that neither class is a subset of the other. To accomplish this, we first give an example of a cycle permutation graph that is not isomorphic to any generalized Petersen graph. We then provide an example of a generalized Petersen graph that is not isomorphic to any cycle permutation graph.

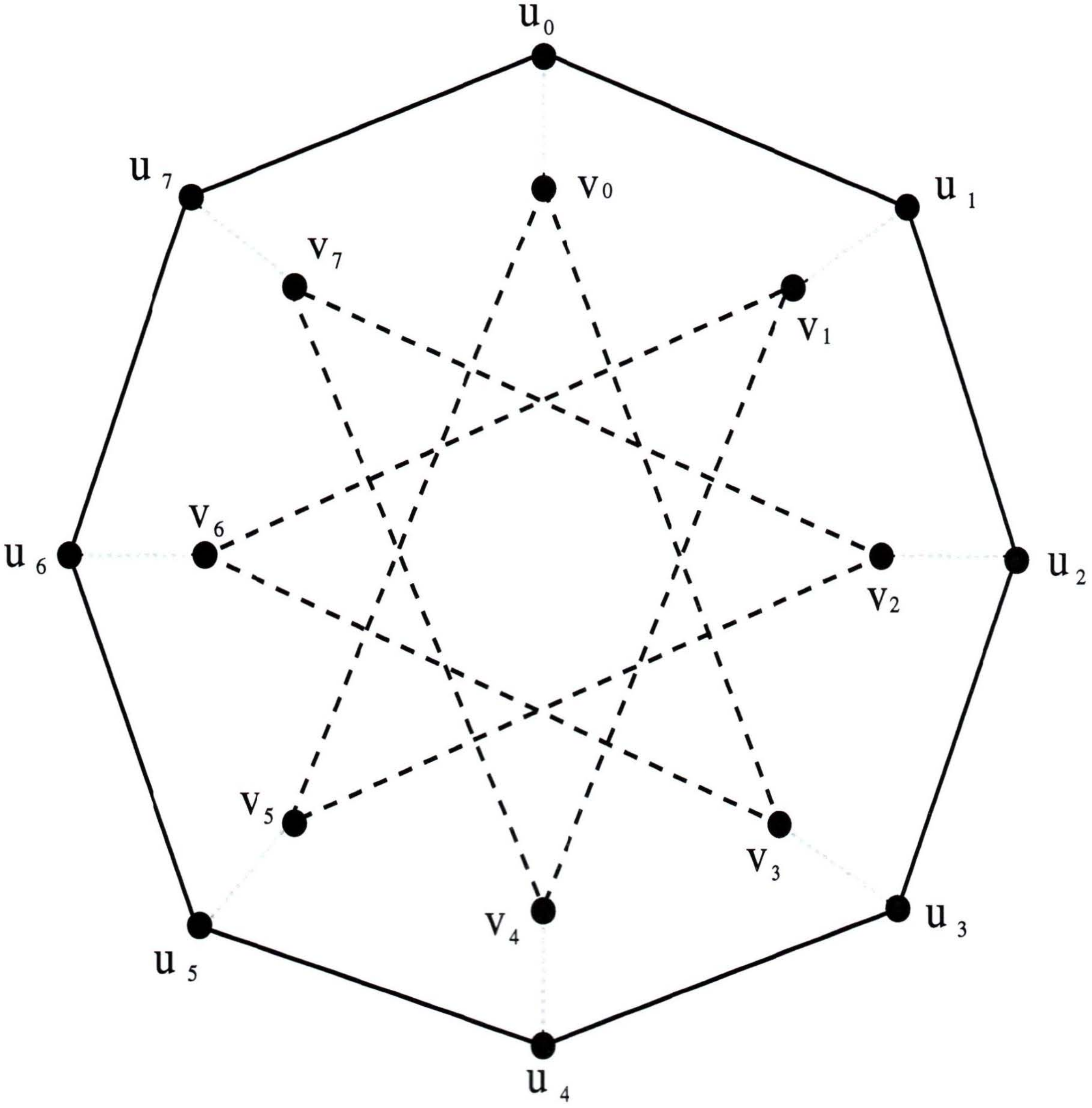

Figure 4 4 Isomorphic graphs $GP(8,3)$ and $(C_8, [0\ 3\ 6\ 1\ 4\ 7\ 2\ 5])$

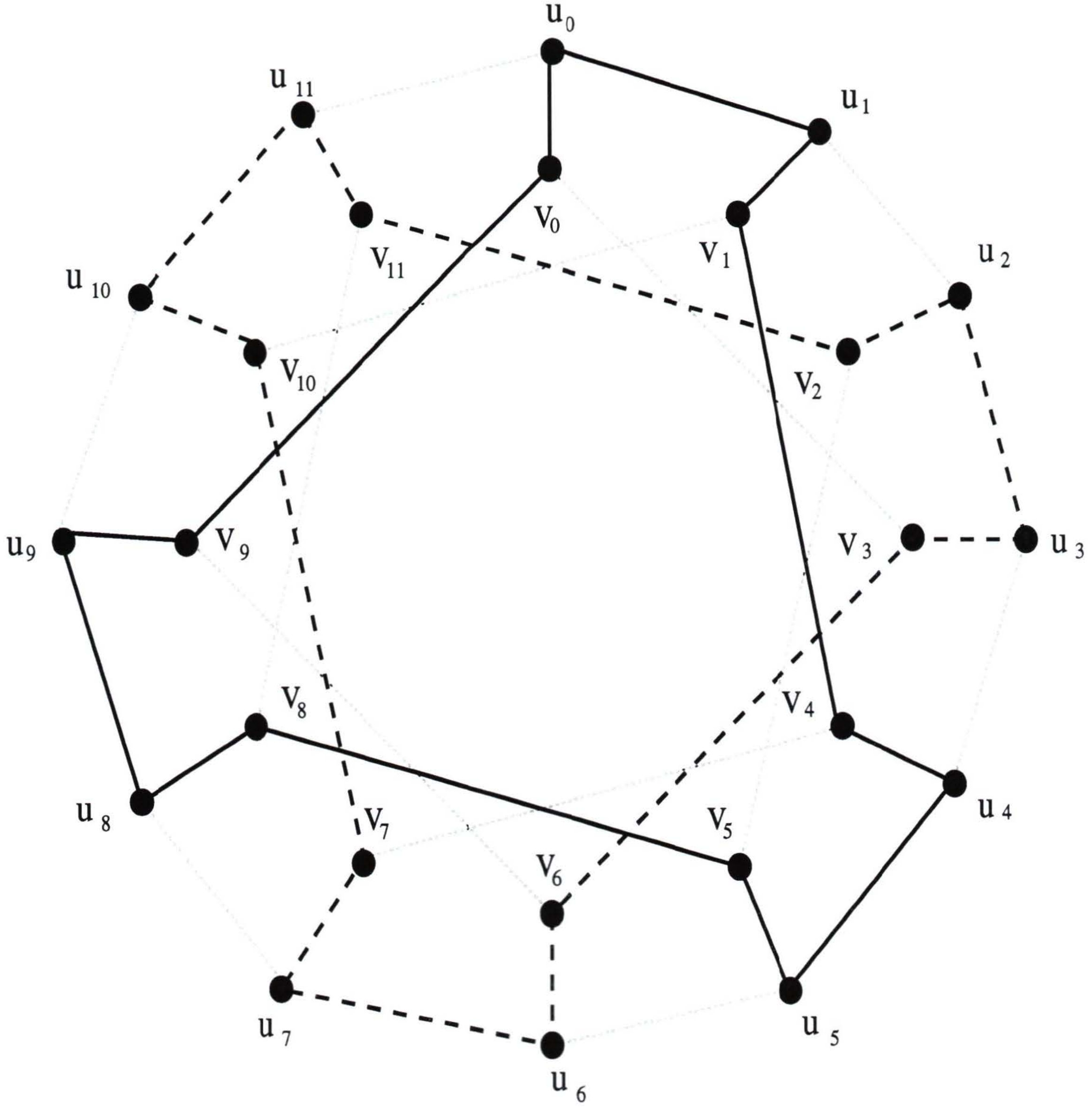

Figure 4.5 Isomorphic graphs $GP(12,3)$ and $(C_{12}, [0\ 3\ 10\ 9\ 4\ 7\ 2\ 1\ 8\ 11\ 6\ 5])$

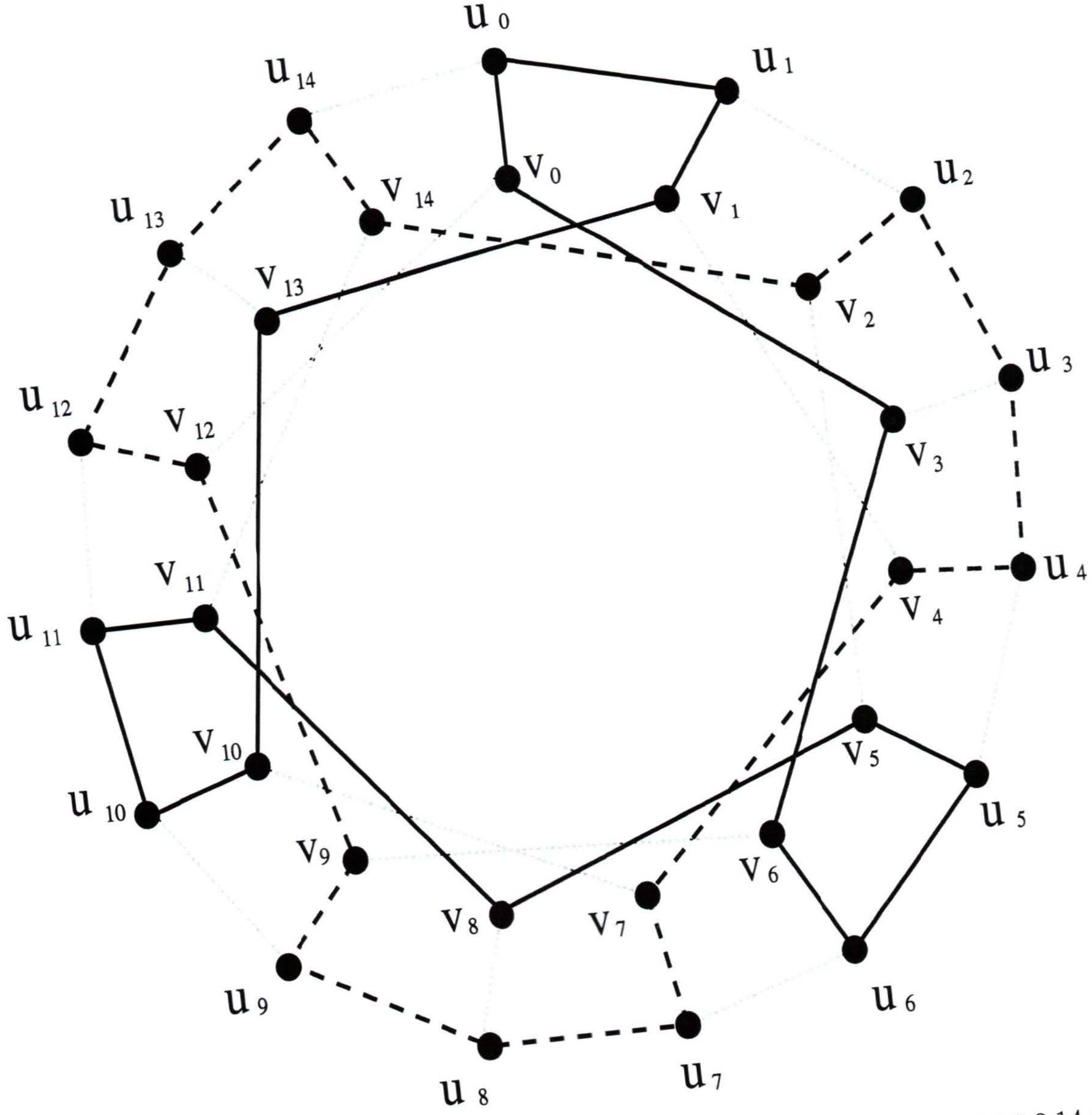

Figure 4 6 Isomorphic graphs $GP(15,3)$ and $(C_{15}, [0\ 12\ 9\ 1\ 3\ 10\ 7\ 4\ 11\ 13\ 5\ 2\ 14\ 6\ 8])$

Example Since $(C_5, [01342])$ has 137 4-edge cutsets, and all of the generalized Petersen graphs on ten vertices have either 135 or 140 4-edge cutsets, $(C_5, [01342])$ is not isomorphic to $GP(5, j)$ for any value of j, $1 \leq j \leq 4$ [See Tables B 5 and C 1]

Example Since $gcd(7, 2) = 2 \neq 1$, $7 \equiv 3 \pmod 4$, and $7 \equiv 2 \pmod 5$, none of the conditions of Lemma 4 3 3 are satisfied Therefore $GP(7, 2)$ is not isomorphic to $(C_7, [\sigma])$ for any $\sigma \in Sym(7)$

4.4 The Big Picture

This chapter contains many results pertaining to the class intersections between quasi-prisms, cycle permutation graphs, circulants, and generalized Petersen graphs The class of quasi-prisms contains all cycle permutation graphs and generalized Petersen graphs However, although the intersection between the class of cycle permutation graphs and the class of generalized Petersen graphs is nonempty, neither of these classes is a subset of the other The class of circulants has a nonempty intersection with each of the aforementioned classes of graphs, but it is not a subset of the quasi-prisms These class interrelationships are summarized in Table 4 1 and, for $n \geq 8$, illustrated in the Venn diagram of Figure 4 7

Table 4.1: Class interrelationships between 3-regular graphs

Graph	*Isomorphism Properties*			
	$Q_k[\pi]$?	(C_k, σ)?	$C_n\langle a, \frac{n}{2}\rangle$?	$GP(k, j)$?
(C_k, σ)	yes	—	$(C_k, \mathcal{I}) \cong C_n\langle 2, \frac{n}{2}\rangle$, $n \geq 6, \frac{n}{2}$ odd	yes, if (1)
$C_n\langle a, \frac{n}{2}\rangle$	$C_n\langle 2, \frac{n}{2}\rangle \cong Q_k[(0\ 1\ \cdots\ k-1)]$, $n \geq 6, \frac{n}{2}$ odd	$C_n\langle 2, \frac{n}{2}\rangle \cong (C_k, \mathcal{I})$, $n \geq 6, \frac{n}{2}$ odd	—	$C_n\langle 2, \frac{n}{2}\rangle \cong GP(k, 1)$, $n \geq 6, \frac{n}{2}$ odd
$GP(k, j)$	yes	yes, if (1)	$GP(k, 1) \cong C_n\langle 2, \frac{n}{2}\rangle$, $n \geq 6, \frac{n}{2}$ odd	—

1. $GP(k, j) \cong (C_k, \sigma)$ if one of the following conditions holds:

- $gcd(k, j) = 1$,
- $k \equiv 0 \pmod 4$, $j \equiv \pm 1 \pmod 4$, and $gcd(\frac{k}{4}, \frac{j \pm 1}{4}) = 1$, or
- $k \equiv 0 \pmod 5$, $j \equiv \pm 2 \pmod 5$, $gcd(\frac{k}{5}, \frac{j \pm 2}{5}) = 1$, and $gcd(\frac{k}{5}, \frac{2j \pm 1}{5}) = 1$.

Note that the above conditions are on k and j, and that we do not state conditions on σ for (C_k, σ) to be isomorphic to some generalized Petersen graph $GP(k, j)$.

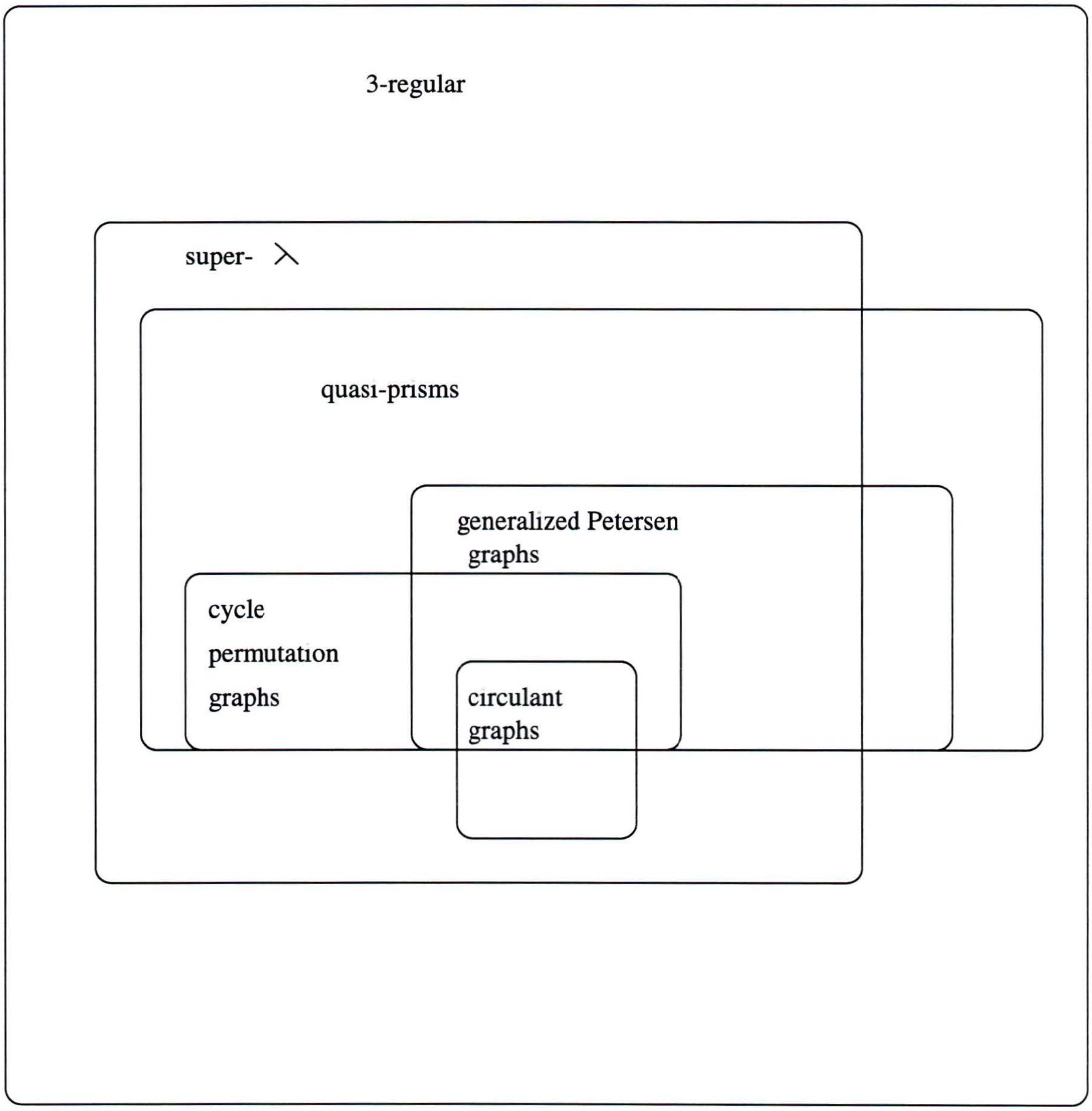

Figure 4.7: Venn diagram of class interrelationships for $n \geq 8$

Chapter 5

Cutsets of Cycle Permutation Graphs

In the previous chapter, we defined cycle permutation graphs (see p 28) and showed that every cycle permutation graph is isomorphic to some quasi-prism Since we want to compare cycle permutation graphs with circulants to find out which graphs have lexicographically smaller cut frequency vectors, we begin by calculating the first five entries of the cut frequency vectors for cycle permutation graphs, for graphs of order ten or greater This chapter contains results which establish that cycle permutation graphs are super-λ under certain restrictions, thus determining N_i for i less than or equal to three We then provide formulas for the number of 4-edge cutsets of a cycle permutation graph These are used to give upper and lower bounds for the value of N_4 for a cycle permutation graph

