Asymptotic Behaviour of Dynamical Systems

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Nolan William Evans
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[redacted]

Professor Reinhard Illner

[redacted]

Professor Roger Davidson

[redacted]

Professor Behrouz Tabarrok

[redacted]

Professor Charles Picciotto

Supervisor: Professor Reinhard Illner

Abstract

The motivation for the study of dynamical systems is shown to have both physical and mathematical aspects resulting from the fact that all of dynamical systems theory is characterisable in terms of a physical/mathematical dichotomy. While the most intuitive approach is a physical one, some of the aspects of dynamical systems discussed are not obvious until they are examined from a mathematical perspective.

It is found that asymptotic invariance is an appropriate method of studying dynamical systems, the resultant discussion centering on stability — resistance to small perturbations of the system. Two types of stability — structural stability and Zeeman stability — are examined and compared.

An important property in the study of dynamical systems is hyperbolicity. The relationship of this property to structural stability is discussed and a characterisation of strange attractors for hyperbolic systems is examined.

In conclusion, some areas which seem to warrant a more detailed examination are mentioned.

Examiners:

[redacted]

Professor Reinhard Illner

[redacted]

Professor Roger Davidson

[redacted]

Professor Behrouz Tabarrok

[redacted]

Professor Charles Picciotto

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1

Introduction

In recent years dynamical systems theory has served as a basis for considerable interdisciplinary study. Many mathematical models, of phenomena ranging from galactic clusters to the ecological interactions of a predator and its prey, can be treated as dynamical systems, and thus it is possible to apply results obtained in one field to another. However, modelling is not the sole motivation for the study of dynamical systems. Many aspects of dynamical systems — strange attractors for example — have a certain aesthetic appeal; one of the best reasons for doing mathematics.

These two motivations characterise the central dichotomy of dynamical systems theory — the purely physical vs. the purely mathematical. It occurs in various contexts, the common theme being dynamical systems and our method of studying them act as a 'middle ground' between the two views. This then is the primary reason for the study of dynamical systems: as a method of resolving the physical/mathematical dichotomy which arises in certain situations.

What precisely is a *dynamical system*? Again the dichotomy. On one hand we have a physical characterisation: a collection of related objects — a *system* — evolving with time — a *dynamical* system. On the other hand, the most general mathematical characterisation of a dynamical system is, [8]

> *"... a flow on a differentiable manifold arising from a vector field, regarded as a map*
>
> $$f : M \to TM$$
>
> *where TM is the tangent bundle of M."*

It is not at all obvious that these two characterisations are equivalent. If we regard them as being valid for different levels of abstraction, the physical the least abstract and the mathematical the most, then an intermediate

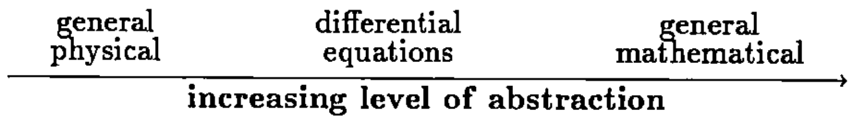

Figure 1.1: Relative level of abstraction of characterisations of dynamical systems.

level of abstraction can be used to provide a link between the two. The characterisation of dynamical systems at this intermediate level is differential equations (see figure 1.1).

It is now straightforward to demonstrate the equivalence of the physical and mathematical characterisations. In the physical characterisation we consider the collection of objects to be explicitly defined by a finite vector of variables — the *system state* — whose evolution with time is described by a finite system of differential equations (there are infinite dimensional results but we shall not be examining them). With the mathematical characterisation, the vector field represents a system of differential equations, the manifold on which it operates a collection of all possible system states — the *state* or *phase space*. Thus differential equations provide us with a basis for our study of dynamical systems from which we can easily move to the realm of the purely physical or more general mathematical.

This physical/mathematical dichotomy is the result of a deeper contrast than physical science vs. abstract mathematics. Consider the universe. It is usually viewed as a collection of physical objects: cars, trees, planets, galaxies, sub-atomic particles, etc., all interacting with each other. But there is another equally valid view of the universe — the universe as information. Everything — physical or not — is information. The information which this page represents includes not only the words printed on it, but the physical structure of the fibres of which it is made as well: their length, shape, colour, etc.. In fact, the entire page could be reduced to a string of symbols existing in some abstract mathematical space which, when properly interpreted, would describe the page explicitly. Thus we may view the universe as information operating on information; this is *complexity theory*. So the role of dynamical systems theory is as a bridge between the information (complexity theory) universe and the physical (physics, chemistry, etc.) universe.

The physical universe is the classical point of view, the information universe is a by-product of the rise of the computer. Previously, mathematical models arose only as simulations of physical situations, but we now study information with models of its complexity, (of course the interesting cases are the more complex ones). There are many definitions of complexity (see Pagels [17] for an excellent treatment), the complexity of a piece of information generally being related to the length of its shortest description. So a quantitative study of information usually involves the manipulation of long strings (for example binary numbers), a procedure which is practical only with the aid of computers. These computer (complexity theory) based models have led to insights into dynamical systems theory just as physically based models have in the past.

As an example consider two systems: a planet orbiting a sun, and a double pendulum in three dimensions. Both systems obey laws of motion which are described by reasonably simple systems of ordinary differential equations. In the case of the orbiting planet the system can be solved to obtain a closed form solution giving the position of the planet as an explicit function of time. However with the pendulum, the exact position of the lower bob at some arbitrary time can be obtained only by following the entire path of the lower bob from the beginning of its motion. There is no closed form solution. Thus we have information of high complexity (the path of the bob, whose shortest description is itself) being generated by information of low complexity (the system of differential equations), an interesting situation which is not obvious from the physical point of view.

While both views of the universe lead to avenues of inquiry in dynamical systems theory, we shall focus on those which originate in the physical realm. This approach is used because it is more intuitive and thus provides a better grasp of the concepts to be discussed. The procedure to be used in our study of dynamical systems is to introduce a concept in general terms by means of a physical motivation, discussing the implications of these physical aspects. Then the concept is defined in precise mathematical terms and the resulting implications discussed. This thesis is divided into two major parts to facilitate this approach. The first part is primarily expository, dealing with background terminology explicitly, while discussing motivations and methods of the study of dynamical systems in rather general terms. The second deals with the mathematical details of our chosen method of studying dynamical systems — stability.

The mathematical background assumed for this thesis is a familiarity with the basics of ordinary differential equations, differential geometry, and topology. Most terms are defined as they are introduced. However there are some for which a proper definition would be far too lengthy, interrupting the flow of the topic being discussed. Books which cover the basic concepts in more detail include: Arnol'd [4], Chillingworth [6], Guckenheimer and Holmes [8], Hirsch [9], Hirsch and Smale [10], Ruelle [34], and Wiggins [41].

The thesis proceeds as follows. Part one introduces the terminology of dynamical systems and provides the physical motivation for studying stability. Part two details two stability classes as well as demonstrating one of the more interesting types of behaviour which can arise in dynamical systems. We conclude with a discussion of several areas of dynamical systems theory which seem worthy of further examination.

Part I

2

Basic Terminology of Dynamical Systems

Unless specified otherwise, notation and symbol usage in this thesis is consistent with first appearances.

2.1 Flows and Maps

Dynamical systems can be studied in terms of either continuous or discrete time. For the continuous case the evolution of the time dependent system state $x = x(t) \in \mathbb{R}^n$, is described by a system of autonomous ordinary differential equations

$$\frac{dx}{dt} \stackrel{\text{def}}{=} \dot{x} = f(x). \tag{2.1}$$

(While it is possible to use non-autonomous or even partial differential equations, we shall not examine such situations.) Here $f \in C^r(M, TM)^\dagger$, $r \geq 1^\ddagger$, is a nonlinear function defined on a compact differentiable manifold, M, which maps M to its tangent bundle TM. f is a *vector field* and as such generates a *flow*, $\phi_t(x) = \phi(t,x) \in C^r(I \times M, M)$; with $I = (a,b) \subseteq \mathbb{R}$ some time interval (not necessarily finite) over which the flow is defined. It obeys the semigroup properties: $\phi_0 = \text{id}$ and $\phi_t \circ \phi_s = \phi_{t+s}$.

The flow is considered to be the set of all solutions to (2.1) in that

$$\left.\frac{d}{dt}(\phi(x,t))\right|_{t=\tau} = f(\phi(x,\tau)), \qquad \tau \in I.$$

With an initial condition $x(0) = x_0 \in M$ the set $\gamma = \{\phi_t(x_0)\}_{t \in I}$ is referred to as a *solution curve* or *trajectory* or *orbit* for the system.

Regarding the existence and uniqueness of ϕ_t we have the following [10, p.162]

---

† $C^r(A,B)$ is the space of r-times continuously differentiable functions from A to B. If $B = A$ we may also use the notation $C^r(A) = C^r(A,A)$.

‡ When r appears in this context we mean for any integer $r \geq 1$ unless otherwise noted.

Theorem 2.1 *Let $U \subset M$ be an open subset of a differentiable manifold M, $f: U \to M$ a C^1 map, and let $x_0 \in M$. Then there exists $c > 0$ and a unique solution $\phi(x_0, \cdot): (-c, c) \to U$ of (2.1) for the initial condition $x(0) = x_0$.*

Thus given our definition of a dynamical system, a unique local solution will always exist. Extension to a global solution is straightforward (see Hirsch and Smale, [10, Ch.8]); and in the case of compact M, automatic [6].

For discrete dynamical systems the system state is written $x_n = x(t_n)$ for some fixed sequence $\{t_0, t_1, t_2, \ldots\} \subset I$ (usually $t_n = n\Delta t$ for some fixed Δt). The system evolution is described by

$$x_{n+1} = F(x_n), \tag{2.2}$$

where $F \in C^r(M, M)$, $r \geq 1$, is a nonlinear map of the state space into itself. It is important with discrete dynamical systems to note whether or not F has an inverse. Most results require $F \in \mathrm{Diff}^r(M)$ ($\mathrm{Diff}^r(M)$ is the space of C^r–diffeomorphisms from M to itself), that is an r–times differentiable inverse. This is not as restrictive as it may seem; in fact all of the discrete dynamical systems in this thesis shall be diffeomorphisms (we shall comment on the non-invertible case in the final chapter).

When referring to continuous dynamical systems (2.1) we use either *'vector field'* or *'flow'*, for discrete dynamical systems (2.2) we use *'diffeomorphism'* or *'map'*, and when either is applicable *'dynamical system'* or just *'system'* is used. Exceptions to this should be obvious from the context.

2.2 The State Space

For convenience we consider the state space M to be a compact differentiable manifold. Most of the results for compact manifolds can also be obtained for the non-compact case however the technical details are more complex. M can always be embedded in $\mathbb{R}^N$, $N = 2m + 1$ (m is the dimension of M) [9], and in fact we shall more often than not assume M to be a bounded subset of $\mathbb{R}^n$. As to differentiability, some results require M to be smooth (where by *smooth* we mean C^∞) however there are results which only require M to be C^r, $r \geq 1$. Unless otherwise noted, the smooth case is assumed.

2.3 Periodicity

Periodic behaviour is a common feature of dynamical systems. The simplest type of periodic behaviour is the ***fixed point***, a point $\bar{x} \in M$ such that $f(\bar{x}) = 0$ (for continuous systems), or $F(\bar{x}) = \bar{x}$ (for discrete systems). Beyond this, for continuous dynamical systems γ_T is a ***periodic orbit*** of period $T \in I$, if $\phi_t(x_0) = \phi_{t+nT}(x_0)$, for any $x_0 \in \gamma_T$, $0 \le t \le T$, and $n \in \mathbb{Z}$. Thus $\gamma_T = \cup_{t\in[0,T]}\phi_t(x_0)$. Similarly, for a discrete system γ_p is a periodic orbit of period $p \in \mathbb{Z}^+$ if $F^p(x_0) = x_0$, where

$$F^p(x_0) = \underbrace{F(F(\cdots F(x_0)))}_{p \text{ times}}.$$

In this case $\gamma_p = \{x_0, x_1, \ldots, x_{p-1}\}$, where $x_i = F^i(x_0)$. Note that we can create a new map $G = F^p$ having p fixed points $\{x_0, x_1, \ldots, x_{p-1}\}$ when the map F is periodic of period p. Note also that the periodic orbit for the discrete system is a finite set of distinct points, while that for the continuous system is a continuous curve in M.

We shall also need the following weak form of periodicity.

Definition 2.1 *An orbit γ is* **recurrent** *if $\gamma \subset \omega(\gamma)$ or $\gamma \subset \alpha(\gamma)$.*

Where we define $\omega(x)$ (the omega limit set) and $\alpha(x)$ (the alpha limit set) of the point x under the flow ϕ_t by

$$\begin{aligned} \omega(x) &= \{y \in M \mid \phi_{t_n}(x) \to y \text{ for some sequence } t_n \to \infty\}, \\ \alpha(x) &= \{y \in M \mid \phi_{t_n}(x) \to y \text{ for some sequence } t_n \to -\infty\}, \end{aligned}$$

and for maps

$$\begin{aligned} \omega(x) &= \{y \in M \mid F^{n_i}(x) \to y \text{ for some sequence } n_i \to \infty\}, \\ \alpha(x) &= \{y \in M \mid F^{n_i}(x) \to y \text{ for some sequence } n_i \to -\infty\}. \end{aligned}$$

Examples

At this point we shall illustrate some of the concepts discussed. The examples used will not necessarily be models of a specific physical system, they are used for their illustrative ability. As more concepts are introduced we shall return to these examples.

EXAMPLE 1: Consider the continuous dynamical system defined by

$$f(x) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 + x_1^2 \end{bmatrix}, \qquad (2.3)$$

on the manifold $M = \mathbb{R}^2$. This system has a single unique fixed point at $\bar{x} = [0,0]^T$. We integrate the system to obtain the flow

$$\phi_t(x_0) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_{1_0}e^t \\ -x_{2_0}e^{-t} + \frac{1}{3}x_{1_0}^2(e^{2t} - e^{-t}) \end{bmatrix}, \quad (2.4)$$

where $x_0 = x(0) = [x_{1_0}, x_{2_0}]^T$. Thus we have $I = \mathbb{R}$. Note that $\phi_t(\bar{x}) = \bar{x}$, $\forall t \in I$.

EXAMPLE 2: Consider the discrete dynamical system

$$F(x_n) = 1 - \frac{9}{10}x_n^2 \quad (2.5)$$

on the manifold $M = [0,1] \subset \mathbb{R}$ (thus $F \in \mathrm{Diff}^\infty(M)$). There is a single fixed point in M, $\bar{x} = -5/9 + \sqrt{115}/9 \simeq 0.636$. We also have a period two orbit $\gamma_2 = \{\sim 0.125, \sim 0.986\}$. Thus the function

$$F^2(x_n) = -\left(\frac{9}{10}\right)^3 x^4 + \frac{81}{50}x^2 + \frac{1}{10} \quad (2.6)$$

has two fixed points $\bar{x}^1 \simeq 0.125$, $\bar{x}^2 \simeq 0.986$. We can generate orbits of arbitrary period by solving $\bar{x} = F^n(\bar{x})$. However for our purposes $n = 2$ is sufficient.

2.4 Equivalence of Flows and Maps

It is often convenient to work with maps instead of flows and vice versa. This is because maps operate in spaces one dimension less than the equivalent flows, while flows — being continuous — are more reasonable models of most physical processes.

The obvious way of defining a diffeomorphism F in terms of a flow ϕ_t is

$$F^n(x_0) = \phi_{n\Delta t}(x_0),$$

that is by sampling a trajectory from the flow at fixed time intervals. A more general method is the *Poincaré* or *first return map*. The Poincaré map can arise in several situations; we shall demonstrate the simplest case — near a periodic orbit.

