Dynamical Classification of the Two-body and Hill's Lunar Problems with Quasi-homogeneous Potentials

by

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ABSTRACT

As seen in many examples, higher order correction added to the Newtonian potential often provides more realistic and accurate model in astrophysics. Such examples include the Schwarzschild potential ($U_{\text{Schwarzschild}}(r) = -\frac{A}{r} - \frac{B}{r^3}$, where r is the mutual distance and A, B are constants) and the Manev potential ($U_{\text{Manev}}(r) = -\frac{A}{r} - \frac{B}{r^2}$).

In the thesis, we study the two-body problem and Hill's lunar problem under quasi-homogeneous potentials.

The quasi-homogeneous two-body problem aims to study the interaction between two point particles under a prescribed potential in the form of $W(r) = U(r) + V(r) = -\frac{A}{r^a} - \frac{B}{r^b}$. It is well known that the number two serves as a threshold value for homogeneous ($U(r) \simeq -\frac{1}{r^a}$) N-body problems: one is able to observe significant difference regarding to the solution dynamics as the power of homogeneous potential exceeds two from below. This phenomenon remains observable for quasi-homogeneous potentials.

In the second chapter, we provide a complete characterization of the whole phase space of the quasi-homogeneous two-body problem in terms of global existence and singularity for all the possible $b > a > 0$. In particular, one is able to generalize the result of Manev two-body problem ($a = 1, b = 2$) to all the quasi-homogeneous potentials with $b = 2 > a$. That is, one is able to show that the initial conditions that lead to finite-time collision has positive measure for all $b = 2 > a$. Two techniques are presented: One is the variational method based on energy and the other is direct computation of collision time based on the integrability of two-body systems.

Hill's lunar equation under the Newtonian or homogeneous potentials has been derived from the Hamiltonian of the three-body problem in a uniform rotating coordinate system with angular speed ω, by using symplectic scaling and heuristic arguments on various physical quantities.

Quasi-homogeneous Hill's lunar problem is the other focus of this thesis. In the third chapter, we first derive Hill's lunar equation for quasi-homogeneous potentials. We then fully characterize the phase space under some energy threshold using an indicator function. This energy threshold is characterized variationally. In particular, our results demonstrate the existence of "black hole effect" for $b > 2 > a$ and ω sufficiently large: invariant sets (in the phase space) with non-zero Lebesgue measure that either contains global solutions and solutions with singularity are constructed, under and at some energy threshold. Next, we apply McGehee-type transform to study near-collision dynamics of the homogeneous Hill's lunar problem. We shall

derive the asymptotic profile near collision for all the strong-force ($a > 2$) potentials.

Finally, in the last chapter, we conclude the advantages and disadvantages of the methods we have used through this thesis, as well as summarizing our future works.

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DEDICATION

Dedicated to everyone who have helped me through my study of Mathematics.

Chapter 1

Introduction

The N-body problem aims to describe the motion of N-point particles

$$(x_1, x_2, ..., x_N) \in (\mathbb{R}^3)^N$$

with masses $m_1, m_2, ..., m_n$ interacting with one another in $\mathbb{R}^3$ under the action of a prescribed potential $W(x_1, x_2, ..., x_N)$, which only depends on each mutual distance formed by each pair of particles. For each particle's location x_i, its equation of motion can be easily obtained by using Newton's second law.

Characterizing initial conditions that correspond to singularities has always been an important concept in the study of differential equations.

For the N-body problem, the collision set is defined to be a subset of $(\mathbb{R}^3)^N$ where at least two particles have the same coordinates in $\mathbb{R}^3$. Symbolically, we have

$$\Delta := \bigcup_{1 \leq i < j \leq N} \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_N) \in (\mathbb{R}^3)^N : x_i = x_j \right\}. \tag{1.0.1}$$

The motion of the Sun-Earth-Moon system is a classical problem in the study of celestial mechanics. Historically, researchers regarded the Sun-Earth-Moon system as two Kepler problems (Earth-Moon and Sun-Earth), the attraction force between the Sun and the Moon was always ignored since it is significantly less noticeable than the force between the Moon and the Earth. Furthermore, unlike three-body problems (with Newtonian potential), where the dynamics is chaotic, the general analytical solution for the two-body system was well-known. However, this two-Kepler-problem formulation has drawbacks: Delaunay used this formulation but his calculation was unable to match the observed lunar motion. (See [15],[10].)

One of the ideas to improve the accuracy of the lunar motion, obtained by solving the two-body problem, is to replace the Newtonian potential $U(r) \sim -\frac{1}{r}$ (r is the mutual distance) by homogeneous potentials.

When equipped with homogeneous potentials in the form of $W \sim -\frac{1}{r^a}$, $a > 0$ (r stands for mutual distance), the equation of motion is locally Lipchitz continuous on $(\mathbb{R}^3)^N \backslash \Delta$, and thus for any given initial condition, there exists a unique solution up to a certain time. It is natural to ask whether this time can be extended to infinity and what kind of initial conditions lead to a collision. Painlevé proved that for Newtonian N-body problem, the distance between the solution and the collision set Δ approaches zero as the solution approaches a singularity. We note that Painlevé's theorem can be easily extended to all homogeneous and quasi-homogeneous potentials by using Cauchy-Lipchitz theorem. (See [17] Theorem 2.2.)

For homogeneous-potential N-body problem, it is well known that the power plays a crucial role in determining the dynamics of each solution. For instance, the improbability theorem [18] states that for the Newtonian N-body problem, the set of initial conditions leading to collisions has Lebesgue measure zero in the phase space. A few years ago, Fleischer and Knauf [9] extended the improbability result to the homogeneous N-body problem with weak force (i.e. $a < 2$) by geometric arguments. Another difficulty of studying weak-force homogeneous potential is that the weak-force N-body problem is chaotic when $N \geq 3$. In particular, the Newtonian N-body problem is known to be chaotic when $N \geq 3$.

On the other hand, for homogeneous Hill's lunar problem (a restricted version of the three-body problem), numerical investigations in [21, 5] showed that as the power a increases, the dynamics becomes more organized in terms of global existence and singularity and the chaotic behavior becomes less prominent. In particular, when a passes the threshold two, the system is no longer chaotic and a clear boundary that distinguishes global and singular solutions emerges ([4]). In addition, for strong-force ($a > 2$) homogeneous N-body problem, the Lagrange-Jacobi identity implies that every solution with negative energy corresponds to a collision and thus the set containing initial conditions that correspond to collision has positive Lebesgue measure ([3]).

Unfortunately, homogeneous potentials sometimes do not provide accurate approximations in some physical situations. For example, even Newton himself realized his Newtonian potential was not perfect, since his classical potential could not explain the apsidal motion of the moon ([7]). Thus he added another term and studied the dynamics of

$$U_{\text{Manev}}(r) = -\frac{A}{r} - \frac{B}{r^2}, \quad A, B > 0.$$

This kind of potential is known as the Manev potential. Clearly, the Newtonian potential plays an indispensable role in the study of non relativistic mechanics. On the other hand, Manev proved that his gravitational model provides a good approximation offered by general relativity. (See [6, 11].) One often interprets the Manev potential as a combination of the Newtonian component with power one and a perturbational component with power two. Another important example of potentials in the form of "Newtonian+Perturbation" would be the Schwarzschild potential

$$U_{\text{Schwarzschild}}(r) = -\frac{A}{r} - \frac{B}{r^3}.$$

This potential is not only able to model astrophysical and stellar dynamics systems in a classical context, but also describes the "black hole" effect: solutions with non-zero angular momentum are possible to reach a collision and the collision set also has positive Lebesgue measure, which makes it distinguishable from its Newtonian counterpart [2]. In fact, Both Manev and Schwarzschild potential are quasi-homogeneous potentials, which, in general, has the form of

$$U(r) = -\frac{A}{r^a} - \frac{B}{r^b}, \quad b > a > 0, \quad A, B \in \mathbb{R}^+.$$

Another example of the quasi-homogeneous potential would be the Lennard-Jones potential

$$U_{\text{Lennard-Jones}}(r) = -\frac{A}{r^6} + \frac{B}{r^{12}}, \quad A, B > 0,$$

which models interactions between atoms and molecules. We notice that this potential is a combination of attracting force and repelling force, which is out of this thesis' scope. More examples of quasi-homogeneous potentials can be found in [19].

The most popular and useful tool for studying near-collision dynamics is the McGehee transformation [13], a change of coordinate method invented in the 70s that blows up the singularities and turns them into equilibria in a new system. Indeed, by applying this method, Xia [20] was able to construct a non-collision singularity for the Newtonian five-body problem and solved the well-known Painlevé conjecture that remained open for about one century. Around the same time, Pérez-Chavela and Vela-Arévalo [16] discovered that McGehee transformation can yet be applied to the collinear quasi-homogeneous three-body problem. In particular, if $b > 2$, then the Lebesgue measure of the triple collision orbits is infinite. For the Manev two-body problem, Diacu et al. [7] showed that the set of initial conditions leading to collisions has positive Lebesgue measure. Moreover, by introducing McGehee-type transformations, Diacu et al. were also able to discover the black hole effect for the Manev problem: If the two particles are close enough and has sufficiently low angular momentum, then they will head for a collision and thus the collision set has non-zero Lebesgue measure.

Meanwhile, for the strong-force ($a > 2$) homogeneous N-body problem, Deng and Ibrahim [3] discovered that it is possible to classify initial conditions that either lead to global or singular solutions below some energy threshold. For the two-body strong-force homogeneous problem, Deng and Ibrahim's method is effective and they were able to construct invariant sets, while for three or more bodies, they were only able to obtain some partial results. The technique used in their paper is to define an indicator function, obtained by scaling method, and then to construct invariant sets based on its sign below some energy threshold called "ground state energy". Is it possible to slightly modify their method and apply it to study quasi-homogeneous two-body problems? The second chapter of this thesis is dedicated to solve this problem.

In addition to the idea of replacing the Newtonian potential, to improve the accuracy of the solution obtained by solving Kepler's two-body problem, it suffices to "unignore" the attraction between the Moon and the Sun. This effectively summarizes the idea of "Hill's lunar problem". The Moon is considered to be an infinitesimal body that is attached to the Earth fixed at the origin. The Moon's motion is described in a rotating coordinate system such that the positive $x-$axis always points to an infinite body (the Sun) that is infinitely far away, while the Sun-(Earth and Moon) system is still governed by the standard Kepler's equation (with Newtonian potential). (See [4],[15].) In fact, Hill's lunar problem can be considered as a restricted version of the three-body problem.

Several decades ago, Meyer [15] became the first researcher to establish the close link between Hill's lunar problem and the three-body problem: he provided a precise derivation of the Hill's lunar equation from the Hamiltonian of the Newtonian three-body system by using symplectic scaling method and heuristic arguments on different physical quantities. Recently, Deng and Ibrahim [4] extended Meyer's result to all the homogeneous potentials.

As mentioned earlier, quasi-homogeneous potentials are among the most important and applicable potentials in the study of natural science. By combining Hill's formulation with quasi-homogeneous potentials, we shall arrive at a more accurate and meaningful model to study the lunar motion. Unfortunately, the derivation of the Hill's lunar equation under quasi-homogeneous potentials needs to be clarified.

The derivations presented in [4, 15] have provided us all the basic frameworks we need for the derivation of Hill's lunar equation under quasi-homogeneous potentials. Indeed, we are able to obtain Hill's lunar equation under quasi-homogeneous potentials and this derivation is included in the first section of Chapter three. As usual, we begin our derivation from the Hamilonian of the Sun-Earth-Moon (three-body) system that is in a uniform rotating frame with arbitrary angular speed (frequency) ω, but with an arbitrary prescribed quasi-homogeneous potential. Following the footsteps of Meyer [15] and Deng–Ibrahim [4], our derivation also adopts all the heuristic arguments and physical assumptions they made, as well as the symplectic scaling method. However, unlike these preceding works, where the frequency is considered to be normalized, we keep the frequency in the equation of motion. Subsequently, we turn our focus to the quasi-homogeneous which consists of one strong and one weak homogeneous potentials, as this type of potentials correspond to the most applications. That is, we study the global existence and singularity of the solutions of Hill's lunar problem when equipped with quasi-homogeneous potentials in the form of

$$U(r) = -\frac{A}{r^a} - \frac{B}{r^b}, \qquad 0 < a < 2 < b, \qquad A, B \in \mathbb{R}^+.$$

We remark that this type of potential contains the Schwarzschild potential.

For homogeneous potentials, the frequency can always be normalized ([4]), this however is not the case of quasi-homogeneous potentials. Nevertheless, we have good reason to keep the angular speed since we wish to study the limiting behavior of solution dynamics when the angular speed $\omega \to \infty$. Akahori et al. [1] studied semi-linear elliptic equation with combined power nonlinearities involving the Sobolev critical exponent and showed that the main behavior of the solutions is given by the Sobolev critical part for high frequency.

This kind of phenomenon is very likely to occur in the case of Hill's lunar problem with weak-strong quasi-homogeneous potential: the behavior of any solution is governed by the strong-force homogeneous potential when frequency is sufficiently large and one may thus be able to classify solutions under and at some energy threshold. Indeed, when the angular speed is large enough, the strong-force homogeneous potential dominates the behavior of solution dynamics and we therefore are able to construct invariant sets with positive measure that lead to finite-time collision, despite the presence of weak-force homogeneous potential.

This thesis is structured as follows.

In the second chapter, we completely characterize the dynamics of any two-body system when equipped with different quasi-homogeneous potentials. For two-body systems, it is well known that besides the energy, the angular momentum is also conserved along any solution. For homogeneous strong-force two-body problems, Deng and Ibrahim [3] were able to construct invariant sets and their main result followed immediately. For the Manev two-body, Llibre et al. [12] gave a complete description of the phase portrait with potentials in the form of $\frac{A}{r} + \frac{B}{r^2}$, $A, B \in \mathbb{R}$.

We attempt to extend Deng-Ibrahim [3] and Diacu's [7] results to all the quasi-homogeneous two-body problems whenever their method is applicable. Inevitably, we are forced to divide into cases depending on the power combinations of the prescribed potential. In each case, we construct invariant sets that either correspond to global solutions or solutions with singularity, based on the energy, angular momentum and the indicator function. The key ingredient is again the idea of energy method proposed in Deng and Ibrahim's paper [3].

Our analysis is divided into five cases and each case correspond to a section. For $b > a > 2$ and $a < b < 2$, everything behaves exactly like the homogeneous case, which is unsurprising. For $b > 2 > a$, the behavior becomes more complicated and it depends on the magnitude of angular momentum c. Nevertheless, we are able to explicitly solve the threshold $c^*(a, b)$. For the limiting cases $a < b = 2$ and $2 = a < b$, the energy method [3] only yields partial results and we seek for analytical method to directly tackle the problem.

For all the possible $b > a > 0$ and the magnitude of angular momentum, we provide a complete description of the phase space in terms of global existence and singularity (See Table 2.1.) As an important consequence, we find that whenever $b \geq 2$, we observe the "black hole effect" and thus we have the following result which concludes this chapter.

Theorem 1.0.1. *For the quasi-homogeneous two-body problem, if $b \geq 2$, then the set of initial conditions that correspond to finite-time collision has positive Lebesgue measure in the phase space $(x, \dot{x})$.*

In the first section of Chapter three, we derive the equation of motion and the effective potential under the quasi-homogeneous potential for Hill's lunar problem by using symplectic scaling and heuristic arguments as proposed in [4, 15]. This section also includes some specific choices and corrections we are forced to make in order to facilitate our study on the qualitative behavior ($\omega \to \infty$) of solution dynamics under strong-weak potentials. Under our settings, and if one assumes the Moon's location is (x, y), then the Hill's lunar equation with quasi-homogeneous potential is derived as

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} &= -\frac{\partial V_\omega}{\partial x}(x, y) \\ \ddot{y} + 2\omega\dot{x} &= -\frac{\partial V_\omega}{\partial y}(x, y). \end{aligned} \tag{1.0.2}$$

Here, the effective potential

$$V_\omega(x, y) := -\frac{a+2}{(x^2+y^2)^{a/2}} - C_\epsilon \frac{1}{(x^2+y^2)^{b/2}} - \left[(1+\frac{b}{2})\omega^2 - C\omega^{\frac{2(a+2)}{b+2}}\right] x^2, \tag{1.0.3}$$

where ω is rotational speed of the uniform rotating coordinate system for the three-body system and both $C_\epsilon \gg 1$ and C are constant. Physically, ϵ represents the mass ratio between the Sun and the Earth-Moon system and it is considered to be an arbitrarily small constant in this section. In particular, when $b = a$ and $\omega = 1$, this effective potential coincides with the homogeneous potential in [4] up to a constant before the first two terms in (1.0.3).

The energy for Hill's lunar problem is given by

$$E_\omega(x, y, \dot{x}, \dot{y}) = \frac{\dot{x}^2 + \dot{y}^2}{2} + V_\omega(x, y). \tag{1.0.4}$$

For each $\omega > (\frac{2C}{2+b})^{\frac{b+2}{2b-2a}} =: \omega^*(a, b)$, this energy attains minimum at the critical points of effective energy with zero speed $\Gamma_0 : (\pm x_\omega, 0, 0, 0)$, where x_ω is the unique solution of

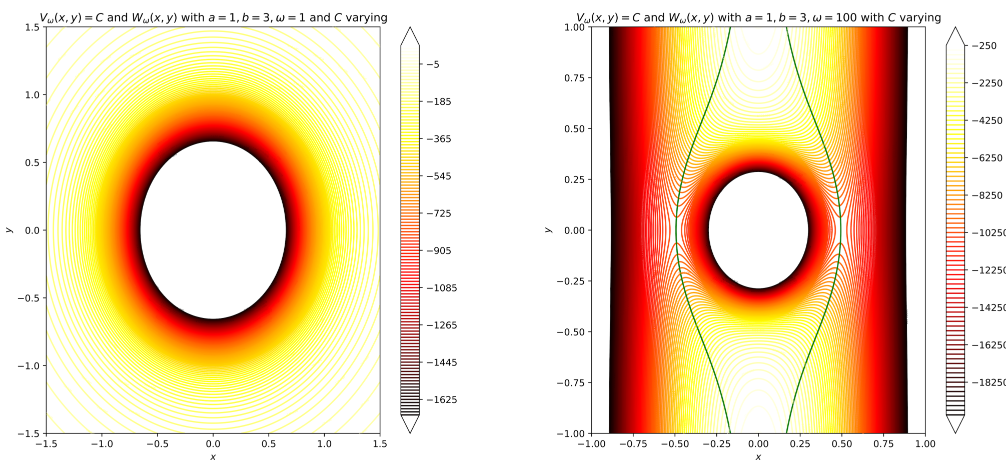
$$(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}} = \frac{C_\epsilon b}{x^{b+2}} + \frac{a(a+2)}{x^{a+2}}. \tag{1.0.5}$$

We define $\Gamma_0^+ = (x_\omega, 0, 0, 0)$ and $\Gamma_0^- = (-x_\omega, 0, 0, 0)$.

The right hand side of the equation of motion (1.0.2) is locally Lipchitz continuous whenever $(x, y) \neq (0, 0)$, hence there exists a unique solution up until some time $T_{\max}$. If $T_{\max} < \infty$, then the solution is said to experience a singularity at $T_{\max}$; otherwise, we say the solution exists globally.

The ultimate goal for the first three sections of Chapter three is to construct invariant sets and demonstrate the presence of “black hole” for $\omega \to \infty$ and $b > 2 > a$.

The main tool we shall deploy is again the energy method, proposed in [4], where Deng and Ibrahim studied the ground state energy problem and used it, as an upper threshold, to construct invariant sets for strong-force homogeneous N-body problem. The ground state energy is defined to be the smallest energy such that its corresponding indicator function vanishes. Roughly speaking, the indicator function for the ground state energy is obtained by scaling the effective potential.



(a) V_ω with $\omega = 1$

(b) V_ω with $\omega = 100$

Figure 1.1: Contour plot of V_ω under Schwarzschild potential
(Green line indicates $W_\omega = 0$)

Formally, for quasi-homogeneous Hill’s lunar problem, for each $\omega > 0$, the indicator function W_ω, which is related to the ground state energy, is defined by scaling on the effective potential:

$$W_\omega(x,y) = -x\frac{\partial V_\omega}{\partial x} - y\frac{\partial V_\omega}{\partial y} = \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right]x^2 - \left(\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b}\right). \tag{1.0.6}$$

The ground state energy for quasi-homogeneous Hill’s lunar problem is defined to be

$$E^*(\omega) = \inf\{E_\omega(x,y,\dot{x},\dot{y}) : W_\omega(x,y) = 0\}, \tag{1.0.7}$$

and we can check that it is finite and attained by the critical points of V_ω: Γ_0 for $b > a$ and $\omega > \omega^*(a,b)$. The study of ground state energy, as well as all the variational properties that are essential to the main theorems are included in the second section of this chapter. The Lagrange-Jacobi identity for our quasi-homogeneous Hill's lunar problem can be calculated by differentiating the moment of inertia $I = \frac{x^2+y^2}{2}$ with respect to time twice:

$$\begin{aligned}\ddot{I} &= \dot{x}^2 + \dot{y}^2 + 2\omega(\dot{y}x - \dot{x}y) + \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right]x^2 - \left[\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b}\right] \\ &=: K_\omega(x, y, \dot{x}, \dot{y}).\end{aligned} \tag{1.0.8}$$

It is well known that the sign of K_ω plays a crucial rule in determining the fate of a solution.

From here, we define the following sets below this ground state energy.

$$\begin{aligned}\mathcal{W}_+^+(\omega) &= \{\Gamma \in \mathcal{W}(\omega) : K_\omega(\Gamma) \geq 0, W_\omega(\Gamma) > 0\} \\ \mathcal{W}_+^-(\omega) &= \{\Gamma \in \mathcal{W}(\omega) : K_\omega(\Gamma) < 0, W_\omega(\Gamma) > 0\} \\ \mathcal{W}_-^+(\omega) &= \{\Gamma \in \mathcal{W}(\omega) : K_\omega(\Gamma) \geq 0, W_\omega(\Gamma) \leq 0\} \\ \mathcal{W}_-^-(\omega) &= \{\Gamma \in \mathcal{W}(\omega) : K_\omega(\Gamma) < 0, W_\omega(\Gamma) \leq 0\},\end{aligned} \tag{1.0.9}$$

where the invariant set

$$\mathcal{W}(\omega) = \{\Gamma = (x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 : E_\omega(\Gamma) < E^*(\omega)\} \tag{1.0.10}$$

and

$$\begin{aligned}\mathcal{W}_+(\omega) &= \mathcal{W}_+^+(\omega) \bigcup \mathcal{W}_+^-(\omega) \\ \mathcal{W}_-(\omega) &= \mathcal{W}_-^+(\omega) \bigcup \mathcal{W}_-^-(\omega).\end{aligned} \tag{1.0.11}$$

Theorem 1.0.2 (Dichotomy for large angular speed below ground state energy)**.** *Let $b > 2 > a$ and ω be sufficiently large. Then, both $\mathcal{W}_+(\omega)$ and $\mathcal{W}_-(\omega)$ are invariant. Furthermore,*

1. *$\mathcal{W}_-^+(\omega)$ is empty.*

2. *$\mathcal{W}_+(\omega)$ contains global solution.*

3. *$\mathcal{W}_-^-(\omega)$ contains solution with singularity.*

At the ground state energy, similar results can be drawn. Let

$$\widetilde{\mathcal{W}}(\omega) = \{\Gamma = (x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 : E_\omega(\Gamma) = E^*(\omega)\} \tag{1.0.12}$$

and $\widetilde{\mathcal{W}}^{\pm}_{\pm}(\omega)$ be defined analogously as in (1.0.9) (1.0.10), and (1.0.11).

Theorem 1.0.3 (Dichotomy for large angular speed at ground state energy)**.** *Let $b > 2 > a$ and sufficiently large ω. Then, both $\widetilde{\mathcal{W}}_+(\omega)$ and $\widetilde{\mathcal{W}}_-(\omega)$ are invariant. Furthermore,*

1. $\widetilde{\mathcal{W}}^+_-(\omega) = \left\{\Gamma^{\pm}_0\right\}$.
2. $\widetilde{\mathcal{W}}_+$ *contains global solutions.*
3. *solutions in the invariant set $\widetilde{\mathcal{W}}^-_-$ either have a finite-time collision or approach $\left\{\Gamma^{\pm}_0\right\}$ as $t \to \infty$.*

Therefore, the "black hole effect" is observable in the strong-weak Hill's lunar problem as long as the angular speed is sufficiently large and $b > 2 > a$. We remark that our classifications are sharp. That is, at and below the ground state energy, based on the invariant sets we construct, we are able to categorize the fate of all the initial conditions that either lead to finite-time collision or exist globally.

In the last section of Chapter three, we apply McGehee transform [14], to study near-collision dynamics of the homogeneous Hill's lunar problem. By introducing a new time we blow up the singularity that corresponds to collision-singularity in the equation of motion. According to [4], for the homogeneous Hill's lunar problem, the equation of motion, which is independent of angular speed ω, is

$$\begin{aligned} \ddot{x} - 2\dot{y} &= -\frac{\partial V}{\partial x}(x, y) \\ \ddot{y} + 2\dot{x} &= -\frac{\partial V}{\partial y}(x, y), \end{aligned} \tag{1.0.13}$$

where the effective potential has the form of

$$V(x, y) = -\frac{\alpha + 2}{2}x^2 - \frac{\alpha + 2}{(x^2 + y^2)^{\frac{\alpha}{2}}},\ \alpha > 0. \tag{1.0.14}$$

We shall show that for strong-force homogeneous Hill's lunar problem, it is possible to derive the asymptotic profile of a collision-bound solution near collision. This is summarized as follows.

Theorem 1.0.4. *Fix $\alpha > 2$, and let $\mathbf{x}(t) = (x, y)(t)$ be a solution of the homogeneous Hill's lunar problem that has a collision-singularity at $t^* < \infty$. Then, as $t \to {t^*}^-$,*

1. *The angular momentum $C(t) := x\dot{y} - y\dot{x} \to C^*$, for some C^*.*

2. *The following asymptotic relation holds.*

$$\mathbf{x}(t) \simeq \begin{cases} 2^{\frac{2}{\alpha+2}}(\frac{\alpha}{2}+1)^{\frac{3}{2\alpha+2}}(t^*-t)^{\frac{2}{\alpha+2}} \exp\left[i\left(\theta^* - \frac{C^*}{\alpha-2}(\frac{\alpha}{2}+1)^{\frac{\alpha-4}{\alpha+2}}(t^*-t)^{\frac{\alpha-2}{\alpha+2}}\right)\right] & C^* \neq 0 \\ 2^{\frac{2}{\alpha+2}}(\frac{\alpha}{2}+1)^{\frac{3}{2\alpha+2}}(t^*-t)^{\frac{2}{\alpha+2}} \exp\left[i\left(\theta^* + (t^*-t)\right)\right] & C^* = 0 \end{cases}$$

Finally, in the last chapter, we conclude this thesis by comparing the pros and cons of the two main method we have used throughout this thesis: the energy method and the McGehee-type transform method. This section also includes some brief discussion of untackled cases and possible future works.

Chapter 2

Quasi-homogeneous two-body problem

2.1 Settings

The quasi-homogeneous potential W, is defined to be a sum of two homogeneous potentials U and V. That is,

$$W(x_1, x_2, ...x_n) = U(x_1, x_2, ...x_n) + V(x_1, x_2, ...x_n),$$

where $x_i \in \mathbb{R}^3$, for $i = 1, 2, ...n$, is the location of each particle, and U and V are defined respectively as

$$\begin{aligned} U(x_1, x_2, ...x_N) &= -\sum_{1\leq i<j\leq N} \frac{m_i m_j}{\left|x_i - x_j\right|^a} \\ V(x_1, x_2, ...x_N) &= -\sum_{1\leq i<j\leq N} \frac{m_i m_j}{\left|x_i - x_j\right|^b} \quad \text{where } 0 < a < b. \end{aligned}$$

For each particle's location x_i, its equation of motion can be obtained by Newton's second law,

$$\begin{aligned} m_i \ddot{x}_i &= -\frac{\partial W}{\partial x_i}(x_1, x_2, ...x_n) \\ &= \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[\frac{-a m_i m_j}{\left|x_i - x_j\right|^{a+2}} + \frac{-b m_i m_j}{\left|x_i - x_j\right|^{b+2}} \right] (x_i - x_j). \end{aligned} \tag{2.1.1}$$

The Quasi-homogeneous N-body problem is an autonomous Hamiltonian system, meaning that the energy is conserved along any solution. It is not difficult to see that the energy can be written as

$$E(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}\sum_{i=1}^{N} m_i |\dot{x}_i|^2 + U(\mathbf{x}) + V(\mathbf{x}), \tag{2.1.2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in (\mathbb{R}^3)^N$.

