GRAPH THEORETIC ASPECTS OF MAXIMIZING THE SPECTRAL RADIUS OF NONNEGATIVE MATRICES

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Abstract

Let $\rho_i(t) = \rho(A + tE_{ii})$ denote the spectral radius of the sum of an irreducible nonnegative matrix A and a matrix tE_{ii} that has a single nonzero entry, namely $t > 0$ in the i, i position. We consider qualitative aspects of maximizing $\rho_i(t)$, especially identifying maximizing indices i, and indices i and j that tie (i.e. $\rho_i(t) = \rho_j(t)$ for all $t > 0$). If the digraph of A is a directed cycle, then all indices tie; whereas if the digraph of A is a star, then the center vertex corresponds to the unique maximizing index. When A is the (0,1) adjacency matrix of a graph, we give sufficient conditions in terms of the orbits of vertices for a tie. For complete bipartite graphs and for paths, vertices i are identified that maximize $\rho_i(t)$ for all $t > 0$. However, even for a tree, it is not in general true that some fixed index i maximizes $\rho_i(t)$ for all $t > 0$.

1 Introduction

Let $A = [a_{ij}]$ be an $n \times n$ irreducible nonnegative matrix with all $a_{ii} = 0$. Let D(A) be the *digraph* associated with A with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E that is a subset of the set of all ordered pairs of V; for $i \neq j$ $(i, j) \in E$ iff $a_{ij} > 0$. In the special case when A is an $n \times n$ (0, 1) symmetric matrix, G(A) is the *graph* associated with A with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E that is a subset of the set of all unordered pairs of V; thus $\{i, j\} \in E$ for $i \neq j$ iff $a_{ij} = a_{ji} = 1$. We write $A^k = [a_{ij}^{(k)}]$ for positive powers of A, and denote by $A(i)$ the $(n-1) \times (n-1)$ matrix obtained from A by deleting row and column i.

The *characteristic polynomial* of A is $\det(\lambda I_n - A)$ and is denoted by $p(A; \lambda)$ where I_n denotes the $n \times n$ identity matrix. Let ρ(A) denote the *spectral radius* (Perron root) of A. For given $G(A)$ or certain $D(A)$, we consider the problem of determining $\max\limits_{1 \leq i \leq n} \rho(A + tE_{ii})$ where $t \geq 0$ and E_{ii} is the $n \times n$ matrix with the i, i entry equal to 1 and every other entry equal to 0. For simplicity we denote $\rho(A + tE_{ii})$ by $\rho_i(t)$. For given A, this problem is considered in [9] where it is shown to be equivalent to maximizing the spectral radius of a trace t, diagonal perturbation of A. We begin with definitions from [9].

DEFINITION 1.1 *If $\rho_i(t) = \rho_j(t)$ for all $t \geq 0$, then index i and index j* tie. *If there exists a $t_1 > 0$ such that $\rho_i(t) > \rho_j(t)$ for all $0 < t < t_1$ and for all j that are not tied with i, then i is called an* initial winner. *Similarly if $\rho_i(t) > \rho_j(t)$ for all $t > t_1$ (for some $t_1 > 0$) and for all j that are not tied with i, then i is called a* terminal winner. *Finally if $\rho_i(t) > \rho_j(t)$ for all $t > 0$ and for all j that are not tied with i, we call i a* universal winner. *Let $0 < t_0 < t_1 < t_2$. If $\rho_i(t) < \rho_j(t)$ for all $t \in (t_0, t_1)$, $\rho_i(t_1) = \rho_j(t_1)$ and $\rho_i(t) > \rho_j(t)$ for all $t \in (t_1, t_2)$, then the functions $\rho_i(t)$ and $\rho_j(t)$ are said to* switch *at t_1. When A is the matrix associated with a digraph or graph, we refer to vertex i rather than index i.*

When some index i is a universal winner, our problem for given $G(A)$ is equivalent to the following graph theoretic problem: determine vertex i such that the spectral radius of the adjacency matrix of $G(A) \cup \{i, i\}$ is maximized.

For terminal and initial winners, our analysis uses results in [9] and [6], which for our purposes can be stated as follows.

THEOREM 1.2 [9, Th. 2.3] *Let $A = [a_{ij}]$ be an $n \times n$ irreducible nonnegative matrix and define $S_k = \left\{ s : a_{ss}^{(k)} = \max_{i \in S_{k-1}} \{a_{ii}^{(k)}\} \right\}$ for $1 \leq k \leq n-1$, where $S_0 = \{1, 2, \ldots, n\}$. If k is the smallest integer such that $|S_k| = 1$, then for all sufficiently large $t > 0$, $\max_{1 \leq i \leq n} \rho_i(t)$ occurs at the index $s \in S_k$. The elements of S_{n-1} are the terminal winners.*

THEOREM 1.3 [6, Th. 3.1; 9, Th. 3.1] *Let $A = [a_{ij}]$ be an $n \times n$ irreducible nonnegative symmetric matrix and $u > 0$ be the left Perron vector of A. If $u_i > u_j$, then $\rho_i(t) > \rho_j(t)$ for all sufficiently small $t > 0$. If u has a unique largest component, then the corresponding index is the initial winner.*

THEOREM 1.4 [9, Th. 4.6] *Let $A = [a_{ij}]$ be an $n \times n$ irreducible nonnegative matrix and i, j fixed. Then the following are equivalent:*

(i) $p(A(i); \lambda) = p(A(j); \lambda)$

(ii) $\rho_i(t) = \rho_j(t)$ *for all* $t \geq 0$

(iii) $a_{ii}^{(l)} = a_{jj}^{(l)}$ *for all* $l \geq 1$.