5.1 Properties of Cycle Permutation Graphs

This section includes proofs that cycle permutation graphs are 3-regular, and that cycle permutation graphs are super-λ under certain conditions To do so, we use earlier results concerning quasi-prisms

Corollary 5 1 1 *Cycle permutation graphs are 3-regular*

Proof This corollary follows from earlier results, since quasi-prisms are 3-regular [Lemma 3 1 1], and cycle permutation graphs are always isomorphic to quasi-prisms [Lemma 4 1 1] □

In [23], Piazza and Stueckle prove that cycle permutation graphs are super-λ We give a different proof of this result below

Lemma 5 1 2 *For $k \geq 4$ and for any permutation $\sigma \in Sym(k)$, (C_k, σ) is super-λ*

Proof By Lemma 4 1 1, the quasi-prism $Q_k[\pi]$ is isomorphic to (C_k, σ) for $\pi = (\sigma(0)\,\sigma(1) \quad \sigma(k-1))$ By Theorem 3 2 4, $Q_k[\pi]$ is super-λ if and only if

(i) π does not contain any cycles of order three or less, and

(ii) there is no nontrivial bipartition of the cycles of π inducing a bipartition $(I, \overline{I})$ of the integers $\{0, 1, \quad, k-1\}$ such that I is contiguous or almost contiguous

Since π consists of only one cycle of order k, and $k \geq 4$, $Q_k[\pi]$ is super-λ Therefore (C_k, σ) is super-λ □

5.2 Enumerating 4-Edge Cuts for Cycle Permutation Graphs

In this section we present results which enable us to determine upper and lower bounds for $N_4((C_k, \sigma))$, for any permutation σ, $k \geq 5$ We first define what is meant by runs and co-runs, and then characterize certain 4-edge cuts in terms of the runs

Let $\pi \in Sym(r)$, where π is expressed in cycle notation and consists of one cycle of order r A *run* is defined to be a contiguous sequence of elements of π whose values can be ordered to form a contiguous sequence of integers modulo r. The *length* of a run is equal to the number of symbols in it. Any run of length zero is considered to be a trivial run, since such a run is not of interest in this thesis, we refer to nontrivial runs as simply runs.

Example The permutation $\pi = (0\ 7\ 1\ 3\ 2\ 5\ 4\ 6)$ has a run $\beta = 6\ 0\ 7\ 1$, since these elements are contiguous in π and the values 6,7,0, and 1 are contiguous modulo 8. Note that a contiguous subsequence of a run is not necessarily a run itself. From the previous example, $\gamma = 7\ 1$ is a contiguous subsequence of β, but since 7 and 1 are not contiguous modulo 8, it is not a run.

Let $\pi \in Sym(r)$, where π is expressed in cycle notation and consists of one cycle of order r, and let β be a run of length s in π. Then the *co-run* of β, denoted by $\pi - \beta$, is defined to be the corresponding run in π which consists of the remaining elements of π once the elements of β are removed. The length of $\pi - \beta$ is $r - s$.

Example If $\pi = (0\ 7\ 1\ 3\ 2\ 6\ 5\ 4)$, then there are exactly three runs of length 3, $\beta_1 = 1\ 3\ 2$, $\beta_2 = 6\ 5\ 4$, and $\beta_3 = 0\ 7\ 1$, whose corresponding co-runs of length 5 are respectively $\beta_4 = \pi - \beta_1 = 6\ 5\ 4\ 0\ 7$, $\beta_5 = \pi - \beta_2 = 0\ 7\ 1\ 3\ 2$, and $\beta_6 = \pi - \beta_3 = 3\ 2\ 6\ 5\ 4$.

Theorem 5.2.1 *Let $k \geq 5$ and $\pi \in Sym(k)$, where π consists of a cycle of order k. Then $(X, \overline{X})$ is a minimal 4-edge cut of the quasi-prism $Q_k[\pi]$, with both $(S, \overline{S})$ and $(T, \overline{T})$ being nontrivial bipartitions, if and only if $(T, \overline{T})$ induces the nontrivial bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is composed of the elements of a run of π.*

Proof. Let the vertex and edge sets of $Q_k[\pi]$ be as follows:

$$V(Q_k[\pi]) = \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and}$$
$$E(Q_k[\pi]) = \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \leq i \leq k-1\}.$$

[$\Rightarrow$] Let $(X, \overline{X})$ be a minimal 4-edge cut of $Q_k[\pi]$, with both $(S, \overline{S})$ and $(T, \overline{T})$ being nontrivial bipartitions. This means that $(X, \overline{X})$ disconnects both the outer and inner cycles of $Q_k[\pi]$. By Lemma 2.1.2, $(X, \overline{X})$ must contain two outer cycle edges and two inner cycle edges, and thus cannot contain any spoke edges. Thus if we let $S = \{s_i, s_{i+1}, \ldots, s_{i+j}\}$ and $\overline{S} = \{s_{i+j+1}, s_{i+j+2}, \ldots, s_{i-1}\}$, then since $(X, \overline{X})$ does not contain any spoke edges, we have $T = \{t_i, t_{i+1}, \ldots, t_{i+j}\}$ and $\overline{T} = \{t_{i+j+1}, t_{i+j+2}, \ldots, t_{i-1}\}$. Then $G[T]$ consists of a path of order $|T|$, where $(t_u, t_v) \in E(G[T])$ implies that $v = \pi(u)$ or $u = \pi(v)$. But $(T, \overline{T})$ induces a nontrivial bipartition $(I, \overline{I})$ on the integers $\{0, 1, \ldots, k-1\}$, where $I = \{i : t_i \in T\}$ and $\overline{I} = \{i : t_i \in \overline{T}\}$, this yields $I = \{i, i+1, \ldots, i+j\}$. Therefore the elements of I form a contiguous sequence of the elements of π. Furthermore, the elements of I can be ordered to form a contiguous sequence of integers modulo k, i.e. $i\ i+1 \ldots i+j$. Therefore the set I is composed of the elements of a run of π.

[$\Leftarrow$] Let $(T, \overline{T})$ be a nontrivial bipartition of the inner cycle vertices of $Q_k[\pi]$ which induces a nontrivial bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$, such that the set I is composed of the elements of a run α of π. Since I is composed of the elements of a run of π, $\overline{I}$ must be composed of the elements of a co-run of π, $\pi - \alpha$. Then $I = \{i, i+1, \ldots, i+j\}$, where the elements of I can be ordered to form a contiguous sequence of integers modulo k, i.e. $i\ i+1 \ldots i+j$, and the elements of I form a contiguous sequence of the elements of π. Since $(T, \overline{T})$ induces $(I, \overline{I})$ we know that $T = \{t_i, t_{i+1}, \ldots, t_{i+j}\}$ and $\overline{T} = \{t_{i+j+1}, t_{i+j+2}, \ldots, t_{i-1}\}$. Then we define $(S, \overline{S})$ to be $S = \{s_i, s_{i+1}, \ldots, s_{i+j}\}$ and $\overline{S} = \{s_{i+j+1}, s_{i+j+2}, \ldots, s_{i-1}\}$. The induced subgraph $G[T]$ consists of a path of order $|T|$, where $(t_u, t_v) \in E(G[T])$ implies that $v = \pi(u)$ or $u = \pi(v)$. Similarly, $G[\overline{T}]$ consists of a path of order $|\overline{T}|$, where $(t_u, t_v) \in E(G[\overline{T}])$ implies that $v = \pi(u)$ or $u = \pi(v)$. Let t_a and t_b be the two vertices of degree one in $G[T]$, and $t_{a'}$ and $t_{b'}$ be the two vertices of degree one in $G[\overline{T}]$. Then without loss of generality, we can assume that the run α begins with

a and ends with b, similarly, we can assume that the run $\pi - \alpha$ begins with a' and ends with b'. Then let $X = S \cup T = \{s_i, s_{i+1}, \ldots, s_{i+j}\} \cup \{t_i, t_{i+1}, \ldots, t_{i+j}\}$, and $\overline{X} = \overline{S} \cup \overline{T} = \{s_{i+j+1}, s_{i+j+2}, \ldots, s_{i-1}\} \cup \{t_{i+j+1}, t_{i+j+2}, \ldots, t_{i-1}\}$. Then $(X, \overline{X}) = \{(s_{i-1}, s_i), (s_{i+j}, s_{i+j+1}), (t_b, t_{a'}), (t_{b'}, t_a)\}$ is a 4-edge cutset of $Q_k[\pi]$, with both $(S, \overline{S})$ and $(T, \overline{T})$ being nontrivial bipartitions. Since $(X, \overline{X})$ does not contain any spoke edges, it must be minimal. □

Example. Consider the quasi-prism $Q_{10}[(0\,3\,6\,4\,5\,7\,1\,9\,2\,8)]$. One run of order 5 in π is $\alpha = 19280$. Letting $(I, \overline{I})$ be a nontrivial bipartition of the integers $\{0, 1, \ldots, 9\}$ such that I contains the elements of α, we have $I = \{i, i+1, \ldots, i+j\} = \{8, 9, 0, 1, 2\}$ and $\overline{I} = \{i+j+1, i+j+2, \ldots, i-1\} = \{3, 4, 5, 6, 7\}$. Therefore $i = 8, i+j = 2, i+j+1 = 3$, and $i - 1 = 7$. We define the nontrivial bipartitions $(S, \overline{S})$ and $(T, \overline{T})$ as follows:

$$\begin{aligned} S &= \{s_8, s_9, s_0, s_1, s_2\}, \\ \overline{S} &= \{s_3, s_4, s_5, s_6, s_7\}, \\ T &= \{t_8, t_9, t_0, t_1, t_2\}, \text{ and} \\ \overline{T} &= \{t_3, t_4, t_5, t_6, t_7\} \end{aligned}$$

We then let $X = S \cup T$ and $\overline{X} = \overline{S} \cup \overline{T}$. Since $\alpha = 1\,9\,2\,8\,0$ starts with $a = 1$ and ends with $b = 0$, and the co-run $\alpha - \pi = 3\,6\,4\,5\,7$ starts with $a' = 3$ and ends with $b' = 7$, the cutset $(X, \overline{X})$ is defined as follows:

$$\begin{aligned} (X, \overline{X}) &= \{(s_{i-1}, s_i), (s_{i+j}, s_{i+j+1}), (t_b, t_{a'}), (t_{b'}, t_a)\} \\ &= \{(s_7, s_8), (s_2, s_3), (t_0, t_3), (t_7, t_1)\} \end{aligned}$$

(See Figure 5.1.)

Corollary 5.2.2 *For $k \geq 5$, $(X, \overline{X})$ is a minimal 4-edge cut of a cycle permutation graph (C_k, σ), with both $(A, \overline{A})$ and $(B, \overline{B})$ being nontrivial bipartitions, if and only*

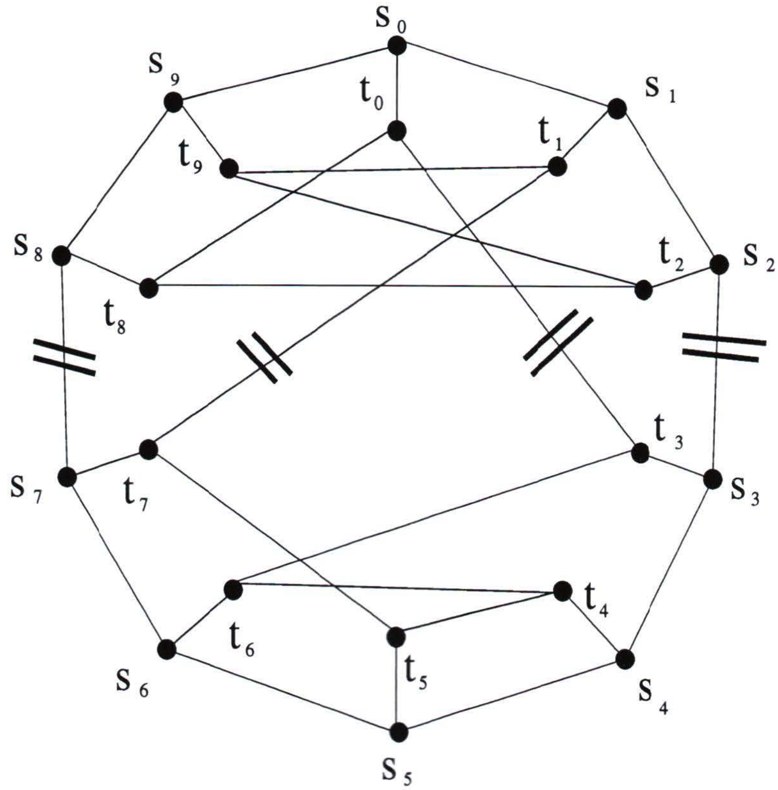

Figure 5.1: Cutset corresponding to run in $Q_{10}[(0\,3\,6\,4\,5\,7\,1\,9\,2\,8)]$

if $(B, \overline{B})$ induces the nontrivial bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is composed of the elements of a run of π, where $\pi = (\sigma(0)\,\sigma(1) \cdots \sigma(k-1))$.

Proof This follows directly from previous results, since (C_k, σ) is isomorphic to $Q_k[\pi]$ if $\pi = (\sigma(0)\ \sigma(1) \cdots \sigma(k-1))$ [Lemma 4 1 1], and $(X, \overline{X})$ is a minimal 4-edge cut of the quasi-prism $Q_k[\pi]$, with both $(S, \overline{S})$ and $(T, \overline{T})$ being nontrivial bipartitions, if and only if $(T, \overline{T})$ induces the nontrivial bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is composed of the elements of a run of π, for $k \geq 5$ [Theorem 5 2 1] □

Consider the cycle permutation graph (C_k, σ), which is isomorphic to the quasi-prism $Q_k[\pi]$, where $\pi = (\sigma(0)\,\sigma(1) \cdots \sigma(k-1))$. The following example shows how a run α in π corresponds to a 4-edge minimal cutset in (C_k, σ).

Example Consider the cycle permutation graph $(C_{10}, [0\,3\,6\,4\,5\,7\,1\,9\,2\,8])$. This cycle permutation graph is isomorphic to the quasi-prism $Q_{10}[(0\,3\,6\,4\,5\,7\,1\,9\,2\,8)]$, which is discussed in the previous example. From that example, we know that the run $\alpha = 1\,9\,2\,8\,0$ corresponds to the cutset

$$(X, \overline{X}) = \{(s_7, s_8), (s_2, s_3), (t_0, t_3), (t_7, t_1)\}$$

Using the bijection from the proof of Lemma 4 1 1, which can be found in Appendix F, we have, for $j = \sigma(i)$,

$$(f(t_j), f(t_{\pi(j)})) = (a_i, a_{i+1}), \text{ and}$$
$$(f(s_i), f(s_{i+1})) = (b_i, b_{i+1})$$

Therefore

$$(f(s_7), f(s_8)) = (b_7, b_8), \text{ and}$$
$$(f(s_2), f(s_3)) = (b_2, b_3)$$

For $(t_0, t_3) = (t_j, t_{\pi(j)})$, we have $j = 0$ and $\pi(0) = 3$. Then since $j = \sigma(i)$, $j = 0 \Rightarrow i = 0$. For $(t_7, t_1) = (t_j, t_{\pi(j)})$, we have $j = 7$ and $\pi(7) = 1$. Then since $j = \sigma(i)$,

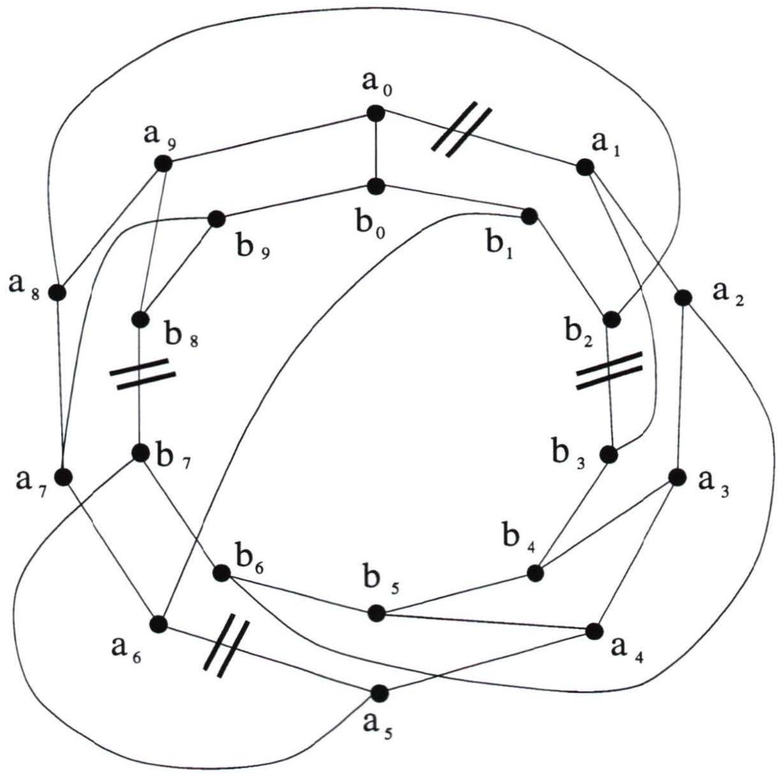

Figure 5.2: Cutset corresponding to run for $(C_{10}, [0\,3\,6\,4\,5\,7\,1\,9\,2\,8])$

$j = 7 \Rightarrow i = 5$. Therefore

$$
\begin{aligned}
(f(t_0), f(t_3)) &= (a_0, a_1), \text{ and} \\
(f(t_7), f(t_1)) &= (a_5, a_6).
\end{aligned}
$$

Thus the 4-edge minimal cutset of (C_k, σ) corresponding to the run α of π is:

$$(X, \overline{X}) = \{(b_7, b_8), (b_2, b_3), (a_0, a_1), (a_5, a_6)\}.$$

(See Figure 5.2.)