The Lagrange-Jocabi identity for the quasi-homogeneous N-body problem can be calculated as

$$\frac{d^2 I}{dt^2}(\mathbf{x}(t)) \equiv \ddot{I}(\mathbf{x}(t)) = 4\left[E(\mathbf{x}(t), \dot{\mathbf{x}}(t)) + (a/2-1)U(\mathbf{x}(t)) + (b/2-1)V(\mathbf{x}(t))\right], \tag{2.1.3}$$

where $E(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \equiv E_{\mathbf{x}}$ is the energy of a solution, independent of time, and I is the moment of inertia, which is defined as

$$I(\mathbf{x}) = \sum_{i=1}^{N} m_i |x_i|^2 .$$

The Lagrange-Jocabi identity is related to the collision singularity in the following sense. Suppose that there exist some time t_0 and some positive number δ such that $\ddot{I}(\mathbf{x}(t)) < -\delta$ for all $t > t_0$, then $I(\mathbf{x}(t))$ is bounded above by a concave down parabola which eventually becomes negative. Since the moment of inertia is always non-negative, $\mathbf{x}(t)$ must have a singularity. In fact, this technique has been repeatedly used in the construction of invariant sets of initial conditions that correspond to finite-time collision ([3]).

For two body-systems, it is well known that any solution is planar, thus the equation of motion and can be reduced to

$$\ddot{x} = (-\frac{a}{|x|^{a+2}} - \frac{b}{|x|^{b+2}})Mx, \tag{2.1.4}$$

where $M = m_1 + m_2$ is the sum of masses and $x = x_1 - x_2 \in \mathbb{R}^2$ is the difference of locations. Without loss of generality, we assume $M = 1$. Using polar coordinates

(r,θ), the previous equation is further reduced to

$$\begin{aligned}\ddot{r} - r\dot{\theta}^2 &= (-\frac{a}{r^{a+2}} - \frac{b}{r^{b+2}})r \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0,\end{aligned} \tag{2.1.5}$$

and the two homogeneous potentials are identified to be $U(r) = -\frac{1}{r^a}$ and $V(r) = -\frac{1}{r^b}$.

The angular momentum

$$L(x) = r^2\dot{\theta}$$

is constant along any solution and we denote its magnitude $c := \big|L(x)\big|$. The total energy $E(x(t),\dot{x}(t))$ is also preserved along any solution $x(t)$ and

$$\begin{aligned}E(x,\dot{x}) = E(r,\dot{r},\dot{\theta}) &= \frac{1}{2}|\dot{x}|^2 + U(x) + V(x) \\ &= \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) + V(r).\end{aligned} \tag{2.1.6}$$

Thus, for fixed c and E,

$$\Gamma_{c,E}: \qquad E = \frac{1}{2}\dot{r}^2 + \frac{c^2}{2r^2} + U(r) + V(r) \tag{2.1.7}$$

determines an orbit in the phase space $(r,\dot{r})$. From here, we recognize the effective potential can be written as

$$U_{\text{eff}}(c,r) = \frac{c^2}{2r^2} + U(r) + V(r).$$

The indicator function [3] is defined by the scaling argument on $U_{\text{eff}}(c,r)$. That is,

$$K(c,r) := -\frac{d}{d\lambda}U_{\text{eff}}(\lambda r, c)|_{\lambda=1} = \frac{c^2}{r^2} + aU(r) + bV(r). \tag{2.1.8}$$

Notice that $K(c,r) = -r\frac{d}{dr}U_{\text{eff}}(c,r)$.

Remark 2.1.1. *In contrast to [3], where the indicator function is defined with respect to angular speed ω, we find it more convenient to define the effective potential and indicator function here with magnitude of angular momentum as a parameter. However, once we know the angular momentum of a relative equilibrium, we know the frequency of it and the converse also holds. Let the angular momentum, radius and frequency of a relative equilibrium be c_0, r_0, ω_0 then for normalized two-body problem* *we have:*

$$\begin{cases} c_0^2 = r_0^2 \omega_0 \\ K(c_0, r_0) = \frac{c^2}{r_0^2} - a\frac{1}{r_0^a} - b\frac{1}{r_0^b} = 0 \end{cases} \tag{2.1.9}$$

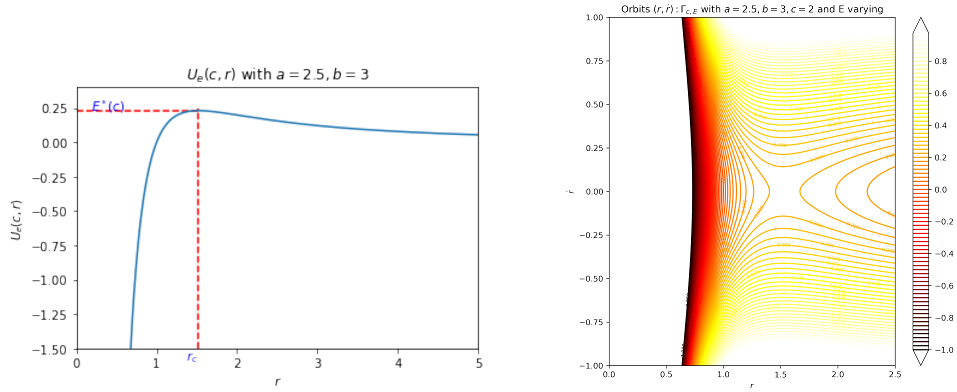
This is a system of equation with four variables, knowing one of c_0, r_0, ω_0 will effectively solve the other two. Notice that in the case of $b > 2 > a$, if we know c_0, the value of r_0 is not uniquely determined by the above equation since we have two relative equilibria that satisfy it. However, since one of them is a local minimum of $U_{eff}(c_0, \cdot)$, we can use second derivative test to determine the unique value of r_0.

2.2 Case by case analysis

2.2.1 Strong-strong potential

This subsection corresponds to the case when $b > a > 2$, and we will see it resembles the homogeneous case with strong potential.

Deng and Ibrahim [3] proved that for strong homogeneous potentials, it is possible to construct invariant sets for each fixed angular momentum c, based on the sign of indicator function $K(c, r)$ below the so-called "ground state energy", which is the global maximum of effective potential. In this case, their method is still applicable and the effective potential has the same shape. (See Figure (2.1)(a).) Before digging into invariant sets, let us first derive some properties about the relative equilibrium points.



(a) effective potential $U_{\text{eff}}(c,\cdot)$ for fixed $c > 0$

(b) phase space $(r,\dot{r})$

Figure 2.1: Case $b > a > 2$

Setting $\frac{d}{dr}U_{\text{eff}}(c,r) = 0$ gives us the unique critical point of $U_{\text{eff}}(c,\cdot) : r_c > 0$ and it satisfies

$$ar_c^{-a+2} + br_c^{-b+2} = c^2,$$

for each $c > 0$ and $b > a > 2$. We also denote the "ground state energy"

$$E^*(c) := U_{\text{eff}}(c, r_c).$$

for each other $c > 0$ whenever $b > a > 2$.

Remark 2.2.1. *For each fixed $c > 0$ and $b > a > 2$, it is straightforward to see that the following properties hold.*

1. $K(c,r) < 0 \iff r < r_c$.

2. $K(c,r) > 0 \iff r > r_c$.

3. $K(c,r) = 0 \iff r = r_c$.

4. $U_{\text{eff}}(c,\cdot)$ *is ascending with respect to c. That is, for each $0 < c_1 < c_2$ and $r > 0$,* $U_{\text{eff}}(c_1,r) < U_{\text{eff}}(c_2,r)$.

Lemma 2.2.1. *Fix $b > a > 2$ and $c > 0$, define*

$$\begin{aligned} K^+(c) &= \left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) < E^*(c), c(x,\dot{x}) \geq c, K(c,r) \geq 0\right\} \\ K^-(c) &= \left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) < E^*(c), c(x,\dot{x}) \geq c, K(c,r) < 0\right\}, \end{aligned} \tag{2.2.1}$$

where $c(x,\dot{x})$ is the angular momentum and $E(x,\dot{x})$ is the energy of initial condition $(x,\dot{x})$ and $r = |x|$. For each $c > 0$, $K^{\pm}(c)$ are invariant.

Proof. We show that $K^-(c)$ is invariant. Let $(x(t),\dot{x}(t))$ be a solution to the two-body problem with $(x(0),\dot{x}(0)) \in K^-(c)$. Since both energy and angular momentum are invariant, $K^+(c) \bigcup K^-(c)$ is invariant and we thus have $c(x,\dot{x}) \geq c$ and $E_x \equiv E(x,\dot{x}) < E^*(c)$. Let $r(t) = |x|(t)$ and suppose that there is t_1 such that $K(c, r(t_1)) = 0$ then $r(t_1) = r_c$. Since U_{eff} is ascending with respect to c, we have

$$E_x \equiv U_{\text{eff}}(c(x,\dot{x}), r(t_1)) \geq U_{\text{eff}}(c, r(t_1)) = E^*(c).$$

On the other hand,

$$E_x < E^*(c),$$

by assumption, and thus we arrive at a contradiction. □

Proposition 2.2.1. *Fix $b > a > 2$ and $c > 0$, any initial condition in $K^+(c)$ corresponds to global solution.*

Proof. Let $(x(t),\dot{x}(t))$ be a solution to the two-body problem with $(x(0),\dot{x}(0)) \in K^+(c)$. The result follows since $K^+(c)$ is invariant and $K(c, |x|(t)) > 0 \iff |x|(t) > r_c$. □

Proposition 2.2.2. *Fix $b > a > 2$ and $c > 0$, any initial condition in $K^-(c)$ corresponds to finite-time collision.*

Proof. Let $(x(t),\dot{x}(t))$ be a solution in $K^-(c)$ and thus there exists $\delta > 0$ such that

$$E_x \equiv E(x,\dot{x}) < E^*(c) - \delta = U_{\text{eff}}(c, r_c) - \delta.$$

Using Lagrange-Jacobi identity, we have

$$\begin{aligned} \ddot{I}(x(t)) &= 4\left[E_x + (1-\frac{b}{2})\frac{1}{r(t)^b} + (1-\frac{a}{2})\frac{1}{r(t)^a}\right] \\ &< 4\left[U_{\text{eff}}(c, r_c) + (1-\frac{b}{2})\frac{1}{r(t)^b} + (1-\frac{a}{2})\frac{1}{r(t)^a}\right] - 4\delta, \end{aligned}$$

where $r = |x|$. Let

$$f(r) = U_{\text{eff}}(c, r_c) + (1-\frac{b}{2})\frac{1}{r^b} + (1-\frac{a}{2})\frac{1}{r^a}.$$

By expanding $U_{\text{eff}}(c, r_c)$, one can check that

$$f(r_c) = \frac{1}{2}K(c, r_c) = 0,$$

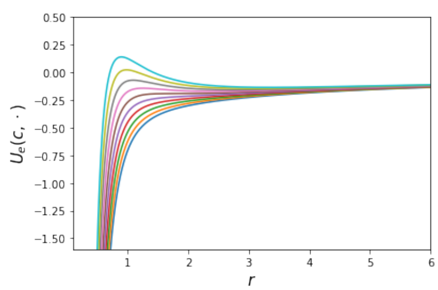
and f is increasing on $r > 0$ for $b > a > 2$. Combining with $K(c, r(t)) < 0 \iff r(t) < r_c$, we conclude that $f(r(t)) < f(r_c) = 0 \quad \forall t \in \mathbb{R}$ and thus

$$\ddot{I}(x(t)) < -4\delta \quad \forall t \in \mathbb{R}.$$

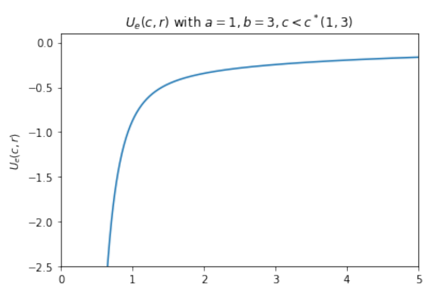
$I(x(t))$ is now bounded above by a concave down parabola which eventually becomes negative for sufficiently large t. On the other hand $I(x)$ is a non-negative function, by definition. Therefore, we conclude the solution x must have a singularity. $\square$

2.2.2 Strong-weak potential

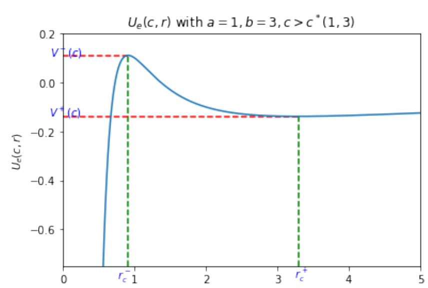
For $b > 2 > a$, a combination of strong and weak potentials, we will see the existence of relative equilibrium depends on the magnitude of angular momentum. Let us first focus on the function $U_{\text{eff}}(c, r)$.



(a) Effective potentials are ascending



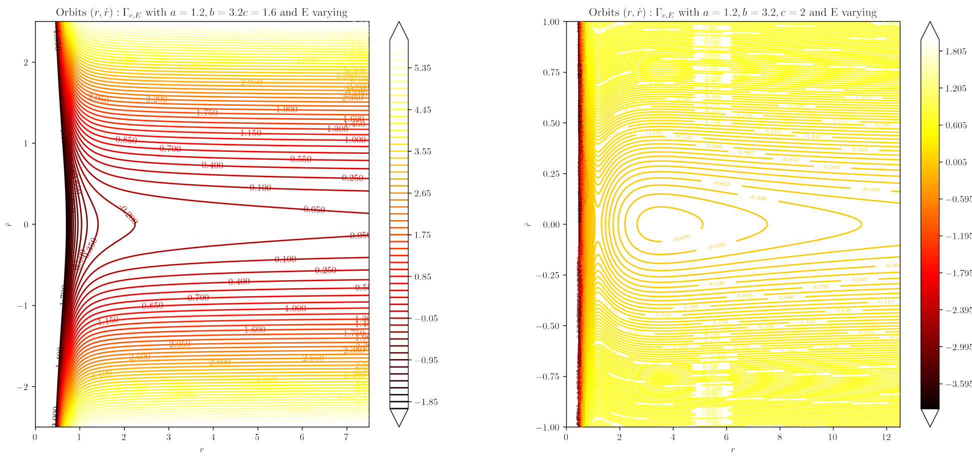
(b) $c < c^*(a, b)$



(c) $c < c^*(a, b)$

Figure 2.2: Effective potentials with different angular momenta

For low angular momentum c, we see that $U_{\text{eff}}(c,\cdot)$ has no critical point, and $U_{\text{eff}}(c,\cdot)$ is increasing on $(0,\infty)$ from $-\infty$ to 0. As c increases, two critical points emerge. One corresponds to its global maximum and we denote its coordinate as $(r_c^-, V^-(c))$. The other one corresponds to its local minimum and we denote its coordinate as $(r_c^+, V^+(c))$.



(a) $c < c^*(a,b)$

(b) $c > c^*(a,b)$

Figure 2.3: Phase space $(r,\dot{r})$ when $b > 2 > a$ with different angular momenta

This phenomenon is clearly illustrated in the phase space $(r,\dot{r})$ (Figure (2.3)): For fixed low c, there is no equilibrium. As c exceeds some threshold, two equilibria emerge. One corresponds to a saddle and the other corresponds to a center. We denote this critical angular momentum $c^* = c^*(a,b)$.

Remark 2.2.2. *We can check the following properties are valid for fixed $b > 2 > a$ and $c > c^*$.*

1. *$U_{\text{eff}}(c,\cdot)$ is increasing on $(0, r_c^-)$ from $-\infty$ to V_c^-, decreasing on (r_c^-, r_c^+) from $V^-(c)$ to $V^+(c)$ and increasing on (r_c^+,∞) from $V^+(c)$ to 0.*

2. *$r_c^- < r_c^+$.*

3. *$V^+(c) < 0$ and $V^-(c) > V^+(c)$.*

2.2.2.1 The angular momentum threshold $c^*(a,b)$

Expanding $\frac{d}{dr}U_{\text{eff}}(c,r) = 0$ gives us

$$ar^{-a+2} + br^{-b+2} - c^2 = 0.$$

If we define $\Phi(r) = ar^{-a+2} + br^{-b+2}$, then the previous equation is reduced to

$$\Phi(r) = c^2. \tag{2.2.2}$$

To find the critical angular momentum c^*, we simply count the number of solutions of the previous equation. Geometrically, in the (r, y) plane, the horizontal line $y = c^2$ originally has no intersection with $\Phi(r)$ for low c, as we increase the value of $c > 0$, the number of intersections becomes one then two. We need to find the c such that $\Phi(r) = c^2$ has only one intersection, which is equivalent to find $\min_{r>0} \Phi(r)$. We denote the coordinate of this global minimum as $(r^*, {c^*}^2)$ and its energy

$$E^* \equiv E^*(a,b) = U_{\text{eff}}(c^*, r^*).$$

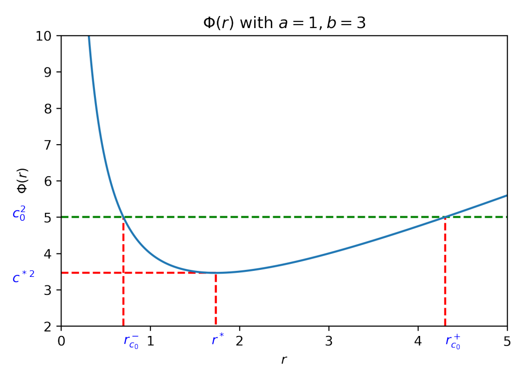


Figure 2.4: The function of $\Phi(r)$, where $c_0 > c^*$

Solving

$$\begin{cases} \Phi(r^*) = c^{*2} \\ \frac{d}{dr}\Phi(r^*) = 0, \end{cases} \tag{2.2.3}$$

we see that

$$(r^*, c^*(a,b)) = \left(\left(\frac{b(b-2)}{a(2-a)} \right)^{\frac{1}{b-a}}, \left[a \left(\frac{b(b-2)}{a(2-a)} \right)^{\frac{2-a}{b-a}} + b \left(\frac{a(2-a)}{b(b-2)} \right)^{\frac{b-2}{b-a}} \right]^{\frac{1}{2}} \right). \tag{2.2.4}$$

Remark 2.2.3. *One can easily verify the following properties for any fixed $b > 2 > a$.*

1. *Equation (2.2.3) is equivalent to*

$$\begin{cases} \frac{d}{dr} U_{\mathit{eff}}(c^*, r^*) = 0 \\ \frac{d^2}{dr^2} U_{\mathit{eff}}(c^*, r^*) = 0. \end{cases}$$

2. *For $c > c^*$, $r_c^- < r^* < r_c^+$.*
3. *For $c \leq c^*$, $U_{\mathit{eff}}(c, \cdot)$ is increasing on $(0, \infty)$ from $-\infty$ to 0.*
4. *For $c > c^*$, we have $U_{\mathit{eff}}(c, r) < V^+(c) \implies r < r_c^-$.*

2.2.2.2 Invariant sets when $c > c^*(a, b)$

For each $c > c^*(a, b)$, the effective potential $U_{\text{eff}}(c, \cdot)$ has two critical points. Thus, according to [3], we have two constructions of invariant sets, based on either the energy $V^-(c)$ or $V^+(c)$.

However, the sign of K will not be included in the construction since it provides insufficient information. For example, when $c > c^*(a, b)$, $K(c, r) < 0$ implies $r < r_c^-$ or $r > r_c^+$.

Proposition 2.2.3. *Fix $b > 2 > a$ and $c_0 > c^*(a, b)$. Define*

$$K_{col}^+(c_0) = \left\{ (x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x, \dot{x}) < V^+(c_0), c(x, \dot{x}) \geq c_0 \right\}.$$

$K_{col}^+(c_0)$ is invariant and each initial condition corresponds to finite-time collision.

Proof. Let $(x(t), \dot{x}(t))$ be a solution in $K_1(c)$ and thus there exists $\delta > 0$ such that

$$E_x \equiv E(x, \dot{x}) < V^+(c_0) - \delta < V^-(c_0) - \delta = U_{\text{eff}}(c_0, r_{c_0}^-) - \delta$$

Using Lagrange-Jacobi identity, we have

$$\begin{aligned}\ddot{I}(x(t)) &= 4\left[E_x + (1-\frac{b}{2})\frac{1}{r(t)^b} + (1-\frac{a}{2})\frac{1}{r(t)^a}\right] \\ &= 4\left[U_{\text{eff}}(c_0, r_{c_0}^-) + (1-\frac{b}{2})\frac{1}{r(t)^b} + (1-\frac{a}{2})\frac{1}{r(t)^a}\right] - 4\delta\end{aligned}$$

where $r = |x|$. Let

$$f(r) = U_{\text{eff}}(c_0, r_{c_0}^-) + (1-\frac{b}{2})\frac{1}{r^b} + (1-\frac{a}{2})\frac{1}{r^a}.$$

By expanding $U_{\text{eff}}(c_0, r_{c_0}^-)$, one can check that

$$f(r_{c_0}^-) = \frac{1}{2}K(c_0, r_{c_0}^-) = 0,$$

and f has a unique critical point at $r = r^*$. More specifically, f is increasing on $(0, r^*)$ and decreasing on (r_*, ∞). In particular, f is increasing on $(0, r_{c_0}^-)$. Moreover, since U_{eff} is ascending with respect to c and K_1 is invariant, we have

$$U_{\text{eff}}(c_0, r(t)) \leq U_{\text{eff}}(c(x,\dot{x}), r(t)) \leq E_x < V^+(c_0) \quad \forall\, t \in \mathbb{R},$$

which implies

$$r(t) < r_{c_0}^- \quad \forall\, t \in \mathbb{R}.$$

Therefore, $f(r(t)) \leq f(r_{c_0}^-) = 0$ and thus

$$\ddot{I}(x(t)) < -4\delta \quad \forall\, t \in \mathbb{R}.$$

$I(x(t))$ is now bounded above by a concave down parabola which eventually becomes negative for sufficiently large t. On the other hand $I(x)$ is a non-negative function, by definition. Therefore, we conclude the solution x must have a singularity. □

Lemma 2.2.2. *Fix $b > 2 > a$ and $c_0 > c^*(a,b)$. Define*

$$\begin{aligned}K_{col}(c_0) &= \left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) < V^-(c_0), c(x,\dot{x}) \geq c_0, r < r_{c_0}^-\right\} \\ K_{glob}(c_0) &= \left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) < V^-(c_0), c(x,\dot{x}) \geq c_0, r \geq r_{c_0}^-\right\}.\end{aligned}$$

Both $K_{col}(c_0)$ and $K_{glob}(c_0)$ are invariant.

Proof. We show that $K_{\text{glob}}(c_0)$ is invariant. Let $(x(t), \dot{x}(t))$ be a solution to the two-body problem with $(x(0), \dot{x}(0)) \in K_{\text{glob}}(c_0)$. Since both energy and angular momentum are invariant, $K_{\text{col}}(c_0) \bigcup K_{\text{glob}}(c_0)$ is invariant and we thus have $c(x, \dot{x}) \geq c_0$ and $E_x \equiv E(x, \dot{x}) < V^-(c_0)$. Let $r(t) = |x|(t)$ and suppose that there is t_1 such that $r(t_1) = r^-_{c_0}$. Since U_{eff} is ascending with respect to c, we have

$$E_x \equiv U_{\text{eff}}(c(x, \dot{x}), r(t_1)) \geq U_{\text{eff}}(c_0, r^-_{c_0}) = V^-(c_0).$$

On the other hand,

$$E_x < V^-(c_0),$$

by assumption, and thus we arrive at a contradiction. □

Proposition 2.2.4. *Fix $b > 2 > a$ and $c_0 > c^*(a, b)$. $K_{col}(c_0)$ corresponds to singular solutions and $K_{glob}(c_0)$ corresponds to global solutions.*

Proof. The proof is similar to the previous proposition. □

2.2.2.3 The degenerate case $c = c^*(a, b), E = E^*(a, b)$.

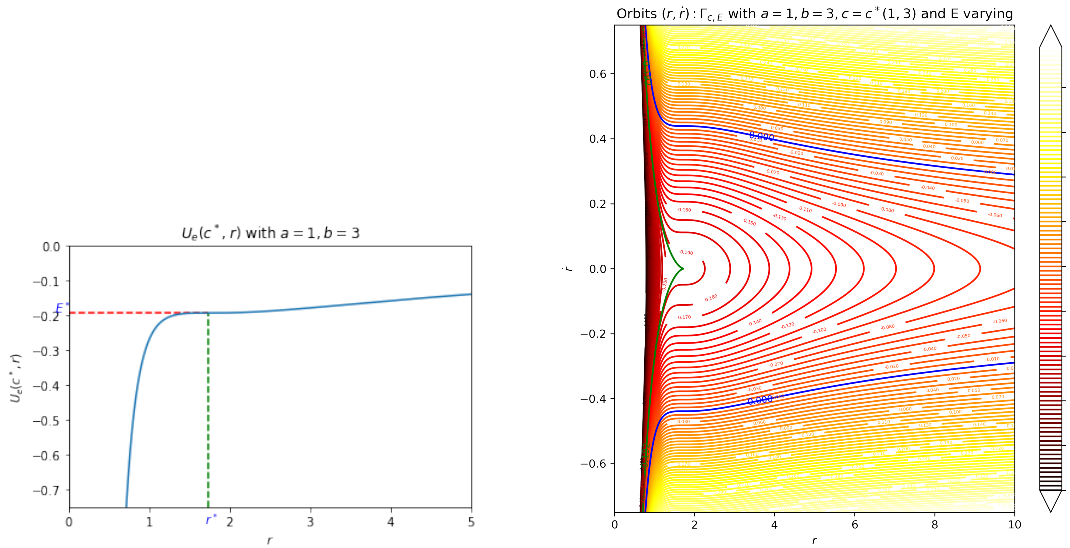
Recall that the function $U_{\text{eff}}(c^*, \cdot)$ is increasing, the unique critical point r^* satisfies $\frac{d}{dr} U_{\text{eff}}(c^*, r^*) = \frac{d^2}{dr^2} U_{\text{eff}}(c^*, r^*) = 0$ and $E^* := U_{\text{eff}}(c^*, r^*)$. Let $(x(t), \dot{x}(t))$ be a solution of the two-body problem that satisfies $E(x, \dot{x}) \equiv E^*$ and $c(x(t), \dot{x}(t)) \equiv c^*$. From energy relationship, the radius of the solution $r = r(t)$ satisfies

$$E^* = U_{\text{eff}}(c^*, r) + \frac{1}{2}\dot{r}^2, \tag{2.2.5}$$

which implies

$$U_{\text{eff}}(c^*, r^*) \geq U_{\text{eff}}(c^*, r),$$

and thus $r \leq r^*$ and $r = r^* \iff \dot{r} = 0$.



(a) Effective potential $U_{\text{eff}}(c^*, r)$

(b) Phase space $(r, \dot{r})$ where green(blue) line indicates $E = E^*(E = 0)$

Figure 2.5: What happens when $c = c^*(a, b)$?

