

THE TREE-CUTTING PROBLEM IN A STOCHASTIC
ENVIRONMENT: THE CASE OF AGE-DEPENDENT GROWTH

By

HARRY R. CLARKE and WILLIAM J. REED

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THE CASE OF AGE-DEPENDENT GROWTH

Harry R. Clarke^(*) and William J. Reed^(**)

^(*)Industrial Engineering and Management Division
Asian Institute of Technology
P.O. Box 2754, Bangkok 10501, Thailand

^(**)Department of Mathematics
University of Victoria
Victoria, British Columbia
Canada V8W 2Y2

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Abstract

Economically optimal harvest policies are derived for growing biological assets (e.g. forests, livestock) subject to stochastic age-dependent growth and price uncertainty. The task posed is analyzed as a continuous-time optimal stopping problem for diffusion processes. Both "single" and "ongoing" rotation problems are considered. Qualitative comparative static and numerical results are provided.

1. Introduction

In recent years a number of studies have examined the Wicksellian "tree cutting" problem in a stochastic context: see e.g. Malliaris and Brock [1982], Brock, Rothschild and Stiglitz [1983], Brock and Rothschild [1984].⁽¹⁾ In the traditional deterministic version of this problem an asset (or "tree") with aggregate intrinsic value $R(t)$ at time t will be sold (or "cut") at the time T when the proportional growth rate in asset value $\dot{R}(T)/R(T)$ equals the market discount rate δ .⁽²⁾ In this event, assuming perfect foresight, the asset's market value at $t \leq T$ is $\exp(\delta(t-T))R(T)$. A key question concerns how these issues of timing and valuation are altered when the asset's unit intrinsic value (i.e. price) and physical growth are stochastic rather than deterministic, and agents are not supposed to be able to foresee the future.

Using discrete time Brock, Rothschild and Stiglitz [1983] show that, if aggregate intrinsic values evolve as a Markov process with additive noise, the effects of uncertainty depend crucially on whether the process is strictly non-decreasing or not. If it is, then uncertainty increases the asset's market value but does not influence the critical value at which the asset should be sold. If however aggregate intrinsic value can either increase or decrease then uncertainty both increases the asset's market value and the critical value at which the asset should be sold. The authors point out that, while the distinction between non-decreasing and possibly decreasing processes is of interest, analytical results for discrete-time processes are not as sharp as one might hope since asset sale dates are constrained to lie at the endpoints of discrete time intervals so the computation of market values is subject to "roundoff" problems.⁽³⁾

Much recent work has thus focused on continuous-time stochastic processes, such as diffusions, where an asset's intrinsic value is modelled by a stochastic differential equation. Sample paths of such processes can both increase and decrease so the discrete-time dichotomy between the effects of uncertainty in non-decreasing and possibly decreasing processes cannot be evoked. A further complication which arises from this is the necessity of specifying additional conditions (e.g. absorbing or reflecting barriers) to prevent intrinsic values from becoming negative and thus implausible. These conditions moreover determine the asset's current market value and the sensitivity effects of parameter changes on issues of timing and valuation.⁽⁴⁾

The fact that these latter issues depend on the specification of boundary conditions is somewhat unsatisfactory since, with a more particular stochastic specification, that seems both theoretically and empirically attractive, intrinsic values will remain non-negative. Specifically, if the random components of price and size are assumed proportional to current price and size respectively then the corresponding variables remain positive with probability one. This idea led Samuelson [1965] to reject "additive" Brownian motion in favour of geometric Brownian motion in his model of warrant pricing. This leads to a lognormal distribution for prices — a specification which seems to have a priori appeal in a variety of other economic contexts as well.⁽⁵⁾ Moreover it has long been recognized that the lognormal distribution can be used as a plausible representation for the weight or size distribution of a biological organism.⁽⁶⁾

The present analysis adopts this general viewpoint and, since it seeks to focus on resource economic issues, distinguishes between the physical size and the unit intrinsic value (i.e. price) of a biological asset. The latter

is assumed to follow geometric Brownian motion, while the former is assumed to follow a stochastic differential equation (a diffusion) in which the infinitesimal variance is proportional to the square of current size. With this specification, analytically tractable rules for determining market valuations and asset harvest policies can be obtained.

The remainder of this paper is structured as follows. Section 2 outlines the asset valuation model used in the analysis as well as the procedure for determining optimal asset harvest policies. Section 3 computes asset values and optimal harvest rules when there is both growth and price uncertainty. Section 4 discusses risk-aversion while Section 5 considers the "ongoing rotations" problem. Finally, Section 6 summarizes the major conclusions and implications of the analysis.

2. Biological Asset Valuation Model

Consider a biological asset subject to random variability in growth for which unit price is exogeneous and random. Suppose the asset's age at time $t = 0$ is $A(t) = 0$ so that it is unnecessary to distinguish between these variables. The key features of the asset's development are described in terms of:

$P(t)$, unit price of the asset at t ,

$q(t)$ = $\log P(t)$,

$X(t)$, asset size at t ,⁽⁷⁾

$y(t)$ = $\log X(t)$,

$R(t)$ = $P(t)X(t)$, the aggregate intrinsic value (revenue yielded by harvesting) at t ,

$g(A(t)) = g(t)$, the deterministic component of the asset's instantaneous proportional growth rate at age $A(t) = t$,

$W(t) \equiv W(t, y, q)$, asset's market value at t when size is e^y and price e^q ,
 δ = positive constant discount rate.

The asset's aggregate intrinsic value at t is then $R(t) = \exp(q(t) + y(t))$ which, we assume, is determined by the stochastic model:

$$dq = bdt + \sigma_q dw_q \quad (1).$$

$$dy = g(t)dt + \sigma_y dw_y. \quad (2).$$

Thus the asset's unit price evolves as geometric Brownian motion with constant drift b and constant variance σ_q^2 . The logarithm of asset size is a diffusion that behaves locally like Brownian motion with drift $g(t)$ and constant variance σ_y^2 ; w_q and w_y are standard Weiner processes.

Because the results of our analysis are sensitive to the precise form of the specification (1), (2) we wish to comment on some aspects of this formulation:

(i) The biological organism's proportional growth rate is supposed to depend purely on its current age rather than its current size. For this reason we refer to the growth process as displaying pure age dependence. The deterministic component of proportional growth $g(t)$ is assumed decreasing and differentiable.

Growth processes which depend on size or density alone we refer to as pure size dependent processes. The stochastic differential equation corresponding to (2) for such processes would be of the form $dy = h(y)dt + \sigma_y dw_y$. In a deterministic setting it is irrelevant whether growth is specified in terms of age or size since in that case these variables are

functionally related. In a stochastic context however the task of selecting optimal harvest rules changes substantially when a size-dependent specification replaces an age-dependent specification since, while size can decrease (e.g. fire, wind or pest infestations can reduce a forest's size) aging processes are irreversible in the sense that age cannot decrease.

It turns out that this characteristic makes the determination of optimal harvest rules with size dependence more complex. For this reason we discuss the case of "age dependence" here and the case of "size dependence" in a separate paper: see Reed and Clarke [1987]. For many biological assets such as a planted stand of trees, or a herd of livestock, the assumption of age-dependent growth seems reasonable. It is important however to emphasize that, in order to determine useful harvest rules, it is essential to initially identify the underlying biological growth mechanism.