Lemma 5.2.3 *The cycle permutation graph* $(C_k, \mathcal{I})$ *is isomorphic to some cycle permutation graph* (C_k, σ) *if and only if* $(\sigma(0)\ \sigma(1)\ \cdots\ \sigma(k-1))$ *is cyclically equivalent to* $(0\,1\,\cdots\,(k-1))$.

Proof [$\Rightarrow$] Let the vertex and edge sets of $(C_k, \mathcal{I})$ be as follows

$$\begin{aligned} V((C_k,\mathcal{I})) &= \mathcal{A} \cup \mathcal{B} = \{a_0, a_1, \ldots, a_{k-1}\} \cup \{b_0, b_1, \ldots, b_{k-1}\}, \text{ and} \\ E((C_k,\mathcal{I})) &= \{(a_i, a_{i+1}), (b_i, b_{i+1}), (a_i, b_i) : 0 \le i \le k-1\} \end{aligned}$$

We use proof by contradiction. Assume that (C_k, σ) is isomorphic to $(C_k, \mathcal{I})$ by a nonnatural isomorphism. Then the vertex and edge sets of $(C_k, \mathcal{I})$ can be decomposed into pairwise disjoint sets as follows:

$$(C_k, \mathcal{I}) = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2) \cup E(H_3)),$$

where

$H_1 = (V(H_1), E(H_1))$ is a cycle of order k corresponding to the outer cycle of (C_k, σ),

$H_2 = (V(H_2), E(H_2))$ is a cycle of order k corresponding to the inner cycle of (C_k, σ), and

$H_3 = (V(H_1) \cup V(H_2), E(H_3))$ is a perfect matching between the vertices of $V(H_1)$ and $V(H_2)$ corresponding to the permutation edges of (C_k, σ).

If $E(H_1)$ does not contain any permutation edges of $(C_k, \mathcal{I})$, then the only possible cycles are either the outer cycle or the inner cycle. Since we are trying to construct a nonnatural isomorphism, $E(H_1)$ must contain at least one permutation edge of $(C_k, \mathcal{I})$.

Let us assume that $E(H_1)$ contains both cycle edges (a_i, a_{i+1}) and (b_i, b_{i+1}); we use proof by contradiction to show that this cannot occur. First assume that one of the permutation edges (a_i, b_i) or (a_{i+1}, b_{i+1}) does not belong to $E(H_1)$. Since that permutation edge is an edge with both endpoints in $V(H_1)$, it cannot belong to $E(H_2)$ or $E(H_3)$. Now assume that both of the permutation edges (a_i, b_i) and (a_{i+1}, b_{i+1}) belong to $E(H_1)$. This results in a 4-cycle. Since $k \ge 5$, such a cycle is not of order k, and therefore is not H_1. Therefore $E(H_1)$ does not contain both cycle edges (a_i, a_{i+1}) and (b_i, b_{i+1}).

Therefore

$$(a_i, a_{i+1}) \in E(H_1) \Rightarrow (b_i, b_{i+1}) \notin E(H_1), \text{ and}$$
$$(b_i, b_{i+1}) \in E(H_1) \Rightarrow (a_i, a_{i+1}) \notin E(H_1)$$

But for H_1 to be a cycle, $E(H_1)$ must contain either (a_i, a_{i+1}) or (b_i, b_{i+1}), but not both, for $0 \leq i \leq k-1$ Since $E(H_1)$ must contain k cycle edges of $(C_k, \mathcal{I})$, and at least one permutation edge of $(C_k, \mathcal{I})$, it is not possible to construct a cycle of order k corresponding to H_1 Therefore $(C_k, \mathcal{I})$ is not isomorphic to (C_k, σ) by a nonnatural isomorphism Hence the only possible isomorphism between (C_k, σ) and $(C_k, \mathcal{I})$ is a natural isomorphism, which implies that $(\sigma(0)\,\sigma(1) \cdots \sigma(k-1))$ is cyclically equivalent to $(0\,1 \cdots (k-1))$.
[$\Leftarrow$] Follows directly, since $(\sigma(0)\,\sigma(1) \cdots \sigma(k-1))$ and $(0\,1 \cdots (k-1))$ are cyclically equivalent. □

Lemma 5.2 4 *For $k \geq 5$, amongst all cycle permutation graphs, $(C_k, \mathcal{I})$ (and all other cycle permutation graphs which are isomorphic to $(C_k, \mathcal{I})$) has the lexicographically largest cut frequency vector*

Proof Let the vertex and edge sets of (C_k, σ) be as follows

$$\begin{aligned} V((C_k, \sigma)) &= \mathcal{A} \cup \mathcal{B} = \{a_0, a_1, \ldots, a_{k-1}\} \cup \{b_0, b_1, \ldots, b_{k-1}\}, \text{ and} \\ E((C_k, \sigma)) &= \{(a_i, a_{i+1}), (b_i, b_{i+1}), (a_i, b_{\sigma(i)}) : 0 \leq i \leq k-1\} \end{aligned}$$

From Lemma 5 1 2 we know that $\lambda((C_k, \sigma))$ is equal to three and $N_\lambda((C_k, \sigma))$ is equal to n From Lemma 2 1 3, we also know that 4-edge cuts are either minimal cuts $(X, \overline{X})$ with $|X|$ even, or else they are cuts of the type $F \cup \{e\}$ which are not minimal, where F is a minimal 3-edge cut and $e \in E((C_k, \sigma)) \setminus F$ Furthermore, from Lemma 2 1 2, any minimal cut $(X, \overline{X})$ contains an even number of inner cycle edges and an even number of outer cycle edges Therefore we have the following possibilities for the 4-edge cutsets, based on whether the cuts are minimal and which edges are contained in the cutsets.

(1) 4-edge cutset is not minimal,

(2) 4-edge cutset $(X, \overline{X})$ is minimal,

(i) $(X, \overline{X})$ contains four outer cycle edges,

(ii) $(X, \overline{X})$ contains two outer cycle edges and two permutation edges,

(iii) $(X, \overline{X})$ contains two outer cycle edges and two inner cycle edges,

(iv) $(X, \overline{X})$ contains four inner cycle edges,

(v) $(X, \overline{X})$ contains two inner cycle edges and two permutation edges, or

(vi) $(X, \overline{X})$ contains four permutation edges.

Case 1: 4-edge cutset is not minimal.

As stated above, 4-edge cutsets which are not minimal are of the type $F \cup \{e\}$, where F is a minimal 3-edge cutset and $e \in E((C_k, \sigma)) \setminus F$. Since the only minimal 3-edge cutsets are those with $|X|$ equal to one, there are $n(m-3)$ 4-edge cutsets which are not minimal.

Case 2: 4-edge cutset $(X, \overline{X})$ is minimal.

Subcase 2(i): $(X, \overline{X})$ contains four outer cycle edges.

Since $(X, \overline{X})$ does not contain any inner cycle edges or permutation edges, there can be no cutset of this type.

Subcase 2(ii): $(X, \overline{X})$ contains two outer cycle edges and two permutation edges.

Without loss of generality, we can assume that the inner cycle vertices are in X. Then for any edge $(a_i, b_i) \in (X, \overline{X})$, $a_i \in \overline{X}$. Furthermore, for any vertex $a_i \in \overline{X}$, $(a_i, b_i) \in (X, \overline{X})$. Since $(X, \overline{X})$ contains two permutation edges, $|\overline{X}|$ is equal to two. This corresponds to $G[\overline{X}] = \{(a_i, a_{i+1})\}$, for $0 \le i \le k-1$. This yields k such cuts of this type.

Subcase 2(iii): $(X, \overline{X})$ contains two outer cycle edges and two inner cycle edges.

Since both $(A, \overline{A})$ and $(B, \overline{B})$ are nontrivial bipartitions, by Corollary 5.2.2, 4-edge cuts of this type can be enumerated by counting the number of runs of the permutation π, where $\pi = (\sigma(0)\,\sigma(1) \dots \sigma(k-1))$. The maximum number of runs occurs when $\sigma(i)$ is the first element of a run, over all possible lengths, for $0 \leq i \leq k-1$. Therefore there are at most k runs of each length in π, where the length r is bounded by $1 \leq r \leq k-1$. This results in $k-1$ possible lengths for a run α. This counts each run twice, since any run α of length r corresponds to the co-run $\pi - \alpha$ of length $k-r$. This upper bound is achieved when $\pi = (0\,1 \dots (k-1))$, or $\sigma = \mathcal{I}$. Since k equals $\frac{n}{2}$, this yields $\frac{n}{4}(\frac{n}{2}-1)$ 4-edge cuts of this type for $(C_k, \mathcal{I})$. Now consider $\sigma' \in Sym(k)$, where (C_k, σ') is not isomorphic to $(C_k, \mathcal{I})$. Then by Lemma 5.2.3, $\pi' = (\sigma'(0)\,\sigma'(1) \dots \sigma'(k-1))$ is not cyclically equivalent to $\pi = (0\,1 \dots (k-1))$. Thus there must exist two adjacent elements in π', say $\sigma'(i)$ and $\sigma'(i+1)$, such that $1 < |\sigma'(i) - \sigma'(i+1)| < k-1$. Then $\sigma'(i)$ and $\sigma'(i+1)$ are not adjacent modulo k, and thus there is no run of length two with $\sigma'(i)$ as the first element. Therefore π' does not achieve the maximum number of runs, and thus (C_k, σ') has fewer than $\frac{n}{4}(\frac{n}{2}-1)$ 4-edge cuts of this type.

<u>Subcase 2(iv)</u>: $(X, \overline{X})$ contains four inner cycle edges.

The argument is the same as Subcase 2(i); there are no cutsets of this type.

<u>Subcase 2(v)</u>: $(X, \overline{X})$ contains two inner cycle edges and two permutation edges.

The argument is the same as Subcase 2(ii); there are k such cutsets.

<u>Subcase 2(vi)</u>: $(X, \overline{X})$ contains four permutation edges.

Since $k \geq 5$, there can be no cutset of this type.

Thus $(C_k, \mathcal{I})$ achieves the maximum number of 4-edge cuts in all of the above cases. Since no other cycle permutation graph (C_k, σ) achieves this upper bound unless (C_k, σ) is isomorphic to $(C_k, \mathcal{I})$, $N_4((C_k, \mathcal{I}))$ is strictly greater than $N_4((C_k, \sigma))$. Therefore $(C_k, \mathcal{I})$ has the lexicographically greatest cut frequency vector amongst all cycle permutation graphs (C_k, σ), given that (C_k, σ) and $(C_k, \mathcal{I})$ are not isomorphic.

□

Corollary 5.2.5 *For $k \geq 5$, the number of 4-edge cuts of the cycle permutation graph $(C_k, \mathcal{I})$ is*

$$n(m-3) + n + \frac{n}{4} \cdot (\tfrac{n}{2} - 1), \text{ where } m = \frac{3n}{2}$$

Proof. Follows directly from the proof of Lemma 5.2.4 □

We now provide a lower bound on the number of 4-edge cutsets for any 3-regular super-λ graph G. In Chapter 7, we show that this lower bound is exact for an infinite family of cycle permutation graphs.

Lemma 5.2.6 *For any 3-regular super-λ graph G,*

$$\begin{aligned} N_4(G) &\geq n(m-3) + n + \tfrac{n}{2} \\ &= n(m-3) + m \end{aligned}$$

Proof. Since G is super-λ, there are n 3-edge cuts. For each such cut $(X, \overline{X})$, $F = (X, \overline{X}) \cup \{e\}$, where $e \in E(G) \setminus (X, \overline{X})$, yields a 4-edge cut which is not minimal. No two such 4-edge cuts are the same, since that implies that G contains multiple edges, and is therefore not super-λ. This results in $n(m-3)$ distinct cuts of this type. Another type of 4-edge cut is obtained by isolating an edge (u, v) from G by removing the four remaining edges adjacent to u and v. There are $m = \frac{3n}{2} = n + \frac{n}{2}$ edges, and therefore $n + \frac{n}{2}$ such 4-edge cuts. Therefore $N_4(G) \geq n(m-3) + n + \frac{n}{2} = n(m-3) + m$
□

Lemma 5.2.7 *Any 3-regular super-λ graph G with $N_4(G) = n(m-3) + n + \frac{n}{2}$ is cyclically 5-edge connected.*

Proof. Since G is a 3-regular super-λ graph, we know that G is cyclically 4-edge connected. What remains to be shown is that every 4-edge cutset results in at most one component containing a cycle. Since G is super-λ, we know that the $n(m-3)$

4-edge cutsets described in Lemma 5.2.6 result in exactly two components, one of which consists of a single vertex. For the same reason, we know that the $n + \frac{n}{2}$ 4-edge cutsets described in Lemma 5.2.6 result in exactly two components, one of which consists of a single edge. This accounts for all of the 4-edge cutsets of G. $\square$

Chapter 6

Cutsets of Circulant Graphs

In Chapter 4 we defined circulant graphs (see p. 31), and presented an isomorphism result which establishes that there are at most two connected 3-regular circulants, up to isomorphism. In this chapter, we reference results which describe when circulants are connected and when they are super-λ. We also enumerate the number of 4-edge cutsets for any connected 3-regular circulant.

6.1 Properties of Circulants

We begin with results which establish when circulants are connected and when they are super-λ. The results are stated for r-regular circulants, but we are only concerned with 3-regular circulants in this thesis.

Lemma 6.1.1 [[5], p. 493] *The circulant graph $C_n\langle a_1, a_2, \ldots, a_j\rangle$ is connected if and only if*

$$gcd(a_1, a_2, \ldots, a_j, n) = 1.$$

Lemma 6.1.2 [[7], p. 95] *The only connected circulants which are not super-λ are the cycles of the form $C_n\langle a\rangle$ and the circulants $C_{2k}\langle 2, 4, \ldots, k-1, k\rangle$ for k odd.*

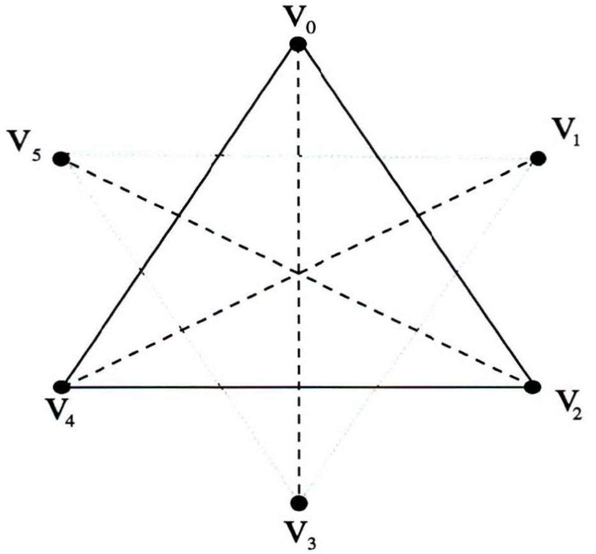

Figure 6 1 Connected circulant $C_6\langle 2, 3\rangle$ which is not super-λ

From Lemma 6 1 1 and Lemma 6 1 2, we see that the only connected 3-regular circulant which is not super-λ is $C_6\langle 2, 3\rangle$, as shown in Figure 6 1 Therefore for $n \geq 8$, all 3-regular connected circulants are super-λ

6.2 Enumeration of 4-edge Cutsets for Circulants

In this section, we determine the number of 4-edge cutsets for any connected 3-regular circulant We know that for $n \geq 8$, connected 3-regular circulants are super-λ, and therefore have cut frequency vectors which are lexicographically smaller than that of a graph which is not super-λ From Chapter 5, we also know that cycle permutation graphs are super-λ for $n \geq 8$ To show that some cycle permutation graphs have cut frequency vectors which are lexicographically smaller than those of circulants, we prove that the number of 4-edge cutsets for those cycle permutation graphs is strictly less than the number of 4-edge cutsets for connected 3-regular circulants. We begin by enumerating $N_4(C_n\langle 1, \frac{n}{2}\rangle)$.

Lemma 6.2.1 *For $n \geq 6$, the circulant $C_n\langle 1, \frac{n}{2}\rangle$ has $n(m-3)+n+\frac{n}{4}(\frac{n}{2}-1)$ 4-edge cuts, where $m = \frac{3n}{2}$.*

Proof. Let the vertex and edge sets of $C_n\langle 1, \frac{n}{2}\rangle$ be as follows:

$$\begin{aligned} V(C_n\langle 1, \tfrac{n}{2}\rangle) &= \{v_0, v_1, \ldots, v_{n-1}\}, \text{ and} \\ E(C_n\langle 1, \tfrac{n}{2}\rangle) &= \{(v_i, v_{i+1}), (v_j, v_{j+\frac{n}{2}}) : 0 \leq i \leq n-1, 0 \leq j \leq \tfrac{n}{2}-1\}. \end{aligned}$$

Since the edges (v_i, v_{i+1}) of $C_n\langle 1, \frac{n}{2}\rangle$, for $0 \leq i \leq n-1$, describe a cycle of order n, by Lemma 2.1.2 we know that any minimal cut $(X, \overline{X})$ of $C_n\langle 1, \frac{n}{2}\rangle$ must contain an even number of cycle edges. We decompose the problem into several cases, based on minimality of the cutset and the composition of the cuts, as follows:

(1) 4-edge cutset is not minimal,

(2) 4-edge cutset $(X, \overline{X})$ is minimal;

(i) $(X, \overline{X})$ contains four chord edges,

(ii) $(X, \overline{X})$ contains two cycle edges and two chord edges, or

(iii) $(X, \overline{X})$ contains four cycle edges.

<u>Case 1</u>: 4-edge cutset is not minimal.

The only possible 4-edge cuts for a super-λ graph which are not minimal are $(X, \overline{X}) \cup \{e\}$, where $(X, \overline{X})$ is a minimal 3-edge cutset and $e \in E(C_n\langle 1, \frac{n}{2}\rangle) \setminus (X, \overline{X})$. Therefore there are $n(m-3)$ such cuts.

<u>Case 2</u>: 4-edge cutset $(X, \overline{X})$ is minimal.