It is an easy computation to show that $p(A + tE_{ii}; \lambda) = p(A; \lambda) - t\, p(A(i); \lambda)$. As in [9], $p(A + tE_{ii}; \lambda) - p(A + tE_{jj}; \lambda) = 0$ iff $p(A(i); \lambda) - p(A(j); \lambda) = 0$, which is equivalent to

$$(E_1(A(j)) - E_1(A(i)))\lambda^{n-2} - \ldots + (-1)^n (E_{n-1}(A(j)) - E_{n-1}(A(i))) = 0, \quad (1)$$

where $E_k(A(i))$ denotes the sum of all $k \times k$ principal minors of $A(i)$. A root of (1) that is greater than $\rho(A)$ gives a $t_0 > 0$ such that $\rho_i(t_0) = \rho_j(t_0)$, i.e. a possible switch at t_0 (see [9, Th. 4.2]). Also if $\rho_i(t) = \rho_j(t)$ for t on some open interval, then $\rho_i(t) = \rho_j(t)$ for all $t \geq 0$, i.e. i and j tie.

In Section 2 we prove qualitative results for digraphs. We characterize digraphs for which all vertices tie, and give a class of digraphs that have a unique universal winner. In Section 3 we consider (0, 1) adjacency matrices of graphs and the orbits of vertices via automorphisms of graphs in order to obtain sufficient conditions for a tie. In Section 4 we investigate two special classes of graphs, K_{n_1,n_2} (the complete bipartite graph) and P_n (the path on n vertices), and determine universal winners. Finally, in Section 5 we give an example of a tree with no universal winner, and state some related open questions.

2 Digraphs

Firstly, we give a necessary and sufficient condition for all vertices in a digraph to tie. Given any $n \times n$ nonnegative matrix A, let D_A denote the set of all nonnegative matrices having the same digraph $D(A)$; i.e., all matrices in D_A have exactly the same (0,+) pattern as A.

THEOREM 2.1 *Let A be an $n \times n$ irreducible nonnegative matrix. Then all vertices in $D(A)$ tie for all $B \in D_A$ iff $D(A)$ is a directed n-cycle.*

Proof. If $D(A)$ is the directed n-cycle, then without loss of generality

$$A = \begin{bmatrix} 0 & a_{12} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{23} & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1,n} \\ a_{n1} & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

It is easily shown that $p(A + tE_{ii}; \lambda) = \lambda^n - t\lambda^{n-1} + (-1)^n E_n(A)$ for each i. Thus by Theorem 1.4 and the discussion after it, all indices tie.

For the converse, suppose that for all $B \in D_A$ all indices tie. Then by Theorem 1.4 and (1), $p(B+tE_{ii};\lambda) = p(B+tE_{jj};\lambda)$, thus $E_l(B(i)) = E_l(B(j))$, for all $1 \le i,j \le n$ and for $l = 1,2,\ldots,n-1$. Now this condition holds for all $B \in D_A$, which is possible only if $E_1(B) = E_2(B) = \cdots = E_{n-1}(B) = 0$, since the condition must hold independently of the magnitudes of the entries of B. So now consider

$$\begin{aligned} p(B;\lambda) &= \lambda^n - E_1(B)\lambda^{n-1} + E_2(B)\lambda^{n-2} - E_3(B)\lambda^{n-3} + \cdots + (-1)^n E_n(B) \\ &= \lambda^n + (-1)^n E_n(B). \end{aligned}$$

Since B is irreducible and nonnegative, $\rho(B)$ is an eigenvalue of B and thus $p(B;\lambda) = \lambda^n - |E_n(B)|$ with $E_n(B) = det(B) \neq 0$, for otherwise $p(B;\lambda)$ has no positive root. Therefore B has n eigenvalues $\sqrt[n]{|E_n(B)|}e^{i\theta_0}$, $\sqrt[n]{|E_n(B)|}e^{i\theta_1}, \cdots,$ $\sqrt[n]{|E_n(B)|}e^{i\theta_{n-1}}$, where $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < 2\pi$. Then (see [2, Th. 2.20, pg 32 with h=n]) there exists a permutation matrix P such that $PBP^T = A$ as above. Thus $D(B)$ and hence $D(A)$ is a directed n-cycle. □

We now give a class of matrices that has a unique universal winner. For $n \ge 3$, let

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & 0 & \cdots & 0 \\ a_{31} & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ a_{n1} & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (2)$$

where a_{1j} and a_{j1} are positive, $2 \leq j \leq n$. Then $D(A)$ is a strongly connected star digraph. A is diagonally symmetrizable via the matrix $D = diag(1, \sqrt{a_{12}/a_{21}}, \cdots, \sqrt{a_{1n}/a_{n1}})$. Thus

$$DAD^{-1} = \begin{bmatrix} 0 & \sqrt{a_{12}a_{21}} & \sqrt{a_{13}a_{31}} & \cdots & \sqrt{a_{1n}a_{n1}} \\ \sqrt{a_{12}a_{21}} & 0 & 0 & \cdots & 0 \\ \sqrt{a_{13}a_{31}} & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ \sqrt{a_{1n}a_{n1}} & 0 & 0 & \cdots & 0 \end{bmatrix} \equiv C \qquad (3)$$

is symmetric and $\rho_i(t) = \rho(C + tE_{ii})$ with $\rho(C) = (\sum_{j \neq 1} a_{1j}a_{j1})^{1/2}$.