Consider a *local cross section* $\Sigma \subset M$ (a cross section is a compact submanifold of codimension one) everywhere *transverse* to the flow ϕ_t. By transverse we mean that $f(x) \cdot n(x) \neq 0$, $\forall x \in \Sigma$ ($n(x)$ is the unit normal to Σ at x). For a closed orbit γ_T of ϕ_t let $p = \gamma_T \cap \Sigma$ be the unique intersection of γ_T with Σ (Σ may be modified, if necessary, to ensure the intersection is unique). For some neighborhood $V \subset \Sigma$ of $p \in \Sigma$ we define the Poincaré map $P\colon V \to \Sigma$ by

$$P(q) = \phi_\tau(q) \quad , \quad q \in V,$$

where $\tau > 0$ is the first time for which $\phi_\tau(q) \cap \Sigma \neq \emptyset$. We do not necessarily have $\tau = T(p)$, the period of γ_T, however $\tau \to T(p)$ as $q \to p$.

A flow is obtained from a diffeomorphism by means of a suspension. On a smooth manifold M the suspension of $F \in \mathrm{Diff}^r(M)$ is a vector field $f \in C^r(\tilde{M}, T\tilde{M})$ [21] on the manifold $\tilde{M} = M \times [0,1]/\sim$ (where $\sim$ means $(x,1) = (F(x),0)$, for each $x \in M$). The resultant flow is

$$\phi_t(x) = \begin{cases} (x,t) & , \ t \in [0,1] \\ (F(x), t-1) & , \ t \in [1,2] \\ (F^{-1}(x), t+1) & , \ t \in [-1,0] \\ \text{etc.} \end{cases} .$$

2.5 Linearised Systems

We study the local behaviour of nonlinear dynamical systems by linearising. If we have a fixed point $\bar{x}$ of the system (2.1) we write $x = \bar{x} + \xi$, $|\xi| \ll 1$ for $x \in U$, $U \subset M$ a small neighborhood of $\bar{x}$. Substituting into (2.1) gives

$$\dot{\xi} = Df(\bar{x})\xi, \tag{2.7}$$

where $Df(\bar{x}) = \left[\frac{\partial f_i}{\partial x_j}\right]$ is the Jacobian matrix of first partials of f and we have ignored second and higher order terms. This allows us to obtain the linearised flow

$$D\phi_t(\bar{x})\xi = e^{tDf(\bar{x})}\xi.$$

Similar results are obtained for (2.2) by letting $x_n = \bar{x} + \xi_n$ for $x_n \in U$, giving

$$\xi_{n+1} = DF(\bar{x})\xi_n. \tag{2.8}$$

The eigenvalues of the linear matrix ($Df(\bar{x})$ or $DF(\bar{x})$) describe the behaviour of the nonlinear system near $\bar{x}$ according to the following theorem (for a proof see [16, p.80]).

Definition 2.2 *If the system (2.7) evaluated at the fixed point $\bar{x}$ has no eigenvalues with zero real part we say $\bar{x}$ is a* **hyperbolic fixed point**. *Similarly for (2.8), $\bar{x}$ is hyperbolic if there are no eigenvalues of modulus one.*

Theorem 2.2 (Hartman – Grobman) *If $\bar{x}$ is a hyperbolic fixed point then there is a homeomorphism h defined on some neighborhood $U \subset M$ of $\bar{x}$ locally taking orbits of the nonlinear flow ϕ_t (nonlinear map F) of (2.1) (resp. (2.2)) to those of the linear flow of (2.7) (linear map (2.8)). The homeomorphism preserves the sense of orbits and can also be chosen to preserve parameterisation by time.*

2.6 Invariant Manifolds and Subspaces

By theorem 2.2 we have a local equivalence between dynamical systems and their linearised versions. This equivalence can be characterised more precisely in terms of *invariant manifolds and subspaces.*

For the linearised system (2.7) there exists a local splitting of the tangent bundle into three invariant eigenspaces

$$\begin{aligned} E^s &= \text{span}\{\text{eigenvectors for eigenvalues with negative real part}\}, \\ E^u &= \text{span}\{\text{eigenvectors for eigenvalues with positive real part}\}, \\ E^c &= \text{span}\{\text{eigenvectors for eigenvalues with zero real part}\}. \end{aligned}$$

Similarly for (2.8) we have

$$\begin{aligned} E^s &= \text{span}\{\text{eigenvectors for eigenvalues with modulus} < 1\}, \\ E^u &= \text{span}\{\text{eigenvectors for eigenvalues with modulus} > 1\}, \\ E^c &= \text{span}\{\text{eigenvectors for eigenvalues with modulus} = 1\}. \end{aligned}$$

These are, respectively, the *stable* (or *contracting*), *unstable* (or *expanding*) and *center* (or *neutral*) *subspaces.*

For the fully nonlinear system, we define *local stable* and *unstable manifolds*, $W^s_{\text{loc}}(\bar{x})$ and $W^u_{\text{loc}}(\bar{x})$, for some suitably small neighborhood $U \subset M$ of $\bar{x}$ by

$$\begin{aligned} W^s_{\text{loc}}(\bar{x}) &= \{x \in U \mid \phi_t(x) \to \bar{x} \text{ as } t \to \infty, \text{and } \phi_t(x) \in U, \forall t \geq 0\}, \\ W^u_{\text{loc}}(\bar{x}) &= \{x \in U \mid \phi_t(x) \to \bar{x} \text{ as } t \to -\infty, \text{and } \phi_t(x) \in U, \forall t \leq 0\}, \end{aligned}$$

for continuous systems, and,

$$\begin{aligned} W^s_{\text{loc}}(\bar{x}) &= \{x \in U \mid F^n(x) \to \bar{x} \text{ as } n \to \infty, \text{and } F^n(x) \in U, \forall n \geq 0\}, \\ W^u_{\text{loc}}(\bar{x}) &= \{x \in U \mid F^{-n}(x) \to \bar{x} \text{ as } n \to \infty, \text{and } F^{-n}(x) \in U, \forall n \geq 0\}, \end{aligned}$$

for discrete systems. We can also define local stable and unstable manifolds for closed orbits. In a suitably small neighborhood U of some closed orbit

γ of the flow ϕ_t we have

$$\begin{aligned} W^s_{\text{loc}}(\gamma) &= \{x \in U \mid |\phi_t(x) - \gamma| \to 0 \text{ as } t \to \infty, \text{and } \phi_t(x) \in U, \forall t \geq 0\}, \\ W^u_{\text{loc}}(\gamma) &= \{x \in U \mid |\phi_t(x) - \gamma| \to 0 \text{ as } t \to -\infty, \text{and } \phi_t(x) \in U, \forall t \leq 0\}, \end{aligned}$$

where $|\phi_t(x) - \gamma| = \min_{y \in \gamma} |\phi_t(x) - y|$. For closed orbits of diffeomorphisms we define the local stable and unstable manifolds in terms of the fixed points of F^p, where p is the period of the orbit.

The equivalence between stable and unstable subspaces and manifolds is given by the following theorem.

Theorem 2.3 (Stable Manifold Theorem for a Fixed Point) *If we suppose the system (2.1) (or (2.2)) has a hyperbolic fixed point $\bar{x}$, then there exist local stable and unstable manifolds $W^s_{\text{loc}}(\bar{x})$, $W^u_{\text{loc}}(\bar{x})$, of the same dimensions n_s, n_u as those of the eigenspaces E^s, E^u of the linearised system (2.7) (resp. (2.8)) tangent to E^s, E^u at $\bar{x}$. $W^s_{\text{loc}}(\bar{x})$, $W^u_{\text{loc}}(\bar{x})$ are as smooth as f (resp. F).*

There is also an equivalent center manifold theorem but, as we shall see, nonhyperbolic fixed points are not common.

We also define *global* stable and unstable manifolds. For flows:

$$\begin{aligned} W^s(\bar{x}) &= \bigcup_{t \leq 0} \phi_t(W^s_{\text{loc}}(\bar{x})), \\ W^u(\bar{x}) &= \bigcup_{t \geq 0} \phi_t(W^u_{\text{loc}}(\bar{x})). \end{aligned}$$

For periodic orbits substitute γ for $\bar{x}$. For maps:

$$\begin{aligned} W^s(\bar{x}) &= \bigcup_{n \geq 0} F^{-n}(W^s_{\text{loc}}(\bar{x})), \\ W^u(\bar{x}) &= \bigcup_{n \geq 0} F^{n}(W^u_{\text{loc}}(\bar{x})). \end{aligned}$$

These are injectively immersed C^r submainfolds of M. In the future when referring to stable or unstable manifolds in the future we shall always mean the global case.

Examples

We now determine the invariant subspaces for our two examples.

EXAMPLE 3: First we linearise the system (2.3) about the fixed point $\bar{x} = [0,0]^T$,

$$Df(\bar{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus the linearised flow is

$$D\phi_t(\bar{x}) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

The linearised system matrix has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ and corresponding eigenvectors $v_1 = [1,0]^T$, $v_2 = [0,1]^T$. This gives us the stable and unstable subspaces

$$\begin{aligned} E^s &= \text{span}\{v_2\} = \{(x,y) \in M \mid x = 0\}, \\ E^u &= \text{span}\{v_1\} = \{(x,y) \in M \mid y = 0\}. \end{aligned}$$

It is also possible to obtain closed form equations for the stable and unstable manifolds of this system. Note that this is not usually possible.

$$\begin{aligned} W^s &= \{(x,y) \in M \mid x = 0\} = E^s, \\ W^u &= \{(x,y) \in M \mid y = \frac{x^2}{3}\}. \end{aligned}$$

Notice that W^u is tangent to E^u at $\bar{x}$ (see figure 2.1).

EXAMPLE 4: The linearised system here is

$$DF(\bar{x}) = \left[-1 + \frac{\sqrt{115}}{5}\right] \simeq [1.14].$$

Thus the eigenvalue at $\bar{x}$ is > 1 and so the local space about $\bar{x}$ is unstable. However if we consider F^2 at $\bar{x}^1$ and $\bar{x}^2$ we have

$$D(F^2(\bar{x}^1)) = \frac{4}{10} = D(F^2(\bar{x}^2)).$$

Thus $\bar{x}^1$ and $\bar{x}^2$ are stable fixed points of F^2 which implies γ_2 is a stable periodic orbit of F.

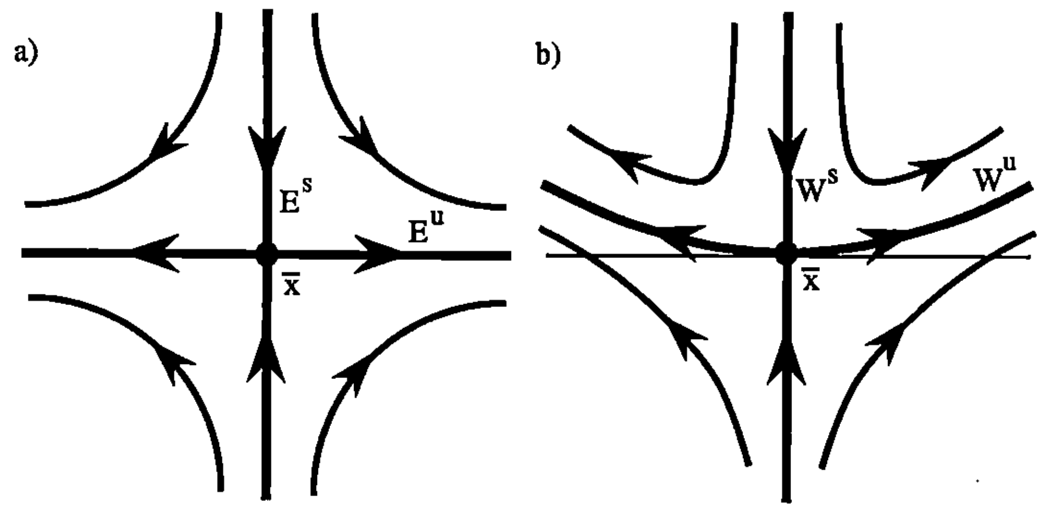

Figure 2.1: The invariant manifolds and subspaces of system (2.3). (a) The linearised system; (b) the non-linear system.

2.7 Intersections of Invariant Manifolds

The global stable and unstable manifolds of hyperbolic fixed points (or periodic orbits) can intersect in various ways. As we shall see, it is these intersections which lead to some of the rich behaviour associated with certain dynamical systems.

First we observe that there are two types of intersections which may occur between manifolds. Let U and V be two submanifolds of the manifold M.

Definition 2.3 *The manifolds U and V have a* **transversal intersection** *if for the set of all points of intersection of U and V, $Z = U \cap V$*

$$T_zU + T_zV = T_zM, \qquad \forall z \in Z,$$

where T_zM is the tangent space of M at z. Otherwise the intersection is **non-transversal.**

It is also useful to define transversality of maps and manifolds. Let $f : N \to M$ be a smooth map and U a submanifold of M.

Definition 2.4 *For a point $x \in N$ we say f is* **transversal** *to U at x if either:*

i) $f(x) \notin U$, or

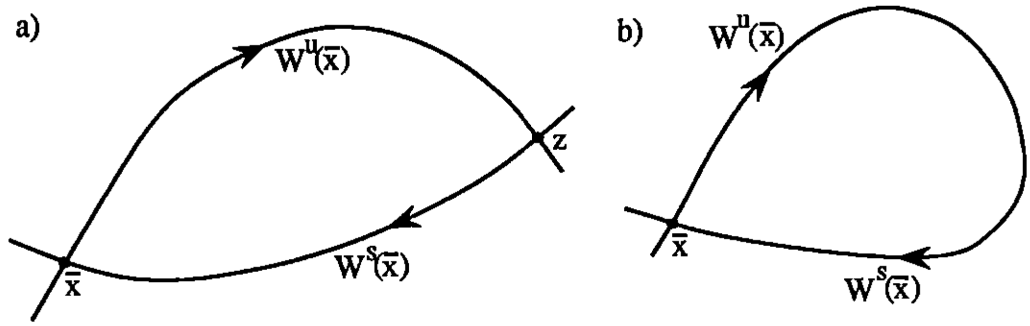

Figure 2.2: Homoclinic points. (a) Transverse; (b) non-transverse.

ii) $T_x f(T_x N) + T_{f(x)} U = T_{f(x)} M$,

where $T_x f : T_x N \to T_{f(x)} M$ *is the tangent map to* f *at* x *[34, p.15].*

Now consider a hyperbolic fixed point $\bar{x} \in M$ with stable and unstable manifolds $W^s(\bar{x})$, $W^u(\bar{x})$.

Definition 2.5 *A point* $z \in W^s(\bar{x}) \cap W^u(\bar{x})$, $z \neq \bar{x}$, *is called a* **homoclinic point**. *The existence of* z *implies a* **homoclinic orbit** *(see figure 2.2).*

If we have a second fixed point $\bar{y} \in M$, $\bar{y} \neq \bar{x}$, then,

Definition 2.6 *A point* $z \in W^s(\bar{x}) \cap W^u(\bar{y})$ *(or* $z \in W^u(\bar{x}) \cap W^s(\bar{y})$*),* $z \notin \{\bar{x}, \bar{y}\}$, *is called a* **heteroclinic point** *(see figure 2.3).*

It is not possible to have stable or unstable manifolds intersecting themselves or others of the same type as this would violate uniqueness of solutions. These results are summarised in table 2.1.

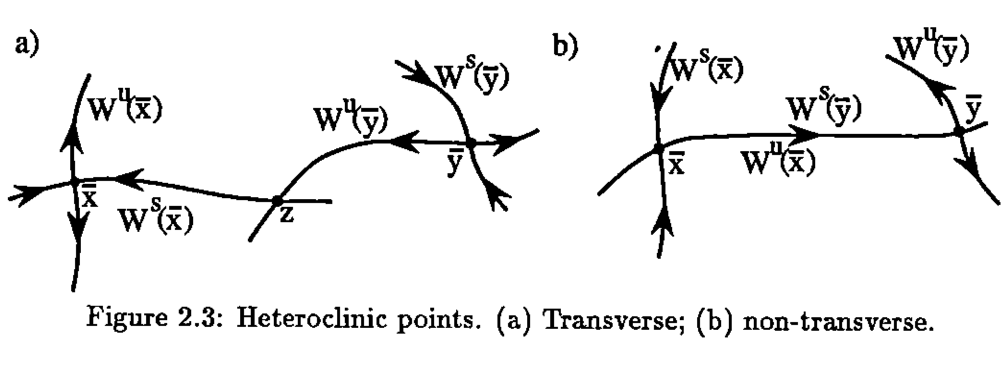

Figure 2.3: Heteroclinic points. (a) Transverse; (b) non-transverse.