Moreover, when $r < r^*$, we see that there are two radial speed with exactly opposite signs corresponding to each r that satisfies Equation (2.2.5). This implies

$$A^+ = \left\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : c(x, \dot{x}) = c^*, E(x, \dot{x}) = E^*, \dot{r} > 0\right\}$$
$$A^- = \left\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : c(x, \dot{x}) = c^*, E(x, \dot{x}) = E^*, \dot{r} < 0\right\}$$

are backward and forward invariant respectively. We show that $r = r^*$ is an attractor for solutions with strictly positive initial radial speed by calculating the time it spends to reach r^* from $r_0 < r^*$. Indeed, the time T can be represented by using the energy relation as

$$T = \int_{r_0}^{r^*} \frac{1}{\sqrt{2E^* + \frac{2}{s^a} + \frac{2}{s^b} - \frac{{c^*}^2}{s^2}}} ds.$$

We notice that the only singularity of this integrand is $s = r^*$ and it can be shown that

$$\frac{1}{r^* - s} = o\left(\frac{1}{\sqrt{2E^* + \frac{2}{s^a} + \frac{2}{s^b} - \frac{{c^*}^2}{s^2}}}\right) \quad \text{as } s \to {r^*}^-,$$

by using the fact $\frac{d}{dr}U_{\text{eff}}(c^*, r^*) = \frac{d^2}{dr^2}U_{\text{eff}}(c^*, r^*) = 0$ and L'Hopital's rule. Indeed,

$$\begin{aligned}
\lim_{s\to r^{*-}} \frac{2E^* + \frac{2}{s^a} + \frac{2}{s^b} - \frac{{c^*}^2}{s^2}}{(r^*-s)^2} &= \lim_{s\to r^*} \frac{2E^* - 2U_{\text{eff}}(s, c^*)}{(r^*-s)^2} \\
&= \lim_{s\to r^{*-}} \frac{-2\frac{d}{ds}U_{\text{eff}}(s, c^*)}{-2(r^*-s)} \\
&= \lim_{s\to r^{*-}} \frac{\frac{d^2}{ds^2}U_{\text{eff}}(s, c^*)}{-1} = 0.
\end{aligned}$$

Since

$$\int_{r_0}^{r^*} \frac{1}{r^*-s} ds = \infty$$

and thus $T = \infty$. By symmetry, for solutions with strictly negative initial radial speed, $r = r^*$ is a repellor.

For any solution starts in A^-, does it take finite time to reach a collision? First of all, we see that there does not exist a lower bound for r since if it does, then the lower bound must be r^*. Likewise, if the initial distance between two particles is $r = r_0 < r^*$ the collision time T_{col} can be represented as

$$T_{\text{col}} = \int_0^{r_0} \frac{1}{\sqrt{2E^* + \frac{2}{s^a} + \frac{2}{s^b} - \frac{{c^*}^2}{s^2}}} ds$$

and we see that the integrand is continuous at $s = 0$, which implies $T_{\text{col}} < \infty$.

Proposition 2.2.5. *Fix $b > 2 > a$. The following statements are valid.*

1. *The set $A^+ = \{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : c(x, \dot{x}) = c^*, E(x, \dot{x}) = E^*, \dot{r} > 0\}$ is invariant and contains global solution. For initial condition $(x(0), \dot{x}(0)) \in A^+$ its corresponding solution $x(t)$ satisfies $r(t) \to r^*(a, b)^-$ as $t \to \infty$.*

2. *The set $A^- = \{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : c(x, \dot{x}) = c^*, E(x, \dot{x}) = E^*, \dot{r} < 0\}$ is invariant and contains solution with finite-time collision. For initial condition $(x(0), \dot{x}(0)) \in A^-$ its corresponding solution $x(t)$ satisfies $r(t) \to r^*(a, b)^-$ as $t \to -\infty$.*

3. *The set $A^0 = \{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x| = r^*(a, b), \dot{r} = 0\}$ is also invariant.*

2.2.2.4 Subcase $c < c^*(a, b)$ or $c = c^*(a, b), E \neq E^*(a, b)$

When $c < c^*(a, b)$, $U_{\text{eff}}(c, \cdot)$ does not have a critical point and it is strictly increasing on $(0, \infty)$ ranging from $(-\infty, 0)$. This case resembles the case when $c \leq \sqrt{2}$ and

$a < b = 2$ and can be argued similarly based on the energy $E > 0$ and $E \leq 0$. For $c = c^*(a, b)$ and $E \neq E^*$, the threshold energy is again zero.

Proposition 2.2.6. *Fix $b > 2 > a$. The following statements are valid.*

1. *Both*
$$\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x, \dot{x}) < 0, 0 < c(x, \dot{x}) < c^*(a, b)\}$$
and
$$\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x, \dot{x}) \geq 0, 0 < c(x, \dot{x}) < c^*(a, b), \dot{r} < 0\}$$
are invariant and contains solution with finite-time collision.

2. $\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x, \dot{x}) \geq 0, 0 < c(x, \dot{x}) < c^*(a, b), \dot{r} > 0\}$ *is invariant and corresponds to global solutions.*

3. $\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : 0 > E(x, \dot{x}) \neq E^*(a, b), c(x, \dot{x}) = c^*(a, b)\}$ *is invariant and contains solution with finite-time singularity.*

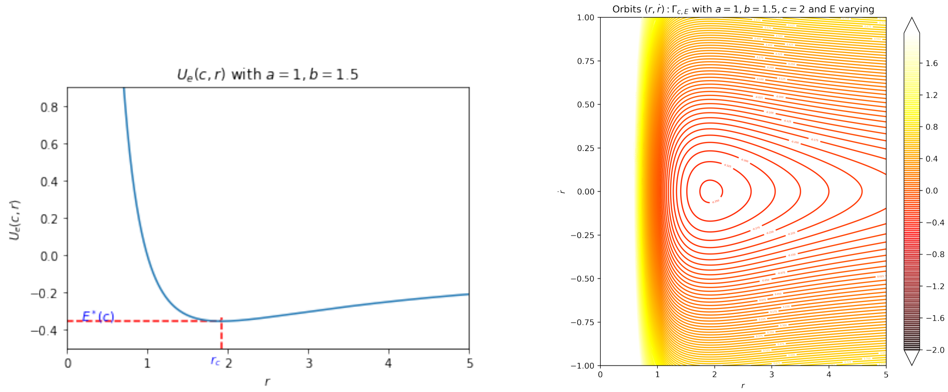
4. $\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x, \dot{x}) \geq 0, c(x, \dot{x}) = c^*(a, b), \dot{r} > 0\}$ *is invariant and corresponds to global solutions.*

5. $\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x, \dot{x}) \geq 0, c(x, \dot{x}) = c^*(a, b), \dot{r} < 0\}$ *is invariant and contains solution with finite-time collision.*

Proof. This proof is similar to the case of $a < 2 = b$ and $c \leq \sqrt{2}$. (Cf 2.2.5.3.) $\square$

2.2.3 Weak-weak potential

The case when $a < b < 2$ is actually quite simple and we will see it resembles the weak homogeneous potential case, where the set of collision initial conditions have zero Lebesgue measure in the phase space $(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2$.



(a) effective potential $U_{\text{eff}}(c, \cdot)$ for fixed $c > 0$

(b) phase space $(r, \dot{r})$

Figure 2.6: Case $2 > b > a$

From energy relation

$$E = U_{\text{eff}}(c, r) + \frac{1}{2}\dot{r}^2 = -\frac{1}{r^a} - \frac{1}{r^b} + \frac{c^2}{2r^2} + \frac{1}{2}\dot{r}^2,$$

we have

$$-r^{2-a} - r^{2-b} + \frac{c^2}{2} \leq Er^2.$$

Letting $r \to 0^+$, we see

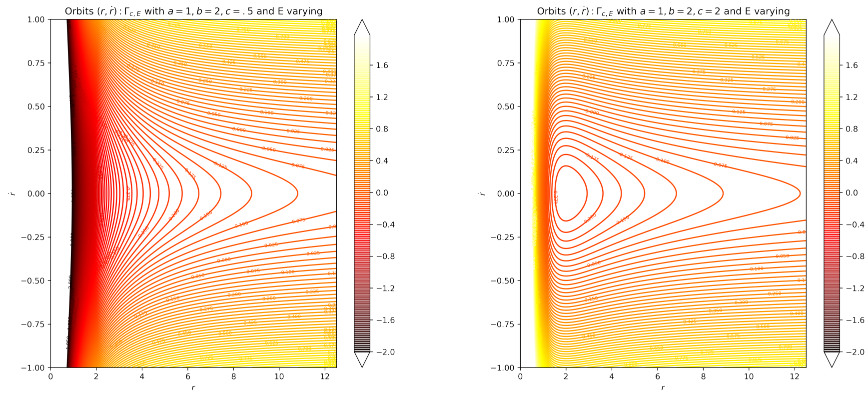
$$\frac{c^2}{2} \leq 0,$$

which is only possible when $c = 0$.

Proposition 2.2.7. *Fix $a < b < 2$ and let $(x(t), \dot{x}(t))$ be a solution of the two-body problem that has non-zero angular momentum. Then x is a global solution.*

2.2.4 Critical angular momentum for the two limiting cases

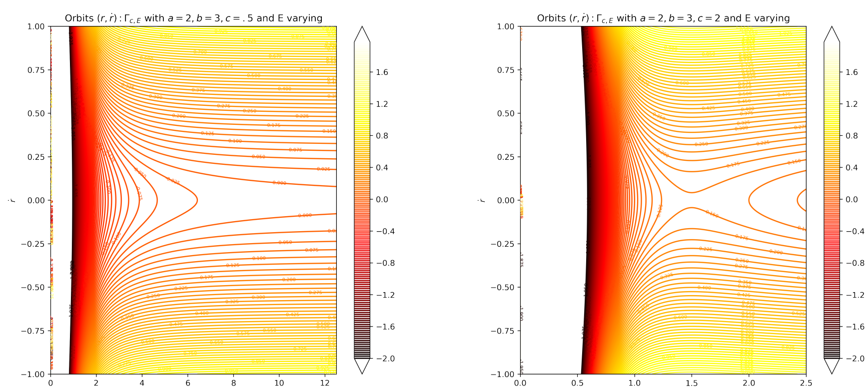
We begin by plotting some orbits in the $(r, \dot{r})$ plane with different angular momenta.



(a) $\Gamma_{c,E}$ with low angular momentum

(b) $\Gamma_{c,E}$ with high low angular momentum

Figure 2.7: Phase space $(r, \dot{r})$ when $2 = b > a$



(a) $\Gamma_{c,E}$ with low angular momentum

(b) $\Gamma_{c,E}$ with high low angular momentum

Figure 2.8: Phase space $(r, \dot{r})$ when $2 = a < b$

There are several observations that can be made from these graphs above. (Rigorous justifications follow in the subsequent subsections.) First, regardless of the power combinations, as long as one power is 2, when the angular momentum is relatively low, there is no relative equilibrium in the $(r, \dot{r})$ plane. As c becomes larger, a critical point emerges in both cases.

However, the potential that does not have power 2 determines the stability of this equilibrium. If the stronger potential has power 2, then the critical point is a center, otherwise, it is a saddle point.

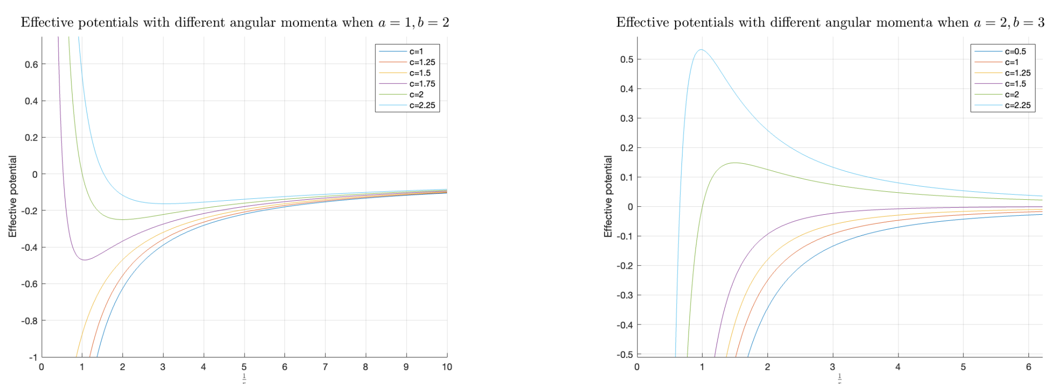
Unlike the case when $a < 2 < b$, where, as c exceeds the threshold $c^*(a,b)$, we always see one stable and one unstable critical points, we only have one equilibrium here when the angular momentum is sufficiently large.

The critical angular momentum is calculated in a similar fashion as in the case of $b > 2 > a$. When $a < b = 2$, we have

$$E = \frac{1}{2}\dot{r}^2 + U_{\text{eff}}(c,r),$$

where the effective potential is

$$U_{\text{eff}}(c,r) = -\frac{1}{r^a} - \frac{1}{r^2} + \frac{c^2}{2r^2}. \tag{2.2.6}$$



(a) Effective potentials as c varies when $a < b = 2$

(b) Effective potentials as c varies when $b > a = 2$

Figure 2.9: Effective potentials of limiting cases

Since this is a Hamiltonian system, critical points have the form $(r, \dot{r}) = (r^*, 0)$, where

$$\frac{d}{dr}U_{\text{eff}}(c, r^*) = (2 - c^2)r^{*-3} + ar^{*-a-1} = 0.$$

In other words, we want to know that under for which values of c, the equation

$$c^2 = ar^{-a+2} + 2 \tag{2.2.7}$$

has a solution. It is easy to see that the answer is if and only if $c^2 > 2$. Furthermore, if the equation has a solution, then it must be unique. Let us denote the critical point of the effective potential associated with an angular momentum by r_c^*, and its corresponding effective potential as

$$r_c^* = \left(\frac{c^2-2}{a}\right)^{\frac{1}{2-a}}, \quad E^*(c) = -(1-\frac{a}{2})\left(\frac{a}{c^2-2}\right)^{\frac{a}{2-a}}$$

whenever they exist. Similarly, when $b > a = 2$, the effective potential is

$$U_{\text{eff}}(c,r) = \frac{c^2-2}{2r^2} - \frac{1}{r^b}.$$

We can also check that $U_{\text{eff}}(c,r)$ has a critical point $\iff c^2 > \sqrt{2}$ and in this case, the unique critical point and its corresponding effective potential satisfy

$$r_c^* = \left(\frac{b}{c^2-2}\right)^{\frac{1}{b-2}}, \quad E^*(c) = (\frac{b}{2}-1)\left(\frac{c^2-2}{b}\right)^{\frac{b}{b-2}}.$$

In the following two sections, we shall show that whenever a solution has angular momentum $c > \sqrt{2}$, the power that is not two dominates the solution dynamics. When $c \leq \sqrt{2}$, the black hole effect is observable: we shall construct positive-measure invariant sets containing initial conditions that lead to finite-time collision.

We discuss the case when $a < b = 2$ first, since the other case can be argued similarly and this case corresponds to numerous physical applications, such as Manev potential.

2.2.5 Manev potential

Potentials that satisfy $b = 2 > a$ should be considered as a generalization of the Manev potential. Indeed, when $b = 2 > a$ and $c > \sqrt{2}$, we find that the weak homogeneous potential dominates the behavior of the solution dynamics: all solutions with non-zero angular momentum are global. While for $c < \sqrt{2}$ and $b = 2 > a$, the "black hole" effect [7] is present: We are able to find invariant sets with positive measure that contains initial conditions leading to collision in the phase space.

We present two constructions of invariant sets when $c < \sqrt{2}$ and $b = 2 > a$. One is based on the energy method [3] and the other one is based on the energy relation and the integrability of the two-body system. We will see that the latter leads to a complete characterization of the phase space $(r, \dot{r})$.

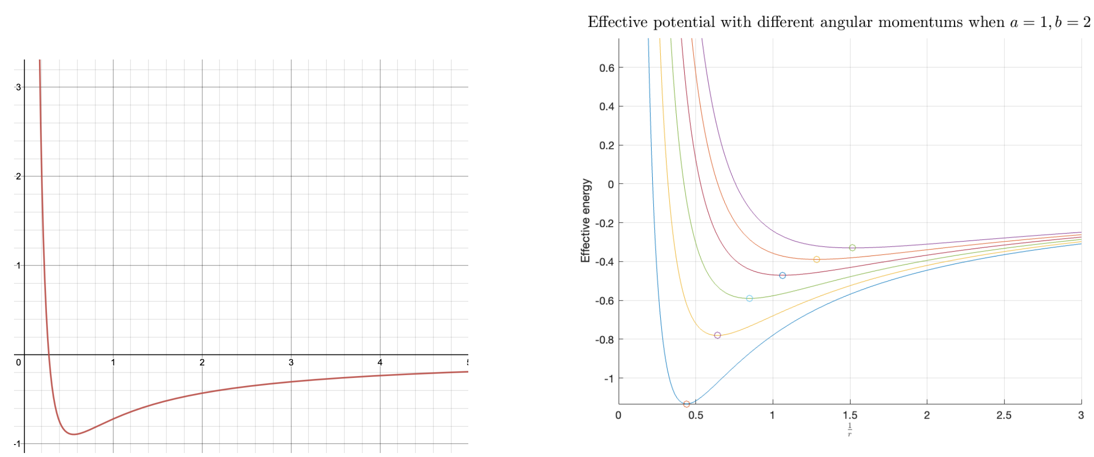
2.2.5.1 Subcase $c > \sqrt{2}$

It is straightforward to observe the following properties for U_{eff} whenever $b = 2 > a$.

1. for $c^2 > 2$ and c fixed, $U_{\text{eff}}(c, \cdot)$ achieves minimum at r_c^* and the minimum is $E^*(c) = -(1 - \frac{a}{2})\left(\frac{a}{c^2-2}\right)^{\frac{a}{2-a}}$.

2. for $c^2 > 2$ and c fixed, $U_{\text{eff}}(c, \cdot)$ is decreasing on $(0, r_c^*)$ from ∞ to $E^*(c) < 0$, and it is increasing (r_c^*, ∞) from $E^*(c)$ to 0.

3. for $c^2 > 2$, r_c^* and $E^*(c) < 0$ increase as c increases. In other words, U_{eff} is ascending with respect to c.



(a) $U_{\text{eff}}(c, \cdot)$ for fixed $c > \sqrt{2}$

(b) $U_{\text{eff}}(c, \cdot)$ with different $c > \sqrt{2}$

Figure 2.10: Effective potentials when $2 = b > a$

From the energy relationship, we have

$$E = \frac{1}{2}\dot{r}^2 + \frac{c^2 - 2}{2r^2} - \frac{1}{r^a},$$

and it implies

$$Er^2 \geq \frac{c^2 - 2}{2} - r^{2-a}.$$

Letting $r \to 0^+$, we see

$$\frac{c^2 - 2}{2} \leq 0,$$

which is only possible when $c \leq \sqrt{2}$. Thus, we conclude

Proposition 2.2.8. *Fix $a < b = 2$ and let $(x(t), \dot{x}(t))$ be a solution of the two-body problem that satisfies $c(x, \dot{x}) > \sqrt{2}$. Then x is a global solution.*

2.2.5.2 Subcase $0 < c \leq \sqrt{2}$ an indirect approach

When $c^2 \leq 2$ and $b = 2 > a$, The function $U_{\text{eff}}(c, \cdot)$ is strictly increasing and it is bounded above by zero. The family of functions $U_{\text{eff}}(c, \cdot)$ is ascending with respect to the angular momentum c.

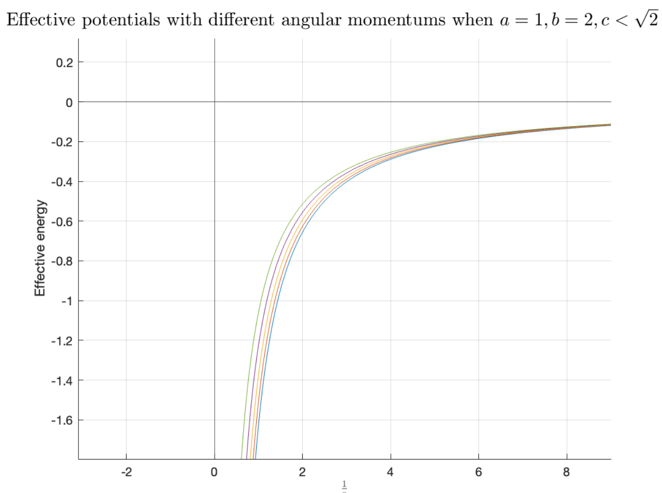


Figure 2.11: $U_{\text{eff}}(c, r)$'s with different $0 < c < \sqrt{2}$

The energy method in [3] requires the effective potential has at least one critical point, thus it is not directly applicable in this case. However, in the same paper, Deng and Ibrahim suggested a different way to construct invariant sets based on a different indicator function.

The Lagrange-Jacobi identity for a solution $(x(t), \dot{x}(t))$ with angular momentum $c(x, \dot{x})$ can be rewritten by plugging in the energy $E(x, \dot{x}) = \frac{1}{2}\dot{r}^2 + \frac{c(x,\dot{x})^2 - 2}{2r^2} - \frac{1}{r^a}$:

$$\begin{aligned} \ddot{I}(x(t)) &= 4\left(E(x, \dot{x}) + (\frac{a}{2} - 1)U(x)\right) \\ &= 4\left(\frac{1}{2}\dot{r}^2 - \frac{2 - c(x, \dot{x})^2}{2r^2} - \frac{a/2}{r^a}\right). \end{aligned}$$

The new indicator function $W(c, x, \dot{x})$

$$W(c, x, \dot{x}) \equiv W(c, r, \dot{r}) := \frac{1}{2}\dot{r}^2 - \frac{2-c^2}{2r^2} - \frac{a/2}{r^a},$$

and a generalized energy

$$E(c, r, \dot{r}) := \frac{1}{2}\dot{r}^2 + \frac{c^2-2}{2r^2} - \frac{1}{r^a}$$

are defined for $r > 0$ and $0 < c \leq \sqrt{2}$.

Remark 2.2.4. *We abuse the notation E here. Given $(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2$, its energy $E(x, \dot{x})$ and its angular momentum $c(x, \dot{x})$ are functions of $(x, \dot{x})$ and they are conserved. On the other hand, $E(c, r, \dot{r})$ is a function of $(c, r, \dot{r}) \in (0, \sqrt{2}] \times (0, \infty) \times \mathbb{R}$ and c and $(r, \dot{r})$ are independent variables. Note that $E(c(x, \dot{x}), r, \dot{r}) = E(x, \dot{x}) \equiv E_x \equiv const$, where $r = |x|$ and $\dot{r} = \dot{r}(x, \dot{x})$.*

In [3], the ground state energy is defined to be

$$E^{**}(c) := \inf_{r>0} \left\{E(c, r, \dot{r}) : W(c, r, \dot{r}) = 0\right\} \equiv \inf_{r>0} \left\{-\frac{1-\frac{a}{2}}{r^a} : W(c, r, \dot{r}) = 0\right\},$$

for each $\sqrt{2} \geq c > 0$. By using the sequence $(r_n, \dot{r}_n) = (1/n, \sqrt{(2-c^2)n^2 + an^a})$, we can easily check that $E^{**}(c) = -\infty$, and thus the ground state energy does not provide us any information. However, for each $\sqrt{2} \geq c > 0$ the problem

$$H(c) = \sup_{r>0} \left\{E(c, r, \dot{r}) : W(c, r, \dot{r}) = 0\right\} \equiv \sup_{r>0} \left\{-\frac{1-\frac{a}{2}}{r^a} : W(c, r, \dot{r}) = 0\right\}$$

is evaluated to be zero, since we can use the sequence $(r_n, \dot{r}_n) = (n, \sqrt{\frac{2-c^2}{n^2} + \frac{a}{n^a}})$. We modify and mimic the construction process of invariant sets proposed in [3], but this time the initial conditions must have energy greater than $H(c) = 0$.

Lemma 2.2.3. *Let $0 < c \leq \sqrt{2}$ and $a < 2$ be fixed. For each $\delta > 0$, the set*

$$U_\delta(c) = \left\{(x, \dot{x}) : E(x, \dot{x}) \geq 0, c(x, \dot{x}) \leq c, W(c, r, \dot{r}) < -\delta\right\}$$

is invariant.

Proof. Let $\phi(t)$ be a solution such that $E_\phi \geq 0$ and $c_\phi \leq c \leq \sqrt{2}$. Suppose there exists t_0 such that

$$W(c, r(t_0), \dot{r}(t_0)) = \frac{1}{2}\dot{r}(t_0)^2 - \frac{2-c^2}{2r(t_0)^2} - \frac{a/2}{r(t_0)^a} = -\delta.$$

This implies

$$\frac{1}{2}\dot{r}(t_0)^2 \leq -\delta + \frac{2-c_\phi^2}{2r(t_0)^2} + \frac{a/2}{r(t_0)^a}.$$

Thus,

$$\begin{aligned}
E(c_\phi, r(t_0), \dot{r}(t_0)) &= \frac{1}{2}\dot{r}(t_0)^2 - \frac{2-c_\phi^2}{2r(t_0)^2} - \frac{1}{r(t_0)^a} \\
&< \frac{1}{2}\dot{r}(t_0)^2 - \frac{2-c_\phi^2}{2r(t_0)^2} - \frac{a/2}{r(t_0)^a} \\
&\leq -\delta + \frac{2-c_\phi^2}{2r(t_0)^2} + \frac{a/2}{r(t_0)^a} - \frac{2-c_\phi^2}{2r(t_0)^2} - \frac{a/2}{r(t_0)^a} = -\delta.
\end{aligned}$$

But $E(c_\phi, r(t_0), \dot{r}(t_0)) \equiv E_\phi \geq 0$, Contradiction. □

Proposition 2.2.9. *Let* $0 < c \leq \sqrt{2}$ *and* $a < 2 = b$ *be fixed and* $U_\delta(c)$ *be the same set as in Lemma* 2.2.3. *Any initial condition in* $U_\delta(c)$ *corresponds to a solution with singularity.*

Proof. Let $\phi(t)$ be a solution such that $E_\phi \geq 0$ and $c_\phi < c < \sqrt{2}$. We know

$$\ddot{I}(x(t)) = 4W(c_\phi, r, \dot{r}) \leq 4W(c, r, \dot{r}) < -4\delta.$$

So the non-negative function I is bounded above a concave down parabola. □

2.2.5.3 Subcase $0 < c \leq \sqrt{2}$ a direct approach

The idea of building invariant sets using energy, some fixed angular momentum, and an indicator function is not effective when the effective potential does not have a critical point. Indeed, we are not satisfied with what we have obtained in the previous section, the construction of invariant sets seems a bit unnatural.

As mentioned in the introduction, Diacu et al. [7] demonstrated the presence of black hole effect in the Manev potential ($a = 1, b = 2$) two-body problem. That is, for sufficiently small angular momentum, as long as both particles are close enough, they shall collide in finite time.

This section aims to generalize Diacu et al.'s result to all the quasi-homogeneous potentials with $a < b = 2$ as well as to give a complete description of the phase space $(r, \dot{r})$ in terms of global existence and singularity when angular momentum is small. The method is simple and only relies on the integrability of the two-body system.

Recall that for $b = 2 > a$, each

$$\Gamma_{c,E}: \quad E = \frac{1}{2}\dot{r}^2 - \frac{2-c^2}{2r^2} - \frac{1}{r^a} \tag{2.2.8}$$

determines an orbit in the phase space $(r, \dot{r})$. When $0 < c < \sqrt{2}$, observe that

1. Each orbit $\Gamma_{c,E}$ is symmetric about the r-axis.

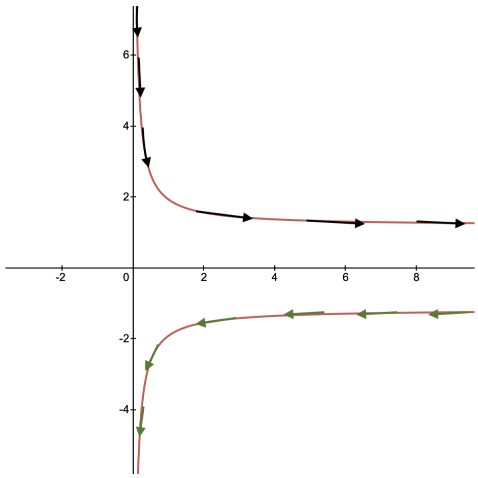
2. If $E < 0$, then $\Gamma_{c,E}$ has a unique intersection with $\dot{r} = 0$. (We denote this intersection by $(r, \dot{r}) = (r_{c,E}, 0)$.) Moreover, for $(r, \dot{r}) \in \Gamma_{c,E}$, $r \leq r_{c,E}$ and $r = r_{c,E} \iff \dot{r} = 0$.