THEOREM 2.2 *Let $n \geq 3$ and let A be the $n \times n$ irreducible nonnegative matrix* (2) *and let $B \in D_A$. Then index 1 is the unique universal winner for* *all* $B \in D_A$.

Proof. As noted above, without loss of generality, we can consider C in (3). It is an easy computation to see that $c_{11}^{(2)} = \sum_{j \neq 1} c_{1j}^2$ and $c_{kk}^{(2)} = c_{1k}^2, k \neq 1$. Then, since $c_{1j} > 0$, $c_{11}^{(2)} > c_{kk}^{(2)}$ for all $k \neq 1$. By Theorem 1.2, $S_2 = \{1\}$ and therefore index 1 is the unique terminal winner. We easily see that the rank of C is at most 2, hence $E_l(C(k)) = 0$ for all $1 \leq k \leq n$ and for $l \geq 3$. Clearly $E_1(C(i)) = 0$ for all $1 \leq i \leq n$, so (1) becomes for $i = 1$ and any $j \neq 1$, $(E_2(C(j)) - E_2(C(1)))\lambda^{n-3} = 0$. Thus for $n \geq 4$ the only root of (1) is $\lambda = 0$, while for $n = 3$ this equation is a nonzero constant and has no root. Hence $\rho_1(t) > \rho_j(t)$ for all $t > 0$, i.e. index 1 is the unique universal winner for C and hence for for all $B \in D_A$. □

Note that for this star digraph, for $j = 2, 3, \ldots, n$, $n \geq 3$, vertex j is in a unique directed cycle and this cycle has length 2. By Theorem 1.2 and a generalization of Theorem 1.3 [6,Th. 3.1;9,Th. 3.1] such a vertex can never be either a terminal or an initial winner. In Proposition 3.1, this is proved for a graph.

The previous results are qualitative in nature. For a particular matrix with a star digraph we note the following quantitative result on the ordering of the functions $\rho_i(t)$, for all $t > 0$ and $i \neq 1$.

COROLLARY 2.3 *Let A be as in (2). Then $\rho_i(t) > \rho_j(t)$ for all $t > 0$ iff $a_{1i}a_{i1} > a_{1j}a_{j1}$ for $i, j \neq 1$.*

Proof. Let $i, j \neq 1$. As before we consider C as in (3). Since C is irreducible nonnegative and symmetric, there exists $x > 0$ such that $Cx = \rho(C)x$. Then the eigenequations are $\sum_{k \neq 1} c_{1k}x_k = \rho(C)x_1$ and $c_{1k}x_1 = \rho(C)x_k$ for $k \neq 1$. If $a_{1i}a_{i1} > a_{1j}a_{j1}$ then clearly $c_{1i} > c_{1j}$, hence by the eigenequations above we see this implies that $\rho(C)x_i > \rho(C)x_j$. But this then implies $x_i > x_j$, then by Theorem 1.3, $\rho_i(t) > \rho_j(t)$ for $t > 0$ sufficiently small. Also, for any i, j, equation (1) has no roots other than (possibly) the zero root. Hence for every pair i, j, $\rho_i(t)$ and $\rho_j(t)$ do not switch. For the converse assume $\rho_i(t) > \rho_j(t)$ for all $t > 0$ and suppose $a_{1i}a_{i1} \leq a_{1j}a_{j1}$. Consider first the case of equality. If $a_{1i}a_{i1} = a_{1j}a_{j1}$, then $c_{1i} = c_{1j}$ but this implies that row and column i are the same as row and column j. Hence $c_{ii}^{(l)} = c_{jj}^{(l)}$ for all $l \geq 1$, then by Theorem 1.4, $\rho_i(t) = \rho_j(t)$, which is a contradiction. Now suppose $a_{1i}a_{i1} < a_{1j}a_{j1}$, then $\rho_i(t) < \rho_j(t)$ but this is a contradiction. □

Let A be an $n \times n$ irreducible nonnegative, tridiagonal matrix. Then $D(A)$ is the path digraph. In contrast with a star digraph, a path digraph has in general no universal winner when $n \geq 5$. This is illustrated in the following example.

EXAMPLE 2.4

$$A = \begin{bmatrix} 0 & 0.7071 & 0 & 0 & 0 \\ 0.7071 & 0 & 0.4564 & 0 & 0 \\ 0 & 0.4564 & 0 & 0.6614 & 0 \\ 0 & 0 & 0.6614 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}.$$

Initially $\rho_3(t) > \rho_2(t) > \rho_4(t) > \rho_1(t) > \rho_5(t)$, *but terminally* $\rho_2(t) > \rho_4(t) > \rho_3(t) > \rho_1(t) > \rho_5(t)$.

3 Graphs

We now restrict consideration to (0, 1) adjacency matrices of graphs. We begin with some additional graph theoretic definitions and notation (see [3]).