(ii) Using Itô's lemma, see e.g. Arnold [1973, Page 90], the price process we have specified as equation (1), can be written:

$$dP = \left[b + \frac{1}{2}\sigma_q^2 \right] P dt + \sigma_q P dw_q. \quad (1A)$$

Thus while $q(t) \sim N[q(0)+bt, \sigma_q^2 t]$, the price level $P(t)$ is lognormally distributed with $E(P(t)) = P(0)\exp\left[bt + \frac{1}{2}\sigma_q^2 t\right]$ which, when $b = 0$, implies $E(P(t)) = P(0)\exp\left[\frac{1}{2}\sigma_q^2 t\right]$. Moreover the median $M(P(t))$ of $P(t)$ is $\exp(M(q(t))) = \exp(q(0)+bt)$ which, when $b = 0$, gives $M(P(t)) \equiv P(0)$.

Thus with $b = 0$, over any time interval, there is an equal probability that price will increase or decrease. The case with $b = 0$ is graphed in Fig. 1(A).

(Figure 1 near here)

Further, although $E(P(t))$ increases exponentially we have

$$\begin{aligned} \text{Prob}\{P(t) < E(P(t))\} &= \text{Prob}\{P(t) < P(0)\exp(bt + \frac{1}{2}\sigma_q^2 t)\} \\ &= \text{Prob}\{q(t) < q(0) + bt + \frac{1}{2}\sigma_q^2 t\} \\ &= \Phi\left[\frac{1}{2}\sigma_q \sqrt{t}\right] \end{aligned}$$

where Φ is the cumulative distribution function of a standard normal random variable. As $t \rightarrow \infty$, this probability approaches 1. Thus although $E(P(t))$ increases exponentially, almost all sample paths of the price process will eventually end up below this exponential curve.

We make these points to emphasize that the price process specified has some rather definite implications. In particular it implies that, with no drift, mean prices will increase exponentially while median prices remain constant so that, looking into the future, there is an equal probability, over any time horizon, that prices will rise or fall.

If, instead of equation (1), prices had been hypothesized to evolve as:

$$dP = bP dt + \sigma_p P dw_q \quad (1B)$$

which is the price specification used in work on warrant pricing (Samuelson [1965]) and natural resource models (Pindyck [1981]), then $q(t) \sim N(q(0) + (b - \frac{1}{2}\sigma_p^2)t, \sigma_p^2 t)$ and price is lognormally distributed with mean $E(P(t)) = P(0)\exp(bt)$ which, with $b = 0$, gives $E(P(t)) \equiv P(0)$. The median of $P(t)$ is now $M(P(t)) = \exp(M(q(t))) = P(0)\exp(b - \frac{1}{2}\sigma_p^2 t)$ which, when $b = 0$, gives $M(P(t)) = P(0)\exp(-\frac{1}{2}\sigma_p^2 t)$. The case with $b = 0$ is graphed in Fig. 1(B).

Note with $b = 0$ that, over any future time period, there is now a smaller chance of a price increase than a decrease. In fact for any $\epsilon > 0$:

$$\begin{aligned} \text{Prob}\{P(t) < \epsilon\} &= \text{Prob}\{q(t) < \log \epsilon\} \\ &= \Phi\{(\log \epsilon - q(0) + \frac{1}{2}\sigma_p^2 t)/\sigma_p \sqrt{t}\} \\ &\rightarrow 1 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Thus almost all sample paths of the price process described by (1B) will end up below any arbitrarily small $\epsilon > 0$ i.e. the price tends to zero "in probability" (see e.g. Arnold [1974, Page 13]). In fact this result is what Samuelson [1965, Page 203] refers to as the "Theorem of virtual certainty of (relative) ruin".

In general the price process specified by (1) seems to make more sense for prices of resource assets, such as timber, than the alternative specification (1B). The latter implies, when drift is zero, that the price will decline to zero "in probability" even while the mean price stays constant. This seems implausible.⁽⁸⁾ The key issue however that we wish to emphasize is that the price specification issue turns out to be important in determining quantitatively the optimal harvest rule. This is discussed further below.

(iii) It should also be noted that a similar dichotomy arises in specifying the biological growth process. Equation (2) expressed in terms of size X is:

$$dX = (g(t) + \frac{1}{2}\sigma_y^2)Xdt + \sigma_y Xdw_y. \quad (2A)$$

An alternative specification is:

$$dX = g(t)Xdt + \sigma_y Xdw_y. \quad (2B)$$

For (2A) the median size $M(X(t)) = X(0)\exp\left\{\int_0^t g(s)ds\right\}$ while the mean size $E(X(t)) = X(0)\exp\left\{\int_0^t g(s)ds + \frac{1}{2}\sigma_y^2 t\right\}$. The median size follows the deterministic trajectory (which would arise with $\sigma_y = 0$), while the mean size has an additional multiplicative exponential growth component. For specification (2B), while the mean size would follow the deterministic trajectory, the median size would eventually exhibit exponential decline and, as with the price specification (1B) would tend to zero "in probability". We feel that (2A) provides a more reasonable representation of reality over economically meaningful time horizons.

Given the model's structure (1), (2) we turn to the formulation of a specific asset valuation procedure. Assuming risk neutrality, at time t , when $y(t) = y$, $q(t) = q$ the biological asset's market value $W(y, q, t)$ is its expected present value assuming that it is harvested at its expected-present-value-maximizing harvest date:

$$\begin{aligned} W &= W(t, y, q) \\ &= \sup_{\tau \geq t} E_t \{ \exp(-\delta(\tau-t) + q(\tau) + y(\tau)) \mid y(t)=y, q(t)=q \} \end{aligned} \quad (3).$$

where E_t denotes mathematical expectation conditional on values at time t .

The problem of determining a closed-form expression for (3) given the dynamics (1) and (2) is simplified once the "optimal stopping rule",

prescribing the conditions under which the asset should be harvested, is determined.

Assuming pure age-dependent growth, the analysis to follow proceeds by deriving a plausible rule for choice of harvest date using a "myopic look ahead" (MLA) procedure. The resulting myopic rule is then shown to imply a market value function consistent with the Hamilton-Jacobi-Bellman equation for market value in the continuation region (i.e. the set of values of (y, q, t) for which the myopic rule is to not harvest the asset). In addition the resulting value function satisfies continuity and "smooth pasting" conditions on the boundary of the continuation region where harvesting of the asset is prescribed. Using results on the optimal stopping of diffusion processes the MLA rule is then seen to be optimal. With this rule a closed-form expression for the asset's market value can be derived and a comparative static analysis of the timing and valuation problem implemented.

