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Research Article

New Relations Involving an Extended Multiparameter Hurwitz-Lerch Zeta Function with Applications

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We derive several new expansion formulas involving an extended multiparameter Hurwitz-Lerch zeta function introduced and studied recently by Srivastava et al. (2011). These expansions are obtained by using some fractional calculus methods such as the generalized Leibniz rules, the Taylor-like expansions in terms of different functions, and the generalized chain rule. Several (known or new) special cases are also given.

1. Introduction

The Hurwitz-Lerch zeta function $\Phi(z, s, a)$ which is one of the fundamentally important higher transcendental functions is defined by (see, e.g., [1, page 121 *et seq.*]; see also [2] and [3, page 194 *et seq.*])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \quad (1)$$

$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1).$

The Hurwitz-Lerch zeta function contains, as its special cases, the Riemann zeta function $\zeta(s)$, the Hurwitz zeta function $\zeta(s, a)$, and the Lerch zeta function $\ell_s(\xi)$ defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1, s, 1) = \zeta(s, 1) \quad (\Re(s) > 1), \quad (2)$$

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1, s, a) \quad (3)$$

$(\Re(s) > 1; a \in \mathbb{C} \setminus \mathbb{Z}_0^-),$

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2\pi n i \xi}}{(n+1)^s} = \Phi(e^{2\pi i \xi}, s, 1) \quad (\Re(s) > 1; \xi \in \mathbb{R}), \quad (4)$$

respectively.

The Hurwitz-Lerch zeta function is connected with other special functions of analytic number theory such as the polylogarithmic function (or *de Jonquière's function*) $Li_s(z)$:

$$Li_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1) \quad (5)$$

$(s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1)$

and the Lipschitz-Lerch zeta function $\phi(\xi, a, s)$ (see [1, page 122, Equation 2.5 (11)])

$$\phi(\xi, a, s) := \sum_{n=0}^{\infty} \frac{e^{2\pi n i \xi}}{(n+a)^s} = \Phi(e^{2\pi i \xi}, s, a) \quad (6)$$

$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \Re(s) > 0 \text{ when } \xi \in \mathbb{R} \setminus \mathbb{Z}; \Re(s) > 1 \text{ when } \xi \in \mathbb{Z}).$

The Hurwitz-Lerch zeta function $\Phi(z, s, a)$ defined in (7) can be continued *meromorphically* to the whole complex s -plane,

except for a simple pole at $s = 1$ with its residue 1. It is well known that

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt$$

(7)

$\Re(a) > 0; \Re(s) > 0$ when $|z| \leq 1 (z \neq 1);$
 $\Re(s) > 1$ when $z = 1).$

Motivated by the works of Goyal and Laddha [4], Lin and Srivastava [5], Garg et al. [6], and other authors, Srivastava et al. [7] (see also [8]) investigated various properties of a natural multiparameter extension and generalization of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ defined by (7) (see also [9]). In particular, they considered the following functions:

$$\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q)}(z, s, a) := \sum_{n=0}^\infty \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{n! \prod_{j=1}^q (\mu_j)_{n\sigma_j}} \frac{z^n}{(n+a)^s}$$

(8)

$(p, q \in \mathbb{N}_0; \lambda_j \in \mathbb{C} (j = 1, \dots, p);$
 $a, \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^- (j = 1, \dots, q);$
 $\rho_j, \sigma_k \in \mathbb{R}^+ (j = 1, \dots, p; k = 1, \dots, q);$
 $\Delta > -1$ when $s, z \in \mathbb{C};$
 $\Delta = -1, s \in \mathbb{C}$ when $|z| < \nabla^*;$
 $\Delta = -1, \Re(\Xi) > \frac{1}{2}$ when $|z| = \nabla^*)$

with

$$\nabla^* := \left(\prod_{j=1}^p \rho_j^{-\rho_j} \right) \cdot \left(\prod_{j=1}^q \sigma_j^{\sigma_j} \right),$$

$$\Delta := \sum_{j=1}^q \sigma_j - \sum_{j=1}^p \rho_j,$$

$$\Xi := s + \sum_{j=1}^q \mu_j - \sum_{j=1}^p \lambda_j + \frac{p-q}{2}.$$

(9)

Here, and for the remainder of this paper, $(\lambda)_\kappa$ denotes the Pochhammer symbol defined, in terms of the gamma function, by

$$(\lambda)_\kappa := \frac{\Gamma(\lambda + \kappa)}{\Gamma(\lambda)}$$

$$= \begin{cases} \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\kappa = n \in \mathbb{N}; \lambda \in \mathbb{C}) \\ 1 & (\kappa = 0; \lambda \in \mathbb{C} \setminus \{0\}); \end{cases}$$

(10)

it is being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ -quotient exists (see, for details, [10, page 21 *et seq.*]).

In their work, Srivastava et al. [7, page 504, Theorem 8] also proved the following relation for the function $\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q)}(z, s, a)$:

$$\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q)}(z, s, a) = \frac{\prod_{j=1}^q \Gamma(\mu_j)}{\prod_{j=1}^p \Gamma(\lambda_j)}$$

$$\cdot \overline{H}_{p+1, q+2}^{1, p+1} \left[-z \mid \begin{matrix} (1 - \lambda_1, \rho_1; 1), \dots, (1 - \lambda_p, \rho_p; 1), (1 - a, 1; s) \\ (0, 1), (1 - \mu_1, \sigma_1; 1), \dots, (1 - \mu_p, \sigma_p; 1), (-a, 1; s) \end{matrix} \right]$$

(11)

provided that both sides of (11) exist.

Definition 1. The $\overline{H}(z)$ involved in the right-hand side of (11) is the generalized Fox's H -function introduced by Inayat-Hussain [11, page 4126]

$$\overline{H}(z) = \overline{H}_{p, q}^{m, n}(z \mid \left[\begin{matrix} (a_j, A_j; \alpha_j)_{j=1}^n, (a_j, A_j)_{j=n+1}^p \\ (b_j, B_j)_{j=1}^m, (b_j, B_j; \beta_j)_{j=m+1}^q \end{matrix} \right])$$

$$:= \frac{1}{2\pi i} \int_{\mathfrak{g}} \chi(s) z^s ds$$

(12)

$(z \neq 0; i = \sqrt{-1};$

$$\chi(s) := \left(\prod_{j=1}^m \Gamma(b_j - B_j s) \cdot \prod_{j=1}^n \{ \Gamma(1 - a_j + A_j s) \}^{\alpha_j} \right)$$

$$\times \left(\prod_{j=n+1}^p \Gamma(a_j - A_j s) \right)$$

$$\cdot \left(\prod_{j=m+1}^q \{ \Gamma(1 - b_j + B_j s) \}^{\beta_j} \right)^{-1}.$$

(12)

Here the parameters

$$A_j > 0 \quad (j = 1, \dots, p), \quad B_j > 0 \quad (j = 1, \dots, q),$$

(13)

and the exponents

$$\alpha_j \quad (j = 1, \dots, p), \quad \beta_j \quad (j = 1, \dots, q) \quad (14)$$

can take noninteger values and $\mathfrak{L} = \mathfrak{L}_{i\tau; \infty}$ is a Mellin-Barnes type contour starting at the point $\tau - i\infty$ and terminating at the point $\tau + i\infty$ ($\tau \in \mathbb{R}$) with the usual indentations to separate one set of poles from the other set of poles.

Buschman and Srivastava [12, page 4708] established that the sufficient conditions for the absolute convergence of the contour integral in (12) are given by

$$\Lambda := \sum_{j=1}^m B_j + \sum_{j=1}^n |\alpha_j| A_j - \sum_{j=m+1}^q |\beta_j| B_j - \sum_{j=n+1}^p A_j > 0 \quad (15)$$

and the region of absolute convergence is

$$|\arg(z)| < \frac{1}{2}\pi\Lambda. \quad (16)$$

Note that when

$$\alpha_1 = \dots = \alpha_n = 1, \quad \beta_{m+1} = \dots = \beta_q = 1, \quad (17)$$

the \overline{H} -function reduces to the well-known Fox's H -function (see [13]).

This paper is devoted to extending several interesting results obtained recently by Srivastava et al. [14] (see also [15, 16]) to the extended multiparameter Hurwitz-Lerch zeta function $\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q)}(z, s, a)$ introduced and studied by Srivastava et al. [7]. In Section 2, we give the representation of the fractional derivatives based on the Pochhammer's contour of integration. Section 3 aims at recalling some major fractional calculus theorems, that is, two generalized Leibniz rules and three Taylor-like expansions as well as a fundamental relation linked to the generalized chain rule for the fractional derivatives. In the two remaining sections, we, respectively, present and prove the main results of this paper and we give some special cases.

