THE N-BODY PROBLEM WITH REPULSIVE-ATTRACTIVE QUASIHOMOGENEOUS POTENTIAL FUNCTIONS

A Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics.

by
© Robert T. Jones

Victoria 2006

THE N-BODY PROBLEM WITH REPULSIVE-ATTRACTIVE QUASIHOMOGENEOUS POTENTIAL FUNCTIONS

by

ROBERT T. JONES
B.Sc. University of Victoria, 2000

*A Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of*

MASTER OF SCIENCE

*in the Department of Mathematics and Statistics.
We accept this Thesis as conforming
to the required standard.*

Dr. F. N. Diacu, Supervisor, Department of Mathematics & Statistics, University of Victoria

---

Dr. R. Edwards, Department of Mathematics & Statistics, University of Victoria

---

Dr. M. Agueh, Department of Mathematics & Statistics, University of Victoria

---

Dr. J. Navarro, Department of Physics and Astronomy, University of Victoria

Supervisor: Dr. F. N. Diacu.

Abstract

This thesis involves the study of a repulsive-attractive N-body problem, which is a subclass of a quasihomogeneous N-body problem [5]. The quasihomogeneous N-body problem is the study of N point masses moving in $\mathbb{R}^{3N}$, where the negative of the potential energy is of the form,

$$\sum_{1\leq i<j\leq N} bm_im_jr_{ij}^{-\beta} + \sum_{1\leq i<j\leq N} am_im_jr_{ij}^{-\alpha}.$$

In the above equation, r_{ij} is the distance between the point mass m_i and the point mass m_j, and a, b, $\alpha > \beta > 0$ are constants. The repulsive-attractive N-body problem is the case where $a < 0$ and $b > 0$.

We start the ground work for the study of the repulsive-attractive N-body problem by defining the first integrals, collisions and pseudo-collisions and the collision set. By examining the potentials where $a < 0$ and $b > 0$, we see that the dominant force is repulsive. This means that the closer two point masses get the greater the force acting to separate them becomes. This property leads to the main result of the first chapter: there can be no collisions or pseudo-collisions for any repulsive-attractive system.

In the next chapter we study central configurations of the system. Quasihomogeneous potentials will have different central configurations than homogeneous potentials [6], thus requiring the classification of two new subsets of central configurations. Loosely speaking, the set of central configurations that are not central configurations for any homogeneous potential are called extraneous. The set of configurations that are central configurations for both homogeneous potentials that make up the quasihomogeneous potential, are called simultaneous configurations.

We also notice that every simultaneous central configuration will be non-extraneous, therefore the two subsets are disjoint.

Next we show the existence of oscillating homothetic periodic orbits associated with non-extraneous configurations. Finally in this chapter, we investigate the polygon solutions for repulsive-attractive N-body problems [11]. In particular we show that the masses need no longer to be equal, for repulsive-attractive potentials. It will be shown that there exists a square configuration with $m_1 = m_2 \neq m_3 = m_4$, that leads to a relative equilibrium. Therefore, for $N = 4$ the set of extraneous configurations is non-empty.

The last chapter deals with the complete analysis of the generalized Lennard-Jones 2-body problem. The generalized Lennard-Jones problem is the subcase of the repulsive-attractive N-body problem, where $a = -1$, $b = 2$, and $\alpha = 2\beta$. We proceed as in [13] by using diffeomorphic transforms to get an associated system thereby generating a picture of the global flow of the system. This gives us the complete flow for the generalized Lennard-Jones 2-body problem.

Examiners:

Dr. F. N. Diacu, Department of Mathematics & Statistics, University of Victoria

---

Dr. R. Edwards, Department of Mathematics & Statistics, University of Victoria

---

Dr. M. Agueh, Department of Mathematics & Statistics, University of Victoria

---

Dr. J. Navarro, Department of Physics and Astronomy, University of Victoria

Contents

List of Figures

Chapter 1

Introduction

1.1 Introduction

Over the last two decades the study of the N-body problem with quasihomogeneous potential functions has been of increasing interest [4] [5] [13]. Quasihomogeneous potential functions are the negative of the potential energy of the system, and are of the form $U(\mathbf{q}) = V(\mathbf{q}) + W(\mathbf{q})$, where $V(\mathbf{q}) = \sum_{1\leq i<j\leq N} bm_i m_j r_{ij}^{-\beta}$ and $W(\mathbf{q}) = \sum_{1\leq i<j\leq N} am_i m_j r_{ij}^{-\alpha}$. In this equation the r_{ij} is the distance that the point mass, m_i, is from the point mass, m_j, and $0 < \beta < \alpha$, and a and b are constants.

For the most part the study of these potentials has been restricted to the case where $a > 0$ and $b > 0$, [6] [13]. However, in physics and chemistry there are cases that have a negative term: Coulomb $b = 0$, $a < 0$, and $\alpha = 1$, Birkhoff $b = 1$, $\beta = 1$, $a < 0$, and $\alpha = 2$, and Lennard-Jones $b = 2$, $\beta = 6$, $a = -1$, and $\alpha = 12$.

Note that the nature of these potentials changes depending on which of the β or α is dominant. If $a < 0$ and $b > 0$, then the repulsive force will dominate, and if $a > 0$ and $b < 0$, then the attractive force will dominate. If $a < 0$ and $b < 0$ there is only a repulsion between the particles, and if $a > 0$ and $b > 0$ there is only an attraction. The complete study of the quasihomogeneous N-body problem can now be reduced to the study of four cases: repulsive-attractive, attractive-repulsive,

repulsive-repulsive, and attractive-attractive, respectively. This thesis is primarily interested in the potentials that are being used to determine the stability of crystal structures in chemistry, particularly the Lennard-Jones potential, as it more closely represents the interaction of atoms.

In chemistry, Newtonian mechanics never gave a good approximation of the interaction of small particles. In the middle of the last century the famous physical chemist Lennard-Jones, started working with a model that more closely represented the interaction of small particles. Instead of a potential function that is strictly attractive, he used a potential function that is both attractive and repulsive. The potential energy that Lennard-Jones used was of the form, $U(\mathbf{r}) = \sum_{1\leq i<j\leq N} \frac{1}{r_{ij}^{12}} - \frac{2}{r_{ij}^{6}}$, where r_{ij} is the relative distance that particle m_i is from particle m_j. So this thesis generalizes this topic to the study of systems where the dominant term is the repulsive force.

Chapter 2 starts by defining all generalized N-body problems. These are the systems where each point mass is accelerated by a force function that is dependent only on the relative mutual distances of the point masses and the masses themselves. We show that the only difference between all the generalized N-body problems is choice of the potential energy of the system. We then introduce the equations of motion for the repulsive-attractive N-body problem, together with their associated integrals of motion. This thesis we will restrict to the subclass of quasihomogeneous potential functions that are repulsive-attractive in nature. We start to lay the groundwork for the study of this subclass of quasihomogeneous potentials. Using the concepts of collisions, pseudo-collisions, and the collision set Δ, we see that for an orbit to tend to Δ would require the total energy of the system to become unbounded. This leads to the fact that there can be no collisions or pseudo-collisions for repulsive-attractive potentials, which shows that the configuration space is $\mathbb{R}^{3N}/\overrightarrow{\mathbf{0}}$. Furthermore, another difference of this subclass is the emergence of equilibrium points, namely the rectilinear 2-body equilibrium, the equilateral 3-body equilibrium, and the tetrahedral non-planar 4-body configuration. This is the concept of crystal structures in chemistry.

In chapter 3, we study the central configurations of the system. These are configurations in which the acceleration of each particle is directly proportional to its position, with the additional restriction that each constant of variation is the same. For homogeneous potential functions we see that all scalar multiples of a central configuration are also central configurations [3], [15]. However, it has been shown [6] that this is not true for quasihomogeneous potentials. This leads to new sub-classes of central configurations, namely extraneous central configurations, and simultaneous central configurations, respectively. An extraneous central configuration is any central configuration which fails to be a central configuration for some constant of variation. A simultaneous central configuration is a configuration which is also a central configuration for each of the homogeneous potentials that make up the quasi-homogeneous potential function [6]. Using properties of the gradient function, we see that all simultaneous configurations are **CC** for the associated repulsive-attractive potential. The next result that is shown is that every simultaneous configuration must be a non-extraneous configuration.

The main purpose of chapter 3 is to find periodic orbits of the system. We start by showing that: if $\mathbf{q}_0$ is a non-extraneous central configuration then the initial value problem of a repulsive-attractive N-body problem, with initial condition that $(\mathbf{q}_0, \overrightarrow{\mathbf{0}})$, will be a homothetic periodic orbit for negative total energy, and will expand without bound, in the future and the past, for non-negative total energy.

Next we show that the regular N-gon, with equal masses is a simultaneous central configuration, thereby finding a class of homothetic periodic orbits of the system. Furthermore, using a diffeomorphic transform, we take the theorem due to Perko, [11], and show that the regular N-gon with equal masses generates a relative equilibrium solution for any repulsive-attractive potential. However, unlike the homogeneous potentials, there is a lower bound on the radius of the configuration. This is due to the repulsive nature of a repulsive-attractive potential function. The last thing in this chapter is that unlike homogeneous potentials, the requirement that the masses be equal is not necessary for the existence of a relative equilibrium solution. In particular there is a square configuration, that leads to a relative equilibrium with $m_1 = m_2 \neq m_3 = m_4$. This is done by noting that, if the mutual distance between two point masses is a particular size, there is no attraction or repulsion between those two particles. This shows that the set of extraneous central configurations is non-empty, for $N = 4$.

The last chapter deals with the idea of the 2-body generalized Lennard-Jones problem, which is a particular repulsive-attractive system. We represent the 2-body system as an equivalent system, namely, the motion of one unit mass particle in a central force field [7]. The potential function for this system has the form $U(\mathbf{x}) = \mid \mathbf{x} \mid^{-2\beta} - 2 \mid \mathbf{x} \mid^{-\beta}$, $\beta > 0$. It will be shown that, unlike the classical and attractive-attractive quasihomogeneous problems, there is a non-collision equilibria, namely at $\mid \mathbf{x} \mid = 1$. Using the diffeomorphism developed by Stoica [13], we transform this system into an associated system that has the same flow. We then use this new system to find bounds on the configuration space as well as a lower bound on the total energy of the system. In particular, we see that the total energy of the system must be greater than or equal to -1. Using the energy manifolds, we see that the flow is restricted to 2-dimensional sub-manifolds. For negative energy, these manifolds are compact, since all motion is bounded. This leads to the global flow on each negative energy manifold. Note that there are only two equilibria on each of these energy manifolds, where each is contained in an invariant sub-manifold, we determine that there are only periodic orbits for negative total energy. This gives the global flow for negative energy.

As for non-negative total energy, we see that the orbits can become unbounded. In order to see how these orbit become unbounded we use the inverse transform, $\rho = r^{-\beta}$. Using this transform we get the near infinity system, which shows the existence of the infinity manifold. Note that for non-negative total energy, each of the near infinity manifolds are compact, invariant manifolds, so using qualitative analysis we can determine the global flow of this system. In particular, we see that for non-negative energy, $\beta = 2$ is a bifurcation value for the system, so the global flow for the Birkhoff potential is quite different than that for the Lennard-Jones potential. Finally, we summarize the global flow, and draw correlations to the overall flow of the generalized Lennard-Jones N-body problem.

Chapter 2

The Repulsive-Attractive N-Body Problem

2.1 The Equations of Motion

The general N-body problem is that of N point masses moving in Euclidean 3-space, where the masses, $m_1, ..., m_N$, are under the influence of a mutually acting force function [1]. Each point mass m_i has i^{th} position vector $q_i \in \mathbb{R}^3$, and i^{th}momentum vector $p_i = m_i\dot{q}_i \in \mathbb{R}^3$, for $i \in \overline{1N} = \big\{1, 2, 3, \ldots, N\big\}$. q_i is a function of time, and $\dot{q}_i$ is the derivative with respect to time. Therefore, there are two vectors

$$\begin{aligned} \mathbf{q} &\in \mathrm{A} \subseteq \mathbb{R}^{3N} \\ \mathbf{p} &\in \mathbb{R}^{3N}, \end{aligned}$$

which are the configuration and momentum vectors, respectively. Furthermore, A is the subset of $\mathbb{R}^3$, where all the possible configuration vectors are contained. The subspace, A, is called the configuration space.

---

[1] by a mutually acting force function, it is meant that each of the point masses is acted upon by a force that is dependent only on the mutual, relative distances of each of the masses, and the masses themselves. There are some papers where this restriction has been removed, [5]

Since $\mathbf{p}$ is the momentum vector, then the kinetic energy of the system is,

$$T(\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{N} m_i^{-1} \mid p_i \mid^2. \tag{2.1}$$

To find the equations that define the motion of the point masses we need to develop the potential energy of the system. Clearly the potential energy is relative to the individual masses and their position vectors. It is here that we are given some latitude in the make up of the system. Since the point masses are viewed as constants, then all that is necessary is that we have a function that is well defined on some subset of $\mathbb{R}^{3N}$. Hence, we can loosely define the potential function of the system as any function of the form,

$$\mathrm{U} : A \subseteq \mathbb{R}^{3N} \longrightarrow \mathbb{R}, \tag{2.2}$$

which is sometimes referred to as the force function mentioned above. Here we have used the negative of the potential energy of the system as $\mathrm{U}(\mathbf{q})$.

Using the potential function (2.2), we get the equation that defines the motion of the point masses, namely, $\mathrm{M}\ddot{\mathbf{q}} = \nabla_q \mathrm{U}(\mathbf{q})$, where $\nabla_{\mathbf{q}} = [\partial_{q_1}, \ldots, \partial_{q_N}]^T$. The subscript $\mathbf{q}$ in $\nabla_{\mathbf{q}}$, will normally be dropped as long as there is no confusion. This leads to the equations of motion for the generalized N-body problem,

$$\begin{aligned} \dot{\mathbf{q}} &= \mathrm{M}^{-1}\mathbf{p} \\ \dot{\mathbf{p}} &= \nabla \mathrm{U}(\mathbf{q}), \end{aligned} \tag{2.3}$$

where the $\mathrm{M} = \mathrm{dia}[\mathrm{m_1}, \mathrm{m_1}, \mathrm{m_1}, \ldots, \mathrm{m_N}, \mathrm{m_N}, \mathrm{m_N}]$. Notice that for each $i \in \overline{1N}$, m_i appears three times in the diagonal, making M a $3N \times 3N$ diagonal matrix. Creating the matrix M in this fashion insures that the inverse matrix mentioned in (2.3) exists. Furthermore, we can see that there is a $6N$ dimensional phase space in which the solutions to (2.3) are contained.

The potential energy function, (2.2), is where all of the N-body problems differ. The classical case, where $\mathrm{U}(\mathbf{q}) = \mathrm{G}\sum_{1\leq i<j\leq N} m_i m_j r_{ij}^{-1}$, where $r_{ij} =\mid q_i - q_j \mid$, and G is the gravitational constant, uses Newton's equations for the gravitational interaction of point masses. This case was the central concern for centuries [15][3].

When dealing with complicated dynamical systems like the N-body problem it is usually necessary to restrict to invariant sets. These are sets, in phase space, that are invariant under the flow.

Definition 2.1 (Invariant Set)**.** *A set I, in the phase space is said to be invariant with respect to the flow, if $\varphi_t(I) \subseteq I$.*

Invariant sets contain complete solution curves to the system (2.3), thus making it more likely to be able to study the nature of the system in general. One such class of invariant sets is that of the first integrals, classically referred to as the conservation integrals.

2.2 First Integrals

First integrals are a collection of invariant manifolds of the system (2.3), that are defined by a certain class of functions. To define this class of functions we need the definition of an orbital derivative.

Definition 2.2 (Orbital Derivative)**.** *Let* $\mathrm{F} : \mathbb{R}^{3N} \longrightarrow \mathbb{R}$ *be a differentiable function and* $\mathbf{x} : \mathbb{R} \longrightarrow \mathbb{R}^{3N}$ *be a time dependent vector function, then the orbital derivative of F along* $\mathbf{x}$*, parameterized by t, is*

$$\mathrm{L_t F} = \frac{\partial \mathrm{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} = \sum_{k=1}^{n} \frac{\partial \mathrm{F}}{\partial x_k}\dot{x}_k.$$

Definition 2.3 (First Integral)**.** *A function* $\mathrm{F}(x)$ *is said to be a first integral of the equation,* $\dot{\mathbf{x}} = f(\mathbf{x})$*, if* $\mathrm{L_t F} = 0$.

Clearly, any level curve $F(\mathbf{x}) = \textit{constant}$ will be invariant with respect to the flow of the system $\dot{\mathbf{x}} = f(\mathbf{x})$. For the system (2.3), we see that the following seven functions are first integrals:

$$F_1(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{N} p_i \in \mathbb{R}^3, \tag{2.4}$$

$$F_2(\mathbf{q}, \mathbf{p}) = \mathbf{q} \times \mathbf{p} \in \mathbb{R}^3, \tag{2.5}$$

$$F_3(\mathbf{q}, \mathbf{p}) = T(\mathbf{p}) - U(\mathbf{q}) \in \mathbb{R}. \tag{2.6}$$

Therefore, the level curves of these functions are invariant manifolds of the system (2.3). Next we need to find the center of mass of the system and show that it defines another invariant manifold. The center of mass of the system is given by

$$F_0(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{N} m_i q_i \in \mathbb{R}^3. \tag{2.7}$$

If we set $F_1 \equiv \mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^3$ is a constant, then

$$\mathrm{L_t F_0} = \sum_{i=1}^{N} m_i \dot{\mathbf{q}}_i = \sum_{i=1}^{N} \mathbf{p}_i = \mathrm{F}_1 = \mathbf{a},$$

and $\mathrm{F}_0(\mathbf{q}) \equiv \mathbf{a}t + \mathbf{b}$. Since the equations (2.3) defines an autonomous system, we have that $\mathrm{F}_0 \equiv \mathbf{c}$ is another integral of motion.

Setting, $F_0 \equiv \mathbf{0}$, $F_1 \equiv \mathbf{0}$, $F_2 \equiv \mathbf{c}$, and $F_3 \equiv h$, gives the center of mass of the system, conservation of momentum, the angular momentum integral, and the conservation of energy of the system. From now on we will assume that all solution curves are contained on both center of mass integral, and the zero momentum integral. These ten integrals are referred to as the *classical integrals*.

The 9 integrals (2.4, 2.5, 2.7) are seen to be independent of $U(\mathbf{q})$, where the equation (2.6) is the only classical integral that is dependent on the potential function. Now that we have the general framework of a N-body system, we can now turn our attention to the systems in question, namely those that may represent the motion of small particles under the laws of chemistry.

For atoms and electrons the dynamics of a system that describes the motion of the point masses will be quite different than that of classical mechanics. For Newtonian mechanics it is assumed that the only force acting to change the momentum of a point mass is that of a strictly attracting force function. This remains the case for both homogeneous and previously studied quasihomogeneous systems [3],[5], [13].

With atoms, however, there is both an attraction and a repulsion between the particles. The repulsive force increases as two or more particles draw nearer to one another, so we need to define a system that attracts at the same time it also repels. The chemist Lennard-Jones came up with the potential function given by $U(\mathbf{q}) = \sum_{1\leq i<j\leq N} m_i m_j \left(\frac{2}{\left(|q_i - q_j|\right)^6} - \frac{1}{\left(|q_i - q_j|\right)^{12}} \right)$, where the term with the exponent 6 is the attraction, and the exponent 12, is the repulsive force. Intuitively, one can expect that the closer two particles get, the more the repulsive force counteracts the attractive force. Using this potential function as a guide, we will create a general class of potential functions that have the same properties. This leads to a new sub-class of a quasihomogeneous N-body system, namely one that is both repulsive and attractive.

Definition 2.4. *Any system of the form (2.3) that has a potential function (2.2) given by*

$$U(\mathbf{q}) = V(\mathbf{q}) + W(\mathbf{q}),$$

where

$$V(\mathbf{q}) = \sum_{1\leq i<j\leq N} \frac{b m_i m_j}{|q_{ij}|^\beta} \qquad\qquad W(\mathbf{q}) = \sum_{1\leq i<j\leq N} \frac{a m_i m_j}{|q_{ij}|^\alpha}, \tag{2.8}$$

and $q_{ij} = q_i - q_j$*,* $a < 0$*,* $0 < b$*, and* $0 < \beta < \alpha$*, is called a repulsive-attractive* N*-body problem.*

Note that the term $|q_i - q_j|^{-\beta}$ is the attractive force and the term $|q_i - q_j|^{-\alpha}$ is the repulsive force. The fact that $0 < \beta < \alpha$, makes it intuitive that as the particles become close together, the repulsive force becomes the dominating term. We will show that this term makes it impossible for two particles to collide. Therefore, if $a < 0$, $b > 0$ and $0 < \beta < \alpha$, then the repulsive force overpowers the attractive force, and this class of problem will be referred to as a repulsive-attractive N-body problem as opposed to attractive-repulsive. The complete study of quasihomogeneous N-body problems can be seen as the study of repulsive-attractive, attractive-repulsive, attractive-attractive, and finally repulsive-repulsive systems. Each of these systems will have dramatically different phase spaces. From now on, repulsive-attractive potential, will mean $a < 0$, $b > 0$ and $\alpha > \beta > 0$.

The Lennard-Jones equation is a repulsive-attractive N-body problem (**R-A**), where $a = -1$, $b = 2$, $\alpha = 12$ and $\beta = 6$. Now, we define the systems that are to be dealt with in this thesis. When we say a generalized Lennard-Jones N-body problem (**GLJ**), it is meant that $a = -1$, $b = 2$, and $\alpha = 2\beta$.

As mentioned above, it appears that for any **R-A**, it is unlikely that any particles will collide. In order to prove this we must first turn our attention to the singularities of the function $U(\mathbf{q})$.

2.3 Singularities

To have solutions to (2.3), we need to determine the domain of $U(\mathbf{q})$. Note that if any two particles collide, then $U(\mathbf{q})$ becomes undefined, so for each $i \neq j$, the set of configurations in which the mass particle m_i collides with m_j is,

$$\Delta_{ij} = \big\{\mathbf{q} \in \mathbb{R} | q_i = q_j\big\}. \tag{2.9}$$

The set of all collisions will be contained in the union of all these sets. This leads to the formation of what is called the collision set,

$$\Delta = \bigcup_{1 \leq i < j \leq N} \Delta_{ij}, \tag{2.10}$$

so Δ is the set of all collisions for the system. Clearly the domain of $U(\mathbf{q})$ is $D = \mathbb{R}^{3N} \backslash \Delta$, since $U(\mathbf{q})$ is not analytic on Δ, but is everywhere else. If $(\mathbf{q}, \mathbf{p}) \in D \bigotimes \mathbb{R}^{3N}$, then by existence and uniqueness theorems, there exists a maximal interval of existence, $(t^-, t^+) \subseteq \mathbb{R}$. For our purposes we will assume the interval is $[0, t^*)$. Proceeding as in [3], we need to define and classify all the possible singularities for the system (2.3).

Definition 2.5. *If $t^* < +\infty$, then t^* is called a finite singularity of (2.3).*

Definition 2.6.

$$\rho\big(\mathbf{q}(t)\big) = \min_{1 \leq i < j \leq N} |q_{ij}(t)|,$$

where $\rho : A \subseteq \mathbb{R}^{3N} \bigotimes \mathbb{R} \longrightarrow [0, \infty)$.

Proposition 2.7. *If $(\mathbf{q}, \mathbf{p})$ is an analytic solution of (2.3), defined for all t in $[0, t^*)$, then t^* is a singularity of (2.3) if and only if*

$$\lim_{t \to t^*} \Big(\inf \rho(\mathbf{q}(t)) \Big) = 0.$$

PROOF:

$(\Rightarrow)$ Let t^* be a singularity of (2.3), and assume $\exists\ c > 0$ such that,

$$\lim_{t \to t^*} \Big(\inf \rho(\mathbf{q}(t)) \Big) \geq c.$$

This implies that $\exists\ t_0 \in [0, t^*)$ and $\gamma_0 \in (0, c)$ such that $|q_{ij}(t)| \geq \gamma_0$ for all $t \in [t_0, t^*)$, when $i \neq j$. We have that

$$m_i \ddot{q}_i = \sum_{1 \leq i \neq j \leq N} m_i m_j q_{ij} \Big(a\alpha |q_{ij}|^{-\alpha-2} - b\beta |q_{ij}|^{-\beta-2} \Big),$$

and by repeated triangle inequalities, and the fact that $0 < \beta < \alpha$ we get,

$$|\ddot{q}_i| \leq \Big(\sum_{j=1, j\neq i}^{N} m_j \Big) \Big(\gamma_0^{-\beta-1} \big(b\beta + a\alpha\gamma_0^{\beta-\alpha} \big) \Big)..$$

Since we have taken γ_0 to be fixed, then $\ddot{\mathbf{q}}$ is bounded. From this point we can proceed as in [15]and [3].

The Taylor expansion of $\mathbf{q}$ at t_0 is

$$\mathbf{q}(t) = \mathbf{q}(t_0) + (t - t_0)\dot{\mathbf{q}}(t_0) + \int_t^{t_0} (t - \tau)\ddot{\mathbf{q}}(t_0) d\tau.$$

Since $\ddot{\mathbf{q}}$ is bounded, then

$$\lim_{t \to t^*} \big(\mathbf{q}(t), \mathbf{p}(t) \big) = (\mathbf{q}^*, \mathbf{p}^*).$$

The solutions depend only on c and not on t_0, so $(\mathbf{q}(t), \mathbf{p}(t))$ is still analytic at $(\mathbf{q}^*, \mathbf{p}^*)$, a contradiction. Therefore, $\liminf \big(\rho(\mathbf{q}(t)) \big) = 0$.

($\Leftarrow$) Suppose that $\liminf \big(\rho(\mathbf{q}(t)) \big) = 0$, but $\ddot{\mathbf{q}} \to \infty$ as $t \to t^*$, so t^* is a singularity. Suppose that $\ddot{\mathbf{q}}$ is bounded on $[0, t^*)$, then the chain implies that

$$\partial_t U(\mathbf{q}(t)) = \Big[\nabla \mathrm{U}(\mathbf{q})(\mathbf{t}) \Big]^{\mathrm{T}} \dot{\mathbf{q}}(t).$$

This implies that $\partial_t U(\mathbf{q}(t))$ is bounded, which in turn implies that $U(\mathbf{q}(t))$ is bounded.

However, $\liminf \big(\rho(\mathbf{q}(t)) \big) = 0 \Rightarrow \limsup \big(U(\mathbf{q}(t)) \big) = \infty$, a contradiction. $\square$

A complete proof of the next proposition can be found in [3], so we will just give a brief outline.

Proposition 2.8. *If* $(\mathbf{q}, \mathbf{p})$ *is an analytic solution of (2.3) on the interval of existence* $[0, t^*)$*, then* t^* *is a singularity if and only if*

$$\lim_{t \to t^*} \rho(\mathbf{q}(t)) = 0.$$

PROOF:(OUTLINE)

($\Leftarrow$) If $\lim(\rho(\mathbf{q}(t))) = 0$ then $\liminf(\rho(\mathbf{q}(t))) = 0$, so by proposition 2.7, t^* must be a singularity.