3. Optimal Harvest Rules

The problem of characterizing the solutions to various optimal stopping tasks has been much discussed. In fact analytical solutions have been derived for only a small class of problems. A number of authors have analyzed the "myopic-look-ahead" rule: see e.g. Ross [1970, Theorem 6.14], Chow, Robbins and Siegmund [1971, Section 3.5], Rosenfield and Shapiro [1981]. In essence the rule proposes that a discrete time process should be stopped when the actual gain from so doing first exceeds the expected gain from stopping one period ahead. The continuous time correlate to this rule would suggest that the optimal harvest time τ , should be characterized by the condition that the asset's intrinsic value at τ (namely $R(\tau) =$

$\exp(y(\tau)+q(\tau))$ should just equal its discounted expected intrinsic value at an infinitesimal time later $\tau + d\tau$.⁽⁹⁾ In the present instance this condition is:

$$\exp(y(\tau)+q(\tau)) = E_{\tau}\{\exp(-\delta d\tau) \cdot \exp(y(\tau)+dy+q(\tau)+dq)\} \quad (4)$$

which implies:

$$\begin{aligned} \exp(y+q) &= E_{\tau}\{\exp(-\delta d\tau) \exp(y+q) \exp(dy+dq)\} \\ \text{or } 1 &= E_{\tau}\{\exp(-\delta d\tau) \exp(dy+dq)\} \\ &= (1-\delta d\tau) E_{\tau}\{1+dy+dq+\frac{1}{2}(dy+dq)^2\} + o(d\tau). \end{aligned}$$

Using (1), (2) and the fact that $E(dw^2) = d\tau$ for a Weiner process we have, in the limit as $d\tau \rightarrow 0$:

$$g(\tau) = \delta - b - \frac{1}{2}(\sigma_y^2 + \sigma_q^2 + 2\sigma_{yq}) \quad (5)$$

where σ_{yq} is the instantaneous covariance of the disturbances in the y and q processes.

The MLA rule here has straightforward implications. With pure age-dependent growth it prescribes harvesting at some fixed age (τ_M say) where the non-random component of the asset's proportional growth equals the R.H.S. of (5). With this rule the size at which the asset is harvested then becomes a random variable. Note that this rule is independent of the absolute price presumed to prevail at the time of harvest. This is readily seen to be a consequence of specifying price as geometric Brownian motion.

When does (5) imply a non-zero bounded harvest date? Clearly when:

$$g(\infty) < \delta - b - \frac{1}{2}(\sigma_y^2 + \sigma_q^2 + 2\sigma_{yq}) \triangleq D < g(0). \quad (6)$$

If $D \geq g(\infty)$ the asset should never be harvested since, in expected value terms, it is appreciating "too quickly".⁽¹⁰⁾ Note that this will occur if there is too much random variability in growth or price. If $D \geq g(0)$ then the asset is growing "too slowly" — it should be harvested at $t = 0$ i.e. never "planted". In what follows condition (6) is assumed to hold so there always exists a finite non-zero harvest date.

Note also using Ito's lemma, since $R = \exp(y+q)$, we have

$$E_t(dR) = \left[(b+g(t))R + \frac{1}{2}(\sigma_y^2 + \sigma_q^2 + 2\sigma_{yq})R \right] dt + o(dt)$$

which when the MLA rule (5) obtains implies, to first order:

$$E_{\tau_M} \{dR(\tau_M)\} / R(\tau_M) = \delta d\tau_M. \quad (7)$$

so the MLA rule provides a stochastic analogue to the deterministic Wicksellian "tree cutting" rule.⁽¹¹⁾

We now investigate the properties of the MLA rule. Is it indeed optimal?⁽¹²⁾ Retaining the assumption of age-dependent growth we prove that τ_M provides the optimal harvest rule. This is a non-trivial result since, as is shown elsewhere (Reed and Clarke [1987]), if growth is respecified to be size-dependent, an MLA rule is non-optimal.

With pure age-dependent growth the dynamics of aggregate intrinsic value ($R(t) = \exp(q(t)+y(t))$) can be expressed in terms of a single variable

$z \triangleq q + y = \log R(t)$ where

$$dz = (b+g(t))dt + \sigma_z dw_z, \quad (8)$$

with $\sigma_z^2 = \sigma_q^2 + \sigma_y^2 + 2\sigma_{yq}$.

To determine the market value of assets with this growth specification the following is required:

Lemma 1: $E_t\{\exp(z(\tau)) | z(t)=\zeta\} = \exp\left\{\zeta + \int_t^\tau (g(s)+b+\frac{1}{2}\sigma_z^2)ds\right\}$.

Proof: From well-known results (see e.g. Arnold [1974, Page 130]) the solution to (8) is:

$$z(\tau | z(t) = \zeta) = \zeta + \int_t^\tau (g(s)+b)ds + \sigma_z w_z(\tau-t)$$

where $w_z(\tau-t) \sim N(0, (\tau-t))$. Using well-known properties of the lognormal distribution (see e.g. Moran [1968, Page 317]) the result follows. ||

The MLA rule suggests the asset should be harvested at the fixed age τ_M . It follows that the asset's value with this rule at $t \leq \tau_M$ when $z(t) = z$ is:

$$\begin{aligned} W^M(t, z) &= W^M(t, y, q) = E_t\{\exp(-\delta(\tau_M-t)+z(\tau_M)) | z(t) = z\} \\ &= \exp\left\{-\delta(\tau_M-t)+z+\int_t^{\tau_M} (g(s)+b+\frac{1}{2}\sigma_z^2)ds\right\}, \quad (\text{on using the lemma}) \\ &= e^z \exp\left\{\int_t^{\tau_M} (g(s)+b+\frac{1}{2}\sigma_z^2-\delta)ds\right\}, \quad (9) \end{aligned}$$

which is the current intrinsic revenue multiplied by a factor reflecting the expected growth in price and size net of discounting effects, over the time until the cutting age is reached. We now prove:

Proposition 1:

The MLA rule of harvesting at age τ_M is the optimal harvest rule for pure age-dependent growth.

Proof: Using the results of Krylov [1980, Pages 35-42], Shirayayev [1978, Pages 157-162] the stopping rule, harvest at τ_M , will be an optimal harvest strategy provided certain regularity conditions are satisfied, and the value function W^M satisfies both the Hamilton-Jacobi-Bellman equation

$$\delta W(t, z) = (g(t)+b)W_z + \frac{1}{2}\sigma_z^2 W_{zz} + W_t$$

on $t < \tau_M$ and the following continuity and smooth-pasting conditions at the stopping boundary $t = \tau_M$:

$$\begin{aligned} W(\tau_M, z) &= R(\tau_M) = e^z \\ W_z(\tau_M, z) &= R_z(\tau_M) = e^z \\ W_t(\tau_M, z) &= R_t(\tau_M) = 0. \end{aligned}$$

The H-J-B equation holds for $t < \tau_M$ since for $W = W^M$ using (9)

$$\begin{aligned} &(g(t)+b)W_z + \frac{1}{2}\sigma_z^2 W_{zz} + W_t \\ &= (g(t)+b)W^M + \frac{1}{2}\sigma_z^2 W^M + (\delta - g(t) - b - \frac{1}{2}\sigma_z^2)W^M \\ &= \delta W^M \text{ as required.} \end{aligned}$$

The continuity condition holds since, from (9),

$$W^M(\tau_M, z) = e^z.$$

Finally the smooth-pasting conditions can be verified from (9):

$$\begin{aligned} W_z^M(\tau_M, z) &= W^M = e^z \\ W_t^M(\tau_M, z) &= (\delta - g(\tau_M) - b - \frac{1}{2}\sigma_z^2)W^M \\ &= 0 \quad (\text{using (5)}). \parallel \end{aligned}$$