2. Pochhammer Contour Integral Representation for Fractional Derivative

The most familiar representation for the fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [17] (see also [18–20]); that is,

$$\mathcal{D}_z^\alpha \{z^p f(z)\} = \frac{1}{\Gamma(-\alpha)} \int_0^z f(\xi) \xi^p (\xi - z)^{-\alpha-1} d\xi \quad (18)$$

$$(\Re(\alpha) < 0; \Re(p) > 1),$$

where the integration is carried out along a straight line from 0 to z in the complex ξ -plane. By integrating by part m times, we obtain

$$\mathcal{D}_z^\alpha \{z^p f(z)\} = \frac{d^m}{dz^m} \{\mathcal{D}_z^{\alpha-m} \{z^p f(z)\}\}. \quad (19)$$

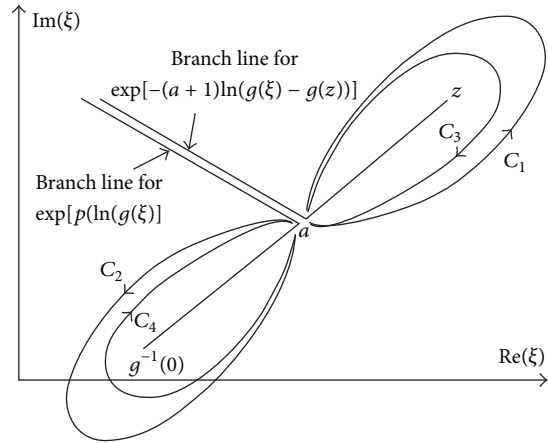


FIGURE 1: Pochhammer's contour.

This allows us to modify the restriction $\Re(\alpha) < 0$ to $\Re(\alpha) < m$ (see [20]).

Another representation for the fractional derivative is based on the Cauchy integral formula. This representation, too, has been widely used in many interesting papers (see, for example, the works of Osler [21–24]).

The relatively less restrictive representation of the fractional derivative according to parameters appears to be the one based on the Pochhammer's contour integral introduced by Lavoie et al. [25] and Tremblay [26].

Definition 2. Let $f(z)$ be analytic in a simply connected region \mathcal{R} of the complex z -plane. Let $g(z)$ be regular and univalent on \mathcal{R} and let $g^{-1}(0)$ be an interior point of \mathcal{R} . Then, if α is not a negative integer, p is not an integer, and z is in $\mathcal{R} \setminus \{g^{-1}(0)\}$, we define the fractional derivative of order α of $g(z)^p f(z)$ with respect to $g(z)$ by

$$D_{g(z)}^\alpha \{[g(z)]^p f(z)\} = \frac{e^{-i\pi p} \Gamma(1 + \alpha)}{4\pi \sin(\pi p)} \times \int_{C(z+, g^{-1}(0)+, z-, g^{-1}(0)-; F(a), F(a))} \frac{f(\xi) [g(\xi)]^p g'(\xi)}{[g(\xi) - g(z)]^{\alpha+1}} d\xi. \quad (20)$$

For nonintegers α and p , the functions $g(\xi)^p$ and $[g(\xi) - g(z)]^{-\alpha-1}$ in the integrand have two branch lines which begin, respectively, at $\xi = z$ and $\xi = g^{-1}(0)$, and both branches pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ at any other point as shown in Figure 1. Here $F(a)$ denotes the principal value of the integrand in (20) at the beginning and the ending point of the Pochhammer contour $P(a)$ which is closed on the Riemann surface of the multiple-valued function $F(\xi)$ (see Figure 2).

Remark 3. In Definition 2, the function $f(z)$ must be analytic at $\xi = g^{-1}(0)$. However, it is interesting to note here that if

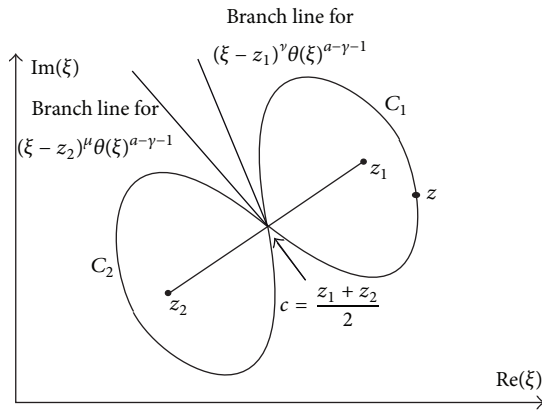


FIGURE 2: Multiloops contour.

we could also allow $f(z)$ to have an essential singularity at $\xi = g^{-1}(0)$, then (20) would still be valid.

Remark 4. In case the Pochhammer contour never crosses the singularities at $\xi = g^{-1}(0)$ and $\xi = z$ in (20), then we know that the integral is analytic for all p and for all α and for z in $\mathcal{R} \setminus \{g^{-1}(0)\}$. Indeed, in this case, the only possible singularities of $D_{g(z)}^\alpha \{g(z)^p f(z)\}$ are $\alpha = -1, -2, \dots$ and $p = 0, \pm 1, \pm 2, \dots$, which can directly be identified from the coefficient of the integral (20). However, by integrating by parts N times the integral in (20) by two different ways, we can show that $\alpha = -1, -2, \dots$ and $p = 0, 1, 2, \dots$ are removable singularities (see, for details, [25]).

In their work, Srivastava et al. [7] made use of the following fractional calculus result obtained by Srivastava et al. [27, page 97, Equation (2.4)]:

$$D_z^\nu \left\{ z^{\lambda-1} \overline{H}_{p,q}^{m,n}(\omega z^\kappa) \right\} = z^{\lambda-\nu-1} \overline{H}_{p+1,q+1}^{m,n+1}(\omega z^\kappa) \times \left[\omega z^\kappa \mid \begin{matrix} (1-\lambda, \kappa; 1), (a_j, A_j; \alpha_j)_{j=1}^n, (a_j, A_j)_{j=n+1}^p \\ (b_j, B_j)_{j=1}^m, (a_j, A_j; \beta_j)_{j=m+1}^q, (1-\lambda+\nu, \kappa; 1) \end{matrix} \right] \quad (\Re(\lambda) > 0; \kappa > 0), \quad (21)$$

in order to derive the following important fractional derivative formula for this work:

$$D_z^{\nu-\tau} \left\{ z^{\nu-1} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} = \frac{\prod_{j=1}^q \Gamma(\mu_j)}{\prod_{j=1}^p \Gamma(\lambda_j)} z^{\tau-1} \cdot \overline{H}_{p+2, q+3}^{-1, p+2} \left[-z^\kappa \mid \begin{matrix} (1-\lambda_1, \rho_1; 1), \dots, (1-\lambda_p, \rho_p; 1), (1-\nu, \kappa; 1), (1-a, 1; s) \\ (0, 1), (1-\mu_1, \sigma_1; 1), \dots, (1-\mu_q, \sigma_q; 1), (1-\tau, \kappa; 1), (-a, 1; s) \end{matrix} \right] = \frac{\Gamma(\nu)}{\Gamma(\tau)} z^{\tau-1} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q, \tau}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z^\kappa, s, a) \quad (\Re(\nu) > 0; \kappa > 0). \quad (22)$$

This fractional calculus formula was obtained by using the Riemann-Liouville representation for the fractional derivative. Adopting the Pochhammer based representation for the fractional derivative, these last restrictions become $\kappa + \nu - 1$, not a negative integer, and $\kappa > 0$.

The parameters involved in the fractional derivative formula (22) can be specialized to deduce other results. For example, setting $p - 1 = q = 1$ in (22) and making the following substitutions $\rho_1 \mapsto \rho$, $\rho_2 \mapsto \sigma$, $\sigma_1 \mapsto \kappa$, $\lambda_1 \mapsto \lambda$, $\lambda_2 \mapsto \mu$, and $\mu_1 \mapsto \nu$ lead to

$$D_z^{\nu-\tau} \left\{ z^{\nu-1} \Phi_{\lambda, \mu; \nu}^{(\rho, \sigma, \kappa)}(z^\kappa, s, a) \right\} = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)} z^{\tau-1} \overline{H}_{4,4}^{-1,4}$$

$$\times \left[-z^\kappa \mid \begin{matrix} (1-\lambda, \rho; 1), (1-\mu, \sigma; 1), (1-\nu, \kappa; 1), (1-a, 1; s) \\ (0, 1), (1-\nu, \kappa; 1), (1-\tau, \kappa; 1), (-a, 1; s) \end{matrix} \right] = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)} z^{\tau-1} \overline{H}_{3,3}^{-1,3} \times \left[-z^\kappa \mid \begin{matrix} (1-\lambda, \rho; 1), (1-\mu, \sigma; 1), (1-a, 1; s) \\ (0, 1), (1-\tau, \kappa; 1), (-a, 1; s) \end{matrix} \right] = \frac{\Gamma(\nu)}{\Gamma(\tau)} z^{\tau-1} \Phi_{\lambda, \mu; \tau}^{(\rho, \sigma, \kappa)}(z^\kappa, s, a) \quad (\nu \text{ not a negative integer; } \kappa > 0). \quad (23)$$