($\Rightarrow$) Suppose that $\limsup(\rho(\mathbf{q}(t))) \geq c$, then there exists a sequence in $\mathbb{R}$, $\{t_n\}_n$ with $t_n \to t^*$ such that $\forall\ n \in \mathbb{Z}^+$ we have $|q_{ij}(t_n)| \geq c\ \forall\ i \neq j$. This implies that $\exists\ B > 0$ such that $U(\mathbf{q}(t_n)) \leq B$, $\forall\ n \in \mathbb{Z}^+$, which in turn implies that $T(\mathbf{p}(t_n)) \leq B + h$. Therefore, $\exists\ \gamma > 0$ such that $|\mathbf{p}(t_m)| \leq \gamma$ for t_m sufficiently close to t^*, where $(\mathbf{q}(t_m), \mathbf{p}(t_m))$ being analytic implies that $(\mathbf{q}, \mathbf{p})$ is analytic at $t = t^*$, a contradiction.

□

Proposition 2.8 means that if t^* is a singularity of (2.3), then $\mathbf{q}(t) \to \Delta$ as $t \to t^*$. We need to classify the two types of finite singularities, namely the orbits that tend to Δ with asymptotic phase, and the orbits that just tend to the set Δ, yet not to a specific configuration in Δ.

Definition 2.9. *If t^* is a singularity of (2.3) and $\exists\ \tilde{\mathbf{q}} \in \Delta$ where $\lim\big(\mathbf{q}(t)\big) = \tilde{\mathbf{q}}$, then t^* is called a collision singularity.*

Definition 2.10. *If t^* is a singularity of (2.3) and not a collision singularity, then t^* is called a pseudo-collision singularity.*

Proposition 2.8 implies that if t^* is a pseudo-collision singularity then $\lim\big(\mathbf{q}(t)\big) \to \Delta$, yet does not tend to a specific configuration in Δ. If t^* is a singularity, then $\mathbf{q}(t) \to \Delta$ and the repulsive term will eventually become the dominating force. It seems intuitive that the closer the particles become the greater the repulsive force acts to separate them. This will lead to the fact that no solution curve of a **R-A** can tend to Δ. We are now prepared to state and prove the main result of this chapter.

Theorem 2.11. *There can be no collision or pseudo-collision singularities for any repulsive-attractive system (2.4).*

PROOF:

By proposition 2.7, if t^* is a collision or pseudo-collision singularity, then $\rho(\mathbf{q}(t)) \to 0$ as $t \to t^*$, where

$$U(\mathbf{q}) = \sum_{1 \le i < j \le N} m_i m_j \Big(b|q_{ij}|^{-\beta} + a|q_{ij}|^{-\alpha} \Big),$$

and $a < 0$ and $b > 0$ are real numbers and $0 < \beta < \alpha$.

If $\rho(\mathbf{q}(t)) \to 0$ as $t \to t^*$ then $\min_{1 \le i < j \le N} |q_{ij}(t)| \to 0$. This implies that $U(\mathbf{q}(t)) \to -\infty$ which implies that $\forall\, M \in \mathbb{R}\ \exists\, t_0 \in [0, t^*)$ such that $\forall\, t \in [t_0, t^*)$, $-U(\mathbf{q}(t)) > M$.

If $\big(\mathbf{q}, \mathbf{p}\big)$ is a solution to (2.3), then there exists a fixed h such that $\big(\mathbf{q}(t), \mathbf{p}(t)\big) \in \big\{(\mathbf{q}, \mathbf{p}) | T(\mathbf{p}) - U(\mathbf{q}) = h\big\}$, $\forall\, t \in [t_0, t^*)$, since this manifold is invariant for the interval of existence. Note that $T(\mathbf{p}) = \frac{1}{2} \sum m_i^{-1} |p_i|^2 \ge 0$, so we can pick an arbitrary solution to (2.3) and set $M = h + 1$, where

$$h = T(\mathbf{p}(t)) - U(\mathbf{q}(t)) \ge 0 + M > h + 1 \text{ for all } t \in [t_0, t^*),$$

a contradiction. Since the solution curve was arbitrary, then no solution curve can have $\mathbf{q}(t) \to \Delta$ □

Theorem 2.11 gives that for each energy level, the mutual distances of the particles is bounded below by a positive number. Caution should be taken when reading this statement. For every solution curve $\phi(t) = \big(\mathbf{q}(t), \mathbf{p}(t)\big)$, $U(\mathbf{q}(t))$ is bounded, because it is bounded on the invariant manifold associated with (2.6). Every energy level h has a conservation of energy manifold defined by

$$\mathfrak{I}_h = \big\{(\mathbf{q}, \mathbf{p}) | T(\mathbf{p}) - U(\mathbf{q}) = h\big\}. \tag{2.11}$$

By equation (2.6), $\mathfrak{I}_h$ is invariant and contains complete solution curves to the system (2.3). Therefore, for every h, $U(\mathbf{q})$ is bounded on $\mathfrak{I}_h$, yet if the reader incorrectly assumes that $U(\mathbf{q})$ is bounded in general, they are assuming that $U(\mathbf{q})$ is bounded on $\bigcup_{h \in \mathbb{R}} \mathfrak{I}_h$. This can be seen to be a false assumption. We can examine a simple similar assumption: $\forall n \in \mathbb{Z}^+$, the function $\frac{1}{x}$ is bounded on the set $\left(\frac{1}{n}, \infty\right)$, yet $\frac{1}{x}$ is not bounded on $\bigcup_{n=1}^{\infty}(\frac{1}{n}, \infty)$. Therefore we only have that the potential function is bounded on each energy level, so we may have that the boundaries of the energy levels may approach Δ as $h \to \infty$, see chapter 4. It can not be assumed that the potential function is bounded in general.

The nice outcome of theorem 2.11 is that the interval of existence given in (2.3) is actually $\mathbb{R}$. If we fix the energy level h and restrict to the invariant manifolds, $\mathfrak{I}_h$, then we have that $U(\mathbf{q})$ is analytic on $\mathbb{R}$.

2.4 Equilibria For Repulsive-Attractive Systems

In studying the N-body problem for homogeneous and attractive-attractive potential functions, it is clear that there are no equilibria for the system (2.3). As for a potential with the difference of two homogeneous functions, the possibility of equilibria emerge. In particular, for a **R-A**, one notices that $\nabla \mathrm{U}(\mathbf{q}) = \overrightarrow{\mathbf{0}}$ when $|q_{ij}| = \sqrt[\alpha-\beta]{\frac{-a\alpha}{b\beta}}$ for all $i \neq j$. Therefore there are equilibria for $N = 2$, 3, and 4. Namely the rectilinear, equilateral, and the tetrahedral configurations. It is clear that these are the only trivial equilibria, however there may be more.

For the 2-body problem we can see that there can only be one class of configurations that will be at equilibrium. This class is a collection of degenerate centers, as will be shown on chapter 4. The process of finding and classifying these equilibria for $N \geq 3$ is quite difficult and is outside the scope of this thesis. In the next chapter it will be shown that for any collection of N similar masses, there will be a planar equilibrium point. These equilibrium points are associated with the regular polygon configuration. Therefore, unlike the homogeneous and previously studied quasihomogeneous cases, for every integer N a repulsive-attractive system will have equilibrium points.

Chapter 3

Central Configurations for the Repulsive-Attractive N-Body Problem

3.1 Central Configurations

In this chapter we will examine particular types of solutions to the repulsive-attractive N-body problem (**R-A**). The type of solutions in question are the ones in which the configuration of the masses remains similar to itself for all time in the interval of existence. The goal is the existence of equilibrium and periodic solutions. We will begin by defining and classifying all solutions that remain self-similar on the interval of existence.

Definition 3.1. *(Homographic,Homothetic,Relative Equilibrium) A solution* $(\mathbf{q}(t), \mathbf{p}(t))$ *of the system (2.3) is called homographic provided that the configuration vectors remain self-similar for all time in the interval of existence. Alternatively, provided there exist functions* $\mathrm{r} : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$*, and* $\Omega : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}^9$ *where* $\Omega(t)$ *is a* 3×3 *orthogonal matrix,* $r(t) > 0\ \forall t \in A$*, and* A *is the interval of existence, such that*

$$q_i(t) = r(t)\Omega(t)q_i(t_0) \qquad \forall i \in \overline{1N} \text{ and } \forall t \in A \subset \mathbb{R}, \tag{3.1}$$

where A is the interval of existence. A homographic solution is called homothetic if $\Omega(t) = \mathrm{I}$ the 3×3 identity matrix. A homographic solution is called a relative equilibrium if $r(t) \equiv 1$.

The solutions that are homothetic or a relative equilibrium are two special classes of homographic solutions. Special note should be made that for a **R-A**, theorem 2.11 gives that the interval of existence, I, is in fact $\mathbb{R}$. There are certain configurations, called central configurations, that can lead to homographic solutions. A central configuration is one for which the change in momentum of any particle is a scalar multiple of the position vector of that particle. That is, for all $i \in \overline{1N}$, $\ddot{q}_i = \lambda q_i$, where λ is independent of i. This can be seen to be equivalent to the following definition.

Definition 3.2. $\mathbf{q} \in \mathbb{R}^{3N}$ *is called a central configuration if there exists some constant λ, such that:*

$$\nabla \mathrm{U}(\mathbf{q}) = \lambda \mathrm{M}\mathbf{q}. \tag{3.2}$$

Central configurations (**CC**) are independent of the coordinate system, and are rotationally invariant, hence they are $SO(3)$. This means that we can use the center of mass as the origin, and determine the classes of central configurations. If the potential function (2.2) is homogeneous, then the **CC** are scale independent (i.e. if $\mathbf{q}$ is **CC**, then for all positive scalars, γ, then $\gamma\mathbf{q}$ is also **CC**). However, for quasihomogeneous potentials, the latter no longer holds [6]. This leads to a new class of central configurations.

Definition 3.3. *Let $\mathbf{q}$ be a central configuration, then if there exists a positive scalar γ such that $\gamma\mathbf{q}$ fails to be a central configuration, then $\mathbf{q}$ is called an extraneous central configuration.*

The subset of the set of **CC** that are extraneous will be denoted by $\mathbf{CC}_e$. Clearly for homogeneous systems $\mathbf{CC}_e = \emptyset$, see [3]. For a **R-A**, with $N = 4$, $\mathbf{CC}_e \neq \emptyset$, see section 3.4.

For the homogeneous cases, finding these configurations was sufficient in order to find homothetic and relative equilibrium solutions, [15], [3]. However, for a **R-A**, this is no longer the case , see section 3.4. What is needed now is a connection between the central configurations for homogeneous potentials and the central configurations for repulsive-attractive potentials.

Lemma 3.4. *If there exist constants* λ_α *and* λ_β *such that for the functions,* W *and* V *given in (2.8) we have:*

$$\nabla \mathrm{W}(\mathbf{q}) = \lambda_\alpha \mathrm{M}\mathbf{q}, \qquad \nabla \mathrm{V}(\mathbf{q}) = \lambda_\beta \mathrm{M}\mathbf{q}, \tag{3.3}$$

then $\mathbf{q}$ *is* ***CC****, for any* ***R-A*** *system as given in definition 2.4.*

PROOF:

Assuming there exists constants λ_α and λ_β such that

$$\nabla \mathrm{W}(\mathbf{q}) = \lambda_\alpha \mathrm{M}\mathbf{q}, \qquad \nabla \mathrm{V}(\mathbf{q}) = \lambda_\beta \mathrm{M}\mathbf{q},$$

where λ_α and λ_β may be dependent on α and β. Then the gradient of $U(\mathbf{q})$ is given by:

$$\begin{aligned} \nabla \mathrm{U}(\mathbf{q}) &= \nabla\big(V(\mathbf{q}) + W(\mathbf{q})\big) \\ &= \nabla \mathrm{V}(\mathbf{q}) + \nabla \mathrm{W}(\mathbf{q}) \\ &= \lambda_\beta \mathrm{M}\mathbf{q} + \lambda_\alpha \mathrm{M}\mathbf{q} \\ &= \big(\lambda_\beta + \lambda_\alpha\big)\mathrm{M}\mathbf{q}. \end{aligned}$$

Setting $\lambda = \lambda_\beta + \lambda_\alpha$, we get that $\mathbf{q}$ is of the form (3.2). $\square$

Therefore, if a configuration $\mathbf{q}$ satisfies the hypothesis of lemma 3.4, then it must be **CC** for any **R-A**. Moreover, one can clearly see in the proof of lemma 3.4 that the result is true for any quasihomogeneous potential function. The converse of lemma 3.4 is false, in particular there exists a square central configuration for any **R-A**, which is not a **CC** for any homogeneous system, as will be shown in section 3.4. Therefore this configuration can not conform to the hypothesis of lemma 3.4. The central configurations that satisfy lemma 3.4 are called simultaneous central configurations [6]. The subset of the set of **CC** that consists of the simultaneous central configurations will be denoted by $\mathbf{CC}_s$.

We have that if the potential function is a homogeneous function, and $\mathbf{q}$ is a **CC**, so for all $\gamma > 0$, $\gamma\mathbf{q}$ is also a **CC**. Lemma 3.4 implies that if $\mathbf{q} \in \mathbf{CC}_s$, then $\mathbf{q} \in \mathbf{CC} \backslash \mathbf{CC}_e$, where $\mathbf{CC} \backslash \mathbf{CC}_e$is the set of non-extraneous **CC**, so we have the following lemma.

Lemma 3.5. $\boldsymbol{CC}_s \subseteq \boldsymbol{CC} \backslash \boldsymbol{CC}_e$.

The class of extraneous central configurations is a particularly interesting collection of configurations. These configurations can lead to rotational periodic orbits of the system, where the stability of these solutions may be quite complicated. However, they may not lead to homothetic solutions as for the homogeneous case. Now for homogeneous potentials, we have the following powerful theorem [3].

Theorem 3.6. *For the homogeneous N-body problem, $(\mathbf{q}(t), \mathbf{p}(t))$ is a homographic solution if and only if $\mathbf{q}(t)$ forms the same central configuration for all $t \in I \subset \mathbb{R}$.*

A detailed proof for the classical N-body problem can be found in [15]. For a **R-A** we can only state that if $\mathbf{q}(t)$ forms the same central configuration for all time t, then $\phi_t = (q(t), p(t))$ is a homographic solution. This is trivially due to the fact that it is by definition self-similar. We now state this as a lemma.

Lemma 3.7. *If* $\mathbf{q}(t)$ *forms the same central configuration for all time* t*, then* $\phi_t = \big(q(t), p(t)\big)$ *is a homographic solution.*

For the classical case, if $\mathbf{q}$ is **CC**, then placing the masses at rest on this configuration would result in the masses homothetically tending to the origin, thus leading to a simultaneous collision [15]. However, for a **R-A**, we get a dramatically different result. We will see in section 3.4, that it is insufficient for the initial configuration to be **CC**. What is needed is that the configuration remains self-similar for all time. Therefore, we have the correlated theorem for a **R-A**.

Theorem 3.8. *If* $\mathbf{q}_0 \in \boldsymbol{CC} \backslash \boldsymbol{CC}_e$ *and* ϕ_t *is the solution to the initial value problem of placing the masses at rest on the configuration,* $\mathbf{q}_0$*, then for non-negative energy the configuration* ϕ_t *becomes unbounded in the past and the future. For negative energy, the solution is either a fixed point or a periodic solution.*

PROOF:

If $\mathbf{q}_0 \in \mathbf{CC} \backslash \mathbf{CC}_e$, and $\phi_t = \Big(\mathbf{q}(t), \mathbf{p}(t)\Big)$ is the solution to the initial value problem (**IVP**), where $\mathbf{p}(0) = \overrightarrow{\mathbf{0}}$, and $\mathbf{q}(0) = \mathbf{q}_0$, then clearly $\mathbf{q}(t)$ remains self-similar. Therefore ϕ_t is a homographic solution to (2.3), moreover, ϕ_t will be a homothetic solution. Proceeding as in [15], if ϕ_t is a homothetic solution then there exists a function $r(t) \in C^2$ such that, for all time t, $\mathbf{q}(t) = r(t)\mathbf{q}_0$.

Let $U(\mathbf{q}) = V(\mathbf{q}) + W(\mathbf{q})$, where $V(\mathbf{q}) = \sum_{1 \le i < j \le N} b m_i m_j |q_i - q_j|^{-\beta}$ and $W(\mathbf{q}) = \sum_{1 \le i < j \le N} a m_i m_j |q_i - q_j|^{-\alpha}$, then V and W are homogeneous functions of degree $-\beta$ and $-\alpha$, respectively. Furthermore, define $U_0 = V_0 + W_0 = V\big(\mathbf{q}_0\big) + W\big(\mathbf{q}_0\big)$ to be the initial value of the potential function. If ϕ_t is a homographic solution, then the following hold,

$$V\big(\mathbf{q}(t)\big) = V_0 r^{-\beta} \quad \text{and} \quad W\big(\mathbf{q}(t)\big) = W_0 r^{-\alpha}, \tag{3.4}$$

$$\partial_i V(\mathbf{q}) = \partial_i V_0 r^{-\beta} \quad \text{and} \quad \partial_i W(\mathbf{q}) = \partial_i W_0 r^{-\alpha}, \tag{3.5}$$

$$\mathbf{q}^{\mathrm{T}}\nabla U = -\beta V_0 r^{-\beta} - \alpha W_0 r^{-\alpha}. \tag{3.6}$$

Since $\mathbf{p}(0) = \overrightarrow{\mathbf{0}}$, then for fixed energy, h, the equation (2.6) becomes $V_0 + W_0 = h$. Finding the times when $\mathbf{p}(\tilde{t}) = \overrightarrow{\mathbf{0}}$ is equivalent to finding the solutions to the equation

$$V_0 + W_0 = V_0 r^{-\beta} + W_0 r^{-\alpha}. \tag{3.7}$$

This is in turn equivalent to finding the roots to the function,

$$s(r) = V_0 r^{-\beta} + W_0 r^{-\alpha} - V_0 - W_0. \tag{3.8}$$

This function has a positive horizontal asymptote at $y = -V_0 - W_0$, and one positive critical number given by $r_c = \sqrt[\alpha-\beta]{\frac{-\alpha W_0}{\beta V_0}}$. By simple calculus we see that there is either one trivial, positive real solution, $r = 1$, or two positive solutions, where the properties change depending on the sign of h.

The Case of Non-negative Energy

If $h \geq 0$, then by equations (3.4) and (2.6) we have that $V_0 + W_0 \geq 0$. This together with the fact that $0 < \beta < \alpha$, gives $r_c > 1$. There is only one positive real solution to (3.7), namely $r = 1$. This means that $\mathbf{p}(\tilde{t}) = \overrightarrow{\mathbf{0}} \iff \mathbf{q}(\tilde{t}) = \mathbf{q}_0 \iff r(\tilde{t}) = 1$. Moreover, $r_c > 1$ implies that $\beta V_0 + \alpha W_0 < 0$, which implies that $\ddot{r}(0) > 0$. Therefore there exists a maximal interval, $(0, t^*)$, where $\mathbf{q}(t)$ is expanding. This is equivalent to saying that $\dot{r}(t) > 0$ for all $t \in (0, t^*)$. If t^* is finite, then $\dot{r} \in C^1$ implies that $\mathbf{p}(t^*) = \overrightarrow{\mathbf{0}}$, but by the invariance of the energy manifold we need that $U\big(\mathbf{q}(t^*)\big) = -h = U\big(\mathbf{q}_0\big)$, where $\mathbf{q}(t^*) = r(t^*)\mathbf{q}_0$. This implies that $r(t^*)$ is a solution to (3.7), which implies that $r(t^*) = 1$. This in turn implies that $\mathbf{q}(t^*) = \mathbf{q}_0$, and thus $\mathbf{q}(t)$ can not be expanding on $(0, t^*)$, a contradiction. Therefore, $t^* = \infty$, and $\mathbf{q}(t)$ expands without bound in the future. The same argument shows that $\mathbf{q}(t)$ also expands without bound in the past.

The Case of Negative Energy

If $h < 0$, then $V_0 + W_0 < 0$, where $0 < r_c < 1$, implies that $\partial_i U_0 < 0$, so there is a maximal interval, $(0, t^*)$ in which $\dot{r}(t) < 0$. Now theorem 2.11, implies that $\mathbf{q}(t)$ must be bounded below. Clearly we have that $r(t^*)$ is the second solution to (3.7) in the interval $(0, 1)$. This fact further implies that $t^* < \infty$, hence there exists a maximal interval, (t^*, t^{**}), on which $\dot{r}(t) > 0$. Similarly, we get that $r(t^{**}) = 1$, and by equation (2.6) and the invariance of the energy manifold $\mathbf{p}(t^{**}) = \overrightarrow{\mathbf{0}}$. therefore, by existence and uniqueness ϕ_t must be a periodic orbit.

For the case where $r_c > 1$, we see that the second solution to (3.7) is in the interval $(1, \infty)$, and $\dot{r}(0) > 0$. Therefore, as before, $\mathbf{q}(t)$ will expand out until $\mathbf{q}(t^*) = r(t^*)\mathbf{q}_0$, and $r(t^*)$ must be a second solution to (3.7). Then $\mathbf{q}(t)$ must contract back to $\mathbf{q}(t^{**}) = \mathbf{q}_0$, and therefore is a periodic orbit, of period t^{**}.

If $r_c = 1$ then $-\alpha W_0 = \beta V_0$ and we have that $\ddot{r}(0) = \dot{r}(0) = 0$, where $\ddot{\mathbf{q}}(0) = \ddot{r}\mathbf{q}(0) = 0$. Therefore ϕ_t is an equilibrium point. □

In the proof of this theorem, we needed the fact that ϕ_t was a homographic solution. This was accomplished by having the initial configuration to be a non-extraneous central configuration. The reason that the corresponding theorem for homogeneous N-body problem does not require this is because placing the masses in the initial position of a **CC** leads to a solution that is **CC** for all time in the interval of existence, so by theorem 3.6 the solution must be homographic.

As for a **R-A**, we will see that this is no longer the case. Yet not all is lost. We can examine each central configuration, and determine restrictions that will guarantee a homographic solution. An example of this procedure will be done in section 3.3, where we get the rotational periodic solutions which are relative equilibrium solutions.

In the proof of theorem 3.8, the fact that $r(t) \in C^2$ shows the existence of a time $\tilde{t}$ such that $\ddot{r}(\tilde{t}) = 0$. Therefore, the initial condition $\left(r(\tilde{t})\mathbf{q}_0, \overrightarrow{\mathbf{0}}\right)$ gives an equilibrium solution to the system. Hence, if $\mathbf{q} \in \mathbf{CC} \backslash \mathbf{CC}_e$, then there is a positive real number r such that $(r\mathbf{q}, \mathbf{0})$ is an equilibrium point of the system (2.3). Theorem 3.8 has the following corollary.

Corollary 3.9. *If* $\mathbf{q} \in \boldsymbol{CC} \backslash \boldsymbol{CC}_e$*, then there exists a positive real number* r*, such that* $(r\mathbf{q}, \mathbf{0})$ *is an equilibrium point to (2.3).*

3.2 The Regular N-gon Configuration

It has been known for some time that the configuration of placing N equal mass at the vertices of a regular N-gon is a central configuration of the homogeneous N-Body problem [3]. By lemma 3.4, and the fact that the regular N-gon configuration is independent of the α and β in (2.8), we get that the regular N-gon configuration is also a **CC** for any quasihomogeneous N-body problem,with potential functions of the form

$$U(\mathbf{q}) = \sum_{1 \le i < j \le N-1} m_i m_j \left(\frac{b}{\left|\mathbf{q}_i - \mathbf{q}_j\right|^\beta} + \frac{a}{\left|\mathbf{q}_i - \mathbf{q}_j\right|^\alpha} \right), \tag{3.9}$$

where a, b are real numbers and $0 < \beta < \alpha$. Lemma 3.5 states that the regular polygon configuration is a non-extraneous central configuration, therefore we get the following theorem for regular polygon configurations.

Theorem 3.10. *The configuration of placing N point masses at the vertices of a regular N-gon, is a non-extraneous central configuration for the system (2.3), where the potential function is of the form (3.9).*

In particular it can be shown explicitly that the λ given in definition 3.2 is,

$$\lambda = -2^{-\alpha} r^{-\alpha-2} \sum_{k=1}^{\left\lfloor \frac{N}{2} \right\rfloor} \left(b\beta (2r)^{\alpha-\beta} \csc^\beta \left(\frac{k\pi}{N}\right) + a\alpha \csc^\alpha \left(\frac{k\pi}{N}\right) \right). \tag{3.10}$$

The next corollary can be easily shown to be true, since the generalized Lennard-Jones problem can be seen as a subcase of the theorem above. So it is stated without proof.

Corollary 3.11. *Placing N equal masses at the vertices of a regular N-gon is a non-extraneous central configuration for the* ***GLJ****.*

Property (A.2) in appendix A states that for **GLJ**, there is a better formula for λ:

$$\lambda = 2^{-2\beta+1}\beta r^{-2\beta-2}\sum_{k=1}^{\lfloor\frac{N}{2}\rfloor}\csc^{\beta}\frac{k\pi}{N}\Big(\csc^{\beta}\frac{k\pi}{N}-(2r)^{\beta}\Big),$$

where r is the radius of the configuration. Proof of these facts can be seen in appendix A. This configuration leads to another collection of non-extraneous central configurations. If N equal masses are placed at the vertices of a regular N-gon and an arbitrary $(N+1)^{th}$ mass is placed at the center of this configuration, then it is a **CC**. This fact is stated in the next corollary.

