

Introduction to Quadratic forms

Fix a field k in which $2 \neq 0$. A *quadratic form over k* is a homogeneous polynomial of degree 2 in finitely many variables with coefficients in k , e.g., the sum of squares $X_1^2 + \cdots + X_n^2$ or the *hyperbolic form* $X^2 - Y^2$.

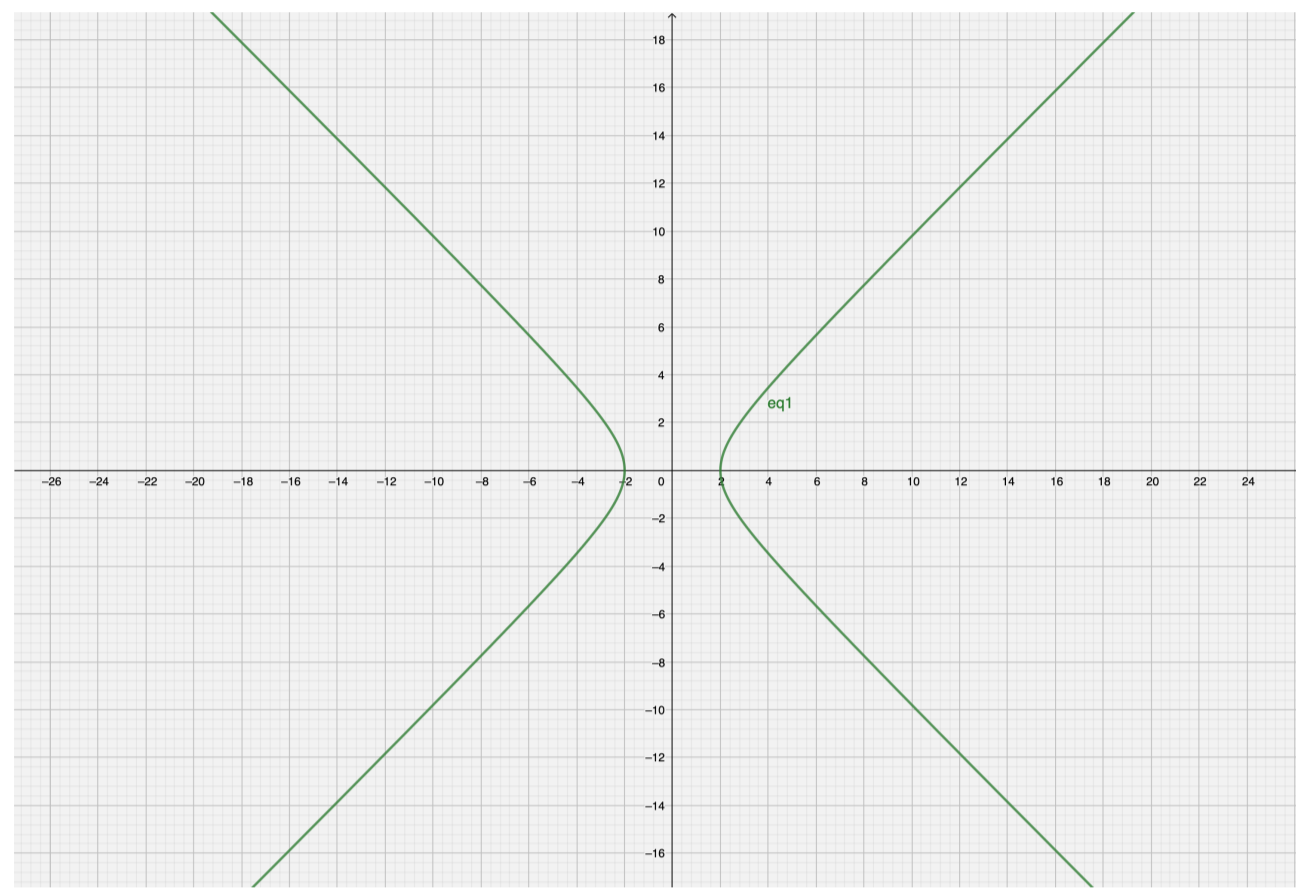


Figure 1: The Graph of $X^2 - Y^2$ over the real plane

The algebraic theory of quadratic forms aims to understand the classification of quadratic forms over k up to isomorphism, i.e., invertible linear change of variables. A more geometric perspective is provided by the following:

