University of Victoria

Testing the Performance of Tensor Network Renormalization Algorithms

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Introduction

Tensor Networks

Tensors can be thought of as matrices with an arbitrary number of dimensions

They can be used to represent information about one part of a system, and joined together to model interactions between connected parts.

A system comprised of many connected tensors is a **tensor network.**

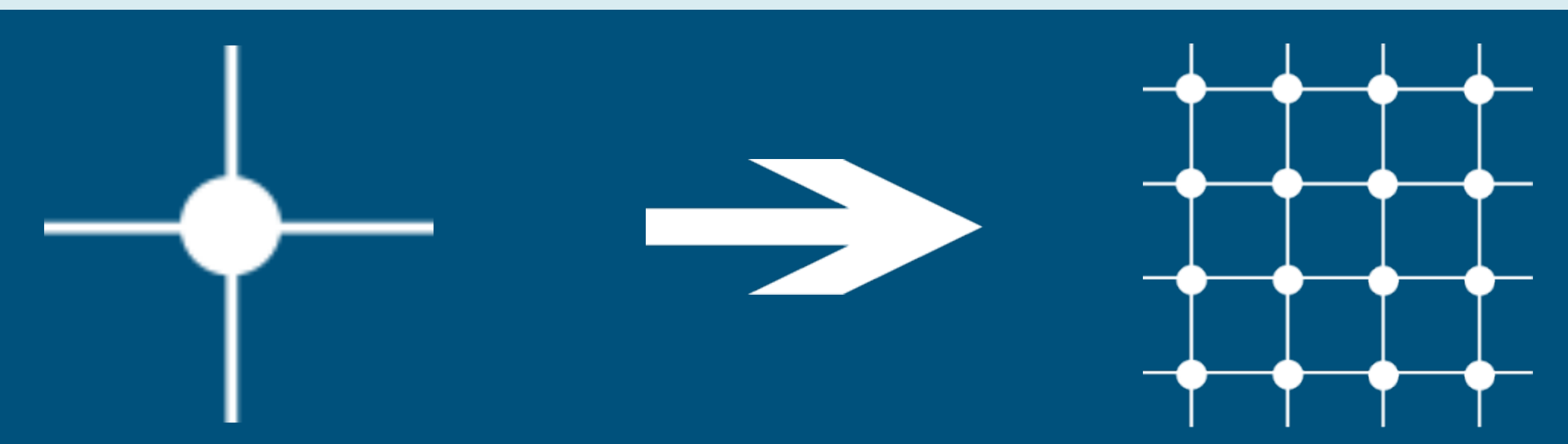

Figure 1. Tensors are represented here as circles with lines indicating the number of indices, or dimensions, of each. Contracting indices together yields a tensor network

Operations called contraction, decomposition, and reshaping allow tensors to be connected together, divided into pieces, or reconfigured while retaining the original system's information

The Partition Function

The properties of a thermodynamic system can be obtained from a quantity called the **partition function**, denoted Z. For instance, it can be used to determine **specific heat capacity** or **free energy density**. Contracting a tensor network onto itself to yield a scalar gives you the value of the partition function for that network. However, systems may be prohibitively large for using this procedure to obtain an exact result.

Renormalization

In tensor networks forming symmetric lattices, a technique called renormalization can be used to obtain a numerical approximation for the partition function. By reconfiguring and contracting groups of sites, the number of sites in the lattice can be reduced while maintaining the lattice structure. Repeating this process increases the amount of information held at each site, until a single tensor can adequately represent the behaviour of the whole network.

Many renormalization algorithms exist, but most have not been compared directly to each other from within the same programming environment.

Testing different renormalization algorithms was the main purpose of this project. Algorithms were from a library written in the julia language by Matt Forbes.

The simplest such algorithm is the Tensor Renormalization Group (**TRG**), whose steps are illustrated below.

