THE GENERAL RELATIVISTIC BALANCE OF A SPHERICALLY SYMMETRIC CORE-AND-SHELL OF CHARGED PERFECT FLUID

by

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ABSTRACT

In this thesis General Relativistic and Newtonian solutions are derived for a spherically symmetric core and shell of charged perfect fluid in balanced equilibrium. The Darmois junction conditions are imposed in order to prevent the occurrence of singular boundary surfaces in the solutions. This enables a simple, logical definition of balance for the system. Newtonian solutions are derived and these show that this system possesses more freedom than that of two balanced particles. No single balance condition is found. Relativistic solutions are derived and compared to the Newtonian system. It is found that the properties of the Relativistic case are similar to, but more restricted than, those of the Newtonian case.

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CHAPTER ONE

INTRODUCTION

1.1 Einstein's Equations in Vacuum

General Relativity is the geometric theory of gravitation devised (largely) by Einstein to correct and extend the Newtonian formulation. The Newtonian concept of a particle, subject only to the force of gravity, following a curve determined by the sum of the attractions of the other bodies in the system is replaced in General Relativity by the idea of a force-free particle moving along a geodesic of a spacetime whose curvature is determined by the configuration of the mass-energy in the system. The three-dimensional Euclidean space of Newtonian theory is replaced by a four-dimensional Riemannian manifold. The theory is formulated in the coordinate-independent mathematics of tensor analysis.

Two conventions in common use in General Relativity will also be used here. The first is the Einstein summation convention:

> If an index occurs twice in one term of an expression, it is always to be summed unless the contrary is expressly stated.
>
> (Einstein, 1952)

Thus where we might once have written

$$R_{jk} = \sum_{i=0}^{3} R^{i}{}_{jik} \quad , \tag{1.1.1}$$

we would now have

$$R_{jk} = R^{i}{}_{jik} \; . \qquad (1.1.2)$$

The second convention is the assumption of the value 1 for the constants c, G, $4\pi\varepsilon_0$. This considerably simplifies many complex expressions but casts all variables in units of length to the appropriate power. In order to familiarize the reader with these new units we present the following examples (Misner, Thorne, and Wheeler, 1973):

$$c = 1 = 2.998 \times 10^{8} \ \mathrm{m\,sec^{-1}} ,$$

$$1 \ \mathrm{sec} = 2.998 \times 10^{8} \ \mathrm{m} ,$$

$$G c^{-2} = 1 = 7.425 \times 10^{-28} \ \mathrm{m\,kg^{-1}} ,$$

$$1 \ \mathrm{kg} = 7.425 \times 10^{-28} \ \mathrm{m} ,$$

electron charge $\quad q_e = 1.381 \times 10^{-36} \ \mathrm{m}$,

therefore $\quad 1 \ \mathrm{coul.} = 8.617 \times 10^{-18} \ \mathrm{m} . \qquad (1.1.3)$

In Special Relativity, the interval ds is given, in an inertial frame, by the invariant expression

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 . \qquad (1.1.4)$$

If we were to make a coordinate transformation to an accelerated (noninertial) frame the form of (1.1.4) would change to the more general

$$ds^2 = g_{ik}\, dx^i dx^k \qquad i,k = 0,1,2,3 \tag{1.1.5}$$

where the metric tensor, though still symmetric, is no longer likely to be diagonal and its components g_{ik} must be considered functions of the coordinates x^i. Motivated by the Equivalence Principle

> ...the properties of the motion in a noninertial system are the same as those in an inertial system in the presence of a gravitational field. In other words, a noninertial reference system is equivalent to a certain gravitational field.
>
> (Landau and Lifshitz, 1979)

Einstein adopted the hypothesis that, in the general case, the metric g_{ik} of equation (1.1.5) determines the effects of the reference frame and of any gravitational fields which might be present. Thus, gravitational effects are manifested through the geometry of spacetime. A particle moving under the influence of gravitational forces in the Newtonian "picture" is now seen to be a force-free particle moving in a curved spacetime described by the metric g_{ik}.

Equation (1.1.5) describes a Riemannian manifold. As the product $dx^i dx^k$ is symmetric, antisymmetric parts of the g_{ik} make no contribution to the interval ds^2, so we may safely assume that the metric itself is symmetric (Papapetrou, 1974). Thus the metric is composed of ten independent functions, the determination of which describes the spacetime in question. Since it is through the curvature of spacetime that gravity acts, we might expect to determine the g_{ik} via some equation involving the Riemann curvature tensor

$$R^{i}{}_{jkm} = \Gamma^{i}{}_{jm,k} - \Gamma^{i}{}_{jk,m} + \Gamma^{i}{}_{nk}\Gamma^{n}{}_{jm} - \Gamma^{i}{}_{nm}\Gamma^{n}{}_{jk} , \qquad (1.1.6)$$

where commas denote partial differentiation and the Christoffel symbols (which are not tensors)

$$\Gamma^{i}{}_{jk} = \tfrac{1}{2} g^{im} (g_{mj,k} + g_{km,j} - g_{jk,m}) , \qquad (1.1.7)$$

are functions of the metric and its derivatives. A guess at the equations which govern gravity in vacuum might be

$$R^{i}{}_{jkm} = 0 . \qquad (1.1.8)$$

Actually these conditions are too strong. Equation (1.1.8) describes only one spacetime, the flat spacetime of Special Relativity. We must use a contraction of the Riemann tensor. The only non-vanishing contraction of the Riemann tensor is the Ricci tensor (Papapetrou, 1974)

$$R_{jk} = R^{i}{}_{jik} , \qquad (1.1.9)$$

and the equations we seek are

$$R_{jk} = 0 . \qquad (1.1.10)$$

These are the Einstein equations in vacuum.

1.2 Matter and the Electric Field

We have derived Einstein's equations in vacuum. However we know that the presence of matter (or some other form of energy) will influence a gravitational field and thus affect our geometry. We must describe such energy distributions in tensor form and, using this tensor, modify equation (1.1.10) accordingly. Such a tensor is known and used in three-dimensional form in Newtonian theory and in the required four-dimensional form in Special Relativity. It is the stress-energy tensor T^{ik} (Spain, 1953; Landau and Lifshitz, 1979). T^{ik} is symmetric. Its definition may be summarized as follows (Ohanion, 1976):

T^{00} = energy density,

$T^{0\alpha}$ = momentum density, or energy flux density,

$T^{\alpha\beta}$ = flux density of α-momentum in β-direction.

One of the simplest tensors T^{ik} is that of a perfect fluid, a matter field under a scalar pressure. In this case T^{ik} takes the form:

$$T^{ik} = (\rho + P)u^i u^k - P g^{ik} , \qquad (1.2.1)$$

where $\rho(x)$ is the mass density, $u^i(x)$ is the four-velocity field, and $P(x)$ is the pressure distribution (Adler, Bazin, and Schiffer, 1975). In the case of static, spherical symmetry, which we will be considering, the nonzero components of T_i^k are

$$T_0^0 = \rho(r) \ , T_1^1 = T_2^2 = T_3^3 = -P(r) \ . \tag{1.2.2}$$

The stress-energy tensor for the electromagnetic field is

$$T^{ik} = \frac{1}{4\pi}\left[-F^{in}F^{k}{}_{n} + \frac{1}{4} g^{ik} F_{mn}F^{mn} \right] \tag{1.2.3}$$

where F_{ik} is the Maxwell tensor

$$F_{ik} = A_{k,i} - A_{i,k} \tag{1.2.4}$$

and A_i is the four-potential (Landau and Lifshitz, 1979). In the case of static, spherical symmetry we have

$$A_i = \left(\Phi(r), 0, 0, 0 \right) \tag{1.2.5}$$

where Φ is the scalar potential, and the nonzero components of T_i^k are

$$T_0^0 = T_1^1 = -T_2^2 = -T_3^3 = \frac{-1}{8\pi g_{00} g_{11}} (\Phi')^2 \ , \tag{1.2.6}$$

where the prime denotes differentiation by r. Following Cooperstock and de la Cruz (1978) we note the Maxwell equation

$$4\pi\sigma^*(-g_{11})^{1/2} = \frac{-1}{r^2}\frac{d}{dr}\left[\frac{r^2\Phi'}{(-g_{00}g_{11})^{1/2}}\right] \tag{1.2.7}$$

where σ^* is the proper charge density, and define the "electric field intensity"

$$E(r) = \frac{-\Phi'}{(-g_{00}g_{11})^{1/2}}, \tag{1.2.8}$$

so that
$$\frac{Q(r)}{r^2} = \frac{1}{r^2}\int_0^r 4\pi r^2\sigma^*(-g_{11})^{1/2}\,dr = E(r) \tag{1.2.9}$$

where Q(r) is the charge contained within a sphere of radius r. It will be convenient later to define a charge density

$$\sigma = \sigma^*(-g_{11})^{1/2} \tag{1.2.10}$$

so that
$$\sigma = \frac{1}{4\pi r^2}\frac{d}{dr}(Er^2),$$

$$Q(r) = \int_0^r 4\pi r^2\sigma\,dr. \tag{1.2.11}$$

We may now write the nonzero components of T_i^k as

$$T_0^0 = T_1^1 = -T_2^2 = -T_3^3 = \frac{E^2}{8\pi}. \tag{1.2.12}$$

The stress-energy tensor has another property which we have not yet considered. It is divergenceless; that is

$$T_i{}^k{}_{;k} = 0 \tag{1.2.13}$$

where the semi-colon denotes covariant differentiation

$$T^i{}_{k;m} = T^i{}_{k,m} - \Gamma^n{}_{km} T^i{}_n + \Gamma^i{}_{nm} T^n{}_k \, . \tag{1.2.14}$$

The Ricci tensor does not exhibit this property and thus we cannot simply place T_{jk} on the right hand side of equation (1.1.10). However, the Einstein tensor

$$G_{ik} = R_{ik} - \tfrac{1}{2} g_{ik} R \quad , \quad R = R^i{}_i \tag{1.2.15}$$

is divergenceless and a simple calculation shows

$$G_{jk} = 0 \tag{1.2.16}$$

are equivalent to the equations (1.1.10). Thus we have the Einstein equations

$$G_{ik} = 8\pi T_{ik} \, . \tag{1.2.17}$$

The constant 8π is determined by requiring that equation (1.2.17) reduce to the Newtonian case in the limit of weak gravitational fields (Papapetrou, 1974).

In the case of static spherical symmetry with the Schwarzschild line element

$$ds^2 = \mu^2 dt^2 - \nu^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 , \tag{1.2.18}$$

the Einstein tensor has the form

$$G_0^{\,0} = \frac{1}{r^2} - \left[\frac{\nu'}{r} + \frac{\nu}{r^2}\right] ,$$

$$G_1^{\,1} = \frac{1}{r^2} - \nu\left[\frac{2}{r}\frac{\mu'}{\mu} + \frac{1}{r^2}\right] ,$$

$$G_2^{\,2} = G_3^{\,3} = -\left[\frac{\mu''}{\mu}\nu + \frac{\mu'}{\mu}\frac{\nu'}{2} + \frac{\mu'}{\mu}\frac{\nu}{r} + \frac{\nu'}{2r}\right] . \tag{1.2.19}$$

Thus the Einstein-Maxwell equations in a static, spherically symmetric, charged perfect fluid are

$$\frac{1}{r^2} - \left[\frac{\nu'}{r} + \frac{\nu}{r^2}\right] = 8\pi\rho + E^2 ,$$

$$\frac{1}{r^2} - \nu\left[\frac{2}{r}\frac{\mu'}{\mu} + \frac{1}{r^2}\right] = -8\pi P + E^2 ,$$

$$\frac{\mu''}{\mu}\nu + \frac{\mu'}{\mu}\frac{\nu'}{2} + \frac{\mu'}{\mu}\frac{\nu}{r} + \frac{\nu'}{2r} = 8\pi P + E^2 , \tag{1.2.20}$$

the third and fourth equations being identical.

If these equations govern all of spacetime we may proceed directly to their solution. However, if we are dealing with bounded regions of matter or matter-energy fields of different types we must solve (1.2.20) or other appropriate equations in each of the bounded regions and "match" these solutions at the boundaries. There are three sets of such junction conditions in use in General Relativity. Bonnor and Vickers (1981) have shown that the Darmois junction conditions are "the most convenient and reliable formulation of junction condition(s) in general relativity." It is the Darmois formulation which we now present.

Following Sarracino (1981), we consider two regions of spacetime, V and $\overline{V}$, with coordinates x^i and $\bar{x}^i$, which are separated by a hypersurface S defined by the functions

$$f(x^i) = 0 \quad , \quad \bar{f}(\bar{x}^i) = 0 \tag{1.2.21}$$

in V and $\overline{V}$, respectively. We define, in V and $\overline{V}$, two parametric representations of S

$$x^i = g^i(u^1, u^2, u^3) \quad , \quad \bar{x}^i = \bar{g}^i(u^1, u^2, u^3) \tag{1.2.22}$$

where S is covered by the same domain u^α (α = 1, 2, 3) in both cases.

Then we say that V and $\bar{V}$ match across S if the first and second fundamental forms of S, calculated as functions of u^α through g^i and $\bar{g}^i$, are identical.