Definition 1. A *quadratic space over k* is a pair (V, φ) , where V is a finite-dimensional k -vector space and $\varphi: V \rightarrow k$ is map satisfying the following conditions:

- $\varphi(\lambda v) = \lambda^2 \varphi(v)$ for all $\lambda \in k$ and $v \in V$;
- The pairing $V \times V \rightarrow k$ that sends $(v, w) \in V \times V$ to $\varphi(v+w) - \varphi(v) - \varphi(w)$ is F -bilinear and non-degenerate.

The basis of V can always be chosen to be orthogonal with respect to the pairing in 2., there by yielding a weighted sum of squares $a_1 X_1^2 + \cdots + a_n X_n^2$. We use the notation $\langle a_1, \dots, a_n \rangle$ for the latter, and the hyperbolic form $\langle 1, -1 \rangle$ is denoted h .

The usual direct sum and tensor product operations for k -vector spaces give rise to orthogonal sum and tensor product operations for quadratic forms over k in an obvious way. These operations, which we denote \perp and \otimes , equip the set of isomorphism classes of quadratic forms over k with the structure of a commutative semiring. The Grothendieck completion of this semiring is the *Grothendieck-Witt ring of k* , denoted $GW(k)$.

A quadratic form φ over k is *isotropic* if there exists a non-zero vector v in its underlying space such that $\varphi(v) = 0$, and *anisotropic* otherwise. We have the following key result:

Theorem 1. If φ is an isotropic quadratic form over k , then $\varphi \cong q \perp m \cdot h$ for a unique $m \in \mathbb{N}$ and unique anisotropic form q over k up to isomorphism. We call q the *anisotropic part of φ* . If $q = 0$, then we say that φ is *split*.

This reduces our classification problem to determination of the quotient $W(k) := GW(k)/(h)$ known as the *Witt ring of k* . The construction of $W(k)$ may be generalized in several directions: On the commutative side, one may define the Witt group of an algebraic variety X over k , $W(X)$, by replacing k -vector spaces with algebraic vector bundles over X ; On the non-commutative side, given a k -algebra A equipped with a non-trivial involution σ , one may define a Witt group $W(A, \sigma)$ by considering hermitian spaces over (A, σ) in lieu of quadratic spaces.

Main Problem

A critical problem in the algebraic theory of quadratic forms is to understand the behaviour of anisotropic quadratic forms under scalar extension to larger fields. If L/k is a field extension, then any form over k may be viewed as a form over L , inducing a ring homomorphism $W(k) \rightarrow W(L)$. The kernel of this homomorphism, called the *Witt kernel of L/k* , consists of the classes of all forms over k that are split over L . A major open problem in the subject, on which many others depend, is that of determining the Witt kernel in the case where L is the function field of an anisotropic projective quadric Q over k , i.e., projective algebraic variety defined by the vanishing of an anisotropic quadratic form q over k . The function field of Q , denoted $k(Q)$, is the “universal” extension of k over which q becomes isotropic. The challenge is thus to understand which forms we must split in order to make a given one isotropic. Despite a great deal of effort, rather little is known about the structure of the Witt kernel of $k(Q)/k$ outside of special cases. As we shall now discuss, however, there is hope to determine a fundamental component of it which is closely related to $W(Q)$, the Witt group of quadratic vector bundles on Q .

Previous Research

Any quadratic form q over k gives rise to a semisimple k -algebra $C_0(q)$ known as its Clifford algebra. This algebra is equipped with a canonical involution, allowing us to consider the Witt group $W(C_0(q))$ of hermitian spaces over $C_0(q)$ (the involution is suppressed from the notation). In 2018, Heng Xie [1] established the following key result:

Theorem 2. If Q is the projective quadric defined by q , then there is a short exact sequence

$$W(C_0(q)) \xrightarrow{tr} W(F) \xrightarrow{c} W(Q)$$

The maps tr and c in the statement are canonical. In particular, tr is explicitly induced by the *reduced trace map* $C_0(q) \rightarrow k$. Prior to this discovery, it was known that the scalar extension homomorphism $W(k) \rightarrow W(k(Q))$ factors through c , and so we have a commutative diagram

$$\begin{array}{ccc} W(C_0(q)) & \xrightarrow{tr} & W(k) & \xrightarrow{c} & W(Q) \\ & \searrow & \downarrow & \swarrow & \\ & & W(k(Q)) & & \end{array}$$

The Witt kernel of $k(Q)/k$ is therefore partly described by the image of the trace homomorphism tr . Moreover, the cokernel of c was also determined explicitly in [2], and so computing the image of tr would also yield a complete computation of the group $W(Q)$.