Furthermore, if we put $\rho = \sigma = \kappa = 1$ in (23), then we obtain

$$\begin{aligned}
 & D_z^{\nu-\tau} \left\{ z^{\nu-1} \Phi_{\lambda, \mu; \nu}^{(1,1,1)}(z, s, a) \right\} \\
 &= \frac{\Gamma(\nu)}{\Gamma(\tau)} z^{\tau-1} \Phi_{\lambda, \mu; \tau}(z, s, a) \\
 &= \frac{\Gamma(\nu)}{\Gamma(\lambda) \Gamma(\mu)} z^{\tau-1} \overline{H}_{3,3}^{1,3} \\
 &\times \left[-z \mid \begin{matrix} (1-\lambda, 1; 1), (1-\mu, 1; 1), (1-a, 1; s) \\ (0, 1), (1-\tau, 1; 1), (-a, 1; s) \end{matrix} \right] \\
 &\quad (\nu \text{ not a negative integer; } \kappa > 0).
 \end{aligned} \tag{24}$$

Finally, letting $\lambda = \nu$ in (24), this yields after elementary calculations

$$\begin{aligned}
 & \Phi_{\mu}^*(z, s, a) \\
 &= \frac{\Gamma(\tau)}{\Gamma(\nu) \Gamma(\mu)} \overline{H}_{3,3}^{1,3} \\
 &\times \left[-z \mid \begin{matrix} (1-\nu, 1; 1), (1-\mu, 1; 1), (1-a, 1; s) \\ (0, 1), (1-\tau, 1; 1), (-a, 1; s) \end{matrix} \right].
 \end{aligned} \tag{25}$$

Another fractional derivative formula that will be very useful in this work is given by the following formula:

$$\begin{aligned}
 & D_z^{\alpha} \left\{ z^{\beta} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^{\kappa}, s, a) \right\} \\
 &= \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} z^{\beta-\alpha} \\
 &\cdot \Phi_{\lambda_1, \dots, \lambda_p, 1+\beta; \mu_1, \dots, \mu_q, 1+\beta-\alpha}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z^{\kappa}, s, a) \\
 &\quad (\beta \text{ not a negative integer, } \kappa > 0).
 \end{aligned} \tag{26}$$

This last result can be established with the help of the following well known formula [28, page 83, Equation (2.4)]:

$$D_z^{\alpha} \{z^p\} = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} \quad (\Re(p) > -1). \tag{27}$$

Adopting the Pochhammer based representation for the fractional derivative modifies the restriction to the case when p is not a negative integer.

3. Some Fundamental Theorems Involving Fractional Calculus

In this section, we recall six fundamental theorems related to fractional calculus that will play central roles in our work. Each of these theorems is the generalized Leibniz rules for fractional derivatives, the Taylor-like expansions in terms of different types of functions, and a fundamental formula related to the generalized chain rule for fractional derivatives.

First of all, we give two generalized Leibniz rules for fractional derivatives. Theorem 5 is a slightly modified theorem obtained in 1970 by Osler [22]. Theorem 6 was given,

some years ago, by Tremblay et al. [29] with the help of the properties of Pochhammer’s contour representation for fractional derivatives.

Theorem 5. (i) Let \mathcal{R} be a simply connected region containing the origin. (ii) Let $u(z)$ and $v(z)$ satisfy the conditions of Definition 2 for the existence of the fractional derivative. Then, for $\Re(p+q) > -1$ and $\gamma \in \mathbb{C}$, the following Leibniz rule holds true:

$$\begin{aligned}
 D_z^{\alpha} \{z^{p+q} u(z) v(z)\} &= \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma+n} D_z^{\alpha-\gamma-n} \\
 &\times \{z^p u(z)\} D_z^{\gamma+n} \{z^q v(z)\}.
 \end{aligned} \tag{28}$$

Theorem 6. (i) Let \mathcal{R} be a simply connected region containing the origin. (ii) Let $u(z)$ and $v(z)$ satisfy the conditions of Definition 2 for the existence of the fractional derivative. (iii) Let $\mathcal{U} \subset \mathcal{R}$ be the region of analyticity of the function $u(z)$ and let $\mathcal{V} \subset \mathcal{R}$ be the region of analyticity of the function $v(z)$. Then, for

$$z \neq 0, \quad z \in \mathcal{U} \cap \mathcal{V}, \quad \Re(1-\beta) > 0, \tag{29}$$

the following product rule holds true:

$$\begin{aligned}
 & D_z^{\alpha} \{z^{\alpha+\beta-1} u(z) v(z)\} \\
 &= \frac{z \Gamma(1+\alpha) \sin(\beta\pi) \sin(\mu\pi) \sin[(\alpha+\beta-\mu)\pi]}{\sin[(\alpha+\beta)\pi] \sin[(\beta-\mu-\nu)\pi] \sin[(\mu+\nu)\pi]} \\
 &\cdot \sum_{n=-\infty}^{\infty} \frac{D_z^{\alpha+\nu+1-n} \{z^{\alpha+\beta-\mu-1-n} u(z)\} D_z^{-1-\nu+n} \{z^{\mu-1+n} v(z)\}}{\Gamma(2+\alpha+\nu-n) \Gamma(-\nu+n)}.
 \end{aligned} \tag{30}$$

Next, in 1971, Osler [30] established the following generalized Taylor-like series expansion involving fractional derivatives.

Theorem 7. Let $f(z)$ be an analytic function in a simply connected region \mathcal{R} . Let α and γ be arbitrary complex numbers and

$$\theta(z) = (z - z_0) q(z) \tag{31}$$

with $q(z)$ a regular and univalent function without any zero in \mathcal{R} . Let a be a positive real number and

$$\begin{aligned}
 & K = \{0, 1, \dots, [c]\} \\
 & \quad ([c] \text{ the largest integer not greater than } c).
 \end{aligned} \tag{32}$$

Let b and z_0 be two points in \mathcal{R} such that $b \neq z_0$ and let

$$\omega = \exp\left(\frac{2\pi i}{a}\right). \tag{33}$$

Then the following relationship holds true:

$$\begin{aligned} & \sum_{k \in K} c^{-1} \omega^{-\gamma k} f(\theta^{-1}(\theta(z)\omega^k)) \\ &= \sum_{n=-\infty}^{\infty} \frac{[\theta(z)]^{cn+\gamma}}{\Gamma(cn+\gamma+1)} \\ & \cdot D_{z-b}^{cn+\gamma} \left\{ f(z)\theta'(z) \left(\frac{z-z_0}{\theta(z)} \right)^{cn+\gamma+1} \right\} \Big|_{z=z_0} \\ & \quad (|z-z_0|=|z_0|). \end{aligned} \tag{34}$$

In particular, if $0 < c \leq 1$ and $\theta(z) = (z - z_0)$, then $k = 0$ and the formula (34) reduces to the following form:

$$f(z) = c \sum_{n=-\infty}^{\infty} \frac{(z-z_0)^{cn+\gamma}}{\Gamma(cn+\gamma+1)} D_{z-b}^{cn+\gamma} \{f(z)\} \Big|_{z=z_0}. \tag{35}$$

This last formula (35) is usually referred to as the Taylor-Riemann formula and has been studied in several papers [23, 31–34].

We next recall that Tremblay et al. [35] obtained the power series of an analytic function $f(z)$ in terms of the rational expression $((z-z_1)/(z-z_2))$, where z_1 and z_2 are two arbitrary points inside the region \mathcal{R} of analyticity of $f(z)$. In particular, they obtained the following result.

