Étale equivalence relations and C^*-algebras for iterated function systems

by

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ABSTRACT

There is a long history of interesting connections between topological dynamical systems and C^*-algebras. Iterated function systems are an important topic in dynamics, but the diversity of these systems makes it challenging to develop an associated class of C^*-algebras. Kajiwara and Watatani were the first to construct a C^*-algebra from an iterated function system. They used an algebraic approach involving Cuntz-Pimsner algebras; however, when investigating properties such as ideal structure, they needed to assume that the functions in the system are the inverse branches of a continuous map. This excludes many famous examples, such as the standard functions used to construct the Siérpinski Gasket. In this thesis, we provide a construction of an inductive limit of étale equivalence relations for a broad class of affine iterated function systems, including the Siérpinski Gasket and its relatives, and consider the associated C^*-algebras. This approach provides a more dynamical perspective, leading to interesting results that emphasize how properties of the dynamics appear in the C^*-algebras. In particular, we show that the C^*-algebras are isomorphic for conjugate systems, and find ideals related to the open set condition. In the case of the Siérpinski Gasket, we find explicit isomorphisms to subalgebras of the continuous functions from the attractor to a matrix algebra. Finally, we consider the K-theory of the inductive limit of these algebras.

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Chapter 1

Introduction

Fractal sets have been a point of interest within pure mathematics, modelling, and engineering, since the 1970s. Though self-similar sets were constructed in a mathematical setting before this, connections to natural processes were first acknowledged in a serious way in Mandelbrot's essays [18]. Here, a fractal is formally defined as a set which has non-integral Hausdorff dimension. See [11] for further examples of how fractals can be used to model structures appearing in nature. A common method for constructing fractals is by iterating a (finite) collection of contractive maps on a compact set; these referred to as iterated function systems. It should be noted that, sometimes, an *iterated function system* is defined without the constraint that the functions be contractive, with contractive iterated function systems being referred to as *hyperbolic* [2]. However, we will always assume the functions to be contractive.

The construction of fractals by iterated function systems was provided by Hutchinson, in [10]. Here, it is shown that every iterated function system admits a unique, non-empty, compact set equal to the union of its images under the functions. This is called the *attractor* of the system. The attractors for systems of similarities are known as self-similar sets, and have the property that its image under each map is a smaller copy sharing the same shape, aside from a combination of rotations, reflections, or translations. Formally, if X is a metric space, then a similarity on X is any map $f : X \to X$ such that there exists a $\lambda > 0$ with $d(f(x), f(y)) = \lambda d(x, y)$ for all $x, y \in X$. In Euclidean space, this is equivalent to f taking the form $f(x) = \lambda Qx + b$, with Q an orthogonal matrix [22]. In particular, a similarity is a specific type of *affine* map. In the case of iterated function systems, we will only consider $\lambda \in (0, 1)$, to ensure that the maps are contractive. It should also be noted that the phrase "self-similar" is often used to describe any set which is the attractor of a (potentially non-affine) iterated function system.

Affine iterated function systems are those for which each map is an affine transformation of $\mathbb{R}^d$; in other words, the functions are given by $f_i(x) = A_i x + b_i$ for matrices $\{A_i\}_{i=1}^m$ in $\mathcal{M}_d(\mathbb{R})$, and vectors $\{b_i\}_{i=1}^m$ in $\mathbb{R}^d$. An interesting application of affine iterated function systems is image processing, as one is often able to find such a system for which the attractor approximates a given set; the precise inverse problem, however, remains unsolved. See [20] for an exposition on this topic. Many affine iterated function systems have also been used in antenna design, and the reader is referred to [15] for details.

Iterated function systems can be interpreted as topological dynamical systems, which are characterized by continuous transformations of a topological space. Most commonly, a single continuous transformation of the space is considered, and the orbit structure of the associated dynamical system is a common point of interest. The orbit structure can reflect qualitative properties of the system, such as periodicity, global symmetry, and stability, and can be reflected in the properties of certain C^*-algebras associated to the system [31]. Cantor minimal systems and the associated C^*-crossed products is an excellent example of this. A Cantor minimal system is a topological dynamical system in which the underlying space is a Cantor set, and the orbit of every point under the continuous transformation is dense in the space [23]. It was shown by

Giordano, Putnam, and Skau that the K-theory of the C^*-crossed product associated to a Cantor minimal system provides a complete invariant for orbit equivalence [8]. In other words, the K-theory classifies the orbit structure up to orbit-preserving homeomorphisms of the Cantor set. This simultaneously enriched the understanding of both Cantor minimal systems and C^*-algebra theory, and examples such as this provide an excellent motivation for building C^*-algebras based on dynamics.

The underlying dynamics used for C^*-algebraic constructions most commonly consist of a single continuous, invertible map on the space. Iterated function systems, on the other hand, can involve any finite number of continuous, potentially non-invertible maps. There is more than one approach to orbits under an iterated function system and equivalence of dynamics. That being said, it seems natural to consider two iterated function systems to be equivalent when there is a homeomorphism of the attractors that is a topological conjugacy for each function in the system. This is discussed in more detail in Section 2.2. Orbit equivalence, however, is not so helpful for iterated function systems. This is because orbit equivalence is uninteresting for connected spaces, and attractors of iterated function systems are often connected. In the case of connected spaces, an orbit equivalence is a topological conjugacy [23].

The first C^*-algebra construction for iterated function systems was provided by Kajiwara and Watatani in [13], as Cuntz-Pimsner algebras. This may be considered an algebraic approach, as opposed to a groupoid C^*-algebra construction arising naturally from the dynamics. Kajiwara and Watatani investigated properties of these algebras, and showed that when the iterated function system satisfies the *open set condition*, these C^*-algebras are simple and purely infinite. The open set condition requires the existence of a non-empty, open subset V of the attractor, such that the images of V under the maps are pairwise disjoint, and the union of these images is contained in V. Kajiwara and Watatani also showed that these C^*-algebras are isomorphic to the Cuntz algebra $\mathcal{O}_m$, where m is the number of functions in the system, when the iterated function system satisfies the *graph separation condition*. The graph separation condition requires that the *cographs* of each of the functions are pairwise disjoint, where the cograph of a function f is the set $\{(x, y) \in K \times K \mid x = f(y)\}$. To demonstrate that these algebras depend on the dynamics, and not just the attractor, Kajiwara and Watatani provided examples of iterated function systems with homeomorphic attractors, but non-isomorphic C^*-algebras. Since then, KMS states on these algebras were considered [12], as well as their relationship to Exel's crossed product [5, 19]. The ideal structure was investigated by Kajiwara and Watatani in [14], but they needed to restrict to the case where the functions in the system are inverse branches of a continuous map on the attractor. This allowed them to realize the Cuntz-Pimsner algebras as groupoid C^*-algebras, using a method inspired by the branch covering method developed in [6]. However, this inverse-branch requirement is fairly restrictive, and excludes many of the standard iterated function systems for classic self-similar sets, such as the Siérpinski Gasket.

The goal of this thesis is to provide a groupoid C^*-algebra construction for iterated function systems in such a way that the functions need not be inverse branches of a continuous map. This has not yet been done, and offers a different perspective than the approach provided in [13]. The approach is based on features of the IFS which are, in some sense, more dynamical than algebraic. Moreover, it is hoped that in some cases, such as the Siérpinski Gasket, the construction may better reflect the dynamics of the underlying system. Specifically, an equivalence relation groupoid construction is provided for a certain class of affine iterated function systems which includes the Siérpinski Gasket. We restrict our attention to what we have called *single-matrix affine iterated function systems*, which are functions on $\mathbb{R}^d$ taking the form $f_i(x) = Ax + b_i$, $i = 1, 2, \ldots, m$, where $A \in \mathcal{M}_d(\mathbb{R})$ is the same for each function, and each b_i is a fixed translation vector. Such single-matrix systems always satisfies the graph separation condition, so we are, in some sense, elaborating on the case where the construction in [13] simply gives the Cuntz algebra generated by m isometries. However, it should be emphasized that providing a new groupoid perspective is interesting in and of itself.

It should be noted that many famous self-similar sets can be described by single-matrix affine iterated function systems. In two dimensions, some examples include the Siérpinski n-gons, the Siérpinski Carpet, the Fudgeflake, and the Twindragon. Single-matrix affine iterated function systems also include self-affine tiles, which is a broad class of self-similar sets, each of which can tile Euclidean space of the appropriate dimension [16]. Both the topological properties of self-affine tiles, and the geometry of the associated tilings are interesting topics; see [17] and the references therein.

We begin by introducing the mathematical background for iterated function systems in Chapter 2, including descriptions of the examples mentioned above. In Chapter 3, some basic facts about C^*-algebras are summarized, followed by the definition of an étale equivalence relation, and the associated C^*-algebra construction. Some relevant facts about inductive limits of both étale equivalence relations and the associated C^*-algebras are considered. Then, in Chapter 4, a countable, increasing sequence of étale equivalence relations is constructed for single-matrix iterated function systems, and we consider its inductive limit. In particular, the method for constructing the étale topology is based on the concept of a local action; we first define the local actions, then show they form a basis for an étale topology on the equivalence relation. We consider properties of the associated C^*-algebras in Chapter 5. Specifically, an increasing sequence of open invariant subsets of each étale equivalence relation is found when the underlying IFS satisfies the open set condition; a comparison of these sets for the Siérpinski Gasket and the Fudgeflake is given. We consider the relationship between a variation of the open set condition, and certain types of ideals in the C^*-algebras. Then, the C^*-algebras for the Siérpinski Gasket iterated function system is considered in detail, by providing explicit isomorphisms to subalgebras of the continuous functions from the attractor to a matrix algebra. Finally, in Chapter 6, we consider the K-theory of the inductive limit of these algebras, restricting to the special case of the Siérpinski Gasket.

Chapter 2

Iterated function systems

In this chapter, we define iterated function systems on $\mathbb{R}^d$, and the notion of an attractor of such a system. We prove a couple of basic facts about attractors, and discuss conjugacy for iterated function systems. Then, we restrict our attention to a specific class of affine iterated function systems; namely, what we will refer to as single-matrix affine iterated function systems. Some nice properties of these systems are presented, as well as an exposition of some famous examples.

2.1 General iterated function systems

We will be considering collections of contractive maps on subsets of $\mathbb{R}^d$, the d-dimensional Euclidean space, with the standard norm $||x||_2 = \left(\sum_{i=1}^d x_i^2\right)^{\frac{1}{2}}$, $x \in \mathbb{R}^d$.

Definition 2.1.1. *Let $X \subseteq \mathbb{R}^d$. A contraction on X is a map $f : X \to X$ for which there exists a constant $\lambda \in (0,1)$ such that $||f(x) - f(y)||_2 \leq \lambda ||x - y||_2$ for all $x, y \in X$. We will refer to the constant λ as a contraction factor.*

We can now present the general definition of an iterated function system on $\mathbb{R}^d$.

Definition 2.1.2 (Iterated Function System)**.** *Let X be a closed subset of $\mathbb{R}^d$, and let $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ be a finite collection of contractions on X. Then, $(X, \mathcal{F})$ is* *an iterated function system. Note that it is common to use the acronym IFS.*

Remark 2.1.3. *The contraction factors for the functions in a general iterated function system need not be the same.*

To every iterated function system $(X, \mathcal{F})$, there is a unique, non-empty, compact subset of X associated to it, called the *attractor* of $(X, \mathcal{F})$. This is the main result of a theorem by J. E. Hutchinson, which can be stated as follows. See [10], p. 724.

Theorem 2.1.4 (Hutchinson, Part I)**.** *Let $(X, \mathcal{F})$ be an iterated function system, with $\mathcal{F} = \{f_1, \ldots, f_m\}$. Then, there exists a unique, non-empty, compact subset $K \subseteq X$ such that*

$$K = \bigcup_{i=1}^{m} f_i(K).$$

There is a second part to this theorem, explaining how one can construct the attractor from any non-empty compact subset of X; however, before stating this result, we should first define *Hutchinson's operator*. Let $\mathcal{S}(X)$ be the set of all non-empty, compact subsets of X. Define the map $F : \mathcal{S}(X) \to \mathcal{S}(X)$ by

$$F(E) = \bigcup_{i=1}^{m} f_i(E).$$

Then, iterating F on a set E will give you the attractor K.