$\cap$	$W^s(\bar{x})$	$W^u(\bar{x})$	$W^s(\bar{y})$	$W^u(\bar{y})$
$W^s(\bar{x})$	not allowed	homoclinic	not allowed	heteroclinic
$W^u(\bar{x})$	homoclinic	not allowed	heteroclinic	not allowed
$W^s(\bar{y})$	not allowed	heteroclinic	not allowed	homoclinic
$W^u(\bar{y})$	heteroclinic	not allowed	homoclinic	not allowed

Table 2.1: Intersections of invariant manifolds of the fixed points $\bar{x}$ and $\bar{y}$ ($\bar{x} \neq \bar{y}$).

2.8 Hyperbolicity

We shall extend the concept of hyperbolicity from fixed points to other objects, beginning with the *hyperbolic periodic orbit.* For continuous systems, a periodic orbit $\gamma \subset M$ is hyperbolic if $\bar{p} \in \gamma \cap \Sigma$ is a hyperbolic fixed point of the Poincaré map $P\colon V \to \Sigma$, $V \subset \Sigma$ a neighborhood of $\bar{p}$. An orbit of period n of a discrete dynamical system is hyperbolic if the fixed points of F^n are hyperbolic.

So we define a *hyperbolic dynamical system* as a dynamical system whose fixed points and periodic orbits are all hyperbolic.

3

The Study of Dynamical Systems

We begin our study with a physical intuition, that the behaviour of systems undergoing deterministic change is in some way predictable. This shall be our goal: *methods of consistently predicting the behaviour of dynamical systems.* We must now decide what is meant by 'predicting' and 'behaviour'.

Predictions may be grouped into two distinct time scales: the immediate future, and the far future. In mathematical terms: transient and asymptotic behaviour, respectively. For the moment we take 'behaviour of a dynamical system' to mean the system state and the way it is evolving at the point in time being examined.

While the study of transient behaviour has its place, asymptotic behaviour provides us with the consistency necessary to make comparisons and generalisations in our study of dynamical systems. For example, consider the behaviour of a dynamical system at two significantly different times, t_0 and t_1, say $t_1 = 10t_0$, $t_0 > 0$. It is entirely possible that the behaviour of this system in some neighborhood of t_0, say $(t_0 - \Delta t, t_0 + \Delta t)$, $\Delta t = 0.1t_0$ — the transient behaviour — will be completely different from that in $(t_1 - \Delta t, t_1 + \Delta t)$. Thus transient behaviour does not allow a consistent characterisation of the system.

3.1 Invariance

When studying dynamical systems, consistent behaviour is synonymous with invariance. In general, *invariance* can refer to any aspect of a dynamical system such as a property, a set of operators, or a subset of the state space, the unifying theme being that the aspect under consideration does not change for all time. Invariance is very important in the study of dynamical systems as, for example, it is not valid to make comparisons under constantly changing terms of reference. As we have demonstrated, asymptotic behaviour is invariant and thus a useful means of comparing dynamical systems.

We now consider how asymptotic behaviour is to be analysed. Again for meaningful comparisons we must use invariance, in this case invariant sets.

Definition 3.1 *An* **invariant set** $\Gamma \subset M$ *is a set for which* $\phi_t(x) \in \Gamma$ *(or* $F^n(x) \in \Gamma$*), for all* $x \in \Gamma$ *and* $t \in I$ *(resp.* $n \in \mathbb{Z}$*).*

The weakest class of invariant set is the nonwandering set.

Definition 3.2 *A point* $x \in M$ *is* **nonwandering** *if for any neighborhood* $V \subset M$ *of* x *and* $T > 0$*, there exists* $t > T$ *such that* $\phi_t(V) \cap V \neq \emptyset$ *(note that this implies that the entire* ω*-limit set of* x *is nonwandering). Similarly for maps,* $x \in M$ *is nonwandering if for any neighborhood* $U \subset M$ *of* x *and* $N > 0$*, there exists* $n > N$ *such that* $F^n(U) \cap U \neq \emptyset$*. We denote the set of all nonwandering points* Ω*. It is closed in* M *[20].*

Another very useful type of invariant set is the attractor. As yet there is no universally accepted rigorous definition of an attractor, however the following is a good general characterisation (Milnor [14] quoting Collet and Eckmann [7])

> "*...the set of points to which most points evolve under iterates of* F".

A rigorous working definition will be given later in the thesis. We shall denote an attractor by Λ, note that $\Lambda \subseteq \Omega$.

An attractor can be classified into one of four categories: fixed point, periodic orbit, quasiperiodic orbit, and strange attractor. We have already encountered fixed points and periodic motion. *Quasiperiodic* motion can be regarded as the motion which results from coupling two (or more) oscillators whose frequencies, ω_1 and ω_2, are not rationally related (i.e. ω_1/ω_2 is not rational). The final class of attractor, the strange attractor, can be regarded as 'everything else'. This, of course, is not a satisfactory definition. We shall endeavour to provide a positive definition later.

We now have two types of invariance to work with, invariant manifolds in the state space and invariant sets with respect to asymptotic behaviour. It should be obvious that these are closely related.

3.1.1 Hyperbolic Invariant Sets

The most important invariant sets (for our purposes) are hyperbolic invariant sets.

Definition 3.3 *A compact invariant set* $\Gamma \subseteq M$ *of the system (2.1) has* **hyperbolic structure** *if there is a continuous invariant direct sum decomposition of the tangent bundle of* M *over* Γ *of the form*

$$T_\Gamma M = W_\Gamma^s \oplus W_\Gamma^u \oplus D_t\phi(t)|_\Gamma,$$

such that for some constants C, $\lambda > 0$, *and some Riemannian metric on* M

i) $\|\phi_t'|_{W_\Gamma^s}\| \leq Ce^{-\lambda t}, \quad t > 0,$

ii) $\|\phi_t'|_{W_\Gamma^u}\| \leq Ce^{\lambda t}, \quad t < 0,$

where $T_\Gamma M$ *is the tangent bundle of* M *over* Γ, *and* $W_\Gamma^s = \cup_{x\in\Gamma} W^s(x)$, $W_\Gamma^u = \cup_{x\in\Gamma} W^u(x)$ *are the stable and unstable manifolds over* Γ. *For the system (2.2) the direct sum decomposition is of the form*

$$T_\Gamma M = W_\Gamma^s \oplus W_\Gamma^u,$$

where for constants $C > 0$, $0 < \lambda < 1$ *and any* $n > 0$, *we have*

i) $\|Df^n(x)v\| \leq C\lambda^n\|v\|, \quad$ *for* $v \in W_x^s$,

ii) $\|Df^{-n}(x)v\| \leq C\lambda^n\|v\|, \quad$ *for* $v \in W_x^u$,

for all $x \in \Gamma$ $(W_x^s = W^s(x),\ W_x^u = W^u(x))$.

We may now define a hyperbolic dynamical system as one with an invariant set (attractor or nonwandering set) with hyperbolic structure. Note that the invariant manifolds — which form a large part of the invariant set — are generated from the fixed points and periodic orbits.

3.2 Stability

Now that we have a basis for the comparison of dynamical systems it is necessary to decide which comparisons are relevant. To decide this we again begin with physical intuition by asking: "In physical terms what is a 'good' dynamical system?" We shall consider a dynamical system 'good' if small changes to the system do not change the overall behaviour. An equivalent definition is that similar systems should exhibit similar behaviour. Systems for which this is true are said to be *stable.*

We have introduced several terms here whose physical, let alone mathematical, meaning is not obvious. Working with the second definition of 'good' we say that two dynamical systems are *similar* if one is the result of a small perturbation of the other. A perturbation can be to initial conditions, parameters of the equations, or the equations themselves.

Similarity of behaviour is always determined with respect to the type of perturbation made, as different perturbations affect different aspects of a systems behaviour. Thus, we define stability classes in terms of the perturbations allowed.

3.2.1 Stability of Solutions

The simplest perturbation — and the one with the most direct physical justification — is a perturbation of initial conditions. The system equations are not changed, thus the range of behaviour which must be examined is reduced drastically. In fact, we need only consider the action along individual solution curves. Accordingly, we refer to this stability class as *stability of solutions.*

Consider a dynamical system represented by the flow ϕ_t and some initial conditions for this system, $x_0,\ y_0 \in M,\ x_0 \neq y_0$. The weakest stability of solutions is for two solutions beginning close together to stay close together.

Definition 3.4 *The flow* ϕ_t *is* **Liapunov stable** *if for any* $\varepsilon > 0$ *there exists* $\delta > 0$*, such that if* $|y_0 - x_0| < \delta$ *then* $|\phi_t(y_0) - \phi_t(x_0)| < \varepsilon$*,* $\forall t \in (0, \infty)$.

A stronger type of stability is when solutions beginning close together approach a common trajectory.

Definition 3.5 *The flow* ϕ_t *is* **asymptotically stable** *if there exists* $\varepsilon > 0$ *such that if* $|x_0 - y_0| < \varepsilon$ *then* $\lim_{t\to\infty} |\phi_t(y_0) - \phi_t(x_0)| = 0$.

For example, an asymptotically stable fixed point, a point $\bar{x} \in M$ for which there exists $\varepsilon > 0$ such that if $|x_0 - \bar{x}| < \varepsilon$ then $\phi_t(x_0) \to \bar{x}$ as $t \to \infty$, is the simplest type of attractor. The main principle of stability of solutions is that small changes in initial conditions will only result in small changes to the trajectory.

As we mentioned there is a physical justification for perturbations of initial conditions. Consider the idea that initial conditions in physical experiments are set by physical measurement, and any physical measurement has some uncertainty associated with it. This is reflected mathematically by introducing small perturbations to the initial conditions of the dynamical system modelling the physical system. Since physical experiments are expected to be repeatable (small uncertainties in the initial set-up should not cause gross deviations in the final results), the same should be true of their models.

In many systems, notably linear ones, this is the case. However, as far back as Poincaré and the three body problem [26] it has been known that there are simple, non-pathological systems for which stability of solutions is not possible. With such a system any two solution curves beginning arbitrarily close together will diverge exponentially with time. This behaviour is termed *sensitive dependence on initial conditions.* Note that the divergence of trajectories is not unbounded as the solutions must remain in the state space, and in fact they are usually drawn to the attractor.

We can now make a preliminary definition of a strange attractor.

Definition 3.6 *A* **strange attractor** *is an attractor on which the system exhibits sensitive dependence on initial conditions.*

The first modern example of a strange attractor came from Lorenz† [11]. Using a very simple model of atmospheric convection cells he showed that solutions beginning arbitrarily close together would eventually evolve to totally unrelated states. These states did not wander through all phase space, instead they remained on an invariant submanifold whose geometric structure was independent of perturbations to the system. Sensitive dependence on initial conditions is wholly a result of nonlinearity in the dynamical system and, due to the difficulty of the calculations involved, it is one of the aspects of dynamical systems theory which was not fully appreciated before the development of the computer.

3.2.2 Structural Stability

As we have found the stability defined by perturbations of initial conditions to have a limited range of applicability we now consider perturbations of the system equations and the stability classes thus defined. The first such stability to be developed was structural stability. A C^1 vector field f on a manifold $M \subset \mathbb{R}^2$ is *structurally stable* if there exists a C^1 neighborhood U of f such that every vector field $g \in U$ is topologically equivalent to f and the homeomorphism defining this equivalence is C^0 close to the identity. By *topological equivalence* we mean there exists a homeomorphism h such that $h \circ \phi_t^f = \phi_t^g \circ h$ (ϕ_t^f, ϕ_t^g are the flows generated by f and g). This is the original definition given by Andronov and Pontrjagin [2]. What makes structural stability interesting is that it preserves certain important topological features, and in certain situations is a generic‡ property.

†Unfortunately the Lorenz attractor is not structurally stable (see below), a somewhat discouraging fact given that it is the canonical example of a strange attractor.

‡A property $\mathcal{P}$ of a set of operators $\mathcal{O}$ on M is *generic* if $\mathcal{O}$ is open dense in the set of all operators on M. That is, almost all operators have this property.

Structural stability has very strong ties with hyperbolicity, which can be characterised in general as a splitting of the manifold into incoming and outgoing invariant manifolds. This is a very strong structure on which to base a study of dynamical systems. We shall see that because of this, structural stability can be considered to be the basis of a class of stabilities characterised by hyperbolicity.

3.2.3 Zeeman Stability

An alternative to structural stability has been proposed recently by Zeeman [43], we shall refer to it as *Zeeman stability*. The definition given in [43] is as follows (symbols have been changed to agree with this thesis).

> *"Given a vector field f on an oriented manifold M, and given $\varepsilon > 0$, let g be the steady state of the Fokker–Planck equation for f with ε-diffusion. Vector fields are defined to be equivalent, or stable, according to whether their steady states are."*

The Fokker–Planck equation is

$$g_t = \varepsilon \Delta g - \nabla \cdot (gf),$$

and the steady state is obtained by solving for $g_t = 0$. Under certain simple conditions on f it will always be the case that the steady state is unique and attracting, and will reflect the behaviour of f on its attractor.

Zeeman stability is a study of a measure on the attractors of a dynamical system, as opposed to the topology of the attractors as in structural stability. The major advantage of Zeeman stability is that on smooth Riemannian manifolds it is a generic property.

Just as hyperbolic systems are important to structural stability, gradient systems provide a basis for Zeeman stability.

Part II

4

Hyperbolic Stabilities

Structural stability and its derivatives have a common basis — hyperbolicity. Thus we refer to this class of stabilities as the ***hyperbolic stabilities.*** The primary reason for focussing on hyperbolicity is that in a certain way, hyperbolicity is generic. Thus if one can find a generic stability for hyperbolic systems of this type it will be generic for all systems†.

When studying stabilities in this class one asks two questions. First, is the stability generic? Second, what are the necessary and sufficient conditions for a system to exhibit this stability? This approach is used because if the answer to the first question is no, the answer to the second may provide a basis for modifying the stability such that the result may be generic.

4.1 Definitions and Terminology

We begin by introducing some of the terminology and notation to be used, starting with a restatement of the definitions of topological equivalence and structural stability.

Notation *$\mathcal{F}^r(M)$ is the space of all C^r vector fields on M with the C^r topology, $r \geq 1$. $\mathrm{F}^r(M)$ is the space of all C^r diffeomorphisms on M with the C^r topology, $r \geq 1$.*

Definition 4.1 *Two vector fields f, $g \in \mathcal{F}^r(M)$ (maps F, $G \in \mathrm{F}^r(M)$) are* **topologically equivalent** *if there exists a homeomorphism $h\colon M \to M$ such that (i) orbits of f, ϕ_t^f, are taken to orbits of g, ϕ_t^g: $h \circ \phi_t^f = \phi_t^g \circ h$ ($h \circ F = G \circ h$) and (ii) orientation is preserved.*

A map $h\colon U \to V$ (U, V open sets in $\mathbb{R}^n$) is *orientation preserving* if the determinant of Dh is positive. Note that topological equivalence is a geometric equivalence between orbits.

†If $\mathcal{O}$ is a set of operators on which the property $\mathcal{P}_1$ is generic, and if the property $\mathcal{P}_2$ is generic on the set of operators $\mathcal{O}_1$ for which $\mathcal{P}_1$ holds, then $\mathcal{P}_2$ is generic on $\mathcal{O}$.

Definition 4.2 *A vector field $f \in \mathcal{F}^1(M)$ (map $F \in \mathrm{F}^1(M)$) exhibits* **basic structural stability** *if there exists a C^1 neighborhood $V \subset \mathcal{F}^1(M)$ of f (neighborhood $U \subset \mathrm{F}^1(M)$ of F) such that f (resp. F) is topologically equivalent to every $g \in V$ (resp. $G \in U$) and the homeomorphism h defining this equivalence is C^0 close to the identity.*

This definition slightly generalises Andronov and Pontrjagin [2] to include manifolds of arbitrary dimension.