Corollary 3.12. *If $\tilde{\mathbf{q}}$ is the central configuration of the N equal masses placed at the vertices of a regular N-gon, and $\mathbf{q}$ is the configuration of $\tilde{\mathbf{q}}$ with an $(N+1)^{th}$ arbitrary mass placed at the origin, then $\mathbf{q}$ is a non-extraneous central configuration of (2.3) with a potential function of the form (2.8).*

PROOF:

Set $m_0 = m_1 = \cdots = m_{N-1} = 1$, where the origin is the center of mass of the system (2.3), then

$$\sum_{k=0}^{N}\mathbf{q}_k = 0.$$

Hence, the following facts are true:

$$\mathbf{q}_N = \overrightarrow{\mathbf{0}},$$

$$\sum_{k=0}^{N-1} m_k \mathbf{q}_k = 0, \tag{3.11}$$

$$\mathbf{q}_{Nk} = -\mathbf{q}_k = -\tilde{\mathbf{q}}_k \quad \forall k \in \overline{0N-1}.$$

Therefore, we get $\left|\mathbf{q}_{kN}\right| = \left|\mathbf{q}_k\right| = \left|\mathbf{q}_0\right| = r, \quad \forall k \in \overline{0N-1}$, where r is the radius of the configuration. Furthermore, we have

$$\partial jU(\mathbf{q}) = -m_j \sum_{i=0, i\neq j}^{N} m_i \Big(b\beta \big|\mathbf{q}_{ji}\big|^{-\beta-2} + a\alpha \big|\mathbf{q}_{ji}\big|^{-\alpha-2}\Big)\mathbf{q}_{ji}. \tag{3.12}$$

Properties (3.11) gives that for $j \neq N$, the equations (3.12) become,

$$\partial jU(\mathbf{q}) = \partial j\tilde{U}(\tilde{\mathbf{q}}) - m_N \Big(b\beta|\mathbf{q}_{jN}|^{-\beta-2} + a\alpha|\mathbf{q}_{jN}|^{-\alpha-2}\Big)\mathbf{q}_{jN}$$

$$= \lambda\tilde{\mathbf{q}}_j - m_N \big(b\beta\big|\mathbf{q}_0\big|^{-\beta-2} + a\alpha\big|\mathbf{q}_0\big|^{-\alpha-2}\big)\mathbf{q}_j = \Big(\lambda - m_N \big(b\beta|\mathbf{q}_0|^{-\beta-2} + a\alpha|\mathbf{q}_0|^{-\alpha-2}\big)\Big)\mathbf{q}_j.$$

$$\partial jU(\mathbf{q}) = \Big(\lambda - m_N \big(b\beta r^{-\beta-2} + a\alpha r^{-\alpha-2}\big)\Big)\mathbf{q}_j.$$

The case where $j = N$ we see that,

$$\partial NU(\mathbf{q}) = -m_N \sum_{k=0}^{N-1} \Big(b\beta|\mathbf{q}_{jN}|^{-\beta-2} + a\alpha|\mathbf{q}_{kN}|^{-\alpha-2}\Big)\mathbf{q}_{kN}$$

$$= -m_N \sum_{k=0}^{N-1} \Big(b\beta|\mathbf{q}_0|^{-\beta-2} + a\alpha|\mathbf{q}_0|^{-\alpha-2}\Big)\mathbf{q}_k.$$

The first term in the sum is a constant, therefore we have that

$$\partial_N \tilde{U}(\mathbf{q}) = -m_N \Big(b\beta r^{-\beta-2} + a\alpha r^{-\alpha-2}\Big) \sum_{k=0}^{N-1} \mathbf{q}_k = \vec{0} = \mathbf{q}_N.$$

Set

$$\lambda_0 = \lambda - m_N \big(b\beta r^{-\beta-2} + a\alpha r^{-\alpha-2}\big), \tag{3.13}$$

where r is the radius of the configuration, then we have

$$\nabla \mathrm{U}(\mathbf{q}) = \lambda_0 \mathrm{M}\mathbf{q},$$

and the corollary is proven. □

To end this section we will show that we have a new type of periodic orbit that emerges for the regular N-gon configuration. Theorem 3.8, showed that if $\mathbf{q} \in \mathbf{CC} \backslash \mathbf{CC}_e$, then we have a homothetic solution to the **IVP**. Therefore we get, as a direct result, the following lemma.

Lemma 3.13. *Let ϕ_t be the solution to the **IVP** of placing N equal masses at rest on the vertices of a regular N-gon, then for any **R-A**, if $h \geq 0$, then ϕ_t expands without bound in the past and the future, and if $h < 0$, then either ϕ_t is an equilibrium point or ϕ_t is a periodic orbit.*

Moreover, by corollary 3.12, the same result holds for the regular N-gon configuration plus a $(N+1)^{th}$ arbitrary mass at the origin.

The main purpose of this section was to show that for all positive radii, the regular N-gon configuration is **CC**. Looking at the formulas for λ given by (A.2) and (3.13), it is apparent that the λ's are not necessarily negative as with the homogeneous and quasihomogeneous cases, with $a > 0$ and $b > 0$. In the next section it will be shown that for the regular N-gon configuration, if λ is positive then no rotational periodic solutions exist.

3.3 Regular Polygon Solutions of a Repulsive-Attractive System

In this section we will determine which of the N-gon configurations given above will have rotational periodic solutions to (2.3). To achieve this we will require that the square of the angular velocity is positive. That is the same as saying that there needs to be a pull towards the center of mass of the system. In the previous section we determined that all regular N-gon configurations are **CC**, yet for the **R-A**, this is insufficient to guarantee a rotational periodic orbit. This is due to the fact that the λ given by (3.10) is not necessarily negative. In order to see when a relative equilibrium orbit is possible, we restrict our attention to planar solutions and proceed as in [11].

For the planar N-body problem we can take the equations of motion (2.3) and express them as:

$$\ddot{q}_k = -\sum_{j=1, j\neq k}^{N} m_j \left(\frac{b\beta}{|q_k - q_j|^{\beta+2}} + \frac{a\alpha}{|q_k - q_j|^{\alpha+2}} \right) \bigl(q_k - q_j\bigr). \tag{3.14}$$

Let ρ_k to be the N^{th} roots of unity, $\mu = \sum_{j=1}^{N} m_j$, and use the transforms

$$d\tau = r^{\frac{\alpha}{2}} dt, \qquad\qquad z_k = r^{-1} q_k, \tag{3.15}$$

then the equations of motion (3.14) then become,

$$\ddot{z}_k = -\sum_{j=1, j\neq k}^{N} m_j \left(\frac{b\beta r^{\beta-\alpha}}{|z_k - z_j|^{\beta+2}} + \frac{a\alpha}{|z_k - z_j|^{\alpha+2}} \right) \bigl(z_k - z_j\bigr). \tag{3.16}$$

Liberty was taken by using $\ddot{z}_k$ to now refer to the rate of change of the velocity, with respect to the new time variable τ. Under these transforms, every configuration is viewed as being on the unit circle, and the $r^{\beta-\alpha}$ term is the correction factor for the force function. This term is what determines if any rotational periodic orbits exist.

The center of mass of the system is given by,

$$z_0 = \mu^{-1} \sum_{j=1}^{N} m_j \rho_j. \tag{3.17}$$

The functions that describe the simple periodic orbits that rotate about z_0 with angular momentum ω are given by,

$$z_k(\tau) = (\rho_k - z_0) e^{\omega \tau i}. \tag{3.18}$$

In (3.18) the term $\omega \in \mathbb{C}$ is the system's angular velocity, and is in the complex plane. The main theorem can now be stated.

Theorem 3.14. *For $N \geq 3$ and $m_1 = m_2 = \cdots = m_N > 0$, then the functions $z_k(\tau) = (\rho_k)e^{\omega\tau i}$ given by (3.18), are solutions to (3.16), provided that $\omega^2 = m_1\sigma \geq 0$. Where ω^2 is uniquely determined by:*

$$\sigma = 2^{-\alpha-1}\sum_{j=1}^{N-1}\csc^{\beta}\left(\frac{j\pi}{N}\right)\left(b\beta r^{\beta-\alpha} + a\alpha\csc^{\alpha}\left(\frac{j\pi}{N}\right)\right). \tag{3.19}$$

We must examine the term σ (3.19), to determine which equations produce a rotational periodic orbit. For every integer N there exists an r that makes $\sigma = 0$, which gives a system with zero angular velocity. Hence, for every positive integer N, if all masses are equal, there exists a planar equilibrium. A counter example to the necessity of the masses being equal will be shown in the next section. As for now, the existence of periodic orbits will be shown. The proof of theorem 3.14 will proceed as in [11], and will use results of circulant matrices (see appendix B).

PROOF:

First define 3 circulant matrices:

Definition 3.15. *Let $N \geq 2$ and let A, B, and C_κ be $N \times N$ complex matrices, such that:*

$$A = [a_{kj}], \qquad B = [b_{kj}], \qquad C_\kappa = A + \kappa B,$$

where,

$$a_{kj} = \begin{cases} \left(\frac{b\beta r^{\beta-\alpha}}{|1-\rho_{j-k}|^{\beta+2}} + \frac{a\alpha}{|1-\rho_{j-k}|^{\alpha+2}}\right)(1-\rho_{j-k}) & ,\text{if} \quad j \neq k \\ 0 & ,\text{if} \quad j = k, \end{cases}$$

$$[b_{kj}] = \rho_{j-k},$$

$$C_\kappa = A + \kappa B.$$

Now by the properties of circulant matrices we have eigenvalues and eigenvectors of C_κ are given by:

$$\lambda_k = \sum_{j=1}^{N} c_{1j}\rho_{k-1}^{j-1}, \tag{3.20}$$

$$\vec{v}_k = [\rho_{k-1}, \rho_{k-1}^2, \dots, \rho_{k-1}^N]^T. \tag{3.21}$$

Next is the lemma that connects the solutions of (3.14) the circulant matrices given in definition (3.15), where the proof is quite similar to the one found in [11].

Lemma 3.16. *For $\vec{m} = [m_1, m_2, \dots, m_N]^T \in \mathbb{R}^N$ and angular velocity $\omega \in \mathbb{C}$, the functions (3.18) are solutions to (3.16) if and only if:*

$$\big(A + \omega^2 \mu^{-1} B\big)\vec{m} = \omega^2 \vec{1}. \tag{3.22}$$

<u>PROOF:</u>

Direct substitution into (3.16) shows that (3.18) are solutions, if and only if

$$\big(\rho_k - z_0\big)\omega^2 e^{\omega t i} = \sum_{j \neq k} m_j \left(\frac{b\beta r^{\beta-\alpha}}{|\rho^k - \rho_j|^{\beta+2}} + \frac{a\alpha}{|\rho^k - \rho_j|^{\alpha+2}} \right) \big(\rho_k - \rho_j\big) e^{\omega t i}.$$

Equivalently, if and only if

$$\omega^2 \rho_k = \sum_{j \neq k} m_j \left(\frac{b\beta r^{\beta-\alpha}}{|\rho^k - \rho_j|^{\beta+2}} + \frac{a\alpha}{|\rho^k - \rho_j|^{\alpha+2}} \right) \big(\rho_k - \rho_j\big) + \omega^2 \mu^{-1} \sum_{j=1}^{N} m_j \rho_j.$$

Given the fact that $\rho_k - \rho_j = \rho_k\big(1 - \rho_{j-k}\big)$, we get

$$\omega^2 \vec{1} = \sum_{j \neq k} m_j \left(\frac{b\beta r^{\beta-\alpha}}{|1 - \rho_{j-k}|^{\beta+2}} + \frac{a\alpha}{|1 - \rho_{j-k}|^{\alpha+2}} \right) \big(1 - \rho_{j-k}\big) + \omega^2 \mu^{-1} \sum_{j=1}^{N} m_j \rho_{j-k}.$$

This equation is equivalent to (3.22), and thus the lemma is proven. □

Lemma 3.17. *For any $\kappa \in \mathbb{C}$, the eigenvalue of λ_1 of C_κ is independent of κ and satisfies*

$$\lambda_1 = \sum_{j=1}^{N-1} \left(\frac{b\beta r^{\beta-\alpha}}{|1 - \rho_j|^{\beta+2}} + \frac{a\alpha}{|1 - \rho_j|^{\alpha+2}} \right) (1 - \rho_j).$$

<u>PROOF:</u>

By (3.20) and definition (3.15), we have that

$$\lambda_1 = \sum_{j=2}^{N} \left(\frac{b\beta r^{\beta-\alpha}}{|1 - \rho_{j-1}|^{\beta+2}} + \frac{a\alpha}{|1 - \rho_{j-1}|^{\alpha+2}} \right) (1 - \rho_{j-1}) \rho_0^{j-1} + \kappa \sum_{j=1}^{N} \rho_{j-1} \rho_0^{j-1}.$$

Using the property of the N^{th} roots of unity, $\sum_{k=0}^{N-1} \rho_k = 0$, and that $\rho_0 = 1$, we get

$$\lambda_1 = \sum_{j=1}^{N-1} \left(\frac{b\beta r^{\beta-\alpha}}{|1-\rho_j|^{\beta+2}} + \frac{a\alpha}{|1-\rho_j|^{\alpha+2}} \right)(1-\rho_j),$$

so the proof of lemma 3.17 is complete. □

Note that the $(N-j)^{th}$ term in the above sum is the complex conjugate of the j^{th} term, therefore we have:

$$\lambda_1 = \sum_{j=1}^{N-1} \left(\frac{b\beta r^{\beta-\alpha}}{|1-\rho_j|^{\beta+2}} + \frac{a\alpha}{|1-\rho_j|^{\alpha+2}} \right) Real(1-\rho_j)$$

$$= \sum_{j=1}^{N-1} \left(\frac{b\beta r^{\beta-\alpha}}{\left|2 - 2\cos\left(\frac{2j\pi}{N}\right)\right|^{\frac{\beta+2}{2}}} + \frac{a\alpha}{\left|2 - 2\cos\left(\frac{2j\pi}{N}\right)\right|^{\frac{\alpha+2}{2}}} \right)(1 - \cos\left(\frac{2j\pi}{N}\right))$$

$$\lambda_1 = 2^{-\beta-1} \sum_{j=1}^{N-1} \csc^{\beta}\left(\frac{j\pi}{N}\right)\left(b\beta r^{\beta-\alpha} + a\alpha 2^{\beta-\alpha}\csc^{\alpha-\beta}\left(\frac{j\pi}{N}\right)\right). \tag{3.23}$$

Equation (3.23) is the same as the right side of (3.19), hence $\lambda_1 = \sigma$. Lemma 3.17 and (3.21) imply that,

$$\lambda_1 \vec{1} = \Big(A + \kappa B\Big)\vec{1}. \tag{3.24}$$

Multiplying both sides by $\vec{m} \in \mathbb{R}^N$ and setting $\omega^2 = \lambda_1$, we get (3.24) is in the form (3.22). Therefore, the proof of lemma 3.14 is complete. □

Note that fixing a ,b, α, and β, gives that the righthand side of (3.23) is a well defined function of r,

$$h(r) = 2^{-\beta-1} b\beta \left(\sum_{j=1}^{N-1} \csc^{\beta}\left(\frac{j\pi}{N}\right) \right) r^{\beta-\alpha} + 2^{-\alpha-1} a\alpha \left(\sum_{j=1}^{N-1} \csc^{\alpha}\left(\frac{j\pi}{N}\right) \right). \tag{3.25}$$

The function $h(r)$ given in (3.25) has one positive root, namely,

$$r_0 = 2 \sqrt[\alpha-\beta]{\frac{-b\beta \sum_{j=1}^{N-1} \csc^{\beta}\left(\frac{j\pi}{N}\right)}{a\alpha \sum_{j=1}^{N-1} \csc^{\alpha}\left(\frac{j\pi}{N}\right)}}. \tag{3.26}$$

Clearly (3.25) is positive for all $r \in (0, r_0)$, and negative for all $r \in (r_0, \infty)$. This makes the ω given in theorem 3.14 to be imaginary when r is to the right of this root, and real to the left. This result is stated in the next lemma.

Lemma 3.18. *For ω given in theorem 3.14 we have the following:*

1. *For all $0 < r < r_0$, ω is real.*
2. *For $r = r_0$, $\omega = 0$.*
3. *For all $r < r_0$, ω is imaginary.*

Theorem 3.18 is for the configuration with respect to (**wrt**) $\mathbf{z} = r^{-1}\mathbf{q}$, so when dealing with the configuration space, **wrt q**, you need to take the reciprocal of r_0. This together with theorem 3.14 gives that a **R-A** will have a periodic orbit if $r \geq r_0^{-1}$, where at equality there is an equilibria point for the system. Now we have the second theorem of this section.

Theorem 3.19. *For the system (3.14), the configuration of placing N equal masses at the vertices of a regular N-gon will generate a rotational periodic solution, provided that $r > r_1$, and if $r = r_1$ then the configuration is an equilibrium. r_1 is given by*

$$r_1 = \frac{1}{2}\sqrt[\alpha-\beta]{\frac{-a\alpha\sum_{j=1}^{N-1}\csc^{\alpha}\left(\frac{j\pi}{N}\right)}{b\beta\sum_{j=1}^{N-1}\csc^{\beta}\left(\frac{j\pi}{N}\right)}}. \tag{3.27}$$

Moreover, if $0 < r < r_1$, then this configuration can not produce a rotational periodic solution.

Note that (3.27) will always generate a positive real number, so theorem 3.19 guarantees an equilibrium point for the system. Therefore, theorem 3.19 has a direct corollary.

Corollary 3.20. *For any N, if $m_1 = \cdots = m_N$, then there exists a planar equilibrium point for any repulsive-attractive system given by definition 2.4.*

In the proof of the non-existence of periodic solutions for certain central configurations, we see that the determining factor is the sign of λ. In particular, if $\lambda > 0$ then $\mathbf{q}$ does not lead to a rotational periodic orbit, and if $\lambda < 0$ it does lead to a rotational periodic orbit. The most interesting fact is the emergence of the case where $\lambda = 0$. In this case there will be an equilibrium point at $(\mathbf{q}, \mathbf{0})$, which is the same equilibrium point guarantied by lemma 3.13

Next we turn to the existence of extraneous central configurations for the 4-body problem.

3.4 The Square Configuration Solution with Non-Equal Masses

A counter example to the necessity of the masses being equal, for $N = 4$, in theorem 3.14 is the placement of two pairs of equal masses at the vertices of a square. Let $\mathbf{q}$ be the square configuration, where

$$\mathbf{q}_1 = -\mathbf{q}_3 = \Big(\frac{1}{\sqrt{2}} \sqrt[\alpha-\beta]{\frac{-a\alpha}{b\beta}}, 0\Big),$$

$$\mathbf{q}_2 = -\mathbf{q}_4 = \Big(0, \frac{1}{\sqrt{2}} \sqrt[\alpha-\beta]{\frac{-a\alpha}{b\beta}}\Big),$$

$$m_1 = m_2 = 1, \qquad \text{and} \qquad m_3 = m_4 = m.$$

This construction gives the following properties:

$$|\mathbf{q}_{12}| = |\mathbf{q}_{14}| = |\mathbf{q}_{32}| = |\mathbf{q}_{34}| = \sqrt[\alpha-\beta]{\frac{-a\alpha}{b\beta}},$$

$$\mathbf{q}_{ij} = 2\mathbf{q}_i, \qquad \text{when } j = (i+2)mod(4), i \neq j,$$

$$|\mathbf{q}_{13}| = |\mathbf{q}_{24}| = \sqrt{2} \sqrt[\alpha-\beta]{\frac{-a\alpha}{b\beta}}.$$

Therefore,

$$\begin{aligned}\partial_1 U(\mathbf{q}) &= -\sum_{k=2}^{4} m_k \Big(b\beta |\mathbf{q}_{1k}|^{-\beta-2} + a\alpha |\mathbf{q}_{1k}|^{-\alpha-2} \Big) \\ &= -\sum_{k=2}^{4} m_k |\mathbf{q}_{1k}|^{-\alpha-2} \Big(b\beta |\mathbf{q}_{1k}|^{\alpha-\beta} + a\alpha \Big) \\ &= a\alpha m 2^{\frac{-\alpha-2}{2}} \left(\frac{-a\alpha}{b\beta} \right)^{\frac{-\alpha-2}{\alpha-\beta}} \Big(1 - 2^{\frac{\alpha-\beta}{2}} \Big) \mathbf{q}_1 .\end{aligned}$$

Similarly, we get that for all $i \in \overline{14}$,

$$\partial_i U(\mathbf{q}) = a\alpha m 2^{\frac{-\alpha-2}{2}} \left(\frac{-a\alpha}{b\beta} \right)^{\frac{-\alpha-2}{\alpha-\beta}} \Big(1 - 2^{\frac{\alpha-\beta}{2}} \Big) \mathbf{q}_i .$$

Hence the configuration is **CC**, moreover, since we have taken $0 < \beta < \alpha$, and $a < 0$, then the constant term will always be negative. Therefore we get the next lemma directly from this result.

Lemma 3.21. *There is a square configuration with* $m_1 = m_2 \neq m_3 = m_4$ *that produces a relative equilibrium solution to (2.3).*

To end this chapter we will state a trivial result that actually is a dramatic difference between the classical case and the repulsive-attractive case. In examining the square configuration created above, one notices that this configuration fails to be **CC** if it is dilated by any amount, hence $\mathbf{q}(0) \in \mathbf{CC}_e$. Therefore, placing the masses at rest on this configuration can not be a homothetic solution. This configuration that can not generate a homothetic solution, yet it will produce a relative equilibrium solution. Note that for this particular configuration, if $\mathbf{q}_0$ is dilated by any amount, it fails to be **CC**, which leads to the question: if $\mathbf{q} \in \mathbf{CC}_e$then will $\gamma\mathbf{q}$ fail to be **CC** for any $\gamma > 0$, $\gamma \neq 1$? If this is so, then every non-extraneous **CC**, can not generate a homothetic solution. This question remains open.

Chapter 4

The 2-Body Generalized Lennard-Jones Problem

4.1 The Equations of Motion.

In this chapter we will study the global flow for the 2-body generalized Lennard-Jones problem. As in [13] and [7], we will take our configuration space to be $\mathbb{R}^2$, and our system to be that of a single particle of unit mass in a central force field. For the generalized Lennard-Jones problem (**GLJ**) we will take $\alpha = 2\beta$ and $b = 2$ and $a = -1$. Using these constructions the potential function becomes,

$$U(\mathbf{x}) = 2|\mathbf{x}|^{-\beta} - |\mathbf{x}|^{-2\beta},$$

where $|\mathbf{x}|$ is the distance between the two particles. Therefore the differential equation that defines the motion of the particle is,

$$\ddot{\mathbf{x}} = 2\beta\big(|\mathbf{x}|^{-2\beta-2} - |\mathbf{x}|^{-\beta-2}\big)\mathbf{x}.$$

The equations of motion (2.3) become,

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{\delta H}{\delta y} \\ \dot{\mathbf{y}} &= -\frac{\delta H}{\delta x}, \end{aligned} \tag{4.1}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 - \{(0,0)\}$ and $\mathbf{y} \in \mathbb{R}^2$ and the energy integral is:

$$H(\mathbf{x}, \mathbf{y}) = \frac{1}{2}|\mathbf{y}|^2 - 2|\mathbf{x}|^{-\beta} + |\mathbf{x}|^{-2\beta}.$$

Using the ideas developed in [13], with the coordinate transform,

$$\begin{aligned} \mathbf{x} &= r^{\frac{1}{\beta}} e^{\theta i}, \\ \mathbf{y} &= r^{-1}(v + iu) e^{\theta i}, \end{aligned} \tag{4.2}$$

and with the time transform,

$$d\tau = r^{-\frac{1}{\beta}-1} dt, \tag{4.3}$$

we will get the new equations of motion,

$$\begin{aligned} \dot{r} &= \beta r v \\ \dot{\theta} &= u \\ \dot{v} &= \beta v^2 + u^2 + 2\beta(1 - r) \\ \dot{u} &= (\beta - 1) u v. \end{aligned} \tag{4.4}$$

In the above equations we are taking the derivative with respect to the new time variable τ. The energy integral and the integral of angular momentum will now have the forms,

$$u^2 + v^2 = -2 + 4r + 2r^2 h, \tag{4.5}$$

$$r^{\frac{1}{\beta}-1} u = C. \tag{4.6}$$

Note that unlike the classical and quasihomogeneous N-body problem with $a > 0$ and $b > 0$, [15], [3] it is clear that there are no collision equilibria (see theorem 2.11). Moreover, there is a class of non-collision equilibria,

$$(1, \theta_0, 0, 0). \tag{4.7}$$

(4.7) represents a collection of degenerate equilibria, where $\theta_0 \in S^1$. Note that if $|\mathbf{x}| = 1$, then $\frac{\delta H}{\delta x} = 0$, so this class of equilibria of the system (4.5) is also a class of equilibria of (4.1).

The flow for the system (4.5) will be bounded for certain values of h, where the most notable difference is that the total energy must be greater that or equal to -1. This is unlike the perviously studied systems [7], [13]. Next we will find the bounds on all motions of the system.

Lemma 4.1. *The total energy of the system must be greater than or equal to* -1. *If the total energy of (4.5) is negative, then all motion is bounded above and below. In particular, if* $h = -1$ *then* $r(t) = 1$, *and if* $-1 < h < 0$ *then* $\frac{-1+\sqrt{1+h}}{h} \leq r(t) \leq \frac{-1-\sqrt{1+h}}{h}$, *for any time* t. *For non-negative energy, all motion is bounded below. In particular, if* $h = 0$ *then* $r(t) \geq \frac{1}{2}$, *and if* $h > 0$ *then* $r(t) \geq \frac{-1+\sqrt{1+h}}{h}$, *for all time* t.

PROOF:

For $h \neq 0$, we define the function $f(r)$ to be the righthand side of the equation (4.5),

$$f(r) = -2 + 4r + 2hr^2. \tag{4.8}$$

Note that $v^2 + u^2 \geq 0$, so we must have that $f(r) \geq 0$. The roots of (4.8) are given by,

$$r_{\pm} = \frac{-1 \pm \sqrt{1+h}}{h}. \tag{4.9}$$

If $h < -1$ then $f(r)$ is a quadratic that opens down and has complex roots, therefore $f(r) < 0$ for all r, a contradiction. Hence, the system will have no solutions when $h < -1$.

For $-1 \leq h < 0$, then $f(r)$ is still a quadratic that opens down, so $f(r) \geq 0$ on $\left[\frac{-1+\sqrt{1+h}}{h}, \frac{-1-\sqrt{1+h}}{h}\right]$, where $\frac{-1+\sqrt{1+h}}{h} > 0$. Therefore, all solutions are bounded by these values. For non-negative energy, it is a similar argument, where $f(r)$ opens upwards. $\square$

Lemma 4.1 shows that, for every energy level h, all motion restricted to the energy manifolds is bounded below, and for negative energy all solution curves are bounded above as well as below. Since each of these energy manifolds are invariant, we will study the flow restricted to these manifolds separately.

4.2 The Global Flow for Negative Energy.

Note that the flow is invariant under rotations, so the θ term can be factored out to get the new equations of motion,

$$\begin{aligned} \dot{r} &= \beta r v \\ \dot{v} &= \beta v^2 + u^2 + 2\beta(1-r) \\ \dot{u} &= (\beta - 1)uv. \end{aligned} \tag{4.10}$$

Therefore, any fixed points of (4.10) are either periodic orbits or fixed points of the system (4.5). Lemma 4.1 shows that we can restrict our attention to $h \geq -1$, where we will divide this case into two subcases, namely $h = -1$ and $-1 < h < 0$.

4.2.1 The Case when $h = -1$.

To understand the global flow of the system (4.10) we will need to find any separatrix of the system (See [10]). First we partition the phase space into some invariant manifolds by using the energy integral (4.5). We then have the collection of invariant manifolds,

$$\mathfrak{M}_h = \Big\{(r, v, u) | u^2 + v^2 = f(r), r > 0\Big\}. \tag{4.11}$$

These manifolds can be seen as 2-dimensional surfaces in 3 space, that have different structure depending on the energy level h. The collection $\bigcup_{h \in \mathbb{A}} \mathfrak{M}_h$, where $\mathbb{A} = \{h \in \mathbb{R} | h \geq -1\}$, foliates the phase space. Using the fact that $\dot{u} = 0$ when $u = 0$, then we get the invariant sub-manifolds

$$\mathfrak{U}_h = \Big\{(r, v, u) \in \mathfrak{M}_h | u = 0\Big\}, \tag{4.12}$$

for each energy level h. Moreover, $\mathfrak{U}_h$ partitions $\mathfrak{M}_h$ into two invariant sub-manifolds. To understand the global flow on $\mathfrak{M}_h$ it is sufficient to get the flow on each of these invariant sub-manifolds.

Furthermore, for any $\beta > 0$ and any time t, we have that if $(r(-t), v(-t), u(-t))$ is a solution curve of (4.10), then so is $(r(t), -v(t), u(t))$, and $(r(t), -v(t), -u(t))$. Therefore, the flow is symmetric about both the u and v axes. This means that to see the flow on $\mathfrak{M}_h$ you need only determine the flow on the quadrant where $u \geq 0$. We will now define the invariant sub-manifold of $\mathfrak{M}_h$ where $u \geq 0$ as,

$$\mathfrak{M}_h^+ = \big\{(r, v, u) \in \mathfrak{M}_h | u \geq 0\big\}. \tag{4.13}$$

Now it is sufficient to determine the flow on this sub-manifold.