Theorem 2.1.5 (Hutchinson, Part II)**.** *Let K be the attractor of an iterated function system $(X, \mathcal{F})$. Then, for any non-empty, compact subset $E \subseteq X$ containing K,*

$$K = \bigcap_{k \geq 1} F^k(E).$$

Let us set up some helpful notation, and prove a couple of preliminary results about the attractor of $(X, \mathcal{F})$.

Definition 2.1.6 (Some helpful notation)**.** *Let* $(X, \mathcal{F})$ *be an iterated function system. Define:*

(a) $\Sigma_n = \{1, 2, \ldots, m\}^n$ *for* $n \geq 1$*,*

(b) $f_\xi = f_{\xi_1} \circ \cdots \circ f_{\xi_n}$ *for each* $\xi \in \Sigma_n$*,*

(c) $\mathcal{F}^{-n}(E) = \{z \in K \mid \exists\, \xi \in \Sigma_n \text{ s.t. } f_\xi(z) \in E\}$ *for each* $n \geq 1$*,* $E \subseteq K$*.*

It is a simple induction to see that the attractor K of $(X, \mathcal{F})$ is also the union of the images resulting from iterating the functions n-times. This is stated precisely in the following proposition, which will be of use in Chapter 4, when we want to know that given an $n \geq 1$, any point in K lies in the image $f_\xi(K)$ for a $\xi \in \Sigma_n$.

Proposition 2.1.7. *Let* $(X, \mathcal{F})$*,* $\mathcal{F} = \{f_i\}_{i=1}^m$ *be an iterated function system. Then* $\bigcup_{\xi \in \Sigma_n} f_\xi(K) = K$*.*

Proof. We prove this by induction on n. The base case, when $n = 1$, is exactly a fundamental result for iterated function systems:

$$\bigcup_{j=1}^{m} f_j(K) = K.$$

Now, assume that for some $N \geq 1$,

$$\bigcup_{\xi \in \Sigma_N} f_\xi(K) = K.$$

We have:

$$\begin{aligned}
\bigcup_{\xi \in \Sigma_{N+1}} f_\xi(K) &= \bigcup_{\xi \in \Sigma_N} \left(\bigcup_{j=1}^{m} f_\xi \circ f_j(K) \right) \\
&= \bigcup_{\xi \in \Sigma_N} f_\xi \left(\bigcup_{j=1}^{m} f_j(K) \right).
\end{aligned}$$

Then, using the base case, this simplifies to

$$\bigcup_{\xi \in \Sigma_N} f_\xi(K) = K$$

after applying the induction hypothesis. $\square$

There is one more fact we will use that holds for general iterated function systems. It says that the pre-image $\mathcal{F}^{-1}(E)$ of a set $E \subseteq K$ is simply the pre-images of E under each of the functions in $\mathcal{F}$, intersected with K. This result will be used in verifying that the sequence of equivalence relations defined in Chapter 4 is increasing.

Lemma 2.1.8. *For all $E \subseteq K$, $\mathcal{F}^{-1}(E) = \bigcup_{i=1}^{m} f_i^{-1}(E) \cap K$.*

Proof. By definition, $\mathcal{F}^{-1}(E) = \{z \in K \mid \exists\ i \in \Sigma \text{ such that } f_i(z) \in E\}$. As $\Sigma = \{1, 2, \ldots, m\}$, it is clear that:

$$\{z \in K \mid \exists\ i \in \Sigma \text{ such that } f_i(z) \in E\} = \bigcup_{i=1}^{m} \{z \in K \mid f_i(z) \in E\}.$$

Finally, $\{z \in K \mid f_i(z) \in E\} = f_i^{-1}(E) \cap K$ for each $i \in \Sigma$. $\square$

In Chapter 5, we will consider the relationship between the number of *addresses* of points in the attractor, and the ideals in the associated C^*-algebra. Let us now define this concept of addresses, and establish some helpful terminology for discussing properties of attractors for iterated function systems.

Definition 2.1.9 (The code space)**.** *Let $(X, \mathcal{F})$, $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$, be an iterated function system on $\mathbb{R}^d$. Let $\Sigma = \{1, 2, \ldots, m\}^{\mathbb{N}}$ denote the collection of all infinite sequences in $\{1, 2, \ldots, m\}$. Then, the code space for $(X, \mathcal{F})$ is defined to be the metric space (Σ, d_C), where d_C is given by*

$$d_C(\xi, \eta) = \sum_{n=1}^{\infty} \frac{|\xi_n - \eta_n|}{(m+1)^n}.$$

Remark 2.1.10. (Σ, d_C) *is equivalent to the metric space resulting from the alternative metric d on Σ, where*

$$d(\xi, \eta) = \left| \sum_{n=1}^{\infty} \frac{\xi_n - \eta_n}{(m+1)^n} \right|.$$

The following theorem gives a clear picture of the relationship between the code space and points on the attractor. The reader is referred to [2] for details of the proof.

Theorem 2.1.11. *Let $(X, \mathcal{F})$, $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$, be an iterated function system on $\mathbb{R}^d$. Let K be the attractor of $(X, \mathcal{F})$, and (Σ, d_C) its code space. For each $\xi \in \Sigma$, $n \geq 1$, and $x \in X$, define*

$$\phi(\xi, n, x) = f_{\xi_1} \circ \cdots \circ f_{\xi_n}(x).$$

Then, the following limit exists, is independent of x, and is an element of K.

$$\phi(\xi) = \lim_{n \to \infty} \phi(\xi, n, x).$$

Moreover, the resulting map $\phi : \Sigma \to K$ taking ξ to $\phi(\xi)$ is continuous and surjective.

Now, we can provide a formal definition of addresses of points on the attractor, in terms of the code space.

Definition 2.1.12 (Set of addresses)**.** *Let $(X, \mathcal{F})$, K, and ϕ be as in Theorem 2.1.11. Then the set of addresses of a point $x \in K$ is*

$$\phi^{-1}\{x\} = \{\xi \in \Sigma \mid \phi(\xi) = x\}.$$

Furthermore, any such $\xi \in \phi^{-1}\{x\}$ is referred to as an address of x.

Next, let us establish some terminology for common topological features of attractors for iterated function systems.

Definition 2.1.13 (Separation properties for IFS)**.** *Let* $(X, \mathcal{F})$*,* $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$*, be an iterated function system on* $\mathbb{R}^d$*. Let* K *be the attractor of* $(X, \mathcal{F})$*.*

(i) $(X, \mathcal{F})$ *is said to satisfy the* *strong separation condition* *when*

$$f_i(K) \cap f_j(K) = \emptyset \ \forall \ i \neq j.$$

This is also referred to as *totally disconnected*, *and is equivalent to every point of* K *having a unique address [2].*

(ii) $(X, \mathcal{F})$ *is said to satisfy the* *open set condition* *when there exists a non-empty open subset* $U \subset K$ *such that*

$$\bigcup_{i=1}^{m} f_i(U) \subset U \text{ and } f_i(U) \cap f_j(U) = \emptyset \ \forall \ i \neq j.$$

Note that this means that U *is an open, dense subset of* K*. A proof of this can be found in [7], p.141.*

(iii) If $(X, \mathcal{F})$ *satisfies (ii) but not (i), then it is referred to as* *just-touching*.

(iv) If $(X, \mathcal{F})$ *is neither just-touching nor totally disconnected, it is referred to as* *overlapping*.

(v) $(X, \mathcal{F})$ *is said to satisfy the* *graph separation condition* *when*

$$\mathit{cograph}(f_i) \cap \mathit{cograph}(f_j) = \emptyset \ \forall \ i \neq j$$

where $\mathit{cograph}(f_i) = \{(x, y) \in K \times K \mid x = f_i(y)\}$.

2.2 Dynamics of iterated function systems

The main reason one likes to associate a C^*-algebra to a topological dynamical system is the potential for using properties of the C^*-algebras to inspire classification results for the underlying dynamics. At the very least, we would like the C^*-algebra to reflect the dynamics in some way; therefore, it is necessary to clarify what we mean by the dynamics of an iterated function system. It is natural to begin by deciding when two iterated function systems are equivalent, which is what we will refer to as conjugacy.

Definition 2.2.1 (Conjugate IFS). *Let $(X, \{f_i\}_{i=1}^m)$ and $(Y, \{g_i\}_{i=1}^m)$ be two iterated function systems with attractors $K \subseteq X$ and $K' \subseteq Y$ respectively. We say that $(X, \{f_i\}_{i=1}^m)$ and $(Y, \{g_i\}_{i=1}^m)$ are conjugate if there exists a homeomorphism $h : K \to K'$ such that $h \circ f_i(x) = g_i \circ h(x)$ for each $x \in K$, and for each $i = 1, \ldots, m$.*

In particular, two iterated function systems are *not* conjugate when the number of functions in the systems differ; if the number of functions is the same, but the attractors are non-homeomorphic, then the systems are still automatically non-conjugate. The most ideal situation would be to find a complete invariant of the dynamics, in the sense that the associated property distinguishes any pair of non-conjugate systems. In most cases, one lands somewhere in the middle. At the very least, we would like to have the associated C^*-algebras of conjugate systems to be isomorphic.

It should also be pointed out that because we are dealing with more than one map, this notion of conjugacy for iterated function systems differs from the standard notion of conjugacy for a topological dynamical system, where a single continuous map on a space is considered. There are, in fact, ways to consolidate an iterated function system into a dynamical system consisting of a single map on the attractor, though some require further assumptions on the iterated function system.

In [2], Barnsley provides a notion of shift dynamics in the case where the iterated function system is totally disconnected. In this situation, every point on the attractor lies in exactly one of the images of the functions in the iterated function system. The shift dynamics consist of taking these unique pre-images of points on the attractor. One way to generalize this notion to the cases where the images overlap is to simply take pre-images at random. Barnsley refers to this as the *random shift dynamics*. However, for those who prefer deterministic dynamics, one can use the associated code space to lift the iterated function system to one which is totally disconnected. Moreover, Barnsley shows that the shift dynamics on a totally disconnected iterated function system is topologically conjugate to the shift dynamics on the code space, so conjugacy for the lifted shift dynamics does not distinguish iterated function systems. Therefore, the notion of conjugacy given in Definition 2.2.1 will be more suited to our purpose.

There is another method of creating a dynamical system from an iterated function system which should be mentioned. In the case where the functions are actually the branches of the inverse of a map, one can use this map to define the dynamics. See [7] for details. In such a situation, conjugacy in the sense of Definition 2.2.1 implies that this notion of topological conjugacy also holds.

2.3 Affine iterated function systems

Definition 2.3.1 (Affine IFS)**.** *Let $X \subseteq \mathbb{R}^d$ be closed. An iterated function system $(X, \mathcal{F} = \{f_i\}_{i=1}^m)$ is called affine when each function $f \in \mathcal{F}$ is of the form $f(x) = Ax + b$, were $A \in \mathcal{M}_d(\mathbb{R})$, $b \in \mathbb{R}^d$. In other words, there exist m matrices $\{A_i\}_{i=1}^d$ in $\mathcal{M}_d(\mathbb{R})$ and m vectors $\{b_i\}_{i=1}^d$ in $\mathcal{M}_d(\mathbb{R})$ such that for each i, $f_i(x) = A_i x + b_i$.*

Remark 2.3.2. *We often take $X = \mathbb{R}^d$. In this case, Definition 2.3.1 implicitly assumes that the norm of each matrix is less than one; indeed, a function $f(x) =$*

$Ax + b$ is a contraction on X if and only if $||A|| = \sup_{||x||_2 \leq 1} \{||Ax||_2\} < 1$.

Remark 2.3.3. *Many nice examples come from the special case of $A = \lambda I_d$, where $\lambda \in (0, 1)$, and I_d is the $d \times d$ identity matrix, some of which we will see in Section 2.4.*

Let us restrict our attention to the following class of affine iterated function systems, which have the property that the images of the attractor under each function in the system differ only by a translation.