To recapitulate, the MLA rule is optimal and requires that the asset should be harvested at the fixed age τ_M . Both the price and size at the optimal cutting age are random variables with a lognormal distribution. The expected value of price is $P_0 \exp\left\{\left[b + \frac{1}{2}\sigma_q^2\right]\tau_M\right\}$ while that of size is

$$X_0 \exp\left\{\int_0^{\tau_M} (g(s) + \frac{1}{2}\sigma_y^2) ds\right\}.$$

An interesting fact about the above optimal policy, is that the effects of uncertainty can be accommodated entirely through a modification of the effective discount rate i.e. that the deterministic optimal policy will be optimal in the stochastic case provided a suitable "uncertainty-adjusted" discount rate is employed. For the deterministic case ($\sigma_z = 0$), the intrinsic value as a function of time is (from solving the ordinary differential equation (8)):

$$R(t) = R(0) \exp\left\{\int_0^t [b + g(s)] ds\right\}.$$

From this, the Wicksell rule ($\dot{R}(t)/R(t) = \delta$) for determining the optimal

cutting age, T , gives $b + g(T) = \delta$, with corresponding value function

$$W(t,R) = R(t) \exp \left\{ \int_t^T [b+g(s)-\delta] ds \right\}.$$

On replacing δ by $\gamma = \delta - \frac{1}{2}\sigma_z^2$, this rule and the associated value function are seen to be equivalent to the stochastic rule (5) and value function (9). Thus the presence of uncertainty in price and in age-dependent growth has the same effect as lowering the effective discount rate. In fact the stochastic problem can be solved by a Certainty Equivalence procedure, i.e. by solving the deterministic problem which arises by setting the random variables dw_q and dw_y in (1A) and (2A) equal to their expected values viz. zero. (See Crabbé [1982] for a discussion of Certainty Equivalence and related issues.)

The use of "risk-adjusted" discount rates and the Certainty Equivalence procedure has arisen in resource-management problems in situations where "catastrophic change" can occur at some random time, e.g. when growing trees can be destroyed by random catastrophic fire [Reed, 1984] or when an essential exhaustible resource can be rendered inessential by means of technological change [Dasgupta and Heal, 1974]. Uncertainty of this type can easily be incorporated into the model of this paper. If there is a risk of destruction characterized by a hazard function $h(t)$,⁽¹³⁾ then the optimal harvest time is given by the solution to

$$g(\tau) + b + \frac{1}{2}\sigma_z^2 - h(\tau) = 0$$

and the value function⁽¹⁴⁾ is

$$W(t, z) = e^z \exp \left\{ \int_t^{\tau} [g(s) + b + \frac{1}{2} \sigma_z^2 - \delta - h(s)] ds \right\}.$$

This is equivalent to the solution of the deterministic problem using a non-constant, risk and uncertainty adjusted discount rate

$$\gamma(t) = \delta + h(t) - \frac{1}{2} \sigma_z^2.$$

Thus risk of destruction adds a premium to the effective discount rate, while price and growth uncertainty decreases it.

It should be noted that the use of Certainty Equivalence and an adjusted discount rate holds only for the case of age-dependent growth. It does not do so for size-dependent growth (see [Reed and Clarke, 1987]).

We turn now to the question of how the optimal harvest age and the pre-harvest market valuations of the asset depend on the parameters of the problem. Using (5) and (9):

$$\begin{aligned} \partial \tau_M / \partial \sigma_q &= -(\sigma_q + \rho \sigma_y) / g'(\tau_M) \\ \partial \tau_M / \partial \sigma_y &= -(\sigma_y + \rho \sigma_q) / g'(\tau_M) \\ \partial \tau_M / \partial b &= -\partial \tau_M / \partial \delta = -1 / g'(\tau_M) > 0 \\ \partial W^M / \partial \sigma_q &= (\tau_M - t) (\sigma_y + \rho \sigma_q) W^M \\ \partial W^M / \partial \sigma_y &= (\tau_M - t) (\sigma_q + \rho \sigma_q) W^M \\ \partial W^M / \partial b &= -\partial W^M / \partial \delta = W^M (\tau_M - t) > 0, \end{aligned}$$

where $\rho = \text{correlation}(w_q, w_y)$. It seems most natural to suppose $\rho = 0$ ⁽¹⁵⁾ so since $g' > 0$:

$$\begin{aligned} \frac{\partial \tau_M}{\partial \sigma_q} &> 0 & \frac{\partial \tau_M}{\partial \sigma_y} &> 0 \\ \frac{\partial W^M}{\partial \sigma_q} &> 0 & \frac{\partial W^M}{\partial \sigma_y} &> 0. \end{aligned}$$

Thus both price and growth uncertainty increases the optimum harvesting age and the current market value of biological assets. Either an increase in price drift or a reduction in the discount rate will increase the optimum harvest age and the current market value. These results are entirely consistent with the qualitative conclusions derived, for example, in Malliaris and Brock [1982, Pages 201-205] for a highly stylized "tree cutting" problem, yet they are derived without invoking somewhat arbitrary boundary conditions and with a more plausible and specific model of the underlying biological and economic process.

It is worth noting that the comparative static results here are sensitive to the specification of the price and growth dynamics. For example if the "warrant pricing" price specifications (1B) and (2B) had been used instead of (1) and (2) — this amounts to replacing b by $b - \frac{1}{2}\sigma_q^2$ and $g(t)$ by $g(t) - \frac{1}{2}\sigma_y^2$ throughout the analysis — the optimal MLA rule would prescribe harvesting at age $\hat{\tau}_M$ where

$$g(\hat{\tau}_M) = \delta - b$$

which would give the pre-harvest market valuation function:

$$\hat{W}^M(t, z) = e^z \exp \left\{ \int_t^{\hat{\tau}_M} (g(s) + b - \delta) ds \right\},$$

for which

$$\begin{aligned}\frac{\partial \hat{\tau}_M}{\partial \sigma_y} &= \frac{\partial \hat{W}^M}{\partial \sigma_y} = 0 \\ \frac{\partial \hat{\tau}_M}{\partial \sigma_q} &= \frac{\partial \hat{W}^M}{\partial \sigma_q} = 0.\end{aligned}$$

With this specification uncertainty would have no influence on the optimum harvest age or market value of the asset. The fact that this finding seems less in accord with other results concerning the effects of uncertainty (e.g. Brock, Rothschild & Stiglitz, [1983], Malliaris and Brock [1982, Page 50]) than those using specifications (1) and (2), increases our confidence in the use of the latter for modelling price and growth uncertainty.⁽¹⁶⁾

Finally it is worth noting that the increase in the value of the asset with increases in σ_y and σ_q , using the specifications (1) and (2), can be attributed to the fact that $E(X(t))$ and $E(P(t))$ contain exponential growth components of the form $\exp\left\{\frac{1}{2}\sigma_y^2 t\right\}$ and $\exp\left\{\frac{1}{2}\sigma_q^2 t\right\}$ respectively. While for most sample paths there will be little or no net gain over that earned with no random variation present, the exceptional gain in a few cases, causes an overall gain in the expected value of revenue earned.