Theorem 8. (i) Let c be real and positive and let

$$\omega = \exp\left(\frac{2\pi i}{a}\right). \tag{36}$$

(ii) Let $f(z)$ be analytic in the simply connected region \mathcal{R} with z_1 and z_2 being interior points of \mathcal{R} . (iii) Let the set of curves

$$\{C(t) : C(t) \subset \mathcal{R}, 0 < t \leq r\} \tag{37}$$

be defined by

$$\begin{aligned} C(t) &= C_1(t) \cup C_2(t) \\ &= \left\{ z : \left| \lambda_t(z_1, z_2; z) \right| = \left| \lambda_t\left(z_1, z_2; \frac{z_1+z_2}{2}\right) \right| \right\}, \end{aligned} \tag{38}$$

where

$$\begin{aligned} \lambda_t(z_1, z_2; z) &= \left[z - \frac{z_1+z_2}{2} + t \left(\frac{z_1-z_2}{2} \right) \right] \\ & \cdot \left[z - \left(\frac{z_1+z_2}{2} \right) - t \left(\frac{z_1-z_2}{2} \right) \right], \end{aligned} \tag{39}$$

which are the Bernoulli type lemniscates with center located at $(z_1+z_2)/2$ and with double-loops in which one loop $C_1(t)$ leads around the focus point

$$\frac{z_1+z_2}{2} + \left(\frac{z_1-z_2}{2} \right) t \tag{40}$$

and the other loop $C_2(t)$ encircles the focus point

$$\frac{z_1+z_2}{2} - \left(\frac{z_1-z_2}{2} \right) t, \tag{41}$$

for each t such that $0 < t \leq r$. (iv) Let

$$[(z-z_1)(z-z_2)]^\lambda = \exp(\lambda \ln(\theta((z-z_1)(z-z_2)))) \tag{42}$$

denote the principal branch of that function which is continuous and inside $C(r)$, cut by the respective two branch lines L_\pm defined by

$$L_\pm = \begin{cases} \left\{ z : z = \frac{z_1+z_2}{2} \pm t \left(\frac{z_1-z_2}{2} \right) \right\} & (0 \leq t \leq 1), \\ \left\{ z : z = \frac{z_1+z_2}{2} \pm it \left(\frac{z_1-z_2}{2} \right) \right\} & (t < 0) \end{cases} \tag{43}$$

such that $\ln((z-z_1)(z-z_2))$ is real when $(z-z_1)(z-z_2) > 0$. (v) Let $f(z)$ satisfy the conditions of Definition 2 for the existence of the fractional derivative of $(z-z_2)^p f(z)$ of order α for $z \in \mathcal{R} \setminus \{L_+ \cup L_-\}$, denoted by $D_{z-z_2}^\alpha \{(z-z_2)^p f(z)\}$, where α and p are real or complex numbers. (vi) Let

$$\begin{aligned} K &= \left\{ k : k \in \mathbb{N}, \arg\left(\lambda_t\left(z_1, z_2, \frac{z_1+z_2}{2}\right)\right) \right. \\ & < \arg\left(\lambda_t\left(z_1, z_2, \frac{z_1+z_2}{2}\right)\right) + \frac{2\pi k}{a} \\ & < \arg\left(\lambda_t\left(z_1, z_2, \frac{z_1+z_2}{2}\right)\right) + 2\pi \left. \right\}. \end{aligned} \tag{44}$$

Then, for arbitrary complex numbers μ, ν, γ and for z on $C_1(1)$ defined by

$$\xi = \frac{z_1+z_2}{2} + \frac{z_1-z_2}{2} \sqrt{1+e^{i\theta}} \quad (-\pi < \theta < \pi),$$

$$\begin{aligned} & \sum_{k \in K} \frac{c^{-1} \omega^{-\gamma k}}{z_1-z_2} f(\phi^{-1}(\omega^k \phi(z))) \\ & \times [\phi^{-1}(\phi(z)\omega^k) - z_1]^\nu [\phi^{-1}(\phi(z)\omega^k) - z_2]^\mu \\ &= \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)} \sin[(\mu+cn+\gamma)\pi]}{\sin[(\mu-c+\gamma)\pi] \Gamma(1-\nu+cn+\gamma)} \\ & \cdot D_{z-z_2}^{-\nu+cn+\gamma} \{(z-z_2)^{\mu+cn+\gamma-1} f(z)\} \Big|_{z=z_1} [\phi(z)]^{cn+\gamma}, \end{aligned} \tag{45}$$

where

$$\phi(z) = \frac{z-z_1}{z-z_2}. \tag{46}$$

The case $0 < c \leq 1$ of Theorem 8 reduces to the following form:

$$\begin{aligned} & \frac{c^{-1} f(z) (z-z_1)^\nu (z-z_2)^\mu}{z_1-z_2} \\ &= \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)} \sin[(\mu+cn+\gamma)\pi]}{\sin[(\mu-c+\gamma)\pi] \Gamma(1-\nu+cn+\gamma)} \\ & \cdot D_{z-z_2}^{-\nu+cn+\gamma} \{(z-z_2)^{\mu+cn+\gamma-1} f(z)\} \Big|_{z=z_1} \left(\frac{z-z_1}{z-z_2} \right)^{cn+\gamma}. \end{aligned} \tag{47}$$

Tremblay and Fugère [36] developed the power series of an analytic function $f(z)$ in terms of the function $(z - z_1)(z - z_2)$, where z_1 and z_2 are two arbitrary points inside the analyticity region \mathcal{R} of $f(z)$. Explicitly, they showed the following theorem.

Theorem 9. Under the assumptions of Theorem 8, the following expansion formula holds true:

$$\begin{aligned} & \sum_{k \in K} c^{-1} \omega^{-\gamma k} \\ & \times \left[\left(\frac{z_2 - z_1 + \sqrt{\Delta_k}}{2} \right)^\alpha \left(\frac{z_1 - z_2 + \sqrt{\Delta_k}}{2} \right)^\beta \right. \\ & \cdot f \left(\frac{z_1 + z_2 + \sqrt{\Delta_k}}{2} \right) - e^{i\pi(\alpha-\beta)} \frac{\sin[(\alpha + c - \gamma)\pi]}{\sin[(\beta + c - \gamma)\pi]} \\ & \cdot \left(\frac{z_2 - z_1 - \sqrt{\Delta_k}}{2} \right)^\alpha \left(\frac{z_1 - z_2 - \sqrt{\Delta_k}}{2} \right)^\beta \\ & \left. \times f \left(\frac{z_1 + z_2 - \sqrt{\Delta_k}}{2} \right) \right] \\ & = \sum_{n=-\infty}^{\infty} \frac{\sin[(\beta - cn - \gamma)\pi]}{\sin[(\beta - c - \gamma)\pi]} \frac{e^{-i\pi c(n+1)} [\theta(z)]^{cn+\gamma}}{\Gamma(1 - \alpha + cn + \gamma)} \\ & \cdot D_{z-z_2}^{-\alpha+cn+\gamma} \left\{ (z - z_2)^{\beta-cn-\gamma-1} \right. \\ & \quad \times \left(\frac{\theta(z)}{(z - z_2)(z - z_1)} \right)^{-cn-\gamma-1} \\ & \quad \left. \times \theta'(z) f(z) \right\} \Big|_{z=z_1}, \end{aligned} \tag{48}$$

where

$$\begin{aligned} \Delta_k &= (z_1 - z_2)^2 + 4V(\omega^k \theta(z)), \\ V(z) &= \sum_{r=1}^{\infty} D_z^{r-1} \left\{ [q(z)]^{-r} \right\} \Big|_{z=0} \frac{z^r}{r!}, \end{aligned} \tag{49}$$

$$\theta(z) = (z - z_1)(z - z_2)q((z - z_1)(z - z_2)).$$

As special case, if we set $0 < c \leq 1$, $q(z) = 1$ ($\theta(z) = (z - z_1)(z - z_2)$), and $z_2 = 0$ in (48), we obtain

$$\begin{aligned} f(z) &= cz^{-\beta}(z - z_1)^{-\alpha} \\ & \times \sum_{n=-\infty}^{\infty} \frac{\sin[(\beta - cn - \gamma)\pi]}{\sin[(\beta + c - \gamma)\pi]} \frac{e^{i\pi c(n+1)} [z(z - z_1)]^{cn+\gamma}}{\Gamma(1 - \alpha + cn + \gamma)} \\ & \cdot D_z^{-\alpha+cn+\gamma} \left\{ z^{\beta-cn-\gamma-1} (z + w - z_1) f(z) \right\} \Big|_{z=z_1, (w=z)}. \end{aligned} \tag{50}$$

Finally, Osler [21, page 290, Theorem 2] discovered a fundamental relation from which he deduced the generalized chain rule for the fractional derivatives. This result is recalled here as Theorem 10.

Theorem 10. Let $f(g^{-1}(z))$ and $f(h^{-1}(z))$ be defined and analytic in the simply connected region \mathcal{R} of the complex z -plane and let the origin be an interior or boundary point of \mathcal{R} . Suppose also that $g^{-1}(z)$ and $h^{-1}(z)$ are regular univalent functions on \mathcal{R} and that $h^{-1}(0) = g^{-1}(0)$. Let $\oint f(g^{-1}(z))dz$ vanish over simple closed contour in $\mathcal{R} \cup \{0\}$ through the origin. Then the following relation holds true:

$$D_{g(z)}^\alpha \{f(z)\} = D_{h(z)}^\alpha \left\{ \frac{f(z)g'(z)}{h'(z)} \left(\frac{h(w) - h(z)}{g(w) - g(z)} \right)^{\alpha+1} \right\} \Big|_{w=z} \tag{51}$$

The relation (51) allows us to obtain very easily known and new summation formulas involving special functions of mathematical physics.