If $h = -1$, then $f(r) = -2(r-1)^2$, however $f(r) = u^2 + v^2 \geq 0$, hence $r = 1$ and $u = v = 0$. This implies that there is only one class of solution curves on $\mathfrak{M}_{-1}$, namely $\phi_t = (1, \theta_0, 0, 0)$, where $\theta_0 \in S^1$, for all time t. In other words, $\mathfrak{M}_{-1}$ consists of a point at $(1, 0, 0)$. Therefore a diagram for the flow on $\mathfrak{M}_h$ is omitted, since it would be trivial.

4.2.2 The Case when $-1 < h < 0$.

For the structure of $\mathfrak{M}_h$ we notice that $f(r)$ is a quadratic that opens down and has roots $r_\pm$ given by (4.9). The structure of $\mathfrak{M}_h$ is the top half of a parabola, rotated about the r-axis, where $\mathfrak{M}_h$ is homotopic to a 2-sphere. In order to find if there are any equilibria on $\mathfrak{M}_h$ we first must use the energy integral (4.5) and substitute it into (4.10) to get the restricted equations of motion,

$$\begin{aligned} \dot{r} &= \beta r v \\ \dot{v} &= (\beta - 1)v^2 + f(r) + 2\beta(1 - r) \\ \dot{u} &= (\beta - 1)uv, \end{aligned} \tag{4.14}$$

where $f(r)$ is given by (4.8).

Looking at the equations of motion (4.14), we note that any equilibria must be solutions to the equation,

$$f(r) = 2\beta(r - 1). \tag{4.15}$$

Equation (4.15) has solutions given by,

$$r^{\pm} = \frac{\beta - 1 \pm \sqrt{(\beta - 2)^2 - 4h(\beta - 1)}}{2h}, \tag{4.16}$$

so equilibria on $\mathfrak{M}_h$ will have the form,

$$(r^{\pm}, 0, \pm\sqrt{f(r^{\pm})}). \tag{4.17}$$

We are now prepared to state and prove the lemma that determines the flow on $\mathfrak{M}_h$.

Lemma 4.2. *For negative energy, there are two equilibria on $\mathfrak{M}_h$, namely $R_-^{\pm} = (r^-, 0, \pm\sqrt{f(r^-)})$. Furthermore, both $R_-^{\pm}$ are degenerate centers, and all other orbits are periodic.*

PROOF:

For $0 < \beta < 1$, we have that $\lim_{\beta \to 1^-} \frac{(\beta-2)^2}{4(\beta-1)} = -\infty$, where $\frac{d}{d\beta}\left[\frac{(\beta-2)^2}{4(\beta-1)}\right] < 0$ for $0 < \beta < 1$. Therefore $\frac{(\beta-2)^2}{4(\beta-1)} < -1$, and lemma 4.1 gives that $h \geq -1$, so we have that $(\beta - 2)^2 - 4h(\beta - 1) > 0$. Hence, there are two real solutions to (4.15), where using simple inequalities shows that only one conforms to the restriction given in lemma 4.1, namely $r^- = \frac{\beta-2-\sqrt{(\beta-2)^2-4h(\beta-1)}}{2h}$. Therefore, $\mathfrak{M}_h$ has two equilibria, namely $R_-^{\pm} = \big(r^-, 0, \pm\sqrt{f(r^-)}\,\big)$.

For $\beta = 1$, then $r^{\pm} = \frac{-1\pm1}{2h} = 0$ or $-\frac{1}{h}$, so the only root that conforms to the restrictions given by lemma 4.1, is $r^- = -\frac{1}{h}$. Again, $\mathfrak{M}_h$ will have two equilibria, $R_-^{\pm}$. For $\beta > 1$, again the only root that conforms to the restrictions is r^-, so for negative energy, $\mathfrak{M}_h$, will have two equilibria, namely $R_-^{\pm}$.

Linearizing about the equilibria given by (4.17), gives eigenvalues of $\lambda_1 = 0$ and two other (possibly non-zero) eigenvalues,

$$\lambda^2 = 2\beta\Big((\beta-2)r^{\pm} - 2(\beta-1)\Big). \tag{4.18}$$

Note that for $0 < \beta < 1$ and $-1 < h < 0$, we have that $r^- > \frac{2(\beta-1)}{\beta-2}$. Therefore, $R_-^{\pm} = (r^-, 0, \pm\sqrt{f(r^-)})$ has zero real part eigenvalues. Similarly, for the other values of β. Therefore we can not use $Hartman-Grobman\ Theorem$ to determine the characteristics of these equilibria and we need to use qualitative methods to determine the nature of these equilibria.

First define a function of r,

$$g(r) = \frac{1}{2}\big(f(r) + 2\beta(1-r)\big) = hr^2 - (\beta-2)r + (\beta-1). \tag{4.19}$$

Clearly, the roots of $g(r)$ are the solutions to (4.15). Moreover, this function determines how the flow crosses the $v = 0$ plane, which, in turn actually determines the characteristics of each of the equilibria on $\mathfrak{M}_h$. To determine how this is accomplished, we need to use the concept of the index of an equilibrium [10]. Next we need to state the lemma for which the flow locally about each equilibrium can be determined.

Lemma 4.3. *Suppose that* $\mathfrak{M}_h^+$ *has an equilibrium* R_0*, then if there exists an* $\epsilon > 0$ *where* $g(r) < 0$ *on* $(r_0 - \epsilon, r_0)$ *and* $g(r) > 0$ *on* $(r_0, r_0 + \epsilon)$ *then* R_0 *is a degenerate* *saddle. If $g(r) > 0$ on $(r_0 - \epsilon, r_0)$ and $g(r) < 0$ on $(r_0, r_0 + \epsilon)$ then R_0 is a degenerate center, and if $g(r)$ does not change sign, then R_0 is a cusp.*

PROOF:

Suppose that R_0 is an equilibrium on $\mathfrak{M}_h^+$, clearly this equilibrium is either a degenerate topological saddle or a degenerate center. The following theorem can be found in [10], so it is stated without proof.

Theorem 4.4. *Let R_0 be an isolated equilibrium contained the interior of a 2-dimensional invariant manifold, then if the index of R_0, denoted by $I\Big(R_0\Big)$, is equal to -1 then R_0 is a topological saddle, and if $I\Big(R_0\Big) = 1$ then it can not be a topological saddle.*

Assume that there exist an $\epsilon > 0$ such that $g(r) > 0$ on $(r_0 - \epsilon, r_0)$ and $g(r) < 0$ on $(r_0, r_0 + \epsilon)$, and $sign(\dot{r}) = sign(v)$, so $I\Big(R_0\Big) > 0$. Now if $I\Big(R_0\Big) > 1$. This indicates the existence of another equilibrium in $\overline{B}_\epsilon\Big(R_0\Big) \bigcap \mathfrak{M}_h^+$. Since R_0 is isolated we can choose a smaller ϵ to avoid this second equilibrium point. Therefore, $I\Big(R_0\Big) = 1$ and by theorem 4.4, R_0 can not be a topological saddle. This leaves the only other possibility being that R_0 is a degenerate center. Similar arguments hold for the other cases. $\square$

To complete the proof of lemma 4.2, we only need to find the characteristics of the roots of $g(r)$. For negative energy, $g(r)$ is a quadratic that opens down, and r^- is the larger root. Therefore, $R_-^\pm$ conforms to the hypothesis of theorem 4.4, and must be a degenerate center. By the symmetry of solutions both equilibria are degenerate centers. To complete the proof we need to show that all non-equilibrium orbits on $\mathfrak{M}_h^+$ must be periodic.

Lemma 4.5. *For $-1 < h < 0$, then all non-equilibrium solution curves are periodic.*

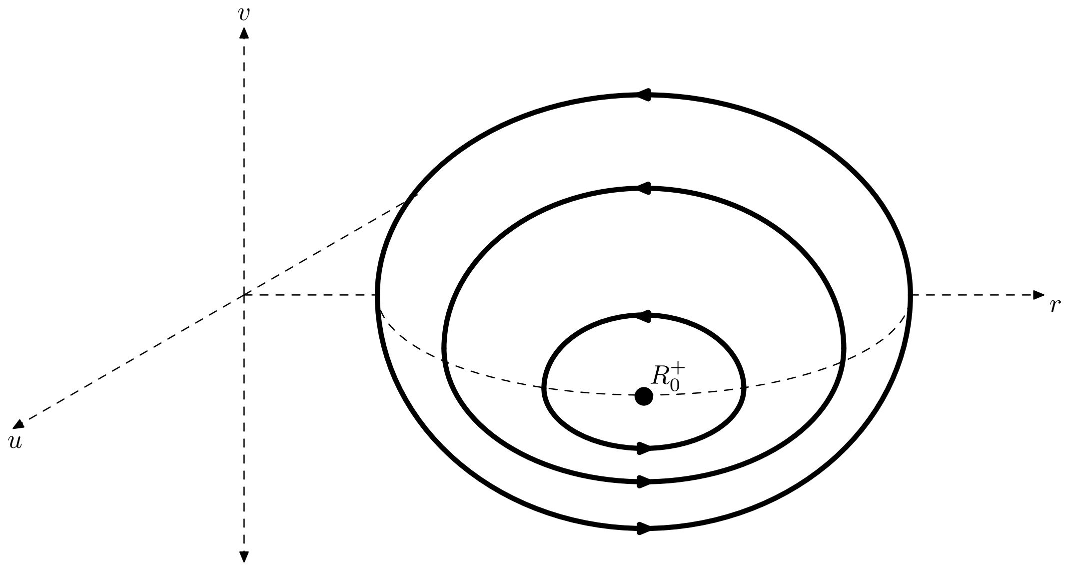

Figure 4.1: Flow on $\mathfrak{M}_h$ for $-1 < h < 0$.

PROOF:

Let $\phi_t = (r(t), v(t), u(t)) \neq R_-^+$ be an orbit on $\mathfrak{M}_h^+$. Now ϕ_t is on a compact, invariant sub-manifold of $\mathfrak{M}_h$, so by *Poincarés Theorem*, $\omega\big(\phi_t\big)$ and $\alpha\big(\phi_t\big)$ must be either a fixed point, periodic orbit, or a graphic. $\mathfrak{M}_h^+$ only has one equilibrium point, namely R_-^+, a degenerate center. Therefore, $\omega\big(\phi_t\big)$ and $\alpha\big(\phi_t\big)$ must be periodic orbits. *Brouwers Fixed Point Theorem* implies that R_-^+ must be on the interior of these limit sets, so they must cross the $v = 0$ plane. By the symmetry of solutions, and the continuity of the flow, $\phi_t = \omega\big(\phi_t\big) = \alpha\big(\phi_t\big)$, and therefore must be a periodic orbit. $\square$

By the symmetry of solutions, we have that $R_-^\pm$ are both degenerate centers, and all other orbits are periodic. Hence the proof of lemma 4.2 is now complete. $\square$

We now have the complete picture of the flow on the compact manifold $\mathfrak{M}_h$ for negative energy. There are two equilibria, $R_-^\pm$ both degenerate centers and $\mathfrak{M}_h$ is densely filled with periodic orbits, see figure 4.1.

4.3 The Flow for Zero Energy.

If $h = 0$, then (4.8) becomes $f(r) = 4r - 2$, so the structure of $\mathfrak{M}_h$ is that of one side of a hyperboloid. In order to find the equilibria, we note that (4.16) leads to $r = \frac{\beta-1}{\beta-2}$, and lemma 4.1 gives that $r \geq \frac{1}{2}$. These facts lead to the next lemma.

Lemma 4.6. *If $h = 0$, and $0 < \beta \leq 2$, then there are no equilibria, and if $\beta > 2$, then there are two equilibria on $\mathfrak{M}_h$, $R^{\pm} = \left(\frac{\beta-1}{\beta-2}, 0, \pm\sqrt{\frac{\beta-1}{\beta-2}}\right)$, both degenerate centers.*

PROOF:

By lemma 4.1 and the fact that for $h = 0$, so (4.16) gives $r_0 = \frac{\beta-1}{\beta-2}$. The necessary and sufficient condition for the existence of equilibria is that $\frac{\beta-1}{\beta-2} \geq \frac{1}{2}$. This has a solution set of $(-\infty, 0] \bigcup (2, \infty)$, where $\beta > 0$ leads to the necessary and sufficient condition being, $\beta > 2$.

If $\beta > 2$ then the equation (4.18) gives that $\lambda^2 = -2\beta(\beta - 1) < 0$. If $h = 0$, then (4.19) becomes $g(r) = (\beta - 1) - (\beta - 2)r$. If $\beta > 2$ then $g(r)$ is a line with negative slope and r-intercept, r_0. Therefore, by lemma 4.3, we have that $r_0^{\pm}$ are both degenerate centers. $\square$

Therefore, $\beta = 2$ is a bifurcation value of the system (4.5), so we will study each case individually.

4.3.1 The Case when $0 < \beta \leq 2$.

To determine the flow on $\mathfrak{M}_h$, we define the different types of orbits that may exist on $\mathfrak{M}_h$.

Definition 4.7 (Escape Orbits, Capture Orbits). *Let $\phi_t = (r(t), v(t), u(t))$ be any orbit on $\mathfrak{M}_h$, then:*

$$\phi_t \text{ is called a Capture Orbit if, } r(t) \to +\infty \text{ as } t \to -\infty,$$
$$\phi_t \text{ is called an Escape Orbit if, } r(t) \to +\infty \text{ as } t \to +\infty.$$

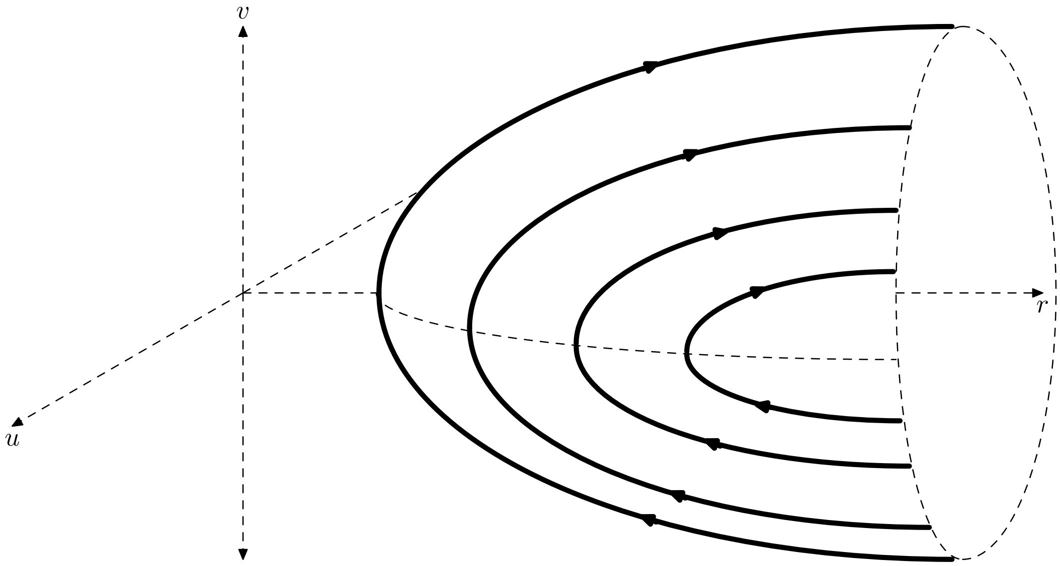

Figure 4.2: Flow on $\mathfrak{M}_0$ for $0 < \beta \leq 2$.

Using these definitions, we can get the general flow for non-negative energy when there are no equilibrium points on $\mathfrak{M}_h$.

Lemma 4.8. *For non-negative energy, if $\mathfrak{M}_h$ has no equilibria, then all orbits must be escape-capture orbits.*

PROOF:

It is sufficient to show that for any orbit $\phi_t = (r(t), v(t), u(t)) \in \mathfrak{M}_h^+$, then $r(t) \to \infty$ as $t \to \pm\infty$. Assume that at time t_0, $v(t_0) > 0$, then by lemma 4.1, there exists a sequence, $\Big\{r(t_i)\Big\}_{i=0}^{\infty}$, that is bounded below, when $t_i > t_{i+1}$. Therefore it must have an accumulation point, say $r(t^*)$. Since $\mathfrak{M}_h$ has no equilibrium points, then the point $(r(t^*), v(t^*), u(t^*))$ must be on the orbit of ϕ_t. Furthermore, $v(t^*) = 0$ and there exist a maximal interval, say (t^{**}, t^*) such that ϕ_t remains in the set $\Big\{(r, v.u) \in \mathfrak{M}_h | v < 0\Big\}$. If $t^{**} > -\infty$, then $v(t^{**}) = 0$, and by the symmetry of solutions ϕ_t would be a periodic orbit. By *Brouwers Fixed Point Theorem*, there would need to be a fixed point on $\mathfrak{M}_h$, a contradiction.

Therefore, $t^{**} = -\infty$ and ϕ_t is a capture orbit. The same argument can be applied to show that ϕ_t must be an escape orbit. Furthermore, if we had assumed that ϕ_{t_0} is in the set $\{(r,v,u) \in \mathfrak{M}_h | v > 0\}$ the sequence used would have the property that $t_j < t_{j+1}$ and the same result would be reached. Hence, every orbit is of the capture-escape type, and the lemma is proved. $\Box$

By lemma 4.6, we see that for $h = 0$, and $0 < \beta \leq 2$, there are no equilibria. Lemma 4.8 shows that all orbits on $\mathfrak{M}_0$ are capture-escape orbits. These facts are stated in the next lemma.

Lemma 4.9. *For zero energy and* $0 < \beta \leq 2$*, there are no equilibrium points and* $\mathfrak{M}_0$ *is densely filled with capture-escape orbits.*

The general flow on $\mathfrak{M}_0$ for $0 < \beta \leq 2$ can be seen in figure 4.2

4.3.2 The Case when $\beta > 2$.

For $h = 0$ and $\beta > 2$, lemma 4.6 we have that $\mathfrak{M}_0^+$ has a degenerate center. Therefore there must be a set of periodic orbits about $R_-^{\pm}$ that has positive measure. Furthermore, the set $\{(r,v,0) \in \mathfrak{M}_h\}$ is an invariant set. This set is a curve on $\mathfrak{M}_h$, that contains no equilibria and therefore must be a solution curve. Clearly, it is a capture-escape orbit. Actually, it is the only capture-escape orbit on $\mathfrak{M}_0$, for $\beta > 2$. In order to show this we need the structure of the invariant manifold near infinity, which will be done in a later section. As for now these results can be stated in the following lemma. For the uniqueness of the capture-escape orbit we will state and show this in the section on the global flow for the system.

Lemma 4.10. *For* $h = 0$ *and* $\beta > 2$*, there two degenerate centers and a set of periodic orbits with positive measure, and there exist a capture-escape orbit.*

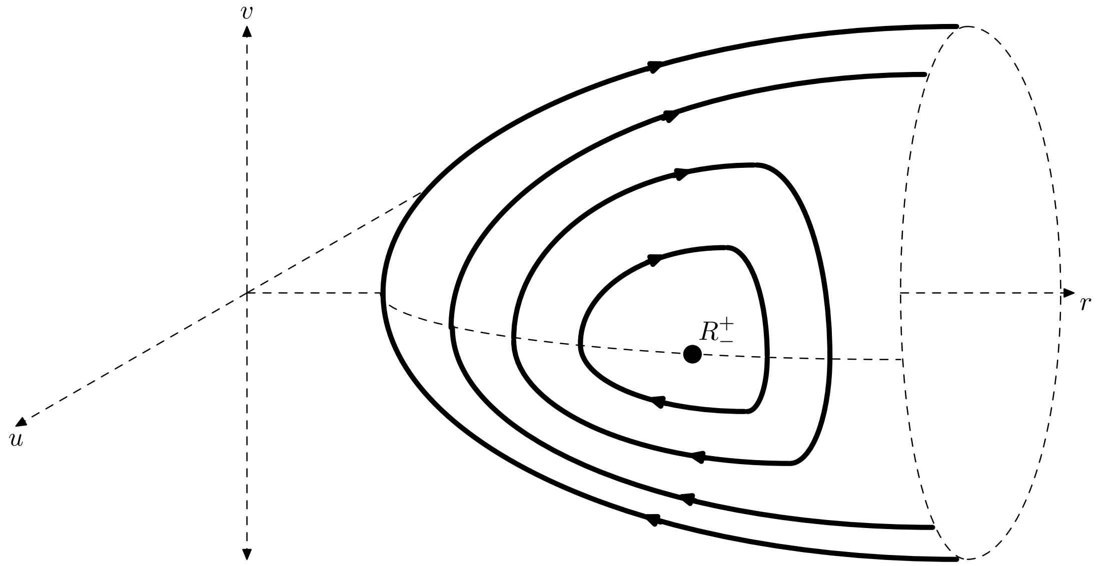

Figure 4.3: Flow on $\mathfrak{M}_0$ for $\beta > 2$.

This lemma gives the general flow on $\mathfrak{M}_0$ when $\beta > 2$, and the flow can be seen in figure 4.3.

4.4 The Flow for Positive Energy.

4.4.1 The Case when $0 < \beta \leq 2$.

For $\beta = 1$, the solutions to (4.16) are $r = 0$ and $r = -\frac{1}{h}$, so there are no positive real solutions, and hence there are no equilibria. For $\beta = 2$, then the only solutions are $r = \pm\sqrt{-\frac{1}{h}}$, therefore not real, and again there are no equilibria.

For $1 < \beta < 2$, then clearly $r^- < 0$, if in fact real to begin with. As for r^+ we see that $\beta - 2 + \sqrt{(\beta-2)^2 - 4h(\beta-1)} < 0$, and again $r^+ < 0$. Therefore, when $1 \leq \beta \leq 2$, there are no equilibria on $\mathfrak{M}_h$.

For $0 < \beta < 1$, then the only possible positive real root is r^+. Fixing the energy level $h > 0$, we see that $\lim_{\beta \to 0^+} r^+ = \frac{-1+\sqrt{1+h}}{h}$, and that r^+ is a strictly decreasing

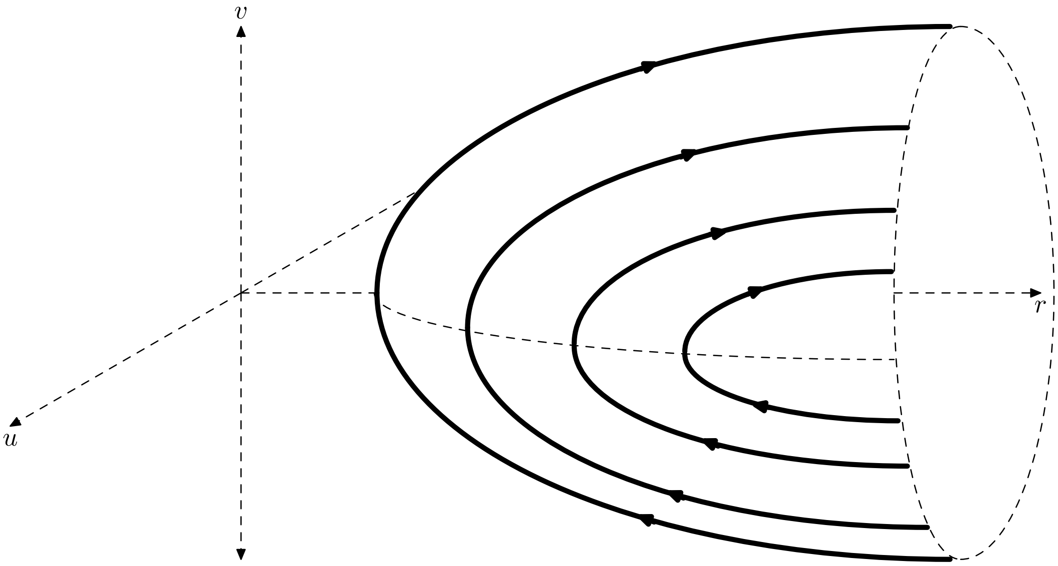

Figure 4.4: Flow on $\mathfrak{M}_h$ for $h > 0$ and $0 < \beta \leq 2$ or for $\beta > 2$ and $h > \frac{(\beta-2)^2}{4(\beta-1)} > 0$.

function, **wrt** β. Therefore, r^+ can not conform the restrictions given in lemma 4.1, so for positive energy and $0 < \beta \leq 2$, there are no equilibria on $\mathfrak{M}_h$.

If $0 < \beta \leq 2$, then lemma 4.8 shows that all finite orbits on $\mathfrak{M}_h$ are capture-escape orbits. This is stated in lemma 4.11, and the flow can be seen in figure 4.4.

4.4.2 The Case when $\beta > 2$ and $h > \frac{(\beta-2)^2}{4(\beta-1)} > 0$.

Clearly there are no real solutions to (4.16), thus there can be no equilibria on $\mathfrak{M}_h$. Lemma 4.8 shows that all orbits are capture-escape orbits. Therefore the flow is the same as in the pervious case, and can be seen in figure 4.4. Thus we have the following lemma.

Lemma 4.11. *For positive energy, and* $0 < \beta \leq 2$ *or,* $\beta > 2$ *and* $h > \frac{(\beta-2)^2}{4(\beta-1)} > 0$*, there are no equilibria, and all orbits are capture-escape orbits.*

4.4.3 The Case when $\beta > 2$ and $h = \frac{(\beta-2)^2}{4(\beta-1)} > 0$.

If $h = \frac{(\beta-2)^2}{4(\beta-1)}$, then there is a multiple root of (4.16), so there are only two equilibria on $\mathfrak{M}_h$, namely $R_0^{\pm} = \left(\frac{2(\beta-1)}{\beta-2}, 0, \pm\sqrt{f\left(\frac{2(\beta-1)}{\beta-2}\right)}\right)$. For the flow on the invariant manifold we state and proof the following lemma.

Lemma 4.12. *For $\beta > 2$ and $h = \frac{(\beta-2)^2}{4(\beta-1)}$, there are two equilibria on $\mathfrak{M}_h$, $R^{\pm}$, which are both cusps. Furthermore, $W^u\left(R^{\pm}\right)$ is an escape orbit, and $W^s\left(R^{\pm}\right)$ is a capture orbit, and all other orbits are capture-escape orbits.*

PROOF:

It is sufficient to show for the flow on $\mathfrak{M}_h^+$. $f(r)$ defined in (4.15) has a multiple real root that conforms to the restriction given in lemma 4.1, so we have two equilibria on $\mathfrak{M}_h^+$. Moreover, $g(r) \geq 0$ and $\dot{v} = (\beta-1)v^2 + 2g(r) \geq 0$, where equality only holds at the equilibrium point. Therefore, the flow on the sub-manifold, $\mathfrak{M}_h^+$, is in the positive v direction. Furthermore, lemma 4.3 gives that R^+ must be a cusp.

Let $\phi_t = (r(t), v(t), u(t))$ be any orbit on this invariant sub-manifold, where if $v(t_0) > 0$ then $r(t)$ is bounded below. If $r(t)$ is bounded below, then there exist a monotonic decreasing sequence, $\left\{r(t_i)\right\}_{i=0}^{\infty}$, $t_j > t_{j+1}$, that is also bounded below. Hence it must have an accumulation point, say $r(t^*)$. If $v(t_0) < 0$, then just use $t_j < t_{j+1}$ and you will still have a sequence that is bounded below.