4. Risk-Averse Management

In view of the fact that future biological growth and future prices are uncertain, the resource owner may well adopt a risk-averse harvesting strategy. Suppose that this can be characterized by an objective of maximizing expected discounted utility,

$$E\left\{e^{-\delta\tau} U(P(\tau)X(\tau))\right\}$$

where τ is the time of harvest and $U(PX)$ represents the utility associated with a revenue of PX dollars. To reflect risk aversion, suppose that $U' > 0$ and $U'' < 0$. Let

$$v = U(e^Z) = U(PX)$$

denote the "intrinsic utility" of a harvest when price is P and size is X . From (8) it follows that this intrinsic utility evolves as a diffusion, characterized by the stochastic differential equation

$$dv = M(v,t)dt + \sigma_Z S(v)dw_Z \quad (10)$$

where, from Itô's lemma,

$$\begin{aligned} M(v,t) = & U'(U^{-1}(v))U^{-1}(v)[b+g(t)] \\ & + \frac{1}{2}\sigma_Z^2[(U'(U^{-1}(v))U^{-1}(v)+U''(U^{-1}(v)))]U^{-1}(v) \end{aligned}$$

and

$$S(v) = U'(U^{-1}(v))U^{-1}(v).$$

The fact that the mean drift term $M(v,t)$ depends upon both v and t suggests that the optimal harvest rule to maximize expected discounted utility will, in general, do likewise, i.e. that it will depend both on age, t and intrinsic value, PX . This is true of the MLA rule which involves harvesting when

$$v(t) = E\left\{e^{-\delta dt}(v+dv)\right\}.$$

This condition gives, in the limit as $dt \rightarrow 0$, after some algebra,

$$g(t) + b + \frac{1}{2}\sigma_z^2 = \delta/\epsilon(PX) + \frac{1}{2}\sigma_z^2\eta(PX) \quad (11)$$

where,

$$\epsilon(PX) = \frac{PX U'(PX)}{U(PX)}$$

is the elasticity of utility, and

$$\eta(PX) = - \frac{PX U''(PX)}{U'(PX)}$$

is the elasticity of marginal utility, which gives a measure of the degree of absolute risk aversion. Clearly the MLA harvest rule given in (11) depends in general on both intrinsic value, PX , and on age, t . There is no reason to believe that it is the optimal harvest rule, although it may well provide a useful approximation to it.

For a positive concave utility function, reflecting risk aversion, the elasticity $\epsilon(PX)$ is less than unity while $\eta(PX)$ is positive. Thus the R.H.S. of (11) is greater than δ . It follows, on comparison with (5), since g is decreasing, that the MLA rule prescribes harvesting at a younger age with risk aversion present than with risk neutrality. The degree to which the MLA harvest is earlier in the risk-averse case, depends in general, on the particular sample paths followed by the size and price processes, and of course on the form of the utility function.

While the MLA rule is not necessarily optimal there is one important special case in which it is. This is the case of an iso-elastic utility,

$$U(PX) = C(PX)^a$$

where $0 < a < 1$, and $C > 0$. In this case

$$\epsilon(PX) \equiv a \quad \text{and} \quad \eta(PX) \equiv 1 - a.$$

It follows from (11) that, for this utility, the MLA harvest rule involves harvesting when

$$g(t) + b + \frac{1}{2}a\sigma_z^2 = \delta/a. \quad (12)$$

This is a barrier rule on age — harvest when the age of the asset reaches τ_a given by (12). By the methods of the last section it is easily shown that this harvest rule is optimal. Indeed this is particularly simple since the logarithm of intrinsic utility is related to the logarithm of intrinsic value, z by

$$\log v = \log C + az.$$

In consequence Lemma 1 can be used to determine the market value (in utility units) explicitly. It is

$$V^a(t, P, X) = C(PX)^a \exp \left\{ -\delta(\tau_a - t) + a \int_t^{\tau_a} (g(s) + b + \frac{1}{2}a\sigma_z^2) ds \right\}.$$

From this it is readily shown that the H-J-B equation and the continuity and smooth-pasting conditions, outlined in the proof of Proposition 1, are satisfied.

It should be noted that the optimum policy will involve harvesting at a non-zero age only if

$$g(0) > \delta/a - b - \frac{1}{2}a\sigma^2.$$

From the fact that the R.H.S. of this expression is greater than D in (6), it can be seen that there may be situations in which, optimally, the risk-averse owner would not "plant", while the risk-neutral one would.

Finally on comparing (12) and (6), it can be seen that the optimal cutting age for the risk-averse owner is lower than that for the risk-neutral owner, and that the optimal cutting age decreases as the degree of absolute risk aversion ($\eta \equiv 1-a$) increases.

5. The "Ongoing Rotations" Problem

It has long been recognized in the forestry literature that the determination of desired cutting ages for trees involves consideration of ongoing forest rotations. In simple terms, once trees are cut, land becomes available for new forest growth so, the longer felling is delayed, the longer it takes to acquire revenues from future harvests. (17)

Consider a sequence of times $T_1 < T_2 < T_3 \dots$ such that at each T_k a tree is cut and a new tree planted. Ignoring harvest and restocking costs and assuming prices and environmental productivities remain constant, the optimization task traditionally considered, is to select T_1, T_2, \dots to maximize the present value of all future harvests. Using the notation of previous sections this task is to maximize:

$$\sum_{i=1}^{\infty} \exp(-\delta T_i) R(T_i - T_{i-1}); \quad T_0 = 0. \quad (13)$$

This task is straightforward since, given constant prices and environmental productivities, all rotations must be of equal length T . This common rotation period is determined by the well-known Faustmann formula:

$$R'(T)/R(T) = \delta / (1 - e^{-\delta T}). \quad (14)$$

Since $\delta > 0$, application of this rule involves harvesting trees earlier than would occur with use of the Wicksell "tree cutting" rule. Further, with this rule, the market value of all future harvests V , can be computed as:

$$\begin{aligned} V &= \sum_{i=1}^{\infty} e^{-\delta i T} \cdot R(T) \\ &= R(T) / \{1 - e^{-\delta T}\}. \end{aligned} \quad (15)$$

Can this type of analysis be extended to account explicitly for price and growth uncertainty? In a stochastic context it is natural to suppose that resource owners seek to select a sequence of harvest times T_1, T_2, \dots to maximize the expected present value of total future returns. Denote:

$$J(P_0, X_0) = E \left\{ \sum_{i=1}^{\infty} \exp(-\delta \tau_i) P(T_i) \hat{X}(T_i) \mid P(0) = P_0, X(0) = X_0 \right\} \quad (16)$$

where $\hat{X}(T_i) = X(T_i - T_{i-1})$ is the size of a stand of trees at absolute time T_i and $X(0) = X_0$ is the initial size of a newly-planted stand.

The task envisaged is the maximization of J . Assuming such a maximum exists we follow the forestry literature and refer to the maximized present value as the land expectation value $V(P_0, X_0)$. This quantity possesses a simple homogeneity property:

Proposition 2:

The land expectation value of a site is proportional to the current resource price. Thus $V(P_0, X_0) = \theta P_0$ where $\theta > 0$ is a constant of proportionality, depending only on X_0 .