3. If $E \geq 0$, $\Gamma_{c,E}$ has no intersection with the axis $\dot{r} = 0$.

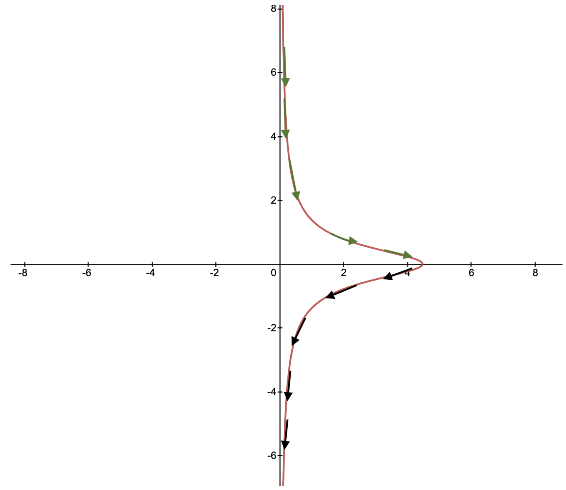
Whenever $a < b = 2$, we define the following sets in the phase space $(x, \dot{x})$.

$$\begin{aligned}
N^+ &= \left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) < 0, \dot{r} > 0, 0 < c(x,\dot{x}) \leq \sqrt{2}\right\} \\
N^- &= \left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) < 0, \dot{r} < 0, 0 < c(x,\dot{x}) \leq \sqrt{2}\right\} \\
P^+ &= \left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) \geq 0, \dot{r} > 0, 0 < c(x,\dot{x}) \leq \sqrt{2}\right\} \\
P^- &= \left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) \geq 0, \dot{r} < 0, 0 < c(x,\dot{x}) \leq \sqrt{2}\right\} \\
A_{\text{glob}} &= \left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : c(x,\dot{x}) > \sqrt{2}\right\}
\end{aligned}$$

We also set $N := N^+ \bigcup N^-$ and $P := P^+ \bigcup N^+$. We have already proved that A_{glob} corresponds to global solutions and it follows immediately from the third observation above that both $P^\pm$ are invariant sets and thus P^+ corresponds to global solutions. Furthermore, N^- is forward invariant by the second observation. What remains unclear is whether the same holds for N^+. In other words, for solution starts at N^+, we need to check how long it takes to arrive at $(r, \dot{r}) = (r_{c,E}, 0)$.



(a) $\Gamma_{c,E}$ for fixed $0 < c \leq \sqrt{2}$ and $E \geq 0$



(b) $\Gamma_{c,E}$ for fixed $0 < c \leq \sqrt{2}$ and $E < 0$

Figure 2.12: Phase space $(r, \dot{r})$ for $b = 2 > a$

Let $(x(t), \dot{x}(t))$ be a solution that starts at N^+ with $r(0) = r_0$, $\dot{r} > 0$, $E < 0$ and angular momentum $0 < c \leq \sqrt{2}$. Then, we use the energy relationship to get

$$\dot{r} = \sqrt{2E + \frac{2 - c^2}{r^2} + \frac{2}{r^a}}.$$

If we set the final position to be $r_{c,E} - \epsilon$ $(0 < \epsilon \ll 1)$ then the time T_ϵ it takes to reach $r_{c,E} - \epsilon$ satisfies

$$T_\epsilon = \int_{r_0}^{r_{c,E}-\epsilon} \frac{dr}{\sqrt{2E + \frac{2-c^2}{r^2} + \frac{2}{r^a}}}.$$

Consider the improper integral

$$\lim_{\epsilon \to 0^+} \int_R^{r_{c,E}-\epsilon} \frac{dr}{\sqrt{2E + \frac{2-c^2}{r^2} + \frac{2}{r^a}}} = \int_R^{r_{c,E}} \frac{dr}{\sqrt{2E + \frac{2-c^2}{r^2} + \frac{2}{r^a}}},$$

and we claim that

$$\frac{1}{\sqrt{2E + \frac{2-c^2}{r^2} + \frac{2}{r^a}}} = O\left(\frac{1}{\sqrt{r_{c,E} - r}}\right) \qquad \text{as } r \to r_{c,E}^-.$$

Indeed, by L'Hôpital's rule, we see

$$\begin{aligned}
0 \leq \lim_{r\to r_{c,E}^-} \frac{r_{c,E}-r}{2E+\frac{2-c^2}{r^2}+\frac{2}{r^a}} &= \lim_{r\to r_{c,E}^-} \frac{r_{c,E}r^2-r^3}{2Er^2+2-c^2+2r^{2-a}} \\
&= \lim_{r\to r_{c,E}^-} \frac{2r_{c,E}r-3r^2}{4Er+2(2-a)r^{1-a}} \\
&= \frac{-r_{c,E}^2}{4Er_{c,E}+2(2-a)r_{c,E}^{1-a}}.
\end{aligned}$$

Notice that the denominator is nonzero, since, from the definition of $r_{c,E}$, we have

$$\begin{aligned}
0 \geq -2\frac{2-c^2}{r_{c,E}} &= 4Er_{c,E}+4r_{c,E}^{1-a} \\
&> 4Er_{c,E}+2(2-a)r_{c,E}^{1-a}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_R^{r_{c,E}} \frac{dr}{\sqrt{2E+\frac{2-c^2}{r^2}+\frac{2}{r^a}}} &= C\int_R^{r_{c,E}} \frac{1}{\sqrt{r_{c,E}-r}} \\
&= C\int_0^{r_{c,E}-R} \frac{dr}{\sqrt{r}} < \infty.
\end{aligned}$$

for some constant C. Thus, it only takes finite time to reach $r_{c,E}$ from any position $r_0 < r_{c,E}$. Combining with symmetry or reversibility, it is not difficult to see a solution that starts at N^+ will reach $(r,\dot{r}) = (r_{c,E},0)$ in finite time and enters N^-. Therefore, solutions start in $N = N^+ \bigcup N^-$ share the same fate in terms of global existence and singularity. Next, we check whether there is a uniform lower bound for solutions starting in P^- or N.

Lemma 2.2.4. *Let $x(t)$ be a solution starts in invariant set P^- or N then $r(t) \to 0$ as $t \to t_{max^-}$, where t_{max} is the maximal time of existence of $x(t)$.*

Proof. We prove this lemma by contradiction. Suppose that the solution $r(t)$ is bounded below for all time. Then it is a global solution and

$$\inf_{t\geq 0} r(t) = \lim_{t\to\infty} r(t) = \epsilon,$$

for some $\epsilon > 0$. From here, we conclude that $\dot{r}(t) \to 0$ as $t \to \infty$. Therefore, from energy relation,

$$\frac{1}{2}\dot{r}(t)^2 = E + \frac{1 - c^2/2}{r(t)^2} + \frac{1}{r(t)^a} \to 0 \qquad \text{as } t \to \infty.$$

If the solution starts in P^-, however, the right hand of the previous equation approaches some strictly positive number. Therefore, there does not exist a strictly positive uniform bound lower for solutions that start in P^-.

Otherwise, we know that for sufficiently large t, $(x(t), \dot{x}(t)) \in N^-$ and it suffices to prove this lemma holds for N^-. On N^-, we have

$$\dot{r}(t) = -\sqrt{2E + \frac{2 - c^2}{r(t)^2} + \frac{2}{r(t)^a}}.$$

Hence,

$$2E + \frac{2 - c^2}{\epsilon^2} + \frac{2}{\epsilon^a} = 0,$$

which implies $\epsilon = r_{c,E}$. But $r(0) = r_{c,E} - \delta$ for some $\delta > 0$ and $\dot{r}(t) < 0$ for all $t > 0$ by the second observation above. Therefore, there does not exist a strictly positive uniform bound lower for solutions that start in N.

Hence, for solutions that start in N or P^-, $r(t)$ must approach zero as $t \to t_{\max}^-$. □

Proposition 2.2.10. *Let $x(t)$ be a solution that starts at invariant sets P^- or N then $x(t)$ has a finite-time collision.*

Proof. Since solutions start at N^+ will reach N^- in finite time, it suffices to only consider solutions start in N^- or P^-. Let $x(t)$ be a solution in either N^- or P^-. Then, from energy relationship we have

$$\frac{dr}{dt} = -\sqrt{2E + \frac{2 - c^2}{r^2} + \frac{2}{r^a}}.$$

Let T_ϵ be the time it takes to reach $0 < r = \epsilon \ll 1$ from some fixed $r = r_0$. (If solution starts at N^-, we need to further impose $r_0 < r_{c,E}$.) By the previous lemma, we know that T_ϵ exists and it can be written as

$$T_\epsilon = \int_\epsilon^{r_0} \frac{ds}{\sqrt{2E + \frac{2-c^2}{s^2} + \frac{2}{s^a}}}.$$

We see that the integrand is continuous at $s = 0$ since

$$\frac{s}{\sqrt{2Es^2 + 2 - c^2 + 2s^{2-a}}} \to 0 \qquad \text{as } s \to 0^+.$$

Thus,

$$\lim_{\epsilon \to 0^+} T_\epsilon < \infty,$$

which means it takes finite time to reach for a collision. □

Proposition 2.2.11 (Complete characterization for $a < b = 2$)**.** *Fix $a < b = 2$, the following statements are valid.*

1. *Solutions in $\left\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : c(x, \dot{x}) > \sqrt{2}\right\}$ are global solutions.*
2. *Solutions in $\left\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x, \dot{x}) \geq 0, \dot{r} \geq 0, 0 < c(x, \dot{x}) \leq \sqrt{2}\right\}$ is invariant and it contains global solutions.*
3. *The set $\left\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x, \dot{x}) \geq 0, \dot{r} < 0, 0 < c(x, \dot{x}) \leq \sqrt{2}\right\}$ is invariant and it contains solution with finite-time singularity .*
4. *Solutions in $\left\{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x, \dot{x}) < 0, 0 < c(x, \dot{x}) \leq \sqrt{2}\right\}$ is invariant and it contains solution with finite-time singularity.*

2.2.6 Strong perturbation of Manev potential

The other limiting case we need to consider is when $b > a = 2$. When the angular momentum is relatively large, it can be shown that the strong homogeneous potential dominates the solution dynamics by an almost identical proof as in the case of $b > a > 2$. On the other hand, when $0 < c \leq \sqrt{2}$, the dynamics behaves similarly to the case $a < b = 2, c < \sqrt{2}$.

2.2.6.1 Subcase $c > \sqrt{2}$

The case $c > \sqrt{2}$ and $b > a = 2$ resembles the homogeneous potential case and the energy method [3] applies. The indicator function in this case becomes

$$K(c, r) = \frac{c^2 - 2}{r^2} - \frac{b}{r^b}.$$

Lemma 2.2.5. *Fix $b > a = 2$ and $c > \sqrt{2}$. The following sets are invariant.*

$$K^+(c) = \{(x, \dot{x}) : E(x, \dot{x}) < E^*(c), c(x, \dot{x}) \geq c, K(c, r) \geq 0\}$$
$$K^-(c) = \{(x, \dot{x}) : E(x, \dot{x}) < E^*(c), c(x, \dot{x}) \geq c, K(c, r) < 0\}$$

Furthermore, each solution in $K^+(c)$ is global solution and each solution in $K^-(c)$ is singular.

Proof. The proof is similar to the case of two strong homogeneous potentials. (Cf. Lemma 2.2.1 and Proposition 2.2.1) □

2.2.6.2 Subcase $0 < c \leq \sqrt{2}$

The case $0 < c \leq \sqrt{2}$ and $b > a = 2$ resembles the Manev-potential case when angular momentum is relatively small. We see that each solution with energy E and angular momentum c satisfies

$$\Gamma_{c,E} : E = \frac{1}{2}\dot{r}^2 + \frac{c^2 - 2}{2r^2} - \frac{1}{r^b} \tag{2.2.9}$$

in the $(r, \dot{r})$ plane . Furthermore, we observe that (Figure 2.13)

1. For $E < 0$, we see that $\Gamma_{c,E}$ plane always has one intersection with $\dot{r} = 0$ in the $(r, \dot{r})$ plane and we denote this unique intersection as $r_{c,E}$. For $(r, \dot{r}) \in \Gamma_{c,E}$, $r \leq r_{c,E}$ and $r = r_{c,E} \iff \dot{r} = 0$.
2. For $E \geq 0$, $\Gamma_{c,E}$ has no intersection with the axis $\dot{r} = 0$.
3. Each $\Gamma_{c,E}$ is symmetric about the r axis.

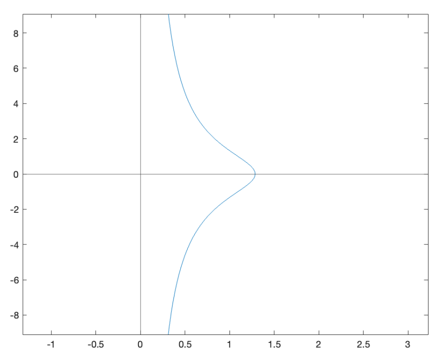
Both situations can be handled analogously as we did in the case where $b = 2 > a$ and $0 < c \leq \sqrt{2}$. For non-negative energy, we see that both

$$P_- = \left\{(x, \dot{x}) : E(x, \dot{x}) \geq 0, \dot{r} < 0, 0 < c(x, \dot{x}) \leq \sqrt{2}\right\}$$

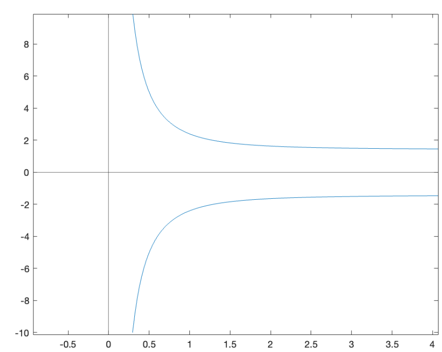
and

$$P_+ = \left\{(x, \dot{x}) : E(x, \dot{x}) \geq 0, \dot{r} > 0, 0 < c(x, \dot{x}) \leq \sqrt{2}\right\}$$

are invariant, since for $E \geq 0$, $\Gamma_{c,E}$ does not have an intersection with $\dot{r} = 0$ plane.



(a) $\Gamma_{c,E}$ for fixed $0 < c \leq \sqrt{2}$ and $E < 0$



(b) $\Gamma_{c,E}$ for fixed $0 < c \leq \sqrt{2}$ and $E \geq 0$

Figure 2.13: Phase space $(r, \dot{r})$ for $b > a = 2$

For strictly negative energy, we define

$$N_- = \left\{(x, \dot{x}) : E(x, \dot{x}) < 0, \dot{r} < 0, 0 < c(x, \dot{x}) \le \sqrt{2}\right\}$$
$$N_+ = \left\{(x, \dot{x}) : E(x, \dot{x}) < 0, \dot{r} > 0, 0 < c(x, \dot{x}) \le \sqrt{2}\right\}.$$

Similarly to the case where $b = 2 > a$, we see N_- is forward invariant since one can apply the same argument as we did in **??**.

Lemma 2.2.6. *Let $x(t)$ be a solution that starts in invariant set P_- or N_- then $r(t) \to 0$ as $t \to t_{max}^-$, where t_{max} is the maximal time of existence of $x(t)$.*

Proof. This proof is similar to the case where $b = 2 > a$ and $0 < c \le \sqrt{2}$. ☐

For solution that starts in P_- or N_-, we have $\dot{r}(t) < 0$ for $t > 0$, and thus, from the energy relation (2.2.9), we see

$$\int_{r(0)}^{0} -\frac{ds}{\sqrt{2E + \frac{2-c^2}{s^2} + \frac{2}{s^b}}} = \int_0^{t_{\text{col}}} d\tau,$$

where t_{col} is the time it takes to reach collision: $r = 0$. Now, for $r(0) \neq 0$, the integrand is continuous on $[0, r(0)]$ and therefore $t_{\text{col}} < \infty$. Another immediate observation is that P_+ contains global solutions.

Let us focus on the set N_+. We need to check whether it is forward invariant. In other words, let $(x(t), \dot{x}(t))$ be a solution that starts in N_+, we need to compute the time it takes to reach $r_{c,E}$. The energy relation implies that we can write

$$\int_{r(0)}^{r_{c,E}} \frac{ds}{\sqrt{2E + \frac{2-c^2}{s^2} + \frac{2}{s^b}}} = \int_0^{t} d\tau$$

where t is the time $(x(t), \dot{x}(t))$ takes from $r(0)$ to $r_{c,E}$. We claim that $\frac{1}{\sqrt{2E+\frac{2-c^2}{s^2}+\frac{2}{s^b}}} =$

$O(\frac{1}{\sqrt{r_{c,E}-s}})$ as $s \to r_{c,E}^-$. Indeed, L'Hôpital's rule implies that

$$\begin{aligned} 0 \leq \lim_{r\to r_{c,E}^-} \frac{r_{c,E}-s}{2E+\frac{2-c^2}{s^2}+\frac{2}{s^b}} &= \lim_{r\to r_{c,E}^-} \frac{r_{c,E}s^b - s^{b+1}}{2Es^b+(2-c^2)s^{b-2}+2} \\ &= \lim_{r\to r_{c,E}^-} \frac{br_{c,E}s^{b-1}-(b+1)s^b}{2bEs^{b-1}+(2-c^2)(b-2)s^{b-3}} \\ &= \frac{-br_{c,E}^b}{2bEr_{c,E}^{b-1}+(2-c^2)(b-2)r_{c,E}^{b-3}}. \end{aligned}$$

Notice that the denominator is nonzero, since, from the definition of $r_{c,E}$ we have

$$\begin{aligned} 0 > -\frac{2b}{r_{c,E}} &= 2bEr_{c,E}^{b-1}+(2-c^2)br_{c,E}^{b-3} \\ &\geq 2bEr_{c,E}^{b-1}+(2-c^2)(b-2)r_{c,E}^{b-3}. \end{aligned}$$

Thus, $t < \infty$. By symmetry or reversibility, solutions with initial conditions in N_+ will reach N_- and lead to a collision singularity in finite time. Finally, we conclude this case by stating the following proposition.

Proposition 2.2.12. *Fix $b > a = 2$. Both*

$$\left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) < 0, 0 < c \leq \sqrt{2}\right\}$$

and

$$\left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) \geq 0, 0 < c \leq \sqrt{2}, \dot{r} < 0\right\}$$

are invariant and they contain solutions with singularity. Meanwhile, the invariant set

$$\left\{(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : E(x,\dot{x}) \geq 0, 0 < c \leq \sqrt{2}, \dot{r} > 0\right\}$$

contains global solutions.

2.3 Summary

Let $c_0 > 0$ be a constant and $(x,\dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2$ be an initial condition. The following notations and conventions are adopted for Table 2.1.

1. $K(c,r) := \frac{c^2}{r^2} - a\frac{1}{r^a} - b\frac{1}{r^b}$ is the indicator function and $U_{\text{eff}}(c,r)$ is the effective potential.

2. E is the energy function, c is the magnitude of angular momentum function and $\dot{r}$ is the radial speed.

3. $E^*(c_0) = \max\limits_{r>0} U_{\text{eff}}(c_0, r)$ and it exists and is defined whenever $2 \leq a < b$.

4. As long as $b > 2 > a$,

$$(r^*(a,b), c^*(a,b)) = \left(\left(\frac{b(b-2)}{a(2-a)} \right)^{\frac{1}{b-a}}, \left[a \left(\frac{b(b-2)}{a(2-a)} \right)^{\frac{2-a}{b-a}} + b \left(\frac{a(2-a)}{b(b-2)} \right)^{\frac{b-2}{b-a}} \right]^{\frac{1}{2}} \right)$$

and $E^*(a,b) = U_{\text{eff}}(c^*(a,b), r^*(a,b)) < 0$ are well defined.

5. For $c_0 > c^*(a,b)$ and $b > 2 > a$, $r^-_{c_0} < r^+_{c_0}$ are exactly the two roots of the equation $K(c_0, r) = 0$, and $V^\pm(c_0) = U_{\text{eff}}(c_0, r^\pm_{c_0})$.

6. As long as c_0 is present, that row should be interpreted as "for each c_0 such that". For example, the first row is interpreted as: For $b > a > 2$, for each c_0 such that $c(x, \dot{x}) \geq c_0$, if $E(x, \dot{x}) < E^*(c_0)$ and $K(c_0, |x|) \geq 0$, then $(x, \dot{x})$ leads to a global solution.

7. "N/A" stands for not applicable: One simply needs to ignore this entry.

<table>
<tr><th>Condition on a, b</th><th>Condition on angular momentum</th><th>Condition on energy</th><th>Sign of K or condition on radius</th><th>Sign of $\dot{r}$</th><th>Global or Singular</th></tr>
<tr><td rowspan="2">$b > a > 2$</td><td rowspan="2">$c(x, \dot{x}) \geq c_0$</td><td rowspan="2">$E(x, \dot{x}) < E^*(c_0)$</td><td>$K(c_0, |x|) \geq 0$</td><td rowspan="2">N/A</td><td>Global solution</td></tr>
<tr><td>$K(c_0, |x|) < 0$</td><td>Finite-time collision</td></tr>
<tr><td>$a < b < 2$</td><td>$c(x, \dot{x}) > 0$</td><td colspan="3">N/A</td><td>Global solution</td></tr>
<tr><td rowspan="5">$b > a = 2$</td><td rowspan="2">$c(x, \dot{x}) > c_0 > \sqrt{2}$</td><td rowspan="2">$E(x, \dot{x}) < E^*(c_0)$</td><td>$K(c_0, |x|) \geq 0$</td><td rowspan="2">N/A</td><td>Global solution</td></tr>
<tr><td>$K(c_0, |x|) < 0$</td><td>Finite-time collision</td></tr>
<tr><td rowspan="3">$0 < c(x, \dot{x}) \leq \sqrt{2}$</td><td rowspan="2">$E(x, \dot{x}) \geq 0$</td><td rowspan="2">N/A</td><td>$\dot{r} > 0$</td><td>Global solution</td></tr>
<tr><td>$\dot{r} < 0$</td><td rowspan="2">Finite-time collision</td></tr>
<tr><td>$E(x, \dot{x}) < 0$</td><td colspan="2">N/A</td></tr>
<tr><td rowspan="4">$a < b = 2$</td><td>$c(x, \dot{x}) > \sqrt{2}$</td><td colspan="3">N/A</td><td>Global solution</td></tr>
<tr><td rowspan="3">$0 < c(x, \dot{x}) \leq \sqrt{2}$</td><td rowspan="2">$E(x, \dot{x}) \geq 0$</td><td rowspan="4">N/A</td><td>$\dot{r} > 0$</td><td>Global solution</td></tr>
<tr><td>$\dot{r} < 0$</td><td rowspan="4">Finite-time collision</td></tr>
<tr><td>$E(x, \dot{x}) < 0$</td><td rowspan="4">N/A</td></tr>
<tr><td rowspan="12">$b > 2 > a$</td><td rowspan="3">$c(x, \dot{x}) > c_0 > c^*(a,b)$</td><td>$E(x, \dot{x}) < V^+(c_0)$</td></tr>
<tr><td rowspan="2">$E(x, \dot{x}) < V^-(c_0)$</td><td>$|x| < r^-_{c_0}$</td></tr>
<tr><td>$|x| \geq r^-_{c_0}$</td><td>Global solution</td></tr>
<tr><td rowspan="6">$c(x, \dot{x}) = c^*(a,b)$</td><td rowspan="3">$E(x, \dot{x}) = E^*(a,b)$</td><td>$|x| = r^*(a,b)$</td><td>$\dot{r} = 0$</td><td>Global solution (circular orbit)</td></tr>
<tr><td rowspan="8">N/A</td><td>$\dot{r} > 0$</td><td>Global solution ($r(t) \xrightarrow{t \to \infty} r^*(a,b)$)</td></tr>
<tr><td>$\dot{r} < 0$</td><td>Finite-time collision</td></tr>
<tr><td rowspan="2">$E(x, \dot{x}) \geq 0$</td><td>$\dot{r} > 0$</td><td>Global solution</td></tr>
<tr><td>$\dot{r} < 0$</td><td rowspan="2">Finite-time collision</td></tr>
<tr><td>$E^*(a,b) \neq E(x, \dot{x}) < 0$</td><td>N/A</td></tr>
<tr><td rowspan="3">$0 < c(x, \dot{x}) < c^*(a,b)$</td><td rowspan="2">$E(x, \dot{x}) \geq 0$</td><td>$\dot{r} > 0$</td><td>Global solution</td></tr>
<tr><td>$\dot{r} < 0$</td><td rowspan="2">Finite-time collision</td></tr>
<tr><td>$E(x, \dot{x}) < 0$</td><td>N/A</td></tr>
</table>

Table 2.1: A summary of this chapter

Chapter 3

Quasi-homogeneous Hill's lunar problem

3.1 Derivation of Hill's lunar equation under quasi-homogeneous potentials

As mentioned in the introduction, this part is heavily based on the derivations of [15, 4]. We strongly encourage the readers to consult the Appendix in [4], as it contains all the physical motivations and details.