By applying relation (51), Gaboury and Tremblay [37] proved the following corollary which will be useful in the next section.

Corollary II. Under the hypotheses of Theorem 10, let p be a positive integer. Then the following relation holds true:

$$\begin{aligned} & z^p O_\beta^\alpha \{f(z)\} \\ & = p(z^{p-1})^{-\alpha} \\ & \cdot z O_\beta^\alpha \left\{ f(z) (z^{p-1})^\alpha \prod_{s=1}^{p-1} \left(1 - \frac{z}{w} e^{-2\pi is/p} \right)^{\beta-\alpha-1} \right\} \Big|_{w=z}, \end{aligned} \tag{52}$$

where

$$g(z) O_\beta^\alpha \{\dots\} := \frac{\Gamma(\beta)}{\Gamma(\alpha)} [g(z)]^{1-\beta} D_{g(z)}^{\alpha-\beta} \left\{ [g(z)]^{\alpha-1} \dots \right\}. \tag{53}$$

4. Relations Involving the Extended Multiparameters Hurwitz-Lerch Zeta

Function $\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q)}(z, s, a)$

In this section, we present the new expansion formulas involving the extended multiparameters Hurwitz-Lerch zeta function $\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p; \sigma_1, \dots, \sigma_q)}(z, s, a)$.

Theorem 12. Under the hypotheses of Theorem 5, the following expansion formula holds true:

$$\begin{aligned} & \Phi_{\lambda_1, \dots, \lambda_p; \nu; \mu_1, \dots, \mu_q, \tau}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z^\kappa, s, a) \\ &= \frac{\Gamma(\tau) \Gamma(1 + \nu - \tau) \sin(\gamma\pi)}{\pi} \\ & \cdot \sum_{n=-\infty}^{\infty} \frac{(-1)^n \Phi_{\lambda_1, \dots, \lambda_p; 1; \mu_1, \dots, \mu_q, 1-\gamma-n}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z^\kappa, s, a)}{(\gamma + n) \Gamma(1 + \nu - \tau - \gamma - n) \Gamma(\tau + \gamma + n)}, \end{aligned} \tag{54}$$

provided that both members of (54) exist.

Proof. Setting $u(z) = z^{\nu-1}$ and $v(z) = \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a)$ in Theorem 5 with $p = q = 0$ and $\alpha = \nu - \tau$, we obtain

$$\begin{aligned} & D_z^{\nu-\tau} \left\{ z^{\nu-1} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \\ &= \sum_{n=-\infty}^{\infty} \binom{\nu-\tau}{\gamma+n} D_z^{\nu-\tau-\gamma-n} \left\{ z^{\nu-1} \right\} \\ & \cdot D_z^{\gamma+n} \left\{ \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\}, \end{aligned} \tag{55}$$

which, with the help of (22), (26), and (27), yields

$$\begin{aligned} & D_z^{\nu-\tau} \left\{ z^{\nu-1} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \\ &= \frac{\Gamma(\nu)}{\Gamma(\tau)} z^{\tau-1} \Phi_{\lambda_1, \dots, \lambda_p; \nu; \mu_1, \dots, \mu_q, \tau}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z^\kappa, s, a), \\ & D_z^{\nu-\tau-\gamma-n} \left\{ z^{\nu-1} \right\} = \frac{\Gamma(\nu)}{\Gamma(\tau + \gamma + n)} z^{\tau+\gamma+n-1}, \end{aligned} \tag{56}$$

$$\begin{aligned} & D_z^{\gamma+n} \left\{ \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \\ &= \frac{z^{-\gamma-n}}{\Gamma(1 - \gamma - n)} \Phi_{\lambda_1, \dots, \lambda_p; 1; \mu_1, \dots, \mu_q, 1-\gamma-n}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z^\kappa, s, a). \end{aligned}$$

Combining (56) with (55) and making some elementary simplifications, the asserted result (54) follows. \square

Theorem 13. Under the hypotheses of Theorem 6, the following expansion formula holds true:

$$\begin{aligned} & \Phi_{\lambda_1, \dots, \lambda_p; \nu; \mu_1, \dots, \mu_q, \tau}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z^\kappa, s, a) \\ &= \frac{\Gamma(\tau) \Gamma(1 + \nu - \tau)}{\Gamma(\nu) \Gamma(\tau - \gamma - \theta - 1)} \\ & \cdot (\sin \beta\pi \sin[(\nu - \tau + \beta - \theta)\pi] \sin \theta\pi) \\ & \times (\Gamma(1 + \gamma + \theta) \sin[(\nu - \tau + \beta)\pi] \\ & \quad \times \sin[(\beta - \theta - \gamma)\pi] \sin[(\theta + \gamma)\pi])^{-1} \\ & \cdot \sum_{n=-\infty}^{\infty} \left((\Gamma(\nu - \theta - n) \Gamma(\theta + n) \right. \\ & \quad \times \Phi_{\lambda_1, \dots, \lambda_p; \theta+n; \mu_1, \dots, \mu_q, 1+\theta+\lambda}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z^\kappa, s, a)) \\ & \quad \times (\Gamma(2 + \nu - \tau + \gamma - n) \Gamma(-\gamma + n))^{-1} \Big), \end{aligned} \tag{57}$$

provided that both members of (57) exist.

Proof. Upon first substituting $\mu \mapsto \theta$ and $\nu \mapsto \gamma$ in Theorem 6 and then setting

$$\alpha = \nu - \tau, \quad u(z) = z^{\tau-\beta} \tag{58}$$

$$v(z) = \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a),$$

in which both $u(z)$ and $v(z)$ satisfy the conditions of Theorem 6, we have

$$\begin{aligned} & D_z^{\nu-\tau} \left\{ z^{\nu-1} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \\ &= \frac{z\Gamma(1 + \nu - \tau) \sin \beta\pi \sin \theta\pi \sin(\nu - \tau + \beta - \theta)\pi}{\sin(\nu - \tau + \beta)\pi \sin(\beta - \theta - \gamma)\pi \sin(\theta + \gamma)\pi} \\ & \cdot \sum_{n=-\infty}^{\infty} \frac{D_z^{\nu-\tau+\gamma+1-n} \left\{ z^{\nu-\theta-1-n} \right\}}{\Gamma(2 + \nu - \tau + \gamma - n) \Gamma(-\gamma + n)} \\ & \cdot D_z^{-1-\gamma+n} \left\{ z^{\theta-1+n} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\}. \end{aligned} \tag{59}$$

Now, by using (22), (26), and (27), we find that

$$\begin{aligned} & D_z^{\nu-\tau} \left\{ z^{\nu-1} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \\ &= \frac{\Gamma(\nu)}{\Gamma(\tau)} z^{\tau-1} \Phi_{\lambda_1, \dots, \lambda_p; \nu; \mu_1, \dots, \mu_q, \tau}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z^\kappa, s, a), \\ & D_z^{\nu-\tau+\gamma+1-n} \left\{ z^{\nu-\theta-1-n} \right\} = \frac{\Gamma(\nu - \theta - n)}{\Gamma(\tau - \gamma - \theta - 1)} z^{\tau-\gamma-\theta-2}, \end{aligned} \tag{60}$$

$$\begin{aligned} & D_z^{-1-\gamma+n} \left\{ z^{\theta-1+n} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \\ &= \frac{\Gamma(\theta + n)}{\Gamma(1 + \theta + \gamma)} z^{\theta+\gamma} \\ & \times \Phi_{\lambda_1, \dots, \lambda_p; \theta+n; \mu_1, \dots, \mu_q, 1+\theta+\lambda}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z^\kappa, s, a). \end{aligned}$$

Thus, finally, the result (57) follows by combining (60) and (59). \square

We now shift our focus to the different Taylor-like expansions in terms of different types of functions involving the extended multiparameters Hurwitz-Lerch zeta functions $\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z, s, a)$.

Theorem 14. Under the assumptions of Theorem 7, the following expansion formula holds true:

$$\begin{aligned} & \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \\ &= c \sum_{n=-\infty}^{\infty} \frac{(z_0)^{-cn} (z - z_0)^{cn}}{\Gamma(cn + 1) \Gamma(1 - cn)} \\ & \times \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z_0^\kappa, s, a) \\ & \quad (|z - z_0| = |z_0|; \lambda > 0), \end{aligned} \tag{61}$$

provided that both members of (61) exist.

Proof. Setting $f(z) = \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a)$ in Theorem 7 with $b = \gamma = 0$, $0 < c \leq 1$, and $\theta(z) = z - z_0$, we have

$$\begin{aligned} & \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \\ &= c \sum_{n=-\infty}^{\infty} \frac{D_z^{cn} \left\{ \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \Big|_{z=z_0} (z - z_0)^{cn}}{\Gamma(1 + cn)}, \end{aligned} \tag{62}$$

for $z_0 \neq 0$ and for z such that $|z - z_0| = |z_0|$.