<u>Subcase 2(i)</u>: $(X, \overline{X})$ contains four chord edges.

Since $(X, \overline{X})$ does not disconnect the cycle of order n, there can be no cutset of this type.

<u>Subcase 2(ii)</u>: $(X, \overline{X})$ contains two cycle edges and two chord edges.

The removal of the two cycle edges divides the cycle of order n into two segments,

Y_1 and Y_2, such that $V(Y_1) \subseteq X$ and $V(Y_2) \subseteq \overline{X}$ Then any chord edge (u, v) with $u \in V(Y_1)$ and $v \in V(Y_2)$ must belong to $(X, \overline{X})$ Since $(X, \overline{X})$ contains two chord edges, either $|X|$ or $|\overline{X}|$ must be equal to two Without loss of generality, we can assume that $|X|$ is equal to two This type of 4-edge cut corresponds to $G[X] = \{(v_i, v_{i+1})\}$, for $0 \leq i \leq n-1$ Therefore there are n cuts of this type

Subcase 2(iii) $(X, \overline{X})$ contains four cycle edges

Since no chord edges can be removed, both endpoints of each chord edge must be in either X or $\overline{X}$ Let the cycle edge (v_i, v_{i+1}) be in $(X, \overline{X})$ Without loss of generality, assume that $v_i \in X$ Since $v_i \in X$, $v_{i+\frac{n}{2}} \in X$, similarly, $v_{i+1} \in \overline{X}$ implies that $v_{i+\frac{n}{2}+1} \in \overline{X}$ Therefore $(v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1}) \in (X, \overline{X})$ Thus choosing the cycle edge (v_i, v_{i+1}) to be in the cutset forces the opposite cycle edge $(v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1})$ to be in the cutset as well, as shown in Figure 6 2 Similarly, choosing the cycle edge (v_j, v_{j+1}) to be in the cutset forces the opposite cycle edge $(v_{j+\frac{n}{2}}, v_{j+\frac{n}{2}+1})$ to be in the cutset as well Therefore let $\{v_i, v_{i+\frac{n}{2}}, v_{j+1}, v_{j+\frac{n}{2}+1}\} \subseteq X$ and let $\{v_{i+1}, v_{i+\frac{n}{2}+1}, v_j, v_{j+\frac{n}{2}}\} \subseteq \overline{X}$ If $\{v_{i+1}, v_{i+2}, \ldots, v_j\} \subseteq \overline{X}$, then no cycle edges (v_r, v_{r+1}), $i+1 \leq r \leq j-1$, need to be removed, and therefore $\{v_{i+\frac{n}{2}+1}, v_{i+\frac{n}{2}+2}, \ldots, v_{j+\frac{n}{2}}\} \subseteq \overline{X}$ as well All other vertices are in X The removal of the four cycle edges $(v_i, v_{i+1}), (v_j, v_{j+1}), (v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1})$, and $(v_{j+\frac{n}{2}}, v_{j+\frac{n}{2}+1})$ results in a minimal 4-edge cut, as shown in Figure 6 2 There are $\frac{n(n-2)}{2}$ ways to choose a pair of non-opposite cycle edges Including the opposites of both edges results in four edges in total, and for each such set of four edges, there are four ways to select the pair of non-opposite cycle edges Therefore the number of distinct ways to choose the set of four edges is $\frac{n(n-2)}{2} \cdot \frac{1}{4}$, which equals $\frac{n}{4} \cdot (\frac{n}{2} - 1)$ Therefore there are $\frac{n}{4} \cdot (\frac{n}{2} - 1)$ 4-edge cuts in this case

Thus,

$$N_4(C_n\langle 1, \tfrac{n}{2}\rangle) = n(m-3) + n + \frac{n}{4}(\tfrac{n}{2} - 1)$$

□

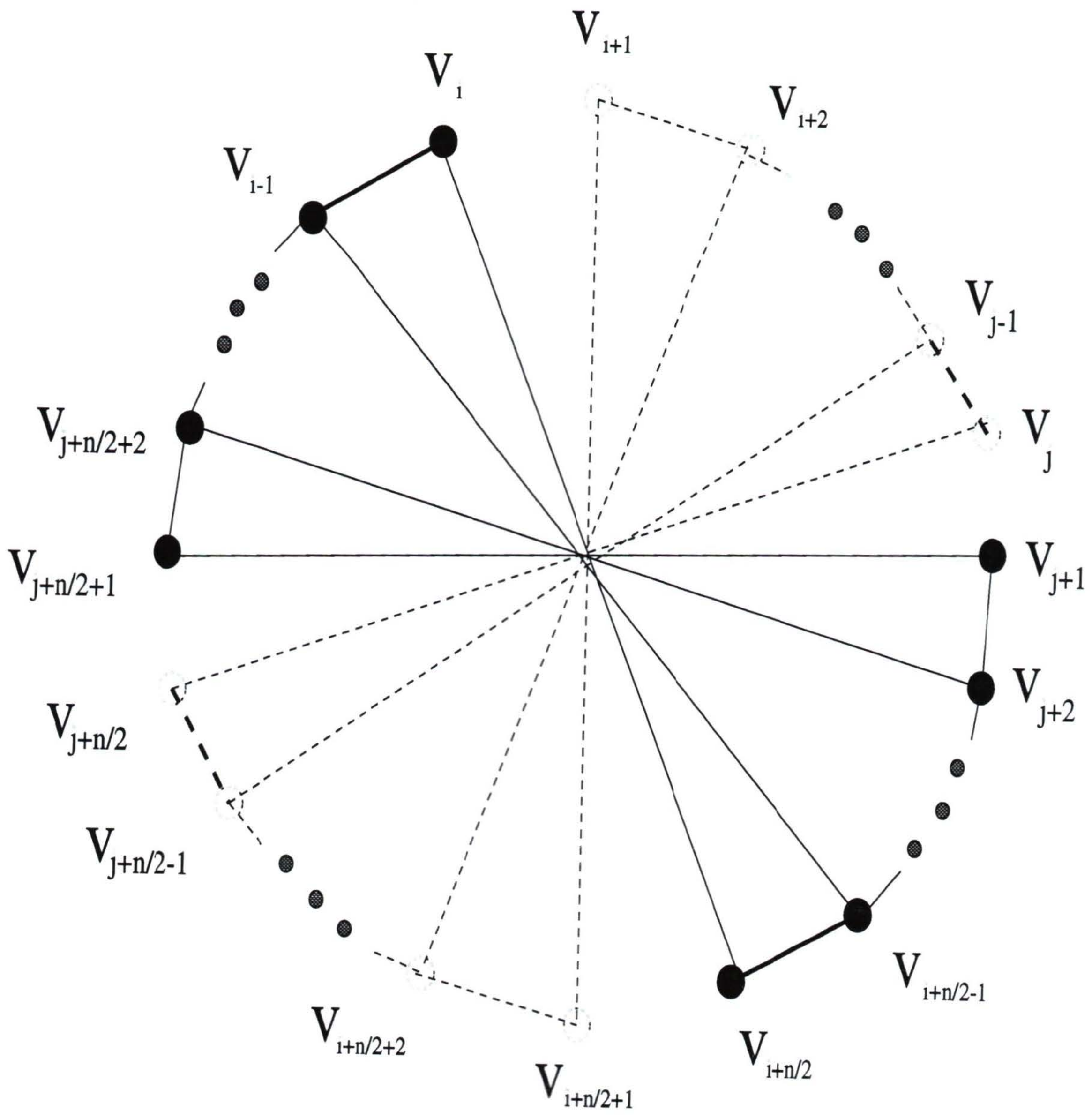

Figure 6.2: $(X, \overline{X})$ of subcase 2(iii) for $C_n\langle 1, \frac{n}{2} \rangle$

Corollary 6 2 2 *Letting G be a connected 3-regular circulant with $n \geq 10$,*

$$\begin{aligned} N_4(G) &= N_4((C_k, \mathcal{I})) \\ &= n(m-3) + n + \frac{n}{4} \cdot (\tfrac{n}{2} - 1) \end{aligned}$$

Proof By Corollary 4 2 5, $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to $(C_k, \mathcal{I})$ for $n \geq 6$ and $\frac{n}{2}$ odd By Lemma 5 2 5, for $n \geq 10$,

$$N_4((C_k, \mathcal{I})) = n(m-3) + n + \frac{n}{4} \cdot (\tfrac{n}{2} - 1)$$

Therefore for $n \geq 10$ and $\frac{n}{2}$ odd,

$$\begin{aligned} N_4(C_n\langle 2, \tfrac{n}{2}\rangle) &= N_4((C_k, \mathcal{I})) \\ &= n(m-3) + n + \frac{n}{4} \cdot (\tfrac{n}{2} - 1) \end{aligned}$$

By Lemma 6 2 1, for $n \geq 6$,

$$N_4(C_n\langle 1, \tfrac{n}{2}\rangle) = n(m-3) + n + \frac{n}{4} \cdot (\tfrac{n}{2} - 1),$$

and by Corollary 4 2 9, for $n \geq 10$, there are at most two connected 3-regular circulants up to isomorphism $C_n\langle 1, \frac{n}{2}\rangle$ for $\frac{n}{2}$ even, and $C_n\langle 1, \frac{n}{2}\rangle$ and $C_n\langle 2, \frac{n}{2}\rangle$ for $\frac{n}{2}$ odd Therefore for $n \geq 10$ and G a connected 3-regular circulant,

$$N_4(G) = N_4((C_k, \mathcal{I}))$$

□

By Lemma 5 2 4, we know that for $n \geq 10$, the cut frequency vector of $(C_k, \mathcal{I})$ is lexicographically larger than that of (C_k, σ), for (C_k, σ) not isomorphic to $(C_k, \mathcal{I})$, and N_4 is the first entry which differs For $n \geq 10$, $(C_k, \mathcal{I})$ has the same number of 4-edge cutsets as any connected 3-regular circulant Thus any cycle permutation graph (C_k, σ), with (C_k, σ) not isomorphic to $(C_k, \mathcal{I})$, has a cut frequency vector which is lexicographically smaller than that of any connected 3-regular circulant

Chapter 7

Cutsets of Generalized Petersen Graphs

Generalized Petersen graphs are defined in Chapter 4 (see p. 34), and in that chapter we showed that the class of generalized Petersen graphs is a subset of the class of quasi-prisms. In this chapter, we use that isomorphism to establish when generalized Petersen graphs are super-λ. We also enumerate the number of 4-edge cutsets for a particular subclass of generalized Petersen graphs. By using a previous result which characterizes the intersection between generalized Petersen graphs and cycle permutation graphs, we are then able to determine N_4 for those cycle permutation graphs which are also generalized Petersen graphs.

7.1 Properties of Generalized Petersen Graphs

In this section we prove that generalized Petersen graphs are 3-regular, and that under certain conditions, generalized Petersen graphs are super-λ. We use earlier results regarding quasi-prisms in these proofs.

Corollary 7.1.1 *Generalized Petersen graphs are 3-regular.*

Proof This corollary follows from previous results, since quasi-prisms are 3-regular [Lemma 3 1 1], and generalized Petersen graphs are always isomorphic to quasi-prisms [Lemma 4 3 1] □

Lemma 7 1 2 *For $k \geq 4$, the generalized Petersen graph $GP(k, j)$ is super-λ if and only if $k \neq 3j$*

Proof [⇒] Assume that $k = 3j$ Let the vertex and edge sets of $GP(k, j)$ be as follows

$$\begin{aligned} V(GP(k,j)) &= \mathcal{U} \cup \mathcal{V} = \{u_0, u_1, \ldots, u_{k-1}\} \cup \{v_0, v_1, \ldots, v_{k-1}\}, \text{ and} \\ E(GP(k,j)) &= \{(u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+j}) \quad 0 \leq i \leq k-1\} \end{aligned}$$

Since $GP(k, j)$ contains the 3-cycle (v_i, v_{i+j}), (v_{i+j}, v_{i+2j}), and (v_{i+2j}, v_i), $F = \{(u_{i+hj}, v_{i+hj}) \quad 0 \leq h \leq 2\}$ is a 3-edge cutset of $GP(k, j)$, and therefore $GP(k, j)$ is not super-λ

[⇐] Assume that $k \neq 3j$ By Lemma 4 3 1, $GP(k, j)$ is isomorphic to the quasi-prism $Q_k[\pi]$ for $\pi(i) = i + j$, $0 \leq i \leq k-1$ Let the vertex and edge sets of $Q_k[\pi]$ be as follows

$$\begin{aligned} V(Q_k[\pi]) &= \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and} \\ E(Q_k[\pi]) &= \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) \quad 0 \leq i \leq k-1\} \end{aligned}$$

By Theorem 3 2 4, for $k \geq 4$, $Q_k[\pi]$ is super-λ if and only if

(i) π does not contain any cycles of order three or less, and

(ii) there is no nontrivial bipartition of the cycles of π inducing a bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is contiguous or almost contiguous

<u>Claim 1</u> π does not contain any cycles of order three or less

Any cycle in π induces a cycle C in $Q_k[\pi]$ on the inner cycle vertices, where $E(C) = \{(v_i, v_{i+j}), (v_{i+j}, v_{i+2j}), \ldots, (v_{i+(h-1)j}, v_i)\}$, for $k = hj$, and all values are taken modulo k. Since $k \neq j$ by definition of generalized Petersen graphs, there are no cycles of order one. Since $k \neq 2j$ by definition of generalized Petersen graphs, there are no cycles of order two. Since $k \neq 3j$ by assumption, there are no cycles of order three. Therefore π does not contain any cycles of order three or less.

Claim 2. There is no nontrivial bipartition of the cycles of π inducing a bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is contiguous or almost contiguous.

Suppose the bipartition $(C, \overline{C})$ of the inner cycles of $Q_k[\pi]$, induced by $(T, \overline{T})$, is nontrivial. If $gcd(k, j) = 1$, then π is a single cycle of order k, and thus any bipartition $(C, \overline{C})$ must be trivial. Therefore let $gcd(k, j) = h$ for some $1 < h \leq j$. Then for $0 \leq i \leq h-1$, each cycle C_i is of the form

$$C_i = (i \;\; i+j \;\; i+2j \;\; \ldots \;\; (i + (\frac{k}{h} - 1)j)),$$

where $i + (\frac{k}{h} - 1)j \cong i - j \pmod{k}$, since h divides j. Therefore we number the cycles $C_0, C_1, \ldots, C_{h-1}$ according to the smallest numbered vertex in the cycle.

Observation. The $\frac{k}{h}$ elements of the cycle C_i belong to the set $L_i = \{i, i+h, i+2h, \ldots, i + (\frac{k}{h} - 1)h\}$.

To see this, let $j = xh$ and $k = yh$. Then

$$C_i = (i \;\; i+xh \;\; i+2xh \;\; \ldots \;\; (i + (y-1)xh)).$$

Since $gcd(xh, yh) = h$, $gcd(x, y) = 1$. By Lemma 2.3.1,

$$\{x \cdot i (\bmod y) : 0 \leq i \leq y-1\} = \{0, 1, \ldots, y-1\}.$$

Therefore for $0 \leq i \leq h-1$,

$$\begin{aligned} L_i &= \{i (\bmod yh), i + xh (\bmod yh), \ldots, (i + (y-1)xh)(\bmod yh)\} \\ &= \{i + 0x (\bmod y)h, i + 1x (\bmod y)h, \ldots, (i + (y-1)x)(\bmod y)h\} \\ &= \{i + 0h, i + 1h, \ldots, (i + (y-1)h)\} \\ &= \{i, i+h, \ldots, (i + (\frac{k}{h} - 1)h)\}. \end{aligned}$$

We now show that there is no nontrivial bipartition of the cycles of π inducing a nontrivial bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is contiguous or almost contiguous. Consider a nontrivial bipartition $(C, \overline{C})$ of the cycles of π which induces a nontrivial bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$. Suppose, without loss of generality, that $C_0 \in C$, and therefore

$$L_0 = \{0, h, 2h, \ldots (\frac{k}{h} - 1)h\} \subseteq I$$

For I to be contiguous or almost contiguous, the intermediate values must be present in all but one (contiguous) or all but two (almost contiguous) of the intervals $[0, h], [h, 2h], \ldots, [(\frac{k}{h} - 1)h, 0]$. Since by definition of a generalized Petersen graph, j is not equal to k or $\frac{k}{2}$, $\frac{k}{h} \geq 3$ and thus there are at least three intervals. Assume without loss of generality that the intermediate values for $[0, h]$ are in I. This implies that $C_0, C_1, \ldots, C_{h-1}$ all belong to C, and thus the cycle bipartition cannot be nontrivial.

Thus $Q_k[\pi]$ is super-λ, and therefore $GP(k, j)$ is also super-λ. □

7.2 Enumeration of 4-Edge Cutsets for Generalized Petersen Graphs

In Chapter 5, we state a lower bound on the number of 4-edge cutsets for a 3-regular super-λ graph. In this section, we show that the lower bound is tight for 3-regular graphs of order ten or greater.

For the following argument, we are interested in enumerating $N_4(GP(k, j))$ for $gcd(k, j) = 1$ and $j \neq \pm 1$. In Chapter 4, we provided a characterization of the intersection between the class of cycle permutation graphs and the class of generalized Petersen graphs. One of the three conditions for isomorphism is for k and j to be relatively prime. In such a case, $GP(k, j)$ is isomorphic to (C_k, σ), where

$\sigma = [0\ j\ 2j\ \ldots\ (k-1)j]$.

For a permutation $\pi \in Sym(k)$, where π is in cycle notation and a is some element in π, we refer to the element $\pi^{-1}(a)$ as the *predecessor of a*. Let $\pi = (\pi_0\ \pi_1\ \pi_2\ \ldots\ \pi_{k-1})$, and let α be some run of length r in π. Then without loss of generality, we can assume that $\alpha = \pi_0\,\pi_1 \ldots \pi_{r-1}$ and $\pi - \alpha = \pi_r\,\pi_{r+1} \ldots \pi_{k-1}$. Then since $\pi^{-1}(\pi_i) = \pi_{i-1}$, where all subscripts are taken modulo k, it is clear that π_r is the only element whose inverse is in α. Thus a co-run $\pi - \alpha$ has exactly one value whose predecessor belongs to the run α.