As usual, we start from the Hamiltonian of the quasi-homogeneous three-body problem

$$E(\widetilde{x}, \dot{\widetilde{x}}) = \frac{1}{2}\sum_{i=0}^{2} m_i|\widetilde{v}_i|^2 + U(\widetilde{x}) + V(\widetilde{x}), \tag{3.1.1}$$

where $\widetilde{x} = (\widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2), \dot{\widetilde{x}} = (\widetilde{v}_0, \widetilde{v}_1, \widetilde{v}_2) \in (\mathbb{R}^2)^3$ are the configuration and its velocity respectively and

$$U(\widetilde{x}) = \sum_{0\leq i<j\leq 2} \frac{m_i m_j}{\left|\widetilde{x}_i - \widetilde{x}_j\right|^a} \qquad V(\widetilde{x}) = \sum_{0\leq i<j\leq 2} \frac{m_i m_j}{\left|\widetilde{x}_i - \widetilde{x}_j\right|^b}.$$

are the weak and strong homogeneous potential ($0 < a < b$), respectively. Here, m_0, m_1, m_2 represent the mass of the Moon, the Earth and the Sun. For each $i =$

$0, 1, 2$, we first introduce uniform rotating coordinate system:

$$\begin{cases} \widetilde{x}_i = \exp{(J\omega t)} x_i \\ v_i = \dot{x}_i \end{cases}$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \exp{(J\omega t)} = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}.$$

Thus the Hamiltonian (3.1.1) becomes

$$H = \frac{1}{2}\sum_{i=0}^{2} \omega^2 m_i |x_i|^2 - \sum_{i=0}^{2} m_i \omega x_i^T J v_i + \frac{1}{2}\sum_{i=0}^{2} m_i |v_i|^2 + U(x) + V(x), \tag{3.1.2}$$

where

$$U(x) = \sum_{0\le i<j\le 2} \frac{m_i m_j}{\left|x_i - x_j\right|^a} \qquad V(x) = \sum_{0\le i<j\le 2} \frac{m_i m_j}{\left|x_i - x_j\right|^b}.$$

Let $y_i = m_i v_i$ and suppose that the moment of inertia is fixed to be zero, then (3.1.2) becomes

$$H = \sum_{i=0}^{2} \frac{|y_i|^2}{2m_i} - \sum_{i=0}^{2} \omega x_i^T J y_i + U(x) + V(x). \tag{3.1.3}$$

To reduce coordinate, we introduce Jacobi coordinates:

$$\begin{aligned} u_0 &= \frac{m_0 x_0 + m_1 x_1 + m_2 x_2}{m_0 + m_1 + m_2} \\ u_1 &= x_1 - x_0 \\ u_2 &= x_2 - \frac{m_0 x_0 + m_1 x_1}{m_0 + m_1} \\ v_0 &= y_0 + y_1 + y_2 \\ v_1 &= \frac{m_0 y_1 - m_1 y_0}{m_0 + m_1} \\ v_2 &= \frac{(m_0 + m_1) y_2 - m_2 (y_0 + y_1)}{m_0 + m_1 + m_2}. \end{aligned}$$

Thus, Equation (3.1.3) is reduced to

$$H = \sum_{i=0}^{2} \left(\frac{|v_i|^2}{2M_i'} - u_i^T J v_i \omega \right) - \frac{m_0 m_1}{|u_1|^a} - \frac{m_1 m_2}{|u_2 - v_0' u_1|^a} - \frac{m_0 m_2}{|u_2 + v_1' u_1|^a} - \frac{m_0 m_1}{|u_1|^b} - \frac{m_1 m_2}{|u_2 - v_0' u_1|^b} - \frac{m_0 m_2}{|u_2 + v_1' u_1|^b}, \tag{3.1.4}$$

where

$$M_0' = m_0 + m_1 + m_2 \qquad M_1' = \frac{m_0 m_1}{m_0 + m_1} \qquad M_2' = \frac{(m_0 + m_1) m_2}{m_0 + m_1 + m_2}$$
$$v_0' = \frac{m_0}{m_0 + m_1} \qquad v_1' = \frac{m_1}{m_0 + m_1}.$$

Without loss of generality, we fix the center of mass (i.e. $u_0 = v_0 = 0$). Thus, the summation in Equation (3.1.4) now starts at 1. Since the Earth and Moon have approximately the same mass and are relatively small compared to the mass of the Sun. We set

$$m_0 = \epsilon^{2\gamma} \mu_0 \qquad m_1 = \epsilon^{2\gamma} \mu_1 \qquad m_2 = \mu_2,$$

where $0 < \epsilon \ll 1$, and $\gamma > 0$ is chosen later. Since $m_0, m_1 = O(\epsilon^{2\gamma})$, if the velocities are of order 1, then the momentum will have the same order. We hence introduce

$$v_1 = \epsilon^{2\gamma} w_1 \qquad v_2 = \epsilon^{2\gamma} w_2.$$

Under this new change of variable, we calculate

$$\left(\frac{|w_1|^2}{2M_1} - u_1^T J w_1 \omega - \frac{\epsilon^{2\gamma} \mu_0 \mu_1}{|u_1|^a} - \frac{\epsilon^{2\gamma} \mu_0 \mu_1}{|u_1|^b} \right) \epsilon^{2\gamma} = \left(\frac{|v_1|^2}{2M_1'} - u_1^T J v_1 \omega - \frac{m_0 m_1}{|u_1|^a} - \frac{m_0 m_1}{|u_1|^b} \right),$$

and

$$\left(\frac{|w_2|^2}{2M_2} \underbrace{\frac{m_0 + m_1 + m_2}{m_2}}_{1 + \epsilon^{2\gamma} \frac{\mu_0 + \mu_1}{\mu_2}} - u_2^T J w_2 \omega - \frac{\mu_1 \mu_2}{|u_2 - k_0 u_1|^a} - \frac{\mu_0 \mu_2}{|u_2 + k_1 u_1|^a} - \frac{\mu_1 \mu_2}{|u_2 - k_0 u_1|^b} - \frac{\mu_0 \mu_2}{|u_2 + k_1 u_1|^b} \right) \epsilon^{2\gamma}$$
$$= \left(\frac{|v_2|^2}{2M_2'} - u_2^T J v_2 \omega - \frac{m_1 m_2}{|u_2 - v_0' u_1|^a} - \frac{m_0 m_2}{|u_2 + v_1' u_1|^a} - \frac{m_1 m_2}{|u_2 - v_0' u_1|^b} - \frac{m_0 m_2}{|u_2 + v_1' u_1|^b} \right),$$

where

$$M_1 = \frac{\mu_0\mu_1}{\mu_0 + \mu_1}, \qquad M_2 = \mu_0 + \mu_1$$
$$k_0 = \frac{\mu_0}{\mu_0 + \mu_1}, \qquad k_1 = \frac{\mu_1}{\mu_0 + \mu_1}.$$

With multiplier $\epsilon^{-2\gamma}$, we obtain the new Hamiltonian from (3.1.4):

$$\begin{aligned} H' &:= \epsilon^{-2\gamma} H = H_1 + H_2 + O(\epsilon^{2\gamma}) \\ H_1 &= \frac{|w_1|^2}{2M_1} - u_1^T J w_1 \omega - \frac{\epsilon^{2\gamma}\mu_0\mu_1}{|u_1|^a} - \frac{\epsilon^{2\gamma}\mu_0\mu_1}{|u_1|^b} \\ H_2 &= \frac{|w_2|^2}{2M_2} - u_2^T J w_2 \omega - \frac{\mu_1\mu_2}{|u_2 - k_0 u_1|^a} - \frac{\mu_0\mu_2}{|u_2 + k_1 u_1|^a} - \frac{\mu_1\mu_2}{|u_2 - k_0 u_1|^b} - \frac{\mu_0\mu_2}{|u_2 + k_1 u_1|^b}. \end{aligned} \tag{3.1.5}$$

The next assumption we shall adopt is that the distance between the earth and the moon $|u_1|$ is relatively small compared to the distance between the Sun and the center of mass of the Earth-Moon system $|u_2|$. We further introduce the change of variable:

$$u_1 = \epsilon^{2\delta}\eta_1,$$

where the positive number δ will be chosen later. Now, apply the assumption $|u_2| \gg |u_1|$ and use Legendre polynomials, we obtain the following series representations in (3.1.5):

$$\begin{aligned} \frac{1}{|u_2 - k_0 u_1|} &= \sum_{k=0}^{\infty} \frac{{k_0}^k |u_1|^k}{|u_2|^{k+1}} P_k(\cos(\theta)) \\ \frac{1}{|u_2 + k_1 u_1|} &= \sum_{k=0}^{\infty} \frac{{k_1}^k |u_1|^k}{|u_2|^{k+1}} P_k(-\cos(\theta)), \end{aligned} \tag{3.1.6}$$

where P_k is the i^{th} Legendre polynomial and θ is the angle between the vectors u_2 and u_1. We also have the binomial expansion

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}x^2 + ... \tag{3.1.7}$$

for $|x| < 1$ and $a \in \mathbb{C}$. Combining (3.1.7) and (3.1.6), we have the following approximations in (3.1.5):

$$\frac{\mu_1\mu_2}{|u_2-k_0u_1|^a} = \frac{\mu_1\mu_2}{|u_2|^a}\left[1+ak_0\left(\frac{|u_1|}{|u_2|}\right)P_1(\cos\theta)+\left(ak_0^2P_2(\cos\theta)+\frac{a(a-1)}{2}k_0^2P_1^2(\cos\theta)\right)\left(\frac{|u_1|}{|u_2|}\right)^2+O\left(\left(\frac{|u_1|}{|u_2|}\right)^3\right)\right].$$

Using the oddness and evenness for Legendre polynomials,

$$\frac{\mu_0\mu_2}{|u_2+k_1u_1|^a} = \frac{\mu_0\mu_2}{|u_2|^a}\left[1-ak_1\left(\frac{|u_1|}{|u_2|}\right)P_1(\cos\theta)+\left(ak_1^2P_2(\cos\theta)+\frac{a(a-1)}{2}k_1^2P_1^2(\cos\theta)\right)\left(\frac{|u_1|}{|u_2|}\right)^2+O\left(\left(\frac{|u_1|}{|u_2|}\right)^3\right)\right].$$

Using the definition of $k_0, k_1, \mu_0, \mu_1, \mu_2$, we see

$$\frac{\mu_1\mu_2}{|u_2-k_0u_1|^a}+\frac{\mu_0\mu_2}{|u_2+k_1u_1|^a} = \frac{\mu_2(\mu_0+\mu_1)}{|u_2|^a}+\frac{\mu_2(\mu_0k_1^2+\mu_1k_0^2)|u_1|^2}{|u_2|^{a+2}}\left[aP_2(\cos\theta)+\frac{a(a-1)}{2}P_1^2(\cos\theta)\right]+\text{h.o.t.}$$

where "h.o.t." stands for $O(\frac{|u_1|^3}{|u_2|^{a+3}})$. The approximations for the $b-$potential are obtained analogously. To summarize, we have

$$\begin{aligned} H_2 = \frac{|w_2|^2}{2M_2} - u_2^TJw_2\omega - &\left[\frac{\mu_2(\mu_0+\mu_1)}{|u_2|^a}+\frac{\mu_2(\mu_0k_1^2+\mu_1k_0^2)|u_1|^2}{|u_2|^{a+2}}\left[aP_2(\cos\theta)+\frac{a(a-1)}{2}P_1^2(\cos\theta)\right]\right] \\ - &\left[\frac{\mu_2(\mu_0+\mu_1)}{|u_2|^b}+\frac{\mu_2(\mu_0k_1^2+\mu_1k_0^2)|u_1|^2}{|u_2|^{b+2}}\left[bP_2(\cos\theta)+\frac{b(b-1)}{2}P_1^2(\cos\theta)\right]\right]+\text{h.o.t.} \end{aligned} \tag{3.1.8}$$

We pull out the lower order terms in (3.1.8) by setting

$$H_3 := \frac{|w_2|^2}{2M_2} - u_2^TJw_2\omega - \frac{\mu_2(\mu_0+\mu_1)}{|u_2|^a} - \frac{\mu_2(\mu_0+\mu_1)}{|u_2|^b}, \tag{3.1.9}$$

and we let

$$z = \begin{bmatrix} u_2 \\ w_2 \end{bmatrix} \in \mathbb{R}^4.$$

The critical points of H_3: z_ω are computed by setting $\nabla_{w_2}H_3 = \nabla_{u_2}H_3 = 0$. We see z_ω has the form of

$$z_\omega = \begin{bmatrix} c_\omega \\ -M_2Jc_\omega \end{bmatrix},$$

where c_ω satisfies

$$\frac{a}{|c_\omega|^{a+2}} + \frac{b}{|c_\omega|^{b+2}} = \frac{\omega^2}{\mu_2}. \tag{3.1.10}$$

Next, we Taylor expand H_3 at z_ω:

$$H_3(z) = H_3(z_\omega) + \frac{1}{2}(z - z_0)^T S(z - z_0) + O(|z - z_0|^3),$$

where S is the Hessian evaluated at z_0. Now we introduce the following change of variables, as in [4]:

$$\begin{aligned} w_1 &= \epsilon^{2(\beta-\delta)}\hat{v}_1 \\ u_1 &= \epsilon^{2\delta}\hat{u}_1 \\ z - z_0 &= \epsilon^{\beta} x. \end{aligned} \tag{3.1.11}$$

Combining (3.1.8) and (3.1.11), (3.1.5) becomes

$$\begin{aligned} H_1 &= \frac{\epsilon^{4(\beta-\delta)}|\hat{v}_1|^2}{2M_1} - \epsilon^{2\delta}\hat{u}_1^T\epsilon^{2(\beta-\delta)}J\hat{v}_1\omega - \frac{\epsilon^{2\gamma}\mu_0\mu_1}{\epsilon^{2\delta a}|\hat{u}_1|^a} - \frac{\epsilon^{2\gamma}\mu_0\mu_1}{\epsilon^{2\delta b}|\hat{u}_1|^b} \\ H_2 &= H_3 - \left[\frac{\mu_2(\mu_0k_1^2 + \mu_1k_0^2)}{|u_2|^{a+2}}\left(aP_2(\cos\theta) + \frac{a(a-1)}{2}P_1^2(\cos\theta)\right)\right]|\hat{u}_1|^2\epsilon^{4\delta} \\ &\quad - \left[\frac{\mu_2(\mu_0k_1^2 + \mu_1k_0^2)}{|u_2|^{b+2}}\left(bP_2(\cos\theta) + \frac{b(b-1)}{2}P_1^2(\cos\theta)\right)\right]|\hat{u}_1|^2\epsilon^{4\delta} + O(\epsilon^{6\delta}). \end{aligned} \tag{3.1.12}$$

Unlike [4], here we can either restrict the kinetic energy to be of the same order of the strong or weak potential in H_1, which gives us the choice between

$$4(\beta - \delta) = 2(\gamma - \delta b) \qquad (\text{“strong restriction”}) \tag{3.1.13}$$

and

$$4(\beta - \delta) = 2(\gamma - \delta a) \qquad (\text{“weak restriction”}). \tag{3.1.14}$$

Regardless of which choice we make, next, we restrict the second and third term of H_2 in (3.1.12) to be of the same order with the Coriolis term in H_1, which gives

$$4\delta = 2\beta.$$

To simplify computation, we thus choose

$$\delta = 1 \qquad \beta = 2 \qquad \gamma = b + 2 \text{ (strong restriction) or } \gamma = a + 2 \text{ (weak restriction).}$$

With the choice above, for strong restriction, the Hamiltonian (3.1.12) is reduced to:

$$\begin{aligned} H'' &= \frac{H' - H_3(z_\omega)}{\epsilon^4} \\ &= \frac{|\hat{v}_1|^2}{2M_1} - \hat{u}_1^T J \hat{v}_1 \omega - \epsilon^{2b-2a} \frac{\mu_0 \mu_1}{|\hat{u}_1|^a} - \frac{\mu_0 \mu_1}{|\hat{u}_1|^b} + \frac{1}{2} x^T S x \\ &\quad - \left[\frac{\mu_2(\mu_0 k_1^2 + \mu_1 k_0^2)}{|u_2|^{a+2}} \left(a P_2(\cos\theta) + \frac{a(a-1)}{2} P_1^2(\cos\theta) \right) \right] |\hat{u}_1|^2 \\ &\quad - \left[\frac{\mu_2(\mu_0 k_1^2 + \mu_1 k_0^2)}{|u_2|^{b+2}} \left(b P_2(\cos\theta) + \frac{b(b-1)}{2} P_1^2(\cos\theta) \right) \right] |\hat{u}_1|^2 + O(\epsilon^2). \end{aligned} \tag{3.1.15}$$

For weak restriction, similarly, we have

$$\begin{aligned} H'' &= \frac{H' - H_3(z_\omega)}{\epsilon^4} \\ &= \frac{|\hat{v}_1|^2}{2M_1} - \hat{u}_1^T J \hat{v}_1 \omega - \frac{\mu_0 \mu_1}{|\hat{u}_1|^a} - \epsilon^{2a-2b} \frac{\mu_0 \mu_1}{|\hat{u}_1|^b} + \frac{1}{2} x^T S x \\ &\quad - \left[\frac{\mu_2(\mu_0 k_1^2 + \mu_1 k_0^2)}{|u_2|^{a+2}} \left(a P_2(\cos\theta) + \frac{a(a-1)}{2} P_1^2(\cos\theta) \right) \right] |\hat{u}_1|^2 \\ &\quad - \left[\frac{\mu_2(\mu_0 k_1^2 + \mu_1 k_0^2)}{|u_2|^{b+2}} \left(b P_2(\cos\theta) + \frac{b(b-1)}{2} P_1^2(\cos\theta) \right) \right] |\hat{u}_1|^2 + O(\epsilon^2). \end{aligned} \tag{3.1.16}$$

As in [4, 15], we make the third assumption that u_2, the center of mass of the Earth-Moon system moves on a nearly circular orbit about the Sun. (i.e. $|u_2| \equiv |c_\omega|$ for each $\omega > 0$). Therefore, the last two terms of (3.1.15) and (3.1.16) can be simplified into

$$\begin{aligned} &\left[\frac{\mu_2(\mu_0 k_1^2 + \mu_1 k_0^2)}{|u_2|^{a+2}} \left(a P_2(\cos\theta) + \frac{a(a-1)}{2} P_1^2(\cos\theta) \right) \right] |\hat{u}_1|^2 \\ &\qquad + \left[\frac{\mu_2(\mu_0 k_1^2 + \mu_1 k_0^2)}{|u_2|^{b+2}} \left(b P_2(\cos\theta) + \frac{b(b-1)}{2} P_1^2(\cos\theta) \right) \right] |\hat{u}_1|^2 \\ &= |\hat{u}_1|^2 \mu_2(\mu_1 k_0^2 + \mu_0 k_1^2) \left[\frac{\omega^2}{\mu_2} \left(P_2(\cos\theta) - \frac{P_1^2(\cos\theta)}{2} \right) + \frac{P_1^2(\cos\theta)}{2} d_\omega \right], \end{aligned}$$

where for each $\omega > 0$,

$$d_\omega = \frac{a^2}{|c_\omega|^{a+2}} + \frac{b^2}{|c_\omega|^{b+2}}. \tag{3.1.17}$$

We shall choose weak restriction for the following reasons.

1. If we choose to adopt strong restriction, then the term $\epsilon^{2b-2a}\frac{\mu_0\mu_1}{|\hat{u}_1|^a}$ in (3.1.15) vanishes as $\epsilon \to 0^+$ and therefore this reduces to the case of homogeneous potential, which has already been studied in [4].

2. Both Manev potential ($U(r) = -\frac{C_1}{r} - \frac{C_2}{r^2}$, where C_1, C_2 are constant and r is the mutual distance) and Schwarzschild potential ($U(r) = -\frac{C_1}{r} - \frac{C_2}{r^3}$) are considered to be perturbation of the Newtonian potential $U(r) \sim -\frac{1}{r}$ and thus the Newtonian (weak) component dominates the solution dynamics.

Combining the definition of $|c_\omega|$:

$$\frac{a}{|c_\omega|^{a+2}} + \frac{b}{|c_\omega|^{b+2}} = \frac{\omega^2}{\mu_2},$$

and the definition of d_ω (3.1.17), we can rewrite

$$d_\omega = \frac{b\omega^2}{\mu_2} - \frac{ab - a^2}{|c_\omega|^{a+2}}, \tag{3.1.18}$$

and we see that the accuracy of the approximation of d_ω thus depends solely on the accuracy of the approximation of $u_\omega := |c_\omega|^{-(a+2)}$. It is clear that u_ω is the unique solution of

$$ax + bx^{\frac{b+2}{a+2}} = \frac{\omega^2}{\mu_2}.$$

Let $v_\omega := (\frac{\omega^2}{b\mu_2})^{\frac{a+2}{b+2}}$ be the unique solution of

$$by^{\frac{b+2}{a+2}} = \frac{\omega^2}{\mu_2}.$$

It is also obvious that both $v_\omega, u_\omega \to \infty$ as $\omega \to \infty$. Thus, Taking $\omega \to \infty$ in the identity

$$au_\omega^{1-\frac{b+2}{a+2}} + b = b(\frac{v_\omega}{u_\omega})^{\frac{b+2}{a+2}} \tag{3.1.19}$$

gives us

$$\lim_{\omega\to\infty} \frac{v_\omega}{u_\omega} = 1.$$

Also, if we let

$$k_\omega = \frac{b\omega^2}{\mu_2} - (ab - a^2)v_\omega,$$

then

$$\frac{k_\omega - d_\omega}{v_\omega} = -\frac{(ab-a^2)(u_\omega - v_\omega)}{v_\omega} \to 0$$

as $\omega \to \infty$, which implies

$$k_\omega = d_\omega + o(\omega^{\frac{2a+4}{b+2}}).$$

Thus, we arrive at an approximation for d_ω:

$$d_\omega = \frac{b\omega^2}{\mu_2} - (ab-a^2)(\frac{\omega^2}{b\mu_2})^{\frac{a+2}{b+2}} + o(\omega^{\frac{2a+4}{b+2}}). \tag{3.1.20}$$

As in [4], we make the use of symplectic algebra to simplify the constants in the Hamiltonian. Set

$$\begin{cases} \hat{u}_1 = K\xi \\ \hat{v}_1 = KM_1\eta \\ x = K\sqrt{M_1}Y \end{cases} \tag{3.1.21}$$

where $K = (\frac{\mu_0+\mu_1}{a+2})^{\frac{1}{a+2}}$. This change of variable is symplectic (symplectic transformation with multiplier $\frac{1}{M_1K^2}$) since the Jacobian

$$A = \mathrm{diag}(K, M_1K, M_1^{1/2}K, M_1^{1/2}K)$$

and it satisfies

$$\frac{1}{M_1K^2}A^TS_2A = S_2,$$

where S_2 is the standard symplectic matrix ($n=2$). Also, we choose $c_\omega = (|c_\omega|, 0)$, which makes u_2 be almost fixed on the $x-$axis and thus

$$\cos\theta = \frac{\xi_1}{|\xi|}, \tag{3.1.22}$$

where $\xi_1 = \xi \cdot (1,0)$.

By combining (3.1.20), (3.1.21) and (3.1.22), we arrive at a more familiar and compact form of Hamiltonian from (3.1.16):

$$\begin{aligned} H''' = \frac{|\eta|^2}{2} - \xi^T J\eta\omega - \frac{a+2}{|\xi|^a} - (\mu_0+\mu_1)^{\frac{a-b}{a+2}}(a+2)^{\frac{b+2}{a+2}}\frac{1}{|\xi|^b}\epsilon^{2a-2b} \\ - \xi_1^2\left((1+\frac{b}{2})\omega^2 - \frac{ab-a^2}{2}b^{-\frac{(a+2)}{b+2}}\mu_2^{\frac{b-a}{b+2}}\omega^{\frac{2(a+2)}{b+2}}\right) + \frac{1}{2}\omega^2|\xi|^2 \\ + \frac{1}{2}Y^TSY + O(\epsilon^2) + o(\omega^{\frac{2a+4}{b+2}}), \end{aligned} \tag{3.1.23}$$

as $\epsilon \to 0^+$ and $\omega \to +\infty$. Thus, we shall consider an approximation of this Hamiltonian

$$\begin{aligned} \hat{H} = \frac{|\eta|^2}{2} - \xi^T J\eta\omega - \frac{a+2}{|\xi|^a} - (\mu_0+\mu_1)^{\frac{a-b}{a+2}}(a+2)^{\frac{b+2}{a+2}}\frac{1}{|\xi|^b}\epsilon^{2a-2b} \\ - \xi_1^2\left((1+\frac{b}{2})\omega^2 - \frac{ab-a^2}{2}b^{-\frac{(a+2)}{b+2}}\mu_2^{\frac{b-a}{b+2}}\omega^{\frac{2(a+2)}{b+2}}\right) + \frac{1}{2}\omega^2|\xi|^2 . \end{aligned} \tag{3.1.24}$$

To avoid repetition, we define

$$C_\epsilon = (\mu_0+\mu_1)^{\frac{a-b}{a+2}}(a+2)^{\frac{b+2}{a+2}}\epsilon^{2a-2b}, \qquad C = \frac{ab-a^2}{2}b^{-\frac{(a+2)}{b+2}}\mu_2^{\frac{b-a}{b+2}},$$

and from now on, we assume C_ϵ is a large but fixed constant. Using the definition of Hamiltonian, we see

$$\begin{aligned} \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} &= \dot{\xi} = \frac{\partial \hat{H}}{\partial \eta} = \eta - J^T\xi\omega \\ \begin{bmatrix}\dot{p}_x\\ \dot{p}_y\end{bmatrix} &= \dot{\eta} = -\frac{\partial \hat{H}}{\partial \xi} = \underbrace{J\eta\omega}_{(p_y,-p_x)^T\omega} - \frac{a(a+2)}{|\xi|^{a+2}}\xi - \frac{bC_1}{|\xi|^{b+2}}\xi + \begin{bmatrix}\left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right]x \\ 0\end{bmatrix} - \omega^2\xi, \end{aligned} \tag{3.1.25}$$

where $\xi = (x, y)$ and $\eta = (p_x, p_y)$. We also see

$$\begin{aligned} p_x + \omega y = \dot{x} \\ p_y - \omega x = \dot{y}. \end{aligned} \tag{3.1.26}$$

We can now rewrite $\hat{H}$ as

$$\hat{H} = \tfrac{1}{2}\left[(p_x + y\omega)^2 + (p_y - x\omega)^2\right] - \tfrac{a+2}{(x^2+y^2)^{a/2}} - C_\epsilon \tfrac{1}{(x^2+y^2)^{b/2}} - \left[(1+\tfrac{b}{2})\omega^2 - C\omega^{\frac{2(a+2)}{b+2}}\right] x^2. \tag{3.1.27}$$

The effective potential is identified to be

$$V_\omega(x,y) := -\frac{a+2}{(x^2+y^2)^{a/2}} - C_\epsilon \frac{1}{(x^2+y^2)^{b/2}} - \left[(1+\frac{b}{2})\omega^2 - C\omega^{\frac{2(a+2)}{b+2}}\right] x^2. \tag{3.1.28}$$

Remark 3.1.1. *When $a = b$ and $\omega = 1$, $V(x,y) = -2\frac{a+2}{(x^2+y^2)^{a/2}} - (1+\frac{b}{2})x^2$ and this coincides with its homogeneous counterpart in [4], up to a constant before the potential component.*

By using (3.1.25), we have

$$\begin{aligned}
\ddot{x} - \dot{y}\omega = \dot{p}_x =& \dot{y}\omega - \frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}x - \frac{C_1 b}{(x^2+y^2)^{\frac{b+2}{2}}}x + \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right] x \\
\ddot{y} + \dot{x}\omega = \dot{p}_y =& -\dot{x}\omega - \frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}y - \frac{C_1 b}{(x^2+y^2)^{\frac{b+2}{2}}}y.
\end{aligned}$$

We calculate partial derivatives of V_ω as

$$\begin{aligned}
\frac{\partial V_\omega}{\partial x} &= \frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}x + C_\epsilon b \frac{1}{(x^2+y^2)^{\frac{b+2}{2}}}x - \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right] x \\
\frac{\partial V_\omega}{\partial y} &= \frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}y + C_\epsilon b \frac{1}{(x^2+y^2)^{\frac{b+2}{2}}}y,
\end{aligned}$$

which implies the equation of motion of the quasi-homogeneous Hill's lunar problem:

$$\begin{aligned}
\ddot{x} - 2\omega\dot{y} &= -\frac{\partial V_\omega}{\partial x}(x,y) \\
\ddot{y} + 2\omega\dot{x} &= -\frac{\partial V_\omega}{\partial y}(x,y).
\end{aligned} \tag{3.1.29}$$

The critical points of the effective potential (3.1.28) have the form of

$$(x,y) = (\pm x_\omega, 0), \tag{3.1.30}$$

where x_ω satisfies

$$(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}} = \frac{C_\epsilon b}{x_\omega^{b+2}} + \frac{a(a+2)}{x_\omega^{a+2}}, \tag{3.1.31}$$

and it exists and is unique if and only if

$$\omega > (\frac{2C}{2+b})^{\frac{b+2}{2b-2a}} =: \omega^*(a,b).$$

We shall call this type of critical points with zero velocity in the phase space as

$$\Gamma_0^\pm = (\pm x_\omega, 0, 0, 0) \in \mathbb{R}^4.$$

In the following section, we shall refer to this type of solutions Γ_0. That is, $\Gamma_0 = \Gamma_0^\pm$.

3.2 Ground state and its properties

The energy E_ω is defined to be

$$E_\omega(x,y,\dot{x},\dot{y}) := \frac{\dot{x}^2+\dot{y}^2}{2} + V_\omega(x,y). \tag{3.2.1}$$

3.2.1 Definition of the ground state energy

Consider the "ground state energy" problem, as introduced in [4, 3]:

$$E^*(\omega) = \inf\{E_\omega(x,y,\dot{x},\dot{y}) : W_\omega(x,y) = 0\}, \tag{3.2.2}$$

where the indicator function W_ω is defined to be

$$\begin{aligned} W_\omega(x,y) &= -(x,y)\cdot\nabla V_\omega(x,y) \\ &= \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right]x^2 - \left[\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b}\right], \end{aligned} \tag{3.2.3}$$

with $r = \sqrt{x^2+y^2}$.