Now, by making use of (26) with $\beta = 0$ and $\alpha = cn$, we find that

$$\begin{aligned} & D_z^{cn} \left\{ \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \Big|_{z=z_0} \\ &= \frac{z_0^{-cn}}{\Gamma(1 - cn)} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q, 1 - cn}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z_0^\kappa, s, a). \end{aligned} \tag{63}$$

By combining (62) and (63), we get result (61) asserted by Theorem 14. \square

Theorem 15. *Under the hypotheses of Theorem 8, the following expansion formula holds true:*

$$\begin{aligned} & \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \\ &= cz^{-\alpha} (z - z_1)^{-\beta} z_1^{\alpha + \beta} \\ & \cdot \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)} \sin[(\alpha + cn + \gamma)\pi] \Gamma(\alpha + cn + \gamma)}{\sin[(\alpha - c + \gamma)\pi] \Gamma(1 - \beta + cn + \gamma) \Gamma(\alpha + \beta)} \\ & \cdot \Phi_{\lambda_1, \dots, \lambda_p, \alpha + cn + \gamma; \mu_1, \dots, \mu_q, \alpha + \beta}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z_1^\kappa, s, a) \left(\frac{z - z_1}{z}\right)^{cn + \gamma} \end{aligned} \tag{64}$$

for $\lambda > 0$ and for z on $C_1(1)$ defined by

$$z = \frac{z_1}{2} + \frac{z_1}{2} \sqrt{1 + e^{i\theta}} \quad (-\pi < \theta < \pi), \tag{65}$$

provided that both sides of (64) exist.

Proof. By taking $f(z) = \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a)$ in Theorem 8 with $z_2 = 0$, $\mu = \alpha$, $\nu = \beta$, and $0 < c \leq 1$, we find that

$$\begin{aligned} & \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \\ &= c(z - z_1)^{-\beta} z^{-\alpha} z_1 \\ & \cdot \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)} \sin[(\alpha + cn + \gamma)\pi]}{\sin[(\alpha - c + \gamma)\pi] \Gamma(1 - \beta + cn + \gamma)} \\ & \times \left(\frac{z - z_1}{z}\right)^{cn + \gamma} \\ & \cdot D_z^{-\beta + cn + \gamma} \left\{ z^{\alpha + cn + \gamma - 1} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \Big|_{z=z_1}. \end{aligned} \tag{66}$$

Now, with the help of relation (26) with $\alpha \mapsto -\beta + cn + \gamma$ and $\beta \mapsto \alpha + cn + \gamma - 1$, we have

$$\begin{aligned} & D_z^{-\beta + cn + \gamma} \left\{ z^{\alpha + cn + \gamma - 1} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \Big|_{z=z_1} \\ &= z_1^{\alpha + \beta - 1} \frac{\Gamma(\alpha + cn + \gamma)}{\Gamma(\alpha + \beta)} \\ & \times \Phi_{\lambda_1, \dots, \lambda_p, \alpha + cn + \gamma; \mu_1, \dots, \mu_q, \alpha + \beta}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z_1^\kappa, s, a). \end{aligned} \tag{67}$$

Thus, by combining (66) and (67), we are led to assertion (64) of Theorem 15. \square

Theorem 16. *Under the hypotheses of Theorem 9, the following expansion formula holds true:*

$$\begin{aligned} & \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \\ &= cz^{-\beta + \gamma} (z - z_1)^{-\alpha + \gamma} z_1^{\beta + \alpha - 2\gamma - 1} \\ & \cdot \sum_{n=-\infty}^{\infty} \frac{\sin[(\beta - cn - \gamma)\pi] e^{i\pi c(n+1)} [z(z - z_1)/z_1^2]^{cn}}{\sin[(\beta + c - \gamma)\pi] \Gamma(1 - \alpha + cn + \gamma)} \\ & \cdot \frac{\Gamma(\beta - cn - \gamma)}{\Gamma(\beta + \alpha - 2cn - 2\gamma)} \\ & \times \left[(z - z_1) \right. \\ & \times \Phi_{\lambda_1, \dots, \lambda_p, \beta - cn - \gamma; \mu_1, \dots, \mu_q, \beta + \alpha - 2cn - 2\gamma}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z_1^\kappa, s, a) \\ & + \frac{(\beta - cn - \gamma) z_1}{(\alpha + \beta - 2cn - 2\gamma)} \\ & \left. \times \Phi_{\lambda_1, \dots, \lambda_p, 1 + \beta - cn - \gamma; \mu_1, \dots, \mu_q, 1 + \beta + \alpha - 2cn - 2\gamma}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)}(z_1^\kappa, s, a) \right] \end{aligned} \tag{68}$$

for $\lambda > 0$ and for z on $C_1(1)$ defined by

$$z = \frac{z_1}{2} + \frac{z_1}{2} \sqrt{1 + e^{i\theta}} \quad (-\pi < \theta < \pi), \tag{69}$$

provided that both sides of (68) exist.

Proof. Setting $f(z) = \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a)$ in Theorem 9 with $z_2 = 0$, $0 < c \leq 1$, $q(z) = 1$, and $\theta(z) = (z - z_1)(z - z_2)$, we find that

$$\begin{aligned} & \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \\ &= cz^{-\beta} (z - z_1)^{-\alpha} \\ & \cdot \sum_{n=-\infty}^{\infty} \frac{\sin[(\beta - cn - \gamma)\pi] e^{i\pi c(n+1)} [z(z - z_1)]^{cn + \gamma}}{\sin[(\beta + c - \gamma)\pi] \Gamma(1 - \alpha + cn + \gamma)} \\ & \cdot D_z^{-\alpha + cn + \gamma} \\ & \times \left\{ z^{\beta - cn - \gamma - 1} (z + w - z_1) \right. \\ & \left. \times \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)}(z^\kappa, s, a) \right\} \Big|_{z=z_1, w=z}. \end{aligned} \tag{70}$$

With the help of the relation in (26), we have

$$\begin{aligned}
 & D_z^{-\alpha+cn+\gamma} \\
 & \times \left\{ z^{\beta-cn-\gamma-1} (z+w-z_1) \right. \\
 & \quad \left. \times \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} (z^\kappa, s, a) \right\} \Big|_{z=z_1, w=z} \\
 & = (z-z_1) D_z^{-\alpha+cn+\gamma} \\
 & \times \left\{ z^{\beta-cn-\gamma-1} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} (z^\kappa, s, a) \right\} \Big|_{z=z_1} \\
 & + D_z^{-\alpha+cn+\gamma} \left\{ z^{\beta-cn-\gamma} \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} (z^\kappa, s, a) \right\} \Big|_{z=z_1} \\
 & = z_1^{\beta+\alpha-2cn-2\gamma} \\
 & \times \left[\left((z-z_1) \Gamma(\beta-cn-\gamma) \right. \right. \\
 & \quad \times \Phi_{\lambda_1, \dots, \lambda_p, \beta-cn-\gamma; \mu_1, \dots, \mu_q, \beta+\alpha-2cn-2\gamma}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)} (z_1^\kappa, s, a) \\
 & \quad \times (z_1 \Gamma(\beta+\alpha-2cn-2\gamma))^{-1} \\
 & \quad + \left(\Gamma(1+\beta-cn-\gamma) \right. \\
 & \quad \quad \times \Phi_{\lambda_1, \dots, \lambda_p, 1+\beta-cn-\gamma; \mu_1, \dots, \mu_q, 1+\beta+\alpha-2cn-2\gamma}^{(\rho_1, \dots, \rho_p, \kappa, \sigma_1, \dots, \sigma_q, \kappa)} (z_1^\kappa, s, a) \\
 & \quad \left. \left. \times (\Gamma(1+\beta+\alpha-2cn-2\gamma))^{-1} \right] \right. \\
 & \tag{71}
 \end{aligned}$$

Thus, by combining (70) and (71), we obtain desired result (68). \square

Finally, from Corollary II given in the preceding section, we obtain the following new relation involving the extended multiparameters Hurwitz-Lerch zeta function $\Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} (z, s, a)$.