Lemma 7.2.1 *Let $k \geq 5$, $1 < j < k-1$, and $j \neq \frac{k}{2}$, such that $gcd(k, j) = 1$. If $\pi = (0\ j\ 2j\ \ldots\ (k-1)j)$, where all values are taken modulo k, then the only runs in π have length r, where r equals 1 or $k-1$.*

Proof: Let $\pi = (0\ j\ 2j\ \ldots\ (k-1)j)$. Then any run α of length r in π must be of the form

$$\alpha = i\ \ i+j\ \ \ldots\ \ i+(r-1)j.$$

Let z be the minimum value of the elements in α. Since we can renumber each element of π from π_i to $\pi_i - z$, where all values are taken modulo k, the elements of α can be normalized so that the new minimum value of the elements in α is zero. Thus, without loss of generality, the elements of the sequence α belong to the set

$$I = \{0, 1, 2, \ldots, r-1\}.$$

Then the co-run $\pi - \alpha$ of length $k - r$ is such that

$$\pi - \alpha = i + rj\ \ i+(r+1)j\ \ \ldots\ \ i+(k-1)j,$$

where the elements of the sequence $\pi - \alpha$ belong to the set

$$\bar{I} = \{r, r+1, r+2, \ldots, k-1\}.$$

We assume that $2 \leq r \leq k-2$, and use proof by contradiction to show that a run having such a value of r cannot occur. We first make the following observation. By assumption, $2 \leq r \leq k-2$, so we know that α and $\pi - \alpha$ both have at least two elements. Since the first two elements of α are i and $i+j$, $i+y \in I$ for $0 \leq y \leq j$, and thus $r \geq j+1$. Similarly, $k - r \geq j+1$. Thus both the run α and the co-run $\pi - \alpha$ contain at least $j+1$ elements.

<u>Claim</u>. The j elements of the co-run $\pi - \alpha$, with values $r+t, 0 \leq t \leq j-1$, have predecessors belonging to the run α.

> Let $i + (r+t')j$ represent the element in the sequence $\pi - \alpha$ whose value is $r+t$, for $0 \leq t \leq j-1$ and $0 \leq t' \leq k-r-1$. Then for each element $i+(r+t')j$, the predecessor of $i+(r+t')j$ is $r+t-j$. We now show that for $0 \leq t \leq j-1$, $0 \leq r+t-j \leq r-1$, and therefore the predecessor of $i+(r+t')j$ is in α.
>
> We use the inequalities $r \geq j+1$ and $t \leq j-1$ to obtain the following:
>
> $$\begin{aligned} r \geq j+1 &\Rightarrow t - j + r \geq t - j + j + 1 = t + 1 \geq 0 \\ &\Rightarrow r + t - j \geq 0 \end{aligned}$$
>
> $$\begin{aligned} t \leq j-1 &\Rightarrow r - j + t \leq r - j + j - 1 = r - 1 \\ &\Rightarrow r + t - j \leq r - 1 \end{aligned}$$
>
> Therefore for j of the elements of the co-run $\pi - \alpha$, the predecessor of that element belongs to α instead of $\pi - \alpha$.

But we know that exactly one of the elements in a co-run can have its predecessor belonging to the run. Since $2 \leq j \leq k-2$, this yields a contradiction. Therefore our assumption was incorrect, and the only possible runs have length equal to 1 or $k-1$. □

Lemma 7 2 2 *Let $k \geq 5$ For a generalized Petersen graph $GP(k, j)$ with $gcd(k, j) = 1$ and $j \neq \pm 1$,*

$$N_4(GP(k, j)) = n(m-3) + n + \tfrac{n}{2}$$

Proof By Corollary 4 3 4, we need only show that for a cycle permutation graph (C_k, σ), with $\sigma = [0 \; j \; 2j \quad (k-1)j]$,

$$N_4((C_k, \sigma)) = n(m-3) + n + \tfrac{n}{2}$$

Let the vertex and edge sets of (C_k, σ) be as follows

$$V((C_k, \sigma)) = \mathcal{A} \cup \mathcal{B} = \{a_0, a_1, \quad , a_{k-1}\} \cup \{b_0, b_1, \quad , b_{k-1}\}, \text{ and}$$
$$E((C_k, \sigma)) = \{(a_i, a_{i+1}), (b_i, b_{i+1}), (a_i, b_{\sigma(i)}) \;\; 0 \leq i \leq k-1\}$$

From the proof of Lemma 5 2 4, we know that (C_k, σ) has $n(m-3)$ 4-edge cuts which are not minimal We also know that (C_k, σ) has k 4-edge minimal cuts with $(X, \overline{X})$ containing two outer cycle edges and two permutation edges, and k 4-edge minimal cuts with $(X, \overline{X})$ containing two inner cycle edges and two permutation edges We need only enumerate the number of 4-edge minimal cutsets with $(X, \overline{X})$ containing two outer cycle edges and two inner cycle edges But these cuts are precisely those which result in both $(A, \overline{A})$ and $(B, \overline{B})$ being nontrivial bipartitions By Corollary 5 2 2, we need only count the runs of π, where $\pi = (0 \; j \; 2j \quad (k-1)j)$ By Lemma 7 2 1, π has no runs of length $2 \leq r \leq k-2$, however it does have k runs of length one Therefore

$$\begin{aligned} N_4((C_k, \sigma)) &= n(m-3) + n + \tfrac{n}{2} \\ &= N_4(GP(k, j)) \end{aligned}$$

□

Therefore by Lemma 5 2 6, the generalized Petersen graphs $GP(k, j)$, with $k \geq 5$, $gcd(k, j) = 1$, and $j \neq \pm 1$, minimize the number of 4-edge cuts over all 3-regular

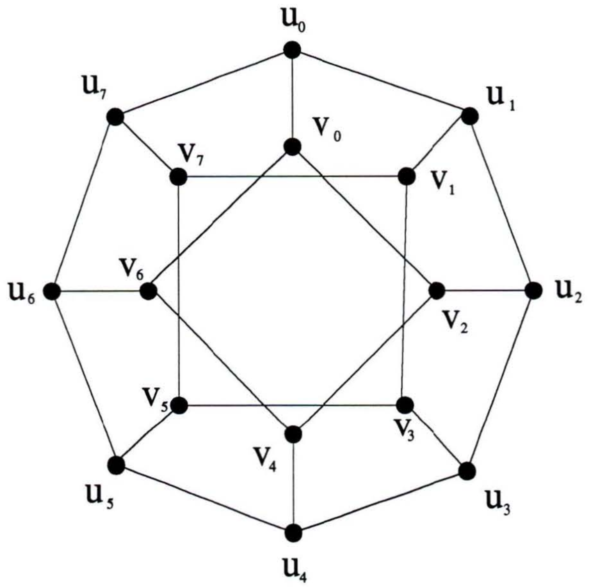

Figure 7.1: Generalized Petersen graph $GP(8,2)$

super-λ graphs. This is not necessarily the case when j divides k, as the following example illustrates.

Example: Consider the graph $GP(8,2)$, as shown in Figure 7.1. Any set of edges containing three edges incident to a single vertex and an additional edge is a cutset; there are $n(m-3)$ such 4-edge cuts. There are also m 4-edge cutsets corresponding to the cutset $(X,\overline{X})$, where, without loss of generality, $G[X] = \{(u,v)\}$, for $(u,v) \in E(GP(k,j))$. There are also two additional 4-edge cutsets: $(X_1,\overline{X_1}) = \{(u_i,v_i) : i = 0,2,4,6\}$, and $(X_2,\overline{X_2}) = \{(u_i,v_i) : i = 1,3,5,7\}$. Therefore

$$\begin{aligned} N_4(GP(8,2)) &= n(m-3)+m+2 \\ &= n(m-3)+n+\tfrac{n}{2}+2 \\ &> n(m-3)+n+\tfrac{n}{2}. \end{aligned}$$

From the proof of Lemma 7.2.1, and from Lemma 7.2.2, which established a lower bound on N_4, we know that the cycle permutation graph (from our family) minimizes the number of 4-edge cutsets over all super-λ 3-regular graphs. This subclass of cycle permutation graphs has cut frequency vectors which are lexicographically smaller than those of 3-regular connected circulants. This subclass is an infinite family, and with the exception of n equal to twelve, at least one member of this family exists for all n greater than or equal to ten.

Chapter 8

Future Research

While working on this thesis, many questions arose which are related to the topic of network reliability and r-regular graphs. The following sections describe some of the problems which we find interesting.

8.1 Reliability of 3-regular Graphs

8.1.1 Improve Bounds

Although we now have an exact lower bound for the number of 4-edge cutsets of a super-λ graph G, we do not have a tight upper bound for $N_4(G)$. When the class of super-λ graphs is restricted to cycle permutation graphs, we have both upper and lower bounds for $N_4((C_k, \sigma))$. We know that this upper bound is an exact one for the cycle permutation graph $(C_k, \mathcal{I})$, and that the lower bound is met when $\sigma = [0\ j\ 2j\ \ldots\ (k-1)j]$, for $gcd(k, j) = 1, 1 < j < k-1$, and $j \neq \frac{k}{2}$. By Corollary 6.2.2, the upper bound for the number of 4-edge cutsets for (C_k, σ) is also an exact one for any connected 3-regular circulant. On the basis of these theoretical results, and also on the computer results which were obtained for n equal to 10,12,14, and 16, we make the following conjectures.

Conjecture 8.1.1.1 *For any 3-regular super-λ graph G with $n \geq 10$ and G not isomorphic to $(C_k, \mathcal{I})$,*

$$N_4(G) \;<\; n(m-3) + n + \frac{n}{4}\;(\tfrac{n}{2} - 1)$$

If the above conjecture is true, then $(C_k, \mathcal{I})$ has the lexicographically largest cut frequency vector amongst all super-λ graphs which are not isomorphic to $(C_k, \mathcal{I})$. We suspect that this result can be expanded to the following conjecture.

Conjecture 8.1.1.2 *For $0 \leq i \leq m$, $N_i((C_k, \mathcal{I}))$ is less than or equal to $N_i(G)$, for any super-λ 3-regular graph G which is not isomorphic to $(C_k, \mathcal{I})$.*

Although our theoretical results only show that 3-regular connected circulants have cut frequency vectors which are lexicographically larger than those of any other cycle permutation graph (C_k, σ), where (C_k, σ) is not isomorphic to $(C_k, \mathcal{I})$, the computer results indicate a stronger result, as reflected in the next conjecture.

Conjecture 8.1.1.3 *For $0 \leq i \leq m$ and G a connected 3-regular circulant, $N_i(G)$ is less than or equal to $N_i(H)$, for any super-λ 3-regular graph H, where H is not isomorphic to $(C_k, \mathcal{I})$.*

8.1.2 Characterization of More Reliable Graphs

Since exact lower bounds exist for the number of 4-edge cutsets of a 3-regular super-λ graph, we can compare two super-λ graphs based on their values of N_4. However, since we do not have exact lower bounds for N_i, for $i \geq 5$, we cannot show which graphs are likely to be uniformly-most reliable for any value of n, or even the existence of such a graph. Even when this problem is restricted to quasi-prisms, we are unable to fully characterize such a graph. We do suspect that the more reliable quasi-prisms take the following form.

Conjecture 8 1 2 1 *The quasi-prisms* $Q_k[\pi]$ *which minimize* N_i *amongst all quasi-prisms, for* $0 \leq i \leq m$, *have* $\pi = (0\ j \quad (k-1)j)$, *with* $gcd(k,j) = 1, 1 < j < k-1$, *and* $j \neq \frac{k}{2}$

In general, the most reliable super-λ graph is not necessarily a quasi-prism Although computer results show that for n equal to 10,12, and 16, the uniformly-most reliable graph is a quasi-prism, for n equal to 14 the uniformly-most reliable graph is the Heawood graph, which is not a quasi-prism (An illustration of the Heawood graph can be found in [10]) We suspect that for larger values of n, while some quasi-prisms may have a very regular structure, there are many other graphs which are more reliable A natural question is to characterize those graphs which are among the most reliable 3-regular graphs

The *girth* of a graph G is the order of the smallest cycle in G A (g,r)-*cage* is a regular graph of degree r and girth g, with the least number of vertices It is known that for any positive integer values of g and r, with both g and r greater than or equal to three, a (g,r)-cage exists For $(10,15)$-graphs, the uniformly-most reliable graph is the Petersen graph, which is also the $(5,3)$-cage For $(14,21)$-graphs, the uniformly-most reliable graph is the Heawood graph, which is also the $(6,3)$-cage We suspect that the following conjecture has already been made, but we state it here for completeness

Conjecture 8 1 2 1 *Let* G *be a* (g,r)-*cage (having order* n*) Then* $N_i(G)$ *is less than or equal to* $N_i(H)$ *for all* $(n, \frac{rn}{2})$-*graphs* H, *for* $0 \leq i \leq m$

A chordal ring, defined previously in Chapter 1 (see p 7) and redefined below, is a 3-regular graph with n vertices on a cycle of order n, labelled consecutively from 0 to $n-1$, and $\frac{n}{2}$ chords $(i, i+h)$, for i and h odd, $1 < h < n-1$ Since the chordal ring $C(14,5)$ is the uniformly-most reliable graph on 14 vertices, and $C(16,5)$ is the uniformly-most reliable graph on 16 vertices, chordal rings appear to be a reasonable family to consider for reliable networks. However, from [16] (p. 490), we know that chordal rings either have girth equal to four or six. Since the (7,3)-cage is of order 24, it is unlikely that any chordal ring has a cut frequency vector which is lexicographically small enough to be near the top of the list for n greater than or equal to 24, unless the largest girth amongst those $(n, \frac{3n}{2})$-graphs happens to be equal to four or six. However, an interesting approach would be to generalize the chordal rings to contain all graphs consisting of a cycle of order n and a set of $\frac{n}{2}$ matching edges. This is the class of all Hamiltonian 3-regular graphs.

This leads us to a related question: can we construct quasi-prisms of arbitrarily large girth? Connected 3-regular circulants of order ten or greater always have girth equal to four. Cycle permutation graphs, generalized Petersen graphs, and quasi-prisms appear to be of variable girth, which increases in range as the value of n increases, but these results have only been observed experimentally for generalized Petersen graphs (from our family) of order 58 and less. Based on these observations, we make the following conjecture.

Conjecture 8.1.2.2 *For all $g \geq 3$, there exists some quasi-prism $Q_k[\pi]$, some cycle permutation graph $(C_{k'}, \sigma)$, and some generalized Petersen graph $GP(k'', j)$, having girth equal to g, where k, k', and k'' are not necessarily equal.*

8.1.3 Spanning Trees

Although the computer results obtained for n equal to 10,12,14, and 16 include all of the values of N_i, for $0 \leq i \leq m$, the theoretical results in this thesis only apply to N_j for $0 \leq j \leq 4$. Therefore if we have two graphs with cut frequency vectors not differing in the first five entries, we cannot yet differentiate between the two of them as to which will have the smaller cut frequency vector.

This observation leads us to consider a different approach, which is described below. A subgraph H of G is *spanning* if $V(H) = V(G)$. A *tree* is a connected graph $G = (V, E)$ with $|E(G)| = |V(G)| - 1$ A subgraph H of G is a *spanning tree* if H is spanning and also a tree Let $C_{m-i}(G)$ denote the number of $(m - i)$-edge connected spanning subgraphs of an (n, m)-graph G There are $\binom{m}{i}$ ways to remove i edges from a graph G The removal of these edges either disconnects the graph, resulting in an i-edge cutset, or leaves the graph connected, resulting in an $(m - i)$-edge connected spanning subgraph of G From this observation, we see that for an (n, m)-graph G, and $0 \le i \le m$,

$$\binom{m}{i} = N_i(G) + C_{m-i}(G)$$

In Chapter 1, the reliability polynomial of a graph G is expressed in terms of the number of i-edge cutsets of G We can also express $Rel(G)$ in terms of the number of i-edge connected spanning subgraphs of G, as follows

$$Rel(G) = \sum_{i=0}^{m} C_i(G) \; p^i \; (1-p)^{m-i}$$

Note that $C_{n-1}(G)$, the number of $(n-1)$-edge connected spanning subgraphs of G, corresponds to the number of spanning trees of G The *connected spanning subgraph vector* of a graph G is given by $(C_0, C_1, C_2, \ldots, C_m)$ Note that C_i equals zero for $0 \le i \le n-2$, since a spanning tree requires $n-1$ edges A natural place to continue this work is to explore the entries of the connected spanning subgraph vectors for 3-regular graphs For the first two nonzero entries, C_{n-1} and C_n, this would involve finding upper and lower bounds on the number of spanning trees and the number of connected spanning unicyclic subgraphs for 3-regular graphs

8.2 Girth

The girth of a graph is defined above, and it has been conjectured that the (g, r)-cages, with order n, are the uniformly-most reliable $(n, \frac{rn}{2})$-graphs For 3-regular graphs with n equal to 10,12,14, and 16, a correlation exists between girth and reliability. Those 3-regular graphs having larger girth tend to be among the more reliable 3-regular graphs. In fact, for the above values of n, there is no case in which some graph G_1 is more reliable than another graph G_2, where the girth of G_1 is strictly lower than the girth of G_2. If this observation is true for r-regular graphs in general, then it is sufficient to show that some r-regular graph G_1 has girth strictly larger than some other r-regular graph G_2 to know that G_1 is more reliable than G_2.

To further examine this correlation between girth and reliability, we expand the notion to the enumeration of cycles of a particular size in a graph G. Letting g_i denote the number of cycles of size i in G, we define the *cycle vector* of G to be the $(n-2)$-tuple $(g_3, g_4, \ldots, g_n)$. We calculated the cycle vectors of 3-regular super-λ graphs of order ten and twelve. Ordering these cycle vectors lexicographically results in an ordering of the graphs which is almost the same as the ordering determined by the cut frequency vectors, with the most reliable graph corresponding to the cycle vector which is lexicographically least. A case in which this ordering does not match that of the cut frequency vectors is given below, for n equal to twelve. Although Table B.6 shows that $(C_6, [014523])$ has a lexicographically smaller cut frequency vector than $(C_6, [013542])$, the following cycle vectors were obtained.

$$(C_6, [014523]) \quad (g_3 = 0, g_4 = 3, g_5 = 0, \ldots), \text{ and}$$
$$(C_6, [013542]) \quad (g_3 = 0, g_4 = 2, g_5 = 6, \ldots).$$

Thus $(C_6, [013542])$ has a lexicographically smaller cycle vector than $(C_6, [014523])$, contrary to our expectation. An interesting approach would be to redefine the cycle vectors in terms of the number of chordless cycles of size i in G, and again perform a lexicographic ordering to see if the ordering corresponds closer to that of the cut frequency vectors.

8.3 r-Regular Graphs

Although we only examine 3-regular graphs in this thesis, a natural extension is to explore r-regular graphs, for $r \geq 4$. The following are some questions that seem interesting.