Remark 3.2.1. *By direct computation, we see for $\omega > \omega^*(a,b)$, $E_\omega(\Gamma_0) = V_\omega(\Gamma_0) < 0$ and $W_\omega(\Gamma_0) = 0$.*

Lemma 3.2.1 (Ground state energy)**.** *For each $\omega > \omega^*(a,b)$ and $b > a$, $E^*(\omega)$ is finite and it is attained exactly by the critical points of $V_\omega(x,y)$.*

Proof. First notice that when $\omega > \omega^*$, $(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}$. Since $E_\omega = \frac{\dot{x}^2+\dot{y}^2}{2} + V_\omega(x,y)$, if the minimum exists, then it must be $\dot{x} = \dot{y} = 0$. Now the problem becomes:

$$E^*(\omega) = \inf\{V_\omega(x,y), W_\omega(x,y) = 0\}, \tag{3.2.4}$$

where

$$V_\omega(x,y) := -\frac{a+2}{r^a} - C_\epsilon \frac{1}{r^b} - \left[(1+\frac{b}{2})\omega^2 - C\omega^{\frac{2(a+2)}{b+2}}\right] x^2, \tag{3.2.5}$$

and

$$W_\omega(x,y) = \left((2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right) x^2 - \left[\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b}\right]. \tag{3.2.6}$$

Suppose $E^*(\omega) = -\infty$, then either $r \to 0^+$ or $x \to \pm\infty$ (mutually exclusive). We see that in either of these two mutually exclusive case, $W_\omega = \pm\infty$. Hence $E^*(\omega) > -\infty$. We can then find a bounded sequence in $\{W_\omega = 0\}$ that approaches its infimum. (If this sequence (x_n, y_n) satisfies $r_n \to \infty$, then from $W_\omega = 0$, we see it must satisfy $|x_n| \to 0$ and hence $\inf\{V_\omega, W_\omega = 0\} = 0$. However, this contradicts Remark 3.2.1.) Next, we can, by using the closedness of $\{W_\omega = 0\}$, obtain a subsequence to see that (3.2.4) is attained. Thus, using Lagrange multiplier, there exists λ such that the minimizer (x,y) of (3.2.4) satisfies

$$\begin{cases} \nabla V_\omega(x,y) = \lambda \nabla W_\omega(x,y) \\ W_\omega(x,y) = 0, \end{cases} \tag{3.2.7}$$

which implies

$$\begin{cases} \left\{\frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}} + C_\epsilon b \frac{1}{(x^2+y^2)^{\frac{b+2}{2}}} - \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right]\right\} x \\ \quad = \lambda\left(2(2+b)\omega^2 - 4C\omega^{\frac{2(a+2)}{b+2}} + \frac{a^2(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}} + \frac{C_\epsilon b^2}{(x^2+y^2)^{\frac{b+2}{2}}}\right) x \\ \left(\frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}} + \frac{C_\epsilon b}{(x^2+y^2)^{\frac{b+2}{2}}}\right) y = \lambda\left(\frac{a^2(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}} + \frac{C_\epsilon b^2}{(x^2+y^2)^{\frac{b+2}{2}}}\right) y \\ \left((2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right) x^2 - \left(\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b}\right) = 0. \end{cases} \tag{3.2.8}$$

Claim: If $\lambda = 0$, then only $x = \pm x_\omega, y = 0$ satisfy (3.2.8).
If $\lambda = 0$, the first equation becomes

$$\left\{ \frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}} + C_\epsilon b \frac{1}{(x^2+y^2)^{\frac{b+2}{2}}} - \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}} \right] \right\} x = 0,$$

and if $x = 0$, then the third equation returns no solution since $a, b, C_\epsilon > 0$. So,

$$\frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}} + C_\epsilon b \frac{1}{(x^2+y^2)^{\frac{b+2}{2}}} - \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}} \right] = 0. \tag{3.2.9}$$

From the second equation of (3.2.8), if $\lambda = 0$,

$$\left(\frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}} + \frac{C_\epsilon b}{(x^2+y^2)^{\frac{b+2}{2}}} \right) y = 0.$$

We know that $\frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}} + \frac{C_\epsilon b}{(x^2+y^2)^{\frac{b+2}{2}}} > 0$, so $y = 0$. Combining with (3.2.9), we see the solution x satisfies

$$(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}} = \frac{C_\epsilon b}{|x|^{b+2}} + \frac{a(a+2)}{|x|^{a+2}},$$

which implies $x = \pm x_\omega$.

Claim: if $\lambda \neq 0$, then (3.2.8) does not have solution.
From the first equation of (3.2.8), if $x = 0$, then the third equation does not have solution. So $x \neq 0$. If $y = 0$, then the third equation can be written as:

$$(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}} = \frac{a(a+2)}{|x|^{a+2}} + \frac{C_\epsilon b}{|x|^{b+2}}.$$

Combing this and $y = 0$, the first equation becomes:

$$0 = \lambda \left(2(2+b)\omega^2 - 4C\omega^{\frac{2(a+2)}{b+2}} + \frac{a^2(a+2)}{(x^2)^{\frac{a+2}{2}}} + \frac{C_\epsilon b^2}{(x^2)^{\frac{b+2}{2}}} \right) x,$$

which is impossible when $x, \lambda \neq 0, \omega > \omega^*$.
Thus, in this case, $x \neq 0 \neq y$ and we can safely cancel x and y from the first and

second equation of (3.2.8) to see

$$\frac{\left\{\frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}+\frac{C_\epsilon b}{(x^2+y^2)^{\frac{b+2}{2}}}-\left[(2+b)\omega^2-2C\omega^{\frac{2(a+2)}{b+2}}\right]\right\}}{\left(2(2+b)\omega^2-4C\omega^{\frac{2(a+2)}{b+2}}+\frac{a^2(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}+\frac{C_\epsilon b^2}{(x^2+y^2)^{\frac{b+2}{2}}}\right)}=\frac{\left(\frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}+\frac{C_\epsilon b}{(x^2+y^2)^{\frac{b+2}{2}}}\right)}{\left(\frac{a^2(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}+\frac{C_\epsilon b^2}{(x^2+y^2)^{\frac{b+2}{2}}}\right)}.$$

Using the third equation of (3.2.8) we can simplify the numerators as

$$\frac{x^2-1}{\left(2(2+b)\omega^2-4C\omega^{\frac{2(a+2)}{b+2}}+\frac{a^2(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}+\frac{C_\epsilon b^2}{(x^2+y^2)^{\frac{b+2}{2}}}\right)}=\frac{x^2}{\left(\frac{a^2(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}+\frac{C_\epsilon b^2}{(x^2+y^2)^{\frac{b+2}{2}}}\right)}. \tag{3.2.10}$$

Now, if $x^2-1<0$, then $LHS<0$ and $RHS\geq 0$. If $x^2-1=0$, then using the right hand side gives $x=0$, which is impossible. If $x^2-1>0$, then both hands are positive. However, the LHS has smaller numerator but larger denominator (since $\omega>\omega^*$), thus the equality is not possible.

□

3.2.2 Variational characterizations of the ground state energy

The moment of inertia is defined as

$$I=\frac{x^2+y^2}{2}. \tag{3.2.11}$$

The second derivative of I (often called Lagrange-Jacobi identity) can be simplified by using the equation of motion:

$$\begin{aligned}\ddot{I}&=\dot{x}^2+\dot{y}^2+2\omega(\dot{y}x-\dot{x}y)+\left[(2+b)\omega^2-2C\omega^{\frac{2(a+2)}{b+2}}\right]x^2-\left[\frac{a(a+2)}{r^a}+\frac{C_\epsilon b}{r^b}\right]\\&=\dot{x}^2+\dot{y}^2+2\omega(\dot{y}x-\dot{x}y)+W_\omega(x,y)\\&=\dot{x}^2+\dot{y}^2+2\omega(\dot{y}x-\dot{x}y)-\left(x\frac{\partial V_\omega}{\partial x}+y\frac{\partial V_\omega}{\partial y}\right)\\&=:K_\omega(x,y,\dot{x},\dot{y}).\end{aligned} \tag{3.2.12}$$

The main purpose of this section is that we want to derive the following variational property that is crucial to the final construction of invariant sets below and at the ground state energy level.

Proposition 3.2.1. *For $b > 2 > a$, there exists $\omega_0 > \omega^*(a,b)$ such that for $\omega > \omega_0$, we have*

$$\inf\{E_\omega : K_\omega \geq 0, W_\omega \leq 0\} = E_\omega(\Gamma_0). \tag{3.2.13}$$

Furthermore, this infimum is attained exactly by the critical points of V_ω: $\Gamma_0^\pm$.

In other words, the ground state energy can be characterized variationally using the two indicator functions K_ω and W_ω.

3.2.2.1 Lagrange equation with one constraint

In this subsection, we study the Lagrange equation

$$\begin{cases} \nabla E_\omega = \lambda \nabla K_\omega \\ K_\omega = 0, \end{cases} \tag{3.2.14}$$

or equivalently,

$$\begin{cases} \frac{\partial V_\omega}{\partial x} = \lambda(2\omega\dot{y} - \frac{\partial V_\omega}{\partial x} - x\frac{\partial^2 V_\omega}{\partial x^2} - y\frac{\partial^2 V_\omega}{\partial x\,\partial y}) \\ \frac{\partial V_\omega}{\partial y} = \lambda(-2\omega\dot{x} - x\frac{\partial^2 V_\omega}{\partial x\,\partial y} - \frac{\partial^2 V_\omega}{\partial y^2} - y\frac{\partial^2 V_\omega}{\partial y^2}) \\ \dot{x} = \lambda(2\dot{x} - 2\omega y) \\ \dot{y} = \lambda(2\dot{y} + 2\omega x) \\ \dot{x}^2 + \dot{y}^2 + 2\omega(x\dot{y} - \dot{x}y) \underbrace{-x\dfrac{\partial V_\omega}{\partial x} - y\dfrac{\partial V_\omega}{\partial y}}_{+W_\omega} = 0. \end{cases} \tag{3.2.15}$$

We shall prove the following proposition that will be useful in the later part of this section (Lemma 3.2.8).

Proposition 3.2.2. *Let $a < 2 < b$ and ω be sufficiently large. Among all the solutions of Equation* (3.2.15)*, Γ_0 has the least energy.*

Notice that if $\lambda = 0$ then the solutions of the Lagrange equation are exactly the critical points of $V_\omega(x,y)$. Let us assume $\lambda \neq 0$, we calculate

$$\begin{aligned}
\frac{\partial V_\omega}{\partial x} &= \frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}x + C_\epsilon b\frac{1}{(x^2+y^2)^{\frac{b+2}{2}}}x - \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right]x \\
\frac{\partial V_\omega}{\partial y} &= \frac{a(a+2)}{(x^2+y^2)^{\frac{a+2}{2}}}y + C_\epsilon b\frac{1}{(x^2+y^2)^{\frac{b+2}{2}}}y \\
\frac{\partial^2 V_\omega}{\partial x^2} &= \frac{a(a+2)}{r^{a+2}} - \frac{a(a+2)^2}{r^{a+4}}x^2 + \frac{C_\epsilon b}{r^{b+2}} - \frac{C_\epsilon b(b+2)x^2}{r^{b+4}} - \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right] \\
\frac{\partial^2 V_\omega}{\partial y^2} &= \frac{a(a+2)}{r^{a+2}} - \frac{a(a+2)^2}{r^{a+4}}y^2 + \frac{C_\epsilon b}{r^{b+2}} - \frac{C_\epsilon b(b+2)}{r^{b+4}}y^2 \\
\frac{\partial^2 V_\omega}{\partial x\,\partial y} &= -\frac{a(a+2)^2 xy}{r^{a+4}} - \frac{C_\epsilon b(b+2)xy}{r^{b+4}}.
\end{aligned}$$

If $\lambda \neq \frac{1}{2}$, then the previous equation becomes:

$$\begin{cases}
\left[\frac{(2+b)\omega^2 - 4\lambda^2\omega^2(1+b) - 2C\omega^{\frac{2(a+2)}{b+2}}}{1-2\lambda} + \frac{a(a+2)(a\lambda-1)}{r^{a+2}} + \frac{C_\epsilon b(b\lambda-1)}{r^{b+2}}\right]x = 0 & \text{(3.2.16a)} \\
\left[\frac{4\omega^2\lambda^2}{1-2\lambda} + \frac{a(a+2)(a\lambda-1)}{r^{a+2}} + \frac{C_\epsilon b(b\lambda-1)}{r^{b+2}}\right]y = 0 & \text{(3.2.16b)} \\
\dot{x} = \frac{-2\lambda\omega y}{1-2\lambda} & \text{(3.2.16c)} \\
\dot{y} = \frac{2\lambda\omega x}{1-2\lambda} & \text{(3.2.16d)} \\
\frac{\lambda(1-\lambda)}{(1-2\lambda)^2}4\omega^2 r^2 + \left((2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right)x^2 - \left(\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b}\right) = 0. & \text{(3.2.16e)}
\end{cases}$$

We prove this proposition by divide it into three cases:

1. $xy \neq 0$,
2. $x = 0$, $y \neq 0$,
3. and $x \neq 0$, $y = 0$.

Lemma 3.2.2. *For $b > a > 0$ and $b \geq 2$, there exists $\omega_0 > \omega^*(a,b)$ such that for any $\omega \geq \omega_0$, $E_\omega(\Gamma_0) < E_\omega(\Gamma_1)$, where $\Gamma_1 = (x_1, y_1, \dot{x}_1, \dot{y}_1) \in \mathbb{R}^4$ is any solution of* (3.2.15) *that satisfies $x_1 y_1 \neq 0$.*

Proof. When $xy \neq 0$, using (3.2.16a) and (3.2.16b), we see the solution

$$\lambda = \lambda_\omega^\pm = \pm\sqrt{\frac{1}{4} - \frac{C}{2(2+b)}\omega^{\frac{2a-2b}{b+2}}}.$$

Plug this λ back into (3.2.16b), we see the solution radius $r_1 = r_1(\omega)$ satisfies

$$\frac{\omega^2 - \frac{2C}{2+b}\omega^{\frac{2a+4}{b+2}}}{1-2\lambda} + \frac{a(a+2)(a\lambda-1)}{r_1^{a+2}} + \frac{C_\epsilon b(b\lambda-1)}{r_1^{b+2}} = 0, \tag{3.2.17}$$

where we have used $4\lambda_\omega^{\pm 2}\omega^2 = \omega^2 - \frac{2C}{2+b}\omega^{\frac{2a+4}{b+2}}$. We first note that $\frac{\omega^2 - \frac{2C}{2+b}\omega^{\frac{2a+4}{b+2}}}{1-2\lambda_\omega^\pm} \to +\infty$, $a\lambda_\omega^\pm - 1 \to \pm\frac{a}{2} - 1$ and $b\lambda_\omega^\pm - 1 \to \pm\frac{b}{2} - 1$ as $\omega \to \infty$. Let us call this type of solutions $\Gamma_1 = (x_1, y_1, \dot{x}_1, \dot{y}_1)$ (which depends on ω) and its radius to be r_1. Similarly, recall that the critical point of V_ω is written as $\Gamma_0 = (\pm r_0, 0, 0, 0)$, with radius r_0 being the unique solution of

$$(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}} = \frac{C_\epsilon b}{x^{b+2}} + \frac{a(a+2)}{x^{a+2}}, \tag{3.2.18}$$

as long as $\omega > \omega^*$. It is not difficult to observe that

$$a(a+2)(1-a\lambda)(1-2\lambda)\left(\frac{r_0}{r_1}\right)^{a+2} r_0^{b-a} + C_\epsilon b(1-b\lambda)(1-2\lambda)\left(\frac{r_0}{r_1}\right)^{b+2} = \frac{1}{2+b}\left(C_\epsilon b + a(a+2)r_0^{b-a}\right) \tag{3.2.19}$$

for $\lambda = \lambda_\omega^\pm$. Letting $\omega \to \infty$, we see

$$\frac{r_1}{r_0} \to \begin{cases} (2+b)^{\frac{2}{b+2}} & \text{if } \lambda = \lambda_\omega^- \\ 0 & \text{if } \lambda = \lambda_\omega^+, \end{cases} \tag{3.2.20}$$

which also implies $r_1 \to 0$ as $\omega \to \infty$ for $\lambda = \lambda_\omega^\pm$, since $r_0 \to 0$. If $\lambda = \lambda_\omega^+$, we immediately see that (3.2.17) does not have a solution for sufficiently large ω and $b > 2$, since the last term of (3.2.17) is positive and dominates as $r_1 \to 0^+$. Using (3.2.16c),(3.2.16d), we see

$$\frac{\dot{x}_1^2 + \dot{y}_1^2}{2} = \frac{4\lambda^2\omega^2 r_1^2}{2(1-2\lambda)^2}, \tag{3.2.21}$$

for $\lambda = \lambda_\omega^\pm$. Thus, the energy The energy of Γ_1 can be re-expressed as

$$\begin{aligned} E_\omega(\Gamma_1) &= \frac{\dot{x}_1^2 + \dot{y}_1^2}{2} + V(x,y) \\ &= \frac{4\lambda^2\omega^2 r_1^2}{2(1-2\lambda)^2} - \frac{a+2}{r_1^a} - \left[(1+\frac{b}{2})\omega^2 - C\omega^{\frac{2a+4}{b+2}}\right] x_1^2 - \frac{C_\epsilon}{r_1^b} \\ &\overset{(3.2.16e)}{=} \frac{4\lambda^2\omega^2 r_1^2}{2(1-2\lambda)^2} - \frac{a+2}{r_1^a} - \left[-2\omega^2 r_1^2 \frac{\lambda(1-\lambda)}{(1-2\lambda)^2} + \frac{a(a+2)}{2r_1^a} + \frac{C_\epsilon b}{2r_1^b}\right] - \frac{C_\epsilon}{r_1^b}, \end{aligned} \tag{3.2.22}$$

where $\lambda = \lambda_\omega^\pm$. Similarly, the energy of Γ_0 is

$$E_\omega(\Gamma_0) = -\frac{1}{r_0^b}\left[\frac{(a+2)^2}{2} r_0^{b-a} + \frac{C_\epsilon(b+2)}{2}\right], \tag{3.2.23}$$

where we have used (3.1.31). For $\lambda = \lambda_\omega^+$ and $b = 2$, using (3.2.18), we can simply (3.2.22) as

$$E_\omega(\Gamma_1) = \left(\frac{1}{\lambda_\omega^+} - 2\right)\frac{C_\epsilon}{r_1^2} + \frac{a(1-a\lambda_\omega^+)(a+2)}{2\lambda r_1^a(1-2\lambda_\omega^+)} - \frac{(a+2)^3}{2r_1^a}.$$

The first term is positive and

$$\frac{\frac{a(1-a\lambda_\omega^+)(a+2)}{2\lambda_\omega^+ r_1^a(1-2\lambda_\omega^+)}}{\frac{(a+2)^3}{2r_1^a}} = \frac{a(1+a\lambda_\omega^+)}{\lambda_\omega^+(1-2\lambda_\omega^+)(a+2)^2} \to \infty \quad \text{as } \omega \to \infty,$$

which, in particular, implies $E_\omega(\Gamma_1) > 0$ for ω sufficiently large and $b = 2, \lambda = \lambda_\omega^+$. Similarly, using (3.2.18), for $\lambda = \lambda_\omega^-$, we have

$$E_\omega(\Gamma_1) = \frac{1 + \frac{4\lambda_\omega^-(1-\lambda_\omega^-)\omega^2}{\omega^2 - \frac{2C}{2+b}\omega^{\frac{2a+4}{b+2}}}}{2(1-2\lambda_\omega^-)}\left[\frac{a(a+2)(1-a\lambda_\omega^-)}{r_1^a} + \frac{C_\epsilon b(1-b\lambda_\omega^-)}{r_1^b}\right] - \frac{(a+2)^2}{2r_1^a} - \frac{C_\epsilon(2+b)}{2r_1^b}.$$

Consider the function

$$f(\omega) = \frac{E_\omega(\Gamma_1)}{E_\omega(\Gamma_0)}$$
$$= \frac{\left(1+\frac{4\lambda_\omega^-(1-\lambda_\omega^-)\omega^2}{\omega^2-\frac{2C}{2+b}\omega^{\frac{2a+4}{b+2}}}\right)\left[a(a+2)(1-a\lambda_\omega^-)(\frac{r_0}{r_1})^a r_0^{b-a}+C_\epsilon b(1-b\lambda_\omega^-)(\frac{r_0}{r_1})^b\right]}{-\frac{(a+2)^2}{2}r_0^{b-a}-\frac{C_\epsilon(b+2)}{2}}$$
$$+\frac{-\frac{(a+2)^2}{2}(\frac{r_0}{r_1})^a r_0^{b-a}-\frac{C_\epsilon(b+2)}{2}(\frac{r_0}{r_1})^b}{-\frac{(a+2)^2}{2}r_0^{b-a}-\frac{C_\epsilon(b+2)}{2}}. \tag{3.2.24}$$

Letting $\omega \to \infty$ and combining (3.2.20) and $r_0 \to 0$ as $\omega \to \infty$, we see

$$\lim_{\omega\to\infty} f(\omega) = \frac{1}{2}(b+2)^{\frac{-b+2}{b+2}} < 1 \qquad \text{as long as } b > 0. \tag{3.2.25}$$

□

Remark 3.2.2. *This limit of ratio resembles the homogeneous case and* $\omega = 1$.

Lemma 3.2.3. *For* $0 < a < 2 \le b$*, the Lagrange equation* (3.2.15) *does not have a solution when* $b = 2$*,* $x = 0$ *and* $y \neq 0$*. When* $b > 2$*, there exists* $\omega_0 > \omega^*(a,b)$ *such that when* $\omega > \omega_0$*,* $E_\omega(\Gamma_2) > 0$*, where* Γ_2 *is any solution of* (3.2.15) *that satisfies* $x = 0$ *and* $y \neq 0$.

Proof. When $x = 0, y \neq 0, |y| = r$ and thus (3.2.16b) and (3.2.16e) become:

$$\begin{aligned} &\frac{4\omega^2\lambda^2}{1-2\lambda} + \frac{a(a+2)(a\lambda-1)}{r^{a+2}} + \frac{C_\epsilon b(b\lambda-1)}{r^{b+2}} = 0 \\ &\frac{\lambda(1-\lambda)}{(1-2\lambda)^2}4\omega^2 r^2 - \frac{a(a+2)}{r^a} - \frac{C_\epsilon b}{r^b} = 0. \end{aligned} \tag{3.2.26}$$

We can simplify them into:

$$\begin{aligned} &\left(-4\omega^2 - \frac{4a(a+2)}{r^{a+2}} - \frac{4C_\epsilon b}{r^{b+2}}\right)\lambda^2 + \left(4\omega^2 + \frac{4a(a+2)}{r^{a+2}} + \frac{4C_\epsilon b}{r^{b+2}}\right)\lambda - \left(\frac{a(a+2)}{r^{a+2}} + \frac{C_\epsilon b}{r^{b+2}}\right) = 0 \\ &\left(4\omega^2 - \frac{2a^2(a+2)}{r^{a+2}} - \frac{2b^2 C_\epsilon}{r^{b+2}}\right)\lambda^2 + \left(\frac{a(a+2)^2}{r^{a+2}} + \frac{C_\epsilon b(b+2)}{r^{b+2}}\right)\lambda - \left(\frac{a(a+2)}{r^{a+2}} + \frac{C_\epsilon b}{r^{b+2}}\right) = 0, \end{aligned} \tag{3.2.27}$$

which gives us:

$$\begin{aligned}&\left[\left(-4\omega^2-\frac{4a(a+2)}{r^{a+2}}-\frac{4C_\epsilon b}{r^{b+2}}\right)\lambda+\left(4\omega^2+\frac{4a(a+2)}{r^{a+2}}+\frac{4C_\epsilon b}{r^{b+2}}\right)\right]\lambda\\&=\left[\left(4\omega^2-\frac{2a^2(a+2)}{r^{a+2}}-\frac{2b^2C_\epsilon}{r^{b+2}}\right)\lambda+\left(\frac{a(a+2)^2}{r^{a+2}}+\frac{C_\epsilon b(b+2)}{r^{b+2}}\right)\right]\lambda.\end{aligned}\tag{3.2.28}$$

Solving this gives $\lambda=0$ or $\frac{1}{2}$ or

$$4\omega^2+\frac{a(2-a)(2+a)}{r^{a+2}}-\frac{2b(b-2)C_\epsilon}{r^{b+2}}=0.\tag{3.2.29}$$

The first case and the second are both impossible under $x=0, y\neq 0$, so let us focus on the third case. Assume $a<2$. When $b=2$, equation (3.2.29) does not have a solution. However, when $b>2$, each ω corresponds to at least one solution and $r_2\to 0$ as $\omega\to\infty$ where we assume the solutions of Lagrange equation $\Gamma_2=(0,y_2,\dot{x}_2,0)$ and this type of solutions satisfy $r_2=|y_2|$. First, we notice $r_2\to 0$ as $\omega\to\infty$. Plugging this radius relation into either of these two equations, we see

$$\lambda^2-\lambda+\frac{\frac{a(a+2)}{r_2^{a+2}}+\frac{C_\epsilon b}{r_2^{b+2}}}{\frac{bC_\epsilon(b+2)}{r_2^{b+2}}+\frac{a(a+2)^2}{r_2^{a+2}}}=0.\tag{3.2.30}$$

When $\omega\to\infty$, the two solutions of this equation converge respectively to $\lambda_\infty^\pm$ where

$$\lambda_\infty^\pm=\frac{1\pm\sqrt{\frac{b-2}{b+2}}}{2}.\tag{3.2.31}$$

We have to compute the energy of $E_\omega(\Gamma_2)$. Since

$$\begin{aligned}E_\omega(\Gamma_2)&=\frac{(\frac{(2\lambda)^2}{(1-2\lambda)^2})\omega^2r_2^2}{2}-\frac{a+2}{r_2^a}-\frac{C_\epsilon}{r_2^b}\\&=\frac{C_\epsilon}{r_2^b}\left(\frac{\lambda^2}{(1-2\lambda)^2}b(b-2)-1\right)-\frac{a+2}{r_2^a}\left(\frac{a(2-a)}{2}\frac{\lambda^2}{(1-2\lambda)^2}+1\right),\end{aligned}\tag{3.2.32}$$

where $\lambda\to\lambda_\infty^\pm$. The only thing we care about is the sign of

$$\frac{\lambda^2}{(1-2\lambda)^2}b(b-2)-1\tag{3.2.33}$$

as $\omega \to \infty$ (or rather $\lambda \to \lambda_\infty^\pm$), since r_3^{-b} dominates as r_3 approaches zero and what is inside each bracket has a limit that is greater than zero. It is not difficult to see

$$\frac{\lambda_\pm^2}{(1-2\lambda_\pm)^2} b(b-2) - 1 \xrightarrow{\lambda \to \lambda_\infty^\pm} \frac{b}{4}(2b \pm 2\sqrt{(b-2)(b+2)}) =: f_\pm(b). \tag{3.2.34}$$

We can easily check the function $\frac{b}{4}(2b \pm 2\sqrt{(b-2)(b+2)}) > 0$ as long as $b > 2$.