To prove this write the expectation (16) as:

$$J(P_0, X_0) = \int_{\Omega} \sum_{i=1}^{\infty} \exp(-\delta T_i) P(t_i; \omega) \hat{X}(T_i; \omega) dF(\omega; P_0, X_0)$$

where $P(T_i; \omega)$ and $\hat{X}(T_i; \omega)$ are sample paths and $F(\omega; P_0, X_0)$ is their distribution function on the sample space Ω . Note that J is also a function of the cutting times T_1, T_2, \dots although this dependence is suppressed in the above notation.

Now consider a change in initial price from P_0^{OLD} to $P_0^{\text{NEW}} = kP_0^{\text{OLD}}$. Since $P(t)$ follows the geometric Brownian motion (1A), the sample paths of $P(t)$, with this change, simply become $P^{\text{NEW}}(t; \omega) = kP^{\text{OLD}}(t; \omega)$. Thus:

$$\begin{aligned}
J(P_0^{\text{NEW}}, X_0) &= \int_{\Omega} \sum_{i=1}^{\infty} \exp(-\delta T_i) P_0^{\text{NEW}}(T_i; \omega) \tilde{X}(T_i; \omega) dF(\omega; P_0^{\text{NEW}}, X_0) \\
&= k \int_{\Omega} \sum_{i=1}^{\infty} \exp(-\delta T_i) P_0^{\text{OLD}}(T_i; \omega) \tilde{X}(T_i; \omega) dF(\omega; P_0^{\text{OLD}}, X_0) \\
&= k J(P_0^{\text{OLD}}, X_0).
\end{aligned}$$

It follows that the maximum of $J(P_0^{\text{NEW}}, X_0)$ is obtained at exactly the same cutting times T_1, T_2, \dots as the maximum of $J(P_0^{\text{OLD}}, X_0)$, and so

$$V(kP_0, X_0) = kV(P_0, X_0)$$

from which it follows that $V(P_0, X_0) = \theta P_0$ with θ independent of P_0 . \parallel

The task of maximizing (16) can now be expressed as

$$\max_{T_1} E\{\exp(-\delta T_1)(P(T_1)X(T_1) + \theta P(T_1)) | P(0)=P_0, X(0)=X_0\}$$

where $\theta P(T_1) = V(P(T_1), X_0)$ since after the first harvest the optimal value of future harvests is the land expectation value. Defining:

$$u = \log(X+\theta) = \psi(y) \tag{17}$$

where, as before, $y(t) = \log X(t)$, and $q(t) = \log P(t)$, the maximization task can be represented compactly as:

$$\max_T E\{\exp(-\delta T + q(T) + u(T)) | q(0), u(0)\}. \tag{18}$$

Furthermore using Ito's lemma

$$\begin{aligned} du &= \left\{ \psi_y g(t) + \frac{1}{2} \sigma_y^2 \psi_{yy} \right\} dt + \psi_y \sigma_y dw_y \\ &= (1 - \theta \exp(-u)) \left\{ g(t) + \frac{1}{2} \sigma_y^2 \theta \exp(-u) \right\} dt \\ &\quad + (1 - \theta \exp(-u)) \sigma_y dw_y. \end{aligned}$$

i.e.
$$du = A(u,t)dt + C(u)dw_y \tag{19}$$

where
$$A(u,t) = (1 - \theta \exp(-u)) \left\{ g(t) + \frac{1}{2} \sigma_y^2 \theta \exp(-u) \right\}$$

$$C(u) = (1 - \theta \exp(-u)) \sigma_y.$$

Thus the ongoing-rotations problem can be solved iteratively by solving single rotation problems of form (18) with dynamic constraints (19) and (1) for various values of θ until a value of θ for which (18) equals $\theta \exp(q(0))$ is found. While an analytic solution does not appear possible some features of the optimal solution can be deduced. We summarize these as:

(i) It seems unlikely that a simple barrier rule on age will provide the solution to this task because the diffusion (19) has drift dependent on both u and t i.e. both on size and age. We conjecture that the optimal stopping barrier will, in general, depend on both size and age. This contrasts to the case of optimization over a single rotation, in the risk-neutral case, but is similar to the general risk-averse case.

(ii) The use of an MLA rule for the present problem, while probably not optimal, yields a solution we believe is quite close to the optimal harvest rule for many practical problems. The rule is unlikely to be optimal because its "stopping set" depends on the "size" variable which can increase as well as decrease with the growth dynamics (2).⁽¹⁸⁾ Thus the MLA "stopping set" is

not "closed" in the sense of Ross [1970, Page 188]. Heuristically however a tree's growth will tend to increase rather than decrease in the neighbourhood of the optimal cutting age so this type of behavior may be of little practical importance. Indeed we have verified numerically (see Reed and Clarke [1987]) that, in the case of a size-dependent growth specification, the MLA rule provides a very close approximation to the optimal cutting rule.

Using the methodology of Section 3 the MLA rule for the present problem can be shown to produce a stopping boundary ∂S for which:

$$A(u,t) = \delta - b - \frac{1}{2}\sigma_q^2 - \frac{1}{2}C^2(u) \quad (20)$$

which, with some manipulation, implies:

$$g(t) = B(X(t)+\theta)/X(t) - \frac{1}{2}\sigma_y^2 \quad (21A)$$

$$\text{or } X(t) = \theta B / \{g(t) + \frac{1}{2}\sigma_y^2 - B\} \quad (21B)$$

where $B = \delta - b - \frac{1}{2}\sigma_q^2$. To avoid trivial situations where it is always optimal to harvest at age zero, the R.H.S. of (21B) must be positive at $t = 0$. This is the case if:

$$X(0) > X^*(0) = \theta B / \{g(0) + \frac{1}{2}\sigma_y^2 - B\} \quad (22)$$

i.e. when the initial size of a newly planted tree exceeds some minimum critical level $X^*(0)$.

Since $g(t)$ is decreasing, and $g(\tau_M) = B - \frac{1}{2}\sigma_y^2$ (from (5)) it follows that the stopping boundary defined by (21B) has X as an increasing function of t with a vertical asymptote at $t = \tau_M$ as illustrated in Fig. 2. Here

∂S denotes the stopping boundary; $X(t, \omega)$ is a typical sample path describing size; τ_M is the cutting age prescribed by rule (5) to be optimal for the single rotation case and $\tau^*(\omega)$ is the (random) optimal cutting age for the ongoing-rotations case.

(Figure 2 near here)

Two features of the MLA rule are apparent from this figure: (i) as can be verified algebraically, one always cuts at a younger age in the ongoing-rotations case than in the single-rotation problem — this is a stochastic generalization of the well-known comparison between the Wicksell and Faustmann solutions as discussed above; (ii) the age at which one cuts depends on the tree's size. If size increases slowly initially then the resource owner tends to cut and abandon that rotation thereby enabling a new rotation to commence. This feature agrees entirely with intuition: it does not arise in the "single" rotation model since no weight is attached there to harvests beyond the first rotation.

Reed and Clarke [1987] show that these types of results are not valid in an ongoing-rotations model with size dependent growth. In fact, if growth depends on size, initially slow growing trees tend to get cut at a later age than initially fast growing trees. Intuitively one does not now cut "slow growers" to try for better growth in subsequent rotations but, rather, one cuts early those trees that attain large sizes early, since these can be profitably replaced with smaller trees which have better growth potential. The issue of selecting between "age" and "size" dependent growth specifications depends, in part, on which of these two possible approaches to a harvest rule seems more plausible for a particular biological asset.