Case I: If $(r(t^*), v(t^*), u(t^*))$ is not on the orbit of ϕ_t, then it must be the equilibrium point R^+, and $\phi_t \subseteq W^u\left(R^+\right)$. Furthermore, for any t^{**} we have that for all time $t \in (t^{**}, \infty)$, ϕ_t must remain in the set $\left\{(r, v, u) \in \mathfrak{M}_h^+ | r > r(t^{**}), v > v(t^{**})\right\}$. Therefore $r(t) \to \infty$ as $t \to \infty$, and ϕ_t must be an escape orbit. If you used that $v(t_0) < 0$ then $\phi_t \subset W^s\left(R^+\right)$ and $r(t) \to \infty$ as $t \to -\infty$. In this case, ϕ_t would have to be a capture orbit.

If ϕ_t was any orbit on $W^u\left(R^+\right)$, then clearly it is completely contained in the upper half of $\mathfrak{M}_h^+$. Assume that ϕ_t is not an escape orbit, then there needs to be a sequence such that $r(t_i) \to r(t^{**}) < \infty$ as $t_i \to \infty$. This requires that $v(t^{**}) = 0$, but $\dot{v} \geq 0$, a contradiction. Hence, any orbit on $W^u\Big(R^+\Big)$ must be an escape orbit. Similarly, if $\phi_t \subset W^s\Big(R^+\Big)$, then it must be a capture orbit. Therefore, $W^u\Big(R^+\Big)$ is a capture orbit and $W^s\Big(R^+\Big)$ is an escape orbit.

Case II: If $(r(t^*), v(t^*), u(t^*))$ is on the orbit of ϕ_t, then for any monotonic decreasing sequence $\Big\{t_i\Big\}_{i=0}^{\infty}$, such that $t_i \to -\infty$, the sequences $\Big\{r(t_i)\Big\}_{i=0}^{\infty}$ and $\Big\{v(t_i)\Big\}_{i=0}^{\infty}$ are both monotonic decreasing sequences. If the former sequence is bounded above, then there would need to be a second equilibrium point, say $(r(t^{**}), v(t^{**}), u(t^{**}))$, with $v(t^{**}) < 0$, a contradiction. Hence, $r(t_i) \to \infty$, and ϕ_t must be an capture orbit. Similarly for $v(t_0) < 0$ we would have that ϕ_t is an escape orbit. The same process would show that ϕ_t is a capture-escape orbit. Therefore, if ϕ_t is any orbit on $\mathfrak{M}_h^+$ that is not in $W^{u,s}\Big(R^+\Big)$, then it must be a capture-escape orbit and the lemma is proved. □

We now have the general flow on $\mathfrak{M}_h$ for positive energy, $\beta > 2$ and $h = \frac{(\beta-2)^2}{4(\beta-1)}$, see figure 4.5.

4.4.4 The Case when $\beta > 2$ and $0 < h < \frac{(\beta-2)^2}{4(\beta-1)}$.

For $\beta > 2$ and $0 < h < \frac{(\beta-2)^2}{4(\beta-1)}$, there are two positive real roots to (4.16). Fixing $\beta > 2$ and viewing the level curves we can see that both of the roots conform to the restrictions given in lemma 4.1. $r^- < r^+$ implies that $g(r) > 0$ on $\Big[\frac{-1+\sqrt{1+h}}{h}, r^-\Big) \bigcup \Big(r^+, \infty\Big)$, and $g(r) < 0$ on (r^-, r^+). By lemma 4.3, we have that $R_-^{\pm}$ are degenerate centers, and $R_+^{\pm}$ are degenerate saddles. This leads to the next lemma that determines the flow on the manifold.

Lemma 4.13. *For positive energy, and $\beta > 2$ and $h < \frac{(\beta-2)^2}{4(\beta-1}$, then there are four equilibria on $\mathfrak{M}_h$, $R_-^{\pm}$ which are degenerate centers, and $R_+^{\pm}$ which are degenerate saddles. There is a set of positive measure of periodic orbits about $R_-^{\pm}$. There is*

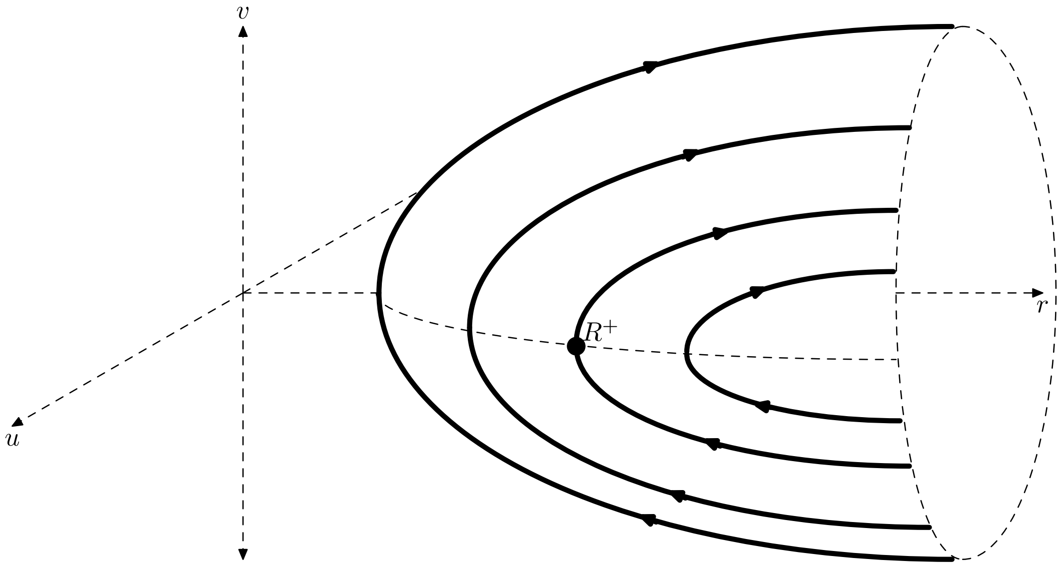

Figure 4.5: Flow on $\mathfrak{M}_h$ for $h > 0$, $\beta > 2$ and $h = \frac{(\beta-2)^2}{4(\beta-1)} > 0$.

also a set of positive measure of capture-escape orbits, and $W^{u,s}\left(R^{\pm}_{+}\right)$ *consists of a homoclinic orbit, and one escape orbit, associated with* $W^{u}\left(R^{\pm}_{+}\right)$ *and one capture orbit associated with* $W^{s}\left(R^{\pm}_{+}\right)$*.*

PROOF:

It is sufficient to prove for the flow on $\mathfrak{M}_h^+$. We have already shown that there is a degenerate center and a saddle on this manifold, R^+_- and R^+_+ respectively. Clearly there needs to be a set of positive measure of periodic orbits about R^+_-. We need to show that these orbits are bounded by a homoclinic orbit contained in $W^{u,s}\left(R^+_+\right)$.

By the continuity of the flow, and $g(r) < 0$ on (r^-, r^+) together with the fact that $sign(\dot{r}) = sign(v)$, $W^s\left(R^+_+\right)$ must have an orbit, say Γ that approaches from the top left side. As before, this orbit must cross the $v = 0$ plane. Moreover, it can not cross at R^+_-, since it is a center, so by symmetry of solutions Γ must be a

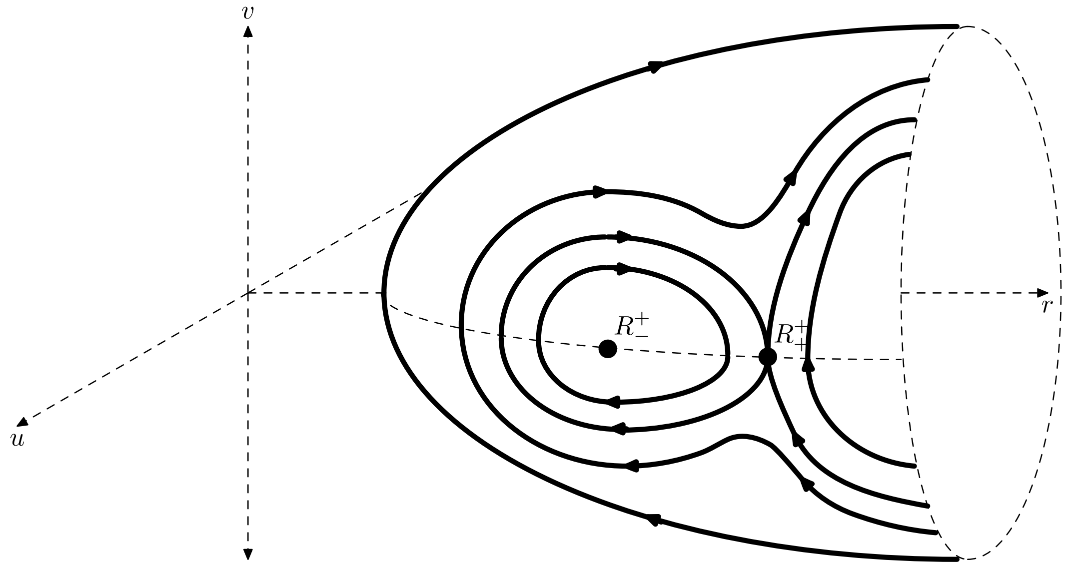

Figure 4.6: Flow on $\mathfrak{M}_h$ for $\beta > 2$ and $0 < h < \frac{(\beta-2)^2}{4(\beta-1)}$.

homoclinic orbit. Suppose that R_-^+ is not in the interior Γ then Γ crosses the $v = 0$ plane within the set $\Big\{(r, 0, u) \in \mathfrak{M}_h | r \in (r^-, r^+)\Big\}$, but this implies that $g(r) > 0$ on some subset of (r^-, r^+), a contradiction. Therefore, R_-^+ must be on the interior of this homoclinic orbit.

The same arguments can be applied to see that $W^s\Big(R_+^+\Big)$ also contains a capture orbit, and $W^u\Big(R_+^+\Big)$ contains an escape orbit. Clearly, all other orbits on the exterior of Γ must be capture-escape orbits. Therefore, there is a set of capture-escape orbits with positive measure. $\square$We now have the general flow on the manifold $\mathfrak{M}_h$, see figure 4.6.

4.5 The Flow Near Infinity.

For negative energy, all orbits are bounded, see lemma 4.1. For negative energy, the global flow of the system can be seen on $\mathfrak{M}_h$ as in section 4.2. However, for positive energy there is a set of initial conditions that has positive measure, where the orbits become unbounded. For zero energy we have seen the existence of a capture-escape orbit. Therefore, to determine the global flow we need to understand how the flow approaches infinity. We will proceed as in [13], where we use the transform $\rho = r^{-\beta}$. Transforming the equations of motion (4.5) by using polar coordinates, we get the equations of motion for the near infinity system,

$$\begin{aligned} \dot{\rho} &= -\beta\rho v \\ \dot{\theta} &= u \\ \dot{v} &= u^2 + 2\beta\rho(1-\rho) \\ \dot{u} &= -uv. \end{aligned} \tag{4.20}$$

The energy integral and angular momentum integral now become,

$$v^2 + u^2 = 2\rho^2 - 4\rho + 2h, \tag{4.21}$$

$$u = 2\rho^{-\beta^{-1}}h. \tag{4.22}$$

To determine the flow near infinity we must first see how the motion would act at infinity, or equivalently , when $\rho = 0$.

4.5.1 The Flow At Infinity.

Since the flow must be continuous, it must approach infinity smoothly. Looking at equations (4.20), we see that $\rho = 0$ implies that $\dot{\rho} = 0$. This makes the following set invariant with respect to the flow,

$$I_h = \Big\{(\rho, \theta, v, u) | \rho = 0, v^2 + u^2 = 2h, \theta \in S^1\Big\}. \tag{4.23}$$

The set (4.23) can be seen as a torus pasted onto the phase space of (4.20), which can be viewed as a cylinder in 3 space, with the boundary of the top identified with the bottom. We will show the flow on the 2 dimensional representation of the torus, namely $\mathbb{Z}_{[0,2\pi]} \bigotimes \mathbb{Z}_{[-b,b]}$.

The equations of motion (4.20) restricted to I_h are,

$$\begin{aligned} \dot{\theta} &= u \\ \dot{v} &= 2h - v^2 \\ \dot{u} &= -uv. \end{aligned} \tag{4.24}$$

The equations (4.24) are seen to have equilibria at $(\theta_0, \pm\sqrt{2h}, 0)$. These equilibrium points have eigenvalues given by:

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= \mp\beta\sqrt{2h}, \\ \lambda_3 &= \mp\sqrt{2h}. \end{aligned} \tag{4.25}$$

If $h \neq 0$, then $(\theta_0, \sqrt{2h}, 0)$ is a collection of degenerate sinks and $(\theta_0, -\sqrt{2h}, 0)$ is a collection of degenerate sources. Let $l^{\pm} = \big\{(\theta, \pm\sqrt{2h}, 0) \in I_h | \theta \in S^1\big\}$ be the two lines containing all these degenerate equilibria, then it can be seen that in the representation of the torus, $\mathbb{Z}_{[0,2\pi]} \bigotimes \mathbb{Z}_{[\pm\sqrt{2h}]}$, the line l^+ is shown twice, at the top and the bottom of the square, and l^- is along the θ axis, see figure 4.7.

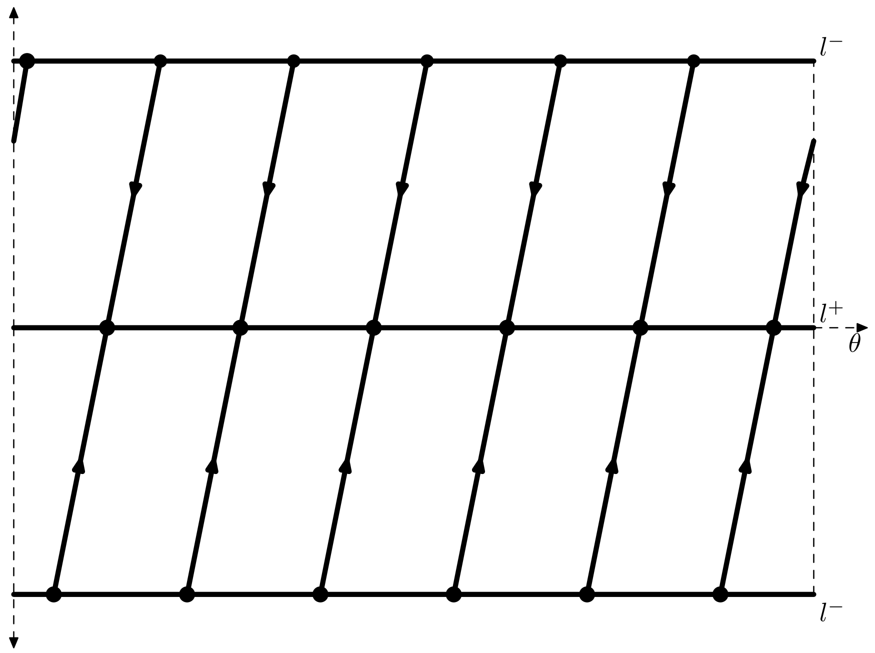

Figure 4.7: Flow on the Infinity Manifold, I_h for $h > 0$.

Since $|v| \leq \sqrt{2h}$, then $\dot{v} \geq 0$ on I_h, where equality holds if and only if $v = \pm\sqrt{2h}$. The set $\left\{(\theta_0, \pm\sqrt{2h}, 0)|\theta_0 \in S^1\right\}$ forms two separatries for the flow restricted to I_h, where $\dot{\theta} = u$ gives that θ is increasing when $u > 0$, decreasing when $u < 0$ and invariant when $u = 0$. Therefore, the lines defined by $v = \pm\sqrt{2h}$ are a collection of degenerate equilibria, and the flow for $h > 0$ can be seen as in figure 4.7.

If $h = 0$, then all the eigenvalues are zero, and the equations of motion become,

$$\begin{aligned} \dot{\theta} &= u \\ \dot{v} &= -v^2 \\ \dot{u} &= -uv. \end{aligned} \tag{4.26}$$

The equilibrium points of (4.26) are $(\theta_0, 0, 0)$, where $\theta_0 \in S^1$. In particular, since $u^2 + v^2 = 0$, then $\dot{\theta} = \dot{v} = \dot{u} = 0$ for all time t. This means that I_0 consists of the points $(\theta_0, 0, 0)$ where $\theta_0 \in S^1$. The flow is just a circle of degenerate fixed points, and the diagram of the flow is omitted since it would be trivial.

4.6 The Global Flow Positive Energy.

To determine the overall flow near the fictitious infinity manifold I_h. we need to define the collection of invariant manifold near infinity. We can proceed as in the previous sections and factor out the θ term. In doing so, we need to remember that all solution curves are actually rotated by $\theta \in S^1$, and are invariant surfaces which contain the actual solution curves to the system (4.20). By defining the near infinity manifold as,

$$N_h = \Big\{(\rho, v, u)|v^2 + u^2 = 2\rho^2 - 4\rho + 2h, \rho \geq 0, \theta \in S^1\Big\}, \tag{4.27}$$

we can determine the flow on this manifold by using the equations of motion,

$$\begin{aligned} \dot{\rho} &= -\beta\rho v \\ \dot{v} &= u^2 + 2\beta\rho(\rho - 1) \\ \dot{u} &= -uv. \end{aligned} \tag{4.28}$$

The equations of motion near infinity (4.28) can be seen to have the equilibria at

$$E^{\pm} = \big(\rho_0, 0 \pm \sqrt{2\beta\rho_0(\rho_0 - 1)}\big). \tag{4.29}$$

Since the rotations have been factored out, the fictitious infinity manifold, I_h (4.23), becomes the circle on the $\rho = 0$ plane, of radius $\sqrt{2h}$. Furthermore, the collection of degenerate equilibria given by $l^{\pm}$ are now the points $(0, \pm\sqrt{2h}, 0)$. Using the energy integral, we define a function of ρ as the right hand side of (4.21),

$$\tilde{f}(\rho) = -2\rho^2 + 4\rho + 2h. \tag{4.30}$$

Following the same reasoning as in lemma 4.1 we have the next lemma that show that all motion is bounded.

Lemma 4.14. *Let* $\phi_t = (\rho(t), v(t), u(t))$ *be a solution to (4.28), then* $0 \leq \rho(t) \leq 1 + \sqrt{1+h}$.

The proof of this lemma is omitted, and the reader is directed to the proof of lemma 4.1. One can clearly see that the restriction is the reciprocal of the restriction given in that lemma. In seeing this connection, we can draw a stronger correlation: the flow on N_h has k non-infinity equilibria, if and only if the flow on $\mathfrak{M}_h$ has k equilibria. The proof of this uses the same ideas as above. Moreover, $\big(r_0, 0, \pm\sqrt{f(r_0)}\big)$ is an equilibrium of the flow of (4.10), if and only if $\Big(r_0^{-1}, 0, \pm\sqrt{\tilde{f}(r_0^{-1})}\Big)$ is a non-infinity equilibrium of (4.28). This fact has strong implications: to determine the global flow for non-negative energy, it is sufficient to study the flow on the compact, invariant manifold N_h, for each energy level $h \geq 0$.

The equilibria of (4.28) has eigenvalues, $\lambda_1 = 0$ and two possible non-zero eigenvalues given by,

$$\lambda^2 = 2\beta\rho_0\Big((\beta - 2) - 2(\beta - 1)\rho_0\Big). \tag{4.31}$$

Clearly all infinity equilibria have three zero eigenvalues, so no information can be obtained. Yet some information can be found by examining the value of $\dot{v}$ locally near I_h. Looking at the equations of motion (4.28) we see that locally near l^+, $\dot{\rho} < 0$. Therefore l^+ must be a sink, and similarly we see that l^- is a source.

The Case when $0 < \beta \leq 2$ or, $\beta > 2$ and $h > \frac{(\beta-2)^2}{4(\beta-1)} \geq 0$.

In sections 4.4.1 and 4.4.2, it was shown that $\mathfrak{M}_h$ has no equilibria. Therefore, N_h can not have any non-infinity equilibria. Under the transforms used in section 4.5, the flow is still symmetric about the $v = 0$ and $u = 0$ planes. Note that the orbit,

$$\phi_0 = \big(\rho(t), v(t), 0\big), \tag{4.32}$$

is a solution curve to the system (4.28) and forms a separatrix for the flow restricted to N_h. Therefore, N_h is a 2 dimensional manifold, and by the *Jordan Curve Theorem*, we have the ϕ_0 partitions N_h into the invariant sub-manifolds,

$$\begin{aligned} N_h^+ &= \big\{(\rho, v, u) \in N_h | u \geq 0, 0 \leq \rho \leq 1 + \sqrt{1+h}\big\}, \\ N_h^- &= \big\{(\rho, v, u) \in N_h | u \leq 0, 0 \leq \rho \leq 1 + \sqrt{1+h}\big\}. \end{aligned} \tag{4.33}$$

Section 4.5.1 gave that l^+ is a sink and l^- is a source. Furthermore, $l^\pm \in \partial N_h^+$, so we can use this to determine the overall flow on N_h^+.

Lemma 4.15. *Let A be a compact, connected, invariant sub-manifold of N_h, where ∂A has two isolated equilibria, l^- a source, and l^+ a sink, and no equilibria on $int\big(A\big)$. For any orbit Γ on A, we have that $\omega\Big(\Gamma\Big) = \big\{l^+\big\}$ and $\alpha\Big(\Gamma\Big) = \big\{l^-\big\}$. Moreover, A is densely filled with heteroclinic orbits that start at l^- and tend to l^+.*

PROOF:

Let Γ be any trajectory on A, where A is a compact, connected, invariant, 2 dimensional manifold. By *Poincaré−Bendixson Theorem*, $\omega\Big(\Gamma\Big)$ is either an equilibrium, a periodic orbit, or a graphic.

If $\omega\Big(\Gamma\Big)$ is a periodic orbit, there would need to be an equilibrium on it interior, but this contradicts the assumption that A has no such equilibrium.

If $\omega\Big(\Gamma\Big)$ is a graphic, it must be a finite collection of heteroclinic orbits, otherwise there would need to be a saddle on A, contradicting our assumptions. Therefore, $\omega\Big(\Gamma\Big) = \bigcup_{i=1}^{n} \Gamma_i$, where Γ_i are all heteroclinic orbits, so for every $i \in \overline{1n}$ we would have $\omega\Big(\Gamma_i\Big) = \big\{l^+\big\}$. This implies that $\omega\Big(\Gamma\Big) = \big\{l^+\big\}$, a contradiction to $\omega\Big(\Gamma\Big)$ being a graphic.

This leaves the only choice being that $\omega\Big(\Gamma\Big)$ being an equilibrium point, thus implying that $\omega\Big(\Gamma\Big) = \big\{l^+\big\}$. A similar argument show that $\alpha\Big(\Gamma\Big) = \big\{l^-\big\}$.

Since the choice of Γ was arbitrary, we have also shown that all trajectories on A are heteroclinic orbits between the two equilibria on ∂A. $\square$

By lemma 4.15, we have that N_h^+ is densely filled with heteroclinic orbits. By symmetry of solutions $N_h$is densely filled with heteroclinic orbits, see figure 4.8. These facts are stated in the next lemma.

Lemma 4.16. *For $h > \frac{(\beta-2)^2}{4(\beta-1)} > 0$, and $0 < \beta \leq 2$, ∂N_h has two equilibria, l^+ which is a sink, l^- which is a source. Moreover, N_h is densely filled with heteroclinic orbits, between l^+ and l^-.*

The Case when $\beta > 2$ and $0 < h = \frac{(\beta-2)^2}{4(\beta-1)}$.

For $\beta > 2$ and $0 < h = \frac{(\beta-2)^2}{4(\beta-1)}$, if $r_0 = \frac{2(\beta-1)}{\beta-2}$, then $\mathfrak{M}_h$ has two equilibrium points. So N_h^+ has only one equilibrium point when $\rho_0 = \frac{\beta-2}{2(\beta-1)}$. Equation (4.31) gives that this equilibrium point has three degenerate eigenvalues, so no information can be obtained in this way. To determine the nature of these equilibrium points we proceed by defining a function of ρ,

$$\tilde{g}(\rho) = (\beta-1)\rho^2 - (\beta-2)\rho + h. \tag{4.34}$$

The roots of $\tilde{g}(\rho)$ are also the critical values of (4.28). Moreover, if $v = 0$ then $\dot{v} = 2\tilde{g}(\rho)$, and $0 < h = \frac{(\beta-2)^2}{4(\beta-1)}$ implies that $\beta > 1$. Therefore, $\tilde{g}(\rho)$ is a quadratic

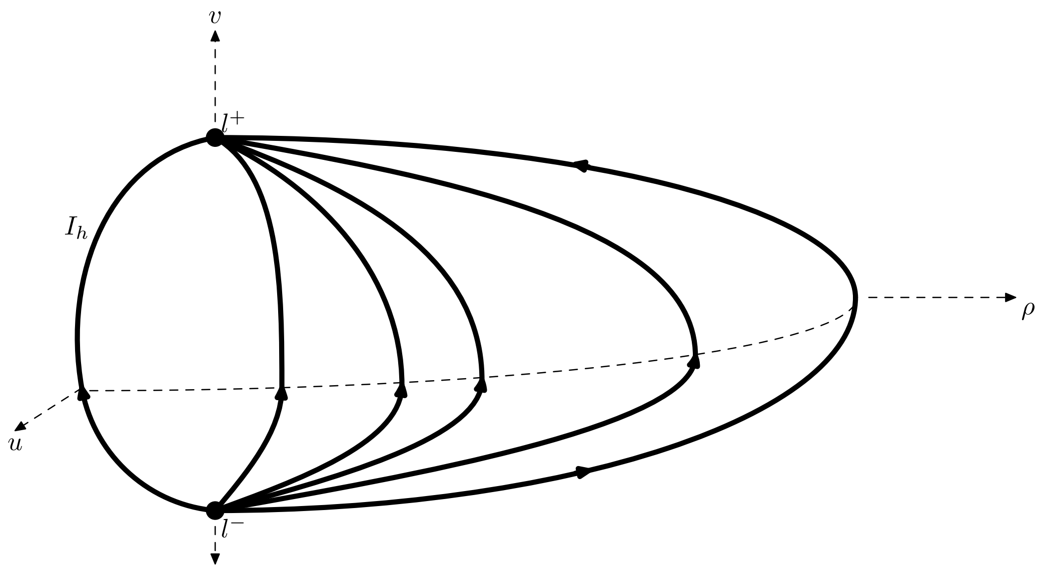

Figure 4.8: Flow on the Near Infinity Manifold, N_h for $h > 0$ and $0 < \beta \leq 2$ or, $\beta > 2$ and $h > \frac{(\beta-2)^2}{4(\beta-1)} > 0$.

that opens up, with one multiple real root that conforms to the restriction given in lemma 4.14. The manifold N_h^+ is a compact, connected, invariant 2-dimensional sub-manifold of $\mathfrak{N}_h$, where $\dot{\rho} < 0$ when $v > 0$ and $\dot{\rho} > 0$ when $v < 0$. Hence, lemma 4.3 has its correlated lemma for the flow on N_h.