There are numerous modifications to basic structural stability (see Anosov [3, Ch.2] for a good overview), perhaps the most common is to not require that h be C^0 close to the identity. This was first proposed by Peixoto [22], the purpose being to ensure that the set of structurally stable systems was open. At the present time, according to Anosov [3], there are no good arguments either for or against the use of this modification. As it is slightly more general we shall use it. Another common generalisation is to consider C^r vector fields or maps with perturbations C^r close on neighborhoods in $\mathcal{F}^r(M)$ or $\mathrm{F}^r(M)$, respectively, for $r \geq 1$. Note that if $f \in \mathcal{F}^r(M)$ then we also have $f \in \mathcal{F}^s(M), 1 \leq s \leq r$, thus $\mathcal{F}^r(M) \subset \mathcal{F}^s(M)$ (similarly $\mathrm{F}^r(M) \subset \mathrm{F}^s(M)$).

Our working definition of structural stability is thus

Definition 4.3 *A vector field $f \in \mathcal{F}^r(M)$ (map $F \in \mathrm{F}^r(M)$) exhibits* **$\mathbf{C^r}$–structural stability** *if there exists a neighborhood $V \subset \mathcal{F}^r(M)$ of f (neighborhood $U \subset \mathrm{F}^r(M)$ of F) such that f (resp. F) is topologically equivalent to every $g \in V$ (resp. $G \in U$).*

When we refer to just *structural stability* we shall usually mean C^1–structural stability, exceptions should be obvious from the context.

We now define some subsets of $\mathcal{F}^r$ and F^r. First we have an axiom of Smale [38]. (Recall that Ω represents the nonwandering set.)

Axiom A *The following conditions hold on Ω*

i) Ω has hyperbolic structure,

ii) the set of periodic and fixed points is dense in Ω.

Notation *$\mathcal{A}^r(M)$ is the set of C^r Axiom A vector fields on M. $\mathrm{A}^r(M)$ is the set of C^r Axiom A diffeomorphisms on M.*

A property often used in conjunction with Axiom A is the transversality condition.

Definition 4.4 *A system satisfies the* **(strong) transversality condition** *if $W^s(x)$ and $W^u(y)$ intersect transversally $\forall x, y \in \Omega$.*

Next the generic type of hyperbolicity.

Definition 4.5 *A system is* **Kupka–Smale** *if*

i) all fixed points and closed orbits are hyperbolic,

ii) the stable and unstable manifolds of the fixed points and closed orbits all intersect transversally.

Notation $\mathcal{K}^r(M)$ *is the set of all* C^r *Kupka–Smale vector fields on* M. $\mathrm{K}^r(M)$ *is the set of all* C^r *Kupka–Smale diffeomorphisms on* M.

There are two important subsets of the Axiom A systems, Morse–Smale systems and Anosov systems.

Definition 4.6 *A system is* **Morse–Smale** *if*

i) Ω *consists only of a finite number of fixed and periodic points, all of which are hyperbolic,*

ii) their stable and unstable manifolds all intersect transversally.

Notation $\mathcal{M}^r(M)$ *is the set of all* C^r *Morse–Smale vector fields on* M. $\mathrm{M}^r(M)$ *is the set of all* C^r *Morse–Smale diffeomorphisms on* M.

Definition 4.7 *A system* $f \in \mathcal{F}^r(M)$ *or* $F \in \mathrm{F}^r(M)$ *is an* **Anosov system** *if the entire manifold* M *has hyperbolic structure under* f *or* F.

Notation $\mathcal{Y}^r(M)$ *is the set of all* C^r *Anosov vector fields on* M. $\mathrm{Y}^r(M)$ *is the set of all* C^r *Anosov diffeomorphisms on* M.

The importance of these subsets comes from the following results.

Theorem 4.1 *Every Morse–Smale system is structurally stable.*

Theorem 4.2 *Every Anosov system is structurally stable.*

It was these which led Smale to develop the Axiom A systems. Note that we have the following relations

$$\mathcal{M}^r, \mathcal{Y}^r \subset \mathcal{A}^r \subset \mathcal{F}^r,$$
$$\mathrm{M}^r, \mathrm{Y}^r \subset \mathrm{A}^r \subset \mathrm{F}^r.$$

As genericity was introduced only in passing, a more detailed discussion is called for.

Definition 4.8 *A* **residual (Baire) subset** *of a topological space is one which can be expressed as a countable intersection of open dense sets.*

Definition 4.9 *A* **generic** *property is one which holds for all members of a residual set of operators.*

Only *properties* are generic, subsets are residual. For example, if structural stability is generic then the set of structurally stable systems is residual.

Once one has a generic property (or equivalently, a residual subset of a set of operators) the Baire category theorem can be invoked.

Theorem 4.3 (Baire Category Theorem) *The union of any countable collection of closed nowhere dense subsets is nowhere dense, and the intersection of any countable collection of open dense subsets is dense.*

For us this means that any system not possessing the desired property can be approximated arbitrarily closely by one which does. Therefore a generic stability provides us with a class of perturbations under which certain aspects of the behaviour of all dynamical systems are invariant.

4.2 Kupka–Smale Systems

Under the following theorem we have a generic hyperbolicity.

Theorem 4.4 (Kupka–Smale) *$\mathcal{K}^r(M)$ ($\mathrm{K}^r(M)$) is residual in $\mathcal{F}^r(M)$ (resp. $\mathrm{F}^r(M)$).*

This is a combination of the Kupka–Smale theorem for vector fields and diffeomorphisms. The proof of the diffeomorphism version — extended to vector fields using suspensions — was first done by Smale [35], while a good reworking of the vector field approach (Kupka's method) was done by Peixoto [25]. For a thorough, modern development of both see Palis and de Melo [20].

We shall only give an outline of the proof of theorem 4.4. As the method is the same for both the vector field and diffeomorphism versions, the following is also applicable to diffeomorphisms.

SKETCH OF PROOF: The proof proceeds by providing residual sets of increasing specificity. The initial set, labelled $\mathcal{G}_1^r$, is the set of all vector fields $f \in \mathcal{F}^r$ with a finite number of fixed points, all of which are hyperbolic. It is shown to be residual in $\mathcal{F}^r$. Next $\mathcal{G}_{12}^r$, the set of all vector fields $f \in \mathcal{G}_1^r$ whose closed orbits are all hyperbolic, is shown to be residual in $\mathcal{G}_1^r$. And finally $\mathcal{G}_{123}^r$, the set of all vector fields $f \in \mathcal{G}_{12}^r$ for which all stable and unstable manifolds intersect transversally, is shown to be residual in $\mathcal{G}^r_{12}$. Thus $\mathcal{G}^r_{123} = \mathcal{K}^r$ is residual in $\mathcal{G}^r_1$, which implies that $\mathcal{K}^r$ is residual in $\mathcal{F}^r$. □

4.3 Structural Stability

4.3.1 Genericity

What is appealing about structural stability is that it preserves certain topological features such as direction of flow and geometric structure of attractors. This is a very strong qualitative consistency for the asymptotic behaviour of similar systems. Unfortunately, structural stability is generic only for flows on a closed surface (M^2), or equivalently maps on a closed interval (M^1). Specifically:

i) C^1–structural stability is generic for flows on M^2 (maps on M^1).

ii) If M^2 (resp. M^1) is orientable† then C^r–structural stability, $r \geq 1$, is generic for flows (resp. maps).

These results were first proved by Peixoto [23,24], the first being a generalisation of the results of Andronov and Pontrjagin [2].

At one time it was hoped that the above could be extended to higher dimensional manifolds. These hopes were ended by Smale's example [37] of an open set of non-structurally stable diffeomorphisms on M^3 (and hence for flows and diffeomorphisms on M^4 and higher) and later Williams' for diffeomorphisms on M^2 [42].

4.3.2 Necessary and Sufficient Conditions

The results here are somewhat incomplete. We have a characterisation of the sufficient conditions for a system to be structurally stable.

Theorem 4.5 (Robbin – Robinson) *If a system satisfies Axiom A and the transversality condition then it is structurally stable.*

This result is the culmination of work by several people, the major contributions being from Robbin [28,29] and Robinson [31,32,33].

It is the question of necessary conditions which has not been fully answered: Does structural stability imply Axiom A and transversality? Actually this can be reduced to asking if structural stability implies Axiom

†This also holds for non-orientable surfaces with Euler characteristic 1 (the projective plane), 0 (Klein bottle), or -1 (projective plane with a handle).

A [30]. This is known as the *Stability Conjecture.* The strongest result to date is due to Mañé [13].

Theorem 4.6 (Mañé) *Every C^1-structurally stable diffeomorphism of a closed manifold satisfies Axiom A.*

Note that we are referring only to diffeomorphisms here; the corresponding formulation for flows fails. In fact there are very few results regarding necessary conditions for the structural stability of flows.

The other obvious limitation of this result — that it does not say anything about the C^r case — is due to the lack of a general Closing Lemma [27].

Theorem 4.7 (Closing Lemma) *Let M be a compact n-dimensional, boundaryless manifold, and γ a recurrent orbit of $f \in \mathcal{F}^1(M)$ ($F \in \mathrm{F}^1(M)$), then given $p \in \gamma$ and $\varepsilon > 0$ there exists $g \in \mathcal{F}^1(M)$ (resp. $G \in \mathrm{F}^1(M)$), with $|f - g| < \varepsilon$ ($|F - G| < \varepsilon$), having the orbit through p closed.*

In other words, any recurrent point can be made fully periodic by means of an arbitrarily small C^1 perturbation of the system. As Mañé's result relies heavily on it, the lack of a C^r-Closing Lemma precludes a more general proof of the Stability Conjecture at this time.

4.4 Ω-Stability

Ω–stability is simply structural stability restricted to the nonwandering set of a system. This is a logical modification as it is the asymptotic behaviour of dynamical systems that we are interested in.

Definition 4.10 *A vector field $f \in \mathcal{F}^r(M)$ (map $F \in \mathrm{F}^r(M)$) is* **$\mathbf{C}^r$ Ω–stable** *if there exists a neighborhood $V \subset \mathcal{F}^r(M)$ (neighborhood $U \subset \mathrm{F}^r(M)$) such that for any $g \in V$ (resp. $G \in U$) there exists a homeomorphism $h\colon \Omega(f) \to \Omega(g)$ (resp. $h\colon \Omega(F) \to \Omega(G)$) with $h \circ \phi_t^f(x) = \phi_t^g \circ h(x)$, $\forall x \in \Omega(f)$ (resp. $h \circ F(x) = G \circ h(x)$, $\forall x \in \Omega(F)$).*

Structural stability implies Ω–stability but not the reverse, thus Ω–stability is a weaker formulation.

Most of the results for Ω-stability originate in a paper of Smale [39], from which we have the following.

Theorem 4.8 (Spectral Decomposition Theorem) *Let $F \in \mathrm{A}^r(M)$. Then we can write $\Omega(F)$ as a finite union of closed, transitive*‡ *sets $\Omega(F) =$*

‡A set is *transitive* if it contains a dense orbit.

$\Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_n$, in which the periodic points are dense. The Ω_i's are called **basic sets.**

Definition 4.11 *A* **cycle** *in Ω is a sequence $\Omega_{i_1}, \Omega_{i_2}, \ldots, \Omega_{i_k}$ with points $x_1, y_1 \in \Omega_{i_1}, \ldots, x_x, y_k \in \Omega_{i_k}$ for which we have $W^s(x_1) \cap W^u(y_2) \neq \emptyset, \ldots$, $W^s(x_k) \cap W^u(y_1) \neq \emptyset$.*

4.4.1 Genericity

Again, early hopes were not borne out as Ω–stability was found to have the same limited genericity as structural stability. Nongenericity was first shown by Abraham and Smale [1].

4.4.2 Necessary and Sufficient Conditions

Note that we defined the basic sets only in terms of diffeomorphisms, because, as with structural stability, there are not many results for flows in the way of necessary and sufficient conditions. The results for Ω-stable diffeomorphisms mirror those for the structurally stable case, we have [39]:

Theorem 4.9 (Ω-Stability Theorem) *If $F \in \mathbb{A}^r(M)$ and has no cycles it is Ω-stable.*

As far as necessary conditions are concerned we are again limited by the Closing Lemma. The following is due to Palis [19] working from the result of Mañé [13].

Theorem 4.10 *If $F \in \mathbb{F}^1(M)$ is Ω-stable then it satisfies Axiom A.*

4.5 Other Hyperbolic Stabilities

Since genericity is unattainable for structural stability the recent emphasis has been towards finding necessary conditions for the full C^r case (sufficient conditions having been found). We have a stability for which the C^r case is completely solved [12].

Definition 4.12 *$F \in \mathbb{F}^r(M)$ is* **C^r-absolutely stable** *if there exists a neighborhood $U \subset \mathbb{F}^r(M)$ of F and a constant $K \in \mathbb{R}^+$ such that if $G \in U$ there exists a homeomorphism $h\colon M \to M$ such that:*

i) $hF = Gh$,

ii) $\|h - I\|_0 < K\|G - F\|_0$,

where $\|\cdot\|_0$ is the C^0 distance and I is the identity map on M.

With the above definition we have:

Theorem 4.11 *$F \in \mathrm{F}^r(M)$ is absolutely stable if and only if F satisfies Axiom A and the strong transversality condition.*

It was thought that one could approach structural stability by means of absolute stability but it seems that Mañé has dropped this method in favour of the Closing Lemma approach [13].

4.6 Overview

While we do not have a generic hyperbolic stability, or necessary and sufficient conditions for the more powerful types (structural and Ω-stability), the usefulness of hyperbolicity is such that these stabilities still constitute an area on which considerable research is being done. We touch on another of the uses of hyperbolicity in the next chapter.

5

Hyperbolic Strange Attractors

The use of hyperbolicity in the study of dynamical systems is not limited to the hyperbolic stabilities. The expanding/contracting splitting provides a structure on which most aspects of dynamical systems are much more easily studied. In this chapter we take advantage of the properties of hyperbolic structure to examine strange attractors.

If we are going to discuss strange attractors in detail it shall be necessary to have a rigorous definition of an attractor.

Definition 5.1 *An* **attractor** $\Lambda \subset M$ *is a closed invariant set such that for some neighborhood* U *of* Λ *we have* $\phi_t(x) \in U$ *(resp.* $F^n(x) \in U$*),* $\forall x \in U$*,* $t \geq 0$ *(resp.* $n \geq 0$*) and* $\phi_t(x) \to \Lambda$ *(resp.* $F^n(x) \to \Lambda$*) as* $t \to \infty$ *(resp.* $n \to \infty$*).*

There is often the additional condition of *indecomposability* imposed (that is for any $x, y \in \Lambda$, and $\varepsilon > 0$ there exist sequences $x = x_0, x_1, \ldots, x_n = y$ and $t_1, \ldots, t_n \geq 1$ such that $d(\phi_{t_i}(x_{i-1}), x_i) < \varepsilon$) however, there are arguments both for and against this condition. We now give an alternative to definition 3.6.

Definition 5.2 *An attractor with hyperbolic structure containing a transverse homoclinic orbit is a* **hyperbolic strange attractor.**

This definition can be applied to certain nonhyperbolic systems also, as will be shown at the end of the chapter.

All of the systems in this chapter are diffeomorphisms. This is because they allow us to demonstrate the desired behaviours in spaces one dimension lower than those required for flows. The results for flows are obtained by means of suspension, the properties demonstrated for diffeomorphisms being unchanged by this process.