The first and second fundamental forms in V and $\bar{V}$ are

$$d\ell^2 = g_{\alpha\beta}\, du^\alpha du^\beta \quad ,$$

$$d\bar{\ell}^2 = \bar{g}_{\alpha\beta}\, du^\alpha du^\beta \quad , \tag{1.2.23}$$

and $$d_{\alpha\beta}\, du^\alpha du^\beta = -n_{i;k} \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta}\, du^\alpha du^\beta \quad ,$$

$$\bar{d}_{\alpha\beta}\, du^\alpha du^\beta = -\bar{n}_{i;k} \frac{\partial \bar{x}^i}{\partial u^\alpha}\frac{\partial \bar{x}^k}{\partial u^\beta}\, du^\alpha du^\beta , \tag{1.2.24}$$

where i, k = 0, 1, 2, 3, α, β = 1, 2, 3, and the unit normals are

$$n_i = f_{,i}\left(-g^{jk} f_{,j} f_{,k}\right)^{-1/2}$$

$$\bar{n}_i = \bar{f}_{,i}\left(-\bar{g}^{jk} \bar{f}_{,j} \bar{f}_{,k}\right)^{-1/2} \quad . \tag{1.2.25}$$

In our case, with the Schwarzschild line element

$$ds^2 = \mu^2 dt^2 - \nu^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 ,$$

the two regions V, $\bar{V}$ use the same coordinates. Thus, for the surface

S, we have

$$f(x^i) = r - a \;, \quad f_{,i} = (0,1,0,0) \tag{1.2.26}$$

and

$$n_i = (-g^{11})^{-1/2} f_{,i} = \nu^{-1/2} f_{,i} \;. \tag{1.2.27}$$

Choosing for S the obvious parametric representation

$$u^1 = x^0 = t \,, \; u^2 = x^2 = \theta \,, \; u^3 = x^3 = \phi \,, \; x^1 = a \tag{1.2.28}$$

we have the first fundamental form

$$d\ell^2 = \mu^2 dt^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \tag{1.2.29}$$

and, after a short calculation, the second fundamental form

$$d_{\alpha\beta}\, du^\alpha du^\beta = \nu^{1/2}\left[\mu'\mu + r(1+\sin^2\theta)\right] . \tag{1.2.30}$$

The Darmois junction conditions require that (1.2.29) and (1.2.30) evaluated in both regions V and $\overline{V}$, are identical at the boundary S. In other words, the functions ν, μ, μ' must be continuous at S. From the second equation of (1.2.20) we see that, for a charged perfect fluid, the continuity of μ' depends on the continuity of η, ν, P, and E^2. Thus for a charged perfect fluid, the Darmois junction conditions require the continuity of the metric, the electric field intensity, and the pressure.

1.3 The Reissner-Nordstrom Solution

As an example of an exact solution of Einstein's equations we will derive the metric of a spherically symmetric electrovac. (electric field in vacuum). This solution was first given by Reissner (1916) and Nordstrom (1918).

We may use equations (1.2.20) if we set ρ =P=0. The electric field is given by

$$E = \frac{q}{r^2} , \tag{1.3.1}$$

as we can see from (1.2.9). Thus the Einstein-Maxwell equations are

$$\frac{1}{r^2} - \left[\frac{\nu'}{r} + \frac{\nu}{r^2}\right] = \frac{q^2}{r^4} , \tag{1.3.2}$$

$$\frac{1}{r^2} - \nu\left[\frac{2}{r}\frac{\mu'}{\mu} + \frac{1}{r^2}\right] = \frac{q^2}{r^4} , \tag{1.3.3}$$

$$\frac{\mu''}{\mu}\nu + \frac{\mu'}{\mu}\frac{\nu'}{2} + \frac{\mu'}{\mu}\frac{\nu}{r} + \frac{\nu'}{2r} = \frac{q^2}{r^4} . \tag{1.3.4}$$

Equation (1.3.2) is easily integrated to give

$$\nu = 1 + \frac{C}{r} + \frac{q^2}{r^2} \tag{1.3.5}$$

where C is the constant of integration. A distant observer will measure a mass of $-C/2$ for the system; thus we have $C=-2m$. Subtracting (1.3.3) from (1.3.2) we have

$$-\frac{\nu'}{r} + \frac{2}{r}\frac{\mu'}{\mu}\nu = 0 \tag{1.3.6}$$

which is easily integrated to give

$$\mu^2 = K\nu \tag{1.3.7}$$

where K is a positive constant. K is determined by the outer boundary condition. If the electrovac region extends to infinite radius we require the metric to be asymptotically flat. That is, as $r\rightarrow\infty$,

$$\mu^2,\ \nu^{-1} \longrightarrow 1\ , \tag{1.3.8}$$

and

$$K = 1\ . \tag{1.3.9}$$

Thus the solution for a charged point mass is

$$E = \frac{q}{r^2}\ ,$$

$$ds^2 = \mu^2 dt^2 - \nu^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2\ ,$$

and $$\mu^2 = \nu = 1 - \frac{2m}{r} + \frac{q^2}{r^2}\,. \tag{1.3.10}$$

1.4 Electrostatic Balance

One of the simplest and most straightforward models illustrating the similarity between the classical gravitational and electric forces is that of two spherically symmetric charged masses whose electrostatic repulsion exactly cancels their gravitational attraction. Such a system is in equilibrium at any separation if

$$m_1 m_2 = q_1 q_2 \tag{1.4.1}$$

where m_1, q_1 and m_2, q_2 are the mass and charge of the first and second body, respectively.

The non-linearities of General Relativity considerably complicate the interaction of gravitational and electric fields and researchers have returned with renewed interest to simple electrostatic systems. The condition for the equilibrium of two charged masses has been investigated by Barker and O'Connell (1977), Kimura and Ohta (1977), and Bonnor (1981) with mixed results, although Bonnor felt confident enough to conjecture that (1.4.1) was, in fact, the relativistic equilibrium condition for two spherically symmetric charged masses. Carminati (1982) has recently renewed doubts of this. Using the Weyl formalism, he has discovered a set of eigensolutions for two charged bodies in which single eigensolutions satisfy (1.4.1) but superpositions do not. However, these eigensolutions seem to possess gravitational multipole structure which could influence the balance condition so the final answer to this question is still unclear.

The fact that even such simple systems as the one above defy exact analysis is a clear indication of the difficulties inherent in the study of the relativistic gravitational field. Such a two-body system is nearly impossible to simplify. Other approaches to the problem involve a trade-off between simplifying and complicating factors. One possibility is to reduce the axial symmetry of the two separated bodies to the spherical symmetry of one body contained within another. In such a system both bodies would share the same center of mass and could no longer be idealized as point sources. Thus the additional complications due to source structure go hand in hand with the advantages of greater symmetry.

It is this type of core-and-shell system which will be investigated in the remainder of this thesis. We will study the electrostatic balance of the system in the relativistic case and compare it to that of the Newtonian case. The system will be assumed to be in balance when the system supports itself "on its own"; that is, when the metric, the electric field and the internal pressure all combine to give a static system without the aid of any supporting "struts" or "skins". We will impose the Darmois junction conditions at the boundaries between the core[1] (Region 0), the inner electrovacuum (Region I), the shell (Region II), and the outer electrovacuum (Region III), so as

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[1]See figure 1.

to avoid any singular surfaces in our solutions. In particular, we will require the pressure to be continuous (i.e. zero) at the boundaries of the core and the shell, eliminating the possibility of pressurized "skins" supporting the system.

To the best of the author's knowledge, the balance of this system has never been considered, either relativistically or classically. Thus we will deal first with the Newtonian solution, in Chapter two, before treating the relativistic case in Chapter three. As it is generally unnecessary in Newtonian theory, and can lead to confusion, we will suspend the Einstein summation convention for the duration of Chapter two.

CHAPTER TWO

THE NEWTONIAN CORE-AND-SHELL

2.1 Integration of the System

We wish to consider, in the Newtonian formalism, a spherically symmetric core and shell of charged perfect fluid in equilibrium with the additional condition that the pressure be zero at all boundaries. The system is illustrated in figure 1, which also introduces the constants r_0, r_1, r_2, the radii of the core, and inner and outer shell boundaries; q_0, q_2, q, the charges of the core and shell, and the total charge; and m_0, m_2, m, the masses of the core and shell, and the total mass.[1] Subscripts on variables and constants refer to the regions of the system in which they are defined.

We will focus our attention on the shell (and hence drop the subscript 2 on variables), primarily because the shell does not affect the core and thus is is in the shell that balance is manifested, but also because Newtonian balls of various types have been extensively studied. Since it is only the overall mass and charge of the core which affects the shell we may use any such ball that satisfies the pressure condition. We define:

$\rho \equiv$ mass density,

$\sigma \equiv$ charge density (generally assumed positive),

$\hat{r} \equiv$ unit radial vector,

[1] Note that in the relativistic case the constants m_0, m_2, and m must be reinterpreted.

$\bar{E}$ ≡ electric field $= E\hat{r}$ (generally assumed positive),

$\bar{g}$ ≡ gravitational field $= g\hat{r}$,

$\bar{f}$ ≡ force per unit volume $= f\hat{r}$,

P ≡ pressure,

M ≡ total mass within a sphere of radius r,

Q ≡ total charge within a sphere of radius r (generally assumed positive).

In the fluid, the force per unit volume

$$f = E\sigma + g\rho . \tag{2.1.1}$$

Also, since we are at equilibrium, we have in any element of volume

$$0 = P r^2 \sin\theta \, d\theta \, d\phi - \left[P + \frac{dP}{dr} dr\right] r^2 \sin\theta \, d\theta \, d\phi + f \, dV . \tag{2.1.2}$$

Therefore $$f = \frac{dP}{dr} = P' , \tag{2.1.3}$$

and from $$\nabla \cdot \bar{E} = 4\pi\sigma ,$$

and $$\nabla \cdot \bar{g} = -4\pi\rho , \tag{2.1.4}$$

we have $$P' = \sigma E + \rho g = \frac{1}{8\pi r^4} \frac{d}{dr}\left[r^4 (E^2 - g^2)\right] . \tag{2.1.5}$$

Thus

$$\rho = \frac{\sigma E - P'}{-g} \tag{2.1.6}$$

and

$$g^2 = E^2 - \frac{8\pi}{r^4}\int r^4 P' \, dr \, . \tag{2.1.7}$$

Combining these we see

$$\rho = \frac{\sigma E - P'}{\left[E^2 - \frac{8\pi}{r^4}\int r^4 P' \, dr\right]^{1/2}} , \tag{2.1.8}$$

where we require the root to be positive.

2.2 A Trial Solution

It seems logical at this point to derive a quick solution which will provide some ideas about the general properties of the shell. It is mathematically convenient here, and necessary in the relativistic case, to specify an appropriate pressure and choose an electric field which will allow a simple determination of the mass density. We will use the simplest expression which results in a continuous pressure at the boundaries; a condition which in the relativistic case is required to eliminate singular supporting "skins" and ensure a proper match to bounding solutions.

Thus, let

$$P = -\frac{A^2}{8\pi}(r-r_1)(r-r_2) \quad , \quad r_1 \leq r \leq r_2 \, , \tag{2.2.1}$$

where A is an arbitrary constant, which for convenience we assume to be positive. Then

$$E^2 - \frac{8\pi}{r^4}\int r^4 P' dr = E^2 + A^2\left[\frac{r^2}{3} - \frac{(r_1+r_2)r}{5} + \frac{k}{r^4}\right] , \tag{2.2.2}$$

where k is the constant of integration. We will choose

$$E^2 = A^2\left[\frac{2}{3}r^2 + \frac{(r_1+r_2)}{5}r - \frac{k}{r^4}\right] \tag{2.2.3}$$

so as to isolate the r^2 term in (2.2.2). The r term gives a rather more complex ρ while the r^{-4} term results in a massless, charged shell (under pressure).

After a simple calculation and the application of boundary conditions we find:

$$A = \frac{m}{r_2^3} \, ,$$

$$\rho = \frac{3m}{4\pi r_2^3} ,$$

$$g = -\frac{M}{r^2} = -\frac{mr}{r_2^3} ,$$

$$\sigma = \frac{m}{8\pi r_2^3} \frac{4r^3 + (r_1+r_2)r^2}{D(r)} ,$$

$$E = \frac{Q}{r^2} = \frac{m}{r_2^3} \frac{D(r)}{r^2} ,$$

where $$D(r) = \left[\frac{q^2}{m^2} r_2^6 - \frac{2}{3}(r_2^6 - r^6) - \frac{(r_1+r_2)(r_2^5 - r^5)}{5}\right]^{1/2} . \qquad (2.2.4)$$

We must require

$$q^2 > d \qquad (2.2.5)$$

where $$d = m^2\left[\frac{2}{3}\left(1 - \frac{r_1^6}{r_2^6}\right) + \frac{1}{5}\left(1 + \frac{r_1}{r_2}\right)\left(1 - \frac{r_1^5}{r_2^5}\right)\right] . \qquad (2.2.6)$$

At this point it is interesting to test the two-particle balance condition.

$$m_0 m_2 = m_0(m - m_0) = m^2 \frac{r_1^3}{r_2^3}\left(1 - \frac{r_1^3}{r_2^3}\right) .$$

$$q_0 q_2 = q_0(q - q_0) = \frac{qm}{r_2^3} D(r_1) - \frac{m^2}{r_2^6}\left[D(r_1)\right]^2 . \tag{2.2.7}$$

A glance at (2.2.4) and (2.2.7) is enough to see there is no reason to expect

$$q_0 q_2 - m_0 m_2 = 0 . \tag{2.2.8}$$

In fact, once r_1, r_2 and m are specified, $m_0 m_2$ is fully determined. However, as

$$q^2 \to d \quad , \quad q_0 = \left[q^2 - d\right]^{1/2} \to 0 ,$$

$$\Longrightarrow q_0 q_2 \to 0 , \tag{2.2.9}$$

and as $\quad q^2 \to \infty \quad , \quad q_0 \to q \quad ,$

$$\Longrightarrow q_0 q_2 = q_0 q - q^2 + d \to d . \tag{2.2.10}$$

Thus, with r_1, r_2, m and hence $m_0 m_2$ specified, we may, by the appropriate choice of q, vary $q_0 q_2$ such that

$$0 < q_0 q_2 < d$$

and $$-m_0 m_2 < q_0 q_2 - m_0 m_2 < d - m_0 m_2 \tag{2.2.11}$$

without altering the equilibrium of the system.

It is apparent that core-and-shell balance allows considerably more freedom in the choice of system parameters than does two-particle balance. A measure of this freedom can be obtained by further investigating the properties of this solution.

A simple check of the limiting inequality (2.2.5) shows that all three conditions

$$q^2 < m^2 \ , \quad q^2 = m^2 \ , \quad q^2 > m^2 \ , \tag{2.2.12}$$

are allowed.

The dependence of the shell on the core can be seen by letting m_0 and q_0 approach zero.

$$\begin{aligned} m_0 \to 0 \Rightarrow m \to 0 \Rightarrow P, \rho, M, g, \sigma &\to 0 \ , \\ Q &\to q \ , \\ E &\to \frac{q}{r^2} \ , \end{aligned} \tag{2.2.13}$$

so the shell disappears and the core becomes a massless point charge.

$$q_0 \to 0 \Rightarrow \text{(i)}\ D(r_1) \to 0 \Rightarrow \sigma(r_1) \to \infty ,$$

$$\text{(ii)}\ m \to 0 , \tag{2.2.14}$$

and the whole system disappears.