Definition 2.3.4 (Single-Matrix Affine IFS)**.** *Let $X \subseteq \mathbb{R}^d$ be closed. Suppose that $A \in \mathcal{M}_d(\mathbb{R})$ is a $d \times d$ real matrix, and $b_1, \ldots, b_m \in \mathbb{R}^d$ are fixed vectors. Further, suppose there is a $\lambda \in (0, 1)$ such that $||Ax||_2 \leq \lambda ||x||_2$ for all $x \in \mathbb{R}^d$. Define:*

$$f_i(x) = Ax + b_i$$

for all $x \in X$, and for all $1 \leq i \leq m$. Let $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$. Then, we will call $(X, \mathcal{F})$ a single-matrix iterated function system. If the matrix A is also invertible, then we will call $(X, \mathcal{F})$ an invertible single-matrix IFS.

Remark 2.3.5. *Note that when A is invertible, every $f \in \mathcal{F}$ is a continuous, invertible function on $\mathbb{R}^d$. In particular, each function is injective, which implies that invertible single-matrix affine IFS always satisfy the graph separation condition.*

The following Lemma verifies that compositions of single-matrix affine functions are also affine, and compositions of n functions all consist of the linear transformation A^n along with a shift.

Lemma 2.3.6. *For each $n \geq 1$, $\xi \in \Sigma_n$, there is a vector v_ξ such that $f_\xi(x) = A^n x + v_\xi$ for all $x \in \mathbb{R}^d$.*

Proof. We prove this by induction. The base case, $n = 1$, follows from the definition of $\mathcal{F}$. Now, suppose this holds for some $k \geq 1$. Let $\xi = (\xi_1, \xi_2, \dots, \xi_k, \xi_{k+1}) \in \Sigma_{k+1}$, and consider f_ξ. For convenience, let $\xi' = (\xi_1, \xi_2, \dots, \xi_k) \in \Sigma_k$. We have:

$$f_\xi(x) = f_{\xi'} \circ f_{\xi_{k+1}}(x).$$

As $\xi' \in \Sigma_k$, $f_{\xi'}(x) = A^k x + v_{\xi'}$ for all $x \in \mathbb{R}^d$. Therefore:

$$\begin{aligned} f_{\xi'} \circ f_{\xi_{k+1}}(x) &= A^k f_{\xi_{k+1}}(x) + v_{\xi'} \\ &= A^k(Ax + b_{\xi_{k+1}}) + v_{\xi'} \\ &= A^{k+1}x + A^k b_{\xi_{k+1}} + v_{\xi'}. \end{aligned}$$

From this, we see that $f_\xi(x) = A^{k+1}x + v_\xi$, where $v_\xi = A^k b_{\xi_{k+1}} + v_{\xi'} \in \mathbb{R}^d$. □

Remark 2.3.7. *From this formula, one can easily see that for each $\xi \in \Sigma_n$, and each $x \in \mathbb{R}^d$,*

$$f_\xi^{-1}(x) = A^{-n}(x - v_\xi).$$

Proposition 2.3.8. *For all $n \geq 1$, and ξ, $\eta \in \Sigma_n$, $f_\xi \circ f_\eta^{-1}$ is a translation. In other words, there is a $v \in \mathbb{R}^d$ such that $f_\xi \circ f_\eta^{-1}(x) = x + v$ for all $x \in K$.*

Proof. Let $\xi, \eta \in \Sigma_n$, and $x \in \mathbb{R}^d$. By Lemma 2.3.6, $f_\xi(x) = A^n x + v_\xi$ and $f_\eta^{-1}(x) = A^{-n}(x - v_\eta)$. Therefore,

$$\begin{aligned} f_\xi \circ f_\eta^{-1}(x) &= A^n(f_\eta^{-1}(x)) + v_\xi \\ &= A^n A^{-n}(x - v_\eta) + v_\xi \\ &= x - v_\eta + v_\xi \end{aligned}$$

which is a translation, with $v = -v_\eta + v_\xi$. □

2.4 Examples of single-matrix affine IFS

In this section, we look at examples of single-matrix affine iterated function systems in $\mathbb{R}^2$. In particular, we will look at the Siérpinski n-gons, the Siérpinski Carpet, the Fudgeflake, and the Twindragon, as a selection of classical examples. Both the Fudgeflake, and the Twindragon, fall into the broad class of examples known as self-affine tiles [16]. A brief exposition on this is provided in Section 2.4.3.

2.4.1 The Siérpinski n-gons

The Siérpinski Gasket, also known as the Siérpinski Triangle, has been an object of interest since the early 1900s, first appearing in the paper [28], by Wacław Siérpinski. The title of this paper reads *on a curve every point of which is a point of ramification*; indeed, Siérpinski showed that, aside from the three main vertices, every point of this curve is a point of ramification [30]. A point of ramification, also referred to as a branch point, is one for which the boundary of any neighborhood intersected with the curve consists of more than two points. Mandelbrot then coined this curve "Siérpinski's Gasket," as inspired by the construction which relies on iteratively removing "tremas" from a triangle [18].

Figure 2.1 shows a decent approximation of the Siérpinski Gasket, using six iterations of the following functions, starting from a solid equilateral triangle of side length 1, centered at $(\frac{1}{2}, \frac{\sqrt{3}}{6})$. Figure 2.2 shows the first three iterations.

$$\begin{aligned} f_1(x) &= \frac{1}{2}x \\ f_2(x) &= \frac{1}{2}x + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \\ f_3(x) &= \frac{1}{2}x + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix} \end{aligned} \tag{2.1}$$

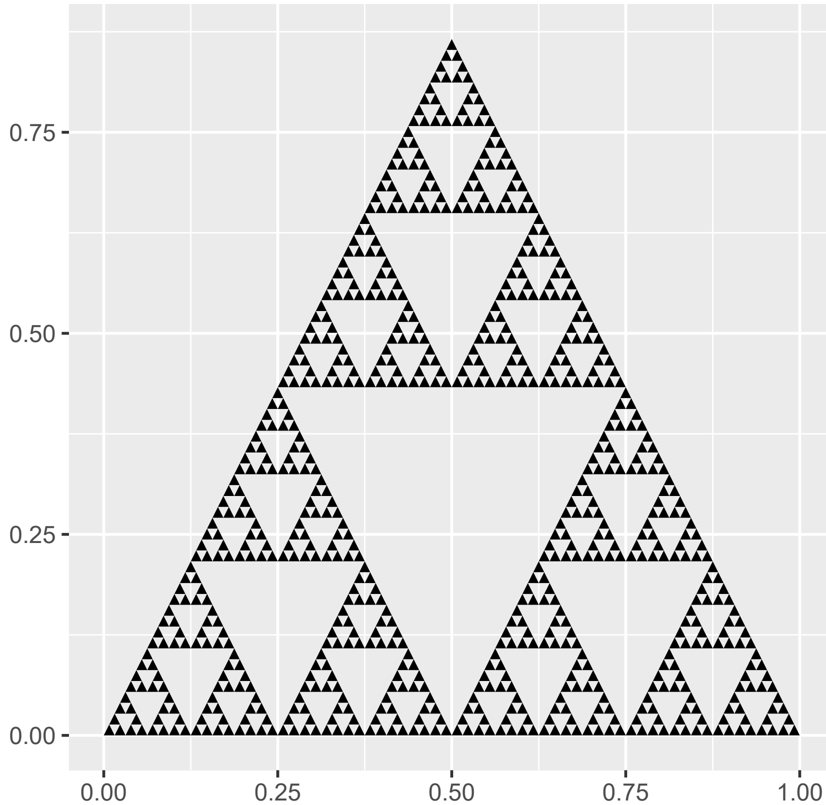


Figure 2.1: The Siérpinski Gasket.

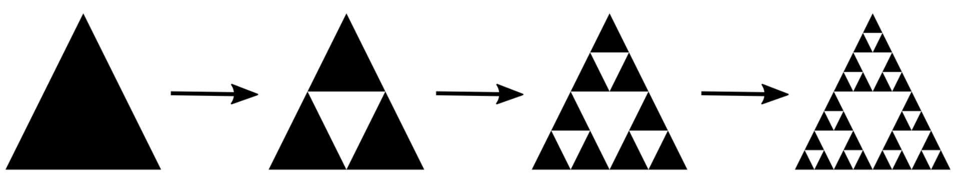


Figure 2.2: Construction of the Siérpinski Gasket.

The IFS in (2.1) can easily be generalized from a 3-function system, to an n-function single-matrix system, by fitting n identical, smaller copies of a regular n-gon into itself. Note that to do this, both the number of functions and the matrix must change with n. The Siérpinski 4-gon, or the Siérpinski square, is simply that: a square. However, for $n \geq 5$, you can always get an interesting, just-touching, self-similar set. In this case, one can even write down a formula for the n-many functions of an IFS that has the Siérpinski n-gon as the attractor. See [27] for details of the construction. See Figures 2.3-2.6 for the Siérpinski Pentagon, and the Siérpinski Hexagon. It may be worth noting that, if one fills in all the central hexagons in the Siérpinski Hexagon, the result will be the Koche Snowflake [25].

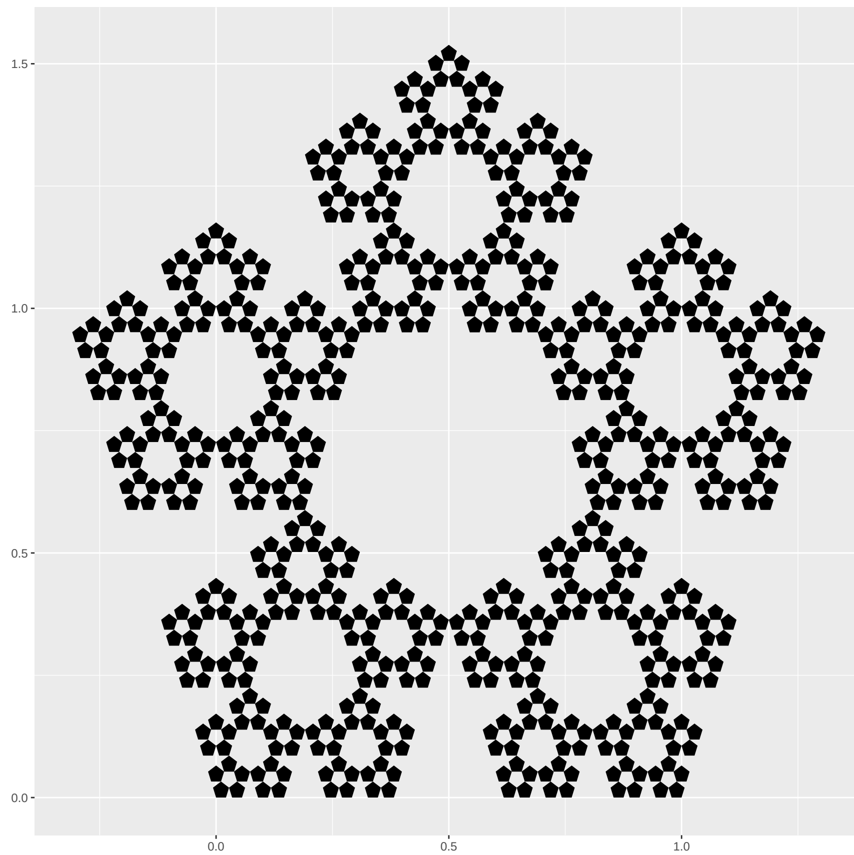


Figure 2.3: The Siérpinski Pentagon.

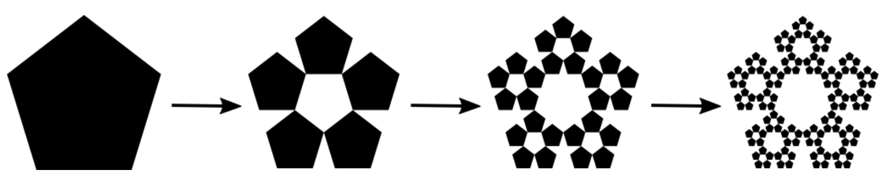


Figure 2.4: Construction of the Siérpinski Pentagon.