A *pendant vertex* is a vertex of degree 1, where $degree(i) = |\{j : \{i,j\} \in E\}|$. Let G be a graph. Then ϕ is an *automorphism* of G if $\phi : V \to V$ is a bijection such that $\{i,j\} \in E$ iff $\{\phi(i), \phi(j)\} \in E$. Let *Aut(G)* denote the set of all automorphisms of G. For a vertex i in G the *orbit* of i, denoted *orb(i)* is $\{\phi(i) : \phi \in Aut(G)\}$. If $j \in orb(i)$, then this is equivalent to saying that there exists a permutation matrix P such that $PAP^T = A$ and $p_{ji} = 1$, where A is the adjacency matrix of G. We say that G is a *vertex transitive* graph if for each pair of distinct vertices i, j there exists $\phi \in Aut(G)$ such that $\phi(j) = i$.

We now present two simple propositions.

PROPOSITION 3.1 *Let G be a connected graph with at least 3 vertices. If G has a pendant vertex, then this vertex is never an initial or terminal winner.*

Proof. Let A be the $n \times n$ adjacency matrix of G. If i is a pendant vertex, then $degree(i) = 1$. Since $a_{ii}^{(2)} = degree(i)$ for (0, 1) adjacency matrices, by Theorem 1.2 it follows that $i \notin S_2$ since $n \geq 3$. Therefore i is not a terminal winner. Since A is nonnegative and irreducible, $\rho(A) \geq 1$, and by the additional assumptions $\rho(A) > 1$. If j is the unique neighbor of

i, the eigenequation corresponding to row i gives $x_j = \rho(A)x_i > x_i$, where $x > 0$ is the Perron vector of A. Thus $\rho_j(t) > \rho_i(t)$ for $t > 0$ sufficiently small (by Theorem 1.3) , showing that i is not an initial winner.□

PROPOSITION 3.2 *Let A be the $n \times n$ adjacency matrix of a connected graph G. If $i \in orb(j)$, then $a_{ii}^{(l)} = a_{jj}^{(l)}$ for all $l \geq 1$.*

Proof. Since $i \in orb(j)$, there exists a permutation matrix P such that $PAP^T = A$ and $p_{ij} = 1$. Thus $A^l = PA^lP^T$ for all $l \geq 1$ and it is a straightforward computation to show that $a_{ii}^{(l)} = a_{jj}^{(l)}$. □

From this we have the following corollaries.

COROLLARY 3.3 *If $i \in orb(j)$, then vertices i and j tie.*

Proof. Use Proposition 3.2, and Theorem 1.4. □

COROLLARY 3.4 *If G is vertex transitive, then all vertices in G tie.*

Proof. Use Corollary 3.3 for all i, j. □

The undirected n-cycle C_n and the complete graph K_n are clearly vertex transitive, and it follows from Corollary 3.4 that all vertices tie in each graph.

We note that even for a tree the converse of Corollary 3.3 is false, as the following example shows.

EXAMPLE 3.5 *See Figure 1. The vertices i=7 and j=14 tie as A(7), A(14) have the same characteristic polynomial (see [7]). However* $7 \notin orb(14)$. □

The converse of Corollary 3.4 is also false, as the following example shows.

EXAMPLE 3.6 *See Figure 2. Here all vertices tie (see [10, pg 188]), as* $p(A(i))$ *are all equal for all i. However it can be easily checked that G is not vertex transitive.* □

To use Corollary 3.4 we introduce some graph product definitions. Let G_1, G_2 be two graphs with vertex sets V_1, V_2 and edge sets E_1, E_2 respectively. The *cartesian product* of G_1, G_2 is denoted by $G_1 \times G_2$ and has vertex set $V_1 \times V_2$ and two vertices xy, $x'y'$ are joined by an edge iff $\{x, x'\} \in E_1$ and

$\{y, y'\} \in E_2$. The *cartesian sum* of G_1, G_2 is denoted by $G_1 \Box G_2$ and has vertex set $V_1 \times V_2$ and two vertices xy, $x'y'$ are joined by an edge iff $x = x'$ and $\{y, y'\} \in E_2$, or $y = y'$ and $\{x, x'\} \in E_1$ (see [1, pg 377]).

LEMMA 3.7 *Let G_1, G_2 be two vertex transitive graphs. Then $G_1 \times G_2$, and $G_1 \Box G_2$ are vertex transitive.*

Proof. To show that $G_1 \Box G_2$ is vertex transitive, just consider the composition of automorphisms of G_1 and G_2 (or vice-versa). For $G_1 \times G_2$ consider the function $\phi : V_1 \times V_2 \to V_1 \times V_2$ given by $\phi(xy) = \phi_1(x)\phi_2(y)$, where $\phi_1 \in Aut(G_1)$ and $\phi_2 \in Aut(G_2)$. Clearly ϕ is a bijection as ϕ_1 and ϕ_2 are bijections. The following shows that edges are mapped to edges.

$$\{\phi(xy), \phi(x'y')\} \in E(G_1 \times G_2)$$

$$\Leftrightarrow \{\phi_1(x)\phi_2(y), \phi_1(x')\phi_2(y')\} \in E(G_1 \times G_2)$$

$$\Leftrightarrow \{\phi_1(x), \phi_1(x')\} \in E(G_1), \{\phi_2(y), \phi_2(y')\} \in E(G_2)$$

$$\Leftrightarrow \{x, x'\} \in E(G_1), \{y, y'\} \in E(G_2)$$

$$\Leftrightarrow \{xy, x'y'\} \in E(G_1 \times G_2).$$

Now let xy; $x'y'$ be any two vertices in $G_1 \times G_2$. We want to show that there exists an automorphism ϕ such that $\phi(xy) = x'y'$. Since G_1, G_2 are vertex transitive, there exist $\phi_1 \in Aut(G_1)$ and $\phi_2 \in Aut(G_2)$ such that $\phi_1(x) = x'$ and $\phi_2(y) = y'$. Then let $\phi = \phi_1\phi_2$. Thus the cartesian product of vertex transitive graphs is vertex transitive. □

Let $\otimes$ denote the Kronecker product of matrices.