However, it would be unreasonable to expect to eliminate either m_0 or q_0 without the continuity of ρ and σ at r_1 since this would leave the inner boundary of the shell without the necessary balance of forces. The continuity of ρ and σ at r_1 is related to the continuity of P' at r_1 as may be seen from (2.1.5). Thus it seems likely that even more freedom may be gained if the system has a pressure which is smooth at r_1.

2.3 A General Solution with P Smooth at the Inner Boundary

In this section we will present a series solution to (2.1.8) in order to illustrate the general nature of the properties of the system. The derivation of the solution is presented in Appendix A.

We assume that the pressure and charge density are given. Thus, let

$$P = \frac{A^2}{8\pi}(r-r_1)^2 \sum_{i=0}^{\infty} a_i r^i , \qquad r_1 \leq r \leq r_2 , \tag{2.3.1}$$

where we require $$P(r_2) = 0 , \tag{2.3.2}$$

and let $$\sigma = \frac{B}{4\pi}\left[(r-r_1)\sum_{i=0}^{\infty} b_i r^i + \Lambda\right] \ , \quad r_1 \leq r \leq r_2 \ , \qquad (2.3.3)$$

where Λ is an arbitrary constant.

We define

$$a_{-1} = b_{-1} = 0 \ ,$$

$$\alpha_i = a_{i-1} - 2a_i r_1 + a_{i+1} r_1^2 \ ,$$

$$\beta_i = b_{i-1} - b_i r_1 \ ,$$

$$\gamma_i = (i+2)a_i - (i+1)a_{i+1} r_1 \ ,$$

$$\delta = \sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i (r_2^{i+5} - r_1^{i+5})$$

$$= (r_2 - r_1)^2 \sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i \, p_{i+3}(r_2) \ ,$$

$$\varepsilon = \sum_{i=0}^{\infty} \frac{\beta_i}{i+3}(r_2^{i+3} - r_1^{i+3}) + \frac{\Lambda}{3}(r_2^3 - r_1^3)$$

$$= (r_2 - r_1)^2\left[\sum_{i=0}^{\infty} \frac{\beta_i}{i+3} p_{i+1}(r_2) + \frac{\Lambda}{3}(r_2 + 2r_1)\right] + (r_2 - r_1)\Lambda r_1^2 \ ,$$

where $$p_n(r) = \sum_{i=0}^{n} (i+1)\, r^{n-i} r_1^i \ . \qquad (2.3.4)$$

Then

$$A^2 = -\frac{(m^2-q^2)-(m_0^2-q_0^2)}{\delta} \ ,$$

$$B = \frac{q_2}{\varepsilon} \ ,$$

$$Q = q_0 + \frac{q_2}{\varepsilon}\left[\sum_{i=0}^{\infty} \frac{\beta_i}{i+3}\left(r^{i+3}-r_1^{i+3}\right) + \frac{\Lambda}{3}(r^3 - r_1^3)\right]$$

$$= q_0 + \frac{q_2}{\varepsilon}\left\{(r-r_1)^2\left[\sum_{i=0}^{\infty} \frac{\beta_i}{i+3} p_{i+1} + \frac{\Lambda}{3}(r+2r_1)\right] + (r-r_1)\Lambda r_1^2\right\} ,$$

$$E = \frac{Q}{r^2} \ ,$$

$$g = -\frac{M}{r^2} \ ,$$

$$M = \left[Q^2 + m_0^2 - q_0^2 + \frac{(m^2-q^2)-(m_0^2-q_0^2)}{\delta} \sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i\left(r^{i+5}-r_1^{i+5}\right)\right]^{1/2}$$

$$= \left[Q^2 + m_0^2 - q_0^2 + \frac{(m^2-q^2)-(m_0^2-q_0^2)}{\delta} (r-r_1)^2 \sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i p_{i+3}\right]^{1/2} ,$$

$$\rho = \left\{ \frac{q_0 q_2}{4\pi\varepsilon}\left[(r-r_1)\sum_{i=0}^{\infty} b_i r^i + \Lambda\right] \right.$$

$$+ \frac{q_2^2}{4\pi\varepsilon^2}\left[(r-r_1)\sum_{i=0}^{\infty} b_i r^i + \Lambda\right]\left[\sum_{i=0}^{\infty}\frac{\beta_i}{i+3}(r^{i+3}-r_1^{i+3}) + \frac{\Lambda}{3}(r^3-r_1^3)\right]$$

$$\left. + \frac{(m^2-q^2)-(m_0^2-q_0^2)}{8\pi\delta}\sum_{i=0}^{\infty}(i+1)\alpha_i r^{i+2} \right\} \Big/ M$$

$$= \left\{ \frac{q_0 q_2}{4\pi\varepsilon}\left[(r-r_1)\sum_{i=0}^{\infty} b_i r^i + \Lambda\right] \right.$$

$$+ \frac{q_2^2}{4\pi\varepsilon^2}\left[(r-r_1)\sum_{i=0}^{\infty} b_i r^i + \Lambda\right]\left[(r-r_1)^2\left(\sum_{i=0}^{\infty}\frac{\beta_i}{i+3}p_{i+1} + \frac{\Lambda}{3}(r+2r_1)\right) + (r-r_1)\Lambda r_1^2\right]$$

$$\left. + \frac{(m^2-q^2)-(m_0^2-q_0^2)}{8\pi\delta}(r-r_1)\sum_{i=0}^{\infty}\gamma_i r^{i+2} \right\} \Big/ M \; . \qquad (2.3.5)$$

The only restrictions we need apply amount to the condition that $M(r)$ be positive everywhere in the shell. Of course there is no guarantee we will not have convergence problems but, as we will be primarily interested in polynomials, these need not concern us.

It may be seen from (2.3.5) that Q is completely independent of m_0 and m; and $M(r_1) = m_0$ and $M(r_2) = m$ are independent of q_0 and q_2. Thus it appears we may find $(m_0 m_2 - q_0 q_2)$ anywhere in the range $(-\infty, \infty)$ and all three cases

$$q < m \; , \quad q = m \; , \quad q > m \qquad (2.3.6)$$

are allowed in these solutions. We will now investigate the dependence of the shell on the core.

Suppose $\Lambda \neq 0$. Then

$$\sigma(r_1) = \frac{q_2 \Lambda}{4\pi\varepsilon}$$

$$\rho(r_1) = \frac{q_0}{m_0} \frac{q_2 \Lambda}{4\pi\varepsilon} \tag{2.3.7}$$

Clearly we may set $q_0=0$ (here and in (2.3.5)) without difficulty. However, unless $q_0=0$, we may not let $m_0 \rightarrow 0$. If $q_0=m_0=0$, then

$$\rho(r_1) = \frac{\left[\frac{q^2}{4\pi\varepsilon^2}\Lambda^2 r_1^2 + \frac{(m^2-q^2)}{8\pi\delta}\sum_{i=0}^{\infty}\gamma_i r_1^{i+2}\right]}{\left[\frac{q^2}{\varepsilon^2}\Lambda^2 r_1^4 + \frac{(m^2-q^2)}{\delta}\sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i p_{i+3}(r_1)\right]^{1/2}} . \tag{2.3.8}$$

Now $$p_{i+3}(r_1) = r_1^{i+3}\sum_{j=1}^{i+4} j = r_1^{i+3}\cdot\tfrac{1}{2}(i+4)(i+5). \tag{2.3.9}$$

Thus $$\sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i p_{i+3}(r_1) = \tfrac{1}{2}\sum_{i=0}^{\infty}(i+1)(i+4)\left[a_{i-1} - 2a_i r_1 + a_{i+1} r_1^2\right] r_1^{i+3}$$

$$= \sum_{j=0}^{\infty} a_j r_1^{j+4} . \tag{2.3.10}$$

Also
$$\sum_{i=0}^{\infty} \gamma_i r^{i+2} = \sum_{i=0}^{\infty} \left[(i+2)a_i - r_1(i+1)a_{i+1}\right] r_1^{i+2}$$
$$= 2\sum_{j=0}^{\infty} a_i r_1^{i+2} \tag{2.3.11}$$

Therefore, with $\Lambda \neq 0$ and $m_0 = q_0 = 0$,

$$\rho(r_1) = \frac{\left[\dfrac{q^2}{4\pi\varepsilon^2}\Lambda^2 r_1^2 + \dfrac{m^2-q^2}{4\pi\delta}\sum_{i=0}^{\infty} a_i r_1^{i+2}\right]}{\left[\dfrac{q^2}{\varepsilon^2}\Lambda^2 r_1^4 + \dfrac{m^2-q^2}{\delta}\sum_{i=0}^{\infty} a_i r_1^{i+4}\right]^{1/2}}$$

$$= \frac{1}{4\pi}\left[\frac{q^2\Lambda^2}{\varepsilon^2} + \frac{m^2-q^2}{\delta}\sum_{i=0}^{\infty} a_i r_1^i\right]^{1/2}, \tag{2.3.12}$$

and $\rho(r_1)$ real requires

$$\frac{q^2\Lambda^2}{\varepsilon^2} > -\frac{m^2-q^2}{\delta}\sum_{i=0}^{\infty} a_i r^i \tag{2.3.13}$$

or
$$\frac{m^2}{q^2} > 1 - \frac{\Lambda^2}{\varepsilon^2}\frac{\delta}{\sum_{i=0}^{\infty} a_i r^i} \qquad \frac{\delta}{\sum_{i=0}^{\infty} a_i r^i} > 0,$$

$$\frac{m^2}{q^2} < 1 - \frac{\Lambda^2}{\varepsilon^2}\frac{\delta}{\sum_{i=0}^{\infty} a_i r^i} \qquad \frac{\delta}{\sum_{i=0}^{\infty} a_i r^i} < 0. \tag{2.3.14}$$

Note that now
$$m^2 = q^2 \Rightarrow P \equiv 0, \tag{2.3.15}$$

and we would no longer have perfect fluid, but dust.

In the case $\Lambda = 0$, we have

$$\sigma(r_1) = \rho(r_1) = 0 \ . \qquad (2.3.16)$$

None of (2.3.5) are unduly disturbed if we let $q_0=0$, but the mass density is sensitive to the absence of core mass.

For $m_0=0$,

$$\rho(r_1) = \frac{\left[\frac{q_0 q_2}{4\pi\varepsilon}\sum_{i=0}^{\infty} b_i r^i + \frac{m^2-(q^2-q_0^2)}{8\pi\delta}\sum_{i=0}^{\infty}\gamma_i r^{i+2}\right]}{\left[\frac{2q_0q_2}{\varepsilon}\sum_{i=0}^{\infty}\frac{\beta_i}{i+3}p_{i+1}(r_1) + \frac{m^2-(q^2-q_0^2)}{\delta}\sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i p_{i+3}(r_1)\right]^{1/2}} \qquad (2.3.17)$$

Now,

$$p_{i+1}(r_1) = r_1^{i+1}\cdot\tfrac{1}{2}(i+2)(i+3) \qquad (2.3.18)$$

so

$$\sum_{i=0}^{\infty}\frac{\beta_i}{i+3}p_{i+1}(r_1) = \tfrac{1}{2}\sum_{i=0}^{\infty}(i+2)\left[b_{i-1}-b_i r_1\right]r_1^{i+1} = \tfrac{1}{2}\sum_{i=0}^{\infty} b_i r_1^{i+2} \ . \qquad (2.3.19)$$

Thus, with $\Lambda = m_0 = 0$,

$$\rho(r_1) = \frac{\left[\frac{q_0q_2}{4\pi\varepsilon}\sum_{i=0}^{\infty} b_i r_1^i + \frac{m^2-(q^2-q_0^2)}{4\pi\delta}\sum_{i=0}^{\infty} a_i r_1^{i+2}\right]}{\left[\frac{q_0q_2}{\varepsilon}\sum_{i=0}^{\infty} b_i r_1^{i+2} + \frac{m^2-(q^2-q_0^2)}{\delta}\sum_{i=0}^{\infty} a_i r_1^{i+4}\right]^{1/2}}$$

or
$$\rho(r_1) = \frac{1}{4\pi r_1}\left[\frac{q_0 q_2}{\varepsilon}\sum_{i=0}^{\infty} b_i r_1^{\,i} + \frac{m^2-(q^2-q_0^2)}{\delta}\sum_{i=0}^{\infty} a_i r_1^{\,i+2}\right]^{1/2} \tag{2.3.20}$$

Note that
$$\frac{m^2-(q^2-q_0^2)}{\delta} = -A^2 < 0 , \tag{2.3.21}$$

and
$$\operatorname{sgn}\left[\sum_{i=0}^{\infty} a_i r_1^{\,i+2}\right] = \operatorname{sgn}\left[P(r_1+dr)\right] . \tag{2.3.22}$$

Thus if we allow negative pressure we may freely set $q_0=0$. If not, then maintaining (r_1) real requires

$$\frac{q_0 q_2}{\varepsilon}\sum_{i=0}^{\infty} b_i r_1^{\,i} \geq -\frac{m^2-(q^2-q_0^2)}{\delta}\sum_{i=0}^{\infty} a_i r_1^{\,i+2} , \tag{2.3.23}$$

$$q_0 q_2\left[\frac{\sum_{i=0}^{\infty} b_i r_1^{\,i}}{\varepsilon\sum_{i=0}^{\infty} a_i r_1^{\,i+2}} - \frac{2}{\delta}\right] \geq -\frac{m^2-q_2^2}{\delta} . \tag{2.3.24}$$

Thus
$$q_0 \geq \frac{m^2-q_2^2}{q_2\left[2-\delta\frac{\sum_{i=0}^{\infty} b_i r_1^{\,i}}{\varepsilon\sum_{i=0}^{\infty} a_i r_1^{\,i+2}}\right]} \qquad \varsigma > 0 ,$$

$$q_0 \leq \frac{m^2-q_2^2}{q_2\left[2-\delta\frac{\sum_{i=0}^{\infty} b_i r_1^{\,i}}{\varepsilon\sum_{i=0}^{\infty} a_i r_1^{\,i+2}}\right]} \qquad \varsigma < 0 , \tag{2.3.25}$$

where $$\varsigma \equiv \frac{\sum_{i=0}^{\infty} b_i r_1^i}{\varepsilon \sum_{i=0}^{\infty} a_i r_1^{i+2}} - \frac{2}{\delta} \quad . \tag{2.3.26}$$

Hence we may set $q_0=0$ only for

$$m^2 = q^2 \Rightarrow P \equiv 0 \tag{2.3.27}$$

and we have dust (this includes the possibility $\varsigma = 0$), or

$$\sum_{i=0}^{\infty} a_i r_1^{i+2} = 0 \Rightarrow P \propto (r-r_1)^n \quad , n \geq 3 \ . \tag{2.3.28}$$