Decomposition of the Clifford Algebra and Hermitian Morita Theory

A natural strategy for computing the image of the trace homomorphism $tr: W(C_0(q)) \rightarrow W(k)$ is to try to reduce it to cases where q involves a small number of variables. One can hope to achieve this by constructing certain orthogonal decompositions of the form q that decompose the Clifford algebra $C_0(q)$ (with its involution) into “simple” pieces. In our work, we aim to explore situations where we can find a subform p of q such that $C_0(q)$ admits a tensor product decomposition $C_0(p) \otimes_k A$, where A is a “split” Clifford algebra, i.e., a matrix algebra or product of two matrix algebras. Using Morita-type equivalences for hermitian Witt groups, we can then reduce the computation of the trace homomorphism to the computation of φ and the trace homomorphism for p :

$$\begin{array}{ccc} W(C_0(q)) & \cong & W(C_0(p) \otimes_k A) \\ \downarrow tr & & \downarrow \sim_{\text{Morita}} \\ W(k) & \xrightarrow{\varphi \otimes tr} & W(C_0(p)) \end{array}$$

Results

Using the ideas discussed above, we are able to explicitly determine the image of the trace homomorphism $tr: W(C_0(q)) \rightarrow W(k)$ for certain families of anisotropic quadratic forms q over k . A major class of forms considered within this are the so-called *excellent forms*:

Definition 2. We say that q is *excellent* if for every field extension L/k , the anisotropic part of q over L is defined over k .

Examples of excellent forms include all quadratic forms over the field of real numbers \mathbb{R} , as well as all sums of squares forms $X_1^2 + \cdots + X_n^2$ over k . The excellent forms of dimension 2^n for a positive integer n are called *n -fold Pfister forms*. They may be described explicitly as the tensor products of n binary forms, i.e., forms of the type $\langle a_1, b_1 \rangle \otimes \cdots \otimes \langle a_n, b_n \rangle$. If q is excellent of dimension lying in the interval $(2^{n-1}, 2^n)$, then q is embedded into a canonical n -fold Pfister form known as its *ambient Pfister form*. Our main result determines the image of the trace homomorphism for any excellent form of *trivial discriminant* in terms of its ambient Pfister form. For each $n \geq 1$, let $I^n(F)$ be the ideal in $W(F)$ generated by the classes of n -fold Pfister forms. The forms of trivial discriminant are then those (that represent an element) in $I^2(k)$. For the excellent forms in $I^3(k)$, our result is the following:

Theorem 3. If q is an anisotropic excellent form in $I^3(k)$ with ambient Pfister form π of dimension 2^n , then the image of the trace homomorphism $tr: W(C_0(q)) \rightarrow W(k)$ is the ideal in $W(k)$ generated by $\pi \otimes \langle 1, \dots, 1 \rangle$, where the number of 1s is $\dim(q)/2 - n - 1$.

For excellent forms in $I^3(k) \setminus I^2(k)$, a similarly simple description of the image holds. As indicated above, these results also determine the Witt group of quadratic vector bundles $W(Q)$ for the projective quadrics Q defined by these excellent quadratic forms.

Future Work

Our immediate objective is to extend the results outlined above to excellent forms of non-trivial discriminant over k , i.e., excellent forms not in $I^2(k)$. The case of a general form q remains wide open and ripe for further investigation. In another direction, we hope to also study the situation for special fields k (e.g., algebraic number fields), where additional arithmetic information may be brought to bear.

References

- [1] Heng Xie. Witt groups of smooth projective quadrics. *Advances in Mathematics*, 346:70–123, 2019.