Theorem 17. Under the hypotheses of Corollary II, let k be a positive integer. Then the following relation holds true:

$$\begin{aligned}
 & \Phi_{\lambda_1, \dots, \lambda_p, \alpha; \mu_1, \dots, \mu_q, \beta}^{(\rho_1, \dots, \rho_p, 1/k, \sigma_1, \dots, \sigma_q, 1/k)} (z, s, a) \\
 & = \frac{k \Gamma(\beta) \Gamma(k\alpha)}{\Gamma(\alpha) \Gamma(\beta + (k-1)\alpha)} \\
 & \cdot \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{\prod_{j=1}^q (\mu_j)_{n\sigma_j}} \frac{(k\alpha)_n}{(\beta + (k-1)\alpha)_n} \frac{z^n}{(n+a)^s} \\
 & \cdot F_D^{(k-1)} \left[k\alpha + n, 1 + \alpha - \beta, \dots, 1 + \alpha - \beta; \right. \\
 & \quad \left. \beta + (k-1)\alpha + n; e^{-2\pi i/k}, \dots, e^{-2(k-1)\pi i/k} \right], \\
 & \tag{72}
 \end{aligned}$$

where $\lambda > 0$ and $F_D^{(n)}$ denotes the Lauricella function of n variables defined by [38, page 60]

$$\begin{aligned}
 & F_D^{(n)} [a, b_1, \dots, b_n; c; x_1, \dots, x_n] \\
 & = \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c)_{m_1+\dots+m_n}} \\
 & \times \frac{x_1^{m_1}}{m_1!} \dots \frac{x_n^{m_n}}{m_n!} \quad (\max \{|x_1|, \dots, |x_n|\} < 1), \\
 & \tag{73}
 \end{aligned}$$

provided that both sides of (72) exist.

Proof. Putting $p = k$ and letting $f(z) = \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} (z, s, a)$ in Corollary II, we get

$$\begin{aligned}
 & z^k O_\beta^\alpha \left\{ \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} (z, s, a) \right\} \\
 & = \frac{k}{(z^{k-1})^\alpha} \\
 & \cdot z O_\beta^\alpha \left\{ \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} (z, s, a) (z^{k-1})^\alpha \right. \\
 & \quad \left. \times \prod_{s=1}^{k-1} \left(1 - \frac{z}{w} e^{-2\pi i s/k} \right)^{\beta-\alpha-1} \right\} \Big|_{w=z}. \\
 & \tag{74}
 \end{aligned}$$

With the help of the definition of $z O_\beta^\alpha$ given by (53), we find for the left-hand side of (74) that

$$\begin{aligned}
 & z O_\beta^\alpha \left\{ \Phi_{\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q}^{(\rho_1, \dots, \rho_p, \sigma_1, \dots, \sigma_q)} (z, s, a) \right\} \\
 & = \Phi_{\lambda_1, \dots, \lambda_p, \alpha; \mu_1, \dots, \mu_q, \beta}^{(\rho_1, \dots, \rho_p, 1/k, \sigma_1, \dots, \sigma_q, 1/k)} (z, s, a). \\
 & \tag{75}
 \end{aligned}$$

We now expand each factor in the product in (74) in power series and replace extended multiparameters Hurwitz-Lerch zeta function by its series representation. We thus find for the right-hand side of (74) that

$$\begin{aligned}
 & \frac{k}{(z^{k-1})^\alpha} z O_\beta^\alpha \\
 & \times \left\{ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{n! \prod_{j=1}^q (\mu_j)_{n\sigma_j}} \frac{z^{n+(k-1)\alpha}}{(n+a)^s} \right. \\
 & \quad \cdot \sum_{m_1, \dots, m_{k-1}=0}^{\infty} (1+\alpha-\beta)_{m_1} \dots (1+\alpha-\beta)_{m_{k-1}} \\
 & \quad \cdot \frac{((z/w) e^{-2\pi i/k})^{m_1}}{m_1!} \\
 & \quad \left. \dots \frac{((z/w) e^{-2(k-1)\pi i/k})^{m_{k-1}}}{m_{k-1}!} \right\} \Big|_{w=z}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{k}{(z^{k-1})^\alpha} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{n! \prod_{j=1}^q (\mu_j)_{n\sigma_j} (n+a)^s} \\
 &\cdot \sum_{m_1, \dots, m_{k-1}=0}^{\infty} \frac{(1+\alpha-\beta)_{m_1} \cdots (1+\alpha-\beta)_{m_{k-1}}}{z^{m_1+\dots+m_{k-1}}} \\
 &\cdot \frac{(e^{-2\pi i/k})^{m_1}}{m_1!} \cdots \frac{(e^{-2(k-1)\pi i/k})^{m_{k-1}}}{m_{k-1}!} \\
 &\times {}_zO_{\beta}^{\alpha} \{z^{m_1+\dots+m_{k-1}+n+(k-1)\alpha}\} \\
 &= k \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{n! \prod_{j=1}^q (\mu_j)_{n\sigma_j}} \frac{z^n}{(n+a)^s} \\
 &\cdot \sum_{m_1, \dots, m_{k-1}=0}^{\infty} (1+\alpha-\beta)_{m_1} \cdots (1+\alpha-\beta)_{m_{k-1}} \\
 &\cdot \frac{(e^{-2\pi i/k})^{m_1}}{m_1!} \cdots \frac{(e^{-2(k-1)\pi i/k})^{m_{k-1}}}{m_{k-1}!} \\
 &\times \frac{\Gamma(\beta) \Gamma(m_1 + \dots + m_{k-1} + n + k\alpha)}{\Gamma(\alpha) \Gamma(\beta + m_1 + \dots + m_{k-1} + n + k\alpha)} \\
 &= \frac{k\Gamma(\beta) \Gamma(k\alpha)}{\Gamma(\alpha) \Gamma(\beta + (k-1)\alpha)} \\
 &\times \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j} (k\alpha)_n}{n! \prod_{j=1}^q (\mu_j)_{n\sigma_j} (\beta + (k-1)\alpha)_n} \frac{z^n}{(n+a)^s} \\
 &\cdot \sum_{m_1, \dots, m_{k-1}=0}^{\infty} \left(((n+k\alpha)_{m_1+\dots+m_{k-1}} \right. \\
 &\quad \times (1+\alpha-\beta)_{m_1} \cdots (1+\alpha-\beta)_{m_{k-1}}) \\
 &\quad \times \left. ((\beta + (k-1)\alpha)_{m_1+\dots+m_{k-1}})^{-1} \right) \\
 &\cdot \frac{(e^{-2\pi i/k})^{m_1}}{m_1!} \cdots \frac{(e^{-2(k-1)\pi i/k})^{m_{k-1}}}{m_{k-1}!}. \tag{76}
 \end{aligned}$$

Finally, by combining (75) and (76), we obtain the result (72) asserted by Theorem 17. \square

Remark 18. Each of the previous theorems can be written in terms of \overline{H} -function given in Definition 1. For instance, if we make use of (11), then Theorem 17 becomes

$$\begin{aligned}
 &\overline{H}_{p+2,q+3}^{1,p+2} \left[-z \mid \begin{matrix} (1-\lambda_1, \rho_1; 1), \dots, (1-\lambda_p, \rho_p; 1), \left(1-\alpha, \frac{1}{k}, 1\right), (1-a, 1; s) \\ (0, 1), (1-\mu_1, \sigma_1; 1), \dots, (1-\mu_p, \sigma_p; 1), \left(1-\beta, \frac{1}{k}, 1\right), (-a, 1; s) \end{matrix} \right] \\
 &= \frac{\prod_{j=1}^p \Gamma(\lambda_j)}{\prod_{j=1}^q \Gamma(\mu_j)} \frac{k\Gamma(k\alpha)}{\Gamma(\beta + (k-1)\alpha)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{n! \prod_{j=1}^q (\mu_j)_{n\sigma_j}} \frac{(k\alpha)_n}{(\beta + (k-1)\alpha)_n} \frac{z^n}{(n+a)^s} \\
 &\cdot F_D^{(k-1)} [k\alpha + n, 1 + \alpha - \beta, \dots, 1 + \alpha - \beta; \beta + (k-1)\alpha + n; e^{-2\pi i/k}, \dots, e^{-2(k-1)\pi i/k}]. \tag{77}
 \end{aligned}$$

5. Corollaries and Consequences

We conclude this paper by presenting some special cases of the main results. These special cases and consequences are given in the form of the following corollaries.

Setting $k = 3$ in Theorem 17, we obtain the following corollary.

Corollary 19. *Under the hypotheses of Theorem 17, the following expansion formula holds true:*

$$\begin{aligned}
 &\Phi_{\lambda_1, \dots, \lambda_p; \alpha; \mu_1, \dots, \mu_q; \beta}^{(\rho_1, \dots, \rho_p, 1/3, \sigma_1, \dots, \sigma_q, 1/3)}(z, s, a) \\
 &= \frac{3\Gamma(\beta) \Gamma(3\alpha)}{\Gamma(\alpha) \Gamma(\beta + \alpha)} \\
 &\cdot \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\lambda_j)_{n\rho_j}}{n! \prod_{j=1}^q (\mu_j)_{n\sigma_j}} \frac{(3\alpha)_n}{(\beta + \alpha)_n} \frac{z^n}{(n+a)^s}
 \end{aligned}$$

$$\cdot F_1 [3\alpha + n, 1 + \alpha - \beta, 1 + \alpha - \beta; \beta + 2\alpha + n; -1, 1], \tag{78}$$

where F_1 denotes the first Appell function defined by [38, page 22]

$$\begin{aligned}
 &F_1 [a, b_1, b_2; c; x_1, x_2] \\
 &= \sum_{m_1, m_2=0}^{\infty} \frac{(a)_{m_1+m_2} (b_1)_{m_1} (b_2)_{m_2}}{(c)_{m_1+m_2}} \frac{x_1^{m_1} x_2^{m_2}}{m_1! m_2!} \tag{79} \\
 &(\max \{|x_1|, |x_2|\} < 1),
 \end{aligned}$$

provided that both sides of (78) exist.