2.4 Sample Solutions

In this section we will present two simple solutions which will be useful for later comparison to the relativistic system. We will assume a positive and negative pressure,

$$P = \mp \frac{A^2}{8\pi} (r-r_1)^2 (r-r_2) \ . \tag{2.4.1}$$

In both cases we will use

$$\sigma = \frac{B}{4\pi} (r-r_1) \ . \tag{2.4.2}$$

We find

$$\delta = \mp\left[\tfrac{3}{7}(r_2^7-r_1^7)-\tfrac{1}{3}(r_2+2r_1)(r_2^6-r_1^6)+\tfrac{1}{5}(2r_2+r_1)r_1(r_2^5-r_1^5)\right] \equiv \mp\eta . \tag{2.4.3}$$

Now $$\frac{105\,\eta}{r_2^7} = 4x^7-7x^6+21x^2-28x+10 \quad , \quad x \equiv \frac{r_1}{r_2} . \tag{2.4.4}$$

Treating x as a variable,

$$\frac{d}{dx}\left[\frac{105\,\eta}{r_2^7}\right] = 14(2x^6-3x^5+3x-2) = 14(x-1)^3(2x^3+3x^2+3x+2)$$

$$< 0 \qquad \text{for} \quad x \in (0,1) . \tag{2.4.5}$$

Thus $\left[\frac{105\,\eta}{r_2^7}\right]$ is everywhere decreasing on (0,1) and since

$$\left[\frac{105\,\eta}{r_2^7}\right](0) = 10 \qquad \text{and} \qquad \left[\frac{105\,\eta}{r_2^7}\right](1) = 0 \quad ,$$

we deduce $$\eta > 0 \qquad \text{for} \quad 0 < r_1 < r_2 .$$

Continuing, we find

$$\varepsilon = (r_2-r_1)^2\,\frac{\left[3r_2^2+2r_2r_1+r_1^2\right]}{12} \equiv \frac{\lambda}{12} . \tag{2.4.6}$$

Clearly $\lambda > 0$ for $0 < r_1 < r_2$. Finally

$$A^2 = -\frac{(m^2-q^2)-(m_0^2-q_0^2)}{\delta} = \pm\frac{(m^2-q^2)-(m_0^2-q_0^2)}{\eta},$$

$$B = \frac{q_2}{\varepsilon} = \frac{12q_2}{\lambda},$$

$$Q = q_0 + \frac{q_2}{\lambda}(r-r_1)^2\left[3r^2+2rr_1+r_1^2\right],$$

$$M = \Big\{ Q^2 + m_0^2 - q_0^2$$

$$+ \frac{(m^2-q^2)-(m_0^2-q_0^2)}{\eta}\left[\tfrac{3}{7}(r^7-r_1^7) - \tfrac{1}{3}(r_2+2r_1)(r^6-r_1^6) + \tfrac{1}{5}(2r_2+r_1)r_1(r^5-r_1^5)\right]\Big\}^{1/2}$$

$$p = \Big\{ \frac{3q_0q_2}{\pi\lambda}(r-r_1) + \frac{3q_2^2}{\pi\lambda^2}(r-r_1)\left[3r^2+2rr_1+r_1^2\right]$$

$$+ \frac{(m^2-q^2)-(m_0^2-q_0^2)}{8\pi\eta}(r-r_1)\left[3r-(2r_2+r_1)\right]\Big\} \Big/ M. \qquad (2.4.7)$$

Thus the sign of the pressure is determined by the sign of

$$\left[(m^2-q^2)-(m_0^2-q_0^2)\right].$$

We may let $q_0=0$ with no difficulty. However, if we let $m_0=0$,

$$\rho(r_1)=\frac{1}{4\pi r_1}\left[\frac{12q_0q_2}{\lambda}+\frac{m^2-(q^2-q_0^2)}{\eta}r_1^2(r_1-r_2)\right]^{1/2}, \tag{2.4.8}$$

so for real $\rho(r_1)$ we must require

$$q_0 \geq \frac{m^2-q_2^2}{2q_2\left[1+\frac{6\eta}{\lambda r_1^2(r_2-r_1)}\right]}. \tag{2.4.9}$$

If we now set $q_0=0$ we need $m^2< q^2$ ($m^2=q^2$ gives dust) and the pressure must be negative.

As one would expect the properties of these solutions reflect those of the more general solution of section 2.3. Q is totally independent of m_0 and m_2, and $M(r_1)=m_0$ and $M(r_2)=m$ are independent of q_0 and q_2. Thus we may expect to find $(m_0m_2-q_0q_2)$ anywhere in the region $(-\infty, \infty)$ and all three cases

$$q<m, \quad q=m, \quad q>m \tag{2.4.10}$$

are allowed. However if we require $P>0$ then we need

$$(m^2-q^2)-(m_0^2-q_0^2)=m_2^2-q_2^2+2(m_0m_2-q_0q_2)>0, \tag{2.4.11}$$

or $$m_0 m_2 - q_0 q_2 > \frac{q_2^2 - m_2^2}{2} \quad . \tag{2.4.12}$$

If we want $(m_0 m_2 - q_0 q_2) \rightarrow -\infty$ we may do so only with care, although it can be done.[2]

We may freely eliminate all charge from the core, but eliminating the core mass introduces a requirement for a minimum core charge unless we permit $m^2 < q^2$ which makes the pressure negative. Thus we may completely eliminate the core, leaving a self-balancing shell with pressure zero at the boundaries; but only if we are willing to permit a negative pressure.

2.5 A Note on the Balance Condition

In closing the chapter, a final note on the balance condition for the system is in order. Recall our definition of balance: the condition that the metric (gravitational field), the electric field, and the internal pressure combine to give a static system free of any supporting struts or skins.

Using $Q=Er^2$ and $M=-gr^2$ we may rewrite equation (2.1.5) in the form

$$dP = \frac{Q\,dQ - M\,dM}{4\pi r^4} \quad . \tag{2.5.1}$$

[2] eg. Make $m_0 = \frac{1}{2} q_2$, and $m_2 = q_0 > 2q_2$. (2.4.12) is satisfied as q_0, and hence m_2 and $-(m_0 m_2 - q_0 q_2)$, go to infinity.

This, the equilibrium condition, might be suggested as a balance condition. It is not. While it is certainly necessary for balance, it is not sufficient. The equation itself does not contain that much information; it is only in the boundary conditions that balance is determined. Unless we specifically arrange it beforehand there is no guarantee that the pressure will vanish at the boundaries of the bodies or that the mass and charge densities will remain nonsingular. Thus some sort of "skin" is likely to exist in order to support the shell or hold it in place, and the system will not be balanced.

To illustrate this one might consider a typical integration of (2.5.1) in the shell. This would involve choosing an M and Q, which must satisfy their own boundary conditions, and integrating the pressure to within an arbitrary constant. One may then apply to the pressure a single boundary condition, which will determine the constant, but the other boundary will not, in general, have zero pressure (even assuming that was our choice for the first boundary). A "skin" supports the shell and the system is not balanced. One might impose the condition that the pressure must be zero there and attempt to solve the resulting (polynomial) equation for a particular "balance condition" (assuming such a solution exists). However, this single condition would, in effect, be a result of restrictive charge and matter distributions and, as we have seen, such a system is merely a special, nonrepresentative case of the more general systems presented earlier. The important point is that, even in this case, it is only in the boundary conditions, and not the equilibrium condition, that balance is manifested. The core-and-shell system seems to have no single balance condition.

CHAPTER THREE

THE GENERALIZED REISSNER-NORDSTROM

3.1 The System

We will consider again the system illustrated in figure 1: a spherically symmetric core and shell of charged perfect fluid in static equilibrium that is "skinless" in that we require the pressure to be continuous at all boundaries. Thus we satisfy the Darmois junction conditions and eliminate any singular supporting "skins" from our solutions. The generalized interior Schwarzschild solution, which is the uncharged version of this system, has been previously investigated (without the "skinless" condition) by Cooperstock, Sarracino and Bayin (1981) and Sarracino (1981).

Once again we will focus our attention on the shell (and drop the subscript 2 on variables). As in the Newtonian case the shell in no way affects the core and thus it is in the shell solutions that balance is manifested. Also, spherically symmetric balls, both charged and uncharged, of perfect fluid and other matter have received considerable attention in the literature (see, for example Cooperstock de la Cruz, 1978; Krori and Paul, 1982; Bonnor, 1982), and since only the overall mass and charge of the core affect the shell we may use any core solution which satisfies the junction conditions.

We will use the Schwarzschild line element

$$ds^2 = g_{ik}dx^i dx^k = \mu^2 dt^2 - \nu^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \qquad (3.1.1)$$

where i and k run over 0-3, μ and ν are functions of radius only, and the Einstein summation convention is again in effect. We define:

$\rho \equiv$ mass density,

$\sigma \equiv$ charge density,

$\sigma^* \equiv$ proper charge density $= \sigma \nu^{1/2}$,

$\hat{r} \equiv$ unit radial vector,

$\bar{E} \equiv$ electric field $= E\hat{r}$ (generally assumed positive)

$P \equiv$ pressure,

$M \equiv$ total mass within a sphere of radius r,

$Q \equiv$ total charge within a sphere of radius r.

It might be appropriate to remind the reader that, despite appearances, many of these definitions are not identical to those of the Newtonian case. In particular, the constants m_0, m_2, and m of figure 1 now "contain" electromagnetic and pressure contributions and may be only loosely associated with their Newtonian values.

Recall equations (1.2.20), the Einstein-Maxwell equations in perfect fluid:

$$\frac{1}{r^2} - \left[\frac{\nu'}{r} + \frac{\nu}{r^2}\right] = 8\pi\rho + E^2 , \tag{3.1.2}$$

$$\frac{1}{r^2} - \nu\left[\frac{2}{r}\frac{\mu'}{\mu} + \frac{1}{r^2}\right] = -8\pi P + E^2 , \tag{3.1.3}$$

$$\frac{\mu''}{\mu}\nu + \frac{\mu'}{\mu}\frac{\nu'}{2} + \frac{\mu'}{\mu}\frac{\nu}{r} + \frac{\nu'}{2r} = 8\pi P + E^2 . \tag{3.1.4}$$

The most straightforward way to solve these equations is to integrate μ and ν in terms of E^2 and r and then to solve for ρ and P. However, even this can be done only in special cases (Kramer, et. al., 1980; Krori and Paul, 1982; Bonnor, 1982) or in terms of special generating functions (Matese and Whitman, 1980; Whitman and Burch, 1981) and such solutions do not generally allow a continuous pressure. Thus we must follow a different path, integrating first in terms of the pressure.

3.2 Integration of the System

The first two Einstein-Maxwell equations may be integrated, and the third simplified, with little real difficulty. This has been presented in Appendix B. We find,

$$\nu = 1 - \frac{2}{r}\int 4\pi r^2 T^0_0 \, dr = 1 - \frac{2M}{r}$$

$$= \frac{k_1}{r} + 1 - \frac{1}{r}\int (8\pi\rho + E^2) r^2 dr \ , \qquad (3.2.1)$$

where k_1 is the constant of integration, and

$$\mu^2 = \frac{k_2^2}{r} \exp \int \frac{(8\pi P - E^2) r^2 + 1}{\nu r} dr \ , \qquad (3.2.2)$$

where k_2 is arbitrary. The third equation becomes

$$uu' = 2XY + \left(rX' - \frac{2Y}{r}\right)u \,, \tag{3.2.3}$$

where
$$a = \int 8\pi P r^2 dr \,, \tag{3.2.4}$$

$$b = \int 8\pi \rho r^2 dr \,, \tag{3.2.5}$$

$$e = \int E^2 r^2 dr \,, \tag{3.2.6}$$

$$u(r) = b + a'r - e'r - k_1 + e \,, \tag{3.2.7}$$

$$\begin{aligned} X(r) &= a' - e' + 1 \\ &= (8\pi P - E^2)r^2 + 1 \,, \end{aligned} \tag{3.2.8}$$

and
$$\begin{aligned} Y(r) &= 2e'r + e''r^2 + 2a'r - a''r^2 \\ &= (E^2 r^4)' - r^4 (8\pi P)' . \end{aligned} \tag{3.2.9}$$

Equation (3.2.3) cannot in general be solved. We may proceed further only through the use of simplifying assumptions that, in effect specify a relation between the pressure and the square of the electric field. Regarding the similarity between the form of Y in (3.2.9) and the Newtonian equilibrium condition (2.1.5) one is tempted to make the assumption 2Y=MM', but unfortunately the resulting equation is much more difficult that (3.2.3). Only the three most basic assumptions result in any progress.

Assumption (a): let

$$Y = 0. \tag{3.2.10}$$

Then

$$E^2 = \frac{k_3}{r^4} + \frac{1}{r^4}\int 8\pi P' r^4 \, dr \tag{3.2.11}$$

where k_3 is the constant of integration, and

$$uu' = rX'u . \tag{3.2.12}$$

The integration of (3.2.12) involves two possible cases:

case (i)

$$u = 0 , \tag{3.2.13}$$

when

$$\rho = -3P - \frac{E^2}{4\pi} , \tag{3.2.14}$$

and case (ii)

$$u' = rX' , \tag{3.2.15}$$

when

$$\rho = -P. \tag{3.2.16}$$

Assumption (b): let

$$X = 0. \tag{3.2.17}$$

Then
$$E^2 = \frac{1}{r^2} + 8\pi P, \tag{3.2.18}$$

and
$$uu' = -\frac{2Y}{r}u. \tag{3.2.19}$$

Again the integration involves two cases

case (i)
$$u = 0, \tag{3.2.20}$$

when
$$\rho = -P, \tag{3.2.21}$$

and case (ii)
$$u' = -\frac{2Y}{r}, \tag{3.2.22}$$

when
$$\rho = \frac{-1}{2\pi r^2} - 9P. \tag{3.2.23}$$

Assumption (c): let

$$2Y = r^2 X'. \tag{3.2.24}$$

Then
$$E^2 = \frac{k_3}{r^{10/3}} + 8\pi P - \frac{8}{3r^{10/3}}\int 8\pi P r^{7/3}\,dr, \tag{3.2.25}$$

and $$uu' = 2XY = r^2X'X \,. \tag{3.2.26}$$

After integrating (3.2.26) we find,

$$\rho = -3P - P'r - \frac{E^2}{4\pi} - \left(\frac{E^2}{8\pi}\right)' r + \frac{1}{8\pi}\frac{X'X}{\left[2k_4 + 2\int X'Xr^2 dr\right]^{1/2}} \,, \tag{3.2.27}$$

where k_4 is another constant of integration, and

$$X = 8\pi P r^2 - E^2r^2 + 1 = 1 - \frac{k_3}{r^{4/3}} + \frac{8}{3r^{4/3}}\int 8\pi P r^{7/3} dr \,. \tag{3.2.28}$$

In the following we will refer to these solutions as solutions a1, a2, b1, b2, and c respectively. Aside from its bizarre appearance, solution c is quickly found to be intractable and will not be considered further.