THEOREM 3.8 *Let G_1, G_2 be two vertex transitive graphs with adjacency matrices A_1, A_2 respectively. Let $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$. Then in $A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2$ and in $A_1 \otimes A_2$, all indices tie.*

Proof. In [4, pg 67], it is shown that $A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2$ and $A_1 \otimes A_2$ are the adjacency matrices of $G_1 \Box G_2$ and $G_1 \times G_2$, respectively. By Lemma 3.7, $G_1 \Box G_2$ and $G_1 \times G_2$ are vertex transitive and hence by Corollary 3.4 all indices tie in $A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2$ and in $A_1 \otimes A_2$. □

Using induction, Lemma 3.7 and hence Theorem 3.8 can be generalized to $G_1 \Box G_2 \Box \cdots \Box G_n$ and $G_1 \times G_2 \times \cdots \times G_n$. For example it is well known that Q_n (the n-hypercube) is isomorphic to $K_2 \Box K_2 \Box \cdots \Box K_2 = K_2^n$, thus since K_2

is vertex transitive, the n-hypercube is vertex transitive. Thus all indices in Q_n tie.

However, it is not true that all graph products preserve vertex transitivity. Consider the *complete product* of two graphs G_1, G_2 denoted $G_1 \nabla G_2$ (see [4, pg 54]). This product is obtained from the union $G_1 \cup G_2$ by adjoining every vertex of G_1 with every vertex of G_2. For example if $G_1 \cong C_4$ (the cycle on 4 vertices) and $G_2 \cong K_2$, where $\cong$ denotes that two graphs are isomorphic, then $G_1 \nabla G_2$ is not vertex transitive as it is not regular. The vertices from G_2 are the terminal winners (by Theorem 1.2). Consider another product in which G_1 and G_2 have the same vertex set $\{1, 2, \ldots, n\}$ and edge sets E_1, E_2 respectively. Then $G_1 \cap G_2$ is the graph with vertex set $\{1, 2, \ldots, n\}$ and $\{x, y\} \in E(G_1 \cap G_2)$ iff $\{x, y\} \in E_1$ and $\{x, y\} \in E_2$ (see [8, pg 116]). If $G_1 \cong C_6$ and $G_2 \cong K_3 \Box K_2$, then $G_1 \cap G_2 \cong P_3 \cup P_3$, where P_3 is the path on 3 vertices (see Section 4.2). Results from Section 4.2 show that all nonpendant vertices in $P_3 \cup P_3$ are universal winners. Note that the adjacency matrix of $G_1 \cap G_2$ is the Hadamard product of the two adjacency matrices of G_1 and G_2.

4 Families of Graphs

In this section we consider in detail two families of graphs. Firstly, we look at K_{n_1,n_2}, the complete bipartite graph with n_1 vertices in one partition and n_2 in the other partition. Secondly, we consider P_n, the path on n vertices.

4.1 The Complete Bipartite Graph K_{n_1,n_2}

We denote the complete bipartite graph by K_{n_1,n_2} with vertex bipartition $V = V_1 \cup V_2$, where $|V_1| = n_1$ and $|V_2| = n_2$. Without loss of generality, let $V_1 = \{1, 2, \ldots, n_1\}$, $V_2 = \{n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 = n\}$. If A denotes the (0,1) adjacency matrix of K_{n_1,n_2} then

$$A = \begin{bmatrix} O_{n_1,n_1} & J_{n_1,n_2} \\ J_{n_2,n_1} & O_{n_2,n_2} \end{bmatrix}, \qquad (4)$$

where J_{n_1,n_2} is the $n_1 \times n_2$ matrix in which every entry is equal to one. It is well known that the characteristic polynomial of A is $p(A; \lambda) = \lambda^{n_1+n_2-2}(\lambda^2 - n_1 n_2)$ (see [4, pg 72]). Hence $\rho(A) = \sqrt{n_1 n_2}$.

THEOREM 4.1.1 *Let K_{n_1,n_2} have vertex partition V_1, V_2 such that $|V_1| = n_1$ and $|V_2| = n_2$, and suppose without loss of generality that $n_1 \geq n_2$. Then the vertices in V_2 are the universal winners. All vertices tie iff $n_1 = n_2$.*

Proof. Let A be as in (4). If $i, j \in V_1$, then clearly there exists a $\phi \in Aut(K_{n_1,n_2})$ such that $\phi(i) = j$, since we can simply permute the labels of vertices in V_1. (A similar argument can be used for $i, j \in V_2$.) Thus from Corollary 3.3 all vertices in V_1 tie. (Similarly all vertices in V_2 tie.) Consider $n_1 > n_2$. Then for $i \in V_2$ and $j \in V_1$, $n_1 = degree(i) > n_2 = degree(j)$. Since $a_{ll}^{(2)} = degree(l)$, by Theorem 1.2 all vertices in V_2 are terminal winners. Since rank(A)= 2, $E_l(A(k)) = 0$ for all $1 \leq k \leq n$ and for $l \geq 3$, and also $E_1(A(k)) = 0$. Now (1) becomes for $i \in V_2$ and $j \in V_1$, $-[E_2(A(i)) - E_2(A(j))]\lambda^{n-3} = 0$. However, for $n_1 > n_2$, there are no nonzero roots of this equation. Hence by comments after Theorem 1.4 there does not exist a $t_0 > 0$ such that $\rho_i(t_0) = \rho_j(t_0)$. Thus since the vertices in V_2 are the terminal winners, $\rho_i(t) > \rho_j(t)$ for $i \in V_2$ and $j \in V_1$ for all $t > 0$. Hence all the vertices in V_2 are universal winners. Now if $n_1 = n_2$, then K_{n_1,n_1} is vertex transitive and hence by Corollary 3.4 all vertices tie. If all vertices tie, then the complete bipartite graph must be regular but then $|V_1| = |V_2|$. □