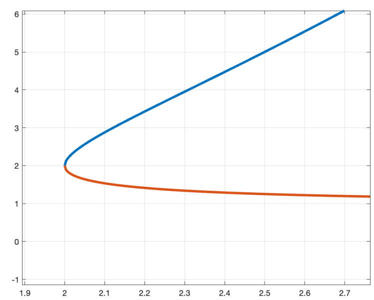


Figure 3.1: $f_\pm(b)$, where the blue (red) line corresponds to $f_{+(-)}(b)$

Thus $E_\omega(\Gamma_2) \to +\infty$ as $\omega \to \infty$. □

Lemma 3.2.4. *For $a < 2 < b$, $y = 0, x \neq 0$, there exists ω_1, such that when $\omega \geq \omega_1$, the Lagrange equation* (3.2.15) *has no solution other than the critical points of V_ω.*

Proof. When $y = 0, x \neq 0$, i.e. $|x| = r$. As usual, we assume $a < 2$. From (3.2.16a) and (3.2.16e) we get:

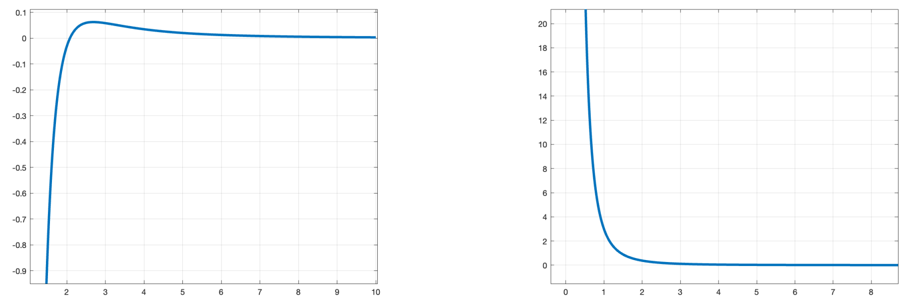
$$\begin{cases} \frac{(2+b)\omega^2 - 4\lambda^2\omega^2(1+b) - 2C\omega^{\frac{2(a+2)}{b+2}}}{1-2\lambda} + \frac{a(a+2)(a\lambda-1)}{r^{a+2}} + \frac{C_\epsilon b(b\lambda-1)}{r^{b+2}} = 0 & \text{(3.2.35a)} \\ \left(\frac{\lambda(1-\lambda)}{(1-2\lambda)^2} 4\omega^2 + (2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right) - \frac{a(a+2)}{r^{a+2}} - \frac{C_\epsilon b}{r^{b+2}} = 0. & \text{(3.2.35b)} \end{cases}$$

It is clear that $\lambda = 0$ is a solution and it corresponds to Γ_0. Performing (3.2.35a)-(3.2.35b) gives us

$$\frac{(8(1+b))\lambda^2 - (8+8b)\lambda + 2b}{(1-2\lambda)^2}\omega^2 - \frac{4C}{1-2\lambda}\omega^{\frac{2a+4}{b+2}} + \frac{a^2(a+2)}{r^{a+2}} + \frac{C_\epsilon b^2}{r^{b+2}} = 0, \tag{3.2.36}$$

and performing 2×(3.2.35b)∓(3.2.36) respectively gives us

$$\begin{cases} \frac{4}{(1-2\lambda)^2}\omega^2 - 4C\omega^{\frac{2a+4}{b+2}}(1 - \frac{1}{1-2\lambda}) = \frac{a(a+2)^2}{r^{a+2}} + \frac{C_\epsilon b(2+b)}{r^{b+2}} & \text{(3.2.37a)} \\ (4+4b)\omega^2 - 4C\omega^{\frac{2a+4}{b+2}}(1 + \frac{1}{1-2\lambda}) + \frac{(a^2-2a)(a+2)}{r^{a+2}} + \frac{C_\epsilon(b^2-2b)}{r^{b+2}} = 0. & \text{(3.2.37b)} \end{cases}$$



(a) $g(r)$ with $a < 2 < b$

(b) $g(r)$ with $a < 2 = b$

Figure 3.2: The function of $g(r)$

We define

$$g(r) = \frac{(2a - a^2)(a+2)}{r^{a+2}} - \frac{C_\epsilon(b^2 - 2b)}{r^{b+2}} \tag{3.2.38}$$

and (3.2.37b) becomes

$$(4 + 4b)\omega^2 - 4\omega^{\frac{2a+4}{b+2}}\left(1 + \frac{1}{1 - 2\lambda}\right) = g(r). \tag{3.2.39}$$

If it has a solution when $\omega \to \infty$, then the term

$$1 + \frac{1}{1 - 2\lambda} = \Theta(f(\omega)), f(\omega) = \Omega(\omega^{\frac{2b-2a}{b+2}}) \text{ for some } f(\omega) > 0.$$

where Θ stands for the same order and Ω stands for "at least the same order". It is easy to see since otherwise the LHS of (3.2.39) goes to plus infinity as $\omega \to \infty$.

We see the LHS of (3.2.37a) has order $\Theta(f^2(\omega)\omega^2)$ and thus $r \to 0$ as $\omega \to \infty$. We can easily verify $\frac{a(a+2)^2}{r^{a+2}} + \frac{C_\epsilon b(2+b)}{r^{b+2}} \simeq \frac{C_\epsilon b(2+b)}{r^{b+2}}$ as $r \to 0$ in (3.2.37a). Thus, $\frac{1}{r^{b+2}} = \Theta(f^2(\omega)\omega^2)$.

In (3.2.39), if $\omega^2 = O(f(\omega)\omega^{\frac{2a+4}{b+2}})$, then the LHS of (3.2.39) has order $O(f(\omega)\omega^{\frac{2a+4}{b+2}})$. If $f(\omega)\omega^{\frac{2a+4}{b+2}} = O(\omega^2)$, then the LHS of (3.2.39) has order $O(\omega^2)$. Similarly, we can easily verify $g(r) = \frac{(2a-a^2)(a+2)}{r^{a+2}} - \frac{C_\epsilon(b^2-2b)}{r^{b+2}} \simeq -\frac{C_\epsilon(b^2-2b)}{r^{b+2}}$ as $r \to 0$, so $\frac{1}{r^{b+2}} = O(\omega^2)$ or $O(f(\omega)\omega^{\frac{2a+4}{b+2}})$. In either case, this leads to a contradiction.

□

3.2.2.2 Lagrange equation with two equality constraints

In this section, we study the Lagrange equation with two constraints. Namely,

$$\begin{cases} \nabla E_\omega = \lambda \nabla K_\omega + \mu \nabla W_\omega \\ K_\omega = 0 \\ W_\omega = 0. \end{cases} \tag{3.2.40}$$

Or equivalently,

$$\begin{cases} \frac{\partial V_\omega}{\partial x} = \lambda(2\omega \dot{y} - \frac{\partial V_\omega}{\partial x} - x\frac{\partial^2 V_\omega}{\partial x^2} - y\frac{\partial^2 V_\omega}{\partial x\, \partial y}) - \mu(\frac{\partial V_\omega}{\partial x} + x\frac{\partial^2 V_\omega}{\partial x^2} + yV\frac{\partial^2 V_\omega}{\partial x\, \partial y}) \\ \frac{\partial V_\omega}{\partial y} = \lambda(-2\omega \dot{x} - x\frac{\partial^2 V_\omega}{\partial x\, \partial y} - \frac{\partial^2 V_\omega}{\partial y^2} - y\frac{\partial^2 V_\omega}{\partial y^2}) - \mu(\frac{\partial V_\omega}{\partial y} + x\frac{\partial^2 V_\omega}{\partial x\, \partial y} + y\frac{\partial^2 V_\omega}{\partial y^2}) \\ \dot{x} = \lambda(2\dot{x} - 2\omega y) \\ \dot{y} = \lambda(2\dot{y} + 2\omega x) \\ \dot{x}^2 + \dot{y}^2 + 2\omega(x\dot{y} - \dot{x}y) \underbrace{-x\frac{\partial V_\omega}{\partial x} - y\frac{\partial V_\omega}{\partial y}}_{+W_\omega} = 0 \\ W_\omega = \left((2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right)x^2 - (\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b}) = 0. \end{cases} \tag{3.2.41}$$

We shall prove the following proposition, which is useful in the proof of the final variational property ((Lemma 3.2.8)).

Proposition 3.2.3. *When* $b > a$ *and* $\omega > \omega^*(a, b)$*, the only solutions of* (3.2.40) *are exactly the critical points of* V_ω*. Moreover,* $\inf\{E_\omega : W_\omega = K_\omega = 0\} = E_\omega(\Gamma_0)$ *as long as* $\omega > \omega^*(a, b)$*.*

Proof. It is clear that if $\lambda = \mu = 0$, the solution of equation (3.2.41) coincides with the critical point of effective potential V_ω. If $\lambda \neq \frac{1}{2}$, then

$$\dot{x} = \frac{-2\lambda\omega y}{1 - 2\lambda} \qquad \dot{y} = \frac{2\lambda\omega x}{1 - 2\lambda}. \tag{3.2.42}$$

Plug this into the second last equation of (3.2.41) and combine with the last equation of (3.2.41), we have

$$\frac{\lambda(1-\lambda)}{(1-2\lambda)^2} 4\omega^2 r^2 = 0, \tag{3.2.43}$$

which implies $\lambda = 0, 1$. Meanwhile, the first and the second equation can be rewritten as:

$$
\begin{aligned}
&-\left[\frac{(2+b)\omega^2 - 4\lambda^2\omega^2(1+b) - 2C\omega^{\frac{2(a+2)}{b+2}}}{1-2\lambda} + \frac{a(a+2)(a\lambda-1)}{r^{a+2}} + \frac{C_\epsilon b(b\lambda-1)}{r^{b+2}}\right] x \\
&\qquad = \mu x \left[2(2+b)\omega^2 - 4C\omega^{\frac{2(a+2)}{b+2}} + \frac{a^2(a+2)}{r^{a+2}} + \frac{C_\epsilon b^2}{r^{b+2}}\right] . \\
&-\left[\frac{4\omega^2\lambda^2}{1-2\lambda} + \frac{a(a+2)(a\lambda-1)}{r^{a+2}} + \frac{C_\epsilon b(b\lambda-1)}{r^{b+2}}\right] y = \mu y \left(\frac{a^2(a+2)}{r^{a+2}} + \frac{C_\epsilon b^2}{r^{b+2}}\right).
\end{aligned}
\tag{3.2.44}
$$

If $\lambda = 1$ and assume $xy \neq 0$, then from these two equations above, we get $\mu = -\frac{3}{2} - \frac{4C}{(2+b)\omega^{\frac{2b-2a}{b+2}} - 2C}$. Plugging this μ and λ back into (3.2.44) and rearranging gives

$$
4\omega^2 + \frac{\frac{1}{2}a(a+2)^2}{r^{a+2}} + \frac{C_\epsilon b(\frac{1}{2}b+1)}{r^{b+2}} = -\frac{4C}{(2+b)\omega^{\frac{2b-2a}{b+2}} - 2C}\left(\frac{a^2(a+2)}{r^{a+2}} + \frac{C_\epsilon b}{r^{b+2}}\right).
\tag{3.2.45}
$$

If $\omega > \omega^*(a, b)$, then the previous equation does not have a solution since both sides have the opposite sign. If $\lambda = 1$ and $x = 0, y \neq 0$, then from the last equation of (3.2.41), we see it is not possible. If $\lambda = 1$ and $x \neq 0, y = 0$, then either (3.2.41) does not have a solution or the solution coincides with the critical points of V_ω.

Similarly, if $\lambda = 0$ and assume $xy \neq 0$, we get $\mu = -\frac{1}{2}$. Plugging λ and μ back into the second equation of (3.2.44) and cancelling y gives

$$
\left(\frac{a(a+2)}{r^{a+2}} + \frac{C_\epsilon b}{r^{b+2}}\right) = -\frac{1}{2}\left[\frac{a^2(a+2)}{r^{a+2}} + \frac{C_\epsilon b^2}{r^{b+2}}\right],
\tag{3.2.46}
$$

which does not have a solution. The remaining two cases ($\lambda = 0, x = 0, y \neq 0$ and $\lambda = 0, x \neq 0, y = 0$) yield the same result as their counterparts when $\lambda = 1$.

For the second part, notice that when $\omega > \omega^*(a, b)$,

$$
\inf\{E_\omega : K_\omega = 0, W_\omega = 0\} \geq \inf\{E_\omega : W_\omega = 0\} = E_\omega(\Gamma_0) > -\infty.
\tag{3.2.47}
$$

and thus we can repeat the argument as we did in the proof of Lemma 3.2.1 to show this infimum is attained by some $\Gamma \in (\mathbb{R}^4)$ and hence it must satisfies the Lagrange equation (3.2.40). □

3.2.2.3 Bounded below lemma

Lemma 2 in [4] can also be extended to quasi-homogeneous case when we have $b > 2$. With this lemma, we will be able to prove the existence of minimizer for the final variational property.

Lemma 3.2.5 (Bounded below lemma). *When $b > 2$ and $b > a$, for each ω there exists $c > 0$ such that $r \geq c$ whenever $E_\omega(\Gamma) < 1$ and $\left|K_\omega(\Gamma)\right| < c$.*

Proof. Suppose not, then there exists a sequence $(\Gamma_n) = (x_n, y_n, \dot{x}_n, \dot{y}_n) \in \mathbb{R}^4$, such that $E(\Gamma_n) < 1$, $\left|K(\gamma_n)\right| < \frac{1}{n}$ but $r(\Gamma_n) = r_n < \frac{1}{n}$. Since

$$K_\omega(\Gamma_n) = (\dot{x}_n - \omega y_n)^2 + (\dot{y}_n + \omega x_n)^2 + \left[(2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right] x_n^2 \\ - \omega^2 r_n^2 - \left(\frac{a(a+2)}{r_n^a} + \frac{C_\epsilon b}{r_n^b}\right) \to 0 \tag{3.2.48}$$

as $r_n \to 0$, then we have

$$\dot{x}_n^2 + \dot{y}_n^2 - \left(\frac{a(a+2)}{r_n^a} + \frac{C_\epsilon b}{r_n^b}\right) \to 0$$

Now, the energy can be rewritten as

$$\begin{aligned} E_\omega(\Gamma_n) &= \frac{\dot{x}_n^2 + \dot{y}_n^2}{2} - \frac{a+2}{r_n^a} - \frac{C_\epsilon}{r_n^b} - \left[(1+\frac{b}{2})\omega^2 - C\omega^{\frac{2(a+2)}{b+2}}\right] x_n^2 \\ &= \frac{1}{2}\left(\dot{x}_n^2 + \dot{y}_n^2 - \frac{C_\epsilon b}{r_n^b} - \frac{a(a+2)}{r_n^a}\right) + \left(\frac{(\frac{b}{2}-1)C_\epsilon}{r_n^b} + \frac{(a+2)(\frac{1}{2}a-1)}{r_n^a}\right) \\ &\quad - \left[(1+\frac{b}{2})\omega^2 - C\omega^{\frac{2(a+2)}{b+2}}\right] x_n^2 \to +\infty \end{aligned} \tag{3.2.49}$$

as $r_n \to 0$, which contradicts the energy assumption. □

Remark 3.2.3. $E_\omega(\Gamma) < 1$ *can be replaced by boundedness from above.*

3.2.2.4 Lagrange equation with inequality constraints

In this section, we study

$$\inf\{E_\omega : K_\omega \geq 0, W_\omega \leq 0\}, \tag{3.2.50}$$

where $\omega > 0$ is considered to be fixed. We shall show that for sufficiently large ω, this infimum is finite and attained exactly by the critical points of V_ω. (Proposition 3.2.1) We divide the proof into three lemmas. We begin by noticing that

Lemma 3.2.6. *When $b > 2$ and $b > a$, for each ω there exists $c > 0$ such that for each $\Gamma \in \mathbb{R}^4$ we have $r \geq c$ whenever $E(\Gamma) < 1$ and $K(\Gamma) \geq 0$.*

Proof. This proof is similar to the bounded below lemma above Lemma 3.2.5. □

Using this lemma, we are able to prove

Lemma 3.2.7. *When $b > 2$ and $b > a$, for each $\omega > \omega^*$,* (3.2.50) *is finite and a minimizer for* (3.2.50) *exists.*

Proof. Let $E'_\omega = \{E_\omega : K_\omega \geq 0, W_\omega \leq 0\}$ and $A_\omega = \left\{\Gamma \in \mathbb{R}^4 : K_\omega \geq 0, W_\omega \leq 0\right\}$. It suffices to prove $E'_\omega > -\infty$ for each fixed $\omega > \omega^*$. Suppose not, then since

$$E_\omega = \frac{\dot{x}^2 + \dot{y}^2}{2} - \frac{a+2}{r^a} - C_\epsilon \frac{1}{r^b} - \left[(1 + \frac{b}{2})\omega^2 - C\omega^{\frac{2(a+2)}{b+2}}\right] x^2,$$

we have two cases: there exists a sequence $(x_n, y_n, \dot{x}_n, \dot{y}_n) \in A_\omega$ such that either $r_n \to 0$ or $x_n^2 \to \infty$. Since

$$W_\omega(x, y) = \left((2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right) x^2 - \left(\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b}\right), \tag{3.2.51}$$

the first case is forbidden by the lemma we just proved. The second case is not possible, because that would imply $W_\omega(x, y) \to \infty$. □

Since the minimum exists, it makes sense to study the Lagrange equation with inequality constraints. We can transform the system of algebraic equations by introducing new variables:

$$\begin{cases} \widetilde{K}_\omega(x, y, \dot{x}, \dot{y}, p, q) = K_\omega(x, y, \dot{x}, \dot{y}) - p^2 \\ \widetilde{W}_\omega(x, y, \dot{x}, \dot{y}, p, q) = W_\omega(x, y) + q^2 \\ \widetilde{E}_\omega(x, y, \dot{x}, \dot{y}, p, q) = E_\omega(x, y, \dot{x}, \dot{y}), \end{cases} \tag{3.2.52}$$

where $p, q \in \mathbb{R}$ and are varying. The new Lagrange equation becomes:

$$\begin{cases} \nabla \widetilde{E}_\omega = \lambda \nabla \widetilde{K} + \mu \nabla \widetilde{W} \\ \widetilde{K}_\omega = 0 \\ \widetilde{W}_\omega = 0. \end{cases} \tag{3.2.53}$$

Lemma 3.2.8. *For $b > 2 > a$, there exists $\omega_0 > \omega^*(a,b)$ such that among all the solutions of* (3.2.53)*, the critical points of V_ω have the least energy.*

Proof. Expanding each term of (3.2.53), we get:

$$\begin{cases} \frac{\partial V_\omega}{\partial x} = \lambda(2\omega\dot{y} - \frac{\partial V_\omega}{\partial x} - x\frac{\partial^2 V_\omega}{\partial x^2} - y\frac{\partial^2 V_\omega}{\partial x \partial y}) - \mu(\frac{\partial V_\omega}{\partial x} + x\frac{\partial^2 V_\omega}{\partial x^2} + yV\frac{\partial^2 V_\omega}{\partial x \partial y}) \\ \frac{\partial V_\omega}{\partial y} = \lambda(-2\omega\dot{x} - x\frac{\partial^2 V_\omega}{\partial x \partial y} - \frac{\partial^2 V_\omega}{\partial y^2} - y\frac{\partial^2 V_\omega}{\partial y^2}) - \mu(\frac{\partial V_\omega}{\partial y} + x\frac{\partial^2 V_\omega}{\partial x \partial y} + y\frac{\partial^2 V_\omega}{\partial y^2}) \\ \dot{x} = \lambda(2\dot{x} - 2\omega y) \\ \dot{y} = \lambda(2\dot{y} + 2\omega x) \\ 0 = 2\lambda p \\ 0 = 2\mu q \\ \dot{x}^2 + \dot{y}^2 + 2\omega(x\dot{y} - \dot{x}y) - x\frac{\partial V_\omega}{\partial x} - y\frac{\partial V_\omega}{\partial y} - p^2 = 0 \\ \left((2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right)x^2 - (\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b}) + q^2 = 0. \end{cases} \tag{3.2.54}$$

We now restrict our focus to $b > 2 > a$.

1. First, we notice if $\lambda = \mu = p = q = 0$, then this equation has solution Γ_0 as long as $\omega > \omega^*$.

2. If $pq \neq 0$, then $\lambda = \mu = 0$ and thus $\frac{\partial V_\omega}{\partial x} = \frac{\partial V_\omega}{\partial y} = 0$. From the second last equation, we have $p = 0$.

3. If $p = 0 = q$, then we have the Lagrange equation with two equality constraints ($\nabla E_\omega = \mu \nabla K_\omega + \lambda \nabla W_\omega, K_\omega = W_\omega = 0$), and we know the only solutions are Γ_0 for $\omega > \omega^*$ (Proposition 3.2.3).

4. If $p = 0, q \neq 0$, we have $\mu = 0$. Plug in these values, we recognize the first four equations and the second last equations are exactly the Lagrange equation with one constraint ($\nabla E_\omega = \lambda \nabla K_\omega, K_\omega = 0$). We know that when ω is sufficiently large, among all the solutions, Γ_0 has the least energy (Proposition 3.2.2).

5. If $p \neq 0, q = 0$, then $\lambda = 0$. The equation becomes

$$\begin{cases} \frac{\partial V_\omega}{\partial x} = -\mu(\frac{\partial V_\omega}{\partial x} + x\frac{\partial^2 V_\omega}{\partial x^2} + yV\frac{\partial^2 V_\omega}{\partial x\,\partial y}) \\ \frac{\partial V_\omega}{\partial y} = -\mu(\frac{\partial V_\omega}{\partial y} + x\frac{\partial^2 V_\omega}{\partial x\,\partial y} + y\frac{\partial^2 V_\omega}{\partial y^2}) \\ \dot{x} = 0 \\ \dot{y} = 0 \\ \dot{x}^2 + \dot{y}^2 + 2\omega(x\dot{y} - \dot{x}y) - x\frac{\partial V_\omega}{\partial x} - y\frac{\partial V_\omega}{\partial y} - p^2 = 0 \\ \left((2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right)x^2 - (\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b}) = 0. \end{cases} \tag{3.2.55}$$

We recognize it contains the equation we studied before. Namely, $\nabla E_\omega = \lambda \nabla W_\omega, W_\omega = 0$. Thus, the only possible solution for $\omega > \omega^*$ in this case is Γ_0 (Lemma 3.2.1).

□

3.3 Invariant sets, global existence, and singularity

3.3.1 Below the ground state energy

For each fixed $\omega > \omega^*(a, b)$, recall that $E^*(\omega) = E(\Gamma_0)$. Define the following sets below this ground state energy.

$$\begin{aligned} \mathcal{W}_+^+(\omega) &= \{\Gamma \in \mathcal{W}(\omega) : K_\omega(\Gamma) \geq 0, W_\omega(\Gamma) > 0\} \\ \mathcal{W}_+^-(\omega) &= \{\Gamma \in \mathcal{W}(\omega) : K_\omega(\Gamma) < 0, W_\omega(\Gamma) > 0\} \\ \mathcal{W}_-^+(\omega) &= \{\Gamma \in W(\omega) : K_\omega(\Gamma) \geq 0, W_\omega(\Gamma) \leq 0\} \\ \mathcal{W}_-^-(\omega) &= \{\Gamma \in W(\omega) : K_\omega(\Gamma) < 0, W_\omega(\Gamma) \leq 0\}. \end{aligned} \tag{3.3.1}$$

Here, the invariant set

$$\mathcal{W}(\omega) := \{\Gamma = (x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 : E_\omega(\Gamma) < E^*(\omega)\}, \tag{3.3.2}$$

and

$$\begin{aligned} \mathcal{W}_+(\omega) &:= \mathcal{W}_+^+(\omega) \bigcup \mathcal{W}_+^-(\omega) \\ \mathcal{W}_-(\omega) &:= \mathcal{W}_-^+(\omega) \bigcup \mathcal{W}_-^-(\omega). \end{aligned} \tag{3.3.3}$$

Proposition 3.3.1. *Let $b > a$. For each $\omega > \omega^*(a,b)$, both $\mathcal{W}_+(\omega)$ and $\mathcal{W}_-(\omega)$ are invariant.*

Proof. We prove $\mathcal{W}_+(\omega)$ is invariant. Let $\phi(t)$ be a solution of quasi-homogeneous Hill's lunar problem. So, the energy $E_\phi < E^*(\omega)$. Notice $\mathcal{W}_+(\omega) = \{\Gamma \in \mathcal{W}(\omega) : W_\omega(\Gamma) > 0\}$. Suppose that there exists t_1 such that $W_\omega(\phi(t_1)) = 0$. Then, the energy $E_\phi \geq \inf\{E_\omega : W_\omega = 0\} = E^*(\omega)$. Contradiction. □

Corollary 3.3.1. *For $b > a$ and $\omega > \omega^*(a,b)$ any solution in $\mathcal{W}_+(\omega)$ exists globally.*

Proof. From $W_\omega(\Gamma) > 0$ we have:

$$\frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b} < \left((2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right) x^2 < (2+b)\omega^2 r^2. \tag{3.3.4}$$

This implies $\frac{C_\epsilon b}{r^b} < (2+b)\omega^2 r^2$ and thus

$$r > \left(\frac{C_\epsilon b}{(2+b)\omega^2}\right)^{\frac{1}{2+b}}.$$

□

Proposition 3.3.2. *Let $b > 2 > a$. For sufficiently large ω, $\mathcal{W}_-^+(\omega) = \emptyset$.*

Proof. Apply Lemma 3.2.8. □

Lemma 3.3.1. *Let $b > 2 > a$ and ω be sufficiently large. Given $\delta > 0$, there exists $k(\delta) > 0$ such that for every $\Gamma \in \mathbb{R}^4$*

$$(E_\omega(\Gamma) \leq E^*(\omega) - \delta,\, W_\omega(\Gamma) \leq 0) \implies \left|K_\omega(\Gamma)\right| \geq k(\delta).$$

Proof. Suppose there exists δ_0 and $(\Gamma_n) = (x_n, y_n, \dot{x}_n, \dot{y}_n) \subseteq \mathbb{R}^4$ such that $E_\omega(\Gamma_n) \leq E^*(\omega) - \delta_0 < 0$, $W_\omega(\Gamma_n) \leq 0$ but $\left|K_\omega(\Gamma_n)\right| \leq \frac{1}{n}$, for ω sufficiently large. Since $W_\omega(\Gamma_n) \leq 0$, we have

$$\left((2+b)\omega^2 - 2C\omega^{\frac{2(a+2)}{b+2}}\right) x_n^2 \leq \frac{a(a+2)}{r_n^a} + \frac{C_\epsilon b}{r_n^b},$$

and therefore

$$
\begin{aligned}
E_\omega(\Gamma_n) &= \frac{\dot{x}_n^2+\dot{y}_n^2}{2} - \frac{a+2}{r_n^a} - \frac{C_\epsilon}{r_n^b} - \left[(1+\frac{b}{2})\omega^2 - C\omega^{\frac{2(a+2)}{b+2}}\right] x_n^2 \\
&\geq \frac{\dot{x}_n^2+\dot{y}_n^2}{2} - \frac{(\frac{a}{2}+1)(a+2)}{r_n^a} - \frac{(\frac{b}{2}+1)C_\epsilon}{r_n^b}.
\end{aligned}
$$

If r_n is unbounded, then we see

$$
\liminf_{n\to\infty} E_\omega(\Gamma_n) \geq 0.
$$

Therefore, r_n is bounded above for every $n \geq 1$. Using Lemma 3.2.5, there exists $M > 0$ such that when $n \geq M$, $r(\Gamma_n) = r_n \geq \frac{1}{M}$. We conclude that there exists $C'' > C' > 0$ such that

$$
C'' < r_n < C' \quad \forall n \geq M.
$$

Since $E_\omega(\Gamma_n) \leq E^*(\omega) - \delta_0$,

$$
\begin{aligned}
\frac{\dot{x}_n^2+\dot{y}_n^2}{2} &\leq E^*(\omega) - \delta_0 + \frac{a+2}{r_n^a} + \frac{C_\epsilon}{r_n^b} + \left[(1+\frac{b}{2})\omega^2 - C\omega^{\frac{2(a+2)}{b+2}}\right] x_n^2 \\
&\leq E^*(\omega) - \delta_0 + \frac{a+2}{r_n^a} + \frac{C_\epsilon}{r_n^b} + (1+\frac{b}{2})\omega^2 r_n^2 := h(r_n).
\end{aligned}
$$

Since h is continuous on compact set $[C'', C']$, thus $\dot{x}_n, \dot{y}_n$ are both bounded for $n \geq M$. Let $\gamma_n = \Gamma_{M+n-1}$, then $(\gamma_n) \subseteq (\Gamma_n)$ is uniformly bounded for $n \geq 1$. Thus there exits a subsequence (γ_{n_k}) and $\gamma_\infty \in \mathbb{R}^4$ such that

$$
\lim_{k\to\infty} \gamma_{n_k} = \gamma_\infty.
$$

Now we have $K_\omega(\gamma_\infty) = 0$, $W_\omega(\gamma_\infty) \leq 0$ and $E_\omega(\gamma_\infty) < E^*(\omega)$. But we already know for sufficiently large ω, the minimum of E_ω under $K_\omega \geq 0$, $W_\omega \leq 0$ is exactly $E^*(\omega)$ (Lemma 3.2.8), which leads to a contradiction. □

Proposition 3.3.3. *Let $b > 2 > a$. For sufficiently large ω, initial conditions in $\mathcal{W}_-^-$ lead to a finite-time collision.*

Proof. Let $\Gamma(t)$ be a solution in $\mathcal{W}_-^-$ and $\delta = E^*(\omega) - E_\Gamma > 0$. Apply the previous lemma, we see

$$
K(\Gamma(t)) = \ddot{I}(\Gamma(t)) \leq -k(\delta) < 0.
$$

Thus,

$$I(\Gamma(t)) \leq -\frac{k(\delta)}{2}t^2 + \dot{I}(\Gamma(0)) + I(\Gamma(0)),$$

which implies $I(\Gamma(t))$ is bounded above by a concave down parabola that becomes negative for some finite time. On the other hand $I = \frac{x^2+y^2}{2}$ is always nonnegative, this implies $\Gamma(t)$ only exists for some finite time. □

3.3.2 At the ground state energy

For each fixed $\omega > \omega^*(a,b)$, again $E^*(\omega) = E(\Gamma_0)$. Define the following sets at this ground state energy.

$$\begin{aligned} \widetilde{\mathcal{W}}_+^+(\omega) &= \{\Gamma \in \widetilde{\mathcal{W}}(\omega) : K_\omega(\Gamma) \geq 0, W_\omega(\Gamma) > 0\} \\ \widetilde{\mathcal{W}}_+^-(\omega) &= \{\Gamma \in \widetilde{\mathcal{W}}(\omega) : K_\omega(\Gamma) < 0, W_\omega(\Gamma) > 0\} \\ \widetilde{\mathcal{W}}_-^+(\omega) &= \{\Gamma \in \widetilde{\mathcal{W}}(\omega) : K_\omega(\Gamma) \geq 0, W_\omega(\Gamma) \leq 0\} \\ \widetilde{\mathcal{W}}_-^-(\omega) &= \{\Gamma \in \widetilde{\mathcal{W}}(\omega) : K_\omega(\Gamma) < 0, W_\omega(\Gamma) \leq 0\} \end{aligned} \tag{3.3.5}$$

Here, the invariant set

$$\widetilde{\mathcal{W}}(\omega) := \{\Gamma = (x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 : E_\omega(\Gamma) = E^*(\omega)\}, \tag{3.3.6}$$

and let

$$\begin{aligned} \widetilde{\mathcal{W}}_+(\omega) &= \widetilde{\mathcal{W}}_+^+(\omega) \bigcup \widetilde{\mathcal{W}}_+^-(\omega) \\ \widetilde{\mathcal{W}}_-(\omega) &= \widetilde{\mathcal{W}}_-^+(\omega) \bigcup \widetilde{\mathcal{W}}_-^-(\omega). \end{aligned} \tag{3.3.7}$$

Using Lemma 3.2.8, we see that for sufficiently large ω and $b > 2 > a$, $\widetilde{\mathcal{W}}_-^+(\omega) = \{\Gamma_0^\pm\}$.