It is interesting to note the appearance of the same equation of state $\rho = -P$ in the two solutions a2 and b1. It is easy to show that, assuming this equation of state, the integration of (3.2.3) results in only these two solutions. Other assumed equations of state have proven either partially or completely unsolvable. Thus the only equation of state for which we have the complete solution is the unusual one, $\rho = -P$.

All three of the equations of state of solutions a1, a2, b1, and b2 are unusual in that the pressure must be negative if the mass density is to be positive and the mass density will then be of the same order of magnitude as the pressure. Thus the energy of interaction of this matter contributes significantly to its mass. Most previous work on relativistic equations of state has dealt with maximum physically attainable densities and superluminal sound velocities (Ruderman, 1968; Bludman and Ruderman, 1968, 1970; Hegyi, et. al., 1975), incompressibility (Cooperstock and Sarracino, 1976; Sarracino, 1977), and special source interiors (Cooperstock and de la Cruz, 1978; Gonzalez-Diaz, 1981) but only particular cases are examined and pressures are assumed positive. However, based on the work of Hawking and Ellis (1973), Bonnor (1982) has given requirements for what he considers physically reasonable matter. These requirements are

$$\rho \geq 0 \;,\; -\rho \leq P_\alpha \leq \rho \;,\; \rho + \sum_\alpha P_\alpha \geq 0 \quad (\alpha = 1,2,3) \tag{3.2.29}$$

where the P_α are the three principal pressures. None of our equations of state satisfy all three of these conditions, although that of solution b2 can be made to do so in the central regions of the shell.

Gonzalez-Diaz (1981) has used a simple $\rho = -P$ equation of state in describing the interior solution of a ball bounded by the event horizon of a Schwarzschild black hole. However the radial pressure is not continuous at this boundary and his solution does not satisfy the Darmois junction conditions. Thus the event horizon is coexistent with a singular skin and as the pressure is negative (inward, as is the gravitational "force") it appears to be this skin which supports his interior.

It is apparent that $\rho = -P$ and the other equations of state we are considering are unlikely to represent any known form of matter and may prove to be completely unrealizable. We proceed only with the hope that, as in the Newtonian case, particular solutions of this system may illustrate general properties.

3.3 Particular Solutions

In this section we will present particular examples for each of the solutions a1, a2, b1 and b2. We assume the simplest pressure which is smooth at the inner boundary

$$P = \frac{A^2}{8\pi}(r-r_1)^2(r-r_2) \quad ,$$

$$P' = \frac{A^2}{8\pi}(r-r_1)(3r-2r_2-r_1) \quad . \tag{3.3.1}$$

The derivatives are given in Appendix C.

Solution a:

$$A^2 = \frac{q^2-q_0^2}{\xi} \quad , \tag{3.3.2}$$

$$E^2 = \frac{Q^2}{r^4} = \frac{q_0^2}{r^4} + \frac{q^2-q_0^2}{r^4}\,\frac{F_a(r)-F_a(r_1)}{\xi}$$

$$= \frac{q^2}{r^4} + \frac{q^2-q_0^2}{r^4}\,\frac{F_a(r)-F_a(r_2)}{\xi} \tag{3.3.3}$$

$$\sigma = \frac{P'}{E} \quad , \quad \sigma \neq 0 \, , \tag{3.3.4}$$

$$F_a(r) = \frac{3}{7} r^7 - \frac{r_2 + 2r_1}{3} r^6 + \frac{(2r_2 + r_1) r_1}{5} r^5 , \tag{3.3.5}$$

$$\xi = F_a(r_2) - F_a(r_1) > 0 \, . \tag{3.3.6}$$

$$E^2 > 0 \quad \Rightarrow \quad \frac{q_0^2}{q^2} > \frac{F_a(r_1) - F_{min}}{F_a(r_2) - F_{min}} \quad , \tag{3.3.7}$$

$$F_{min} = F_a\left(\frac{2r_2 + r_1}{3}\right) \, . \tag{3.3.8}$$

case (i)
$$\rho = -3P - \frac{E^2}{4\pi} \quad , \quad \rho \neq 0 \, , \tag{3.3.9}$$

$$M^2 = k_2^2 \exp \int \frac{-k_1}{\nu r^2} \, dr \, , \qquad \text{(intractable)}, \tag{3.3.10}$$

$$\nu = 1 - \frac{2m_0}{r} + \frac{2}{r}\frac{q_0^2}{r_1} + 8\pi P r^2 - E^2 r^2$$

$$= 1 - \frac{2m}{r} + \frac{2}{r}\frac{q^2}{r_2} + 8\pi P - E^2 r^2 \quad , \tag{3.3.11}$$

$$m - m_0 = \frac{q^2}{r_2} - \frac{q_0^2}{r_1} \quad . \tag{3.3.12}$$

$\mu^2 > 0$ follows from $\nu > 0$. (3.3.11) is too difficult to exactly determine the condition for $\nu > 0$. A weaker condition is

$$m < \frac{r_1^2}{2r_2} + \frac{2r_1 - r_2}{2r_2^2} q^2 \tag{3.3.13}$$

As (3.3.11) proves intractable, no analysis of the singular behaviour of this solution has been made.

case (ii) $$\rho = -P \, , \tag{3.3.14}$$

$$\mu^2 = \nu \, , \tag{3.3.15}$$

$$\nu = 1 - \frac{2m_0}{r} + E^2 r^2 + \frac{q^2 - q_0^2}{\xi} \frac{\Omega(r_1) - \Omega(r)}{r}$$

$$= 1 - \frac{2m}{r} + E^2 r^2 + \frac{q^2 - q_0^2}{\xi} \frac{\Omega(r_2) - \Omega(r)}{r} \, , \tag{3.3.16}$$

$$m - m_0 = \frac{q^2 - q_0^2}{\xi} \frac{\Omega(r_2) - \Omega(r_1)}{2} \, , \tag{3.3.17}$$

$$\Omega(r) = \frac{1}{3}r^6 - \frac{r_2+2r_1}{5}r^5 + \frac{r_2 r_1^2}{3}r^3 \tag{3.3.18}$$

(3.3.16) is too difficult to exactly determine the condition for $\nu > 0$. A weaker condition is

$$m < \frac{r_1^2 + Q_{min}^2}{2r_2} , \tag{3.3.19}$$

$$Q_{min}^2 = Q^2\left(\frac{2r_2 + r_1}{3}\right) . \tag{3.3.20}$$

As the expression for ν again proves intractable, no analysis of the singular behaviour of this solution has been made.

Solution b:

$$E^2 = \frac{Q^2}{r^4} = \frac{1}{r^2} + 8\pi P , \tag{3.3.21}$$

$$\sigma = \frac{\frac{1}{4\pi r^2} + \frac{A^2}{8\pi}\left[7r^3 - 6(r_2+2r_1)r^2 + 5(2r_2+r_1)r_1 r - 4r_2 r_1^2\right]}{\left[1 - A^2 F_b(r)\right]^{1/2}} , \tag{3.3.22}$$

$$q_0^2 = r_1^2 , \quad q^2 = r_2^2 , \tag{3.3.23}$$

$$F_b(r) = -r^2(r-r_1)^2(r-r_2)\,. \tag{3.3.24}$$

$$E^2 > 0 \Rightarrow A^2 < \frac{1}{F_b(r_b)}, \tag{3.3.25}$$

$$r_b = \frac{3r_1 + 4r_2 + (9r_1^2 - 16r_1r_2 + 16r_2^2)^{1/2}}{10}. \tag{3.3.26}$$

case (i) $$\rho = -P, \tag{3.3.27}$$

$$A^2 \quad \text{is undetermined,} \tag{3.3.28}$$

$$\mu^2 = \nu \tag{3.3.29}$$

$$\nu = 2\frac{(q_0 - m_0)}{r} = 2\frac{(q-m)}{r}, \tag{3.3.30}$$

$$m - m_0 = q - q_0 \,. \tag{3.3.31}$$

$$\nu > 0 \Rightarrow q > m \,. \tag{3.3.32}$$

Clearly the behaviour of ν is closely tied to the charge in the system and, recalling (3.3.23), the configuration of the shell. The singularity in ν is much like that of the event horizon in the Schwarzschild geometry. ν behaves properly for $q > m$. For $q < m$, g_{00} and g_{11} switch signs and t and r switch roles; the shell is no longer static and represents a region of collapse. The singularity at $q=m$ is a coordinate singularity and may be removed via an appropriate transformation. For example, with

$$q>m, \quad \bar{t} = \Psi_1(t,r) \quad , \bar{r} = \Psi_2(t,r)$$

$$q<m, \quad \bar{t} = \Psi_2(t,r) \quad , \bar{r} = \Psi_1(t,r)$$

$$\Psi_1(t,r) = 4\,|q-m|^{1/2}\,t + \frac{r^2}{2|q-m|^{1/2}}$$

$$\Psi_2(t,r) = 2\,|q-m|^{1/2}\,t + \frac{r^2}{|q-m|^{1/2}} \tag{3.3.33}$$

the metric becomes

$$ds^2 = \frac{d\bar{t}^2 - d\bar{r}^2}{6r} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \tag{3.3.34}$$

where r is now thought of as a function of $\bar{t}$, $\bar{r}$ given implicitly by (3.3.33). Note that although (3.3.34) is no longer singular at $q=m$, the metric still depends indirectly on $(q-m)$. While the functions Ψ_1, Ψ_2 are quite similar, it is Ψ_1 which is more "timelike". Thus, although we have switched the symbols $\bar{t}$, $\bar{r}$, the $q < m$ case still involves a "timelike" radial component and a "radiuslike" time component.

case (ii) $$\rho = \frac{-1}{2\pi r^2} - qP \quad , \; \rho \neq 0 , \tag{3.3.35}$$

$$A^2 = \frac{m - m_0 + q - q_0}{4\omega} , \tag{3.3.36}$$

$$\mu^2 = \frac{2(q-m)}{r} , \tag{3.3.37}$$

$$\nu = 4 - \frac{2(m+q)}{r} + 2\,\frac{(m+q)-(m_0+q_0)}{r}\,\frac{W(r_2)-W(r)}{\omega}$$

$$= 4 - \frac{2(m_0+q_0)}{r} + 2\,\frac{(m+q)-(m_0+q_0)}{r}\,\frac{W(r_1)-W(r)}{\omega} , \tag{3.3.38}$$

$$W(r) = \frac{-1}{6} r^6 + \frac{(r_2+2r_1)}{5} r^5 - \frac{(2r_2+r_1)}{4} r_1 r^4 + \frac{r_2 r_1^2}{3} r^3 , \tag{3.3.39}$$

$$\omega = W(r_2) - W(r_1) > 0 . \tag{3.3.40}$$

$$\mu^2 > 0 \Rightarrow q > m . \tag{3.3.41}$$

(3.3.38) is too difficult to determind the exact condition for $\nu > 0$. A weaker condition is

$$m < 2q_0 - q . \tag{3.3.42}$$

The singular behaviour of μ^2 is identical to that of μ^2, ν in solution b1; q=m is a coordinate singularity which may be transformed away. ν is not suitable for exact analysis of its singular behaviour. However it is easily seen from (3.3.38) that for q=m, $\nu(r_2)=0$ matching $\mu^2(r_2)$ but $\nu(r_1) \neq \mu^2(r_1)=0$. The behaviour of ν at the inner boundary is no longer locked into its behaviour at the outer edge of the shell.

In all of the cases presented above we had for the mass-constant in the inner electrovac region

$$m_0 = \int_0^{r_1} 4\pi T_0^0 r^2 dr + \frac{q_0^2}{2r_1} = \int_0^{r_0} 4\pi T_0^0 r^2 dr + \frac{q_0^2}{2r_0} . \tag{3.3.43}$$

Thus the "mass" inside the shell is determined as if the shell did not exist and the inner electrovac region extended to infinity.

The solution for the inner electrovac region (I) is

$$E_1 = \frac{q_0}{r^2} , \tag{3.3.44}$$

$$\nu_1 = 1 - \frac{2m_0}{r} + \frac{q_0^2}{r^2} , \tag{3.3.45}$$

$$\mu_1^2 = K\nu_1 \tag{3.3.46}$$

where, for solutions a1: K was not determined

a2,b1: $K = 1$

$$b2: \qquad K = \frac{q-m}{q_0-m_0} \,. \tag{3.3.47}$$

In all cases, the solution for the outer electrovac region (III) is

$$E_3 = \frac{q}{r^2} \,, \tag{3.3.48}$$

$$\nu_3 = \mu_3^2 = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \,. \tag{3.3.49}$$

It should be noted that in region I the condition that $\nu > 0$ may be different that in region III. Consider

$$r^2 \nu_3 = r^2 - 2mr + q^2 < 0 \tag{3.3.50}$$

for
$$r \in \left(m - \sqrt{m^2 - q^2} \,,\; m + \sqrt{m^2 - q^2} \right) . \tag{3.3.51}$$

Thus in region III, $\nu_3 > 0$ requires either

$$m^2 < q^2 \,,$$

or
$$r_2 > m + \sqrt{m^2 - q^2} \,. \tag{3.3.52}$$

while in region I, $\nu_1 > 0$ requires

$$m_0^2 < q_0^2 \;,$$

$$r_0 > m_0 + \sqrt{m_0^2 - q_0^2} \;,$$

or
$$r_1 < m_0 - \sqrt{m_0^2 - q_0^2} \;. \tag{3.3.53}$$

The last condition allows us access to the well-behaved region inside the event horizons of the Reissner-Nordstrom solution.