These arguments generalize to the complete multipartite graph in the sense that if $G \cong K_{n_1,n_2,\ldots,n_k}$, then the vertex set is partitioned into k components $V_1, V_2, \ldots, V_k$, where $|V_i| = n_i$ $(1 \le i \le k)$. Each component V_i consists of one orbit, thus for each i all vertices in V_i tie. Vertices in different components tie iff their respective components have equal cardinality, and all vertices in G tie iff $n_1 = n_2 = \cdots = n_k = n/k$. However, it is not known whether there can be any switches in a complete multipartite graph with more than two components.

EXAMPLE 4.1.2 *The star graph $K_{1,n-1}$ has (0,1) adjacency matrix given by* (2) *with $a_{1j} = a_{j1} = 1$. Thus $\rho(A) = \sqrt{n-1}$, vertex 1 is the universal winner and vertices $2, 3, \ldots, n$ all tie. Note that vertex 1 is the center (cf. Theorem 4.2.6).* $\square$

4.2 The Path P_n

We now consider the second family of graphs. First we state a few more definitions from [3]. The *path* P_n has vertex set $\{1, 2, \ldots, n\}$ and edges $\{i, i+1\}$, for $1 \le i \le n-1$. Let A_n denote the $n \times n$ (0,1) adjacency matrix of P_n. We will determine which indices are the universal winners of A_n. The *distance* between any two vertices i, j, denoted $d(i, j)$, is the length (number of edges) of the shortest path from i to j. A vertex i is called a *center* of a graph if $\max_{v \in V} d(i, v)$ is a minimum. Note that in general a center of a graph is not unique. For example, vertex $(k+1)$ is the center of P_{2k+1}, and vertices $k, k+1$ are centers of P_{2k}.

It is well known that the spectrum of A_n is $\{2\cos(\frac{k\pi}{n+1}) : 1 \leq k \leq n\}$ (see [4, pg 73]), thus $\rho(A_n) = 2\cos(\frac{\pi}{n+1})$. Let $x > 0$ be the Perron vector of A_n. Then for $1 \leq j \leq n$, $x_j = \sin(\frac{j\pi}{n+1})$. We now find an initial ordering of the functions $\rho_i(t)$ for sufficiently small $t > 0$.

THEOREM 4.2.1 *Let $x > 0$ be the Perron vector of A_n, the $n \times n$ (0,1) adjacency matrix of P_n. For sufficiently small $t > 0$,*

1. *if n is odd ($n = 2k+1$), then $\rho_{k+1}(t) > \rho_k(t) > \cdots > \rho_1(t)$,*

2. *if n is even ($n = 2k$), then $\rho_{k+1}(t) = \rho_k(t) > \cdots > \rho_1(t)$.*

Proof. We consider $\rho_i(t)$ for only $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$, since the vertices i and $n - i + 1$ are in the same orbit, and thus these vertices tie. Suppose firstly

$n = 2k + 1$. Then $x_j = \sin(\frac{j\pi}{2k+2})$, which is a strictly increasing function of j when $1 \leq j \leq k+1$ with $x_{k+1} = 1$. From Theorem 1.3 for sufficiently small $t > 0$, $\rho_{k+1}(t) > \rho_k(t) > \cdots > \rho_1(t)$. For $n = 2k$, $x_j = \sin(\frac{j\pi}{2k+1})$, which is symmetrical about $\frac{2k+1}{2}$. Thus $x_k = x_{k+1}$, and by similar arguments and Corollary 3.3 it follows that $\rho_{k+1}(t) = \rho_k(t) > \cdots > \rho_1(t)$. □

We now prove that there are no switches between any of the functions $\rho_i(t)$, $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$, but first we need some lemmas and notation. Denote $A_n(n, n-1, \ldots, n-k)$ by A_{n-k-1}, for $k = 0, 1, \ldots, n-1$, define $p(A_0; \lambda) = 1$, and recall that $p(A_1; \lambda) = \lambda$.

LEMMA 4.2.2 *For* $n \geq 2$, $p(A_n; \lambda) = \lambda p(A_{n-1}; \lambda) - p(A_{n-2}; \lambda)$.

Proof. See [4, pg 59]. □

LEMMA 4.2.3 *For* $n \geq 2$ *and any* $1 \leq j \leq n$, *by symmetry*

$p(A_{n-j}; \lambda) = p(A_n(1, 2, \ldots, j); \lambda) = p(A_n(n-j+1, n-j+2, \ldots, n); \lambda)$.