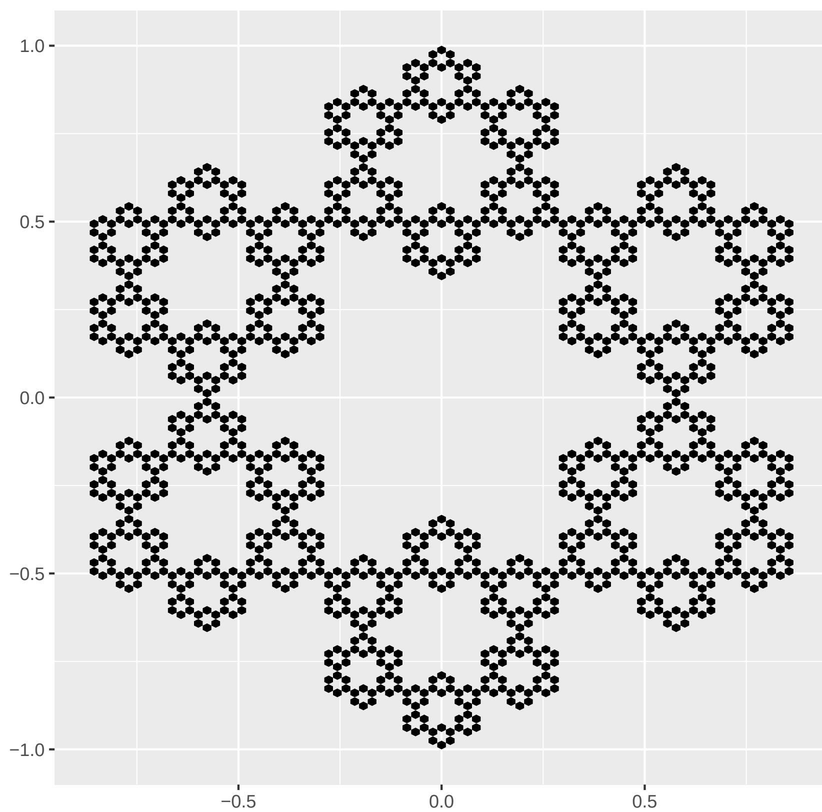


Figure 2.5: The Siérpinski Hexagon.

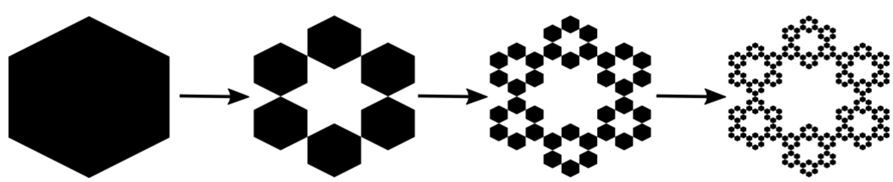


Figure 2.6: Construction of the Siérpinski Hexagon.

2.4.2 The Siérpinski Carpet

Even though the Siérpinski 4-gon is simply a square, by using 8 functions instead of 4, an interesting self-similar set can be constructed from just-touching, congruent squares. This is again due to Siérpinski, and is referred to as the Siérpinski Carpet. Figure 2.7 shows a decent approximation of the Siérpinski Carpet, starting from a solid square of side length 1, centered at $(\frac{1}{2}, \frac{\sqrt{1}}{2})$. Figure 2.8 shows the first three iterations.

$$
\begin{aligned}
f_1(x) &= \frac{1}{3}x & f_5(x) &= \frac{1}{3}x + \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} \\
f_2(x) &= \frac{1}{3}x + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} & f_6(x) &= \frac{1}{3}x + \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \\
f_3(x) &= \frac{1}{3}x + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} & f_7(x) &= \frac{1}{3}x + \begin{bmatrix} 0 \\ 2/3 \end{bmatrix} \\
f_4(x) &= \frac{1}{3}x + \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} & f_8(x) &= \frac{1}{3}x + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}
\end{aligned} \tag{2.2}
$$

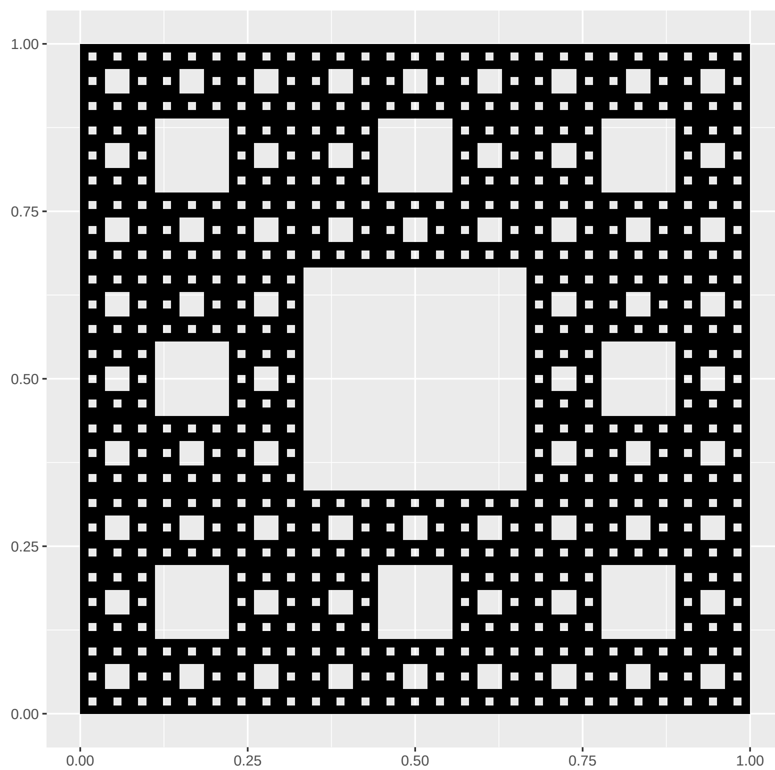


Figure 2.7: The Siérpinski Carpet.

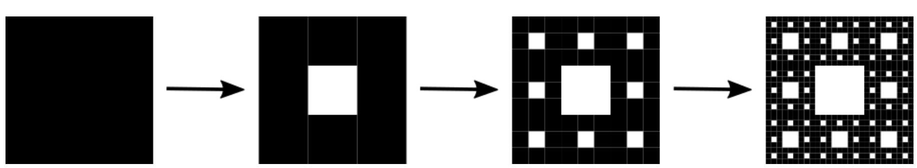


Figure 2.8: Construction of the Siérpinski Carpet.

2.4.3 Self-affine tiles

Let us take a brief excursion from specific, classical examples, and look at the broader collection of examples of single-matrix affine iterated function systems coming from self-affine tiles. Let us begin by defining the notion of self-affine tile, following the set-up in [16].

Definition 2.4.1 (Expanding matrix)**.** *A matrix $B \in \mathcal{M}_d(\mathbb{C})$ is called expanding if all of its eigenvalues have modulus strictly greater than 1.*

Definition 2.4.2 (Self-affine tile)**.** *Let $T \subseteq \mathbb{R}^d$ be compact, with positive Lebesgue measure. If there exists an expanding matrix $B \in \mathcal{M}_d(\mathbb{R})$ such that*

$$B(T) = \bigcup_{i=1}^{m} (T + c_i) \tag{2.3}$$

for a collection $\mathcal{D} = \{c_1, c_2, \ldots, c_m\}$ of vectors in $\mathbb{R}^d$, and $(T + c_i) \cap (T + c_j)$ has measure zero whenever $i \neq j$, then T is a <u>self-affine tile</u>. The set $\mathcal{D}$ is called a <u>digit set</u>.

This definition actually imposes a restriction on the expanding matrix B, and the digit set $\mathcal{D}$, as follows.

Proposition 2.4.3. *If T is a self-affine tile with expanding matrix B and digit set $\mathcal{D}$, then $|\det(B)| = |\mathcal{D}| = m$.*

Proof. Recall that the determinant has the following relationship with Lebesgue measure:

$$\text{Leb}(B(E)) = \det(B)\text{Leb}(E) \text{ for any measurable set } E \subseteq \mathbb{R}^d.$$

Using this fact, and Equation (2.3):

$$\mathrm{Leb}(B(T)) = \det(B)\mathrm{Leb}(T) = \bigcup_{i=1}^{m} \mathrm{Leb}(T + c_i) = m\mathrm{Leb}(T).$$

Then, because $\mathrm{Leb}(T) > 0$, we get that $\det(B) = m$. □

Let us now consider the relationship of self-affine tiles to IFS. The following proposition should feel reminiscent of Hutchinson's theorem on existence and uniqueness of the attractor for IFS. See [16], p. 23.

Proposition 2.4.4. *For any expanding matrix B, and any finite set $\mathcal{D}$ in $\mathbb{R}^d$, there exists a unique compact set T satisfying property* (2.3)*. This set is given by*

$$T = \left\{ \sum_{j=1}^{\infty} B^{-j} c_j \;\middle|\; (c_j)_{j=1}^{\infty} \in \mathcal{D}^{\mathbb{N}} \right\}. \tag{2.4}$$

Remark 2.4.5. *Given a self-affine tile T, there are infinitely many possible choices for an expanding matrix B and digit set $\mathcal{D}$ giving rise to T in this way.*

Proposition 2.4.4 is using that any self-affine tile T with expanding matrix B and digit set $\mathcal{D}$ is the attractor of the following iterated function system:

$$f_i(x) = B^{-1}(x + c_i),\ i = 1, 2, \ldots, m. \tag{2.5}$$

Note that equation (2.3) is satisfied by $T = \bigcup_{i=1}^{m} f_i(T)$.

Remark 2.4.6. *Self-affine tiles in one dimension are related to number systems. Suppose that b is an integer greater than 1, and that $\mathcal{D}$ is a finite subset of $\mathbb{R}$ containing b many non-negative elements. Here, $\mathcal{D}$ is to be interpreted as a candidate digit set to be used to write real numbers in base b. The question of which digits sets $\mathcal{D}$ can be used to represent all real numbers in base b is considered in [21]. It is shown that* *the collection of numbers that can be represented by* $\mathcal{D}$ *is* $\bigcup_{N=-\infty}^{\infty} b^N(E \cup (-E))$, *where* $E = \left\{ \sum_{i=1}^{\infty} c_i b^{-i} \mid c_i \in \mathcal{D} \right\}$ *is a self-affine tile with digit set* $\mathcal{D}$.

2.4.4 The Fudgeflake

If you "fudge" the symmetry of the Koche Snowflake, you can create a different snowflake-like set which can be subdivided into translates of itself; this is how the Fudgeflake, shown in Figure 2.9, got its name [18]. It is the attractor of the following single-matrix affine IFS:

$$\begin{aligned} f_1(x) &= \frac{\sqrt{3}}{3} R_{\frac{\pi}{6}} x \\ f_2(x) &= \frac{\sqrt{3}}{3} R_{\frac{\pi}{6}} x + \begin{bmatrix} 1/2 \\ \sqrt{3}/6 \end{bmatrix} \\ f_3(x) &= \frac{\sqrt{3}}{3} R_{\frac{\pi}{6}} x + \begin{bmatrix} 1/2 \\ -\sqrt{3}/6 \end{bmatrix} \end{aligned} \tag{2.6}$$

where

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

is the rotation matrix for angle $\theta \in [0, 2\pi]$. Figure 2.10 shows a schematic of the first three iterations of this IFS on a hexagon.

It should be noted that the Fudgeflake is fundamentally different from the Siérpinski n-gons and the Koche Snowflake. Indeed, the Fudgeflake has no lines of reflective symmetry; it does, however, have rotation symmetry. If you rotate the Fudgeflake clockwise, or counterclockwise, by an angle of $\frac{\pi}{3}$, it will fall back onto itself. The Fudgeflake provides an excellent example of a single-matrix affine IFS that includes a rotation. The functions in (2.6) map the Fudgeflake into three identical, smaller, rotated copies of itself, overlapping only on their boundaries, as shown in Figure 2.11. This makes it a self-affine tile.