3.4 Properties of the Solutions

We will now investigate and compare the properties of the solutions for the shell presented in the last section. As in the Newtonian example, our pressure is smooth at the inner boundary of the shell. However, here the mass and charge densities are not all zero at r_1. We have

a1: $$\sigma(r_1) = 0 \;, \quad \rho(r_1) = \frac{-q_0^2}{4\pi r_1^4} \;,$$

a2: $$\sigma(r_1) = 0 \;, \quad \rho(r_1) = 0 \;,$$

b1: $$\sigma(r_1) = \frac{1}{4\pi r_1^2} \;, \quad \rho(r_1) = 0 \;,$$

b2: $$\sigma(r_1) = \frac{1}{4\pi r_1^2} \;, \quad \rho(r_1) = \frac{-1}{2\pi r_1^2} \;. \tag{3.4.1}$$

In the Newtonian example we could, with a negative pressure, dispense with the core entirely. We may not do so here.

a1: $q_0 = 0 \Rightarrow E^2 < 0$ in much of the shell.

a2: $q_0 = 0 \Rightarrow E^2 < 0$ in much of the shell.

b1: $q_0 = 0 \Rightarrow$ (i) $r_1 = 0$ and the core becomes a point mass,

(ii) $\nu < 0$.

b2: $q_0 = 0 \Rightarrow$ (i) $r_1 = 0$ and the core becomes a point mass,

$$\text{(ii)} \quad \nu < 0. \tag{3.4.2}$$

Thus we need a charge on the core. It would appear, however, that in all of solutions a1, a2, b1, b2 we may freely let $m_0=0$. But, unless we also have $q_0=0$, this is misleading. Both the electromagnetic field and the matter distribution contribute to m_0 so, if $q_0 > 0$, making $m_0=0$ would require a negative (core) mass density.

In the Newtonian example we were allowed all of the possibilities $q < m$, $q=m$, $q > m$. Here we may have

a1: $q < m, q = m, q > m$ and still have (3.3.13),

a2: $q < m, q = m, q > m$ and still have (3.3.19), b1: $q > m$,

b2: $q > m$. (3.4.3)

Finally, in the Newtonian example we had no particular balance condition and, in fact, might find $(m_0m_2-q_0q_2)$ anywhere in the range $(-\infty, \infty)$. In the relativistic case m_0, m are no longer the mass of the core and the core plus shell respectively. Each contains energy contributions from both the pressure distribution and the electric field and direct comparison to their Newtonian counterparts is simply not valid. We can only loosely associate m_0 with the "mass" of the core and m with the "mass" of the system. It is only with this in mind that we define

$$m_2 = m - m_0 \tag{3.4.4}$$

and think about $(m_0m_2-q_0q_2)$. Thus we find

a1: $-\infty < (m_0m_2 - q_0q_2) < \infty$,

a2: $-\infty < (m_0m_2 - q_0q_2) < \infty$,

b1: $-\infty < (m_0m_2 - q_0q_2) < 0$,

b2: $-\infty < (m_0m_2 - q_0q_2) < 0$. (3.3.5)

As a last note it is interesting to observe that μ^2, ν in solution b1 and μ^2 in solution b2 are totally independent of the functional form of the pressure. It appears that assumption b (3.2.17) results in an electric field which is unusually attuned to the pressure.

3.5 Newtonian Limits

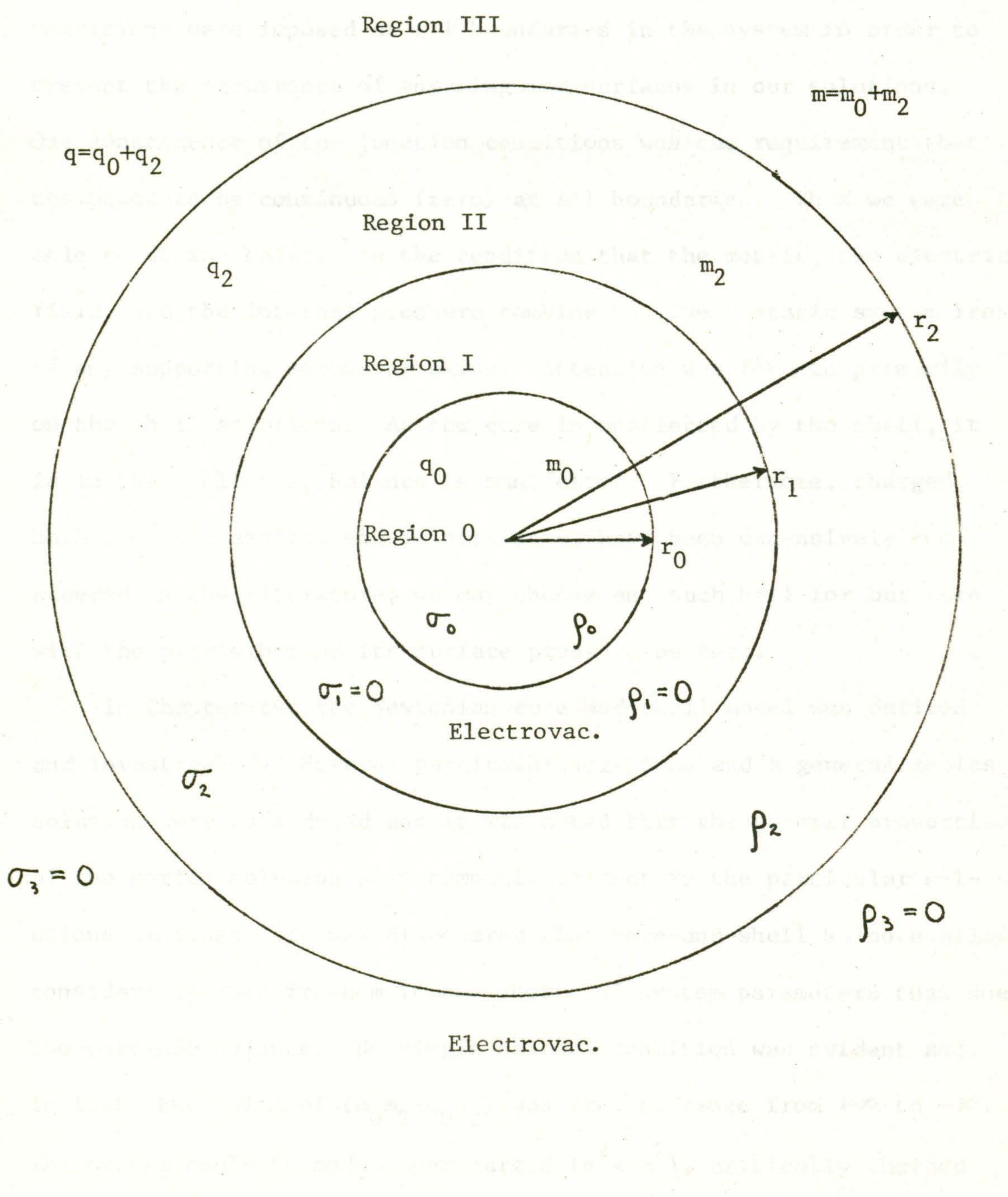
Now that we have gone from the Newtonian case to the general relativistic one, it is tempting to complete the circle, taking the Newtonian limits of the solutions we have investigated. Unfortunately the Newtonian limit requires (Misner, Thorne, and Wheeler, 1973)

$$T_\alpha^\beta \ll T_0^0 \quad , \quad (\alpha, \beta = 1, 2, 3) \tag{3.5.1}$$

a condition which we cannot satisfy. Our solutions are highly relativistic, and have no Newtonian limits.

Figure 1

The Core-and-Shell Model



SUMMARY AND CONCLUSIONS

In this thesis we have derived and compared General Relativistic and Newtonian solutions for a spherically symmetric core and shell of charged perfect fluid in balanced equilibrium. The Darmois junction conditions were imposed at all boundaries in the system in order to prevent the occurrence of any singular surfaces in our solutions. One consequence of the junction conditions was the requirement that the pressure be continuous (zero) at all boundaries. Thus we were able to define balance as the condition that the metric, the electric field, and the internal pressure combine to give a static system free of any supporting struts or skins. Attention was focused primarily on the shell solutions. As the core is unaffected by the shell, it is in the shell that balance is manifested. Furthermore, charged balls, both classical and relativistic, have been extensively considered in the literature; we may choose any such ball for our core with the provisio that its surface pressure be zero.

In Chapter two the Newtonian core-and-shell model was derived and investigated. Several particular solutions and a general series solution were considered and it was noted that the general properties of the series solution were commonly evident in the particular solutions examined. It was discovered that core-and-shell balance allows considerably more freedom in the choice of system parameters than does two-particle balance. No single balance condition was evident and, in fact, the value of $(m_0 m_2 - q_0 q_2)$ was free to range from $+\infty$ to $-\infty$. The system could be made undercharged ($q^2 < m^2$), critically charged

($q^2=m^2$), or overcharged ($q^2 > m^2$), as desired. The shell was not strongly dependent on the core. The core charge could be freely eliminated. For positive pressure, elimination of the core mass required a minimum core charge. If negative pressures were permitted, the entire core could be freely eliminated.

In Chapter three the General Relativistic core-and-shell model was investigated. The Einstein-Maxwell equations could not be solved generally. Only four usable solutions were derived, all with negative pressure. All had unusual equations of state which are likely to be unphysical, the most reasonable one being $\rho=-P$. No single balance condition was evident for any of these solutions. Two of these solutions permitted the reinterpreted ($m_0m_2-q_0q_2$) to take any value in the range $(-\infty, \infty)$, the other two permitted only the more restricted range $(-\infty, 0)$. Two of the solutions permitted the system to be undercharged, critically charged, or overcharged; the other two required an overcharged system. All of the solutions required a core charge, although we could freely eliminate the core mass if we allowed the core to have a negative mass density.

The results presented here are weakened by the lack of a more general, relativistic solution and by the unphysical nature of the particular solutions which we do have. However, we may hope that, as in the Newtonian case, our particular solutions exhibit quite general properties. In any case, the nature of our results prevent quantitative conclusions. The relativistic solutions, though more prone to restrictions, tend to exhibit the general properties of the Newtonian solutions, except that they show a definite dependence on a core charge and mass (for realistic matter). It is interesting that it is the relativistic system which appears to have the least freedom.

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APPENDIX A

We have

$$P(r) = \frac{A^2}{8\pi}(r-r_1)^2 \sum_{i=0} a_i r^i \quad , \quad P(r_2) = 0 \quad . \tag{A.1}$$

Therefore $\frac{8\pi}{A^2} P' = 2(r-r_1)\sum_{i=0}^{\infty} a_i r^i + (r^2 - 2rr_1 + r_1^2)\sum_{i=0} i a_i r^{i-1}$

$$= (r-r_1)\left\{\sum_{i=0}^{\infty}(i+2)a_i r^i - r_1 \sum_{j=0}^{\infty}(j+1)a_{j+1}r^j\right\}$$

$$= (r-r_1)\sum_{i=0}^{\infty}\left[(i+2)a_i - r_1(i+1)a_{i+1}\right]r^i \quad . \tag{A.2}$$

We define $\gamma_i = \left[(i+2)a_i - r_1(i+1)a_{i+1}\right]$. (A.3)

Alternatively $\frac{8\pi}{A^2} P' = \sum_{i=0}^{\infty}(i+2)a_i r^{i+1} - \sum_{i=0}^{\infty} 2(i+1)a_i r_1 r^i + \sum_{i=0}^{\infty} i a_i r_1^2 r^{i-1}$

$$= \sum_{j=1}^{\infty}(j+1)a_{j-1}r^j - \sum_{i=0}^{\infty} 2(i+1)a_i r_1 r^i + \sum_{j=0}^{\infty}(j+1)a_{j+1}r_1^2 r^j \tag{A.4}$$

If we define $a_{-1} \equiv 0$,

then $\frac{8\pi}{A^2} P' = \sum_{i=0}^{\infty}(i+1)\left[a_{i-1} - 2a_i r_1 + a_{i+1}r_1^2\right]r^i$. (A.5)

Let $$\alpha_i = [a_{i-1} - 2a_i r_1 + a_{i+1} r_1^2] \; . \tag{A.6}$$

Then $$\frac{8\pi}{A^2}\int r^4 P' dr = \sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i r^{i+5} + k \, , \tag{A.7}$$

where k is the constant of integration. Thus,

$$g^2 = E^2 - \frac{8\pi}{r^4}\int r^4 P' dr = E^2 - A^2\left\{\sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i r^{i+1} + \frac{k}{r^4}\right\} . \tag{A.8}$$

Applying boundary conditions,

$$g^2(r_1) = \frac{m_0^2}{r_1^4} = \frac{q_0^2}{r_1^4} - A^2\left\{\sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i r_1^{i+1} + \frac{k}{r_1^4}\right\}$$

$$\Rightarrow -k = \frac{m_0^2 - q_0^2}{A^2} + \sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i r_1^{i+5} \; , \tag{A.9}$$

so $$g^2 = E^2 + \frac{m_0^2 - q_0^2}{r^4} - \frac{A^2}{r^4}\sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i (r^{i+5} - r_1^{i+5}) \; . \tag{A.10}$$

Thus $$g^2(r_2) = \frac{m^2}{r_2^4} = \frac{q^2}{r_2^4} + \frac{m_0^2 - q_0^2}{r_2^4} - \frac{A^2}{r_2^4}\sum_{i=0}^{\infty}\left(\frac{i+1}{i+5}\right)\alpha_i (r_2^{i+5} - r_1^{i+5})$$

$$\Rightarrow -A^2 = \frac{(m^2 - q^2) - (m_0^2 - q_0^2)}{\delta} \tag{A.11}$$

where we have defined

$$\delta = \sum_{i=0}^{\infty} \left(\frac{i+1}{i+5}\right) \alpha_i \left(r_2^{i+5} - r_1^{i+5}\right). \tag{A.12}$$

Now,
$$(r^n - r_1^n) = (r - r_1)^2 p_{n-2}(r) + (r - r_1) n r_1^{n-1} \tag{A.13}$$

where
$$p_n(r) = \sum_{i=0}^{n} (i+1) r^{n-i} r_1^i . \tag{A.14}$$

Thus
$$\sum_{i=0}^{\infty} \left(\frac{i+1}{i+5}\right) \alpha_i (r^{i+5} - r_1^{i+5})$$

$$= \sum_{i=0}^{\infty} \left(\frac{i+1}{i+5}\right) \alpha_i \left[(r-r_1)^2 p_{i+3} + (r-r_1)(i+5) r_1^{i+4}\right] . \tag{A.15}$$