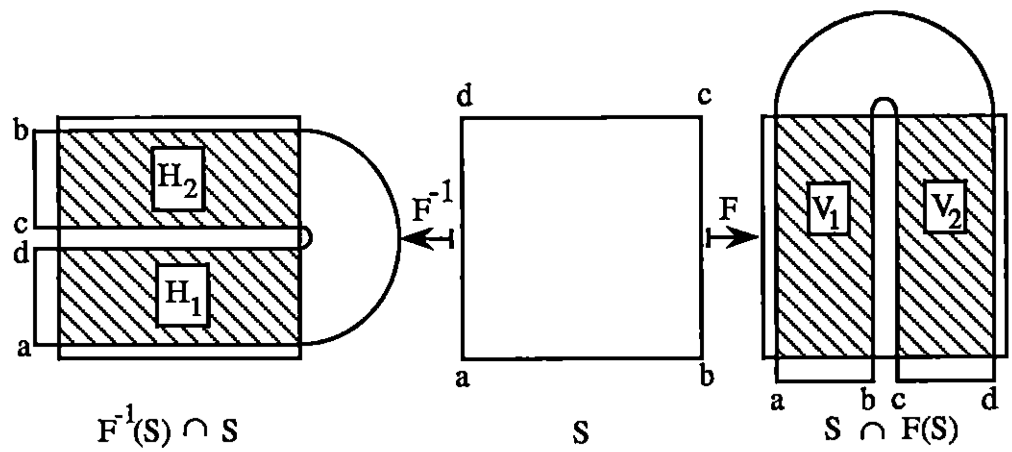

Figure 5.1: The Smale horseshoe map.

5.1 The Smale Horseshoe

The horseshoe map (which first appeared Smale's seminal 1967 paper [38]) has been the motivation for much of the work in modern dynamical systems theory. While this map may seem somewhat contrived its purpose is not to serve as a model in the physical sense, rather it is an idealised version of behaviour which may occur in physical systems. It allows us to examine this behaviour in a linear setting so that we do not have to deal with the complications which arise in non-linear physical models.

The Smale horseshoe is a planar map $F\colon S \to \mathbb{R}^2$, where $S = [0,1]\times[0,1]$ is the unit square in $\mathbb{R}^2$. It is invertible. In fact we have $F \in \mathrm{Diff}^\infty(S)$. The action of F is best understood by referring to figure 5.1. F stretches S vertically by a factor $\mu > 2$, contracts horizontally by a factor $0 < \lambda < 1/2$, and bends the resulting rectangle about its center to obtain a horseshoe shape.

It is important to keep in mind that we are interested only in the invariant set Λ[†] $\subset S$ of F, defined by

$$\Lambda = \{x \mid F^n(x) \in S,\ -\infty < n < \infty\} = \bigcap_{n=-\infty}^{\infty} F^n(S).$$

All other points are eventually mapped outside of S and thus out of the domain of F.

[†] Note that Λ is not an attractor, but standard notation for the horseshoe map labels the invariant set Λ.

Referring to figure 5.1, observe that by limiting the map to $S \cap F^{-1}(S)$ — the rectangles H_1 and H_2 — we have a linear map with constant Jacobian

$$DF = \begin{bmatrix} \pm\lambda & 0 \\ 0 & \pm\mu \end{bmatrix}, \qquad (+ \text{ on } H_1, \; - \text{ on } H_2),$$

such that $F(H_i) = V_i$, $i = 1, 2$. Similarly, F^{-1} is linear when restricted to $S \cap F(S)$, with $F^{-1}(V_i) = H_i$, $i = 1, 2$.

A second forward and backward iteration of F gives us the situation in figure 5.2, with $F^{-2}(S) \cap F^{-1}(S) \cap S$ (resp. $S \cap F(S) \cap F^2(S)$) composed of four horizontal (resp. vertical) strips. Another iteration will double this to eight. It is obvious (by induction) that the number of strips in $F^{-n}(S) \cap F^{-(n-1)}(S) \cap \cdots \cap S$ or $S \cap F(S) \cap \cdots \cap F^n(S)$ is 2^n, as the number of strips doubles with each iteration. Additionally, it is straightforward to show that the width of each vertical strip at the n^{th} iteration is λ^n, and the height of each horizontal strip at the $-n^{\text{th}}$ iteration is, μ^{-n}. In the limit $n \to \infty$ these strips form Cantor sets of horizontal and vertical segments, the intersection of which gives us Λ. Thus Λ is a Cantor set of points in the form of a two dimensional array. Figure 5.3 gives an indication of the eventual structure of Λ.

If we pick some point $x \in \Lambda$ and apply F to it the point will move around in Λ. Since we are dealing with a bi-infinite array of points it is difficult to visualise the path of an arbitrary point (other than the fixed point at the lower left hand corner). Thus we require a labelling scheme so that we can easily distinguish each point in Λ.

Consider the labels in figure 5.3. The symbol to the right of the dot corresponds to the horizontal strip (H_1 or H_2) in which that rectangle lies — those in the upper half of S have a 2 and those in the lower a 1. Now apply F. If we shift the dot one place to the right the new label again corresponds to the horizontal strip in which the new rectangle lies. Similarly, after applying F^{-1} we shift the dot one position to the left to get the labels agree.

It is this method which we use to label the elements of Λ. A point $x \in \Lambda$ is uniquely characterised by a *bi-infinite sequence* $\mathbf{a} = \{a_j\}_{j=-\infty}^{\infty} = \ldots a_{-2}a_{-1}{\cdot}a_0a_1a_2\ldots$ (the dot acts only as a placekeeper, it has no other significance), where $a_j = i$ for $F^j(x) \in H_i$, $i = 1, 2$. We define this characterisation formally as the map $\psi\colon \Lambda \to \Sigma^2$ (Σ^2 is defined in the following section) such that $\mathbf{a} = \psi(x)$.

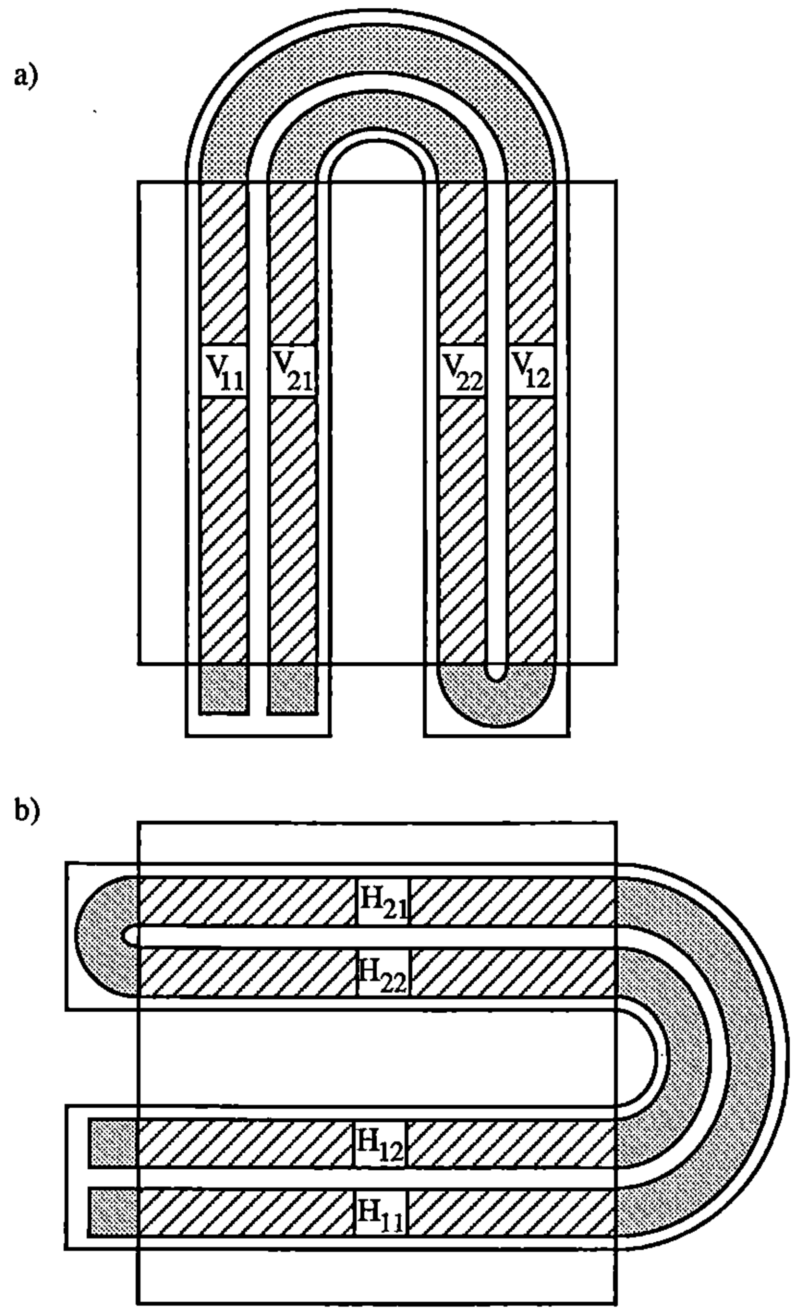

Figure 5.2: The horseshoe map. (a) $\bigcap_{n=0}^{2} F^n(S)$; (b) $\bigcap_{n=-2}^{0} F^n(S)$.

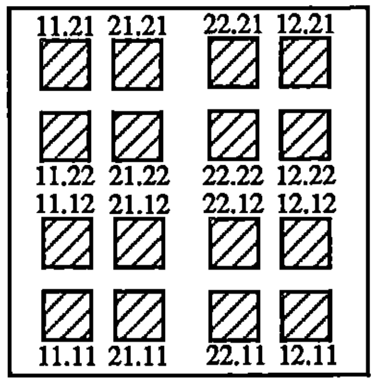

Figure 5.3: The components of $\bigcap_{n=-2}^{2} F^n(S)$.

5.2 Symbolic Dynamics

It is necessary at this point to discuss the space in which the symbol sequences **a** exist. Define Σ^N, $N \geq 2$, to be the set of all bi-infinite sequences of N symbols. Our purposes require only Σ^2 — the symbols being the digits 1 and 2 — but it is important to know that this is not the only possibility.

Define a metric for Σ^2 by

$$d(\mathbf{a},\mathbf{b}) = \sum_{j=-\infty}^{\infty} \delta_j 2^{-|j|}, \qquad \delta_j = \begin{cases} 0 & , \quad a_j = b_j \\ 1 & , \quad a_j \neq b_j \end{cases} .$$

We also define a function on this space, the shift map $\sigma\colon \Sigma^2 \to \Sigma^2$ by $\sigma(\mathbf{a}) = \mathbf{b}$ with $b_j = a_{j+1}$, $j \in \mathbb{Z}$.

With the shift map it is possible to refer to a point $\mathbf{a} \in \Sigma^2$ as a fixed point ($\sigma(\mathbf{a}) = \mathbf{a}$, e.g. $\mathbf{a} = \ldots 111{\cdot}111\ldots$), a periodic orbit ($\sigma^n(\mathbf{a}) = \mathbf{a}$, $n \in \mathbb{Z}^+$, e.g. $\mathbf{a} = \ldots 12221{\cdot}2221\ldots$), or a non-periodic orbit ($\sigma^n(\mathbf{a}) \neq \mathbf{a}$, $\forall n \in \mathbb{Z}$, e.g. $\mathbf{a} = \ldots 211121121{\cdot}2121121112\ldots$). There are only two fixed points in Σ^2 but we have the following concerning periodic and non-periodic orbits.

Theorem 5.1 *The shift map σ on Σ^2 has*

i) a countable number of periodic orbits,

ii) an uncountable number of non-periodic orbits, and

iii) a dense orbit.

PROOF: *i)* There exist a countable number of possible periods (the integers in $\mathbb{Z}^+$ to be precise) and each specific period has a finite number of unique orbits associated with it. Thus the total number of unique periodic orbits is countable.

ii) If we look at just one side of the sequence it can be thought of as a binary expansion of a real number between zero and one (use the change of symbols: $2 \to 1$ and $1 \to 0$, for this to make sense). A binary expansion of an irrational number is an infinite, non-repeating sequence. There are an uncountable number of unique irrational numbers in the interval $[0, 1]$. Thus there are an uncountable number of unique non-periodic sequences, corresponding to the irrational numbers.

iii) We define a dense orbit as a bi-infinite sequence $\mathbf{a} \in \Sigma^2$ such that for any $\varepsilon > 0$ and $\mathbf{b} \in \Sigma^2$, there exists $n \in \mathbb{Z}$ for which $d(\sigma^n(\mathbf{a}), \mathbf{b}) < \varepsilon$. We proceed by constructing such a sequence.

Given an $\varepsilon > 0$, $\exists m \in \mathbb{Z}^+$ such that $\varepsilon > 2^{-(m-2)}$. There are a finite number (2^{2m+1} to be precise) of finite central sequences of the form

$$\mathbf{a}^{m,i} = a_{-m}a_{-(m-1)}\ldots a_{-1}\cdot\dot{a}_0 a_1 \ldots a_m, i = 1, \ldots, 2^{2m+1}.$$

We form another finite sequence $\mathbf{a}^{m,0}$ by concatenating every $\mathbf{a}^{m,i}$:

$$\mathbf{a}^{m,0} = \mathbf{a}^{m,1}\mathbf{a}^{m,2}\cdots\mathbf{a}^{m,2^{2m+1}}$$

(the position of the dot is not important).

Now repeat the above construction for all integers greater than m and concatenate the resulting $\mathbf{a}^{m+1,0}$, $\mathbf{a}^{m+2,0}, \ldots$ to $\mathbf{a}^{m,0}$ as follows

$$\mathbf{a} = \ldots\mathbf{a}^{m+3,0}\mathbf{a}^{m+1,0}\mathbf{a}^{m,0}\mathbf{a}^{m+2,0}\mathbf{a}^{m+4,0}\ldots$$

Thus for any $\varepsilon > 0$ there exists $n \in \mathbb{Z}$ such that $d(\sigma^n(\mathbf{a}), \mathbf{b}) < \varepsilon$. □

The structure and properties of the Σ^N spaces are a field of study in their own right, an excellent introduction to which is found in Wiggins [41, §2.2]. We also mention that the shift map σ is more formally known as a *subshift of finite type* when operating on spaces for which $N < \infty$, and a *subshift of infinite type* when $N = \infty$.

5.3 Equivalence of the Horseshoe Map and a Subshift of Finite Type

The following result was first obtained by Smale [36], it allows us to fully understand the dynamics of the horseshoe map.

at the very least, they must agree with each other on a finite central subsequence of length m in both directions, with m being the smallest integer such that $\varepsilon > 2^{-(m-1)}$. If **a** and **b** are such that they disagree completely everywhere but this central string then successive applications of σ will mean that $d(\sigma^n(\mathbf{a}), \sigma^n(\mathbf{b})) = 2^{-(m-1)+n} = Ke^{n\ln 2}$.

5.4 The Existence of Horseshoes in Nonlinear Systems

We have yet to answer the question: "Of what use is the Smale horseshoe in nonlinear dynamical systems?" After all, the horseshoe map is (effectively) linear and does not seem to have any physical basis. As we shall show, the significance of the horseshoe is not that it is a map which commonly arises in dynamical systems, rather it is a structure which is embedded in the invariant set. This embedding carries with it all of the properties associated with the original horseshoe.

For now we shall be dealing exclusively with diffeomorphisms of the plane, $\mathbb{R}^2$. Higher dimensional results shall be discussed at the end of the chapter.

5.4.1 The Generalised Horseshoe

First it is necessary to provide conditions which imply a horseshoe which are less restrictive than those used to construct the Smale horseshoe. The primary features of the Smale horseshoe are: (i) horizontal and vertical strips which are mapped onto each other, and (ii) a strong expansion in one direction and a strong contraction in the other. These features are characterised as a (redundant) set of three properties.

We shall be dealing with nonlinear maps so we shall need nonlinear horizontal and vertical strips in the unit square, S.