Lemma 3.3.2. *Let $b > a$ and $\omega > \omega^*(a,b)$, both $\widetilde{\mathcal{W}}_\pm$ are invariant.*

Proof. We prove $\widetilde{\mathcal{W}}_+$ is invariant. Let $\Gamma(t)$ be a solution that starts in $\widetilde{\mathcal{W}}_+$. Thus, $E_\omega(\Gamma(t)) \equiv E^*(\omega)$. Suppose there exists t_1 such that $W_\omega(\Gamma(t_1)) = 0$, then by Lemma 3.2.1, $E_\omega(\Gamma(t_1)) > E^*(\omega)$. Contradiction. □

Remark 3.3.1. *This lemma also implies that for sufficiently large ω and $b > 2 > a$, $\widetilde{\mathcal{W}}_-^-$ is invariant.*

Corollary 3.3.2. *Let $b > a$ and $\omega > \omega^*(a,b)$ be fixed, then $\widetilde{\mathcal{W}}_+$ contains global solutions.*

Proof. Notice that $W_\omega > 0$ implies

$$(2+b)\omega^2 r^2 > \frac{a(a+2)}{r^a} + \frac{C_\epsilon b}{r^b},$$

from which we conclude that r must be uniformly bounded below. □

Proposition 3.3.4. *For $b > 2 > a$ and sufficiently large ω, solutions in the invariant set $\widetilde{\mathcal{W}}_-^-$ either have a finite-time collision or approach $\{\Gamma_0^\pm\}$ as $t \to \infty$.*

Proof. Let $\Gamma(t)$ be a solution in $\widetilde{\mathcal{W}}_-^-$. If $K_\omega(\Gamma(t))$ is uniformly bounded above below zero, then from the proof of Proposition 3.3.3, we see $\Gamma(t)$ must collide in finite time. Otherwise, there exists a sequence (t_n) such that $K_\omega(\Gamma(t_n)) \to 0^-$ as $t_n \to \sigma$, where σ is the maximal time of existence for Γ. Recall that $K_\omega(\Gamma(t)) = \ddot{I}(\Gamma(t))$ and thus $\dot{I}(\Gamma(t))$ is decreasing. Let $\lim_{t\to\sigma^-} \dot{I}(\Gamma(t)) = a$. If $-\infty \le a < 0$, then we must have $\sigma < \infty$, since if $\sigma = \infty$, that would imply $I(\Gamma(t))$ is bounded above by a linear function with negative slope, which eventually becomes negative in finite time. If $a \ge 0$, then $\dot{I}(\Gamma(t)) > 0$ for all time and thus $\sigma = \infty$. Since $r(\Gamma)(t)$ is always bounded in $\widetilde{\mathcal{W}}_-^-$, $a = 0$ and hence $K_\omega(\Gamma(t)) \to 0$ as $t \to \infty$. □

3.4 Near-collision dynamics on strong homogeneous potentials

In this section, we apply McGehee-type [14] transform to study the near-collision dynamics of the homogeneous Hill's lunar problem.

Let $\mathbf{x} = (x, y)$ and $\mathbf{u} = (\dot{x}, \dot{y})$, then according to [4], the equation of motion of the homogeneous Hill's lunar problem can be written as

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{u} \\ \dot{\mathbf{u}} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{u} - \begin{bmatrix} V_x \\ V_y \end{bmatrix}, \end{cases} \tag{3.4.1}$$

where the effective potential has the form of

$$V(x,y) = -\frac{\alpha+2}{2}x^2 - \frac{\alpha+2}{(x^2+y^2)^{\frac{\alpha}{2}}},\ \alpha > 0. \tag{3.4.2}$$

The conservation of energy holds for the homogeneous Hill's lunar problem. That is,

$$\frac{\dot{x}^2+\dot{y}^2}{2}+V(x,y)\equiv h. \tag{3.4.3}$$

We introduce McGehee-type change of variable [14]

$$\begin{cases} \mathbf{x}=r^{\gamma}e^{i\theta} \\ \mathbf{u}=r^{-\beta\gamma}(v+iw)e^{i\theta}, \end{cases} \tag{3.4.4}$$

where

$$\gamma=\frac{1}{\beta+1},\ \beta=\frac{\alpha}{2}, \tag{3.4.5}$$

and the equation of motion reads

$$\begin{cases} \dot{r}=(\beta+1)v \\ \dot{\theta}=\frac{1}{r}w \\ \dot{w}=-2v-(\beta+1)r\sin(2\theta)+wv(\beta-1)\frac{1}{r} \\ \dot{v}=2w+\frac{1}{r}(\beta v^2+w^2-4\beta(\beta+1))+2(\beta+1)r\frac{1}{1+\tan^2(\theta)}. \end{cases} \tag{3.4.6}$$

To regularize the singularity at the origin, we further introduce a new time τ such that

$$dt=rd\tau. \tag{3.4.7}$$

Thus, the previous equation of motion becomes

$$\begin{cases} r'=(\beta+1)vr \\ \theta'=w \\ w'=-2vr-(\beta+1)r^2\sin(2\theta)+wv(\beta-1) \\ v'=2wr+(\beta v^2+w^2-4\beta(\beta+1))+2(\beta+1)r^2\cos^2(\theta), \end{cases} \tag{3.4.8}$$

where differentiation is with respect to the new time τ. The energy relation (3.4.3) becomes

$$\frac{v^2+w^2}{2}-(\beta+1)r^2\cos^2(\theta)-2(\beta+1)=hr^{\frac{2\beta}{\beta+1}}. \tag{3.4.9}$$

We define the collision manifold N as

$$N = \left\{(r, \theta, w, v) : r = 0, v^2 + w^2 = 4(\beta + 1)\right\}, \tag{3.4.10}$$

and thus the flow on collision manifold N is

$$\begin{cases} \theta' = w \\ w' = wv(\beta - 1) \\ v' = \beta v^2 + w^2 - 4\beta(\beta + 1)). \end{cases} \tag{3.4.11}$$

We see that $\nabla V(x, y)$ in the new coordinate system becomes

$$4\beta(\beta + 1)r^{\frac{2\beta+1}{\beta+1}} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} - (2\beta + 2)r^{\frac{1}{\beta+1}} \begin{bmatrix} \cos(\theta) \\ 0 \end{bmatrix} \tag{3.4.12}$$

and the configuration $\mathbf{x}$ becomes

$$r^{\frac{1}{\beta+1}} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$

Thus, we immediately observe that

Corollary 3.4.1. *For $\beta > 0$, as $r \to 0^+$, $\nabla V(\mathbf{x}) \parallel \mathbf{x}$.*

It is not difficult to see that one can obtain a Hill's lunar version of Lemma 4.1 and 4.2 in [14]. In particular, with very little modification, the proofs are still valid for the case of homogeneous Hill's lunar problem. Here, we restate one of them for readers' reference, and the other one is slightly and accordingly modified for our case.

Lemma 3.4.1 ([14])**.** *Suppose that $(r, \theta, w, v)(t)$ satisfies* (3.4.6) *and that $r \to 0$ as $t \to t^*\pm$, where t^* is the collision time. Then $\tau(t) \to \mp\infty$.*

Define a subset of N

$$S^{\pm} = \left\{(r, \theta, w, v) : r = 0, w = 0, v = \pm 2\sqrt{\beta + 1}\right\}. \tag{3.4.13}$$

Lemma 3.4.2 ([14])**.** *Assume $\beta \neq 1$. Let $\mathbf{p}(\tau) = (r, \theta, w, v)(\tau)$ be a solution of* (3.4.8)*, and write $\mathbf{p}_0 = \mathbf{p}(0)$. Then*

1. *$r \to 0$ as $\tau \to \infty \iff \omega(\mathbf{p}_0) \subseteq S^-$, and*

2. $r \to 0$ *as* $\tau \to -\infty \iff \alpha(\mathbf{p}_0) \subseteq S^+$.

Combining both lemmas, we see that for any collision-bound solution $(r, \theta, w, v)(t)$, $v(t) \to -2\sqrt{\beta+1}$, $w(t) \to 0$ as $t \to t^{*^-}$, as long as $\beta \neq 1$.

Corollary 3.4.2. *For* $\beta \neq 1$*, let* $(r, \theta, w, v)(t)$ *be a solution that achieves a collision at* $t = t^*$*, then its moment of inertia*

$$I(t) \simeq 2^{\frac{1-\beta}{\beta+1}} (\beta+1)^{\frac{3}{\beta+1}} (t^* - t)^{\frac{2}{\beta+1}} \text{ as } t \to t^{*^-}. \tag{3.4.14}$$

Proof. The moment of inertia I in the (r, θ, w, v) coordinate system is $I = \frac{1}{2} r^{\frac{2}{\beta+1}}$. Using the equation of motion (3.4.6), we see

$$I_t^2 = r^{\frac{2-2\beta}{\beta+1}} v^2.$$

Thus, the following identity holds

$$\frac{I_t^2}{v^2} = 2^{1-\beta} I^{1-\beta}. \tag{3.4.15}$$

Since $v \to -2\sqrt{\beta+1}$ and $sign(I_t) = sign(\dot{r}) = sign(v) = -1$ as $t \to t^{*^-}$, we square root both sides of (3.4.15) to see that

$$I^{\frac{\beta-1}{2}} dI = 2^{\frac{1-\beta}{2}} v(t) dt. \tag{3.4.16}$$

Integrating from t to t^* we see

$$\int_{I(t)}^{0} J^{\frac{\beta-1}{2}} dJ = 2^{\frac{1-\beta}{2}} \int_t^{t^*} v(s) ds. \tag{3.4.17}$$

Using mean value theorem for integrals, there exists $t_0 \in (t, t^*)$ such that

$$-\frac{2}{\beta+1} I(t)^{\frac{\beta+1}{2}} = 2^{\frac{1-\beta}{2}} \int_t^{t^*} v(s) ds = 2^{\frac{1-\beta}{2}} v(t_0)(t^* - t).$$

Or equivalently,

$$\frac{I(t)}{(t^* - t)^{\frac{2}{\beta+1}}} = \left(-v(t_0) 2^{\frac{1-\beta}{2}} \frac{\beta+1}{2} \right)^{\frac{2}{\beta+1}}$$

Taking $t \to {t^*}^-$, we get

$$I(t) \simeq 2^{\frac{1-\beta}{\beta+1}} (\beta+1)^{\frac{3}{\beta+1}} (t^* - t)^{\frac{2}{\beta+1}} \tag{3.4.18}$$

□

Remark 3.4.1. *Effectively, for $\beta \neq 1$, any collision-bound solution $(r, \theta, w, v)(t)$ satisfies*

$$r(t) = (2I)^{\frac{\beta+1}{2}} \simeq 2^{\frac{\beta+1}{2}} (\beta+1)^{\frac{3}{2}} (t^* - t) \tag{3.4.19}$$

as $t \to {t^*}^-$.

Corollary 3.4.3. *For $\beta > 1$, let $(r, \theta, w, v)(t)$ be a solution that achieves a collision at $t = t^*$, if $\theta \to \theta^*$ for some θ^*, then the angular momentum*

$$C(t) := w(t) r(t)^{-\frac{\beta-1}{\beta+1}} \to C^*, \tag{3.4.20}$$

for some C^*. Furthermore, as $t \to {t^*}^-$,

$$\theta^* - \theta(t) \simeq \begin{cases} -(t^* - t) & C^* = 0 \\ \frac{C^*}{2(\beta-1)} (\beta+1)^{\frac{\beta-2}{\beta+1}} (t^* - t)^{\frac{\beta-1}{\beta+1}} & C^* \neq 0. \end{cases} \tag{3.4.21}$$

Proof. Divide the third by the first equation of (3.4.6), we see

$$\frac{dw}{dr} = -\frac{2}{\beta+1} - \underbrace{\frac{r}{v}\sin(2\theta)}_{f(r)} + \frac{\beta-1}{\beta+1}\frac{w}{r}. \tag{3.4.22}$$

Since $\dot{r} = (\beta+1)v$ and $v \to -2\sqrt{\beta+1}$ as $t \to {t^*}^-$, there exists $t_1 < t^*$ and $r_1 = r(t_1) > 0$ such that $r(t)$ is strictly decreasing from r_1 to 0. Therefore, the inverse function $t(r)$ exists and it is continuous on $[0, r_1]$. We then regard

$$\frac{r}{v(t(r))} \sin(2\theta(t(r))) =: f(r),$$

on $[0, r_1]$. We also know that $f(r) \to 0$ as $r \to 0^+$. Moreover, since $\frac{f(r)}{r} \to \frac{\sin(2\theta^*)}{-2\sqrt{\beta+1}}$,

$$f(r) = O(r) \qquad \text{as } r \to 0^+.$$

Hence, there exists $K > 0$ and $r_2 > 0$ such that $|f(r)| < Kr$, whenever $r < r_2$.

Moreover, (3.4.22) is equivalent to

$$\frac{d}{dr}(w(r)r^{-\frac{\beta-1}{\beta+1}}) = r^{-\frac{\beta-1}{\beta+1}}\left(-\frac{2}{\beta+1} - f(r)\right) \tag{3.4.23}$$

Fix $r_3 = \min\{r_1, r_2\}$ and let $R < r_3$ we integrate (3.4.23) from r_3 to R,

$$\begin{aligned} w(r_3)r_3^{-\frac{\beta-1}{\beta+1}} - w(R)R^{-\frac{\beta-1}{\beta+1}} &= -\int_R^{r_3} \frac{\frac{2}{\beta+1} + f(\xi)}{\xi^{\frac{\beta-1}{\beta+1}}} d\xi \\ &= -\int_R^{r_3} \frac{\frac{2}{\beta+1}}{\xi^{\frac{\beta-1}{\beta+1}}} d\xi - \int_R^{r_3} \frac{f(\xi)}{\xi^{\frac{\beta-1}{\beta+1}}} d\xi \end{aligned} \tag{3.4.24}$$

The first integral exists as $R \to 0^+$, since $0 < \frac{\beta-1}{\beta+1} < 1$. To see the existence of the second one, we observe

$$\int_0^{r_3} \left|\frac{f(\xi)}{\xi^{\frac{\beta-1}{\beta+1}}}\right| d\xi \le \int_0^{r_3} K\xi^{\frac{2}{\beta+1}} d\xi.$$

Define

$$h(R) := w(R)R^{-\frac{\beta-1}{\beta+1}}$$

and let $R \to 0^+$ on (3.4.24), we see $h(0) = \lim_{R\to 0^+} h(R)$ exists. Let $r < r_3$ and we integrate (3.4.23) from 0 to r,

$$\begin{aligned} w(r) &= h(0)r^{\frac{\beta-1}{\beta+1}} - r^{\frac{\beta-1}{\beta+1}} \int_0^r \frac{\frac{2}{\beta+1} + f(\xi)}{\xi^{\frac{\beta-1}{\beta+1}}} d\xi \\ &= h(0)r^{\frac{\beta-1}{\beta+1}} - r^{\frac{\beta-1}{\beta+1}} \left(\int_0^r \frac{\frac{2}{\beta+1}}{\xi^{\frac{\beta-1}{\beta+1}}} d\xi + \int_0^r \frac{f(\xi)}{\xi^{\frac{\beta-1}{\beta+1}}} d\xi\right) \\ &= h(0)r^{\frac{\beta-1}{\beta+1}} - r - r^{\frac{\beta-1}{\beta+1}} \int_0^r \frac{f(\xi)}{\xi^{\frac{\beta-1}{\beta+1}}} d\xi, \end{aligned} \tag{3.4.25}$$

where we have used the function $\frac{\frac{2}{\beta+1}}{\xi^{\frac{\beta-1}{\beta+1}}}$ is integrable on $[0, r]$ (Since $0 < \frac{\beta-1}{\beta+1} < 1$.)

Rearranging the previous equation, we see

$$\begin{aligned}\left|w(r)+r-h(0)r^{\frac{\beta-1}{\beta+1}}\right| &= \left|r^{\frac{\beta-1}{\beta+1}}\int_0^r \frac{f(\xi)}{\xi^{\frac{\beta-1}{\beta+1}}}d\xi\right| \\ &\leq r^{\frac{\beta-1}{\beta+1}}\int_0^r \left|\frac{f(\xi)}{\xi^{\frac{\beta-1}{\beta+1}}}\right| d\xi \\ &\leq r^{\frac{\beta-1}{\beta+1}}\int_0^r K\xi^{\frac{2}{\beta+1}}d\xi \\ &= K\frac{\beta+1}{3+\beta}r^2.\end{aligned}$$

If $h(0)=0$ then

$$w(r)\simeq -r \qquad \text{as } r\to 0^+,$$

and thus

$$\dot{\theta}=\frac{w}{r}\to -1 \qquad \text{as } r\to 0^+(\text{or } t\to {t^*}^-).$$

we use L'hopital's rule to see

$$\lim_{t\to {t^*}^-}\frac{\theta^*-\theta(t)}{t^*-t}=\lim_{t\to {t^*}^-}\frac{-\dot{\theta}(t)}{-1}=-1, \tag{3.4.26}$$

which implies

$$\theta^*-\theta(t)\simeq -(t^*-t). \tag{3.4.27}$$

If $h(0)\neq 0$, then

$$w(r)\simeq h(0)r^{\frac{\beta-1}{\beta+1}} \qquad \text{as } r\to 0^+,$$

and thus

$$\dot{\theta}=\frac{w}{r}\simeq h(0)r^{\frac{-2}{\beta+1}}\simeq h(0)2^{-1}(\beta+1)^{\frac{-3}{\beta+1}}(t^*-t)^{\frac{-2}{\beta+1}} \qquad \text{as } r\to 0^+(\text{or } t\to {t^*}^-).$$

we use L'hopital's rule to see

$$\lim_{t\to {t^*}^-}\frac{\theta^*-\theta(t)}{(t^*-t)^{\frac{\beta-1}{\beta+1}}}=\lim_{t\to {t^*}^-}\frac{-\dot{\theta}(t)}{-\frac{\beta-1}{\beta+1}(t^*-t)^{\frac{-2}{\beta+1}}}=\frac{\beta+1}{\beta-1}h(0)2^{-1}(\beta+1)^{\frac{-3}{\beta+1}}, \tag{3.4.28}$$

which implies

$$\theta^*-\theta(t)\simeq\frac{h(0)}{2(\beta-1)}(\beta+1)^{\frac{\beta-2}{\beta+1}}(t^*-t)^{\frac{\beta-1}{\beta+1}}. \tag{3.4.29}$$

□

Lemma 3.4.3. *Let $\beta > 1$ and $(r, \theta, w, v)(t)$ be a solution such that $r \to 0$ as $t \to t^{*^-}$. Then as $t \to t^{*^-}$,*

$$w(t) \to 0, \qquad v(t) \to -2\sqrt{\beta+1}, \qquad \textit{and} \quad \theta(t) \to \theta^*, \tag{3.4.30}$$

for some constant θ^*.

Proof. By Lemma 3.4.1 and Lemma 3.4.2, it suffices to show these limits hold for all the points in the fixed energy set

$$M(h) = \left\{(r, \theta, w, v) : F(r, \theta, w, v) := \frac{v^2 + w^2}{2} - (\beta+1) r^2 \cos^2(\theta) - 2(\beta+1) - h r^{\frac{2\beta}{\beta+1}} = 0\right\} \tag{3.4.31}$$

on the stable manifold of S^-, as $\tau \to \infty$. Let $\mathbf{s} \in S^-$, then the tangent space of the manifold of $M(h)$ at $\mathbf{s}$ is

$$T_{\mathbf{s}} F = \left\{(r, \theta, w, v) : \nabla F(\mathbf{s}) \cdot \mathbf{s} = 0\right\}. \tag{3.4.32}$$

While for $\beta > 1 \iff \frac{2\beta}{\beta+1} > 1$,

$$\begin{aligned} \nabla F|_{\mathbf{s}} &= (-(\beta+1) 2r \cos^2(\theta) - h \frac{2\beta}{\beta+1} r^{\frac{2\beta}{\beta+1}-1}, 2\sin(\theta)\cos(\theta)(\beta+1) r^2, w, v)|_{\mathbf{s}} \\ &= (0, 0, 0, -2\sqrt{\beta+1}). \end{aligned} \tag{3.4.33}$$

Thus, we get

$$T_{\mathbf{s}} F = \left\{(r, \theta, w, v) : v = 0\right\}. \tag{3.4.34}$$

The Jacobian matrix associated to the global flow (3.4.8) evaluated at $\mathbf{s}$ has the form of

$$J = \begin{bmatrix} -2(\beta+1)^{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4\sqrt{\beta+1} & 0 & -2\sqrt{\beta+1}(\beta-1) & 0 \\ 0 & 0 & 0 & -4\beta\sqrt{\beta+1} \end{bmatrix}, \tag{3.4.35}$$

and its restriction on $T_{\mathbf{s}}F$ is

$$J|_{T_{\mathbf{s}}F} = \begin{bmatrix} -2(\beta+1)^{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4\sqrt{\beta+1} & 0 & -2\sqrt{\beta+1}(\beta-1) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.4.36}$$

Since $T_{\mathbf{s}}F$ is spanned by the basis

$$\left\{\xi_1 = (1,0,0,0), \xi_2 = (0,1,0,0), \xi_3 = (0,0,1,0)\right\},$$

A representative of $J|_{T_{\mathbf{s}}F}$ in this basis is

$$\begin{bmatrix} -2(\beta+1)^{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 1 \\ 4\sqrt{\beta+1} & 0 & -2\sqrt{\beta+1}(\beta-1) \end{bmatrix} \tag{3.4.37}$$

and it has eigenvalues $-2(\beta+1)^{\frac{3}{2}}, 0, -2\sqrt{\beta+1}(\beta-1)$. For $\beta \neq 1$, S^- has hyperbolic normal structure and hence the stable manifold of S^- is the union of the stable manifolds of each points on S^- ([14],[8]). □

Finally, we conclude this section by stating everything in the original $(x,y)-$coordinate system and parameters.

Proposition 3.4.1. *Fix $\alpha > 2$, and let $\mathbf{x}(t) = (x,y)(t)$ be a solution of the homogeneous Hill's lunar problem that has a collision-singularity at $t^* < \infty$. Then, as $t \to {t^*}^-$,*

1. *The angular momentum $C(t) := x\dot{y} - y\dot{x} \to C^*$, for some C^*.*
2. *The following asymptotic relation holds.*

$$\mathbf{x}(t) \simeq \begin{cases} 2^{\frac{2}{\alpha+2}}(\frac{\alpha}{2}+1)^{\frac{3}{2\alpha+2}}(t^*-t)^{\frac{2}{\alpha+2}} \exp\left[i\left(\theta^* - \frac{C^*}{\alpha-2}(\frac{\alpha}{2}+1)^{\frac{\alpha-4}{\alpha+2}}(t^*-t)^{\frac{\alpha-2}{\alpha+2}}\right)\right] & C^* \neq 0 \\ 2^{\frac{2}{\alpha+2}}(\frac{\alpha}{2}+1)^{\frac{3}{2\alpha+2}}(t^*-t)^{\frac{2}{\alpha+2}} \exp\left[i\left(\theta^* + (t^*-t)\right)\right] & C^* = 0 \end{cases}$$

Chapter 4

Conclusions

The main purpose of this thesis is to extend some of the results that hold for homogeneous potentials to quasi-homogeneous potentials.

Throughout this thesis, we implemented the energy method [3, 4] to construct invariant sets that either contain initial conditions that correspond to global solution or solutions with singularity, whenever the energy method is applicable. However, the energy method is not perfect.

As mentioned in the introduction, the McGehee transform method [13] is a standard and the most popular way to study near-collision dynamics and classification of initial conditions. Indeed, for the strong-force homogeneous Hill's lunar problem, we were able to obtain the asymptotic profile near collision by introducing one type of McGehee coordinate system and a new time.

As an alternative to the McGehee transform, the energy method have both advantages and disadvantages.

The advantage of the energy method [3, 4] is its simplicity and explicitness. We have seen that for both quasi-homogeneous two-body and Hill's lunar problems, the invariant sets constructed based on the indicator function were expressed in the original coordinate system, which is not the case of McGehee transform. Furthermore, the energy method usually does not involve a lot of computations.

The disadvantage of the energy method is its limitation. In the study of quasi-homogeneous two-body problem, it only yielded partial results or did not apply when we have either strong-weak quasi-homogeneous potentials or Manev-type potentials. Luckily, based on the integrability of the two-body system, we were still able to complete all the cases even when it was not applicable. It is, however, fully functional when we have strong-strong quasi-homogeneous potentials. In contrast, many sources have indicated McGehee transform is versatile. For example, for Newtonian potential, Xia's [20] construction of non-collision singularity was based on McGehee transform.

Recall that in the study of quasi-homogeneous Hill's lunar problem, we chose to restrict the power combination to $a < 2 < b$, since one of the lemmas fails for the limiting case ($b = 2$). This excludes the case of Manev potential, which itself corresponds to numerous applications in physics. A rational, yet feasible, option would be to implement McGehee transform.

For quasi-homogeneous Hill's lunar problem, we did not apply McGehee transform since we had the impression that it would involve too much computation, given that the equation of motion for quasi-homogeneous potentials is already a little complicated. On the other hand, the equation of motion of the homogeneous Hill's lunar problem is simple and compact and we thus anticipated that it would lead to nice results. It is indeed true for strong-force homogeneous Hill's lunar problem, but we were not able to draw any similar conclusion for all the weaker potentials ($\alpha \leq 2$).

By combining McGehee transform, would it be possible to overcome the shortcomings of the energy method? Our next goal would be to improve the energy method so that it can be, hopefully, applied to finish all the aforementioned incomplete cases.

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