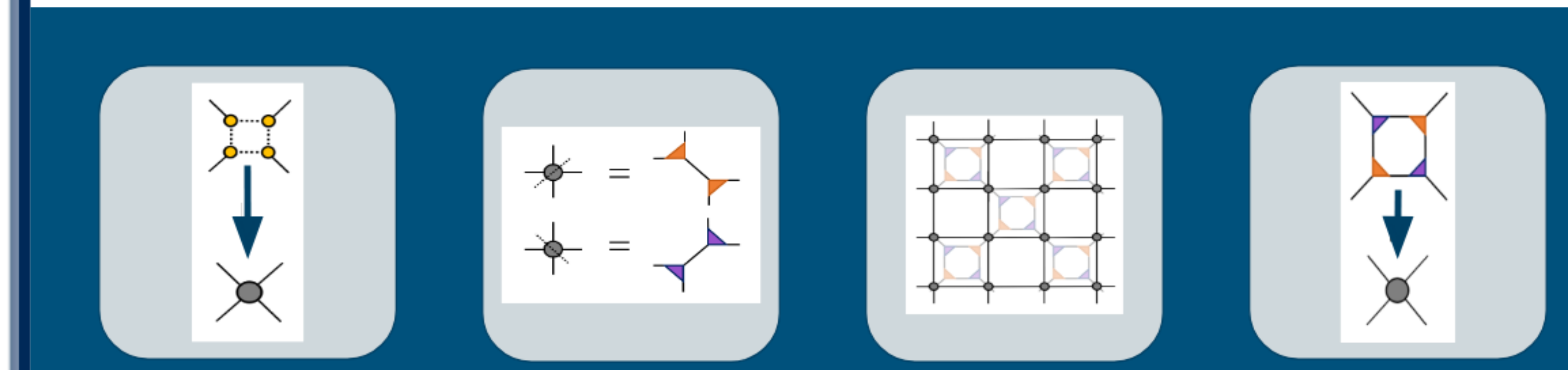

Figure 2. The TRG algorithm on a square lattice. The interaction between 4 adjacent particles is recorded in a rank-4 tensor. A system of these forms a square lattice. Each rank-4 tensor can be split diagonally, and 4 neighbouring half-tensors contracted together to create a new square lattice with the number of sites reduced by a factor of 2.

The **Ising model** was used to compare the performance of different algorithms. This "toy model" has particles with spin up or down interacting only with their nearest neighbours. Square, triangular, and hexagonal 2-dimensional lattices have exact solutions for the Ising model, which provided known benchmarks against which to compare the algorithms.

A Foray into Graph Theory

A **planar graph** is a 2D graph in which no edges overlap. Using tensor operations, any tensor network forming a planar graph can be transformed into any other planar graph. This means that the optimal renormalization algorithm for any particular planar lattice should be the optimal method for *any* 2D graph. But 3D lattices cannot be transformed into planar graphs, so they must be tested separately.

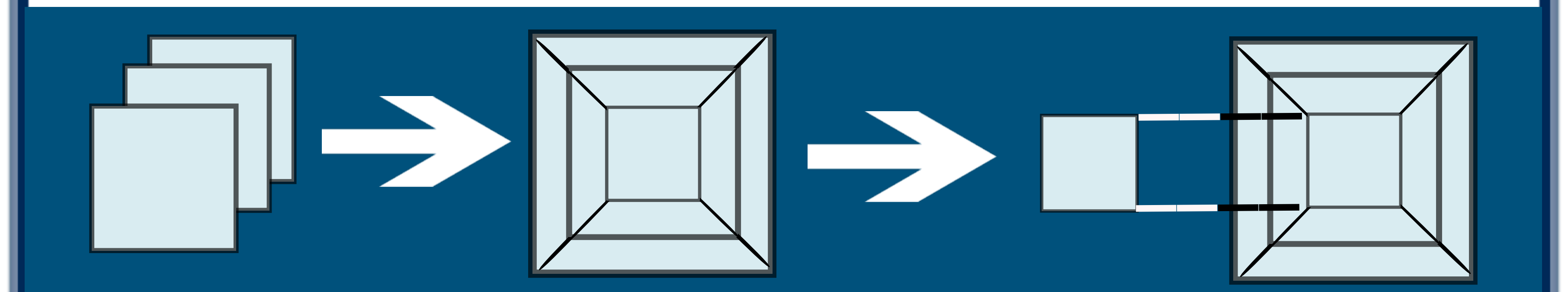

Figure 3. Demonstrating that 3D tensor lattices cannot be reshaped into planar graphs. To make an infinite column of cubes into a planar graph, each square layer would have to be nested inside the next. But adding a cube to the side of the column to form a 3D cubic lattice would require edges crossing into the stacked square.