Definition 5.3 *A* **vertical curve** *is a curve $x = v(y) \subset S$ such that*

$$|v(y_1) - v(y_2)| \leq \mu |y_1 - y_2|, \qquad 0 \leq y_1 \leq y_2 \leq 1,$$

for some $0 < \mu < 1$. A **horizontal curve** *is a curve $y = h(x) \subset S$ such that*

$$|h(x_1) - h(x_2)| \leq \mu |x_1 - x_2|, \qquad 0 \leq x_1 \leq x_2 \leq 1.$$

Definition 5.4 *A* **vertical strip** *is the region between two nonintersecting vertical curves (that is $v_1(y) < v_2(y)$, $\forall y \in [0,1]$):*

$$V = \{(x,y) \mid x \in [v_1(y), v_2(y)],\ y \in [0,1]\},$$

its width is

$$d(V) = \max_{y \in [0,1]} |v_1(y) - v_2(y)|.$$

A **horizontal strip** *is the region between two nonintersecting horizontal curves (that is $h_1(x) < h_2(x)$, $\forall x \in [0,1]$):*

$$H = \{(x,y) \mid y \in [h_1(x), h_2(x)],\ x \in [0,1]\},$$

its width is

$$d(H) = \max_{x \in [0,1]} |h_1(x) - h_2(x)|.$$

We now state the properties required for a map $F\colon S \to \mathbb{R}^2$ to exhibit a horseshoe. Recall, linearising the map $F \in \mathrm{F}^r(M^2)$, $r \geq 1$ gives

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} \mapsto \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = DF \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

P1 Let $\mathcal{S} = \{1,2,\ldots,N\}$, $N \in \mathbb{Z}^+$. Then there exist disjoint horizontal and vertical strips H_i and V_i, such that $F(H_i) = V_i$, $\forall i \in \mathcal{S}$.

P2 If we have a vertical substrip $V_i' \subseteq V_i$ then $F(V_i') \cap V_j$ is a vertical strip with

$$d(F(V_i') \cap V_j) \leq \nu d(V_j),$$

for some $0 < \nu < 1$, and $i, j \in \mathcal{S}$. Similarly if we have a horizontal substrip $H_i' \subseteq H_i$ then $F^{-1}(H_i') \cap H_j$ is a horizontal strip such that

$$d(F^{-1}(H_i') \cap H_j) \leq \nu d(H_j).$$

In other words, F and F^{-1} uniformly contract vertical and horizontal strips respectively.

P3 There exist sets (called *sector-bundles*) $S^u = \{(\xi,\eta) \mid |\xi| < \mu|\eta|\}$ and $S^s = \{(\xi,\eta) \mid |\eta| < \mu|\xi|\}$ $0 < \mu < 1$ defined over $\cup_{i\in\mathcal{S}} V_i$ and $\cup_{i\in\mathcal{S}} H_i$ respectively, such that $DF(S^u) \subset S^u$ and $DF^{-1}(S^s) \subset S^s$. Additionally, if $DF(\xi_0,\eta_0) = (\xi_1,\eta_1)$ and $DF^{-1}(\xi_0,\eta_0) = (\xi_{-1},\eta_{-1})$ then $|\eta_1| \geq (1/\mu)|\eta_0|$ and $|\xi_{-1}| \geq (1/\mu)|\xi_0|$.

Note that P3 is a local condition. Thus as we shall only be concerned with hyperbolicity in the neighborhood of Λ, we need only check that P3 holds on the "squares": $\cup_{i,j\in\mathcal{S}}(V_i \cap H_j)$ rather than the full strips.

We now state some results without providing proofs.

Theorem 5.4 *P1 and P3, with $0 < \mu < 1/2$ imply P2 with $\nu = \mu(1-\mu)$.*

Theorem 5.5 *If $F \in \mathbb{F}^r(M^2)$, $r \geq 1$, satisfies P1 and P2 then there exists an invariant set Λ of F which is topologically equivalent to a subshift of finite type σ on Σ^N, $N \geq 2$. Moreover, if $0 < \mu < 1/2$ (thus we can equivalently use P1 and P3) and $|\mathrm{Det}(DF)|$, $|\mathrm{Det}(DF)|^{-1} \leq \mu^{-2}/2$, then Λ is hyperbolic.*

Thus P1, P2, and P3 provide a characterisation of the generalised horseshoe.

5.4.2 The Smale–Birkhoff Homoclinic Theorem

The Smale–Birkhoff homoclinic theorem (S–B) is the impetus for our definition of a hyperbolic strange attractor. It describes the conditions necessary for the existence of a horseshoe in the invariant set of a diffeomorphism.

We begin by quoting a local result of Palis [18].

Lemma 5.6 (The λ-Lemma) *Let $\bar{x} \in M$ be a hyperbolic fixed point of $F \in \mathbb{F}^1(M)$, $M \subseteq \mathbb{R}^n$, $n > 1$, with stable and unstable manifolds of dimension s and u respectively ($s + u = \mathrm{dim}(M)$). Also let D be a u-disk (a u-dimensional disk) in $W^u(\bar{x})$, and Δ be a u-disk meeting $W^s(\bar{x})$ transversely at a point z. Then $\cup_{n \geq 0} F^n(\Delta)$ contains u-disks arbitrarily C^1 close to D.*

The significance of the λ-lemma is more obvious if Δ is chosen to be in $W^u(\bar{x})$, or in other words, z is a transverse homoclinic point. In this case the λ-lemma implies that $W^u(\bar{x})$ accumulates on itself. For example if $s = u = 1$ we have a planar *homoclinic tangle* (see figure 5.4).

We emphasise that for purposes of clarity we are examining S–B in the planar case only. The full result, by Smale [36], covers n-dimensional manifolds. The following proof is adapted from Moser [15, pp.181–188].

Theorem 5.7 (Smale–Birkhoff Homoclinic Theorem) *If we have a system $F \in \mathbb{F}^\infty(M^2)$ with a hyperbolic fixed point $\bar{x} \in M^2$ and a transverse homoclinic point $z \in M^2$, $z \neq \bar{x}$, then there exists an invariant set $\Lambda \subset M^2$ on which F is topologically equivalent to a subshift of finite type.*

PROOF: In other words the existence of a transverse homoclinic point implies that the invariant set of the map has a horseshoe embedded in it. To prove the existence of this horseshoe we shall use theorem 5.5, that is: F satisfies P1, P3 with $0 < \mu < 1/2$.

By now it should be obvious that the proof involves reduction to a local problem. Consider the behaviour of F in a neighborhood $U \subset M^2$

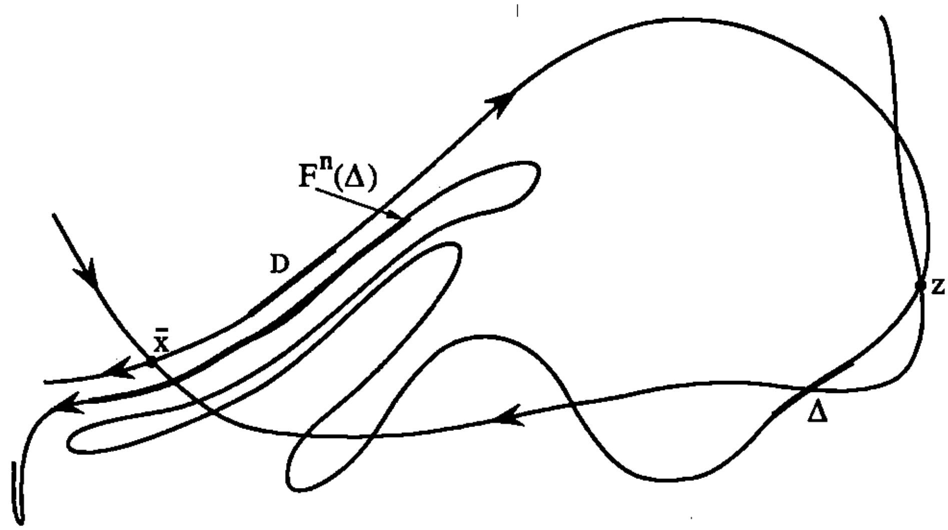

Figure 5.4: A homoclinic tangle.

of $\bar{x}$. Assume the eigenvalues of $DF(\bar{x})$ are λ_1, λ_2 where $0 < \lambda_1 < 1 < \lambda_2$. The invariant manifolds, $W^s(\bar{x})$ and $W^u(\bar{x})$ are C^∞ curves so we may introduce local coordinates (x, y) under which $\bar{x} = (0,0)$ and the invariant manifolds define the coordinate axes ($x \leftrightarrow$ stable, $y \leftrightarrow$ unstable). With this coordinate system write $F = (f_s, f_u)$ such that in U we have:

i) $x_{n+1} = f_s(x_n, y_n), \qquad y_{n+1} = f_u(x_n, y_n),$

ii) $f_s(0, y) = f_u(x, 0) = 0,$

iii) $\left.\dfrac{\partial f_s}{\partial x}\right|_{(0,0)} = \lambda_1, \qquad \left.\dfrac{\partial f_u}{\partial y}\right|_{(0,0)} = \lambda_2.$

We now define U precisely as $U = \{(x, y) \mid 0 \le x, y \le a\}$, for suitably small a.

To begin with, for the linearised map DF given by:

$$\begin{bmatrix} \xi_{n+1} \\ \eta_{n+1} \end{bmatrix} = DF \begin{bmatrix} \xi_n \\ \eta_n \end{bmatrix} = \begin{bmatrix} \frac{\partial f_s}{\partial x} & \frac{\partial f_s}{\partial y} \\ \frac{\partial f_u}{\partial x} & \frac{\partial f_u}{\partial y} \end{bmatrix} \begin{bmatrix} \xi_n \\ \eta_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \xi_n \\ \eta_n \end{bmatrix},$$

we have the following result.

Lemma 5.8 *For sufficiently small $a > 0$ and any sequence of iterates*

$(x_k, y_k) \in U$, $k = 0, 1, \ldots, n$, if

$$|\xi_0| \leq \sqrt{\frac{x_0}{y_0}}|\eta_0|$$

then

$$|\xi_k| \leq \sqrt{\frac{x_k}{y_k}}|\eta_k|,$$

and

$$|\eta_k| \leq \sqrt{\frac{y_k}{y_0}}|\eta_0|.$$

PROOF: We define new coordinates (u, v) by $x = u^2$, $y = v^2$. Write the map under these new coordinates as $G = (g_s, g_u)$. Then if $0 \leq u, v \leq \sqrt{a}$ we have:

i) $u_{n+1} = g_s(u_n, v_n), \qquad v_{n+1} = g_u(u_n, v_n),$
ii) $g_s(0, v) = g_u(u, 0) = 0,$
iii) $\left.\frac{\partial g_s}{\partial u}\right|_{(0,0)} = \sqrt{\lambda_1}, \qquad \left.\frac{\partial g_u}{\partial v}\right|_{(0,0)} = \sqrt{\lambda_2}.$

Note that $g_s,\ g_u \in \mathrm{F}^\infty$.

We linearise G:

$$\begin{bmatrix} \gamma_{n+1} \\ \zeta_{n+1} \end{bmatrix} = DG \begin{bmatrix} \gamma_n \\ \zeta_n \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} \begin{bmatrix} \gamma_n \\ \zeta_n \end{bmatrix}.$$

So for $|\gamma_0| \leq |\zeta_0|$ we can see that

$$|\zeta_1| \geq (\sqrt{\lambda_2} - O(a))|\zeta_0|,$$
$$|\gamma_1| \leq (\sqrt{\lambda_1} + O(a))|\zeta_0|,$$

and thus (for sufficiently small a)

$$|\gamma_1| \leq |\zeta_1|, \qquad |\zeta_1| \geq |\zeta_0|.$$

Or in general

$$|\gamma_k| \leq |\zeta_k|, \quad |\zeta_{k+1}| \geq |\zeta_k|, \qquad k = 0, 1, \ldots, n,$$

if $(\gamma_k, \zeta_k) \in U$, $\forall k = 0, 1, \ldots, n$. Transforming back to original coordinates gives

$$\gamma_k = \frac{\xi_k}{2\sqrt{x_k}} \quad , \qquad \zeta_k = \frac{\eta_k}{2\sqrt{y_k}},$$

from which we can obtain the desired results. ∇

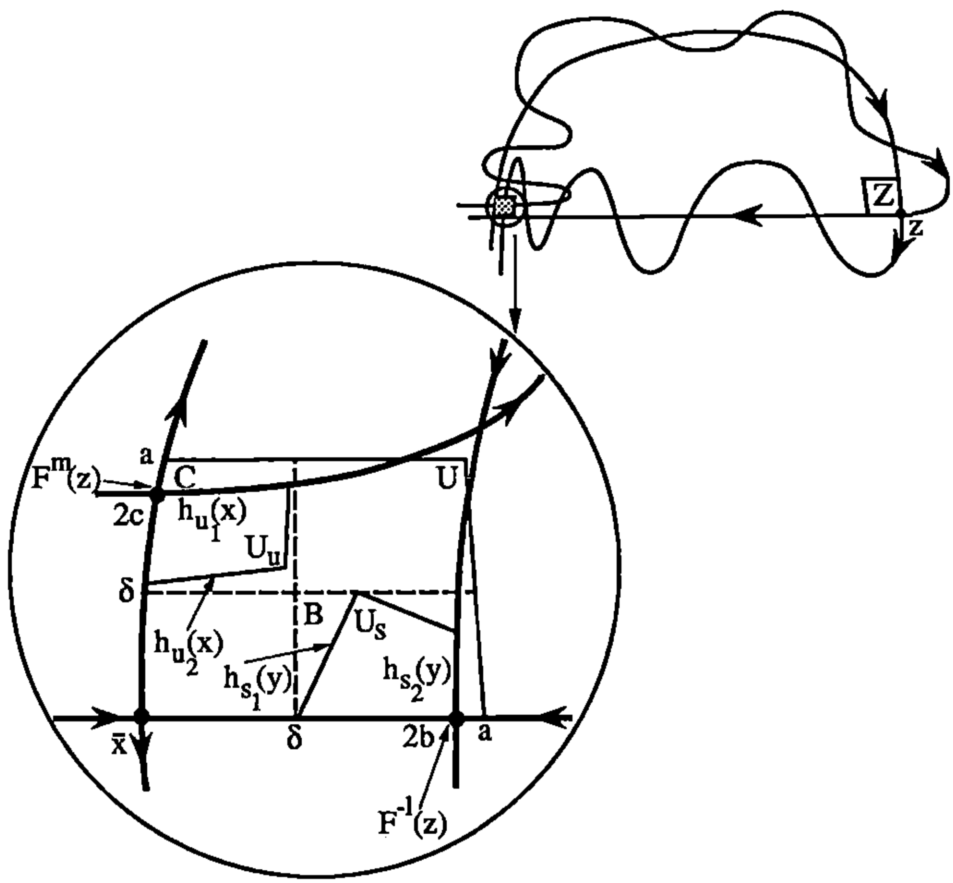

Figure 5.5: Local structure for the Smale–Birkhoff homoclinic theorem.

In other words, in a sufficiently small neighborhood U the sector-bundle $|\xi| \le \sqrt{x/y}|\eta|$ is mapped into itself under DF. Using similar arguments it can be shown that $|\eta| \le \sqrt{y/x}|\xi|$ is mapped into itself by DF^{-1}.

As we are dealing with a local problem, we now define the specific local regions on which the action of F shall be studied. Construct a quadrilateral Z at the homoclinic point z, such that two sides lie on $W^s(\bar{x})$ and $W^u(\bar{x})$ and the other two are parallel to the tangents to $W^s(\bar{x})$ and $W^u(\bar{x})$ at z (see figure 5.5). Z is not of any specific shape or size, the important feature is where it lies in relation to $W^s(\bar{x})$ and $W^u(\bar{x})$.

Since $z \in W^s(\bar{x})$ and $z \in W^u(\bar{x})$, both invariant manifolds, it is true that for any $k \in \mathbb{Z}$ we have $F^k(z) \in W^s(\bar{x})$ and $F^k(z) \in W^u(\bar{x})$, or

$F^k(z) \in W^s(\bar{x}) \cap W^u(\bar{x})$ — that is $F^k(z)$ is also a homoclinic point. In addition, there exist l, $m \in \mathbb{Z}^+$ for which $F^l(z) \in U$, $F^{-m}(z) \in U$ (by travelling along $W^s(\bar{x})$ and $W^u(\bar{x})$ respectively). Fix l, m and a, and define b, $c \in \mathbb{R}$ such that $F^l(z) = (2b, 0)$ and $F^{-m}(z) = (0, 2c)$ — thus $0 < b, c < a/2$.