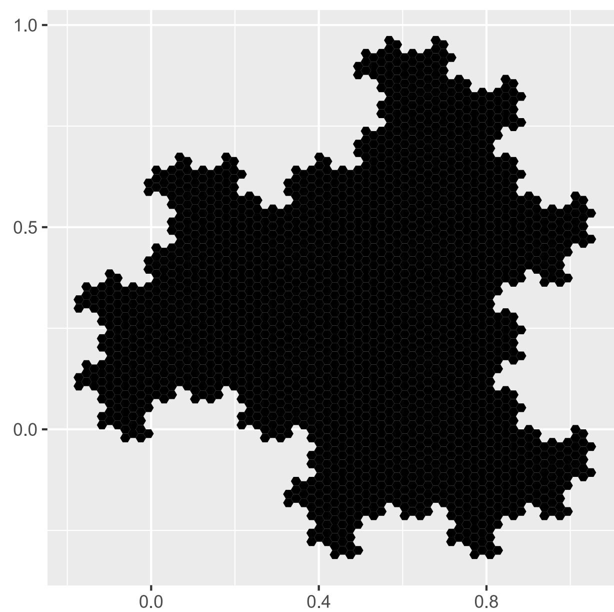


Figure 2.9: The Fudgeflake.

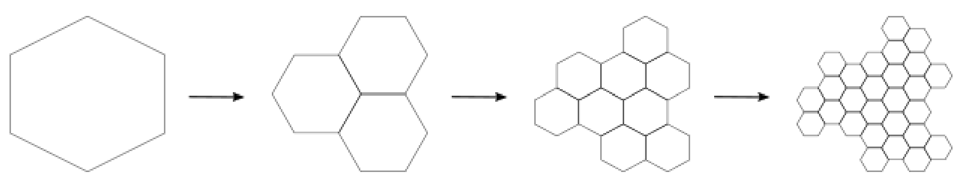


Figure 2.10: Construction of the Fudgeflake.

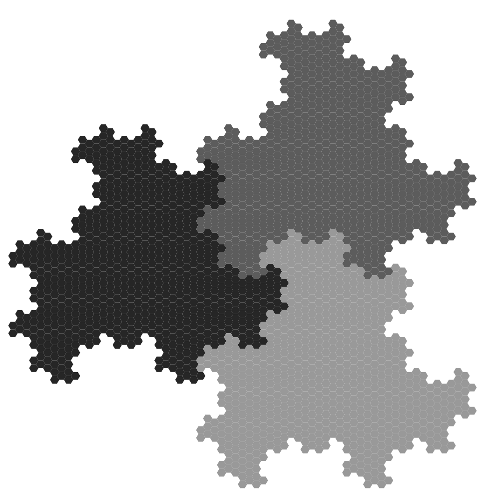


Figure 2.11: Images for the Fudgeflake IFS.

2.4.5 The Twindragon

The Twindragon, like the Fudgeflake, is a self-affine tile, coming from an IFS involving a rotation. It is depicted in Figure 2.12. Indeed, you can separate the Twindragon into two identical, smaller Twindragons, by separating across its middle. See (2.7) for the maps in the IFS used to generate it. Figure 2.13 shows a schematic of the first three iterations of this IFS on a square, and Figure 2.14 shows the two images of the Twindragon under the maps.

$$\begin{aligned} f_1(x) &= \frac{\sqrt{2}}{2} R_{\frac{\pi}{4}} x \\ f_2(x) &= \frac{\sqrt{2}}{2} R_{\frac{\pi}{4}} x + \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \end{aligned} \tag{2.7}$$

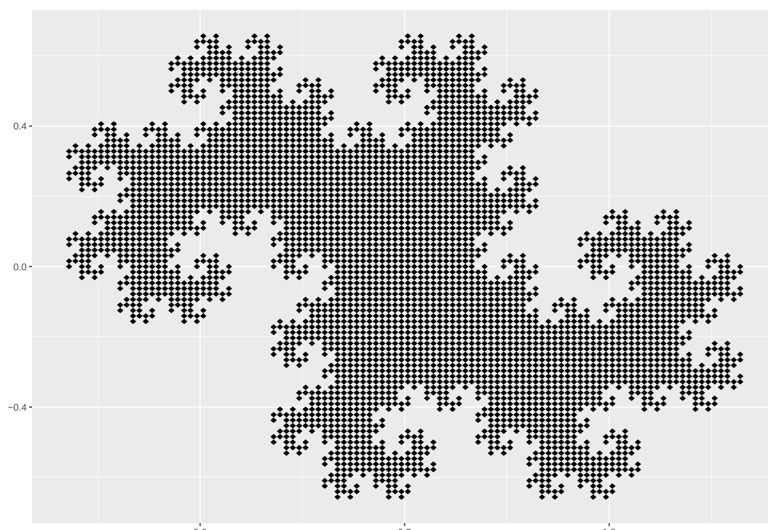


Figure 2.12: The Twindragon.

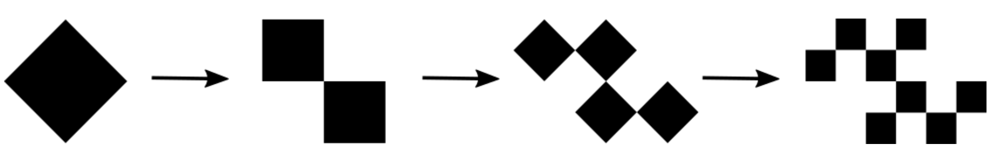


Figure 2.13: Construction of the Twindragon.

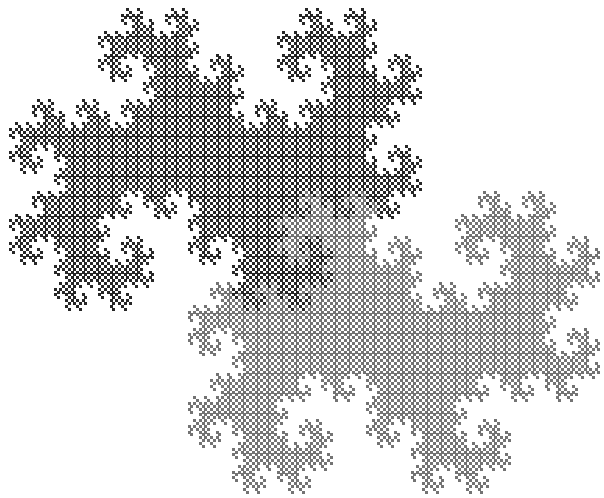


Figure 2.14: Images for the Twindragon IFS.

As suggested by its name, the Twindragon can be divided into two copies of the Harter-Heighway dragon [18]. The Harter-Heighway dragon, however, does not seem to come from an affine IFS with a single matrix, as the two functions used differ by a rotation. Its standard IFS construction is give in (2.8).

$$\begin{aligned} f_1(x) &= \frac{\sqrt{2}}{2} R_{\frac{\pi}{4}} x \\ f_2(x) &= \frac{\sqrt{2}}{2} R_{\frac{3\pi}{4}} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned} \tag{2.8}$$

Chapter 3

Equivalence relation C^*-algebras

In this chapter, we switch our perspective from dynamics to operator algebras. In particular, we review the standard construction of a C^*-algebra from an étale equivalence relation. Though this construction generalizes to étale groupoids, the case of equivalence relations comes with some particularly nice features, some of which will be presented in Sections 3.2 and 3.3. We also introduce inductive limits of C^*-algebras, with a particular focus on those coming from increasing sequences of open sub-equivalence relations.

3.1 Brief introduction to C^*-algebras

Definition 3.1.1 (C^*-algebra)**.** *A C^*-algebra is a (non-empty) vector space A over $\mathbb{C}$, with an associative multiplication operation $A \times A \to A$, which distributes over addition, and also satisfies*

(i) $\lambda(ab) = (\lambda a)b = a(\lambda b)$ *for all* $\lambda \in \mathbb{C}$, $a, b \in A$.

In addition to this, A must also have a conjugate linear involution $a \mapsto a^*$, meaning

(ii) $(a^*)^* = a$ *for all* $a \in A$

(iii) $(\lambda a + b)^* = \bar{\lambda} a^* + b^*$ *for all* $\lambda \in \mathbb{C}$, $a, b \in A$.

This involution must also satisfy the following rule for products:

(iv) $(ab)^* = b^*a^*$ *for all* $a, b \in A$.

Finally, A *must also be equipped with a norm* $||\cdot||$ *in which it is complete, and*

(v) $||ab|| \leq ||a||\,||b||$ *for all* $a, b \in A$

(vi) $||a^*a|| = ||a||^2$ *for all* $a \in A$.

The involution is often referred to as the *adjoint*, or *conjugate* operation. Also, note that property (v) makes A into a Banach algebra. Property (vi) is referred to as the C^*-condition, and has some surprisingly strong consequences. We will take a look at some of these shortly, but first, it should be pointed out that any $a \in A$ must have the same norm as its adjoint a^*; indeed, the C^*-condition combined with property (v) gives us:

$$||a||^2 = ||a^*a|| \leq ||a^*||\,||a|| \implies ||a|| \leq ||a^*||$$

and $(a^*)^* = a$ gives us the reverse inequality:

$$||a^*||^2 = ||aa^*|| \leq ||a||\,||a^*|| \implies ||a^*|| \leq ||a||.$$

A C^*-algebra A is called *unital* if it contains a multiplicative identity, usually denoted by 1, or 1_A. Many interesting C^*-algebras are actually *non-unital*, and we will encounter some examples in Chapters 5 and 6. Let us define a few more useful properties for elements of a C^*-algebra.

Definition 3.1.2. *Let* a *be an element of a* C^**-algebra* A*. Then* a *is called a*

(i) *self-adjoint* *element if* $a^* = a$

(ii) <u>*normal*</u> *element if* $a^*a = aa^*$

(iii) <u>*projection*</u> *if* a *is self-adjoint and* $a^2 = a$

(iv) <u>*unitary*</u> *if* A *is unital and* $a^*a = aa^* = 1_A$.

Before proceeding to give a few interesting consequences of the C^*-condition, we will need the notions of *spectrum*, and *spectral radius* for unital C^*-algebras.

Definition 3.1.3. *Let* A *be a unital* C^**-algebra. Then the spectrum of an element* $a \in A$ *is defined to be the following subset of* $\mathbb{C}$*:*

$$\operatorname{spec}(a) = \{\lambda \in \mathbb{C} \ : \ \lambda 1_A - a \text{ is not invertible}\}.$$

The spectral radius of a *is then the largest modulus value in the spectrum:*

$$r(a) = \sup\{|\lambda| \ : \ \lambda \in \operatorname{spec}(a)\}.$$

In fact, if the element a is normal, then its spectral radius is equal to its norm. See [4, Chapter 8], p. 234.

Theorem 3.1.4. *Let* A *be a unital* C^**-algebra, and let* $a \in A$ *be normal. Then* $r(a) = ||a||$.

A nice consequence of this theorem is that the norm on a C^*-algebra is unique. This does not mean that a given $*$-algebra can only have one norm defined on it; we make the precise statement below.

Corollary 3.1.5. *Suppose that* A *is a unital* C^**-algebra with norm* $||\cdot||$*, and* $||\cdot||_1$ *is another norm on* A *under which* A *is a* C^**-algebra. Then,* $||\cdot||_1 = ||\cdot||$.

Proof. If $||\cdot||_1$ satisfies the C^*-condition, then for any $a \in A$, $||a||_1^2 = ||a^*a||_1$. The element a^*a is normal, so $||a^*a||_1 = r(a^*a)$. However, the spectral radius is a purely algebraic property, having to do with the invertibility of elements. Therefore, $r(a^*a)$ is independent of the norm on A, and $||a||_1^2 = r(a^*a) = ||a^*a|| = ||a||^2$. □

In particular, if you have a norm $||\cdot||$ in which A is a C^*-algebra, and $||\cdot||_1 \neq ||\cdot||$ is another $*$-algebra norm on A satisfying the C^*-condition, then A cannot be complete with respect to $||\cdot||_1$.

In the case of a non-unital C^*-algebra, one can still consider its spectrum by looking at its *unitization*. This is also an important tool used in the K-theory for C^*-algebras [26]. It will be helpful to first introduce the notions of *ideals* and *quotients* of C^*-algebras, which will also be widely used in Chapter 6.

Definition 3.1.6. *Let A be a C^*-algebra, and let I be a closed vector subspace of A. Then, I is an ideal in A if for every $b \in I$ and $a \in A$, both ba and ab are elements of I.*

We will also need the following result on the quotient of a C^*-algebra by an ideal. See [4, Chapter 8], pp. 246-247.