Circulants have been proposed as a good network topology for r-regular graphs in general. With few exceptions, they are almost always super-λ, and therefore have cut frequency vectors which are lexicographically smaller than those of graphs which are not super-λ. Results from this thesis show that 3-regular circulants have cut frequency vectors which are lexicographically larger than those of cycle permutation graphs which are not isomorphic to some circulant, but this leads to a natural question. Are the cut frequency vectors of r-regular circulants always amongst the largest for super-λ $(n, \frac{rn}{2})$-graphs for any value of r, or does that only hold for certain values of r?

Once it was discovered that the cut frequency vectors of 3-regular circulants are larger than that of any cycle permutation graph which is not a circulant, we examined the 3-regular graphs to find out which graphs have cut frequency vectors which are amongst the lexicographically smallest. Cycle permutation graphs which are isomorphic to generalized Petersen graphs, in the case where k and j are relatively prime, tend to have small cut frequency vectors, excluding $(C_k, \mathcal{I})$, but these are 3-regular graphs with no clear extension to r-regular graphs in general. Their 4-regular counterpart might consist of two cycles connected by two sets of matching edges, where each collection of matching edges is described by some permutation. It might also consist of three cycles, every two of which are connected by a set of matching edges, where the collection of matching edges is described by some permutation. Although various possibilities have been considered, we have yet to obtain any computer results which give the cut frequency vector for some 4-regular graph, for any value of n. Therefore a related question is which classes of graphs tend to be among the most reliable super-λ 4-regular graphs? What about for r-regular super-λ graphs, for $r \geq 5$? The least reliable super-λ graphs?

8.4 Other Criteria

The *geodesic* is a shortest path between a pair of vertices. The longest geodesic over all pairs of vertices in a graph G is the *diameter* of G. Boesch and Wang suggest in [6] that any model of network reliability which does not take diameter into account is inadequate. One of the reasons why circulants have been proposed as a good choice for a reliable network topology is that they have a great variation in their diameters. Therefore another area of future research is to examine the diameters of quasi-prisms, especially for the subclass of generalized Petersen graphs described in Lemma 7.2.2.

It is possible for one topology to be more reliable than another, and still have the latter be a more desirable choice for a network topology. This is because routing algorithms are required for working with a given topology as a network. A relatively unreliable graph may have simple routing algorithms, which could make it more desirable than some other more reliable topology. Therefore another area for further study is to determine routing algorithms for the subclass of generalized Petersen graphs described in Lemma 7.2.2.

Appendix A

Summary of Computer Search

We used the computer to calculate the cut frequency vectors of all 3-regular graphs of order less than or equal to 16. We obtained the 3-regular graphs of order less than or equal to 16 via anonymous ftp from a database maintained by Gordon Royle [19]. MAPLE was used to calculate the cut frequency vectors. We also used the computer to determine graph isomorphisms, this involved using "nauty" (version 1.6), a set of procedures for determining the automorphism group of a graph, and optionally for canonically labelling it. The whole family of nauty programs was written by Brendan McKay of The Australian National University.

Table A 1 Number of 3-regular graphs and super-λ graphs

n	*# 3-regular graphs*	*# super-λ graphs*
4	1	1
6	2	1
8	5	2
10	19	5
12	85	18
14	509	84
16	4082	607

Table A 2 Most and least reliable super-λ graphs for $n = 10, 12, 14, 16$

n	*least reliable super-λs*		*most reliable*
	least	*second least*	*super-λ*
10	$(C_5, [\mathcal{I}])$ $\cong C_{10}\langle 2,5\rangle$	$C_{10}\langle 1,5\rangle$	$(C_5, [02413])$
12	$(C_6, [\mathcal{I}])$	$C_{12}\langle 1,6\rangle$	$(C_6, [024153])$
14	$(C_7, [\mathcal{I}])$ $\cong C_{14}\langle 2,7\rangle$	$C_{14}\langle 1,7\rangle$	$C(14,5)*$
16	$(C_8, [\mathcal{I}])$	$C_{16}\langle 1,8\rangle$	$(C_8, [03614725])$ $\cong C(16,5)*$

The $*$ emphasizes those graphs which are chordal rings

Table A.3: Super-λ properties of 3-regular graphs

Graph	*Super-λ?*	*Result*
$Q_k[\pi]$	see (1)	Theorem 3.2.4, p. 23
(C_k, σ)	yes, if $n \geq 8$	Lemma 5.1.2, p. 44
$C_n\langle a, \frac{n}{2}\rangle$	yes, if $C_n\langle a, \frac{n}{2}\rangle$ $\not\cong C_n\langle 2, 3\rangle$	Lemma 6.1.2, p. 57
$GP(k, j)$	yes, iff $n \geq 8$ and $k \neq 3j$	Lemma 7.1.2, p. 64

(1) Let $n \geq 8$ and $\pi \in Sym(k)$, where π is expressed in cycle notation. Then the quasi-prism $Q_k[\pi]$ is super-λ if and only if

(i) π does not contain any cycles of order less than or equal to three, and

(ii) there is no nonempty bipartition of the cycles of π inducing a bipartition $(I, \overline{I})$ of the integers $\{0, 1, \ldots, k-1\}$ such that I is contiguous or almost contiguous.

Appendix B

Cycle Permutation Graphs

Appendices B through D contain tables of cut frequency vectors for cycle permutation graphs, generalized Petersen graphs, and circulants. In these tables, the values for N_0, N_1, N_2, and N_3 are omitted, since they are identical for (n,m)-graphs which are super-λ. The values are as follows:

$$\begin{aligned} N_0 &= 0, \\ N_1 &= 0, \\ N_2 &= 0, \\ N_3 &= n. \end{aligned}$$

Exceptions occur in Table C.2, since $GP(6,2)$ is not super-λ, and in Tables B.1 through B.4, since not all of the cycle permutation graphs are super-λ and since the values of n are so small. In these tables, all values of $N_i(G)$ are provided for clarity.

The ranking of each graph is also provided in the following tables, where the graphs are ranked according to the ordering of the cut frequency vectors. The ordering chosen is performed lexicographically on the values of N_i, from the smallest values of i to the largest values of i.

Table B 1 Numbers of cutsets of the cycle permutation graph for $n = 2$

σ	$N_i((C_k, \sigma))$			*rank*
	N_0	N_1	N_2	
0	0	1	2	1

Table B 2 Numbers of cutsets of the cycle permutation graph for $n = 4$

σ	$N_i((C_k, \sigma))$				*rank*
	N_0	N_1	N_2	N_3	
01	0	0	0	4	1

Table B 3 Numbers of cutsets of the cycle permutation graph for $n = 6$

σ	$N_i((C_k, \sigma))$					*rank*
	N_0	N_1	N_2	N_3	N_4	
012	0	0	0	7	51	2

This cycle permutation graph is ranked second, since the other 3-regular graph of order six, $K_{3,3}$, is super-λ

Table B.4: Numbers of cutsets of cycle permutation graphs for $n = 8$

σ	$N_i((C_k, \sigma))$						*rank*
	N_0	N_1	N_2	N_3	N_4	N_5	
0132	0	0	0	8	86	400	1
0123	0	0	0	8	87	408	2

Table B.5: Numbers of cutsets of cycle permutation graphs for $n = 10$

σ	$N_i((C_k, \sigma))$			*rank*
	N_4	N_5	N_6	
02413	135	831	3005	1
01342	137	855	3085	2
01243	138	867	3124	3
01234	140	891	3200	5

Table B.6: Numbers of cutsets of cycle permutation graphs for $n = 12$

σ	$N_i((C_k, \sigma))$				*rank*
	N_4	N_5	N_6	N_7	
024153	198	1520	7120	22024	1
013524	199	1536	7215	22256	3
014523	201	1566	7376	22608	7
013542	201	1568	7403	22654	8
015342	202	1584	7496	22824	9
013254	202	1584	7496	22864	10
012453	202	1584	7496	22874	11
012543	203	1598	7562	23004	13
012354	204	1616	7680	23244	16
012345	207	1662	7926	23724	18

Table B.7: Numbers of cutsets of cycle permutation graphs for $n = 14$

σ	$N_i((C_k,\sigma))$					*rank*
	N_4	N_5	N_6	N_7	N_8	
0246135	273	2506	14294	56008	155267	5
0251364	273	2507	14310	56093	155295	6
0241635	273	2508	14328	56210	155599	7
0146253	274	2526	14458	56700	156478	10
0135264	274	2529	14511	57031	157239	14
0136425	274	2530	14528	57130	157410	17
0135624	275	2546	14624	57418	157923	22
0136254	275	2547	14641	57517	158082	24
0146352	275	2549	14675	57663	158134	30
0134625	275	2549	14676	57729	158535	31
0124635	276	2570	14856	58464	159714	43
0143652	277	2586	14951	58720	160002	47
0136452	277	2588	14985	58874	160150	51
0135642	277	2588	14986	58888	160247	52
0125634	278	2604	15080	59208	160930	55
0124653	278	2607	15131	59455	161223	59
0135462	278	2608	15148	59502	161126	60
0156342	278	2608	15148	59532	161250	61
0132564	278	2608	15149	59556	161415	62
0126453	279	2625	15258	59857	161671	66
0125643	279	2625	15260	59927	162043	68

Table B.8: Numbers of cutsets of cycle permutation graphs for $n = 14$ (cont.)

σ	$N_i((C_k, \sigma))$					*rank*
	N_4	N_5	N_6	N_7	N_8	
0124563	279	2627	15294	60129	162423	70
0123564	280	2647	15455	60733	163311	74
0132465	280	2648	15472	60768	163170	76
0124365	280	2648	15472	60808	163362	77
0123654	281	2663	15548	60989	163659	78
0123465	283	2705	15902	62353	165647	82
0123456	287	2779	16436	64177	168203	84

Table B.9: Numbers of cutsets of cycle permutation graphs for $n = 16$

σ	$N_i((C_k, \sigma))$						*rank*
	N_4	N_5	N_6	N_7	N_8	N_9	
03614725	360	3840	25696	120240	413559	1058672	1
02573164	360	3844	25781	120966	416823	1065584	9
02573614	360	3846	25824	121340	418314	1067744	19
02461375	360	3847	25847	121555	419439	1070582	25
02517364	360	3848	25868	121730	420086	1071340	32
02537164	360	3848	25868	121730	420090	1071296	33
02471635	360	3848	25868	121734	420134	1071554	34
02416375	360	3850	25911	122104	421599	1073732	42
02415736	360	3851	25933	122299	422479	1075476	45
02417536	360	3852	25956	122514	423498	1077596	47
01537264	361	3868	26034	122492	422319	1075184	54
01462735	361	3868	26035	122514	422496	1075844	55
01462753	361	3870	26078	122886	423959	1077962	74
01473625	361	3870	26078	122886	423965	1077962	75
01357264	361	3871	26099	123059	424690	1079322	84
01573624	361	3872	26119	123214	425108	1079168	90
01375264	361	3872	26119	123214	425128	1079372	91
01364725	361	3872	26120	123234	425267	1079756	94
01475263	361	3872	26120	123236	425265	1079636	96
01357246	361	3872	26122	123272	425707	1081446	98
01374625	361	3874	26163	123604	426814	1082396	119
01357426	361	3875	26185	123797	427680	1084092	123
01362574	361	3875	26185	123799	427678	1084036	124
02574613	361	3876	26206	123900	427367	1081712	128
01362475	361	3876	26206	123972	428327	1084956	129

Table B 10 Numbers of cutsets of cycle permutation graphs for $n = 16$ (cont)

σ	$N_i((C_k,\sigma))$						rank
	N_4	N_5	N_6	N_7	N_8	N_9	
01352746	361	3876	26207	123992	428466	1085332	130
01537624	362	3892	26285	123974	427299	1082864	140
01472563	362	3892	26288	124036	427539	1082864	146
01365274	362	3894	26329	124366	428927	1085540	161
01362745	362	3894	26330	124384	429114	1086266	163
01367425	362	3895	26350	124539	429630	1086802	172
01537462	362	3896	26370	124640	429612	1085436	184
01463752	362	3896	26370	124642	429644	1085642	185
01374265	362	3896	26372	124732	430400	1087900	191
01356274	362	3896	26372	124734	430550	1088714	192
01362754	362	3896	26373	124754	430647	1088866	196
01352764	362	3898	26416	125120	432058	1090876	220
01573642	362	3899	26435	125203	431921	1089252	223
01475362	362	3900	26457	125396	432727	1090704	227
01374526	362	3900	26457	125446	433251	1092464	228
01346275	362	3900	26458	125466	433436	1093116	232
01367254	363	3916	26539	125512	432690	1091674	243
01367245	363	3916	26539	125512	432788	1092212	244
01436725	363	3916	26542	125570	432970	1092032	245
01257364	363	3918	26577	125750	433218	1091474	255
01376245	363	3918	26582	125878	434199	1094192	263
01365724	363	3919	26600	125943	433895	1092200	270
01357624	363	3919	26601	125959	434056	1092762	272
01347265	363	3919	26602	126031	434715	1094832	274
01375624	363	3920	26621	126114	434506	1092908	281

Table B.11. Numbers of cutsets of cycle permutation graphs for $n = 16$ (cont.)

σ	$N_i((C_k,\sigma))$						*rank*
	N_4	N_5	N_6	N_7	N_8	N_9	
01372564	363	3920	26622	126134	434647	1093292	288
01367524	363	3920	26622	126134	434695	1093606	289
01372645	363	3920	26623	126152	434842	1094042	293
01346725	363	3920	26625	126242	435606	1096320	296
01574623	363	3922	26664	126468	435703	1094512	311
01354726	363	3922	26665	126498	436126	1095756	312
01372465	363	3922	26666	126518	436237	1095996	313
01357462	363	3923	26685	126601	436050	1094704	319
01247536	363	3924	26706	126848	437627	1098504	326
01246375	363	3924	26707	126868	437742	1098716	327
01246735	364	3942	26830	127260	438392	1099634	355
01247635	364	3943	26850	127413	438898	1100186	364
01257463	364	3944	26870	127512	438720	1098704	379
01247365	364	3944	26872	127604	439622	1101148	383
01245736	364	3947	26936	128139	441732	1104492	404
01746352	364	3948	26956	128192	441154	1101704	407
01325746	364	3948	26956	128230	441552	1103104	408
01547623	365	3964	27036	128296	441095	1102704	414
01437562	365	3966	27079	128618	442192	1103634	433
01367452	365	3966	27079	128620	442074	1103392	434
01436752	365	3966	27080	128636	442305	1103958	438
01375462	365	3968	27119	128882	442738	1103552	451
01365742	365	3968	27120	128902	442927	1104182	454
01357642	365	3968	27122	128938	443201	1104966	456
01257634	366	3987	27264	129529	444886	1107796	472

Table B.12: Numbers of cutsets of cycle permutation graphs for $n = 16$ (cont.)

σ	$N_i((C_k, \sigma))$						rank
	N_4	N_5	N_6	N_7	N_8	N_9	
01465723	366	3988	27285	129654	444939	1106864	479
01326745	366	3988	27287	129728	445619	1108976	481
01254763	366	3988	27288	129756	445651	1108592	483
01247563	366	3990	27326	129980	446164	1108736	494
01246753	366	3990	27328	130018	446466	1109642	496
01375642	366	3991	27347	130101	446361	1108326	497
01325764	366	3991	27348	130161	446868	1109866	498
01326574	366	3991	27349	130179	447029	1110342	500
01364572	366	3992	27368	130272	446898	1109152	503
01356472	366	3992	27368	130274	446926	1109306	504
01354762	366	3992	27370	130340	447466	1110432	506
01342675	366	3992	27370	130350	447632	1111154	508
01435762	366	3992	27371	130358	447695	1110944	509
01342756	366	3992	27371	130368	447745	1111462	510
01674523	366	3992	27372	130408	447751	1110896	513
01235746	366	3994	27410	130704	449162	1113704	514
01256734	367	4008	27451	130478	447808	1112276	517
01735462	367	4012	27530	130916	448231	1109936	530
01274653	367	4012	27531	130988	448692	1111424	531
01327564	367	4012	27532	130998	448923	1112036	532
01267453	367	4012	27532	131038	449195	1112752	533
01247653	367	4012	27532	131058	449451	1113584	535
01267534	367	4012	27534	131076	449575	1113704	537
01326754	367	4012	27535	131094	449712	1114304	538
01257643	367	4012	27535	131104	449694	1114106	539

Table B.13 Numbers of cutsets of cycle permutation graphs for $n = 16$ (cont.)

σ	$N_i((C_k,\sigma))$						*rank*
	N_4	N_5	N_6	N_7	N_8	N_9	
01246573	367	4014	27575	131366	450370	1114266	544
01245763	367	4014	27577	131444	450956	1115808	547
01274563	368	4032	27696	131740	450687	1113968	556
01236745	368	4032	27700	131876	451959	1117424	558
01256743	368	4032	27702	131896	451983	1117424	560
01235764	368	4036	27780	132434	453608	1118922	568
01324675	368	4039	27843	132883	454829	1119862	573
01243675	368	4039	27844	132941	455374	1121234	574
01675342	368	4040	27865	133050	455419	1120304	575
01325476	368	4040	27865	133090	455675	1121136	576
01237564	369	4056	27941	133058	454724	1119456	581
01236754	369	4056	27945	133176	455614	1121792	583
01275643	369	4058	27985	133426	455954	1121098	585
01243765	369	4058	27987	133504	456654	1122874	586
01235674	369	4060	28028	133844	458007	1124976	587
01237654	371	4100	28352	135192	461071	1128304	593
01234675	371	4106	28477	136158	464264	1132106	596
01324576	371	4108	28518	136404	464695	1131888	598
01243576	371	4108	28518	136404	464751	1131944	599
01235476	371	4108	28518	136444	465079	1132784	600
01234765	372	4124	28600	136568	464971	1132784	601
01234576	375	4196	29329	140394	475388	1144544	605
01234567	380	4304	30296	144944	486799	1156976	607

Appendix C

Generalized Petersen Graphs

Table C.1: Number of cutsets of generalized Petersen graphs for $n = 10$

$GP(k,j)$	$N_i(GP(k,j))$			rank
	N_4	N_5	N_6	
$GP(5,2)$	135	831	3005	1
$GP(5,1)$	140	891	3200	5

Table C.2: Number of cutsets of generalized Petersen graphs for $n = 12$

$GP(k,j)$	$N_i(GP(k,j))$								rank
	N_0	N_1	N_2	N_3	N_4	N_5	N_6	N_7	
$GP(6,1)$	0	0	0	12	207	1662	7926	23724	18
$GP(6,2)$	0	0	0	14	237	1842	8473	24474	35

Note that for $n = 12$, the highest ranked generalized Petersen graph is $GP(6,1)$, which has the lexicographically largest cut frequency vector amongst the super-λ graphs. The other generalized Petersen graph, $GP(6,2)$, is not super-λ.