Lemma 4.17. *Suppose that N_h has an equilibrium R_0, and if there exists an $\epsilon > 0$ where $\tilde{g}(\rho) < 0$ on $(\rho_0 - \epsilon, \rho_0)$ and $\tilde{g}(\rho) > 0$ on $(\rho_0, \rho_0 + \epsilon)$, then R_0 is a degenerate center. If $\tilde{g}(\rho) > 0$ on $(\rho_0 - \epsilon, \rho_0)$ and $\tilde{g}(\rho) < 0$ on $(\rho_0, \rho_0 + \epsilon)$, then R_0 is a degenerate saddle, and if $\tilde{g}(\rho)$ does not change sign, then R_0 is a cusp.*

The proof of this lemma is identical to the proof of lemma 4.3 and therefore is omitted. Since $\tilde{g}(\rho)$ has a multiple root, then by lemma 4.17 R_0 must be a cusp. This leads to the next lemma which completely determines the flow on N_h.

Lemma 4.18. *If* $0 < h = \frac{(\beta-2)^2}{4(\beta-1)}$*, then* N_h *has two equilibria,* $R_0^{\pm}$*, both cusps. Both* $W^{u,s}\Big(R_0^{\pm}\Big)$ *are heteroclinic orbits, where* $\omega\Big(W^u\Big(R_0^{\pm}\Big)\Big) = \big\{l^+\big\}$ *and* $\alpha\Big(W^s\Big(R_0^{\pm}\Big)\Big) = \big\{l^-\big\}$*. All other orbits are of a heteroclinic between the two equilibria on* I_h*.*

PROOF:

If $v = 0$, then $\dot{v} = \tilde{g}(\rho) \geq 0$, so the sets $\big\{(\rho, v, u) \in N_h^+ | v \leq 0\big\}$ and $\big\{(\rho, v, u) \in N_h | v \geq 0\big\}$ are negatively and positively invariant, respectively. This implies that $W^s\Big(R_0^+\Big) \subseteq \big\{(\rho, v, u) \in N_h^+ | v \leq 0\big\}$ and $W^u\Big(R_0^+\Big) \subseteq \big\{(\rho, v, u) \in N_h^+ | v \geq 0\big\}$. Both of these sets are compact, therefore both the stable and unstable manifolds must tend to a graphic, periodic orbit, or an equilibrium point. The first two choices have been shown to be impossible., so both manifolds must tend to an equilibrium point. By continuity of solutions we must have that $\alpha\Big(W^s\Big(R_0^+\Big)\Big) = \big\{l^-\big\}$ and $\omega\Big(W^u\Big(R_0^+\Big)\Big) = \big\{l^+\big\}$.

Let ϕ_t be any trajectory on N_h that is not on $W^{s,u}\Big(R_0\Big)$., then $\omega\big(\phi_t\big)$ must be an equilibrium point. Clearly it has to be the point l^+. Similarly we get that $\alpha\Big(\phi_t\Big) = \big\{l^-\big\}$, so all orbits are heteroclinic and the lemma is proved. $\square$

We have that $W^s\Big(R_0\Big)$, and $W^s\Big(R_0\Big)$ are heteroclinic orbits that emanates from l^-, and tends to l^+, respectively. All other orbits are heteroclinic orbits between the two equilibria on I_h. Figure 4.9 shows the flow restricted to N_h, when $h = \frac{(\beta-2)^2}{4(\beta-1)}$.

The Case when $\beta > 2$ and $0 < h < \frac{(\beta-2)^2}{4(\beta-1)}$.

Section 4.4.4 showed that there are four equilibria on $\mathfrak{M}_h$, therefore there are four equilibria on $int\Big(N_h^+ \bigcup N_h^-\Big)$. $\tilde{g}(\rho)$ is a quadratic that opens up, so by lemma 4.17 we have that $R_-^{\pm}$ are degenerate saddles and $R_+^{\pm}$ are degenerate centers. Next we will state and prove the lemma that completely determines the flow on N_h.

Lemma 4.19. *For* $\beta > 2$ *and* $0 < h < \frac{(\beta-2)^2}{4(\beta-1)}$*, then* N_h *has two degenerate saddles,* $R_-^{\pm}$ *and two degenerate centers,* $R_+^{\pm}$*, and* $I_h \subset \partial N_h$ *has two equilibria, one*

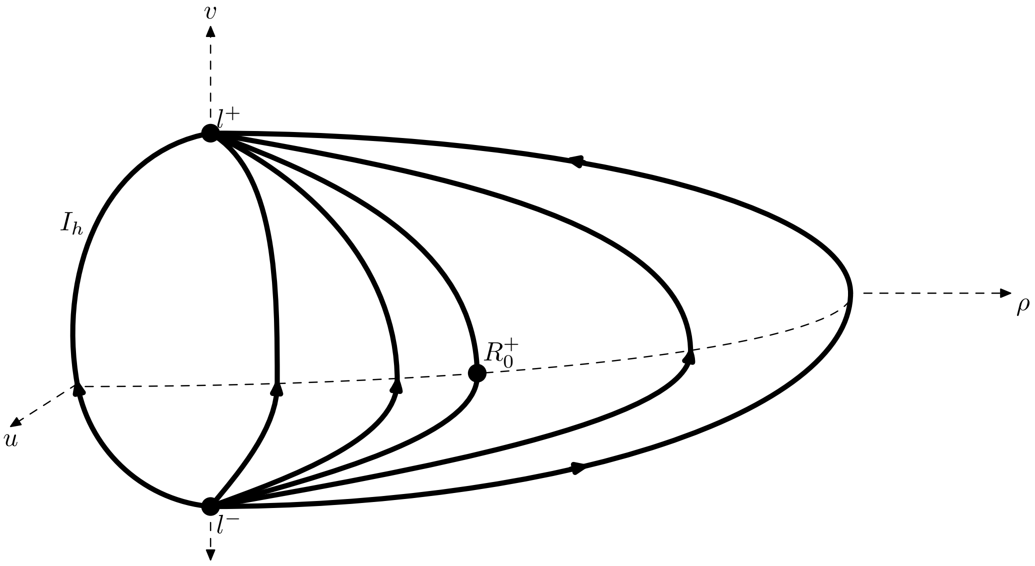

Figure 4.9: Flow on the Near Infinity Manifold, N_h for $\beta > 2$ and $0 < h = \frac{(\beta-2)^2}{4(\beta-1)}$.

source, l^- and one sink, l^+. $W^s\left(R_-^{\pm}\right)$ consists of heteroclinic orbits between $R_-^{\pm}$ and l^- and homoclinic orbits with $R_+^{\pm}$ in its interior. Similarly, $W^u\left(R_-^{\pm}\right)$ consists of heteroclinic orbit between $R_-^{\pm}$ and l^+ and the homoclinic orbits also contained in $W^s\left(R_-^{\pm}\right)$. Furthermore there are sets of initial conditions with positive measure, of periodic orbits, and heteroclinic orbits between the two equilibria on I_h.

PROOF:

We have already shown the existence and nature of the equilibria on N_h, so it is sufficient to show the nature of all other orbits on N_h^+. The roots of $\tilde{g}(\rho)$ are given by,

$$\rho^{\pm} = \frac{\beta - 2 \pm \sqrt{(\beta-2)^2 - 4h(\beta-1)}}{2(\beta-1)}. \tag{4.35}$$

Equation (4.35) gives the eigenvalues of (4.31) as,

$$\lambda^2 = \mp\sqrt{(\beta-2)^2 - 4h(\beta-1)}. \tag{4.36}$$

ρ^+ has two pure imaginary eigenvalues, and ρ^- has two non-zero, real eigenvalues. By $Hartman-Grobmann\ Theorem$, $W^{s,u}\Big(R_-^+\Big)$ both have degree 1 and must be tangential to the associated linear manifolds. Therefore, we have that both the stable and unstable manifolds of R_-^+ must have parts in the positive and negative v quadrants.

The part of $W^u\Big(R_-^+\Big)$, say Γ, that is below the $v = 0$ plane, is an orbit that is bounded above (**wrt** ρ). Clearly we have that there exists a time, t^*, at which Γ touches the $v = 0$ plane. The symmetry of solutions shows that Γ needs to be a homoclinic orbit. Let Γ' be the other section of $W^u\Big(R_-^+\Big)$, then it must be completely contain in the set $\big\{(\rho, v, u) \in N_h^+ | v \geq 0, \rho \leq \rho^-\big\}$, since this set is positive invariant.This set must be a compact subset of N_h^+, therefore $\omega\Big(\Gamma'\Big)$ has to be an equilibrium point. This only leaves the point l^+, which is a sink, so $W^u\Big(R_-^+\Big)$ consist of a homoclinic orbit and a heteroclinic orbit between R_-^+ and l^+. By symmetry of solution we have that $W^s\Big(R_-^+\Big)$ is a homoclinic orbit about R_-^+, and a heteroclinic orbit between R_-^+ and l^-.

In order to see that R_+^+ is in the interior of the homoclinic orbit, we note that if $v = 0$ then $\dot{v} = 2\tilde{g}(\rho)$ where on (ρ^-, ρ^+) then $\tilde{g}(\rho) < 0$. The rest follows trivially. □

The flow on N_h for $\beta > 2$ and $0 < h < \frac{(\beta-2)^2}{4(\beta-1)}$ can be seen in figure 4.10.

4.7 The Global Flow for Zero Energy.

4.7.1 The Case when $0 < \beta \leq 2$.

If $h = 0$ then section 4.5.1 showed that I_0 consisted of one point, $l_0 = (0, 0, 0)$. Moreover, in section 4.3.1 we saw that $\mathfrak{M}_h$ has no equilibria, therefore l_0 is the only equilibria of N_0^+.Therefore, N_0^+ is a compact, connected, invariant sub-manifold of a 2 dimensional manifold with only one equilibrium on the boundary, ∂N_0. To provide the overall flow on N_0 we need the next lemma.

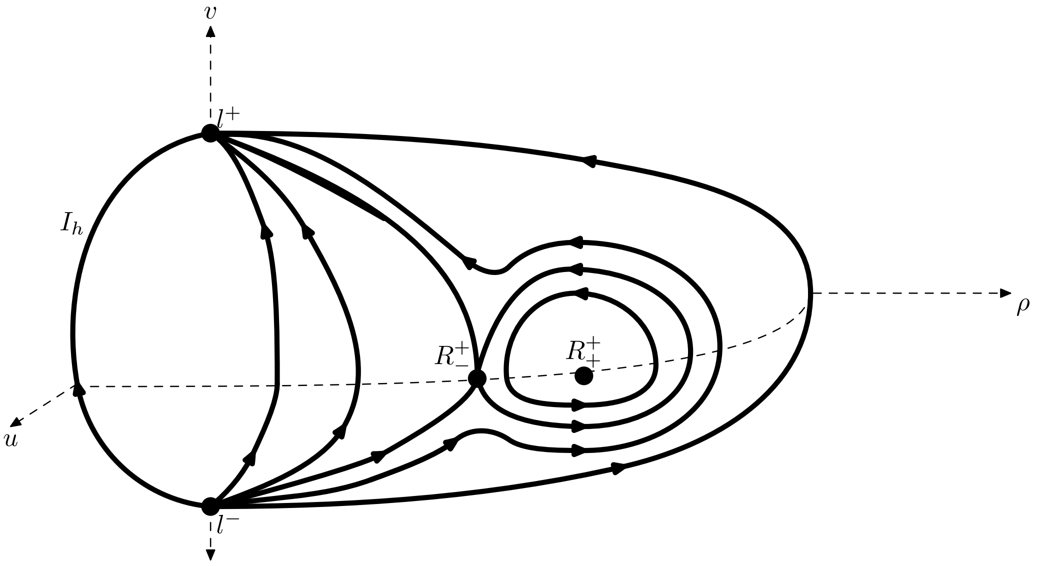

Figure 4.10: Flow on the Near Infinity Manifold, N_h for $\beta > 2$ and $0 < h < \frac{(\beta-2)^2}{4(\beta-1)}$.

Lemma 4.20. *Let A be a compact, connected, 2 dimensional manifold, where ∂A has one equilibria, R_0, and no other equilibria on $int\Big(A\Big)$. Then, for any orbit on A, say Γ, we have that $\omega\Big(\Gamma\Big) = \alpha\Big(\Gamma\Big) = \Big\{R_0\Big\}$. Moreover, A is densely filled with homoclinic orbits emanating from R_0.*

<u>PROOF:</u>

Let Γ be any trajectory on A, where A is a compact, connected, 2 dimensional, invariant manifold, then *Poincaré−Bendixson Theorem*, states that $\omega\Big(\Gamma\Big)$ is either an equilibrium, a periodic orbit, or a graphic.

If $\omega\Big(\Gamma\Big)$ is a periodic orbit, then $int\Big(A\Big)$ would need to contain a fixed point, but this contradicts the assumption that A has no such equilibrium.

If $\omega\Big(\Gamma\Big)$ is a graphic, then it must be a finite collection of homoclinic orbits, otherwise there would need to be a second equilibrium point on A, which contradicts

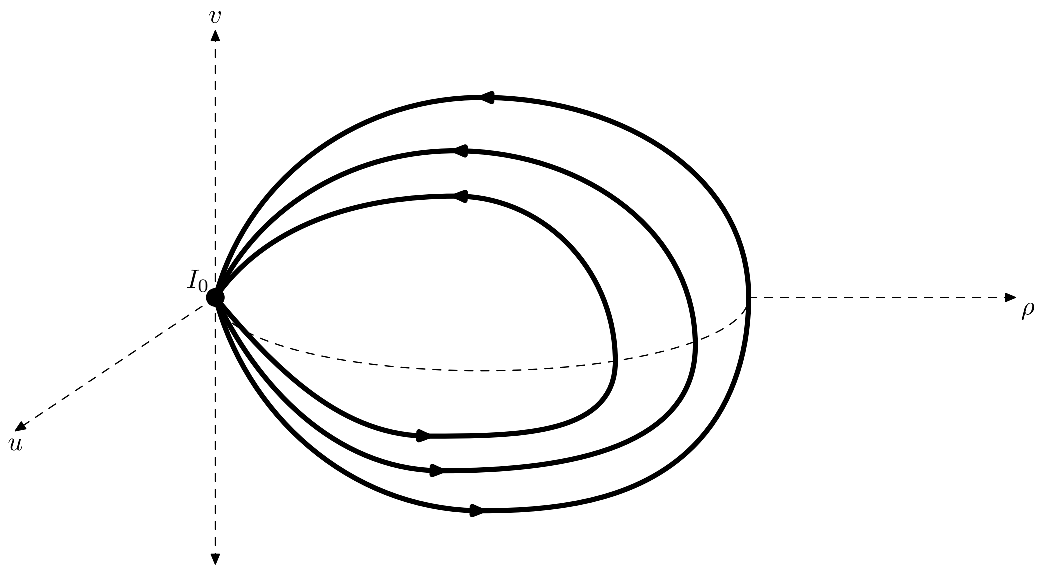

Figure 4.11: Flow on N_0 and $0 < \beta \leq 2$.

our assumptions. Therefore, $\omega\Big(\Gamma\Big) = \bigcup_{i=1}^{n} \Gamma_i$, where Γ_i are homoclinic orbits, thus for every $i \in \overline{1n}$, we have $\omega\Big(\Gamma_i\Big) = \big\{R_0\big\}$. There are only two possible out comes of this, $\omega\Big(\Gamma\Big) = \big\{R_0\big\}$, or Γ cycles around inside a homoclinic orbit. The latter implies the existence of a second equilibrium point, and the former indicates that $\omega\Big(\Gamma\Big)$ is not a graphic. Both of these facts contradict our assumption.

Therefore we have that $\omega\Big(\Gamma\Big) = \big\{R_0\big\}$, and A must be densely filled with homoclinic orbits emanating from R_0. $\square$

The manifold, N_0^+, conforms to the hypothesis of lemma 4.20, and therefore is densely filled with heteroclinic orbits, see figure 4.11.

Looking at the construction of the manifolds $\mathfrak{M}_h$ for $-1 \leq h < 0$, and $\mathfrak{N}_h$ for $0 < h$, and using the continuity of solutions, we can study the structure of $\mathfrak{N}_0$. Taking $h \to 0^-$, on $\mathfrak{M}_h$ and $h \to 0^+$ on N_h, we can see where the equilibrium points and the boundary values go.

For $-1 < h < 0$, we see that $\mathfrak{M}_h$ is homotopic to a 2 sphere, with r boundary points $\left[\frac{-1+\sqrt{1+h}}{h}, \frac{-1-\sqrt{1+h}}{h}\right]$. Note that

$$\lim_{h\to 0^-} \frac{-1-\sqrt{1+h}}{h} = \infty,$$

$$\lim_{h\to 0^-} \frac{-1+\sqrt{1+h}}{h} = \frac{1}{2},$$

so the r boundary of $\mathfrak{M}_h$ smoothly approaches, from the left, the boundary of N_0. Moreover, we have that,

$$\lim_{h\to 0} r^- = \infty, \tag{4.37}$$

so the only equilibria of $\mathfrak{M}_h^+$ approaches the only equilibrium of ∂N_0. The equilibrium points of $\mathfrak{M}_h$ go from being centers into being saddles that emanates homoclinic orbits.

The manifold, N_h for $h > 0$, has two equilibria on I_h, where N_h is densely filled with heteroclinic orbits between these two equilibria. $I_h \to (\infty, 0, 0)$ as $h \to 0^+$ and

$$\lim_{h\to 0} \frac{-1+\sqrt{1+h}}{h} = \frac{1}{2}, \tag{4.38}$$

so the two equilibria on ∂N_h merge into I_0. Moreover, the r boundary approaches $\left[\frac{1}{2}, \infty\right)$. Therefore, the phase spaces of $\mathfrak{M}_h$, $h < 0$, and of N_h, $h > 0$, both tend smoothly to the phase space of N_0. The global flow on $\mathfrak{M}_h$ is now completely known and can be stated in the next lemma.

Lemma 4.21. *For $0 < \beta \leq 2$, then all orbits on $\mathfrak{N}_0$ are heteroclinic and emanate from the infinity equilibrium point, I_0.*

4.7.2 The Case when $\beta > 2$.

If $h = 0$ and $\beta > 2$, then $\mathfrak{M}_h^+$ has only one equilibrium point, so N_0^+ has one non-infinity equilibrium point associated with $\rho_0 = \frac{\beta-2}{\beta-1}$. The eigenvalues given by (4.36) become, $\lambda^2 = \frac{2\beta(\beta-2)^2}{1-\beta} < 0$, so by lemma 4.17, we get R_0^+ is a degenerate center. Note that for $\beta > 2$, so we have

$$\lim_{h \to 0^-} \rho^- = \frac{\beta - 2}{\beta - 1},$$

where (4.37) and (4.38), show that the boundaries of M_h tend to the boundaries of N_0. Furthermore, the two equilibrium points on $\mathfrak{M}_h$, $-1 < h < 0$, tend to the one equilibrium point on ∂N_0. The flow on $\mathfrak{M}_h$ also tends smoothly to the flow on N_0.

The flow on N_h, for $h > 0$, can be seen to tend to N_0 in the same manner, where the two equilibria on I_h, together with $R_-^{\pm}$, merge into the equilibrium point I_0. The two degenerate centers on $\mathfrak{N}_h$ both remain on the interior of N_0. Since $\mathfrak{M}_h$ is filled with periodic orbits and the interior of the heteroclinic orbits on $\mathfrak{N}_h$ are densely filled with periodic orbits, all which all remain on N_0, then there is a unique capture-escape orbit, $\phi_t = (r(t), v(t), 0)$ on $\mathfrak{N}_0$. This orbit is a homoclinic orbit emanating from I_0, so we have the complete flow on N_0, and is stated in the next lemma. The flow can be seen in figure 4.12.

Lemma 4.22. *If $\beta > 2$, then $\mathfrak{N}_0$ is densely filled with periodic orbits. Moreover, there exists a unique escape-capture orbit, $\phi_t = (r(t), v(t), 0)$ which is a heteroclinic orbit emanating for the infinity equilibrium point I_0.*

4.8 The Global Flow.

The global flow for the **GLJ** can be seen in the diagrams where the manifolds are compact. This means that for negative energy the manifolds $\mathfrak{M}_h$ can be used to represent the global flow for the system and for non-negative energy, we can use the near infinity manifolds, $\mathfrak{N}_h$.

For non-negative energy we will represent the flow by using the infinite energy manifold, I_h, to be the right hand limit, so all the phase flow in these cases will

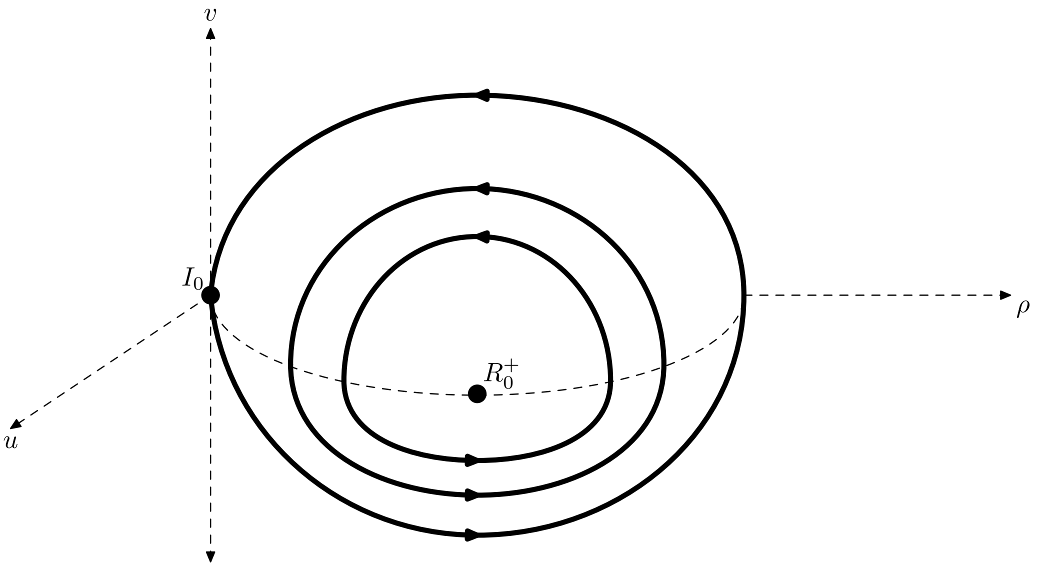

Figure 4.12: Flow on N_0 and $\beta > 2$.

be determined off the diagrams of N_h. The flow will be the mirror image of N_h, where $\rho = 0$ will correspond with $r = \infty$. One must be cautious when looking at these diagrams, in particular, do not enter into the belief that the u and v values are bounded, quite the contrary. For non-negative energy, the v and u values can grow without bound. Explicitly, the orbit $\phi_t = (r(t), v(t), 0)$ has that $v(t) \to \pm\infty$ as $t \to \pm\infty$. We have seen that $\beta = 2$ is a bifurcation value of the system (4.5), so we will deal with these cases separately.

4.8.1 The Global Flow for $0 < \beta \leq 2$.

The Global Flow for Negative Energy and $0 < \beta$.

For negative energy, there is no bifurcation for the flow, and we will give the general theorem for the flow with negative energy for all $\beta > 0$. Lemma 4.2 gives the following theorem.

Theorem 4.23. *(The Global Flow for Negative Energy for $\beta > 0$)*
For $-1 \leq h < 0$, there are two possible cases for the motion for the generalized Lennard-Jones 2-body problem:

1. *If $h = -1$, then there is no motion, and there is a collection of degenerate equilibria at $(1, \theta_0, 0, 0)$, where $\theta_0 \in S^1$.*
2. *If $-1 < h < 0$, then there are two degenerate centers, $R_0^{\pm}$, and all other motion is periodic. The flow can be seen as in figures 4.13 and 4.16.*

Recall that the flow restricted to $\mathfrak{M}_h$ had $\theta \in S^1$ factored out, so the equilibrium points on $\mathfrak{M}_h$, $-1 < h < 0$, are in fact periodic orbits of the system (4.5). Moreover, the periodic orbits are invariant torii of the overall system. In particular, the torus associated with the periodic orbit, $\phi_t = (r(t), v(t), 0)$, is densely filled with periodic orbits, since $\dot{\theta} = u = 0$ in equations (4.5). The reader should keep these constructions in mind when viewing the restricted phase portraits. The global flow, **wrt** (4.5), consists of two collections of nested torii, in $\mathbb{R}^4$. Both collections of torii are bounded above by the torus $\phi_t \bigotimes S^1$, where $\phi_t = (r(t), v(t), 0)) \in \mathfrak{M}_h$.

Finally, $\mathfrak{M}_{-1} \bigotimes S^1$ is a collection of degenerate equilibria of the overall flow of (4.5). By the continuity of the flow one can see that these equilibria are Lyapunov stable (see page 66 [14]).

The Global Flow for Zero Energy and $0 < \beta \leq 2$.

For zero energy, we have that I_0 consists of a point at $r = \infty$, where this equilibrium point was seen to be a degenerate topological saddle. Lemma 4.21, gives the following theorem about the global flow for zero energy when $0 < \beta \leq 2$.

Theorem 4.24. *(The Global Flow for Zero Energy and $0 < \beta \leq 2$)*
For $h = 0$ and $0 < \beta \leq 2$, then I_0 consists of the fictitious point at $(\infty, 0, 0)$, which

is a degenerate topological saddle. All finite orbits are capture-escape, heteroclinic orbits, that start at I_0 *attain a* $r_{min} > 0$ *value and then tend back to* I_0*. See figure 4.14.*

Since $\theta \in S^1$, then each heteroclinic orbit generates an invariant torus that meets in the center, at $I_0 \bigotimes S^1$, so the associated flow for $h = 0$ is two collections of imbedded torii in $\mathbb{R}^4$. The circle $I_0 \bigotimes S^1$ is the inner circle of all the torii, and consists of a collection of degenerate equilibria. The flow on each of the torii spin out from $I_0 \bigotimes S^1$, (counter-clockwise for $u > 0$, and clockwise for $u < 0$, no spin for $u = 0$) then spin back towards $I_0 \bigotimes S^1$.

The Global Flow for Positive Energy and $0 < \beta \leq 2$.

For positive energy and $0 < \beta \leq 2$, we have that all non-infinity orbits are capture-escape orbits. Moreover, I_h consists of the circle $v^2 + u^2 = \sqrt{2h}$ centered at the fictitious point $r = \infty$. Figure 4.8 shows the global flow for positive energy where $0 < \beta \leq 2$, and we have the following theorem.

Theorem 4.25. *(The Global Flow for Positive Energy and* $0 < \beta \leq 2$*)*
For $h > 0$ *and* $0 < \beta \leq 2$ *, then* I_h *is the circle* $v^2 + u^2 = 2h$ *centered at* $r = \infty$*, consisting of two heteroclinic orbits between* l^+ *and* l^-*. The energy manifolds are densely filled with heteroclinic orbits between these two equilibria, and all non-infinity orbits attain an* r_{min} *value. See figure 4.15.*

For each $h > 0$, $I_h \bigotimes S^1$ is a torus in the phase space of (4.5), where every non-infinity orbit on $\mathfrak{M}_h$, generates a surface that intersects this torus. This surface can be constructed by starting with a torus and removing the inner circle and replacing it with the torus $I_h \bigotimes S^1$. Furthermore, as above, the orbits on these torii spin out from the collection of degenerate equilibria, $l^- \bigotimes S^1$, attain a r_{min} value, then spin back towards $l^+ \bigotimes S^1$.