Also
$$\sum_{i=0}^{\infty} (i+1)(r-r_1) r_1^{i+4} \alpha_i$$

$$= (r-r_1) \sum_{i=0}^{\infty} \left\{ (i+1) a_{i-1} r_1^{i+4} - 2(i+1) a_i r_1^{i+5} + (i+1) a_{i+1} r_1^{i+6} \right\}$$

$$= (r-r_1) \left\{ \sum_{j=0}^{\infty} (j+2) a_j r_1^{j+5} - \sum_{i=0}^{\infty} 2(i+1) a_i r_1^{i+5} + \sum_{j=1}^{\infty} j a_j r_1^{j+5} \right\}$$

$$= (r-r_1) \left\{ 2a_0 r_1^5 - 2a_0 r_1^5 + \sum_{j=1}^{\infty} a_j r_1^{j+5} \left[j+2 - 2(j+1) + j \right] \right\}$$

$$= 0 . \tag{A.16}$$

Therefore
$$\sum_{i=0}^{\infty} \left(\frac{i+1}{i+5}\right) \alpha_i (r^{i+5} - r_1^{i+5}) = (r-r_1)^2 \sum_{i=0}^{\infty} \left(\frac{i+1}{i+5}\right) \alpha_i p_{i+3} \tag{A.17}$$

and $$\delta = \sum_{i=0}^{\infty}\binom{i+1}{i+5}\alpha_i\left(r_2^{i+5}-r_1^{i+5}\right) = (r_2-r_1)^2\sum_{i=0}^{\infty}\binom{i+1}{i+5}\alpha_i\, p_{i+3}(r_2) \;. \tag{A.18}$$

Thus $$g^2 = E^2 + \frac{m_0^2-q_0^2}{r^4} + \frac{(m^2-q^2)-(m_0^2-q_0^2)}{r^4}\,\frac{(r-r_1)^2}{\delta}\sum_{i=0}^{\infty}\binom{i+1}{i+5}\alpha_i\, p_{i+3} \;,$$

$$P = -\,\frac{(m^2-q^2)-(m_0^2-q_0^2)}{8\pi\delta}\,(r-r_1)^2\sum_{i=0}^{\infty} a_i r^i \;,$$

$$P' = -\,\frac{(m^2-q^2)-(m_0^2-q_0^2)}{8\pi\delta}\sum_{i=0}^{\infty}(i+1)\,\alpha_i r^i$$

$$= -\,\frac{(m^2-q^2)-(m_0^2-q_0^2)}{8\pi\delta}\,(r-r_1)\sum_{i=0}^{\infty}\gamma_i r^i \;. \tag{A.19}$$

It is convenient here to define $\sigma(r)$ and derive E(r).

Let $$4\pi\sigma(r) = B\left[(r-r_1)\sum_{i=0}^{\infty} b_i r^i + \Lambda\right]$$

$$= B\left[\sum_{i=0}^{\infty}\beta_i r^i + \Lambda\right] \quad , r_1 \le r \le r_2 \;, \tag{A.20}$$

where $$\beta_i = b_{i-1} - r_1 b_i \qquad , b_{-1} \equiv 0 \;, \tag{A.21}$$

and Λ is an arbitrary constant.

Then $$Q = r^2 E = B\int\left[\sum_{i=0}^{\infty}\beta_i r^i + \Lambda\right] r^2\, dr$$

or $$Q = B\left[\sum_{i=0}^{\infty} \frac{\beta_i}{i+3} r^{i+3} + \frac{\Lambda}{3} r^3 + C\right] . \tag{A.22}$$

where C is the constant of integration. Applying boundary conditions

$$Q(r_1) = q_0 = B\left[\sum_{i=0}^{\infty} \frac{\beta_i}{i+3} r_1^{i+3} + \frac{\Lambda}{3} r_1^3 + C\right]$$

$$\Longrightarrow C = \frac{q_0}{B} - \left[\sum_{i=0}^{\infty} \frac{\beta_i}{i+3} r_1^{i+3} + \frac{\Lambda}{3} r_1^3\right] , \tag{A.23}$$

so $$Q = q_0 + B\left[\sum_{i=0}^{\infty} \frac{\beta_i}{i+3} (r^{i+3} - r_1^{i+3}) + \frac{\Lambda}{3} (r^3 - r_1^3)\right] . \tag{A.24}$$

Thus $$Q(r_2) = q \Rightarrow B = \frac{q}{\mathcal{E}} \tag{A.25}$$

where we have defined

$$\mathcal{E} = \sum_{i=0}^{\infty} \frac{\beta_i}{i+3} (r_2^{i+3} - r_1^{i+3}) + \frac{\Lambda}{3} (r_2^3 - r_1^3) . \tag{A.26}$$

Also $$\frac{\beta_i}{i+3}(r^{i+3} - r_1^{i+3}) = \frac{\beta_i}{i+3}\left[(r-r_1)^2 p_{i+1} + (r-r_1)(i+3) r_1^{i+2}\right] , \tag{A.27}$$

and $$(r-r_1)\sum_{i=0}^{\infty}\beta_i r_1^{i+2} = (r-r_1)\sum_{i=0}^{\infty}(b_{i-1}-r_1 b_i)r_1^{i+2}$$

$$= (r-r_1)\left\{\sum_{j=-1}^{\infty} b_j r_1^{j+3} - \sum_{i=0}^{\infty} b_i r_1^{i+3}\right\}$$

$$= 0 \quad . \qquad \text{(A.28)}$$

Therefore $$\sum_{i=0}^{\infty}\frac{\beta_i}{i+3}(r^{i+3}-r_1^{i+3}) + \frac{\Lambda}{3}(r^3-r_1^3)$$

$$= (r-r_1)^2\left[\sum_{i=0}^{\infty}\frac{\beta_i}{i+3}p_{i+1} + \frac{\Lambda}{3}(r+2r_1)\right] + (r-r_1)\Lambda r_1^2 \qquad \text{(A.29)}$$

and $$\varepsilon = \sum_{i=0}^{\infty}\frac{\beta_i}{i+3}(r_2^{i+3}-r_1^{i+3}) + \frac{\Lambda}{3}(r_2^3-r_1^3)$$

$$= (r_2-r_1)^2\left[\sum_{i=0}^{\infty}\frac{\beta_i}{i+3}p_{i+1}(r_2) + \frac{\Lambda}{3}(r_2+2r_1)\right] + (r_2-r_1)\Lambda r_1^2 \quad . \qquad \text{(A.30)}$$

Thus $$\sigma = \frac{q_2}{4\pi\varepsilon}\left[(r-r_1)\sum_{i=0}^{\infty} b_i r^i + \Lambda\right] \quad ,$$

$$Q = q_0 + \frac{q_2}{\varepsilon}\left[\sum_{i=0}^{\infty}\frac{\beta_i}{i+3}(r^{i+3}-r_1^{i+3}) + \frac{\Lambda}{3}(r^3-r_1^3)\right]$$

$$= q_0 + \frac{q_2}{\varepsilon}\left\{(r-r_1)^2\left[\sum_{i=0}^{\infty}\frac{\beta_i}{i+3}p_{i+1} + \frac{\Lambda}{3}(r+2r_1)\right] + (r-r_1)\Lambda r_1^2\right\} \quad ,$$

$$E = \frac{Q}{r^2} ,$$

$$M = \left[Q^2 + m_0^2 - q_0^2 + \frac{(m^2-q^2)-(m_0^2-q_0^2)}{\delta} \sum_{i=0}^{\infty} \left(\frac{i+1}{i+5}\right) \alpha_i (r^{i+5} - r_1^{i+5}) \right]^{1/2}$$

$$= \left[Q^2 + m_0^2 - q_0^2 + \frac{(m^2-q^2)-(m_0^2-q_0^2)}{\delta} (r-r_1)^2 \sum_{i=0}^{\infty} \left(\frac{i+1}{i+5}\right) \alpha_i p_{i+3} \right]^{1/2} ,$$

$$g = -\frac{M}{r^2} . \tag{A.31}$$

Now $$\rho = \frac{\sigma E - P'}{-g} = \frac{\sigma Q - r^2 P'}{M} . \tag{A.32}$$

Thus

$$\rho = \left\{ \frac{q_0 q_2}{4\pi\varepsilon} \left[(r-r_1) \sum_{i=0}^{\infty} b_i r^i + \Lambda \right] \right.$$

$$+ \frac{q_2^2}{4\pi\varepsilon^2} \left[(r-r_1) \sum_{i=0}^{\infty} b_i r^i + \Lambda \right] \left[\sum_{i=0}^{\infty} \frac{\beta_i}{i+3} (r^{i+3} - r_1^{i+3}) + \frac{\Lambda}{3} (r^3 - r_1^3) \right]$$

$$\left. + \frac{(m^2-q^2)-(m_0^2-q_0^2)}{8\pi\delta} \sum_{i=0}^{\infty} (i+1) \alpha_i r^{i+2} \right\} \Big/ M$$

or

$$\rho = \Bigg\{ \frac{q_0 q_1}{4\pi\varepsilon}\left[(r-r_1)\sum_{i=0}^{\infty} b_i r^i + \Lambda\right]$$

$$+ \frac{q_1^2}{4\pi\varepsilon^2}\left[(r-r_1)\sum_{i=0}^{\infty} b_i r^i + \Lambda\right]\left[(r-r_1)^2\left(\sum_{i=0}^{\infty}\frac{\beta_i}{i+3}p_{i+1} + \frac{\Lambda}{3}(r+2r_1)\right) + (r-r_1)\Lambda r_1^2\right]$$

$$+ \frac{(m_1^2-q_1^2)-(m_0^2-q_0^2)}{8\pi\delta}(r-r_1)\sum_{i=0}^{\infty}\gamma_i r^{i+2}\Bigg\} \Big/ M \quad . \qquad \text{(A.33)}$$

APPENDIX B

The Einstein-Maxwell equations are:

$$\frac{1}{r^2} - \left[\frac{\nu'}{r} + \frac{\nu}{r^2}\right] = 8\pi\rho + E^2 \quad , \tag{B.1}$$

$$\frac{1}{r^2} - \nu\left[\frac{2}{r}\frac{\mu'}{\mu} + \frac{1}{r^2}\right] = -8\pi P + E^2 \quad , \tag{B.2}$$

$$\frac{\mu''}{\mu}\nu + \frac{\mu'}{\mu}\frac{\nu'}{2} + \frac{\mu'}{\mu}\frac{\nu}{r} + \frac{\nu'}{2r} = 8\pi P + E^2 \quad . \tag{B.3}$$

Let
$$S = 8\pi P - E^2 \quad . \tag{B.4}$$

Differentiating (B.2) we find

$$\frac{\mu''}{\mu} = \frac{1}{4\nu^2}\left[\frac{1}{r} + Sr\right]^2 - \frac{1}{2\nu}\left[\frac{1}{r} + \frac{\nu'}{\nu}\right]\left[\frac{1}{r} + Sr\right] + \frac{1}{2\nu}\left[\frac{1}{r} + Sr\right]' + \frac{3}{4r^2} \quad . \tag{B.5}$$

Substituting this into (B.3) gives

$$\frac{1}{4\nu}\left[\frac{1}{r} + Sr\right]^2 - \frac{\nu'}{4\nu}\left[\frac{1}{r} + Sr\right] + \frac{1}{2}\left[\frac{1}{r} + Sr\right]' + \frac{\nu}{4r^2} + \frac{\nu'}{4r} = 8\pi P + E^2 \quad . \tag{B.6}$$

We may solve (B.1) for ν' and substitute this into (B.6). After simplifying and rearranging we arrive at

$$8\pi(\rho+P)(Sr^2+1-\nu)=(8E^2-2S'r)\nu \quad . \tag{B.7}$$

From (B.1) we see

$$\nu = 1-\frac{2}{r}\int 4\pi r^2 T^0_0\, dr = 1-\frac{2M(r)}{r} \tag{B.8}$$

so g_{11} is "quasi-Schwarzschild". It will be convenient in the following to explicitly consider the constant of integration in (B.8). Thus

$$\nu = \frac{k_1}{r}+1-\frac{1}{r}\int(8\pi\rho+E^2)r^2\, dr \tag{B.9}$$

where k_1 is arbitrary. Hence

$$8\pi(P+\rho)\left[(8\pi P-E^2)r^2-\frac{k_1}{r}+\frac{1}{r}\int(8\pi\rho+E^2)r^2\, dr\right]$$

$$=\left[8E^2-2r(8\pi P-E^2)'\right]\left[\frac{k_1}{r}+1-\frac{1}{r}\int(8\pi\rho+E^2)r^2\, dr\right] . \tag{B.10}$$

Let $\qquad a=\int 8\pi P r^2\, dr \quad ,$

$$b = \int 8\pi\rho r^2 dr ,$$

and
$$e = \int E^2 r^2 dr . \tag{B.11}$$

Then (B.10) becomes

$$\frac{a'+b'}{r^2}\left[a'-e'-\frac{k_1}{r}+\frac{b+e}{r}\right]$$

$$=\left[\frac{4e'}{r^2}+\frac{2e''}{r}+\frac{4a'}{r^2}-\frac{2a''}{r}\right]\left[\frac{k_1}{r}+1-\frac{b+e}{r}\right] . \tag{B.12}$$

We may conceive of (B.12) as an equation for $e=e(a,b)$ or an equation for $b=b(a,e)$. As the first case is quite intractable we must attempt to solve for $b=b(a,e)$. Rewriting (B.12) in more appropriate form we have

$$\left[(a'r-e'r-k_1+e)+b\right]b' = \left[4e'+2e''r+4a'-2a''r\right]\left[k_1+r-e\right]$$

$$-a'\left[a'r-e'r-k_1+e\right]-\left[4e'+2e''r+5a'-2a''r\right]b . \tag{B.13}$$

This is an Abel equation of the second kind, which simplifies under the substitution

$$u = b + a'r - e'r - k_1 + e ,$$

$$X(r) = a' - e' + 1 \qquad = (8\pi P - E^2)r^2 + 1 \ ,$$

$$Y(r) = 2e'r + e''r^2 + 2a'r - a''r^2$$

$$= (e'r^2)' - r^4\left(\frac{a'}{r^2}\right)' = (E^2r^4)' - r^4(8\pi P)' \ , \tag{B.14}$$

to

$$uu' = 2XY + \left(rX' - \frac{2Y}{r}\right)u \ . \tag{B.15}$$

This equation cannot be solved in general, although assumptions on X and Y permit specific integrations.