LEMMA 4.2.4 *For* $n \geq 2$ *and any* $1 \leq q \leq n-1$,

$$[p(A_q; \lambda)]^2 - p(A_{q-1}; \lambda)p(A_{q+1}; \lambda) = 1. \tag{5}$$

Proof. The proof uses induction on q for fixed n. For $q = 1$,

$[p(A_1;\lambda)]^2 - p(A_0;\lambda)p(A_2;\lambda) = \lambda^2 - 1(\lambda^2 - 1) = 1.$

Assume (5) is true for $q = k$. By Lemma 4.2.2.

$$[p(A_{k+1};\lambda)]^2 - p(A_k;\lambda)p(A_{k+2};\lambda)$$

$$= p(A_{k+1};\lambda)[\lambda p(A_k;\lambda) - p(A_{k-1};\lambda)] - p(A_k;\lambda)[\lambda p(A_{k+1};\lambda) - p(A_k;\lambda)]$$

$$= [p(A_k;\lambda)]^2 - p(A_{k-1};\lambda)p(A_{k+1};\lambda)$$

$= 1$, by the induction hypothesis.□

THEOREM 4.2.5 *For $n \geq 3$, and for fixed (but arbitrary) integer s, where $2 \leq s \leq \frac{n+1}{2}$,*

$$p(A_n(s);\lambda) - p(A_n(s-1);\lambda) = p(A_{n+1-2s};\lambda). \tag{6}$$

Proof. We note here that using Lemma 4.2.3, (6) can also be written as

$$p(A_{n+1-2s};\lambda)$$

$$= p(A_{s-1};\lambda)p(A_n(1,2,\ldots,s);\lambda) - p(A_{s-2};\lambda)p(A_n(1,2,\ldots,s-1);\lambda)$$

$$= p(A_{s-1};\lambda)p(A_{n-s};\lambda) - p(A_{s-2};\lambda)p(A_{n-s+1};\lambda).$$

We use induction on n.

Consider the case $n = 3$. Then $s = 2$ and $p(A_3(2);\lambda) - p(A_3(1);\lambda) = \lambda^2 - (\lambda^2 - 1) = 1$, and recall $p(A_0;\lambda) = 1$. Assume now that (6) is true for $n = 1, 2, \ldots, k$ and for $2 \leq s \leq \frac{k+1}{2}$. The left hand side of (6) is

$$p(A_{k+1}(s);\lambda) - p(A_{k+1}(s-1);\lambda)$$
$$= p(A_{s-1};\lambda)p(A_{k+1-s};\lambda) - p(A_{s-2};\lambda)p(A_{k-s};\lambda).$$

Using Lemma 4.2.2, this is equivalent to

$$p(A_{s-1};\lambda)[\lambda p(A_{k-s};\lambda) - p(A_{k-s-1};\lambda)] - p(A_{s-2};\lambda)[\lambda p(A_{k-s-1};\lambda) - p(A_{k-s-2};\lambda)]$$
$$= \lambda[p(A_{s-1};\lambda)p(A_{k-s};\lambda) - p(A_{s-2};\lambda)p(A_{k-s-1};\lambda)]$$
$$-[p(A_{s-1};\lambda)p(A_{k-s-1};\lambda) - p(A_{s-2};\lambda)p(A_{k-s-2};\lambda)]$$
$$= \lambda p(A_{k+1-2s};\lambda) - p(A_{k-2s};\lambda), \quad \text{by the induction hypothesis}$$
$$= p(A_{k+2-2s};\lambda), \quad \text{by Lemma 4.2.2.}$$

There is just one difficulty: when $k = 2m$, then $s \leq \frac{2m+1}{2}$ so that $s \leq m$, but when $k = 2m + 1$, we get $s \leq m + 1$ so there is a possibility of one case not being covered in the above induction step. When $k = 2m$ we need to consider $s = m + 1$. In this case for $k + 1$,

$$p(A_{2m+1}(m+1);\lambda) - p(A_{2m+1}(m);\lambda)$$
$$= [p(A_m;\lambda)]^2 - p(A_{m-1};\lambda)p(A_{m+1};\lambda)$$

$= 1$, by Lemma 4.2.4.

But $1 = p(A_0; \lambda) = p(A_{2m+2-2(m+1)}; \lambda)$, thus completing the proof. □

THEOREM 4.2.6 *The vertices that are centers of P_n are the universal winners, and the functions $\rho_i(t)$ are ordered according to the shortest distance of vertex i to a center (and there are no switches).*

Proof. By Theorem 4.2.1 for t sufficiently small, the $\rho_i(t)$ are ordered as stated. Thus if there is a switch, it must be between a vertex s and a neighbor $(s-1)$. But from (1) and Theorem 4.2.5, such a ρ value at a switch must be a zero of some $p(A_{n+1-2s}; \lambda)$ with $2 \leq s \leq \lfloor \frac{n+1}{2} \rfloor$. But $n+1-2s < n$, thus A_{n+1-2s} is a proper principal submatrix of A_n. As A_n is irreducible, $\rho(A_{n+1-2s}) < \rho(A_n)$ (see [2, Cor. 1.6, pg 28]). Thus there are no roots of $p(A_{n+1-2s}; \lambda)$ that are bigger than $\rho(A_n)$. Thus there are no switches and the functions $\rho_i(t)$ are ordered for all $t > 0$ as stated. □

5 Discussion

We have presented many results pertaining to universal winners for different families of graphs. For both examples of graphs that were trees, namely $K_{1,n-1}$ and P_n, there is a universal winner(s) and this winner(s) is the center of the respective graphs. We note that amongst all trees on n vertices, the spectral radius of the adjacency matrices is minimized and maximized by P_n and $K_{1,n-1}$ respectively (see [5, Th. 3.1]). However, it is not true in general for trees that there is always a universal winner. Consider the following example.