Now for a small parameter δ, $0 < \delta < a$, choose Z to be such that $F^l(Z) = U_s \subset B = \{(x,y) \mid b \le x \le a, 0 \le y \le \delta\}$, and $F^{-m}(Z) = U_u \subset C = \{(x,y) \mid 0 \le x \le \delta, c \le y \le a\}$. By our choice of the orientation of Z with respect to $W^s(\bar{x})$ and $W^u(\bar{x})$, a simple continuity argument shows that such conditions are easily satisfied. We also have, trivially, that one side of U_s lies on $y = 0$ (and one side of U_u lies on $x = 0$) and an adjacent side intersects $y = 0$ (resp. $x = 0$) transversally. Finally, shrink Z further (if necessary) such that the two sides of U_s adjacent to $y = 0$ (and the two sides of U_u adjacent to $x = 0$) can be written in the form $x = h_{s_1}(y)$, $x = h_{s_2}(y)$, $h_{s_{1,2}} \in C^1(M, M)$ (resp. $y = h_{u_1}(x)$, $y = h_{u_2}(y)$, $h_{u_{1,2}} \in C^1(M, M)$).

We can now define a local version of F to be the map $\Psi \colon U_s \to U_u$ in the following manner: if $p \in U_s$, and there exists $k \in \mathbb{Z}^+$ such that $F^k(p) \in U_u$ and $F^i(p) \in U, \forall i \in \{1, 2, \ldots, k-1\}$, then p is in the domain of Ψ, $D(\Psi)$. So we set

$$\Psi(p) = F^k(p) \in U_u,$$

with k being the smallest integer for which the above holds. That such a k exists is a result of the λ-lemma.

Now define the map $\bar{F} \colon Z \to Z$ by

$$\bar{F} = F^m \Psi F^l.$$

It is this map which produces the horseshoe.

First show that P1 is satisfied by $\bar{F}$, that is there exist horizontal strips $\bar{H}_i \subset Z$, $i = 1, 2, \ldots$, such that $\bar{F}(\bar{H}_i) = \bar{V}_i \subset Z$, where $\bar{V}_i$ are vertical strips. We again employ the λ-lemma. Using the notation of the λ-lemma, we have $\Delta_j = h_{s_j}$, $j = 1, 2$, being two u-disks transverse to $W^s(\bar{x})$ (in this case a u-disk is just a curve segment), and D to be the side of U_u which lies on $W^u(\bar{x})$. Then we have an infinite number of maps of Δ_j — $F^n(\Delta_j)$, $n \in \mathbb{Z}^+$, curve segments between Δ_j and D — which can be found arbitrarily close to D. A continuity argument gives us that the interior of U_s is mapped between $F^n(\Delta_1)$ and $F^n(\Delta_2)$, thus $\cup_{n>0}(F^n(U_s) \cap U_u)$ will consist of an infinite number of vertical strips, $V_i \subset U_u$, $i = 1, 2, \ldots$.

Let us examine in detail the action of F on the unstable manifold. As previously stated the set $\{F^n(z)\}_{n=0}^{\infty}$is a sequence of homoclinic points approaching $\bar{x}$ along $W^s(\bar{x})$ (figure 5.6). Consider the half-open segment of $W^u(\bar{x})$, ω_0, defined to be the points from z to $F(z)$. As we iterate $F(z)$ the distance between the iterates as measured along $W^s(\bar{x})$ shrinks while the distance along $W^u(\bar{x})$ — the length of $\omega_i = F^i(\omega_0)$ — grows, the result being a homoclinic tangle.

Now consider the region U. We can see that the V_i connect the sides ($h_{u_{1,2}}$) of U_u as for sufficiently small a we have

$$x_1 \leq \lambda_1^{\frac{1}{2}} x_0 \quad , \quad y_1 \geq \lambda_2^{\frac{1}{2}} y_0,$$

$x_0, x_1, y_0, y_1 \in U$. So $\Delta_{1,2}$ will eventually span the distance from h_{u_1} to h_{u_2}.

Now, using our definition of Ψ we define horizontal strips as sets $H_i \subset D(\Psi) \subset U_s$ such that $\Psi(H_i) = V_i$, $i = 1, 2, \ldots$. Finally we define $\tilde{H}_i = F^{-l}(H_i)$, and $\tilde{V}_i = F^m(V_i)$. Then $\tilde{H}_i$, $\tilde{V}_i \subset Z$, $i = 1, 2, \ldots$ and $\tilde{F}(\tilde{H}_i) = \tilde{V}_i$ by construction. Thus P1 is satisfied (see figure 5.7).

To show that P3 is satisfied we employ lemma 5.4.2. In fact if we define $\mu = \sqrt{x_0/y_0} < 1/2$ for S^s, and $\mu = \sqrt{y_0/x_0} < 1/2$ for S^u then the conditions of P3 follow directly from the results of the lemma.

Thus we have an invariant set

$$\Lambda = \bigcap_{n>0} \tilde{F}^n(\bigcup_{i,j=1}^{\infty} \tilde{V}_i \bigcap \tilde{H}_j),$$

with the structure of the Smale horseshoe map. □

Remarks

- The above is a weak version of the Smale–Birkhoff homoclinic theorem, chosen for its illustrative value rather than its completeness. In Guckenheimer and Holmes [8] it is shown that S–B holds for systems of arbitrary dimension, and that Λ is always hyperbolic. The structure of Λ in the n-dimensional case is an n-dimensional Cantor array of points.

- A further generalisation is found in Wiggins [41] where he considers hyperbolic invariant tori rather than hyperbolic fixed points. Λ in this case is a Cantor set of tori.

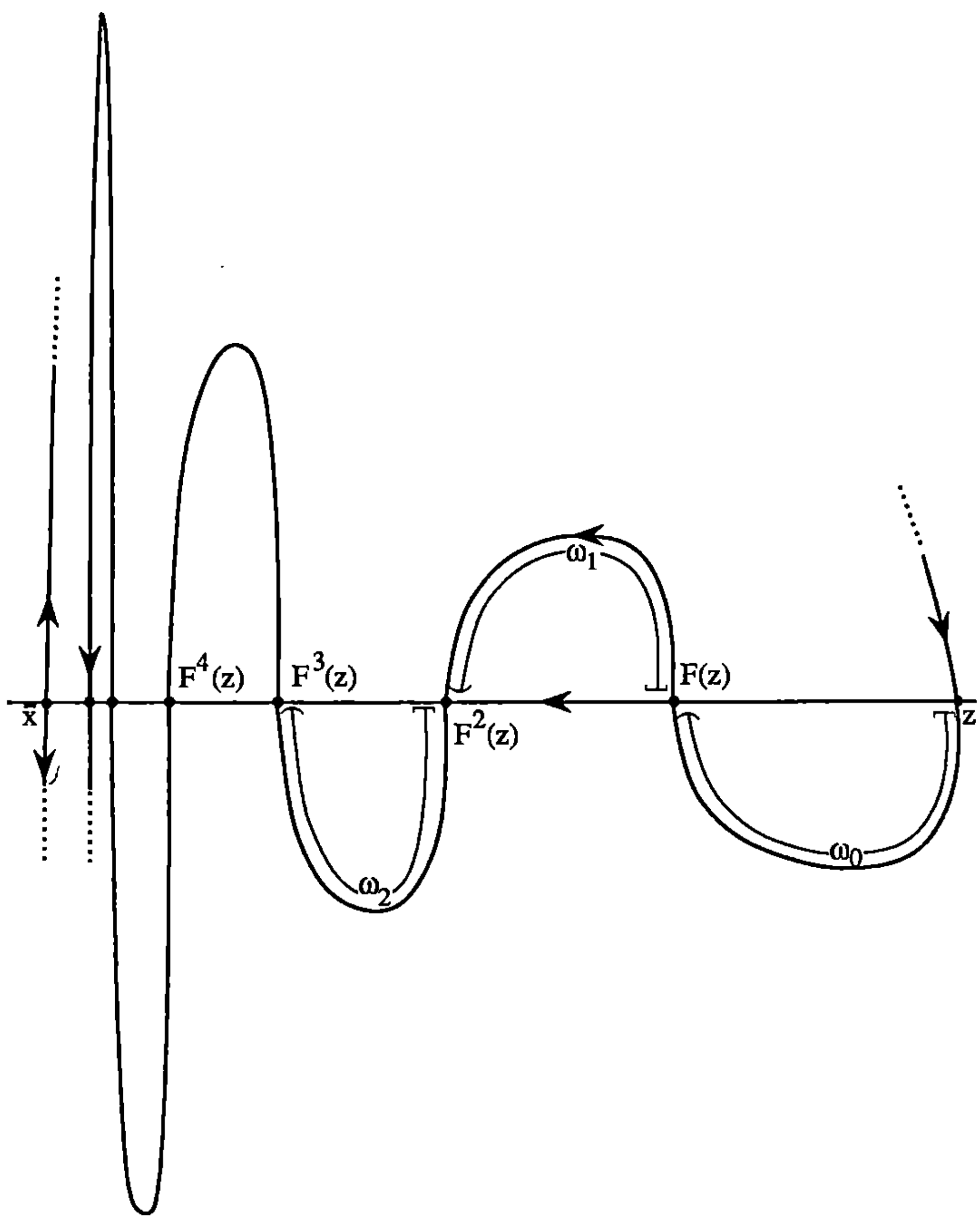

Figure 5.6: The structure of the homoclinic tangle.

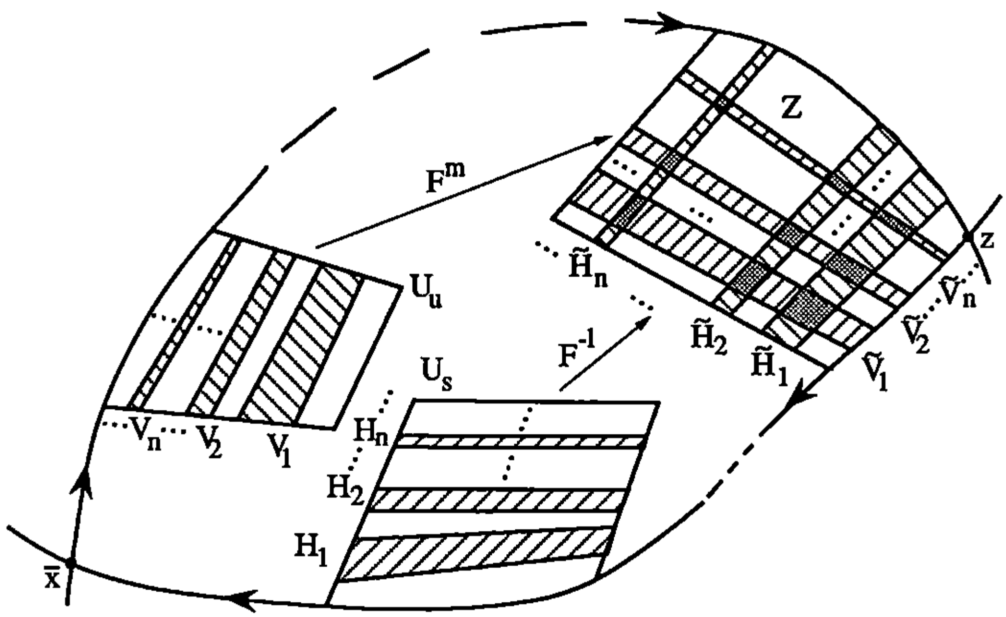

Figure 5.7: Horizontal and vertical strips for the Smale–Birkhoff horseshoe.

- Cantor structure seems to be very closely tied to 'strange' behaviour (i.e. sensitive dependence on initial conditions). The question of whether or not Cantor structure is a generic property of 'strange' invariant sets, however, does not seem to have been addressed. For those sets which are the result of a derivative of the horseshoe map Cantor structure is implicit; beyond this less is known.

  One can also ask the question: "Does Cantor structure imply the existence of a generalised horseshoe map?" The stretching and folding action of a horseshoe map being a natural origin for a Cantor set.

  The answers to these questions would provide an important insight into the general qualitative behaviour of dynamical systems.

- We reiterate that S–B does not imply the existence of a strange attractor. It only provides an invariant set on which the system exhibits strange behaviour. If however, the hyperbolic fixed point with transverse homoclinic orbit is part of an attractor, then that attractor will be a strange attractor. So, while for any hyperbolic dynamical system, S–B implies strange behaviour, it is uncertain as to whether the inverse is true. That is, does a hyperbolic system with strange behaviour always have a transverse homoclinic orbit? Again, we have

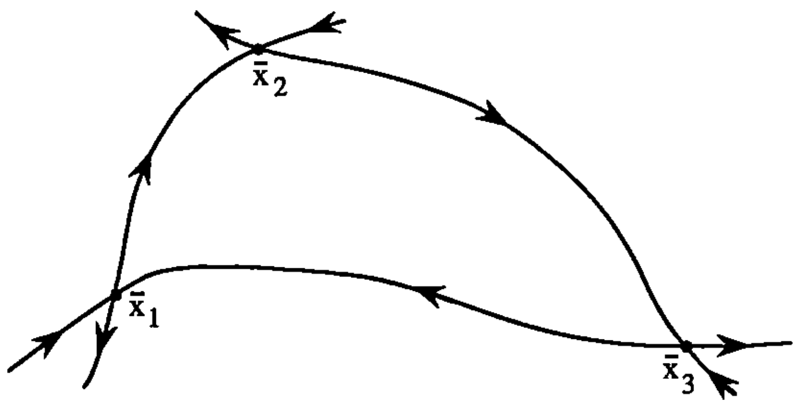

Figure 5.8: A heteroclinic cycle.

sufficient but not necessary conditions.

- If, instead of a homoclinic orbit, the system contains a nontransverse heteroclinic cycle (figure 5.8) it is possible to show that similar behaviour — *hetero*clinic tangles, horseshoes, etc. — shall exist. See Wiggins [41] for details.

5.5 Nonhyperbolic Strange Attractors

Recall theorem 5.5. The properties P1, P2, and P3 are a characterisation of hyperbolicity, thus the theorem is valid only for hyperbolic systems. If we have a nonhyperbolic system then, in addition to W^s and W^u, we have a center manifold W^c. The center manifold is characterised by eigenvalues which are imaginary (for flows) and of modulus one (for maps), thus on this manifold the system exhibits neutral growth behaviour.

With this in mind it is fairly straightforward to create an analogue of theorem 5.5 for the nonhyperbolic case (see Wiggins [41, §2.4] for details). We create the horizontal and vertical sets as before, but now they are extended into the domain of W^c. For example, if we are in $\mathbb{R}^3$ with $\dim W^s = \dim W^u = \dim W^c = 1$, the H_i's and the V_i's, instead of being rectangles in the plane, are slabs with rectangular cross-section in the W^s, W^u plane extending along W^c. Other than this modification to the structure of the horizontal and vertical sets the properties carry over directly.

With this modification, instead of a Cantor set of points we end up with a Cantor set of p-dimensional surfaces (p being the dimension of W^c).

Actually this is not strictly true. It is necessary that the symbol sequences in the Σ^N to which the invariant set is equivalent have a minimum diversity. Consider the matrix A defined by: $(A)_{ij} = 1$, $i, j = 1, 2, \ldots, N$ if the symbol combination ij is allowed in some symbol sequence for this system, and 0 otherwise. Then if for some $k > 0$, $(A^k)_{ij} = 1$, $\forall i, j = 1, 2, \ldots, N$ the invariant set will be a Cantor set. In other words, if enough symbol combinations are allowed, a Cantor set emerges.

This condition is also required of the hyperbolic case. We were working in Σ^2, the space of all possible sequences of two symbols. Thus A was

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

If we only allowed the symbol combinations 11 and 22, A would be

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and obviously, no Cantor set would be possible.

6

Zeeman Stability

The basis for Zeeman stability is very different from that for structural stability. The physical motivation is not as strong as we are now working with algebraic rather than geometric concepts. This is because Zeeman stability operates on a measure on the attractor of the dynamical system in question — requiring algebraic manipulation — while structural stability is concerned exclusively with the topology of the orbits of the system — a geometric point of view.

We shall not attempt a detailed treatment of Zeeman stability (for that see Zeeman's paper [43]) instead we focus on the aspects which are most easily compared to structural stability. These include genericity results, and types of behaviour conserved.