Theorem 3.1.7. *Let A be a C^*-algebra, and let I be an ideal in A. Then, the quotient space A/I is a C^*-algebra when the involution and norm are defined as follows, for each element $a + I \in A/I$:*

$$(a+I)^* = a^* + I, \quad ||a+I|| = \inf_{b \in I}\Big\{||a+b||\Big\}.$$

Though this definition of ideal is closed and two-sided, it should be mentioned that one-sided ideals, and open ideals, can also be considered; however, we will only be looking at closed, two-sided ideals. Related to the notion of ideals is that of a simple C^*-algebra, which is one containing only trivial ideals.

Definition 3.1.8. *A C^*-algebra A is simple if the only ideals in A are $\{0\}$ and A.*

We are now ready to define the unitization of a C^*-algebra, which will be given in the following theorem. See [4, Chapter 8], pp. 233-234.

Theorem 3.1.9. *Let A be a (potentially non-unital) C^*-algebra. Define $A^\sim$ to be the vector space $\mathbb{C} \oplus A$, and give it the following product and conjugation operations:*

$$(\alpha, a)(\beta, b) = (\alpha\beta, \alpha b + \beta a + ab) \quad \textit{for all } (\alpha, a), (\beta, b) \in A^\sim,$$
$$(\alpha, a)^* = (\bar{\alpha}, a^*) \quad \textit{for all } (\alpha, a) \in A^\sim.$$

Then $A^\sim$ is a unital $*$-algebra, with unit $(1, 0)$. Furthermore, the following defines a norm on $A^\sim$, in which it is a C^*-algebra:

$$||(\alpha, a)|| = \sup_{b \in A, ||b|| \leq 1} ||ab + \alpha b|| \quad \textit{for all } (\alpha, a) \in A^\sim.$$

If we identify $(0, a) \in A^\sim$ with $a \in A$, then $A^\sim$ is the unique unital C^*-algebra containing A as an ideal in $A^\sim$, such that $A^\sim/A$ is one-dimensional. $A^\sim$ is referred to as the <u>*unitization*</u> *of A.*

If a C^*-algebra A is non-unital, we can define the spectrum of an element $a \in A$ to be the spectrum of $(0, a) \in A^\sim$. Under this notion of spectrum, Theorem 3.1.4 holds in the non-unital case.

Another important concept is that of equivalence for C^*-algebras. Suppose we have two C^*-algebras, A and B, and a map $\rho : A \to B$. We call ρ a $*$-homomorphism if it preserves the algebraic structure of the C^*-algebras; namely, ρ should be linear, and satisfy the following:

(a) $\rho(ab) = \rho(a)\rho(b)$ for all $a, b \in A$

(b) $\rho(a^*) = \rho(a)^*$ for all $a \in A$.

If, in addition to being a $*$-homomorphism, ρ is also a bijection, then it is called a $*$-isomorphism. When there is a $*$-isomorphism between C^*-algebras, we say that they are isomorphic. This is how we interpret equivalence for C^*-algebras; however, we would also like ρ to be isometric. In other words, we would like to have $||\rho(a)|| = ||a||$ for all $a \in A$. Another nice consequence of the C^*-condition is that a *-isomorphism as defined above is always isometric. To show this, let us first see what Theorem 3.1.4 can tell us about $*$-homomorphisms.

Proposition 3.1.10. *Any $*$-homomorphism between C^*-algebras is contractive.*

Proof. Let us consider the unital case. See [4, Chapter 8] pp. 234-235 for a treatment of the non-unital case. Let A and B be unital C^*-algebras, and let $\rho : A \to B$ be a $*$-homomorphism such that $\rho(1_A) = 1_B$. Let a be an element of A. Notice that a $*$-homomorphism must map invertible elements to invertible elements. So, if $\lambda 1_A - a$ is invertible in A, then $\rho(\lambda 1_A - a) = \lambda 1_B - \rho(a)$ is invertible in B. Therefore, $\text{spec}(B) \subseteq \text{spec}(A)$, and $r(\rho(a)) \leq r(a)$. So, by Theorem 3.1.4, and the C^*-condition,

$$||\rho(a)||^2 = ||\rho(a)^*\rho(a)|| = r(\rho(a)^*\rho(a)) = r(\rho(a^*a)) \leq r(a^*a) = ||a^*a|| = ||a||^2$$

as desired. □

In fact, it turns out that any injective $*$-homomorphism between C^*-algebras is isometric, even if the map is not surjective. See [4, Chapter 8], p. 247.

Theorem 3.1.11. *Every injective $*$-homomorphism $\rho : A \to B$ between C^*-algebras is isometric. Moreover, the image $\rho(A)$ is closed in B, and is therefore a C^*-subalgebra of B.*

Finally, we should note that the quotient map sending $a \in A$ to $a + I \in A/I$ as in Theorem 3.1.7 is a $*$-homomorphism.

3.1.1 Inductive limits of C^*-algebras

The construction provided in Chapter 4 will hand us an inductive limit of C^*-algebras, so let us first understand what this means in a general context. Then we will look at the specific situation encountered in Chapters 4 and 5.

Definition 3.1.12. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of C^*-algebras, and suppose that there is a corresponding collection of $*$-homomorphisms $\phi_{n,m} : A_n \to A_m$ for all $1 \leq n \leq m$, satisfying the following two properties:*

(1) $\phi_{n,n} = Id_{A_n}$,

(2) $\phi_{n,m} = \phi_{k,m} \circ \phi_{n,k}$ *whenever* $n \leq k \leq m$.

Then, $(A_n, \phi_{n,m})$ is referred to as a directed system. A C^*-algebra A is called an <u>inductive limit</u> of the directed system $(A_n, \phi_{n,m})$ if there exists a collection of $*$-homomorphisms $\phi_n : A_n \to A$ that satisfy the following two properties:

(i) $\phi_n = \phi_m \circ \phi_{n,m}$ *(compatibility),*

(ii) If B is another C^*-algebra with compatible $*$-homomorphisms $\psi_n : A_n \to B$, then there exists a unique $*$-homomorphism $\rho : A \to B$ such that $\psi_n = \rho \circ \phi_n$ (universality).

Inductive limits for C^*-algebras always exist, and are unique See, for example, [26] pp. 94-96. Let us denote the inductive limit of a directed system $(A_n, \phi_{n,m})$ by $\varinjlim A_n$. In particular, we will be interested in the following situation where we have an increasing sequence of inclusions of C^*-algebras. See, for example, the survey [9] pp. 646-647, which treats the more general case of unital injective $*$-homomorphisms.

Theorem 3.1.13. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of C^*-algebras such that each A_n is a C^*-subalgebra of A_{n+1}. Let A be the completion of $\bigcup_{n=1}^{\infty} A_n$. Then, A is a C^*-algebra* *containing each A_n as a C^*-subalgebra. For simplicity, we write:*

$$A = \overline{\bigcup_{n=1}^{\infty} A_n}.$$

Moreover, if for each $n \geq 1$ we set ι_n to be the inclusion map of A_n in A, then $A = \varinjlim A_n$ with respect to the compatible maps $\{\iota_n\}_{n \in \mathbb{N}}$.

Remark 3.1.14. *If one replaces the C^*-algebras with abelian groups, and the $*$-homomorphisms with group homomorphisms, one encounters the notion of an inductive limit of abelian groups. Like C^*-algebras, inductive limits of groups always exist [26]. We will find an inductive limit of abelian groups in Section 6.4.*

3.2 Étale equivalence relations

In this section, we define the notion of an étale equivalence relation. There is a generalization of this notion for groupoids, and the reader is referred to [29] for a treatment of the general case. First, let us recall that an *equivalence relation* on a set X is a subset R of $X \times X$ for which the following three properties hold:

(i) $(x, x) \in R$ for all $x \in X$,

(ii) $(x, y) \in R \iff (y, x) \in R$ for all $x, y \in X$,

(iii) $(x, y), (y, z) \in R \implies (x, z) \in R$ for all $x, y, z \in X$.

For a point $x \in X$, we call the set $[x]_R = \{y \in X \mid (x, y) \in R\}$ the equivalence class of x. We will also need the following.

Definition 3.2.1. *Let X and Y be topological spaces. Then, a function $f : X \to Y$ is called a local homeomorphism if*

(a) f maps open sets in X to open sets in Y, and

(b) for all $x \in X$*, there exists a neighborhood* U *of* x *such that* $f\big|_U$ *is a homeomorphism from* U *to* $f(U)$*.*

Now we are ready to define the étale property for equivalence relations equipped with a topology.

Definition 3.2.2. *Let* R *be an equivalence relation on a set* X*, and let* τ *be a topology on* R*. Then,* (R, τ) *is called* <u>*étale*</u> *if*

(i) the set $R^2 = \{((x, y_1), (y_2, z)) \in R \times R \mid y_1 = y_2\}$ *is closed in the relative topology of* $R \times R$*,*

(ii) $((x, y), (y, z)) \mapsto (x, z)$ *is a continuous map from* R^2 *to* R*,*

(iii) $(x, y) \mapsto (y, x)$ *is a continuous map from* R *to* R*,*

and the following two maps are local homeomorphisms:

$$r : R \to R,\ (x, y) \mapsto (x, x),$$
$$s : R \to R,\ (x, y) \mapsto (y, y).$$

The maps r *and* s *are commonly referred to as the* <u>*range*</u> *and* <u>*source*</u> *maps for* R*. If* (R, τ) *is étale, we will refer to* τ *as an étale topology for* R*, or say that* R *is étale in the topology* τ*.*

3.2.1 Inductive limits of étale equivalence relations

In this section, we consider the general situation of an increasing sequence of étale sub-equivalence relations R_n, $n \geq 1$, each étale in the topology τ_n. We will look at the étale topology on $R = \bigcup_{n\geq 1} R_n$, where each R_n is an an open subset of R_{n+1}, and $\tau_n \subseteq \tau_{n+1}$. We will show that R is étale in the topology given by the base $\bigcup_{n\geq 1} \tau_n$.

Proposition 3.2.3 (Inductive Limit of Topological Spaces)**.** *Suppose that* (X_n, τ_n)*,* $n \geq 1$*, is a sequence of topological spaces such that for all* n*,* $X_n \subseteq X_{n+1}$ *and* $\tau_n \subseteq \tau_{n+1}$*. Let* $X = \bigcup_{n\geq 1} X_n$*, and let* $\tau = \bigcup_{n\geq 1} \tau_n$*. Then,* τ *is a base for a topology on* X*. Moreover, for all* n*,* X_n *is an open subset of* X*.*

Proof. If $x \in X$, then there is an $n \geq 1$ such that $x \in X_n$. As τ_n covers X_n, there is a $U \in \tau_n \subseteq \tau$ such that $x \in U$. Therefore, τ covers X. Next, let $U, V \in \tau$ be such that $U \in \tau_n$, $V \in \tau_m$. Without loss of generality, take $n \geq m$. Then, $V \in \tau_m \subseteq \tau_n$. If $x \in U \cap V$, then $U \cap V \in \tau_n \subseteq \tau$ contains x. Therefore, τ is a base for a topology on X. Moreover, $X_n = \bigcup \tau_n$, so X_n is open in X for each $n \geq 1$. □

Proposition 3.2.4. *Let* R_n*,* $n \geq 1$ *be a sequence of equivalence relations, each étale in the topology* τ_n*, such that for all* $n \geq 1$*,* R_n *is an open subset of* R_{n+1}*, and* $\tau_n \subseteq \tau_{n+1}$*. Then,* $R = \bigcup_{n\geq 1} R_n$ *is an étale equivalence relation in the topology given by the base* $\tau = \bigcup_{n\geq 1} \tau_n$*.*

Proof. We will show that conditions (i)-(iii) of Definition 3.2.2 hold in the topology given by τ, and that the maps r and s are local homeomorphisms. Using the fact that $R_n \subseteq R_{n+1}$ for all $n \geq 1$, we have:

$$\begin{aligned} R^2 &= \{((x,z),(z,y)) \mid (x,z),(z,y) \in R\} \\ &= \{((x,z),(z,y)) \mid (x,z) \in R_n, (z,y) \in R_m \text{ for some } n, m \geq 1\} \\ &= \{((x,z),(z,y)) \mid (x,z),(z,y) \in R_n \text{ for some } n \geq 1\} \\ &= \bigcup_{n\geq 1} R_n^2. \end{aligned}$$

So, to show that R^2 is closed in R, it is enough to show that each R_n^2 is closed in R. By Corollary 4.3.3, we have that each R_n^2 is closed in R_n, so $R_n \setminus R_n^2$ is in $\tau_n \subseteq \tau$. So, $R \setminus (R_n \setminus R_n^2) = R_n^2 \bigcup (R \setminus R_n)$ is closed in R. As this is a disjoint union, and $R \setminus R_n$ is closed in R, R_n^2 must also be closed in R.