Setting $p - 1 = q = 1, \kappa = 1$, and making the following substitutions $\rho_1 \mapsto \rho, \rho_2 \mapsto \sigma, \sigma_1 \mapsto \eta, \lambda_1 \mapsto \lambda, \lambda_2 \mapsto \mu$, and $\mu_1 \mapsto \nu$ in Theorem 14 lead to the following expansion formula given recently by Srivastava et al. [14].

Corollary 20. Under the hypotheses of Theorem 14, the following expansion formula holds true:

$$\begin{aligned} &\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\eta)}(z, s, a) \\ &= c \sum_{n=-\infty}^{\infty} \frac{(z_0)^{-cn} (z - z_0)^{cn}}{\Gamma(cn + 1) \Gamma(1 - cn)} \Phi_{\lambda,\mu,1;\nu,1-cn}^{(\rho,\sigma,1,\eta,1)}(z_0, s, a) \end{aligned} \tag{80}$$

for z such that $|z - z_0| = |z_0|$ and provided that both members of (80) exist.

Remark 21. The function $\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a)$, which occurs in Corollary 20 above, as well as its multiparameter generalizations, was introduced and investigated in a series of papers by Srivastava et al. (see [7, page 491, Equation (1.20)]; see also [8, 9, 39]) and is defined as follows:

$$\begin{aligned} \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z, s, a) &:= \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n} (\mu)_{\sigma n}}{n! (\nu)_{\kappa n}} \frac{z^n}{(n + a)^s} \\ &(\lambda, \mu \in \mathbb{C}; a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \rho, \sigma, \kappa \in \mathbb{R}^+; \\ &\kappa - \rho - \sigma > -1 \text{ when } s, z \in \mathbb{C}; \\ &\kappa - \rho - \sigma = -1, s \in \mathbb{C} \text{ when } |z| < \delta^* := \rho^{-\rho} \sigma^{-\sigma} \kappa^{\kappa}; \\ &\kappa - \rho - \sigma = -1, \Re(s + \nu - \lambda - \mu) > 1 \text{ when } |z| = \delta^*). \end{aligned} \tag{81}$$

Putting $p - 1 = q = 1$, $\kappa = 1$, and setting $\rho_1 = \rho_2 = \sigma_1 = 1$, $\lambda_1 = \mu$, and $\lambda_2 = \mu_1$ in Theorem 15 reduce to the following expansion formula given recently by Gaboury [15, Equation (4.4)].

Corollary 22. Under the hypotheses of Theorem 15, the following expansion formula holds true:

$$\begin{aligned} &\Phi_{\mu}^*(z, s, a) \\ &= cz^{-\alpha} (z - z_1)^{-\beta} z_1^{\alpha+\beta} \\ &\cdot \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)} \sin[(\alpha + cn + \gamma)\pi] \Gamma(\alpha + cn + \gamma)}{\sin[(\alpha - c + \gamma)\pi] \Gamma(1 - \beta + cn + \gamma) \Gamma(\alpha + \beta)} \\ &\cdot \Phi_{\mu,\alpha+cn+\gamma;\alpha+\beta}^{(1,1,1)}(z_1, s, a) \left(\frac{z - z_1}{z}\right)^{cn+\gamma} \end{aligned} \tag{82}$$

for z on $C_1(1)$ defined by

$$z = \frac{z_1}{2} + \frac{z_1}{2} \sqrt{1 + e^{i\theta}} \quad (-\pi < \theta < \pi), \tag{83}$$

provided that both sides of (82) exist.

Remark 23. The function $\Phi_{\mu}^*(z, s, a)$, which occurs in Corollary 22 above, was introduced and studied by Goyal and Laddha [4, page 100, Equation (1.5)] and is defined as follows:

$$\Phi_{\mu}^*(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n + a)^s} \tag{84}$$

($\mu \in \mathbb{C}; a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}$ when $|z| < 1$;

$\Re(s - \mu) > 1$ when $|z| = 1$).

Setting $s = 0$, $\kappa = 1$, $p \mapsto p + 1$, and $q \mapsto q + 1$ with

$$\begin{aligned} \rho_1 = \dots = \rho_p = 1, \quad \lambda_{p+1} = \rho_{p+1} = 1, \\ \sigma_1 = \dots = \sigma_q = 1, \quad \mu_{q+1} = \beta, \quad \sigma_{q+1} = \alpha \end{aligned} \tag{85}$$

in Theorem 16, we deduce the following expansion formula.

Corollary 24. Under the hypotheses of Theorem 16, the following formula holds true:

$$\begin{aligned} &\frac{1}{\Gamma(\beta)^{p+1}} \Psi_{q+1}^* \left[\begin{matrix} (\lambda_1, 1), \dots, (\lambda_p, 1), (1, 1); \\ (\mu_1, 1), \dots, (\mu_q, 1), (\beta, \alpha) \end{matrix} \middle| z \right] \\ &= cz^{-\beta+\gamma} (z - z_1)^{-\alpha+\gamma} z_1^{\beta+\alpha-2\gamma-1} \\ &\cdot \sum_{n=-\infty}^{\infty} \frac{\sin((\beta - cn - \gamma)\pi) e^{i\pi c(n+1)} [z(z - z_1)/z_1^2]^{cn}}{\sin((\beta + c - \gamma)\pi) \Gamma(1 - \alpha + cn + \gamma)} \\ &\times \frac{\Gamma(\beta - cn - \gamma)}{\Gamma(\beta + \alpha - 2cn - 2\gamma)} \\ &\cdot \left[(z - z_1) \Phi_{\lambda_1, \dots, \lambda_p, 1, 1+\beta-cn-\gamma; \mu_1, \dots, \mu_q, \beta, 1+\beta+\alpha-2cn-2\gamma}^{(1, \dots, 1, 1, 1, \dots, 1, \alpha, 1)} \right. \\ &\times (z_1, 0, a) + \frac{z_1(\beta - cn - \gamma)}{(\alpha + \beta - 2cn - 2\gamma)} \\ &\left. \times \Phi_{\lambda_1, \dots, \lambda_p, 1, \beta-cn-\gamma; \mu_1, \dots, \mu_q, \beta, \beta+\alpha-2cn-2\gamma}^{(1, \dots, 1, 1, 1, \dots, 1, \alpha, 1)}(z_1, 0, a) \right] \end{aligned} \tag{86}$$

for z on $C_1(1)$ defined by

$$z = \frac{z_1}{2} + \frac{z_1}{2} \sqrt{1 + e^{i\theta}} \quad (-\pi < \theta < \pi), \tag{87}$$

provided that both sides of (86) exist.

The function ${}_l\Psi_m^*$ involved in the left-hand side of (86) is the Fox-Wright function ($l, m \in \mathbb{N}_0$) with l numerator parameters a_1, \dots, a_l and m denominator parameters b_1, \dots, b_m such that

$$\begin{aligned} a_j &\in \mathbb{C} \quad (j = 1, \dots, l), \\ b_j &\in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j = 1, \dots, m), \end{aligned} \tag{88}$$

defined by (see, for details, [17, 40])

$$\begin{aligned}
 {}_l\Psi_m^* & \left[\begin{matrix} (a_1, A_1), \dots, (a_l, A_l); \\ (b_1, B_1), \dots, (b_m, B_m); \end{matrix} z \right] \\
 & := \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n} \cdots (a_l)_{A_l n}}{(b_1)_{B_1 n} \cdots (b_m)_{B_m n}} \frac{z^n}{n!} \\
 & \left(A_j > 0, (j = 1, \dots, l); B_j > 0, (j = 1, \dots, m); \right. \\
 & \left. 1 + \sum_{j=1}^m B_j - \sum_{j=1}^l A_j \geq 0 \right), \tag{89}
 \end{aligned}$$

where the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \left(\prod_{j=1}^l A_j^{-A_j} \right) \left(\prod_{j=1}^m B_j^{B_j} \right). \tag{90}$$

In our series of forthcoming papers, we propose to consider and investigate analogous expansion formulas and other results involving the λ -extensions of the generalized Hurwitz-Lerch zeta functions. These functions have been investigated recently by Srivastava et al. [41] (see also [42]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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