In this chapter we focus on the continuous case. Most of the results discussed have equivalent formulations for the discrete case. However there are some which do not. As the parallel results do not contain any new features and we are involved only in a qualitative examination, we do not cover them.

6.1 Definitions

First we define the space we are to be working in. M is as in §2.2 with the added requirements that it is smooth and without boundary. For a vector field $f \in \mathcal{F}^\infty(M)$ we have:

Definition 6.1 *Given* $\varepsilon > 0$, *the* **Fokker–Planck equation for** f **with** ε**-diffusion** *is*

$$g_t = \varepsilon \Delta g - \nabla \cdot (gf) \tag{6.1}$$

where $g\colon M \times \mathbb{R}^+ \to \mathbb{R}^+$, $g(x,t) \in M \times \mathbb{R}^+$ *and* $\int_M g = 1$, $\forall t \in \mathbb{R}^+$. *Also* $g_t = \partial g / \partial t$, *and the div,* $\nabla\cdot$, *and Laplacian,* $\Delta = \nabla^2$, *are determined by the structure of* M *(see [43, Appendix]).*

Thus g is, effectively, a probability measure on M. However we also shall be treating it as a function. The validity of this is discussed by Zeeman [43, §3.5], the main idea being g is a smooth measure and thus corresponds to a smooth function. The switch from g as a measure to g as a function occurs as we go from the construction of g to its application.

If we denote the steady state solution of (6.1) as

$$g^{f,\varepsilon}: M \to \mathbb{R}^+,$$

found by solving for $g_t = 0$, then we have the following [43, §5:Theorem 3].

Theorem 6.1 *Let $f \in \mathcal{F}^\infty(M)$ and $\varepsilon > 0$. Then there exists a unique steady state $g^{f,\varepsilon}$ to which all solutions of (6.1) tend.*

Recall that we have been defining stability classes in terms of a perturbation. Thus our requirement for equivalence of functions is:

Definition 6.2 *Two smooth functions g, g': $M \to \mathbb{R}^+$ are* **equivalent** *(write $g \sim g'$) if there exist diffeomorphic functions α of M and β of $\mathbb{R}^+$ such that the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{g} & \mathbb{R}^+ \\ \downarrow \alpha & & \downarrow \beta \\ M & \xrightarrow{g'} & \mathbb{R}^+ \end{array}$$

Definition 6.3 *g is a* **stable function** *if there exists a neighborhood $V \subset C^\infty(M, \mathbb{R}^+)$ of g such that $g \sim g'$, $\forall g' \in V$.*

Definition 6.4 *Two vector fields f, $f' \in \mathcal{F}^\infty(M)$ are* **ε-equivalent** *(write $f \overset{\varepsilon}{\sim} f'$) if*

$$g^{f,\varepsilon} \sim g^{f',\varepsilon}.$$

Definition 6.5 *The vector field $f \in \mathcal{F}^\infty(M)$ is* **ε-stable** *if there exists a neighborhood $U \subset \mathcal{F}^\infty(M)$ of f such that $f \overset{\varepsilon}{\sim} f'$, $\forall f' \in U$.*

Definition 6.6 *The vector field $f \in \mathcal{F}^\infty(M)$ is* **Zeeman stable** *if it is ε-stable for arbitrarily small $\varepsilon > 0$*[†]

We provide one example of the application of Zeeman stability. For others see [43, §2.4].

[†]By arbitrarily small ε we mean $\forall \varepsilon' > 0$, $\exists \varepsilon$ such that $0 < \varepsilon < \varepsilon'$. This is weaker than requiring it to hold $\forall \varepsilon > 0$.

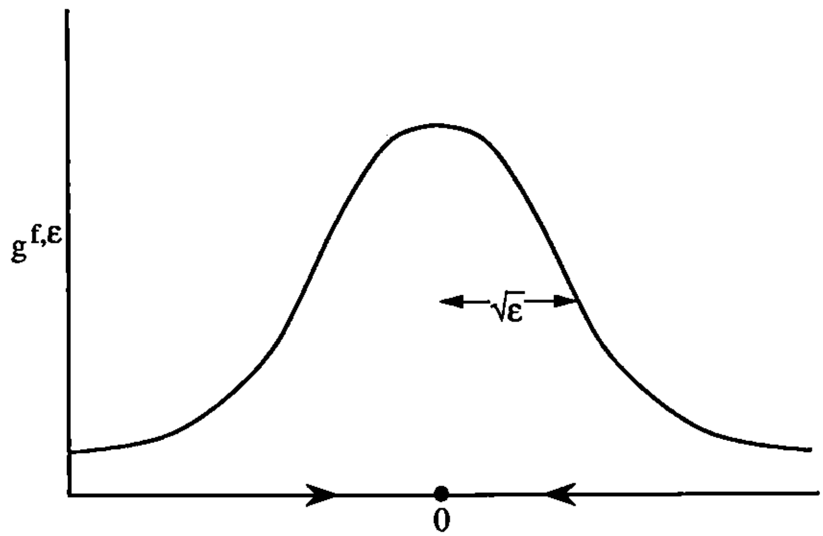

Figure 6.1: The Fokker–Planck steady state for the vector field $f(x) = -x$.

EXAMPLE 1: Let $f(x) = -x$, thus $M = \mathbb{R}$. The steady state of equation (6.1) is found from

$$\varepsilon g_{xx} + (gx)_x = 0.$$

This can be solved directly to obtain

$$g^{f,\varepsilon} = (A \int_0^x e^{y^2/2\varepsilon} dy + B)e^{-x^2/2\varepsilon},$$

where A, $B \in \mathbb{R}$ are constants. Note that if $A < 0$ we have $g < 0$ for sufficiently large x, and similarly if $A > 0$, $g < 0$ for sufficiently large $-x$. We require $g \geq 0$, thus $A = 0$. As we require $\int g = 1$, B is also determined. So the steady state is

$$g^{f,\varepsilon}(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-x^2/2\varepsilon}.$$

This is a normal distribution with mean 0 and variance ε (see figure 6.1). Note that as $\varepsilon \to 0$ the steady state approaches a Dirac δ function at $x = 0$.

The above example is a good illustration of the mechanism of Zeeman stability. In a general sense we have a direct correspondence between the shape of the steady state $g^{f,\varepsilon}$ and the attractor of f, $\Lambda(f)$. The steady state can be considered as the attractor with a normal distribution centered above each point in the attractor.

In the case where f is a gradient vector field, that is we can write $f = -\nabla\phi$ for some function $\phi\colon M \to \mathbb{R}$, the steady state is given by $g^{f,\varepsilon} = ke^{-\phi/\varepsilon}$ (the constant k is obtained from solving $\int g = 1$). In this situation we have the following [43, p.121].

Theorem 6.2 *$f = -\nabla\phi$ is Zeeman stable if and only if ϕ is a stable function.*

Thus we have straightforward necessary and sufficient conditions for a vector field to be Zeeman stable.

6.2 Genericity Results

Zeeman stability is attractive because it is generic and provides one with a complete classification of vector fields. We have the following theorems [43, §6] (corollaries in the original paper).

Theorem 6.3 *Zeeman stable vector fields are residual in $\mathcal{F}^\infty(M)$.*

Theorem 6.4 *ε-stable vector fields are classified by Morse functions*† *ε-unstable vector fields of codimension r are classified by the elementary catastrophes of codimension r.*

With the above we have that a vector field f is ε-stable if and only if $g^{f,\varepsilon}$ is a Morse function. This classification is a direct lifting of Thom's classification of the elementary catastrophes [40].

Note that the above theorems are subject to several limitations. One is that the genericity result holds only for C^∞ systems. Another is there are no equivalent genericity or classification theorems for diffeomorphisms. Concerning the latter, a recent note by Chaperon, de Medrano, López, Watts, and Zeeman [5] lays the ground work for the diffeomorphism case. As far as a full proof is concerned there is only a citation of a preprint in [44].

† A function f is a *Morse function* if the singularities of ∇f are all hyperbolic. The set of Morse functions is open dense in $C^r(M, \mathbb{R}^+)$, $r \geq 1$.

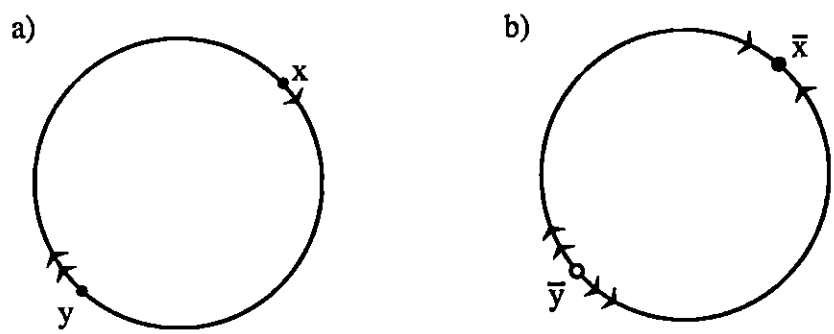

Figure 6.2: ε-equivalent flows on a circle.

6.3 Comparison with Structural Stability

The structure of a flow can be separated into global and local features. The qualitative properties of a flow are dependent on both however the stabilities we are examining do not preserve the same aspects of these features. The global flow is the overall shape of the trajectories, i.e. the closed orbits, intersections of invariant manifolds, etc.. In other words the shape of the surface on which the system operates. The local flow is the details — everything else. This includes: the direction of the flow, the type of fixed points and periodic orbits (the *critical elements*), and the speed of the flow.

Both structural and Zeeman stability preserve the global flow; it is local flow on which they differ. Structural stability — being a geometric construction — preserves the local geometric structure: the critical elements and direction of flow; Zeeman stability, the speed of the flow.

For example, the flows in figure 6.2 are ε-equivalent because the speed of the flow at an attractor $\bar{x}$ and repellor $\bar{y}$ (6.2(b)) is equivalent to a slow flow at x and fast flow at y (6.2(a)). They are not topologically equivalent because: (i) one flow has two fixed points, the other none, and (ii) the flow changes direction in one and not the other. However figure 6.3 demonstrates topologically equivalent (both have a single repelling fixed point and a unidirectional cycle), not ε-equivalent (the flow speed pattern is different).

Another local aspect which must be emphasised is smoothness. With Zeeman stability, equivalence is obtained by means of diffeomorphic functions; structural stability requires a homeomorphism. Homeomorphisms are necessary to ensure that structurally stable flows are dense for two dimensional manifolds. This, according to Zeeman [43, p.130], is *" the beginning of the rot"*, because, as we have seen, structural stability is no longer generic when we go to three dimensions.

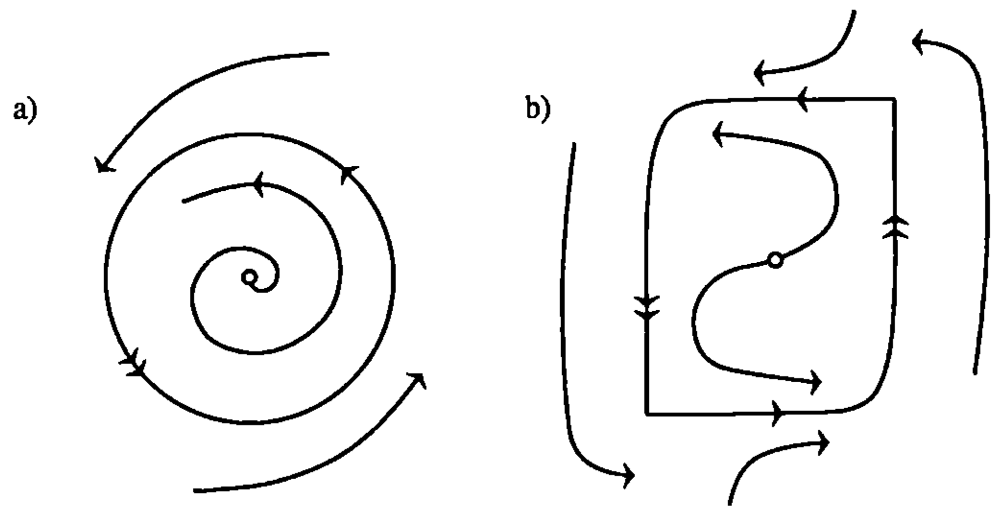

Figure 6.3: Topologically equivalent flows.

This is not the case for ε-equivalence, as Zeeman stability is generic for manifolds of arbitrary dimension. However there have been trade-offs made for genericity here as well, the most serious being with Zeeman stability one loses the dynamics of the system. There have been suggestions made [44] which may allow the incorporation of the dynamics of the system into Zeeman stability but as yet there are no concrete results.

Another problem with Zeeman stability is when dealing with strange attractors — particularly those with Cantor structure — the structural aspects which made them interesting in the first place are obscured by the smoothing process. In fact, in his original paper [43] Zeeman states that he has not as yet been able to demonstrate a Zeeman stable strange attractor. In [44] Zeeman uses the solenoid as an example of the effect of the Fokker–Planck equation on a strange attractor but he does not specifically state that it is Zeeman stable. Thus, although the implication of genericity is that most strange attractors are Zeeman stable, we have no concrete examples, and if we did they would not necessarily exhibit the behaviour associated with strange attractors.

So, while Zeeman stability provides us with the genericity results we desire, the smoothing required to obtain them is so strong as to make it unsuitable for our purposes — the study of the asymptotic behaviour of dynamical systems. So as yet there is no completely satisfactory stability class.

7

Concluding Comments

In this thesis we have attempted to provide the reader with an intuitive grasp of the basic features of dynamical systems, mathematical rigour being limited to those situations where it did not obscure the concepts under discussion. On the whole however, the topics discussed were those which have been most commonly studied. To conclude, we would like to mention some of the aspects of dynamical systems theory which have seen relatively little development.

A major gap in the theory of dynamical systems is for *non-invertible systems*. A non-invertible system is a map $F\colon M \to M$ for which $F^{-n}(x)$, $n \in \mathbb{Z}^+$, $x \in M$, is not unique. For example, the map

$$x_{n+1} = \mu(x_n - x_n^2)$$

is non-invertible (depending, of course, on M). One dimensional non-invertible maps have been studied but most of the interesting behaviour begins in two dimensions. The lack or results here is significant. For example there is no equivalent of the Smale horseshoe and related theorems.

There has also been relatively little work done on nonhyperbolic systems. The results in Wiggins' [41] cover what could be termed the "weakly nonhyperbolic" case. There is little said about the behaviour of the system on the center manifold and, in fact, if there is no stable and unstable manifold it says nothing at all. There is no characterisation of nonhyperbolic behaviour.

Perhaps the most serious shortcoming of dynamical systems theory is the lack of a satisfactory stability. Structural stability is not generic, Zeeman stability disguises the dynamics of the attractor, and other stabilities fail as well. Guckenheimer and Holmes [8] suggest that a modification of the *stability dogma* — the idea that a model is good only if its qualitative properties are resistant to small perturbations — is in order (note, this was taken to be our characterisation of structural stability).

Their argument against the stability dogma is: while it may be nice for a system to be resistant to perturbations in all of its variables — that is, the model has stable qualitative properties — there may be some perturbations which do not make physical sense. Thus Guckenheimer and Holmes propose allowing only those perturbations which are *physically relevant*. The difficulty is to translate "physically relevant" into mathematical terms, applicable to dynamical systems.

Thus, while there are deep and elegant results in dynamical systems theory, the basic physical/theoretical duality is unresolved. And in fact it is still the source of the questions with the most wide-ranging implications.

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VITA

Surname: Evans Given Names: Nolan William
Place of Birth: Leduc, Alberta Date of Birth: June 7, 1964

Educational Institutions Attended, with Dates of Entering and Leaving:

UNIVERSITY OF VICTORIA, BC	1982 to 1987
UNIVERSITY OF VICTORIA, BC	1987 to 1989

Degrees, Diplomas, Etc., Awarded, with Dates and Names of Institutions:

B.Sc. (honours)	1987	University of Victoria, BC

Honours and Awards:

University of Victoria Graduate Scholarship, 1987/88 and 1988/89

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