4.8.2 The Global Flow for $\beta > 2$.

The Global Flow for Negative Energy and $\beta > 2$.

For $-1 \leq h < 0$, and $\beta > 2$ the global flow is the same as when $0 < \beta \leq 2$. Therefore the manifold $\mathfrak{M}_h$ is densely filled with periodic orbits and r^- tends to the equilibrium of N_0 as $h \to 0^-$. The upper boundary point of $\mathfrak{M}_h$ tends to ∞, so we can use theorem 4.23. The flow can be seen as in figure 4.16.

The Global Flow for Zero Energy and $\beta > 2$.

For zero energy, and $\beta > 2$,lemma 4.22 gives us the following theorem for the global flow on N_0.

Theorem 4.26. *(The Global Flow for Zero Energy and $\beta > 2$)*
For zero energy, and $\beta > 2$, then N_0 has three equilibrium points, $R_0^\pm$ both degenerate centers and I_0 a degenerate saddle. N_0 is densely filled with periodic orbits, and there is one capture-escape orbit, $\phi_t = (r(t), v(t), 0)$. See figure 4.17.

The general flow with respect to (4.5) can be seen to be two collections of nested torii in $\mathbb{R}^4$, where the torii are bounded above by the torus associated with $u = 0$. This torus has an inner circle consisting of a collection of degenerate equilibria on $I_0 \bigotimes S^1$. The flow on these torii spiral out from $I_0 \bigotimes S^1$ and then spiral back towards $I_0 \bigotimes S^1$. Therefore, each of these torii is densely filled with heteroclinic orbits.

The Global Flow for Positive Energy and $\beta > 2$.

For positive energy we have $I_h \equiv S^1$, with equilibria $l^\pm$, so the infinity manifold becomes the torus $I_h \bigotimes S^1$, and is pasted on the left side of the phase space. The flow on N_h was divided into three cases, so we get the following theorem.

Theorem 4.27. *(The Global Flow for Positive Energy and $\beta > 2$)*
If $h > 0$ then I_h consist of two heteroclinic orbits between the two fictitious points

$(\infty, \pm\sqrt{2h}, 0)$, *where* $\beta > 2$ *divides into three cases as* h *moves in the positive direction.*

1. *If* $0 < h < \frac{(\beta-2)^2}{4(\beta-1)}$, *then there are two degenerate centers,* $R_0^{\pm}$ *and two degenerate saddles,* $R_1^{\pm}$. $W^u\left(R_1^{\pm}\right)$ *consists of two heteroclinic obits between* $R_1^{\pm}$ *and* l^+ *and two homoclinic orbits, where* $R_0^{\pm}$ *are in the interiors of each of these orbits, respectively. All other orbits are capture-escape, heteroclinic orbits between the equilibria on* I_h. *See figure 4.18.*

2. *If* $0 < h = \frac{(\beta-2)^2}{4(\beta-1)}$, *then there are two equilibria,* $R_0^{\pm}$, *both cusps, and the energy manifold is densely filled with capture-escape, heteroclinic orbits. Moreover,* $W^u\left(R_0^{\pm}\right)$ *are heteroclinic orbits between* $R_0^{\pm}$ *and* l^+, *and* $W^s\left(R_0^{\pm}\right)$ *are heteroclinic orbits between* $R_0^{\pm}$ *and* l^-. *See figure 4.19.*

3. *If* $0 < \frac{(\beta-2)^2}{4(\beta-1)} < h$, *then are no non-infinity equilibria, where the energy manifolds are densely filled with capture-escape, heteroclinic orbits between the two equilibria on* I_h. *See figure 4.20.*

4.8.3 The Global Flow for $\beta > 2$ and $0 < h < \frac{(\beta-2)^2}{4(\beta-1)}$.

In the above section we saw how the periodic orbits, fixed point and escape-capture orbits are associated with the overall flow of (4.5). In the case where $0 < h$, I_h becomes a torus in $\mathbb{R}^4$, where the top circle corresponds to the collection of degenerate equilibria $l^+ \bigotimes S^1$ and the bottom circle corresponds to the collection of degenerate equilibria $l^- \bigotimes S^1$.

Each escape-capture orbit on N_h corresponds to a 2 dimensional invariant submanifold that is homotopic to an annulus. The outer circle and inner circle are identified with the collections of degenerate equilibria, $l^+ \bigotimes S^1$ and $l^- \bigotimes S^1$, respectively. The flow on this annulus spirals out from $l^- \bigotimes S^1$, attains an $r_{min} > 0$ value, then spirals back towards $l^+ \bigotimes S^1$ ($u = 0$, there is no spin, see section 4.8.1).

The manifolds $W^{u,s}\big(R_1^+\big)\backslash W^{s,u}\big(R_1^+\big)$ generates two surfaces that are homotopic to annuli. On $W^s\big(R_1^\pm\big)\backslash W^u\big(R_1^\pm\big)$, the inner and outer circles correspond with $l^- \bigotimes S^1$ and $R_1^+ \bigotimes S^1$, respectively. The flow spirals out from $l^- \bigotimes S^1$ towards $R_1^+ \bigotimes S^1$, where we have similar constructions for the sub-manifold $W^u\big(R_1^\pm\big)\backslash W^s\big(R_1^\pm\big)$.

The sub-manifold $W^{u,s}\big(R_1^\pm\big) \bigcap W^{s,u}\big(R_1^\pm\big)$ generates two surfaces that are homotopic to a torii, where the inner circles of the torii are identified with $R_1^\pm \bigotimes S^1$. The flow on $\Big(W^{u,s}\big(R_1^\pm\big) \bigcap W^{u,s}\big(R_1^\pm\big)\Big) \bigotimes S^1$ spirals out from $R_1^\pm \bigotimes S^1$, attains an r_{min} value then spirals back towards $R_1^\pm \bigotimes S^1$.

$\Big(W^{u,s}\big(R_1^\pm\big) \bigcap W^{u,s}\big(R_1^\pm\big)\Big) \bigotimes S^1$ is homotopic to a torus, where the flow spirals out from $R_1^\pm$, attains an r_{min} value ($0 < r_{min} < r_1$), then spirals back towards $R_1^\pm$. The interior of $\Big(W^{u,s}\big(R_1^\pm\big) \bigcap W^{u,s}\big(R_1^\pm\big)\Big) \bigotimes S^1$, consists of nested, invariant torii that are associated with the set of positive measure of periodic orbits about $R_0^\pm$. The center of these torii is the periodic orbits $R_0^\pm \bigotimes S^1$.

4.8.4 The Global Flow for $\beta > 2$ and $0 < h = \frac{(\beta-2)^2}{4(\beta-1)}$.

The above constructions show that the global flow of (4.5) can be seen as two collections of nested torii, where the inner circle is removed and the infinity manifold, $I_h \bigotimes S^1$ is pasted in it place. The manifolds that correspond to the stable and unstable manifolds of $R_0^\pm$, namely $W^{s,u}\big(R_0^\pm\big) \bigotimes S^1$, are homotopic to annuli. Where the flow spirals out from $l^- \bigotimes S^1$ towards $R_0^\pm \bigotimes S^1$ on $W^s\big(R_0^\pm\big) \bigotimes S^1$. Similarly for the flow on $W^u\big(R_0^\pm\big) \bigotimes S^1$, however the flow spirals from $R_0^\pm \bigotimes S^1$ towards $l^+ \bigotimes S^1$.

Each capture-escape orbit also correspond to a surface that is homotopic to an annulus. The orbits on the annulus spiral out from $l^- \bigotimes S^1$, then spiral back towards $l^+ \bigotimes S^1$ (for $u = 0$ there is no spin).

4.8.5 The Global Flow for $\beta > 2$ and $0 < \frac{(\beta-2)^2}{4(\beta-1)} < h$.

$W^s\big(R^{\pm}_{+}\big)\backslash W^u\big(R^{\pm}_{+}\big)$ consists of a heteroclinic orbit between $R^{\pm}_{+}$ and l^-. This orbit corresponds to a surface that is equivalent to an annulus, where the outer boundary, is associated with the periodic orbit $R^{\pm}_{+} \bigotimes S^1$, and the inner boundary is associated with $l^- \bigotimes S^1$. Furthermore, all orbits spiral (clockwise for R^{-}_{+}, counterclockwise for R^{+}_{+}) from $l^- \bigotimes S^1$, out towards $R^{\pm}_{+} \bigotimes S^1$. Similar constructions can be made for the flow associated with $W^u\big(R^{\pm}_{+}\big)\backslash W^s\big(R^{\pm}_{+}\big)$.

Each capture-escape orbit also correspond to a surface that is homotopic to an annulus. The orbits on the annulus spiral out from $l^- \bigotimes S^1$, then spiral back towards $l^+ \bigotimes S^1$ (for $u = 0$ there is no spin).

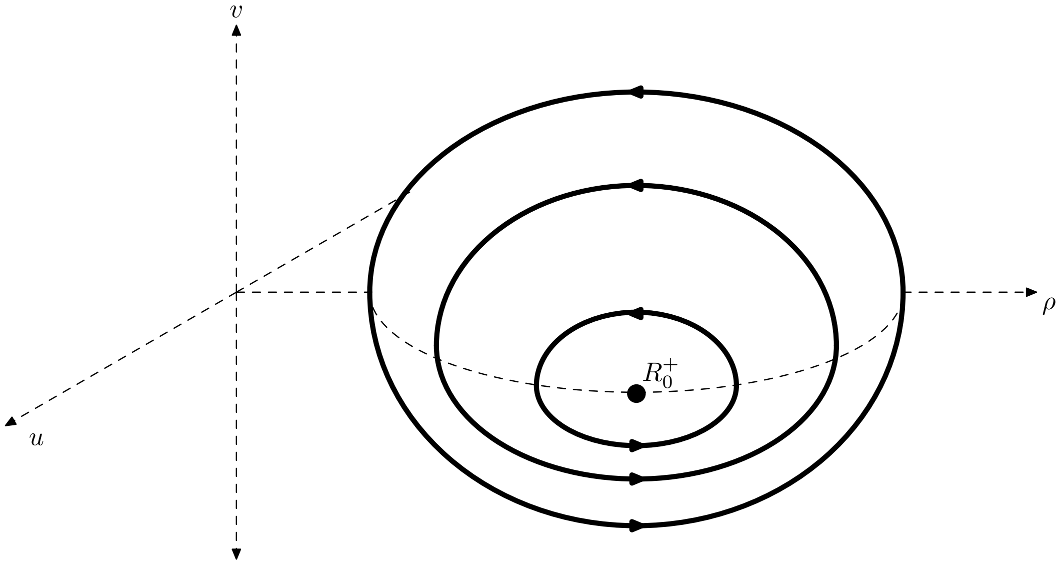

Figure 4.13: Global Flow for $-1 < h < 0$ and $0 < \beta \leq 2$.

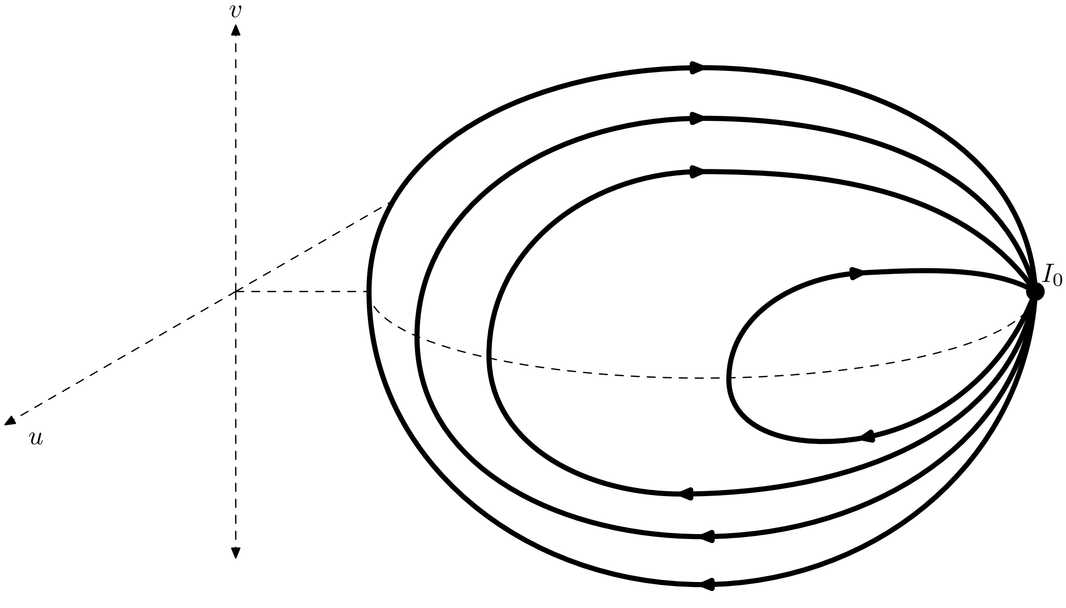

Figure 4.14: Global Flow for $h = 0$ and $0 < \beta \leq 2$.

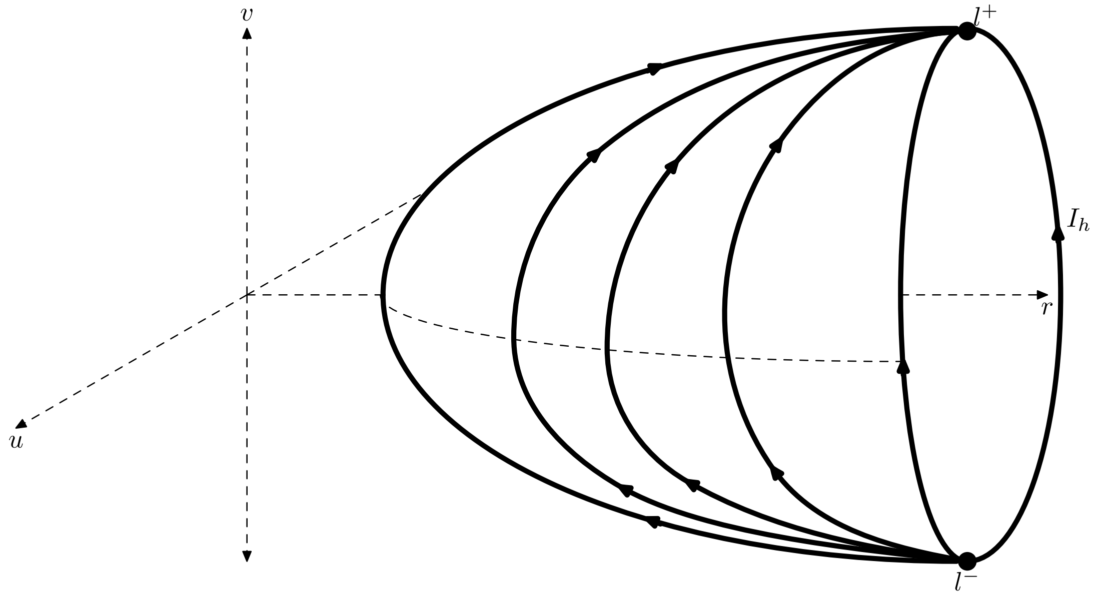

Figure 4.15: Global Flow for $h > 0$ and $0 < \beta \leq 2$.

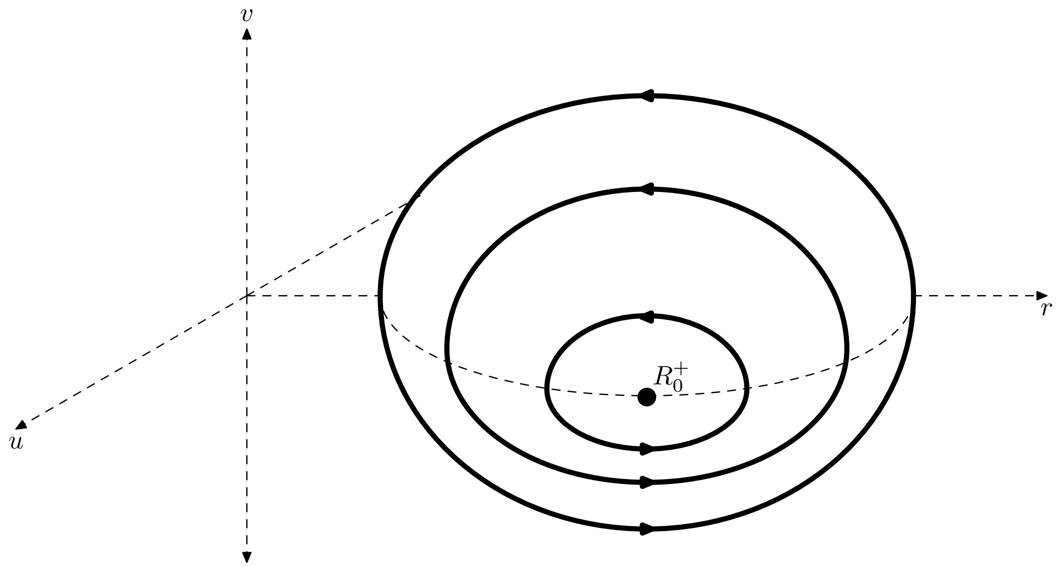

Figure 4.16: Global Flow for $-1 < h < 0$ and $\beta > 2$.

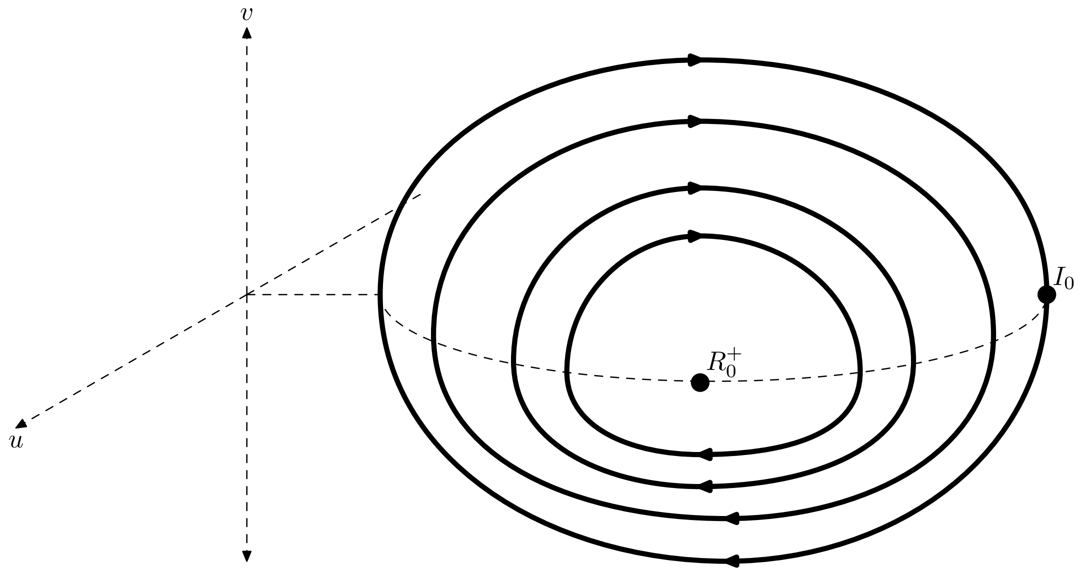

Figure 4.17: Global Flow for $\beta > 2$ and $h = 0$.

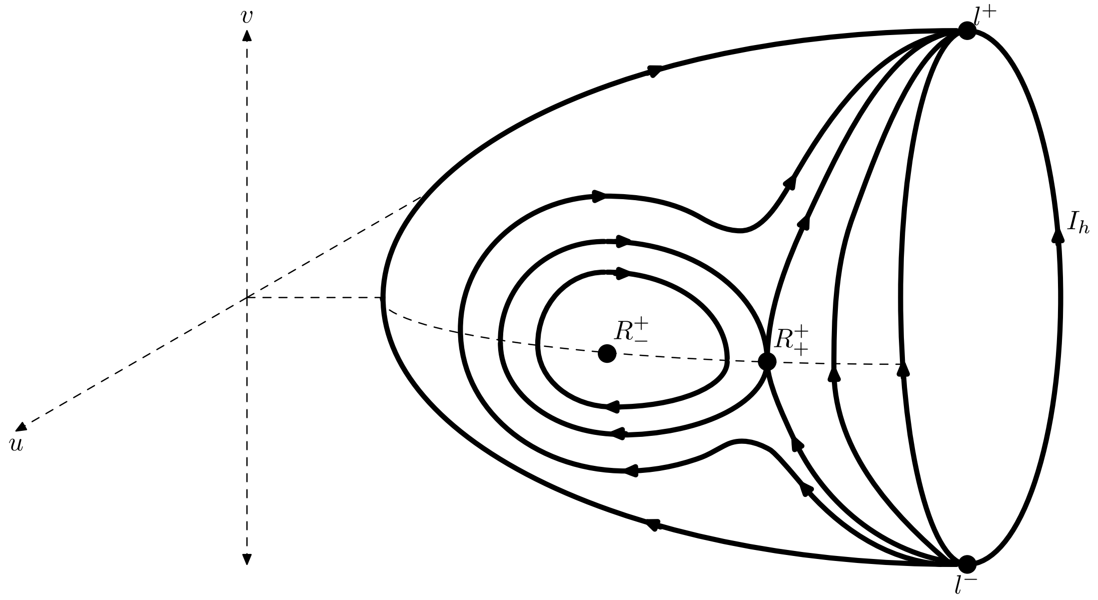

Figure 4.18: Global Flow for $\beta > 2$ and $0 < h < \frac{(\beta-2)^2}{4(\beta-1)}$.

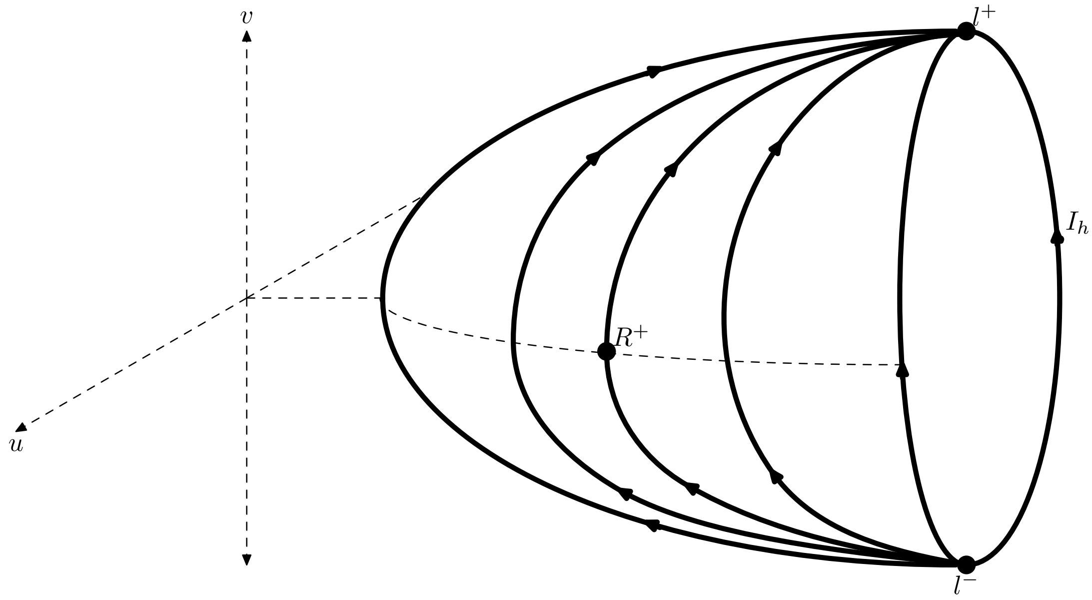

Figure 4.19: Global Flow for $\beta > 2$ and $0 < h = \frac{(\beta-2)^2}{4(\beta-1)}$.

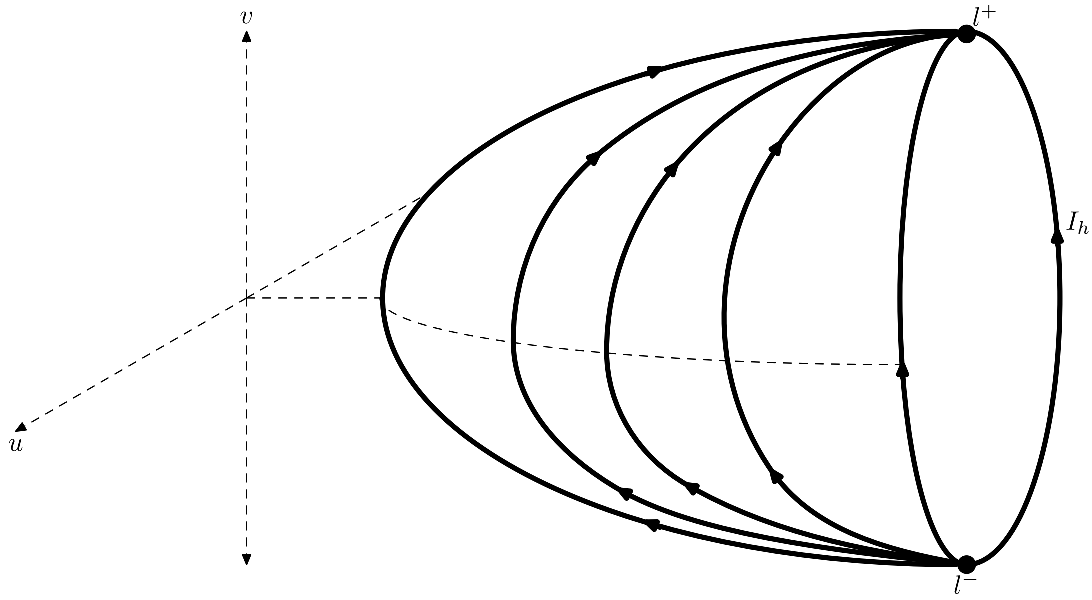

Figure 4.20: Global Flow for $\beta > 2$ and $0 < \frac{(\beta-2)^2}{4(\beta-1)} < h$.

Chapter 5

Conclusions and Discussions

5.1 Conclusions and Discussions

This thesis started by generalizing the study of a quasihomogeneous N-body problem, where we saw that the properties for solutions is dramatically different that for homogeneous potentials. In particular, when restricted to the repulsive-attractive case, we saw that there can be no collisions or pseudo-collisions, theorem 2.11. This means that the potential function is analytic on each of the invariant energy manifolds $\mathfrak{I}_h = \big\{(\mathbf{q}, \mathbf{p}) | T(\mathbf{p}) - U(\mathbf{q}) = h\big\}$. It can be easily seen in the proof of theorem 2.11 that this property will remain for a repulsive-repulsive system. However, there will be collisions for a attractive-repulsive and attractive-attractive systems. This is due to the fact that the requirement that the total energy becomes unbounded is removed. Therefore, the study of all quasihomogeneous potentials will have to take into account the bifurcation values of $a = 0$ and $b = 0$.