Results

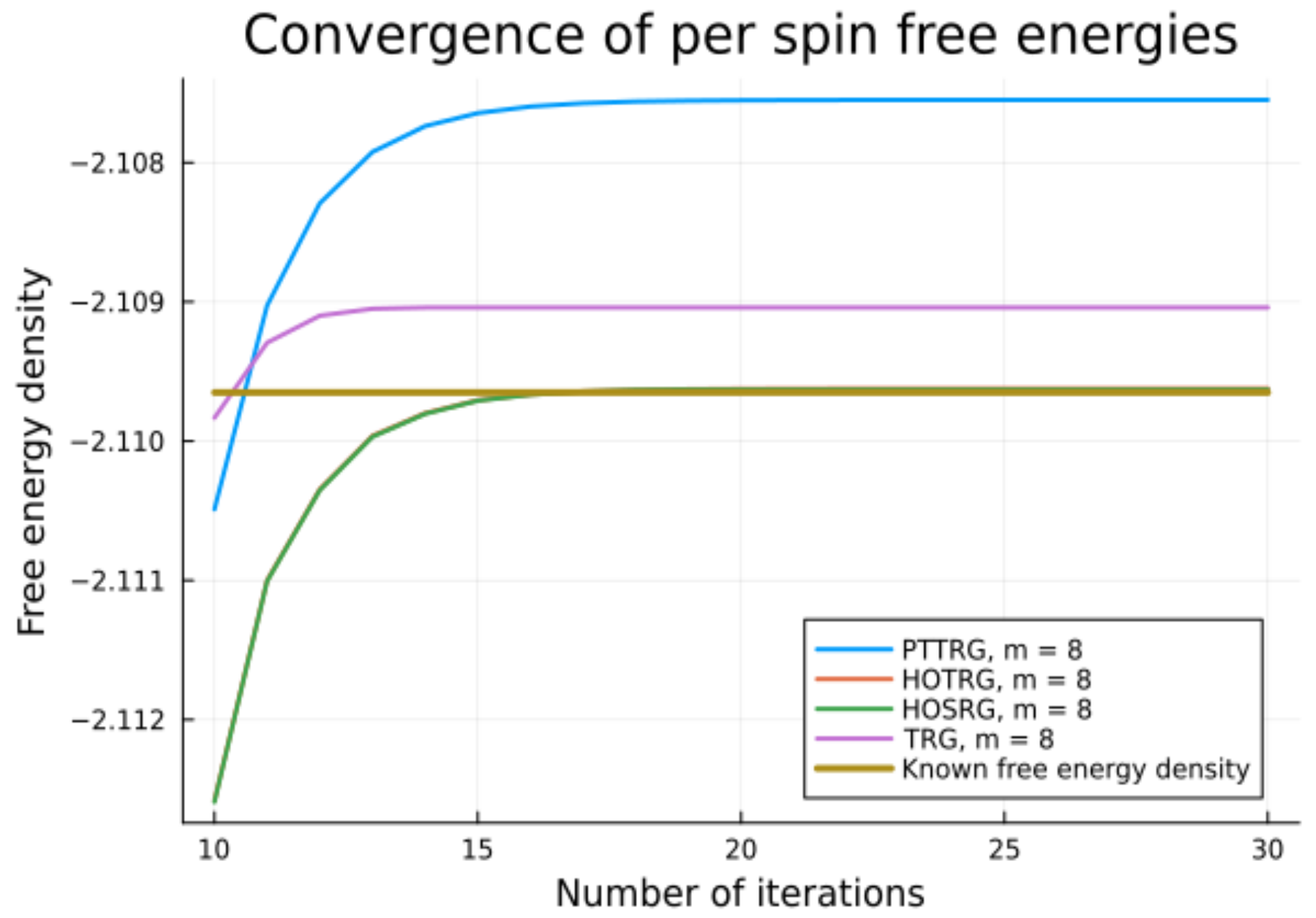

Figure 4. The convergence of free energy densities to the known value at the critical temperature is compared for 4 algorithms. At this bond dimension (8), HOSRG and HOTRG both perform well

At lower bond dimensions, HOTRG and HOSRG performed well.

At high bond dimensions,**TRG** was among the fastest to run and soonest to converge, while some algorithms that fared well at lower bond dimensions became too time-and-memory-intensive.

Future Research Directions

- **Apply the same methods to 3D graphs**
- **Look for further connections to graph theory**
- **Compare convergence times between algorithms**
- **Test remaining algorithms from the library**

References

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