EXAMPLE 5.1 *See Figure 3. Since degree*$(1) >$ *degree*(i), *for all* $i \neq 1$, *vertex 1 is the terminal winner. However numerically it can be shown that the center vertex 7 is the initial winner. Thus there must be at least one switch; numerically it can be shown that there is exactly one switch that occurs at* $t \approx 0.2$. □

If there is a universal winner, then it is a vertex of maximum degree. We also believe when G is a tree, then if the center is a vertex of maximum degree (not necessarily the only one), then the center is a universal winner.

It is not known whether any $\rho_j(t)$ can switch with a $\rho_i(t)$ where i is a center vertex of maximum degree when G is a tree.

In Proposition 3.1, we noted that a pendant vertex of a graph is never a terminal or initial winner. We have numerical evidence that in fact $\rho_j(t)$ cannot switch with $\rho_i(t)$, where j is a pendant vertex and i is its unique neighbor. We can prove this only in the case in which $degree(i) = 2$. The coresponding problem for digraphs is also open, see the note following Theorem 2.2.

For graphs $G(A)$ in general, some components of the Perron vector may be equal for vertices not in the same orbit, so that Theorem 1.3 needs to be generalized to identify initial winners. This involves the Moore-Penrose inverse of $\rho(A)I_n - A$, see [9].

EXAMPLE 5.2 *See Figure 4. As the graph $G(A)$ is regular of degree 3, thus $\rho(A) = 3$ and all components of the Perron vector are equal (as A is row sum constant). However, after numerically computing the Moore-Penrose inverse of $\rho(A)I_n - A$, we see that indices 1,5,8,9 are the initial winners and in fact the universal winners. Notice here that the centre of $G(A)$ is $\{3, 6\}$ and vertices 3,6 are not universal winners.* □

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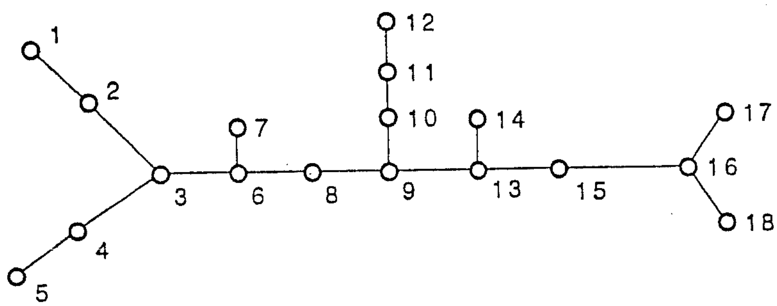

Figure 1.

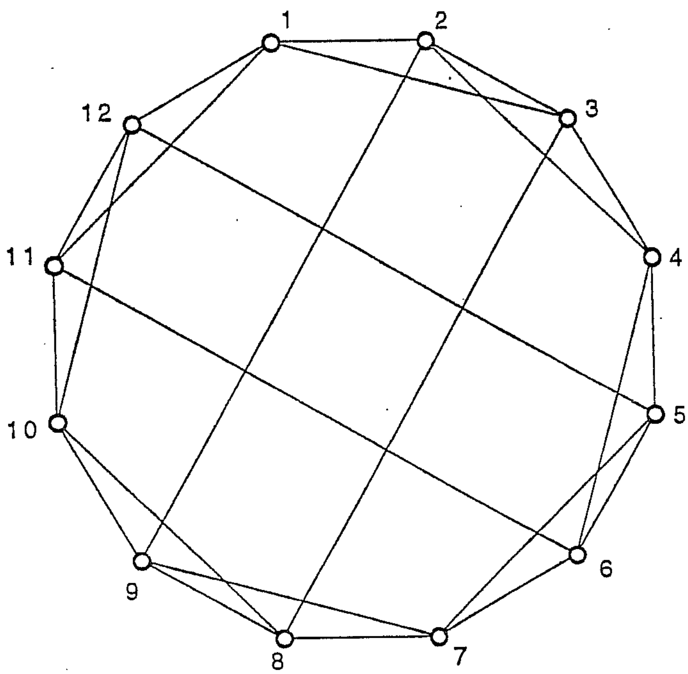

Figure 2.

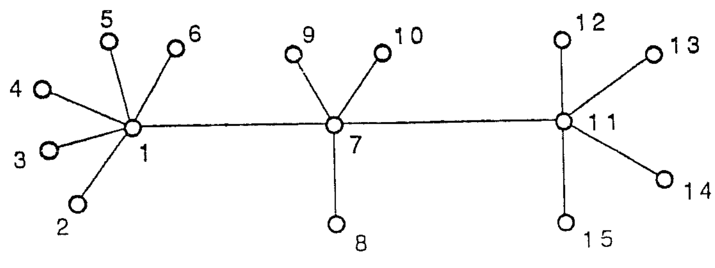

Figure 3.

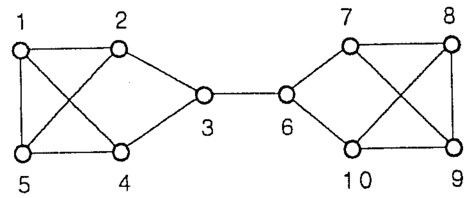

Figure 4.