In the next chapter we studied the central configurations of a repulsive-attractive N-body problem. The purpose of this was to find fixed points and periodic orbits. The main result in this chapter was theorem 3.8, where we found a special class of homothetic periodic orbits associated with the non-extraneous **CC**. The implications in chemistry is that these periodic orbits correspond to molecules that vibrate with a frequency that is dependent on the periodicity, and the fixed configurations correspond to crystal structures. This theorem seems to have an easy extension to a complete theorem for any quasihomogeneous potential function, which will be done in a subsequent article. The next result is the existence of rotational periodic orbits given by the regular polygon configuration, where the possibility of a generalization for any quasihomogeneous potential is evident. The cases with the attraction being the dominate force may not have a lower bound on the radius and the repulsive-repulsive case may have no rotational periodic orbits. The final, and greatest possibility exposed in this thesis is the connection between simultaneous and extraneous configurations. We saw in lemma 3.5 that $\mathbf{CC}_s \subseteq \mathbf{CC} \backslash \mathbf{CC}_e$, so if we could show that $\mathbf{CC} \backslash \mathbf{CC}_e \subseteq \mathbf{CC}_s$, and that $\mathbf{CC}_e$is nowhere dense, then the set of all central configurations of a quasihomogeneous N-body problem would be nowhere dense. However, it seems that for now the reverse inclusion is not such a trivial question, yet I hope to achieve both of these results in the near future.

Finally we obtained the complete flow for a generalized Lennard-Jones 2-body problem. One notable connection to the previous chapters is that the rectilinear homothetic solutions guaranteed by theorem 3.8 are associated with the invariant manifolds $\mathfrak{U}_h = \left\{(r, v, u) \in \mathfrak{M}_h \middle| u = 0\right\}$. Moreover, for non-negative energy, these invariant manifolds are the unbounded rectilinear homothetic solutions also given by theorem 3.8. A final comment on chapter 4 is that the methods used in obtaining the global flow for the generalized 2-body problem could also work in obtaining the global flow for any quasihomogeneous 2-body problem. This will be done in a subsequent article, where the cases with the attraction being the dominate term will have to deal with collisions by block regularization [7], [13]. Stoica has already achieved the global flow for the attractive-attractive case [13], so we can use the same methods for the attractive-repulsive case. One can see that the complete analysis of the global flow for a quasihomogeneous 2-body problem will be quite lengthy, since it will have to deal with the bifurcation values of $a = 0$ and $b = 0$, yet the analysis should be straight forward.

Appendix A

The Regular N-gon Configuration

It has been known for some time that the configuration of placing N equal mass at the vertices of a regular N-gon is a central configuration of the classical N-body problem [11]. Now it will be shown that this configuration is a **CC** for any potential functions of the form

$$W(\mathbf{q}) = \sum_{1\leq i<j\leq N-1} m_i m_j \left(\frac{a}{|\mathbf{q}_i - \mathbf{q}_j|^{\alpha}} + \frac{b}{|\mathbf{q}_i - \mathbf{q}_j|^{\beta}} \right), \tag{A.1}$$

where a, b are real numbers and $0 < \beta < \alpha$. The case for the generalized Lennard-Jones problem will be a corollary of this theorem. The theorem is stated as follows.

Theorem A.1. *The configuration of placing N-point masses at the vertices of a regular N-gon, is a central configuration for the system (2.3), where the potential function is of the form (A.1).*

PROOF:
The equations of motion from Chapter 2 give the relation

$$\dot{\mathbf{p}}_{\mathbf{j}} = m_j \sum_{k=0\ \ k\neq j}^{N-1} m_k \Big(\alpha' |q_{jk}|^{-\alpha-2} + \beta' |q_{jk}|^{-\beta-2} \Big),$$

where $\alpha' = -a\alpha$ and $\beta' = -b\beta$.

wlog the system can be configured by ordering the N vertices in a clockwise manner, starting with $k = 0$ and ending with $k = N - 1$. Furthermore, rotate the coordinate system so that $\mathbf{q}_0$ is on the x-axis. So the individual coordinates can be seen tho have the form;

$$\mathbf{q}_k = re^{\frac{2k\pi}{N}i} \qquad \forall k \in \overline{0N-1}$$

where $r = \left|\mathbf{q}_0\right|$

We will use some interesting properties which emerge from the symmetries of the regular N-gon, namely:

(i) $\left|q_j\right| = \left|q_k\right| \qquad \forall\, j\; k \in \overline{0N-1}$,

(ii) $q_{(j+k)\mathrm{mod(N)}} + q_{(j-k)\mathrm{mod(N)}} = 2\cos\left(\frac{2k\pi}{N}\right)q_j$
where $2\cos\left(\frac{2k\pi}{N}\right)$ is independent of j,

(iii) $\left|q_{0k}\right| = \left|q_{j\left((j+k)\mathrm{mod(N)}\right)}\right| = \left|q_{j\left((j-k)\mathrm{mod(N)}\right)}\right| \qquad \forall j, k \in \overline{0N-1}.$

(i) is clearly true, since the center of our system has been taken to be the center of the regular N-gon. (ii) can be seen from the simple identity that

$$e^{i\theta_{j\pm k}} = e^{i\frac{2(j\pm k)\pi}{N}} = e^{i\frac{2j\pi}{N}}e^{i\frac{\pm 2k\pi}{N}},$$

and that

$$e^{\gamma i} + e^{-\gamma i} = 2\cos\gamma.$$

Taking all the masses to be equal , then we may assume that $m_j = 1 \;\; \forall j \in \overline{0N-1}$. So by (A.1) we have,

$$\partial_j W(\mathbf{q}) = \sum_{k=0,k\neq j}^{N-1} q_{jk}\Big(\alpha'\left|q_{jk}\right|^{-\alpha-2} + \beta'\left|q_{jk}\right|^{-\beta-2}\Big).$$

Furthermore, defining

$$\lfloor \tfrac{n}{2} \rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$$\lceil \tfrac{n}{2} \rceil = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases} \quad \text{and;}$$

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

the problem divides into two cases:

Case I If $i \le \lfloor \frac{n-1}{2} \rfloor \quad \Rightarrow \quad i+1 \le \lfloor \frac{n-1}{2} \rfloor + i \le n-1$ for $n \ge 3$

We have,

$$\begin{aligned}
\partial_i W(\mathbf{q}) &= \sum_{k=0}^{i-1} \mathbf{q}_{ik} \big(\alpha' \big|\mathbf{q}_{ik}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{ik}\big|^{-\beta-2}\big) \\
&+ \sum_{k=i+1}^{\lfloor \frac{n-1}{2} \rfloor + i} \mathbf{q}_{ik} \big(\alpha' \big|\mathbf{q}_{ik}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{ik}\big|^{-\beta-2}\big) \\
&+ \sum_{k=\lceil \frac{n+1}{2} \rceil + i}^{n-1} \mathbf{q}_{ik} \big(\alpha' \big|\mathbf{q}_{ik}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{ik}\big|^{-\beta-2}\big) \\
&+ f(n) \mathbf{q}_{i\left(\lfloor \frac{n}{2} \rfloor + i\right)} \big(\alpha' \big|\mathbf{q}_{i\left(\lfloor \frac{n}{2} \rfloor + i\right)}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{i\left(\lfloor \frac{n}{2} \rfloor + i\right)}\big|^{-\beta-2}\big).
\end{aligned}$$

Setting $k' = (i+k)\mathrm{mod(N)}$ in the above sums gets

$$\begin{aligned}
\partial_i W(\mathbf{q}) = &\sum_{k=n-i}^{n-1} \mathbf{q}_{i(i+k)\mathrm{mod(N)}} \big(\alpha' \big|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\big|^{-\beta-2}\big) + \\
&+ \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \mathbf{q}_{i(i+k)\mathrm{mod(N)}} \big(\alpha' \big|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\big|^{-\beta-2}\big) + \\
&+ \sum_{k=\lceil \frac{n+1}{2} \rceil}^{n-i-1} \mathbf{q}_{i(i+k)\mathrm{mod(N)}} \big(\alpha' \big|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\big|^{-\beta-2}\big) \\
&+ f(n) \mathbf{q}_{i\left(\lfloor \frac{n}{2} \rfloor + i\right)\mathrm{mod(N)}} \big(\alpha' \big|\mathbf{q}_{i\left(\lfloor \frac{n}{2} \rfloor + i\right)\mathrm{mod(N)}}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{i\left(\lfloor \frac{n}{2} \rfloor + i\right)\mathrm{mod(N)}}\big|^{-\beta-2}\big)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \mathbf{q}_{i(i+k)\mathrm{mod(N)}} \big(\alpha' \big|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\big|^{-\beta-2}\big) + \\
&+ \sum_{k=\lceil \frac{n+1}{2} \rceil}^{n-1} \mathbf{q}_{i(i+k)\mathrm{mod(N)}} \big(\alpha' \big|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\big|^{-\beta-2}\big)
\end{aligned}$$

$$+f(n)\mathbf{q}_{i\left(\left\lfloor \frac{n}{2}\right\rfloor+i\right)\mathrm{mod(N)}}\left(\alpha'\left|\mathbf{q}_{i\left(\left\lfloor \frac{n}{2}\right\rfloor+i\right)\mathrm{mod(N)}}\right|^{-\alpha-2}+\beta'\left|\mathbf{q}_{i\left(\left\lfloor \frac{n}{2}\right\rfloor+i\right)\mathrm{mod(N)}}\right|^{-\beta-2}\right)$$

Now setting $k = k'\mathrm{mod(N)}$ in the second sum to get;

$$\partial_i W(\mathbf{q}) = \sum_{k=1}^{\left\lfloor \frac{n-1}{2}\right\rfloor} \mathbf{q}_{i(i+k)\mathrm{mod(N)}}\left(\alpha'\left|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\right|^{-\alpha-2}+\beta'\left|\mathbf{q}_{i(i+k)\mathrm{mod(N)}}\right|^{-\beta-2}\right)+$$

$$\sum_{k=1}^{\left\lfloor \frac{n+1}{2}\right\rfloor} \mathbf{q}_{i(i-k)\mathrm{mod(N)}}\left(\alpha'\left|\mathbf{q}_{i(i-k)\mathrm{mod(N)}}\right|^{-\alpha-2}+\beta'\left|\mathbf{q}_{i(i-k)\mathrm{mod(N)}}\right|^{-\beta-2}\right)$$

$$f(n)\mathbf{q}_{i\left(\left\lfloor \frac{n}{2}\right\rfloor+i\right)\mathrm{mod(N)}}\left(\alpha'\left|\mathbf{q}_{i\left(\left\lfloor \frac{n}{2}\right\rfloor+i\right)\mathrm{mod(N)}}\right|^{-\alpha-2}+\beta'\left|\mathbf{q}_{i\left(\left\lfloor \frac{n}{2}\right\rfloor+i\right)\mathrm{mod(N)}}\right|^{-\beta-2}\right)$$

Using the property (iii) above the sums become,

$$\partial_i W(\mathbf{q}) = \sum_{k=1}^{\left\lfloor \frac{n-1}{2}\right\rfloor} \left(\mathbf{q}_{i(i+k)\mathrm{mod(N)}}+\mathbf{q}_{i(i-k)\mathrm{mod(N)}}\right)\left(\alpha'\left|\mathbf{q}_{0k}\right|^{-\alpha-2}+\beta'\left|\mathbf{q}_{0k}\right|^{-\beta-2}\right)$$

$$+f(N)\mathbf{q}_{i\left(\left\lfloor \frac{N}{2}\right\rfloor+i\right)\mathrm{mod(N)}}\left(\alpha'\left|\mathbf{q}_{0\left\lfloor \frac{N}{2}\right\rfloor}\right|^{-\alpha-2}+\beta'\left|\mathbf{q}_{0\left\lfloor \frac{N}{2}\right\rfloor}\right|^{-\beta-2}\right)$$

Using property (ii) form above we have that:

$$\partial_i W(\mathbf{q}) = 2\sum_{k=1}^{\left\lfloor \frac{n-1}{2}\right\rfloor} \left(1-\cos\frac{k\pi}{N}\right)\left(\alpha'\left|\mathbf{q}_{0k}\right|^{-\alpha-2}+\beta'\left|\mathbf{q}_{0k}\right|^{-\beta-2}\right)\mathbf{q}_i$$

$$+f(N)\mathbf{q}_{i\left(\left\lfloor \frac{N}{2}\right\rfloor+i\right)\mathrm{mod(N)}}\left(\alpha'\left|\mathbf{q}_{0\left\lfloor \frac{N}{2}\right\rfloor}\right|^{-\alpha-2}+\beta'\left|\mathbf{q}_{0\left\lfloor \frac{N}{2}\right\rfloor}\right|^{-\beta-2}\right)$$

$$= 4\sum_{k=1}^{\left\lfloor \frac{n-1}{2}\right\rfloor} \sin^2\left(\frac{k\pi}{N}\right)\left(\alpha'\left|\mathbf{q}_{0k}\right|^{-\alpha-2}+\beta'\left|\mathbf{q}_{0k}\right|^{-\beta-2}\right)\mathbf{q}_i$$

$$+f(N)\mathbf{q}_{i\left(\left\lfloor \frac{N}{2}\right\rfloor+i\right)\mathrm{mod(N)}}\left(\alpha'\left|\mathbf{q}_{0\left\lfloor \frac{N}{2}\right\rfloor}\right|^{-\alpha-2}+\beta'\left|\mathbf{q}_{0\left\lfloor \frac{N}{2}\right\rfloor}\right|^{-\beta-2}\right)$$

Noticing the fact that if N is even we have

$$\mathbf{q}_{i(\left\lfloor \frac{N}{2}\right\rfloor+i)\mathrm{mod(N)}} = 2\mathbf{q}_i \qquad \text{and} \qquad \cos\frac{\left\lfloor \frac{N}{2}\right\rfloor\pi}{N} = 0$$

and if N is odd, then $\left\lfloor \frac{N-1}{2} \right\rfloor = \left\lfloor \frac{N}{2} \right\rfloor$, so the right side of the equation above becomes the sum,

$$\partial_i W(\mathbf{q}) = 4\Bigg(\sum_{k=1}^{\left\lfloor \frac{N}{2} \right\rfloor} \sin^2\left(\frac{k\pi}{N}\right)\left(\alpha' \big|\mathbf{q}_{0k}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{0k}\big|^{-\beta-2}\right)\Bigg)\mathbf{q}_i$$

Case II: The proof for $\left\lfloor \frac{N-1}{2} \right\rfloor < i \leq N-1$ proceeds in the same way as above.

Setting

$$\lambda = 4 \sum_{k=1}^{\left\lfloor \frac{N}{2} \right\rfloor} \sin^2\left(\frac{k\pi}{N}\right)\left(\alpha' \big|\mathbf{q}_{0k}\big|^{-\alpha-2} + \beta' \big|\mathbf{q}_{0k}\big|^{-\beta-2}\right),$$

we get that $\partial_i W(\mathbf{q}) = \lambda \mathbf{q}_i \;\;, \;\; \forall i \in \overline{0N-1}$, where λ is independent of i. Hence equations in (A.1) become

$$\nabla \mathrm{W}(\mathbf{q}) = \lambda \mathbf{I}\mathbf{q},$$

where $\mathrm{I}^{=}dia[1, \ldots, 1]$ is the identity matrix. Therefore the regular N-gon is a **CC**

□

Moreover we have the condition that $\big|\mathbf{q}_{0k}\big| = 2\big|\mathbf{q}_0 \sin \frac{k\pi}{N}\big|$, where the orientation of the system we have taken $\mathbf{q}_0$ to be on the positive x-axis and $0 \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor$. Therefore, setting $\mathbf{q}_0 = r$, the radius of the configuration, we get

$$\lambda = -2^{-\alpha} r^{-\alpha-2} \sum_{k=1}^{\left\lfloor \frac{N}{2} \right\rfloor} \left(a\alpha \csc^{\alpha}\left(\frac{k\pi}{N}\right) + b\beta (2r)^{\alpha-\beta} \csc^{\beta}\left(\frac{k\pi}{N}\right)\right) \tag{A.2}$$

Set $\alpha = 2\beta$ and $a = -1$, $b = 2$, then for the generalized Lennard-Jones problem we have,

$$\lambda = 2^{-2\beta+1} \beta r^{-2\beta-2} \sum_{k=1}^{\left\lfloor \frac{N}{2} \right\rfloor} \left(\csc^{2\beta} \gamma_k - (2r)^{\beta} \csc^{\beta} \gamma_k \right), \tag{A.3}$$

where $\gamma_k = \frac{k\pi}{N}$ and r is the radius of the configuration.

Appendix B

Properties of Circulant Matrices

B.1 Circulant Matrices

Definition B.1. *Let* $\mathbf{P}$ *be an* $N \times N$ *matrix, defined as follows*

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then $\mathbf{P}$ *is called the Primitive* N *Circulant Matrix.*

Clearly we have that

$$P^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Looking at the structure of the matrices, one notices that multiplying $\mathbf{P}^m$ by $\mathbf{P}$ shifts all the columns one place to the right (taking the N^{th} column and placing it in the 1^{st} column). Therefore,

$$P^{N-1} = P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

which leads to the result

$$P^{N} = P^{0} = I = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

These properties give us a new class of matrices known as circulant matrices.

Definition B.2. *Given any $\vec{a} = [a_0, a_1, \dots, a_{N-1}, a_{N-1}] \in \mathbb{R}^N$ we can define an $N \times N$ matrix by*

$$\mathbf{A} = a_0\mathbf{P}^0 + a_1\mathbf{P} + a_2\mathbf{P}^2 + \cdots + a_{N-1}\mathbf{P}^{N-1} = \sum_{j=0}^{N-1} a_j\mathbf{P}^j,$$

where $\mathbf{A}$ is called the circulant matrix with base vector $\vec{a}$.

The circulant matrix, with base vector $[a_0, a_1, \dots, a_{N-1}]$ has the form

$$\mathbf{A} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_{N-3} & a_{N-2} & a_{N-1} \\ a_{N-1} & a_0 & a_1 & a_2 & \dots & a_{N-4} & a_{N-3} & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & a_1 & \dots & a_{N-5} & a_{N-4} & a_{N-3} \\ \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\ a_3 & a_4 & a_5 & a_6 & \dots & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & \dots & a_{N-1} & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_4 & \dots & a_{N-2} & a_{N-1} & a_0 \end{pmatrix}.$$

Lemma B.3. *For the primitive N circulant matrix* $\mathbf{P}$*, then the eigenvalues of* $\mathbf{P}$ *are the N, N^{th} roots of unity, which are the solutions to the equation $x^N - 1 = 0$.*

PROOF:

Note that if $\mathbf{B}$ is an upper triangular matrix, then $det\big(\mathbf{B}\big) = dia\big(\mathbf{B}\big)$. Therefore, $det\big(\mathbf{P} - \lambda\mathbf{I}\big) = -\lambda det\big(\mathbf{B}\big) - det\big(\mathbf{C}\big)$, where $\mathbf{B}$ is a $N-1 \times N-1$, upper triangular matrix, with $dia\big(\mathbf{B}\big) = [-\lambda, -\lambda, \ldots, -\lambda]$. Furthermore, we have

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 1 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & -\lambda \end{pmatrix},$$

so clearly we can see that $det\big(\mathbf{C}\big) = -(-1)^{N-1}$, and $-\lambda det\big(\mathbf{B}\big) = (-\lambda)^N$. Therefore, we have that λ is an eigenvalue, if and only if it solves the equation $(-\lambda)^N - (-1)^N = 0$. This property is equivalent to solving the equation $\lambda^N - 1 = 0$. Hence, the eigenvalues of the primitive N circulant matrix, $\mathbf{P}$, are the N^{th} roots of unity, $\lambda_j = e^{\frac{2j\pi}{N}i}$, $j \in \overline{1N}$. $\square$

We can use this result to show that in fact the eigenvalues of any circulant matrix are the N, N^{th} roots of unity.

Lemma B.4. *If λ_j is an eigenvalue of* $\mathbf{P}$*, the primitive N circulant matrix, then λ_j^m is an eigenvalue of* $\mathbf{P}^m$*, for $m = 1, \ldots, N-1$. Furthermore, if $\vec{v}_j$ is an eigenvector of* $\mathbf{P}$ *then it is also an eigenvector of* $\mathbf{P}^m$ *for $m = 1, \ldots, N-1$*

PROOF:

Let $\vec{v}_j$ is the eigenvector associated with λ_j and proceed by induction on m. Clearly the induction step for $m = 1$ is true directly from lemma B.3, so assume true for $m-1$. Therefore, we assume that λ_j^{m-1} and $\vec{v}_j$ are an eigenvalue and an associated eigenvector of $\mathbf{P}^{m-1}$. This leads to the equations,

$$\mathbf{P}^m\vec{v}_j = \mathbf{P}^{m-1}\mathbf{P}\vec{v}_j$$
$$\mathbf{P}^{m-1}\lambda_j\vec{v}_j = \lambda_j\mathbf{P}^{m-1}\vec{v}_j$$

By our induction hypothesis, we have that $\mathbf{P}^{m-1}\vec{v}_j = \lambda^{m-1}\vec{v}_j$, hence,

$$\begin{aligned}\mathbf{P}^m\vec{v}_j &= \lambda_j\mathbf{P}^{m-1}\vec{v}_j \\ &= \lambda^m\vec{v}_j\end{aligned}$$

Thus proving the lemma. $\square$

Lemma B.5. *Let* $\mathbf{A}$ *be the circulant matrix generated by* $\vec{a} = [a_0, a_1, \ldots, a_{N-1}] \in \mathbb{R}^N$*, then the* N *eigenvalues of* $\mathbf{A}$ *are given by:*

$$\tilde{\lambda}_j = \sum_{k=0}^{N-1} a_k \lambda_j^k = \sum_{k=0}^{N-1} a_k e^{\frac{2jk\pi}{N} i}$$

PROOF:

The proof is quite easy. We start by noticing that $\mathbf{A} = \sum_{k=0}^{N-1} a_k \mathbf{P}^k$, which gives,

$$\mathbf{A}\vec{v}_j = \big(a_0\mathbf{P}^0 + a_1\mathbf{P}^1 + \cdots + a_{N-1}\mathbf{P}^{N-1}\big)\vec{v}_j$$

$$= a_0\mathbf{P}^0\vec{v}_j + a_1\mathbf{P}^1\vec{v}_j + \cdots + a_{N-1}\mathbf{P}^{N-1}\vec{v}_j$$

Define the N^{th} roots of unity by,

$$\rho_j = e^{\frac{2j\pi}{N} i}. \tag{B.1}$$

Lemma B.4 shows that $\vec{v}_j$ is an eigenvector of $\mathbf{P}^m$ associated with λ_j^m for $m = 1, \ldots, N-1$. Therefore, we have

$$\mathbf{A}\vec{v}_j = a_0\lambda_j^0\vec{v}_j + a_1\lambda^1\vec{v}_j + \cdots + a_{N-1}\lambda^{N-1}\vec{v}_j = \left(\sum_{k=0}^{N-1} a_k\lambda_j^k\right)\vec{v}_j,$$

and $\tilde{\lambda}_j = \sum_{k=0}^{N-1} a_k\lambda_j^k = \sum_{k=0}^{N-1} a_k\rho_j^k$ is the eigenvalue of $\mathbf{A}$, associated with the eigenvector $\vec{v}_j$. Thus completing the proof. □

Lemma B.6. *The eigenvector of* $\mathbf{A}$ *associated with the eigenvalue* $\tilde{\lambda}_j$ *is* $\vec{v}_j = [\rho_j, \rho_j^2, \ldots, \rho_j^{N-1}, 1]$*, for all* $j \in \overline{0N-1}$*, where* ρ_j *is given by (B.1)*

PROOF:

By lemma B.4, it is sufficient to show that the lemma is true for the primitive circulant matrix $\mathbf{P}$. Let $\overrightarrow{v}_j^T = [v_0, v_1, \ldots, v_{N-1}]$ be the eigenvector associated with the eigenvalue λ_j, then we have

$$\mathbf{P}\overrightarrow{v}_j = \lambda_j\overrightarrow{v}_j$$

$$\mathbf{P}\overrightarrow{v}_j = e^{\frac{2j\pi}{N} i}\overrightarrow{v}_j$$

This is equivalent to solving the equation,

$$v_k = e^{\frac{2j\pi}{N} i} v_{k-1} \qquad k \in \overline{0N-1}. \tag{B.2}$$

Define the vector $\overrightarrow{v}_j$, by setting each of its coordinates v_k given above as,

$$v_k = \rho_j^{k+1} \qquad k \in \overline{0n-1}. \tag{B.3}$$

Using these constructions, we get

$$e^{\frac{2j\pi}{N}i} v_{k-1} = e^{\frac{2j\pi}{N}i} \left(e^{\frac{2j\pi}{N}i} \right)^k,$$

$$e^{\frac{2j\pi}{N}i} v_{k-1} = \left(e^{\frac{2j\pi}{N}i} \right)^{k+1},$$

$$e^{\frac{2j\pi}{N}i} v_{k-1} = v_k.$$

Therefore, (B.3) is a solution to (B.2), and therefore must be an eigenvector associated with the eigenvalue λ_j. $\square$

Bibliography

[1] Delgado et al. (1996) *The Global Flow of the Manev Problem.* Journal of Mathematical Physics **37** (6) June 2748-2761

[2] Diacu, F.N. (1990). *The Masses in a Symmetric Centered Solution of the n-Body Problem.* Proceedings of the American Mathematical Society **109** (4) August 1079-1085

[3] Diacu, F.N. (1991). *Singularities of the N-Body Problem.* Montrél: Les Publications CRM.

[4] Diacu, F. (1993) *The Planar Isosceles Problem for Maneff's Gravitational Law.* Journal of Mathematical Physics **34** (12) 5671-5690

[5] Diacu, F. (1996) *Near Collision Dynamics for Particle Systems with Quasihomogeneous Potentials.* Journal of Differential Equations **128** (1) 58-77

[6] Diacu, F.,et al (to appear) *Central Configurations and Total Collisions for Quasihomogeneous N-Boby Problems.* Nonlinear Analysis

[7] McGehee, R. (1981). *Double Collisions for a classical partivle system with non-gravitational interactions.* Comment Mathematical Helvetici bf 56 524-557

[8] Moeckel, R. (1990) *On Central Configurations.* Mathematishe Zeitschrift **205** 499-517.

[9] Moulton, F.R. (1910) *The Straight Line Solutions of the N-Body Problem.* Annauls of Math **297** 41-52.

[10] Perko, L. (1996). *Differential Equations and Dynamical Systems* 2^{nd} *Edition.* New York: Springer-Verlag.

[11] Perko, L. et al.(1985)*Regular Polygon Solutions of the N-Body Problem.* Proceedings of the American Mathematical Society **94** (2) June 301-309

[12] Riahi, H. (1995). *Periodic Orbits of n-Body Type Problems: The Fixed Period Case*. Transactions of the American Mathematical Society **347** (12) December 4663-4685

[13] Stoica, C. (2000). *Particle Systems with Quasi-homogeneous Interaction.* Phd Thesis, University of Victoria, Publication Number AATNQ52774

[14] Verhulst, F. (1985). *Nonlinear Differential Equations and Dynamical Systems.* Berlin: Springer-Verlag.

[15] Wintner, A. (1941). *The Analytical Foundations of Celestial Mechanics.* Princeton: Princeton University Press.

Partial Copyright License

TITLE OF THESIS:

THE N-BODY PROBLEM
WITH REPULSIVE-ATTRACTIVE
QUASIHOMOGENEOUS POTENTIAL FUNCTIONS

Author: ________________________________

ROBERT T. JONES

____

(Date)