(iii) One case where problem (18) can be solved analytically occurs when growth is deterministic and only price evolves stochastically. This is an important practical situation.⁽¹⁹⁾ In this case the dynamics (19) simplify to an ordinary differential equation:

$$du = A(u,t)dt. \quad (23)$$

The MLA rule in this event implies a stopping boundary:

$$g(t) = B(X(t) + \theta)/X(t). \quad (24)$$

Since the R.H.S. of (24) is a function of time, the MLA rule implies cutting at some fixed age $\bar{\tau}^*$. Furthermore by similar methods to those used in the single rotation case, this rule can be shown to be optimal. Note the stopping set (the set $\tau: \tau \geq \bar{\tau}^*$) is a "closed" set in the sense of Ross [1970, Page 188].

To determine the value of $\bar{\tau}^*$, consider the land expectation value using a cutting age τ , assuming initial size X_0 :

$$\begin{aligned} V(P_0, X_0) &= E \left\{ \sum_{i=1}^{\infty} \exp(-\delta i \tau) P(i\tau) X(\tau) \mid P(0) = P_0 \right\} \\ &= X(\tau) E \left\{ \sum_{i=1}^{\infty} \exp(-\delta i \tau + q(i\tau)) \mid P(0) = P_0 \right\} \end{aligned}$$

$$\begin{aligned}
&= X(\tau) \sum_{i=1}^{\infty} \exp(-\delta i\tau + q_0 + b(i\tau) + \frac{1}{2}\sigma_q^2(i\tau)) \\
&= P(0)X(\tau)e^{-B\tau}/(1-e^{-B\tau}).
\end{aligned} \tag{25}$$

Setting the derivative of this equal to zero gives the following condition for the optimal cutting age τ^* :

$$X'(\tau^*) = BX(\tau^*)/(1-\exp(-B\tau^*)) \tag{26}$$

which, by comparison with (11), is immediately seen to be a version of the Faustmann formula where the discount rate δ is replaced by the "uncertainty adjusted" discount rate $B = \delta - b - \frac{1}{2}\sigma_q^2 < \delta$.

To sum up: with deterministic growth the optimal cutting rule can be expressed as a barrier on age or, because growth is non-random, as a barrier on size. In each case the barrier is independent of current prices. Such cutting rules can be interpreted as stochastic versions of the Faustmann rule where the effective discount rate is reduced to $B = \delta - b - \frac{1}{2}\sigma_q^2$. In fact, as in the single-rotation case price uncertainty only influences the cutting age and the land expectation value through its effect on the effective discount rate. (20)

Empirically the relative effects of price uncertainty on cutting age and land expectation values can be evaluated using growth function data. This is carried out in Tables 1, 2 for forestry data using the net stumpage value figures for a "typical" stand of British Columbia Douglas Fir trees provided by Pearse [1967]. (21) The general conclusion is that increased price uncertainty can substantially increase land expectation values even though the effect on optimal cutting ages is small.

6. Conclusions and Final Remarks

This paper derives harvest rules for biological assets exhibiting age-dependent growth under conditions of price and growth uncertainty. Over a single rotation a myopic look ahead rule is shown to be optimal and to be equivalent to the familiar Wicksell rule using an "uncertainty adjusted" discount rate. The rule implies biological assets should be harvested at a fixed age τ_M with the size of the asset and its price at harvest time being lognormally distributed random variables. The rule provides an analytically tractable expression for the asset's expected value prior to harvest, and sensitivity results are derived which show how harvest age and market value vary with discount and growth rates, and with levels of price and growth uncertainty. The general results are consistent with earlier work in the field and yet involve no arbitrary boundary conditions. Moreover the results, by distinguishing between price and growth uncertainty, are less restrictive than studies which focus on a single random revenue variable, even though specific forms of the individual processes are now used.

For ongoing rotations our results are less complete since, in general, harvest decisions will depend on both the size and age of an asset. Using a myopic-look-ahead rule as a heuristic approximation to the optimal harvest rule enables this dependence on size and age to be illustrated. One now cuts trees at a random age $\tau^*(\omega) < \tau_M$. Initially slow-growing assets tend to be harvested quickly so that resource managers can try their luck with succeeding rotations. In the particular, though important case, where only price is random the myopic-look-ahead rule becomes optimal. Furthermore this rule is independent of current price (as is the optimal cutting rule in the single-rotation case)⁽²²⁾ and is determined by a stochastic extension of the Faustmann rule.

The analysis throughout has emphasized the crucially sensitive dependence of harvest rules on the stochastic specification of price and growth processes. In particular, as argued in a companion paper (Reed and Clarke [1987]), the results above change fundamentally if a size-dependent growth specification is adopted. The present study also shows that considerable care needs to be exercised in specifying the price and growth processes.

While the issue of specification might seem to be purely an empirical issue it is hoped that the analysis given here offers some useful insights. Since very definite harvest rules are implied by particular specifications, the plausibility of the specifications themselves can, in part, be evaluated in terms of the plausibility of the associated harvest rule. For example, if an age-dependent/single-rotation model does not prescribe harvesting "runts" while common sense suggests that they should be harvested to make way for potentially more productive organisms, then one might consider either reformulating the growth specification or perhaps reformulating the policy task to account for ongoing rotations. Another example of this type of reasoning is our rejection of the "warrant pricing" specification (LB). This seems implausible for resource assets because it implies that in the absence of positive mean drift, asset prices and asset sizes will tend in probability to zero. It also suggests that price uncertainty has no effect on asset harvest policies or market valuations.

In fact it is the interplay between the prescribed form of policy rule and the model specification that can stimulate endeavours to derive improved understanding both of biological growth processes themselves and the efficient utilization of biological assets.

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Footnotes

- (1) These studies are essentially applications of optimal stopping techniques to a stylized "tree cutting" problem. Earlier applications of such techniques to asset management issues are given by McKean [1965], Samuelson [1965], Boyce [1970]. For a useful discussion of one-dimensional continuous stopping problems see Van Moerbeke [1975], [1976]. Multi-dimensional problems are discussed in Krylov [1980], Shiriyayev [1978].
- (2) This statement takes for granted the relevant existence, second-order and uniqueness conditions.
- (3) The ability to distinguish between non-decreasing and possibly decreasing processes is a strength of discrete-time models that is of particular interest to resource economists. Over economically interesting time horizons many biological assets do not decrease in size through, of course, their value may decrease. If size does diminish this may be due to catastrophic environmental events (e.g. forest fires) rather than random decrements in growth.
- (4) These points are emphasized in Malliaris and Brock [1982, Pages 199-205].
- (5) The assumption of geometric Brownian motion for price implies that at any time price will be lognormally distributed. This is the natural specification to make in econometric studies which assume constant elasticities and multiplicative risk. The assumption has thus become popular

in the "price stabilization" literature: see e.g. Turnovsky [1976], Newbery and Stiglitz [1981]. The assumption that asset prices follow geometric Brownian motion lies at the heart of much recent financial economics literature: see e.g. the option pricing models of Black and Scholes [1973], Merton [1973].