We still have μ^2 to integrate. From (B.2)

$$\ln\mu = \int \frac{Sr^2+1}{2\nu r}\,dr - \tfrac{1}{2}\ln r + \ln k_2 \ , \tag{B.16}$$

where $\ln k_2$ is the constant of integration. Thus

$$\mu^2 = \frac{k_2^2}{r}\exp\int \frac{(8\pi P - E^2)r^2+1}{\nu r}\,dr \ . \tag{B.17}$$

APPENDIX C

We know, for our pressure

$$8\pi P = A^2 (r-r_1)^2 (r-r_2) ,$$

$$8\pi P' = A^2 (r-r_1)(3r-2r_2-r_1) . \tag{C.1}$$

Solution a:

From equation (3.2.11) we see

$$P' = \frac{1}{8\pi r^4} \frac{d}{dr} (E^2 r^4) = \sigma E , \tag{C.2}$$

so $$\sigma = \frac{P'}{E} . \tag{C.3}$$

Note that for $$r_1 < r < \frac{2r_2 + r_1}{3} , \quad \sigma < 0 .$$

Letting $$F_a(r) = \frac{1}{A^2} \int 8\pi P' r^4 \, dr$$

$$= \frac{3}{7} r^7 - \frac{r_2 + 2r_1}{3} r^6 + \frac{(2r_2 + r_1) r_1}{5} r^5 , \tag{C.4}$$

we have $$E^2 = \frac{k_3}{r^4} + \frac{A^2}{r^4} F_a(r) \quad . \tag{C.5}$$

Since $$E^2(r_1) = \frac{q_0^2}{r_1^4} \quad , \quad E^2(r_2) = \frac{q^2}{r_2^4} \quad , \tag{C.6}$$

we find $$k_3 = q_0^2 - A^2 F_a(r_1) \quad , \tag{C.7}$$

and $$A^2 = \frac{q^2 - q_0^2}{\xi} \quad , \tag{C.8}$$

where we have defined

$$\xi = F_a(r_2) - F_a(r_1) \quad . \tag{C.9}$$

Thus $$E^2 = \frac{q_0^2}{r^4} + \frac{q^2 - q_0^2}{r^4} \frac{F_a(r) - F_a(r_1)}{\xi}$$

$$= \frac{q^2}{r^4} + \frac{q^2 - q_0^2}{r^4} \frac{F_a(r) - F_a(r_2)}{\xi} \quad . \tag{C.10}$$

It is easy to show that $\xi > 0$. We require

$$E^2 > 0 \quad . \tag{C.11}$$

Analysis of $F_a(r)$ shows its minimum value in the shell

$$F_{min} = F_a\left(\frac{2r_2+r_1}{3}\right) . \tag{C.12}$$

Equation (C.11) implies

$$F_a(r) > F_a(r_1) - \frac{q_o^2 \xi}{q^2 - q_o^2} , \tag{C.13}$$

so our requirement is

$$\frac{q_o^2}{q^2} > \frac{F_a(r_1) - F_{min}}{F_a(r_2) - F_{min}} . \tag{C.14}$$

case (j): $$\rho = -3P - \frac{E^2}{4\pi} . \tag{C.15}$$

From (3.2.1) we have

$$\nu = \frac{k_1}{r} + 1 - \frac{k_3}{r^2} + \frac{4A^2}{r^2}\int\left[r^6 - (r_2+2r_1)r^5 + (2r_2+r_1)r_1r^4 - r_2r_1^2r^3\right]dr$$

$$= \frac{k_1}{r} + 1 - \frac{q_o^2}{r^2} + \frac{q^2-q_o^2}{\xi}\left[\frac{8\pi P r^2}{A^2} - \frac{F_a(r) - F_a(r_1)}{r^2}\right] , \tag{C.16}$$

Thus $$\nu = \frac{k_1}{r} + 1 + 8\pi P r^2 - E^2 r^2 . \tag{C.17}$$

Applying $\quad \nu(r_1) = 1 - \frac{2m_0}{r_1} + \frac{q_0^2}{r_1^2} \;, \quad \nu(r_2) = 1 - \frac{2m}{r_2} + \frac{q^2}{r_2^2} \;,$ (C.18)

we find $\quad k_1 = \frac{2q_0^2}{r_1} - 2m_0$ (C.19)

and $\quad m = m_0 + \frac{q^2}{r_2} - \frac{q_0^2}{r_1} \;.$ (C.20)

Thus $\quad \nu = 1 - \frac{2m_0}{r} + \frac{2}{r}\frac{q_0^2}{r_1} + 8\pi P r^2 - E^2 r^2$

$$= 1 - \frac{2m}{r} + \frac{2}{r}\frac{q^2}{r_2} + 8\pi P r^2 - E^2 r^2 \;, \tag{C.21}$$

and since $\quad m = \int_0^\infty 4\pi T_0^0 r^2 dr = \int_0^{r_1} 4\pi T_0^0 r^2 dr + \frac{q^2}{r_2} - \frac{q_0^2}{2r_1} \;,$ (C.22)

we have $\quad m_0 = \int_0^{r_1} 4\pi T_0^0 r^2 dr + \frac{q_0^2}{2r_1} = \int_0^{r_0} 4\pi T_0^0 r^2 dr + \frac{q_0^2}{2r_0} \;.$ (C.23)

From (3.2.2) and (3.2.11) we have

$$\mu^2 = \frac{k_2^2}{r} \exp \int \frac{\nu - k_1/r}{\nu r} dr = k_2^2 \exp \int \frac{-k_1}{\nu r^2} dr \tag{C.24}$$

which cannot be further simplified. We must require

$$\mu^2 > 0 \;, \quad \nu > 0 \;. \tag{C.25}$$

A glance at (C.24) shows that the first of these inequalities should follow from the second. Unfortunately the expression for ν is too complicated for exact analysis. However, a weaker condition is easily obtained by requiring

$$(\nu r^2)_{min} \geq r_1^2 - 2mr_2 + 2\frac{q^2}{r_2}r_1 - q^2 > 0$$

or $$m < \frac{r_1^2}{2r_2} + \frac{2r_1 - r_2}{2r_2^2}q^2 \quad . \tag{C.26}$$

case (ii) $$\rho = -P \quad . \tag{C.27}$$

From (3.2.1) we have

$$\nu = \frac{k_1}{r} + 1 + \frac{1}{r}\int (8\pi P - E^2) r^2 dr \quad . \tag{C.28}$$

A simple integration by parts will show

$$\int (8\pi P - E^2) r^2 dr = E^2 r^3 + \int 8\pi (P - P'r) r^2 dr$$

$$= E^2 r^3 + \Omega(r) \tag{C.29}$$

where $$\Omega(r) = \frac{1}{3} r^6 - \frac{r_2 + 2r_1}{5} r^5 + \frac{r_2 r_1^2}{3} r^3 \quad . \tag{C.30}$$

Thus $$\nu = \frac{k_1}{r} + 1 + E^2 r^2 + \frac{q^2 - q_0^2}{\xi} \frac{\Omega(r)}{r} \quad . \tag{C.31}$$

Applying (C.18) we find

$$k_1 = -2m + \frac{q^2 - q_0^2}{\xi} \Omega(r_2) \quad , \tag{C.32}$$

and $$m = m_0 + \frac{q^2 - q_0^2}{2\xi} \left[\Omega(r_2) - \Omega(r_1) \right] \quad . \tag{C.33}$$

Thus $$\nu = 1 - \frac{2m_0}{r} + E^2 r^2 + \frac{q^2 - q_0^2}{\xi} \frac{\Omega(r_1) - \Omega(r)}{r}$$

$$= 1 - \frac{2m}{r} + E^2 r^2 + \frac{q^2 - q_0^2}{\xi} \frac{\Omega(r_2) - \Omega(r)}{r} \quad , \tag{C.34}$$

and since $$m = \int_0^\infty 4\pi T_0^0 r^2 dr$$

$$= \int_0^{r_1} 4\pi T_0^0 r^2 dr + \frac{q_0^2}{2r_1} + \frac{q^2 - q_0^2}{2\xi} \left[\Omega(r_2) - \Omega(r_1) \right] \tag{C.35}$$

we again have (C.23). It is not difficult to show first that $\Omega(r)$ decreases from r_1, attains its minimum and increases to r_2, and then that $\Omega(r_2) - \Omega(r_1) > 0$. From (3.2.2) and (3.2.11) we have

$$\mu^2 = \frac{k_2^2}{r}\exp\int\frac{(\nu r)'}{\nu r}dr = k_2^2\nu. \tag{C.36}$$

Boundary conditions require $\mu^2(r_2) = \nu(r_2)$, so $k_2^2=1$,

and $$\mu^2 = \nu\ . \tag{C.37}$$

We need $$\nu > 0\ . \tag{C.38}$$

Unfortunately the expression for ν is once again too complicated for exact analysis. As before we may obtain a simpler but weaker condition by requiring

$$(\nu r^2)_{min} \geq r_1^2 - 2mr_2 + Q^2_{min} > 0$$

or $$m < \frac{r_1^2 + Q^2_{min}}{2r_2}\ , \tag{C.39}$$

where Q^2_{min}, the minimum value of Q^2, is just E^2r^4 evaluated at $r=(2r_2+r_1)/3$.

Solution b:

We have $$E^2 = \frac{1}{r^2} + 8\pi P\ . \tag{C.40}$$

Thus $$\sigma E = \frac{1}{8\pi r^4}\frac{d}{dr}(E^2r^4) = \frac{1}{4\pi r^3} + \frac{1}{r^4}\frac{d}{dr}(Pr^4)\ , \tag{C.41}$$

or
$$\sigma = \frac{\dfrac{1}{4\pi r^2} + \dfrac{A^2}{8\pi}\left[7r^3 - 6(r_2+2r_1)r^2 + 5(2r_2+r_1)r_1 r - 4r_2 r_1^2\right]}{\left[1+A^2r^2(r-r_1)^2(r-r_2)\right]^{1/2}} \tag{C.42}$$

Applying (C.6) we find

$$q_o^2 = r_1^2 \quad , \quad q^2 = r_2^2 \, . \tag{C.43}$$

We again require $E^2 > 0$. It is not difficult to determine that the maximum value of

$$F_b(r) = -r^2(r-r_1)^2(r-r_2) \tag{C.44}$$

is at
$$r_b = \frac{3r_1 + 4r_2 + (9r_1^2 - 16r_1r_2 + 16r_2^2)^{1/2}}{10} \, . \tag{C.45}$$

Then our requirement is

$$A^2 < \frac{1}{F_b(r_b)} \, . \tag{C.46}$$

case (i):
$$\rho = -P. \tag{C.47}$$

From (3.2.1) we quickly see

$$\nu = \frac{k_1}{r} \, . \tag{C.48}$$

Applying (C.18) we find

$$k_1 = 2(\frac{q-m}{r}) = 2(\frac{q_0-m_0}{r}) \ . \tag{C.49}$$

Thus $$\nu = 2\,\frac{q_0-m_0}{r} = 2\,\frac{q-m}{r} \ , \tag{C.50}$$

and since $$m = \int_0^\infty 4\pi T_0^0\, r^2 dr = \int_0^{r_1} 4\pi T_0^0\, r^2 dr + q - \frac{q_0}{2} \ , \tag{C.51}$$

comparison to (C.49) shows that we again have (C.23). From (3.2.2) and (C.40) we see

$$\mu^2 = \frac{k_2^2}{r} \ . \tag{C.52}$$

Applying $$\mu^2(r_2) = \nu(r_2) \tag{C.53}$$

we find $$\mu^2 = \nu \ . \tag{C.54}$$

Clearly, $\mu^2, \nu > 0$ require $q > m$. (C.55)

It should be noted that we have not used the explicit forms of either P or E^2 in deriving μ^2 and ν. Thus the metric components, for solution b1, are totally independent of the functional form of the pressure.

case (ii): $$\rho = \frac{-1}{2\pi r^2} - 9P \quad . \tag{C.56}$$

From (3.2.1) we find

$$\nu = \frac{k_1}{r} + 4 + \frac{8}{r}\int 8\pi P r^2 dr$$

$$= \frac{k_1}{r} + 4 - 8A^2 \frac{W(r)}{r} \quad , \tag{C.57}$$

where $$W(r) = -\frac{1}{6} r^6 + \frac{r_2 + 2r_1}{5} r^5 - \frac{(2r_2 + r_1)r_1}{4} r^4 + \frac{r_2 r_1^2}{3} r^3 \, . \tag{C.58}$$

Applying (C.18) gives

$$k_1 = -2r_2 - 2m + 8A^2 W(r_2) \quad , \tag{C.59}$$

and $$A^2 = \frac{m - m_0 + q - q_0}{4\omega} \tag{C.60}$$

where $$\omega = W(r_2) - W(r_1) \quad . \tag{C.61}$$

Thus $$\nu = 4 - 2\frac{m+q}{r} + 2\frac{(m+q) - (m_0+q_0)}{r} \frac{W(r_2) - W(r)}{\omega}$$

or $$\nu = 4 - 2\,\frac{m_0+q_0}{r} + 2\,\frac{(m+q)-(m_0+q_0)}{r}\,\frac{W(r_1)-W(r)}{\omega} \; . \tag{C.62}$$

It is easy to show that W is strictly increasing in the shell and thus $\omega > 0$. Also

$$m = \int_0^\infty 4\pi T_0^0 r^2 dr = \int_0^{r_1} 4\pi T_0^0 r^2 dr + m - m_0 + \frac{q_0}{2} \; , \tag{C.63}$$

so we again have (C.23). From (3.2.2) and (C.40) we have

$$\mu^2 = \frac{k_2^2}{r} \; . \tag{C.64}$$

Applying (C.53) and

$$\mu^2(r_1) = K\,\nu(r_1) \; , \tag{C.65}$$

we find $$k_2^2 = 2(q-m) \; , \tag{C.66}$$

and $$K = \frac{q-m}{q_0-m_0} \; . \tag{C.67}$$

Thus $$\mu^2 = 2\,\frac{(q-m)}{r} \; . \tag{C.68}$$

We require $\mu^2, \nu > 0$. An exact expression for the requirement on ν cannot be obtained. A weaker condition is found by requiring

$$(\nu r)_{min} \geq 4r_1 - 2(m+q) > 0$$

or $$m < 2q_0 - q. \tag{C.69}$$

Clearly $\mu^2 > 0$ for $q > m$. (C.70)

Note that once again μ^2 is totally independent of the functional form of the pressure.

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