Table C.3: Number of cutsets of generalized Petersen graphs for $n = 14$

$GP(k,j)$	$N_i(GP(k,j))$					rank
	N_4	N_5	N_6	N_7	N_8	
$GP(7,2)$	273	2506	14294	56008	155267	5
$GP(7,1)$	287	2779	16436	64177	168203	84

note: $GP(7,3) \cong GP(7,2)$

Table C.4: Number of cutsets of generalized Petersen graphs for $n = 16$

$GP(k,j)$	$N_i(GP(k,j))$						rank
	N_4	N_5	N_6	N_7	N_8	N_9	
$GP(8,3)$	360	3840	25696	120240	413559	1058672	1
$GP(8,2)$	362	3904	26548	126280	437098	1100136	237
$GP(8,1)$	380	4304	30296	144944	486799	1156976	607

Appendix D

Circulant Graphs

Table D.1: Numbers of cutsets of circulants for $n = 10$

$C_n\langle a, \frac{n}{2}\rangle$	$N_i(C_n\langle a, \frac{n}{2}\rangle)$			*rank*
	N_4	N_5	N_6	
$C_{10}\langle 1,5\rangle$	140	890	3190	4
$C_{10}\langle 2,5\rangle$	140	891	3200	5

Table D.2: Numbers of cutsets of the circulant for $n = 12$

$C_n\langle a, \frac{n}{2}\rangle$	$N_i(C_n\langle a, \frac{n}{2}\rangle)$				*rank*
	N_4	N_5	N_6	N_7	
$C_{12}\langle 1,6\rangle$	207	1662	7925	23712	17

Table D.3: Numbers of cutsets of circulants for $n = 14$

$C_n\langle a, \frac{n}{2}\rangle$	$N_i(C_n\langle a, \frac{n}{2}\rangle)$					rank
	N_4	N_5	N_6	N_7	N_8	
$C_{14}\langle 1, 7\rangle$	287	2779	16436	64176	168189	83
$C_{14}\langle 2, 7\rangle$	287	2779	16436	64177	168203	84

Table D.4: Numbers of cutsets of the circulant for $n = 16$

$C_n\langle a, \frac{n}{2}\rangle$	$N_i(C_n\langle a, \frac{n}{2}\rangle)$						rank
	N_4	N_5	N_6	N_7	N_8	N_9	
$C_{16}\langle 1, 8\rangle$	380	4304	30296	144944	486798	1156960	606

Appendix E

A Nonnatural Isomorphism

This appendix contains an example of two cycle permutation graphs which are isomorphic, but not naturally isomorphic. This isomorphism is explained with figures, as follows. Figure E.1 depicts (C_{13}, [0 11 2 1 4 3 9 7 5 8 6 12 10]), and Figure E.2 depicts (C_{13}, [0 11 2 1 4 3 8 6 9 7 5 12 10]). The cycle of order 13 in (C_{13}, [0 11 2 1 4 3 9 7 5 8 6 12 10]) which is in bold solid line is mapped to the outer cycle of (C_{13}, [0 11 2 1 4 3 8 6 9 7 5 12 10]), and is labelled as such in Figure E.1. The cycle of order 13 in (C_{13}, [0 11 2 1 4 3 9 7 5 8 6 12 10]) which is in thin solid line is mapped to the inner cycle of (C_{13}, [0 11 2 1 4 3 8 6 9 7 5 12 10]), and is labelled as such in Figure E.1. Finally, the set of permutation edges in (C_{13}, [0 11 2 1 4 3 9 7 5 8 6 12 10]) which are in dotted line are mapped to the set of permutation edges of (C_{13}, [0 11 2 1 4 3 8 6 9 7 5 12 10]).

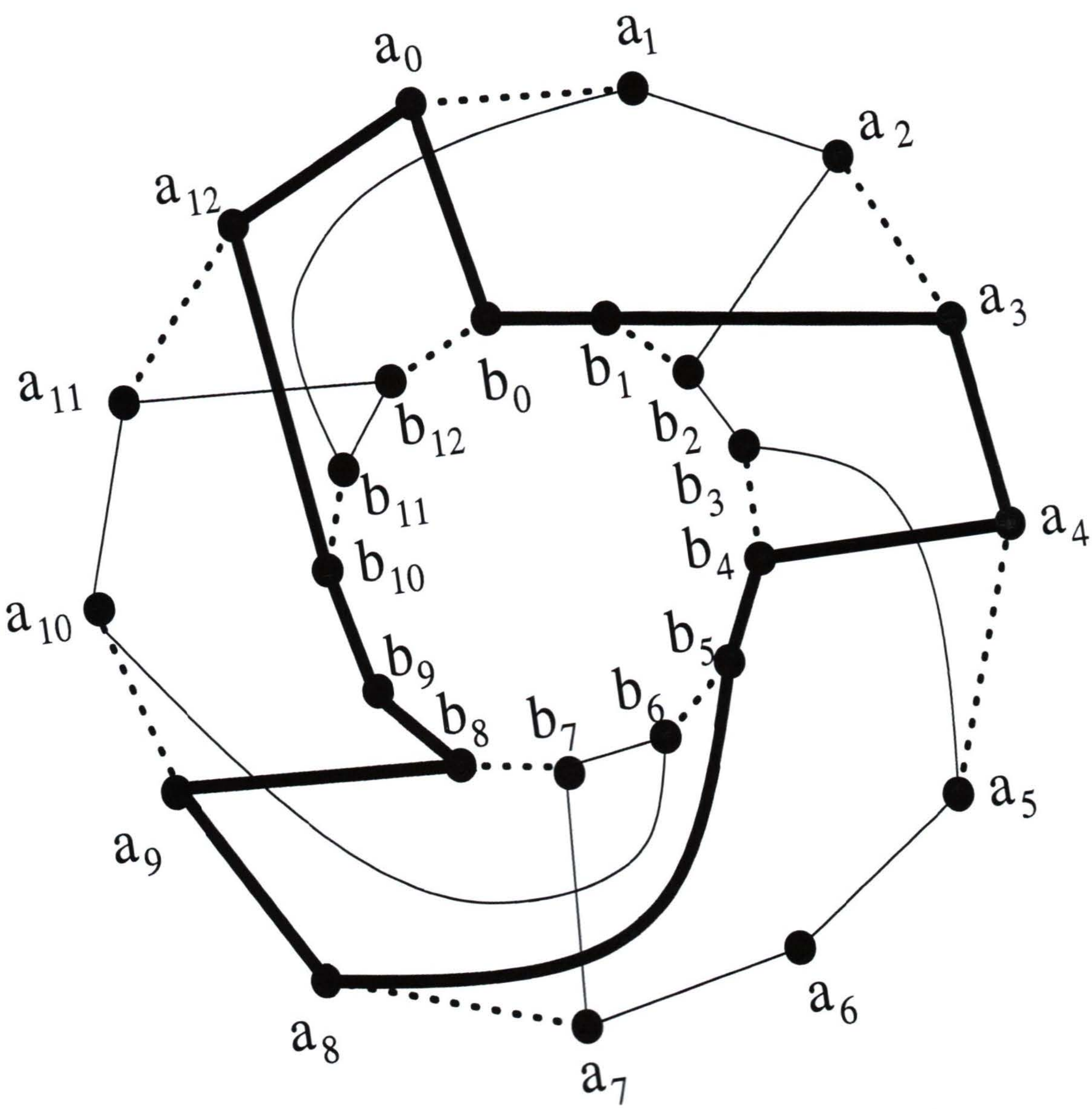

Figure E.1: $(C_{13}, [0\ 11\ 2\ 1\ 4\ 3\ 9\ 7\ 5\ 8\ 6\ 12\ 10])$

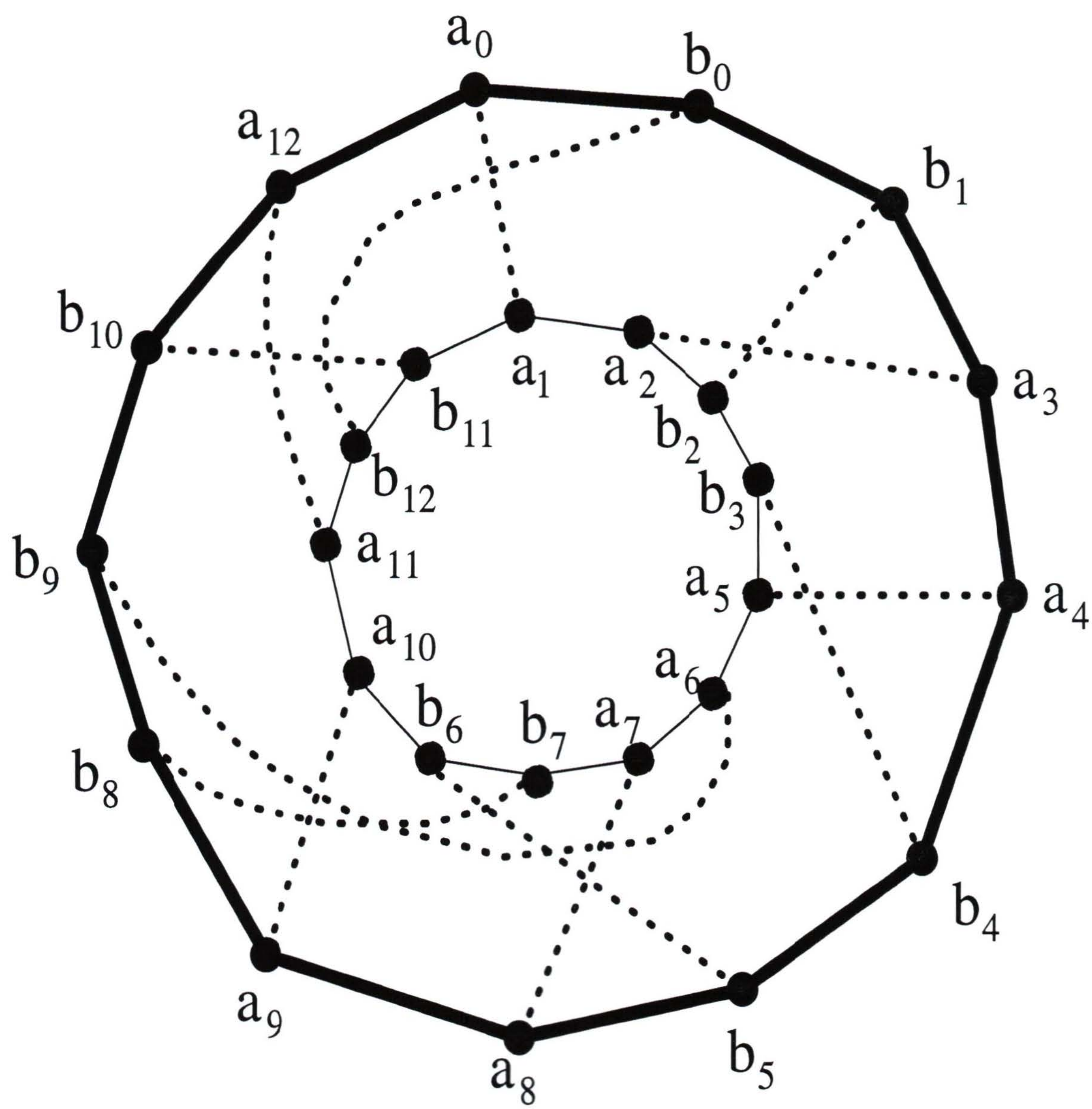

Figure E.2: $(C_{13}, [0\ 11\ 2\ 1\ 4\ 3\ 8\ 6\ 9\ 7\ 5\ 12\ 10])$

Appendix F

Isomorphism Proofs

This appendix contains expanded proofs of some of the lemmas in Chapter 4. The numbering of results in this chapter corresponds to the original numbering of the lemmas in Chapter 4.

Lemma 4.1.1 *For $\sigma \in Sym(k)$, the cycle permutation graph (C_k, σ) is isomorphic to the quasi-prism $Q_k[\pi]$ if $\pi = (\sigma(0)\ \sigma(1) \cdots \sigma(k-1))$.*

Proof: Let the vertex and edge sets of $Q_k[\pi]$ be as follows:

$$\begin{aligned} V(Q_k[\pi]) &= \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and} \\ E(Q_k[\pi]) &= \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \le i \le k-1\}. \end{aligned}$$

Let the vertex and edge sets of (C_k, σ) be as follows:

$$\begin{aligned} V((C_k, \sigma)) &= \mathcal{A} \cup \mathcal{B} = \{a_0, a_1, \ldots, a_{k-1}\} \cup \{b_0, b_1, \ldots, b_{k-1}\}, \text{ and} \\ E((C_k, \sigma)) &= \{(a_i, a_{i+1}), (b_i, b_{i+1}), (a_i, b_{\sigma(i)}) : 0 \le i \le k-1\}. \end{aligned}$$

Let the bijection $f : V((C_k, \sigma)) \to V(Q_k[\pi])$ be defined as follows, for $0 \le i \le k-1$:

$$\begin{aligned} f(a_i) &= t_{\sigma(i)} \text{ and} \\ f(b_i) &= s_i. \end{aligned}$$

We now show that for $(u,v) \in E((C_k,\sigma))$, $(f(u),f(v)) \in E(Q_k[\pi])$. For $(a_i,a_{i+1}) \in E((C_k,\sigma))$,

$$(f(a_i), f(a_{i+1})) = (t_{\sigma(i)}, t_{\sigma(i+1)})$$

Since

$$\pi = (\sigma(0)\ \sigma(1)\ \dots\ \sigma(i)\ \sigma(i+1)\ \dots\ \sigma(k-1)),$$

$\pi(\sigma(i)) = \sigma(i+1)$. Therefore for $j = \sigma(i)$,

$$\begin{aligned}(t_{\sigma(i)}, t_{\sigma(i+1)}) &= (t_{\sigma(i)}, t_{\pi(\sigma(i))}) \\ &= (t_j, t_{\pi(j)}) \in E(Q_k[\pi])\end{aligned}$$

For $(b_i, b_{i+1}) \in E((C_k,\sigma))$,

$$(f(b_i), f(b_{i+1})) = (s_i, s_{i+1}) \in E(Q_k[\pi])$$

For $(a_i, b_{\sigma(i)}) \in E((C_k,\sigma))$ and $j = \sigma(i)$,

$$\begin{aligned}(f(a_i), f(b_{\sigma(i)})) &= (t_{\sigma(i)}, s_{\sigma(i)}) \\ &= (s_j, t_j) \in E(Q_k[\pi])\end{aligned}$$

It follows directly that all of the edges of $Q_k[\pi]$ are included in this mapping □

Lemma 4 2 3 *Any 3-regular connected circulant* $C_n\langle a, \frac{n}{2}\rangle$ *is isomorphic to either* $C_n\langle 1, \frac{n}{2}\rangle$ *or* $C_n\langle 2, \frac{n}{2}\rangle$

Proof Let $C_n\langle a, \frac{n}{2}\rangle$ be a 3-regular connected circulant Then either $gcd(a,n) = 1$ or $gcd(a,n) = h \neq 1$ We prove this result by using case analysis, based on the value of $gcd(a,n)$

<u>Case 1</u> Assume that $gcd(a,n) = h \neq 1$

For this case, h is either even or odd

<u>Subcase (i)</u> Assume that h is odd

We use proof by contradiction to show that this case does not occur From this assumption, we obtain

$$\begin{aligned} gcd(a,n) &= gcd(a, \tfrac{n}{2}) \\ &= gcd(a, \tfrac{n}{2}, n) \\ &= h \\ &\neq 1 \end{aligned}$$

Therefore by Lemma 6 1 1, $C_n\langle a, \frac{n}{2}\rangle$ is disconnected, which contradicts our assumption that $C_n\langle a, \frac{n}{2}\rangle$ is connected

<u>Subcase (ii)</u> Assume that h is even

Since $C_n\langle a, \frac{n}{2}\rangle$ is connected, $gcd(a, \frac{n}{2}, n) = 1$, and therefore $\frac{n}{2}$ is odd and $gcd(a, \frac{n}{2}) = 1$ Furthermore, $gcd(a,n) = gcd(2,n) = 2$, and thus by Lemma 4 2 2, $C_n\langle a, \frac{n}{2}\rangle \cong C_n\langle 2, \frac{n}{2}\rangle$

<u>Case 2</u> Assume that $gcd(a,n) = 1$

For this case, $gcd(a,n) = gcd(1,n) = 1$ Therefore by Lemma 4 2 2, $C_n\langle a, \frac{n}{2}\rangle \cong C_n\langle 1, \frac{n}{2}\rangle$ □

Lemma 4 2 4 *For $n \geq 6$ and $\frac{n}{2}$ odd, the circulant $C_n\langle 2, \frac{n}{2}\rangle$ is isomorphic to the quasi-prism $Q_k[\pi]$ if $\pi = (0\ 1\ 2\ \ldots\ (k-1))$*

Proof Let the vertex and edge sets of $Q_k[\pi]$ be as follows

$$\begin{aligned} V(Q_k[\pi]) &= \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and} \\ E(Q_k[\pi]) &= \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \leq i \leq k-1\} \end{aligned}$$

Let the vertex and edge sets of $C_n\langle a, \frac{n}{2}\rangle$ be as follows

$$\begin{aligned} V(C_n\langle a, \tfrac{n}{2}\rangle) &= \{v_0, v_1, \ldots, v_{n-1}\}, \text{ and} \\ E(C_n\langle a, \tfrac{n}{2}\rangle) &= \{(v_i, v_{i+a}), (v_j, v_{j+\frac{n}{2}}) : 0 \leq i \leq n-1, 0 \leq j \leq \tfrac{n}{2} - 1\} \end{aligned}$$

Let the bijection $f : V(Q_k[\pi]) \rightarrow V(C_n\langle 2, \frac{n}{2}\rangle)$ be defined as follows, for $0 \leq i \leq k-1$

$$\begin{aligned} f(s_i) &= v_{2i} \text{ and} \\ f(t_i) &= v_{2i+\frac{n}{2}} \end{aligned}$$

We now show that for $(u, v) \in E(Q_k[\pi])$, $(f(u), f(v)) \in E(C_n\langle 2, \frac{n}{2}\rangle)$ For $(s_i, s_{i+1}) \in E(Q_k[\pi])$, and $0 \leq i \leq k-1$,

$$(f(s_i), f(s_{i+1})) = (v_{2i}, v_{2i+2}) \in E(C_n\langle 2, \tfrac{n}{2}\rangle)$$