(6) Note e.g. the comments in Moran [1968, Page 318].

(7) "Size" here might indicate useful weight (e.g. kilograms of biomass) or useful volume (e.g. board feet of lumber). Alternatively, for assets where unit value varies with age (e.g. timber), it might reflect "size" adjusted for variation in value with age.

(8) Note that if the Stratonovich definition of a stochastic integral (see e.g. Arnold [1974, Page 167]) is employed the equations (1) and (1B) are equivalent, (while the equations (1) and (1A) differ). thus if we start off with the price equation

$$dP = bP dt + \sigma_P P dw_q$$

the choice between the two specifications (1A) and (1B) is equivalent to that of deciding whether the Stratonovich or Itô integral should be employed. Our argument that the former is more appropriate in this context is phenomenological. For a further discussion of this issue see Reed and Clarke [1987]. For a more general discussion of the modelling uses of Itô and Stratonovich calculi see Mortensen [1969], Turelli [1977] and Sethi and Lehoczky [1981].

(9) The MLA rule is discussed by Ross [1970, Page 188]. It does not in general determine the optimal stopping time. Counterexamples to the

optimality of such rules are well known: see e.g. Chow, Robbins and Siegmund [1971, Pages 54-55]. See also Reed and Clarke [1987], where it is shown that, for the case of size-dependent growth, the MLA rule is not optimal.

(10) More precisely an optimal stopping rule does not exist in this case. Never stopping the process is non-optimal since a positive return is yielded by stopping the process in finite time. Yet, it is never optimal to stop in any finite time T since it always pays to wait an additional period.

(11) In fact, the stochastic Wicksellian "tree cutting" rule is equivalent to the myopic rule (5) in the sense that it is both implied by and can be shown to imply the myopic rule.

(12) A theorem of Ross [1970, Page 188] would indicate that the MLA rule is optimal. The theorem (stated without proof by Ross) says that if the MLA stopping set for a continuous-time problem is closed, in the sense that once the process has entered the set it can never leave it again, then subject to certain regularity conditions holding, it is optimal. However since no precise statement or proof of the theorem appear to have been published we eschew its use and prove the optimality of the MLA rule directly. Some of the intermediate results used in the proof are useful in deriving comparative static results.

(13) The hazard function $h(t)$ is defined as

$$\lim_{\Delta \rightarrow 0} \{P(\text{asset destroyed in } (t, t+\Delta) | \text{alive at } t) / \Delta\}.$$

(14) It can be verified directly that this value function satisfies the relevant H-J-B equation

$$(h(t)+\delta)W = \frac{1}{2}\sigma_z^2 W_{zz} + g(t)W_z + W_t$$

and the continuity and smooth-pasting conditions $W = e^z$, $W_z = e^z$ and $W_t = 0$ on the boundary $t = \tau$. Thus $t = \tau$ is the optimal stopping boundary.

(15) There seems no a priori reason to assume correlation between prices and quantities here. While the price prevailing post-harvest might be influenced by quantity (though this is not modelled here) it seems tenuous to suppose the pre-harvest price is influenced by quantity.

(16) It should be emphasized that the form of the specification (1A) or (1B) is important only for analytic purposes — specifically for determining the effects of increased random variability on the harvest age and market value of the asset. From an operational viewpoint the parameters b and σ_q^2 would statistically be estimated simultaneously by fitting a discretized form of (1) or (1B) to an observed time series of $\log(\text{price})$. The estimate \hat{b} of b using specification (1) would be the same as that of $\left[b - \frac{1}{2}\sigma_q^2\right]$ using (1B). A similar situation prevails with respect to estimating the growth parameters. Thus, regardless of the specification chosen, the estimated optimal harvest age would be the same, as would be estimated market value.

(17) The seminal study here is Faustmann [1849]. For a modern treatment see Samuelson [1976].

(18) It can be shown from a theorem of Miroshnichenko [1975, Page 388] that the MLA stopping set contains the optimal stopping set, and that the optimal stopping set is connected.

(19) In fact, for most biological assets with which we are familiar, growth variation seems to be considerably less than price variation.

(20) Miller and Voltaire [1980], [1983] provide an "ongoing rotations" extension to the Brock, Rothschild and Stiglitz [1983] capital theory model. They do not, however, distinguish asset size and unit intrinsic value. The barrier rules they derive depend on the stochastic revenue obtained rather than "age" or "size".

(21) The smoothed data used here is cited in Clark [1976, Page 261]. We fit the polynomial regression:

$$X(T) = 233.02 - 21.99T + 0.534T^2 - 0.00245T^3$$

(1.9) (3.8) (6.5) (6.8)

where t-statistics are in brackets. The polynomial had good fit properties $R^2 = .998$, S.E. = 20.1, D.W. = 1.63 but gave rise to slightly different optimal rotation lengths to those provided by the Clark analysis.

(22) The fact that the optimal cutting rule is independent of current price is a consequence of using a geometric Brownian motion specification, of assuming risk-neutrality (or at least assuming an iso-elastic utility) and of ignoring costs. With fixed harvest costs we conjecture that the optimal cutting rule will depend on price together with age (single rotation) or price together with both age and size (ongoing rotations). With constant

carrying costs for the asset (e.g. as for livestock) we conjecture the optimal stopping rule will depend on price, age and size for the single rotation problem, but that for the ongoing rotations problem will depend on size and age but not on price. These conjectures are based on the use of myopic-look-ahead rules for the respective optimal stopping problems.

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Table 1: Optimal Rotation Length (Years)^(*)

	$\delta = .02$.04	.06	.08	.10
$\sigma_q^2 = 0$	81.0	64.3	55.1	49.7	46.2
.0001	81.1	64.4	55.1	49.7	46.2
.001	81.6	64.6	55.3	49.8	46.9
.01	86.8	67.6	57.0	50.8	46.9
.05	—	86.8	67.6	57.0	50.8

Table 2: Land Expectation Values^(*)

	$\delta = .02$.04	.06	.08	.10
$\sigma_q^2 = 0$	161.9	31.2	8.9	3.0	1.2
.0001	162.8	31.3	8.9	3.1	1.2
.001	170.4	32.3	9.2	3.1	1.2
.01	279.8	44.7	11.9	3.9	1.5
.05	—	279.8	44.7	11.9	3.9

(*) Prices are assumed to evolve with zero drift, $b = 0$. For $\sigma_q^2 \geq 25$ the relevant land expectation values diverge so solutions exist only when $\sigma_q^2 < 25$. Optimal rotations and land expectation values are computed using a Newton algorithm applied to (26) with $X(t)$ a third-degree polynomial in t fitted by least squares. The data are net stumpage values for a "typical" stand of British Columbia Douglas Fir (compare Pearse [1967]).

Figure Captions

Figure 1. Time paths of expected and median price, under alternative specifications (equations (1A) and (1B)), assuming that the drift coefficient b is zero.

Figure 2. Application of the MLA procedure to the ongoing rotations problem. The heavy curve ∂S shows the MLA stopping boundary, while $X(t, \omega)$ and $\tau^*(\omega)$ show respectively a particular sample path for size, and the corresponding (random) harvest age. Also shown is the optimal single-rotation cutting age τ_M .