For $(s_i, t_i) \in E(Q_k[\pi])$, and $0 \leq i \leq k-1$,

$$(f(s_i), f(t_i)) = (v_{2i}, v_{2i+\frac{n}{2}}) \in E(C_n\langle 2, \tfrac{n}{2}\rangle)$$

For $(t_i, t_{\pi(i)}) \in E(Q_k[\pi])$, and $0 \leq i \leq k-1$,

$$\begin{aligned} (f(t_i), f(t_{\pi(i)})) &= (v_{2i+\frac{n}{2}}, v_{2(\pi(i))+\frac{n}{2}}) \\ &= (v_{2i+\frac{n}{2}}, v_{2(i+1)+\frac{n}{2}}) \\ &= (v_{2i+\frac{n}{2}}, v_{2i+\frac{n}{2}+2}) \in E(C_n\langle 2, \tfrac{n}{2}\rangle) \end{aligned}$$

Then since the edges (v_{2i}, v_{2i+2}) correspond to jumps of size two between vertices with even subscripts, the edges $(v_{2i+\frac{n}{2}}, v_{2i+\frac{n}{2}+2})$ correspond to jumps of size two between vertices with odd subscripts, and the edges $(v_{2i}, v_{2i+\frac{n}{2}})$ correspond to all jumps of size $\frac{n}{2}$, all of the edges of $C_n\langle 2, \frac{n}{2}\rangle$ are included in this mapping □

Lemma 4 2 6 *For* $n \geq 10$, $C_n\langle 1, \frac{n}{2}\rangle$ *is not isomorphic to* $Q_k[\pi]$ *for any permutation* π, $\pi \in Sym(k)$

Proof Let the vertex and edge sets of $Q_k[\pi]$ be as follows

$$V(Q_k[\pi]) = \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and}$$
$$E(Q_k[\pi]) = \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \leq i \leq k-1\}$$

Let the vertex and edge sets of $C_n\langle a, \frac{n}{2}\rangle$ be as follows

$$V(C_n\langle a, \tfrac{n}{2}\rangle) = \{v_0, v_1, \ldots, v_{n-1}\}, \text{ and}$$
$$E(C_n\langle a, \tfrac{n}{2}\rangle) = \{(v_i, v_{i+a}), (v_j, v_{j+\frac{n}{2}}) : 0 \leq i \leq n-1, 0 \leq j \leq \tfrac{n}{2}-1\}$$

The proof is divided into two cases, depending on whether $\frac{n}{2}$ is odd or even In both cases, we demonstrate that the circulant has no chordless cycle of order $\frac{n}{2}$, and hence there is no subgraph mapping to the outer cycle of the quasi-prism

<u>Case 1</u> Let $\frac{n}{2}$ be odd

First we demonstrate that this graph is bipartite by describing a bipartitioning of the vertices We bipartition the vertices of $C_n\langle 1, \frac{n}{2}\rangle$ into two sets, E and O, as follows Let the vertices v_{2j} be in E and the vertices v_{2j+1} be in O, for $0 \leq j \leq \frac{n}{2}-1$ Then the edges (v_i, v_{i+1}) have one endpoint in E and one endpoint in O, for $0 \leq i \leq \frac{n}{2}-1$ The edges $(v_i, v_{i+\frac{n}{2}})$ also have one endpoint in E and one endpoint in O, for $0 \leq i \leq \frac{n}{2}-1$ Therefore $C_n\langle 1, \frac{n}{2}\rangle$ is bipartite But a quasi-prism graph cannot be bipartite because the outer cycle is an odd cycle of order $\frac{n}{2}$, and by Lemma 2 1 1 we know that bipartite graphs have no odd cycles Therefore $C_n\langle 1, \frac{n}{2}\rangle$ cannot be isomorphic to $Q_k[\pi]$ for any permutation π

<u>Case 2</u> Let $\frac{n}{2}$ be even

If there is an isomorphism between $C_n\langle 1, \frac{n}{2}\rangle$ and $Q_k[\pi]$, then the vertex and edge sets of $C_n\langle 1, \frac{n}{2}\rangle$ can be decomposed into pairwise disjoint sets as follows

$$C_n\langle 1, \tfrac{n}{2}\rangle = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2) \cup E(H_3)),$$

where

$H_1 = (V(H_1), E(H_1))$ is a cycle of order $\frac{n}{2}$ corresponding to the outer cycle of $Q_k[\pi]$,

$H_2 = (V(H_2), E(H_2))$ is a union of cycles, with $|V(H_2)|$ equal to $\frac{n}{2}$, corresponding to the inner cycles of $Q_k[\pi]$, and

$H_3 = (V(H_1) \cup V(H_2), E(H_3))$ is a perfect matching between the vertices of $V(H_1)$ and $V(H_2)$, corresponding to the spoke edges of $Q_k[\pi]$.

If $C_n\langle 1, \frac{n}{2}\rangle$ contains three edge-disjoint subgraphs corresponding to H_1, H_2, and H_3 as described above, then $C_n\langle 1, \frac{n}{2}\rangle$ is isomorphic to $Q_k[\pi]$ for some permutation π. We now use proof by contradiction to show that the set $E(H_1)$ must contain at least one chord edge of $C_n\langle 1, \frac{n}{2}\rangle$. Assume that $E(H_1)$ does not contain any chord edges of $C_n\langle 1, \frac{n}{2}\rangle$. Then for H_1 to be a cycle, $E(H_1)$ must contain all of the cycle edges of $C_n\langle 1, \frac{n}{2}\rangle$. This results in H_1 being a cycle of order n, not of order $\frac{n}{2}$. Therefore $E(H_1)$ must contain at least one chord edge of $C_n\langle 1, \frac{n}{2}\rangle$.

<u>Observation 1</u>. If the cycle edge (v_i, v_{i+1}) belongs to $E(H_1)$, then its opposite cycle edge $(v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1})$ does not belong to $E(H_1)$.

> Let the cycle edge (v_i, v_{i+1}) be in $E(H_1)$. We use proof by contradiction. Assume that the opposite cycle edge $(v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1})$ is in $E(H_1)$. Then consider the two chord edges $(v_i, v_{i+\frac{n}{2}})$ and $(v_{i+1}, v_{i+\frac{n}{2}+1})$. If either one of them does not belong to $E(H_1)$, then it is an edge joining two vertices which belong to $V(H_1)$, and therefore cannot belong to $E(H_2)$ or $E(H_3)$. If both chord edges $(v_i, v_{i+\frac{n}{2}})$ and $(v_{i+1}, v_{i+\frac{n}{2}+1})$ belong to $E(H_1)$, then H_1 is a 4-cycle. But for $n \geq 10$, such a cycle H_1 cannot correspond to the outer cycle of $Q_k[\pi]$. Therefore if the cycle edge (v_i, v_{i+1}) belongs to $E(H_1)$, then its opposite cycle edge $(v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1})$ does not belong to $E(H_1)$. (See Figure F.1.)

<u>Observation 2</u> If the cycle edge (v_i, v_{i+1}) does not belong to $E(H_1)$, then its opposite cycle edge $(v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1})$ belongs to $E(H_1)$

Let the cycle edge (v_i, v_{i+1}) belong to either $E(H_2)$ or $E(H_3)$. Since $E(H_1)$ must contain some cycle edges from $C_n\langle 1, \frac{n}{2}\rangle$, assume that none of the vertices v_{i+h}, $1 \leq h \leq j-1$, belong to $V(H_1)$, but that the cycle edge (v_{i+j}, v_{i+j+1}) belongs to $E(H_1)$. Then since H_1 is a cycle and $v_{i+j-1} \notin V(H_1)$, $(v_{i+j}, v_{i+j+\frac{n}{2}})$ must belong to $E(H_1)$. Since the vertices v_{i+h} do not belong to $V(H_1)$, for $1 \leq h \leq j-1$, none of the chord edges incident with the vertices v_{i+h}, $1 \leq h \leq j-1$, can belong to $E(H_1)$ either. Therefore the path of order $j+1$ from $v_{i+\frac{n}{2}}$ to $v_{i+\frac{n}{2}+j}$ must be a subgraph of H_1. Thus $(v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1})$ belongs to $E(H_1)$. (See Figure F.2.)

By Observation 1 and Observation 2,

$$(v_i, v_{i+1}) \in E(H_1) \Rightarrow (v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1}) \notin E(H_1), \text{ and}$$
$$(v_i, v_{i+1}) \notin E(H_1) \Rightarrow (v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1}) \in E(H_1).$$

Therefore $\frac{n}{2}$ of the n cycle edges must belong to $E(H_1)$. Since $E(H_1)$ must also contain at least one chord edge, the cycle H_1 is of order strictly greater than $\frac{n}{2}$. Therefore $C_n\langle 1, \frac{n}{2}\rangle$ is not isomorphic to $Q_k[\pi]$ for any permutation π. □

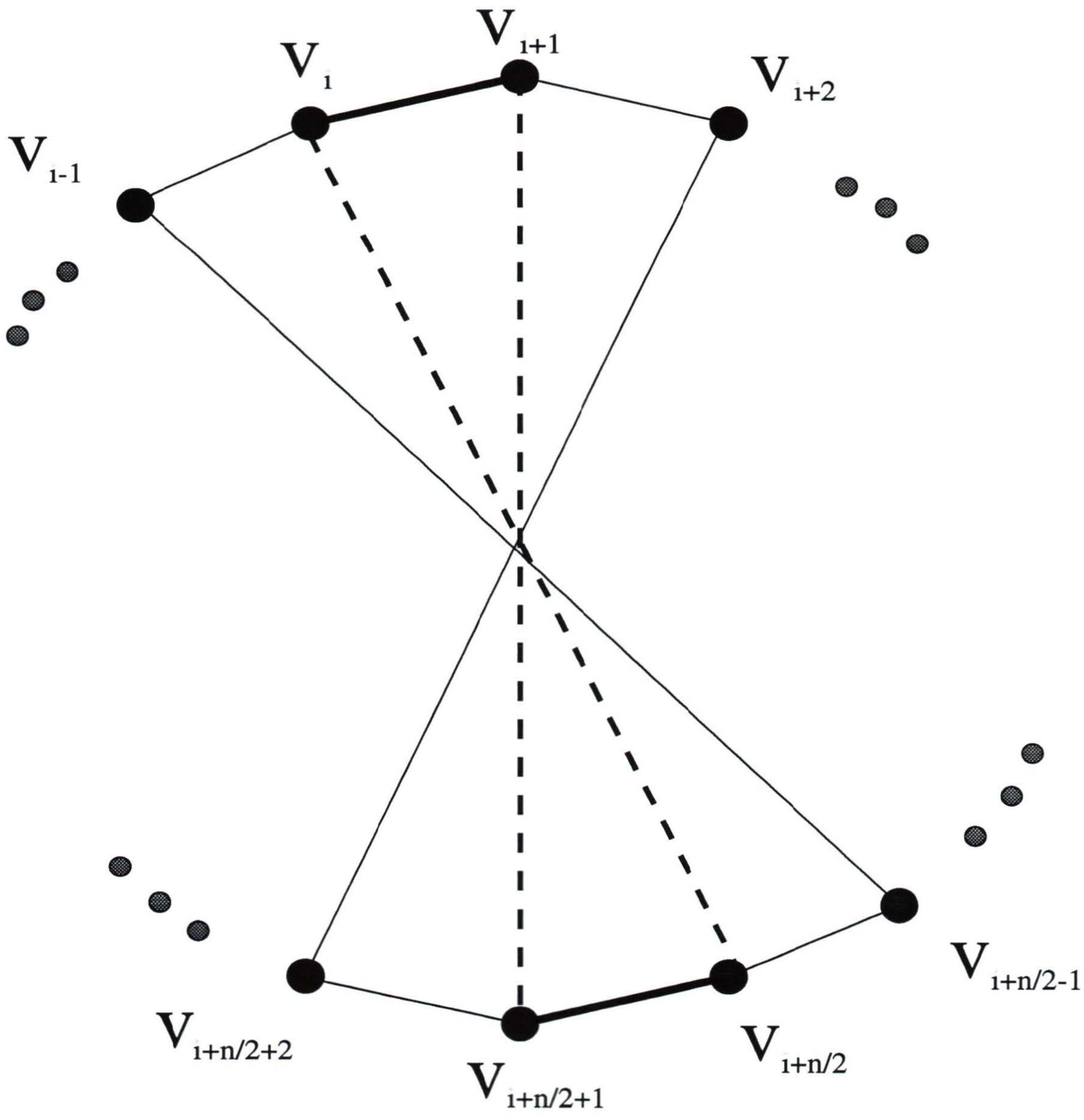

Figure F.1: Observation 1

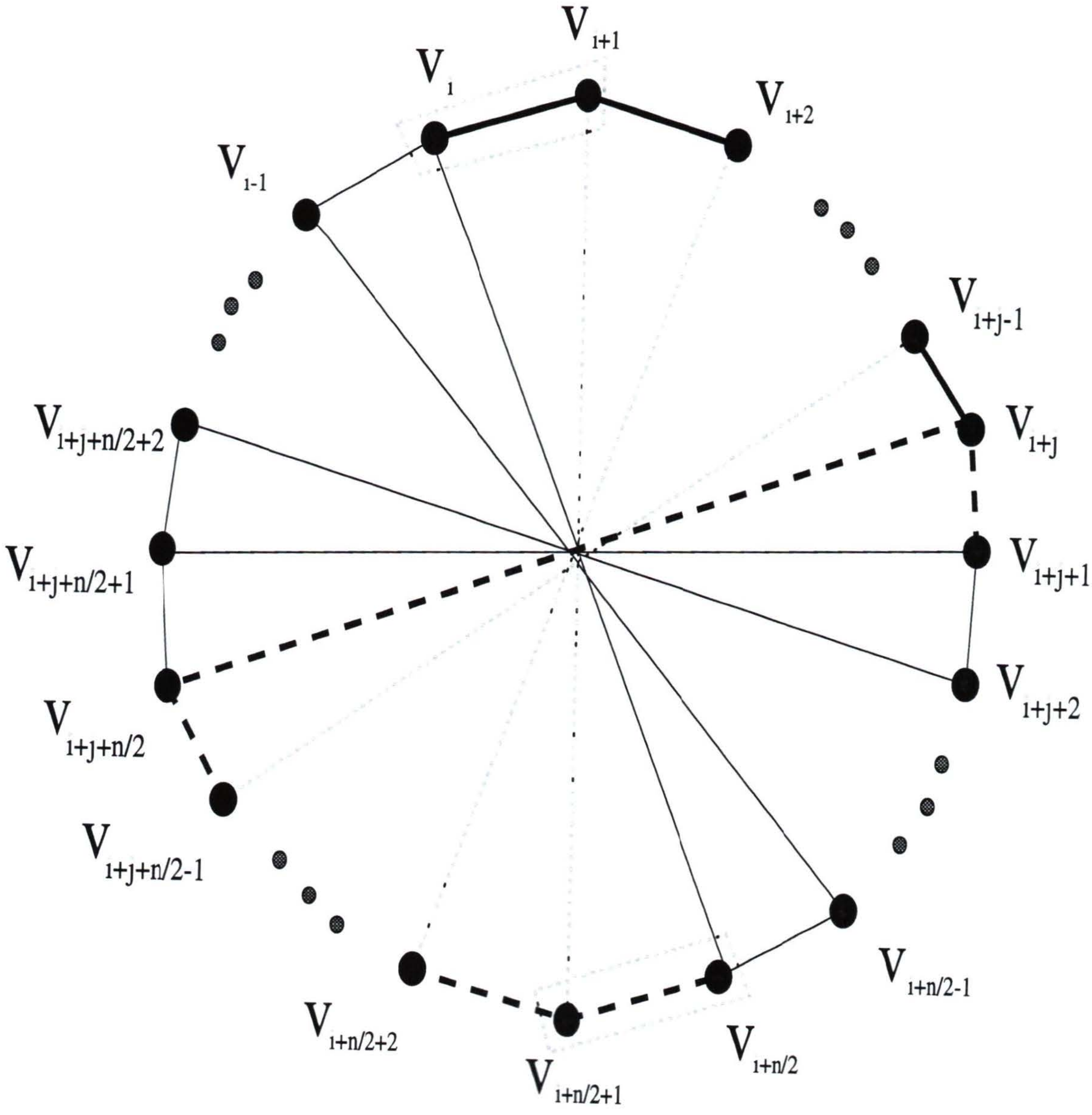

Figure F.2: Observation 2

Lemma 4.3.1 *The quasi-prism $Q_k[\pi]$ is isomorphic to the generalized Petersen graph $GP(k,j)$ if the permutation π has $\pi(i) = i + j$, for $1 \le j \le k-1$ and $j \neq \frac{k}{2}$.*

Proof. Let the vertex and edge sets of $Q_k[\pi]$ be as follows:

$$V(Q_k[\pi]) = \mathcal{S} \cup \mathcal{T} = \{s_0, s_1, \ldots, s_{k-1}\} \cup \{t_0, t_1, \ldots, t_{k-1}\}, \text{ and}$$
$$E(Q_k[\pi]) = \{(s_i, s_{i+1}), (s_i, t_i), (t_i, t_{\pi(i)}) : 0 \le i \le k-1\}.$$

Let the vertex and edge sets of $GP(k,j)$ be as follows:

$$V(GP(k,j)) = \mathcal{U} \cup \mathcal{V} = \{u_0, u_1, \ldots, u_{k-1}\} \cup \{v_0, v_1, \ldots, v_{k-1}\}, \text{ and}$$
$$E(GP(k,j)) = \{(u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+j}) : 0 \le i \le k-1\}.$$

Let the bijection $f : V(Q_k[\pi]) \to V(GP(k,j))$ be defined as follows, for $0 \le i \le k-1$:

$$f(s_i) = u_i \text{ and}$$
$$f(t_i) = v_i.$$

We now show that for $(u,v) \in E(Q_k[\pi])$, $(f(u), f(v)) \in E(GP(k,j))$. For $(s_i, s_{i+1}) \in E(Q_k[\pi])$, and $0 \le i \le k-1$,

$$(f(s_i), f(s_{i+1})) = (u_i, u_{i+1}) \in E(GP(k,j)).$$

For $(s_i, t_i) \in E(Q_k[\pi])$, and $0 \le i \le k-1$,

$$(f(s_i), f(t_i)) = (u_i, v_i) \in E(GP(k,j)).$$

For $(t_i, t_{\pi(i)}) \in E(Q_k[\pi])$, and $0 \le i \le k-1$,

$$(f(t_i), f(t_{\pi(i)})) = (v_i, v_{\pi(i)})$$
$$= (v_i, v_{i+j}) \in E(GP(k,j)).$$

It follows directly that all of the edges of $GP(k,j)$ are included in this mapping. □

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