To see that the map $((x, y), (y, z)) \mapsto (x, z)$ is a continuous map from R^2 to R, let n be such that $V \in \tau_n$. By Corollary 4.3.3, the pre-image of V under the product map is open in R_n^2. This means that it is of the form $R_n^2 \cap U$, for some U open in $R_n \times R_n$. As $U \subseteq R_n \times R_n$ is open in $R_n \times R_n$, U is also open in $R \times R$. Therefore, the pre-image can be written as $R^2 \cap U$, which is open in R^2. Similarly, to see that the map $(x, y) \mapsto (y, x)$ is continuous on R, let n be such that $V \in \tau_n$. By Corollary 4.3.3, the pre-image of V under the inverse map is open in R_n, so it is open in R, as desired.

Lastly, let r and s denote the range and source maps on R. To see that R is étale when given the topology from the base τ, let us first show that r and s satisfy (a) of Definition 3.2.1. Recall that we are assuming the maps $r, s : R_n \to R_n$ are local homeomorphisms for each $n \geq 1$. Then, (a) follows directly from the observation that $r|_{R_n}$ and $s|_{R_n}$ are the range and source maps for R_n, as a set is open in R if and only if it open in R_n for some n. Next, if we consider these maps at a point $(x, y) \in R$, notice that we can pick an n such that $(x, y) \in R_n$. As the restrictions of r and s to R_n are local homeomorphisms, there exists a neighborhood of (x, y) in R_n for which $r|_U$ and $s|_U$ are homeomorphisms from U to $r(U)$ and $s(U)$, respectively. As open set in R_n is an open set in R, r and s satisfy (b) of Definition 3.2.1. Therefore, the étale property extends nicely to R. □

3.3 C^*-algebras of étale equivalence relations

In this section, we define two C^*-algebras associated to an étale equivalence relation: the *reduced* C^*-algebra, and the *universal* C^*-algebra. These are both built from a $*$-algebra based on matrix-like operations for continuous functions of compact support. First, let us recall the definition of the support of a function.

Definition 3.3.1. *Let X be a topological space, and let $f : X \to \mathbb{C}$ be a complex-valued function. Then, the support of f is the set of points of X where f takes non-zero values:*

$$supp(f) = \{x \in X \mid f(x) \neq 0\}.$$

The collection of all continuous functions for which the closure of the support is compact is denoted by $C_c(X)$; more specifically:

$$C_c(X) = \{f \in C(X) \mid \overline{supp(f)} \text{ is compact}\}$$

where $C(X)$ denotes the set of all continuous functions from X into $\mathbb{C}$.

We will be interested in the continuous functions of compact support on an étale equivalence relation. In particular, we take $X = R$ to be an étale equivalence relation, and define matrix-like operations on $C_c(R)$, which make it into a $*$-algebra. See [29], p. 15.

Theorem 3.3.2. *Let R be an étale equivalence relation. Define the following operations on $C_c(R)$:*

(a) $f^*(x,y) = \overline{f(y,x)}$ *for all* $(x,y) \in R$

(b) $(fg)(x,y) = \sum_{z \in [x]_R} f(x,z) \cdot g(z,y)$ *for all* $(x,y) \in R$.

Then, $C_c(R)$ is a $*$-algebra, with the usual linear structure from $C(R)$; in other words, it satisfies properties $(i)-(iv)$ of Definition 3.1.1.

Defining a norm in which the completion of $C_c(R)$ becomes a C^*-algebra is challenging, and there is more than one way to do this. Here, we will define the *reduced* and *universal* norms. Let us start with the reduced norm, as it is the easier to compute of the two. Given an étale equivalence relation R on $X \times X$, and a point $y \in X$, we use $\ell^2([y]_R)$ to denote the Hilbert space

$$\ell^2([y]_R) = \left\{ \xi : [y]_R \to \mathbb{C} \;\middle|\; \sum_{z \in [y]_R} |\xi(z)|^2 < \infty \right\}$$

with the inner product $\langle \xi, \eta \rangle = \sum_{z \in [y]_R} \xi(z)\overline{\eta(z)}$ for each $\xi, \eta \in \ell^2([y]_R)$. For the equivalence relations we are interested in, the equivalence classes $[y]_R$ will be countable. The reader is referred to [29] for a more general treatment of the following theorem.

Theorem 3.3.3. *Let R be an étale equivalence relation on X. For each fixed $y \in X$, define the map $\pi_\lambda^y : C_c(R) \to$ into $\mathcal{B}\left(\ell^2\left([y]_R\right)\right)$ by:*

$$\left(\pi_\lambda^y(g)\xi\right)(x) = \sum_{z \in [y]_R} g(x, z) \cdot \xi(z)$$

for each $g \in C_c(R)$, $\xi \in \ell^2\left([y]_R\right)$, and $x \in [y]_R$. Then, this is a well-defined and bounded operator, and the following is a norm on $C_c(R)$:

$$||g||_r = \sup_{y \in X} \left\{ \left|\left|\pi_\lambda^y(g)\right|\right| \right\}$$

using the standard operator norm on $\mathcal{B}\left(\ell^2\left([y]_R\right)\right)$. Moreover, the completion of $C_c(R)$ in this norm is a C^*-algebra. This completion is denoted by $C_r^*(R)$, and is referred to as the reduced C^*-algebra of R.

If one takes the direct sum of the maps π_λ^y over $y \in X$, you get what is referred to as the *left regular representation* of the equivalence relation. Let us clarify what that means, as it will lead nicely into the *universal* C^*-algebra of an étale equivalence relation.

Definition 3.3.4. *Let A be a $*$-algebra, and let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a $*$-homomorphism, where $\mathcal{B}(\mathcal{H})$ denotes the bounded linear operators on a Hilbert space $\mathcal{H}$, with the usual* *operations and norm. Then, π is called a representation of A on $\mathcal{H}$.*

We can define another norm on $C_c(R)$ by considering *all the possible representations* of it on Hilbert spaces. Once again, the reader is referred to [29] for a more general treatment of the following theorem.

Theorem 3.3.5. *Let R be an étale equivalence relation on X. Then, the following supremum exists and defines a non-trivial norm on $C_c(R)$:*

$$||g|| = \sup \{||\pi(g)|| \mid \pi \text{ is a representation of } C_c(G)\} .$$

Moreover, the completion of $C_c(R)$ in this norm is a C^*-algebra. This completion is denoted by $C^*(R)$, and is referred to as the universal C^*-algebra of R.

Remark 3.3.6. *There is a property one can define for étale equivalence relations known as amenability. One of the nice features of amenable étale equivalence relations is that their reduced and universal C^*-algebras are equal. The reader is referred to [29] for a definition of this property. The equivalence relations we consider will all be amenable, and we will provide a brief justification for this in Section 3.3.1. We will make use of the reduced norm, as opposed to the universal norm, as this is most suitable to the application at hand.*

Let us introduce a notion that will be helpful in understanding the C^*-algebras to come, regarding the relationship between open invariant sets for an étale equivalence relation and the ideal structure of the associated C^*-algebra. Let us first define what it means for a set to be invariant under an equivalence relation.

Definition 3.3.7. *Let R be an equivalence relation on X. Then, a subset $Y \subseteq X$ is said to be R-invariant if for each $y \in Y$, $[y]_R \subseteq Y$.*

The following theorem says that all ideals of the reduced C^*-algebra come from open invariant sets. See [24], p. 103.

Theorem 3.3.8. *Let R be an étale equivalence relation. Then, there is a bijection between the open invariant subsets of R and the ideals of $C_r^*(R)$.*

The same holds for the universal C^*-algebra. In this case, you also get a short exact sequence coming from the inclusion of the ideal in $C^*(R)$, with a nice description of the quotient. However, in general, you cannot be sure that these give you all the ideals. We refer the reader to [29], pp. 34-35.

Theorem 3.3.9. *Let $R \subseteq X \times X$ be an étale equivalence relation. If $U \subseteq X$ is an open R-invariant subset, then $C^*\left(R|_U\right)$ is an ideal in $C^*(R)$, giving rise to the following short exact sequence*

$$0 \to C^*\left(R|_U\right) \xrightarrow{\iota_U} C^*(R) \xrightarrow{\pi_U} C^*\left(R|_{X \setminus U}\right) \to 0$$

where ι_U is inclusion, and π_U is restriction. In other words, ι_U is injective, π_U is surjective, and $C^*\left(R|_{X \setminus U}\right)$ is the quotient of $C^*(R)$ by $C^*\left(R|_U\right)$.

Remark 3.3.10. *In general, the above maps may not form an exact sequence for the reduced C^*-algebra [29]. However, if the equivalence relation in question is amenable, then the map sending U to the corresponding ideal $C^*\left(R|_U\right)$ defines a bijection between the open R-invariant subsets and the ideals in $C^*(R) = C_r^*(R)$ of Theorem 3.3.8.*

Finally, let us consider a nice consequence of this correspondence related to simplicity of the associated C^*-algebras.

Definition 3.3.11. *Let R be an étale equivalence relation on X. Then, R is minimal if for any $x \in X$, the equivalence class $\overline{[x]_R} = X$. This is equivalent to the only open invariant subsets of R being $\emptyset$ and X.*

Theorem 3.3.12. *If R is a minimal étale equivalence relation, then $C_r^*(R)$ is simple.*

Proof. If the only open invariant subsets of R are $\emptyset$ and X, then by Theorem 3.3.8, there are only two ideals of $C^*(R)$. As $\{0\}$ and $C_r^*(R)$ are two distinct ideals of $C^*(R)$, there can be no other ideals. □

3.3.1 An inductive limit of equivalence relation C^*-algebras

Now, let us consider the special case of C^*-algebras coming from an increasing sequence of étale equivalence relations. In particular, let us verify that the inductive limit of an increasing sequence of these equivalence relation C^*-algebras agrees with the C^*-algebra of the inductive limit of the equivalence relations as topological spaces.

Proposition 3.3.13. *Let $\{R_n\}_{n\geq 1}$ be a sequence of étale equivalence relations such that R_n is an open subset of R_{n+1} for each $n \geq 1$. Let $R = \bigcup\limits_{n\geq 1} R_n$, with the inductive limit topology. Then, $\{C_c(R_n)\}_{n\geq 1}$ is an increasing sequence of inclusions of $*$-algebras, and $\bigcup_{n\geq 1} C_c(R_n) = C_c(R)$. Furthermore,*

$$C_r^*(R) = \overline{\bigcup_{n\geq 1} C_r^*(R_n)} = \varinjlim C_r^*(R_n).$$

Proof. To show that $C_c(R_n)$ is a $*$-subalgebra of $C_c(R_{n+1})$, we must first view it as a subset of $C_c(R_{n+1})$ by extending the functions to be zero outside of R_n. More specifically, if $g \in C_c(R_n)$ is a continuous function of compact support, then g must vanish outside of a compact subset of R_n. As R_n is an open subset of R_{n+1}, this means that the function

$$\tilde{g}(x,y) = \begin{cases} g(x,y) & (x,y) \in R_n \\ 0 & (x,y) \in R_{n+1} \setminus R_n \end{cases}$$

will be continuous on R_{n+1}, and will also have compact support on R_{n+1} because it vanishes outside of the compact support of g. Moreover, the map $g \mapsto \tilde{g}$ is an injective $*$-homomorphism from $C_c(R_n)$ to $C_c(R_{n+1})$, and in this way, we can view $C_c(R_n)$ as a $*$-subalgebra of $C_c(R_{n+1})$. To see that the map is a $*$-homomorphism, we must verify that it is linear, and compatible with conjugation, and products. Linearity follows from the following:

$$
\begin{aligned}
(\widetilde{\alpha g + h})(x,y) &= \begin{cases} \alpha g(x,y) + h(x,y) & (x,y) \in R_n \\ 0 & (x,y) \in R_{n+1} \setminus R_n \end{cases} \\
&= \begin{cases} \alpha g(x,y) + h(x,y) & (x,y) \in R_n \\ \alpha \cdot 0 + 0 & (x,y) \in R_{n+1} \setminus R_n \end{cases} \\
&= \alpha \tilde{g}(x,y) + \tilde{h}(x,y)
\end{aligned}
$$

where $\alpha \in \mathbb{C}$, $g, h \in C_c(R_n)$, and $(x,y) \in R_{n+1}$. For conjugation, we have:

$$
\begin{aligned}
(\widetilde{g^*})(x,y) &= \begin{cases} (g^*)(x,y) & (x,y) \in R_n \\ 0 & (x,y) \in R_{n+1} \setminus R_n \end{cases} \\
&= \begin{cases} \overline{g(y,x)} & (x,y) \in R_n \\ 0 & (x,y) \in R_{n+1} \setminus R_n \end{cases} \\
&= \begin{cases} \overline{g(y,x)} & (y,x) \in R_n \\ \bar{0} & (y,x) \in R_{n+1} \setminus R_n \end{cases} \\
&= (\tilde{g})^*(x,y)
\end{aligned}
$$

where we used that $(x,y) \in R_n \iff (y,x) \in R_n$. For the product, we have:

$$\widetilde{(gh)}(x,y) = \begin{cases} (gh)(x,y) & (x,y) \in R_n \\ 0 & (x,y) \in R_{n+1} \setminus R_n \end{cases}$$
$$= \begin{cases} \sum_{z \in [x]_{R_n}} g(x,z) \cdot h(z,y) & (x,y) \in R_n \\ 0 & (x,y) \in R_{n+1} \setminus R_n \end{cases}$$

and

$$(\tilde{g}\tilde{h})(x,y) = \sum_{z \in [x]_{R_{n+1}}} \tilde{g}(x,z) \cdot \tilde{h}(z,y)$$

the terms of which will be equal to zero unless both (x,z) and (z,y) are in R_n, which can only happen if $(x,y) \in R_n$. Therefore, if $(x,y) \in R_{n+1} \setminus R_n$, the result is zero. If $(x,y) \in R_n$, the only non-zero terms in the sum over $z \in [x]_{R_{n+1}}$ are when $z \in [x]_{R_n}$. Therefore, we have $\widetilde{(gh)}(x,y) = (\tilde{g}\tilde{h})(x,y)$.

Let us also show that the inclusion map $g \mapsto \tilde{g}$ is an isometry. To this end, let us first notice that for any $y \in X$, there exists a collection of R_n-equivalence classes $[y_i]_{R_n}$, $i \in \mathcal{I}$ such that $[y]_{R_{n+1}}$ is equal to the disjoint union:

$$[y]_{R_{n+1}} = \dot{\bigcup_{i \in \mathcal{I}}} [y_i]_{R_n}. \tag{3.1}$$

Indeed, If $x \in [y]_{R_{n+1}}$ and $z \in [x]_{R_n}$, then $R_n \subseteq R_{n+1}$ tells us that $z \in [x]_{R_{n+1}} = [y]_{R_{n+1}}$. Therefore, $x \in [y]_{R_{n+1}} \implies [x]_{R_n} \subseteq [y]_{R_{n+1}}$. Then, (3.1) tells us that

$$\ell^2([y]_{R_{n+1}}) = \bigoplus_{i \in \mathcal{I}} \ell^2([y_i]_{R_n}). \tag{3.2}$$

Note that we can view this as R_{n+1} inducing an equivalence relation on the R_n-equivalence classes. If $\mathcal{J}$ is the set of equivalence classes in R_n, then (3.2) can also be written as

$$\ell^2([y]_{R_{n+1}}) = \bigoplus_{[z]_{R_n} \in \mathcal{J}} \ell^2([z]_{R_n}).$$

We claim that for any $g \in C_c(R_n)$, $\ell^2([y_i]_{R_n})$ is invariant under $\pi_\lambda^y(\tilde{g})$, and the result of applying $\pi_\lambda^y(\tilde{g})$ to an element $\xi \in \ell^2([y_i]_{R_n})$ will agree with the result of applying $\pi_\lambda^y(g)$. To see this, let $x \in [y]_{R_{n+1}}$, and let $\xi \in \ell^2\left([y_i]_{R_n}\right)$. We have:

$$\left(\pi_\lambda^y(\tilde{g})\xi\right)(x) = \sum_{z \in [y]_{R_{n+1}}} \tilde{g}(x,z) \cdot \xi(z) = \sum_{z \in [y_i]_{R_n}} \tilde{g}(x,z) \cdot \xi(z)$$

as the terms in the sum over $z \in [y]_{R_{n+1}}$ will be equal to zero unless $z \in [y_i]_{R_n}$, by (3.2). Then, as $\tilde{g}(x,z)$ is equal to zero unless $(x,z) \in R_n$, the remaining terms are zero unless $x \in [z]_{R_n} = [y_i]_{R_n}$. This tells us that $\pi_\lambda^y(\tilde{g})\xi \in \ell^2([y_i]_{R_n})$, and

$$\left(\pi_\lambda^y(\tilde{g})\xi\right)(x) = \sum_{z \in [y_i]_{R_n}} \tilde{g}(x,z) \cdot \xi(z) = \sum_{z \in [y_i]_{R_n}} g(x,z) \cdot \xi(z) = \left(\pi_\lambda^{y_i}(g)\xi\right)(x)$$

because $\tilde{g}$ agrees with g on R_n. This means that

$$||\pi_\lambda^y(\tilde{g})|| = \max_{i \in \mathcal{I}} ||\pi_\lambda^{y_i}(g)|_{\ell^2([y_i]_{R_n})}||.$$

Therefore, the supremum of $||\pi_\lambda^y(\tilde{g})||$ over y agrees with that of $||\pi_\lambda^y(g)||$, and the map $g \mapsto \tilde{g}$ preserves the reduced norm.

Note that we can take this map $g \mapsto \tilde{g}$ one step further, and view each $C_c(R_n)$ as a $*$-subalgebra of $C_c(R)$, by extending the functions to be zero on $R \setminus R_n$. The arguments above work just as well in this case, so we get an increasing sequence of $*$-subalgebras. By completing in the reduced norm, we get the desired increasing sequence of C^*-subalgebras.

The inductive limit $\varinjlim C_r^*(R_n)$ matches the description in Theorem 3.1.13. From this result, we have that

$$\varinjlim C_r^*(R_n) = \overline{\bigcup_{n\geq 1} C_r^*(R_n)}.$$

Next, recall from Proposition 3.2.4 that R itself will be an étale equivalence relation, where the base for its topology is the union of topologies of each R_n. To see that $C_r^*(R) = \overline{\bigcup_{n\geq 1} C_r^*(R_n)}$, it is sufficient to show that $\bigcup_{n\geq 1} C_c(R_n) = C_c(R)$, because taking the norm closure of two equal $*$-algebras, in the same norm, will give the same result. To clarify the meaning of equal here, we can view the elements of $C_c(R_n)$ as elements of $C_c(R)$ when taking the union. So, as this union uses the inclusion of $C_c(R_n)$ in $C_c(R)$, we have that $\bigcup_{n\geq 1} C_c(R_n) \subseteq C_c(R)$.

For the reverse inequality, let $g \in C_c(R)$. Then, the closure of the support of g, $\overline{\{(x, y) \in R \mid g(x, y) \neq 0\}}$, is compact. The sets R_n are an open cover of R by the definition of R, so there exists a finite subcover of these; label it by $R_{k_1}, R_{k_2}, \ldots, R_{k_\ell}$, in such a way that $R_{k_1} \subseteq R_{k_2} \subseteq \cdots \subseteq R_{k_\ell}$. Let $n = k_\ell$, and consider $g|_{R_n}$. By construction, R_n contains the closure of the support of g, so the support of $g|_{R_n}$ is equal to that of g. Therefore, $g|_{R_n} \in C_c(R_n)$. It remains to show that $\widetilde{g|_{R_n}} = g$, where $\widetilde{g|_{R_n}}$ denotes the inclusion of $g|_{R_n}$ in $C_c(R)$. Suppose first that $(x, y) \in R_n$. Then, $\widetilde{g|_{R_n}}(x, y) = g|_{R_n}(x, y) = g(x, y)$. Otherwise, if $(x, y) \in R_{n+1} \backslash R_n$, then $\widetilde{g|_{R_n}}(x, y) = 0$, and as (x, y) is not in the support of g, $g(x, y)$ must also be zero. Therefore, $g = \widetilde{g|_{R_n}}$, and $C_c(R) \subseteq \bigcup_{n\geq 1} C_c(R_n)$. □

Remark 3.3.14. *If, for each $n \geq 1$, there is a uniform upper bound on the number of members in each equivalence class, then R_n is an amenable. This can be seen from Proposition 4.1.9 in [29], which, when interpreted for equivalence relations, states that if R is an étale equivalence relation on X, and is a union of countably many closed sets in $X \times X$, then R is amenable if and only if each equivalence class $[x]_R$ is locally closed. This is certainly true if each equivalence class has only finitely many points, under the assumption that X is Hausdorff. See also [1] for more on details* *on amenability for groupoids.*

If an étale equivalence relation is amenable, then $C^*(R_n) = C_r^*(R_n)$*, and this will be a nuclear* C^**-algebra [3]. Moreover, because* $C^*(R)$ *and* $C_r^*(R)$ *are equal to the corresponding inductive limits, and inductive limits are unique, you also get that* $C^*(R) = C_r^*(R)$*; this will also be nuclear, as nuclearity is preserved in inductive limits [3].*

Chapter 4

Étale equivalence relations for IFS

In this chapter, we construct étale equivalence relations for iterated function systems consisting of invertible affine maps on $\mathbb{R}^d$, satisfying the single-matrix property as in Definition 2.3.4. The attractor of the iterated function system will be denoted by K. We will also use the notation of Definition 2.1.6 throughout. In particular, we will use the following notion of pre-image of a point $x \in K$ to define an equivalence relation for each positive integer n:

$$\mathcal{F}^{-n}\{x\} = \{z \in K \mid \exists\, \xi \in \Sigma_n \text{ such that } f_\xi(z) = x\}.$$

We will use the idea of a local action to build an étale topology on these equivalence relations. Finally, we will verify that these equivalence relations are homeomorphic when the underlying iterated function systems are conjugate.

4.1 Defining the equivalence relation

Proposition 4.1.1. *For each $n \geq 1$, define*

$$R_n = \{(x, y) \in K \times K \mid \mathcal{F}^{-n}\{x\} = \mathcal{F}^{-n}\{y\}\}.$$

Then, R_n is an equivalence relation on K, and $R_n \subseteq R_{n+1}$ for all $n \geq 1$.

Proof. To see that R_n is an equivalence relation, simply notice that reflexivity, symmetry, and transitivity, all hold for equalities. To see that $R_n \subseteq R_{n+1}$, we will first show that for any $x \in K$, $\mathcal{F}^{-(n+1)}\{x\} = \mathcal{F}^{-1}\left(\mathcal{F}^{-n}\{x\}\right)$. To this end,

$$\begin{aligned}
\mathcal{F}^{-1}\left(\mathcal{F}^{-n}\{x\}\right) &= \{z \in K \mid \exists\, i \in \Sigma \text{ such that } f_i(z) \in \mathcal{F}^{-n}\{x\}\} \\
&= \{z \in K \mid \exists\, i \in \Sigma,\ \xi \in \Sigma_n \text{ such that } f_\xi \circ f_i(z) = x\} \\
&= \{z \in K \mid \exists\, \xi \in \Sigma_{n+1} \text{ such that } f_\xi(z) = x\} \\
&= \mathcal{F}^{-(n+1)}\{x\}
\end{aligned}$$

as desired.

Now, suppose that $(x, y) \in R_n$. Then, $\mathcal{F}^{-n}\{x\} = \mathcal{F}^{-n}\{y\}$. From this, we also get that

$$\bigcup_{i=1}^{m} f_i^{-1}\left(\mathcal{F}^{-n}\{x\}\right) \cap K = \bigcup_{i=1}^{m} f_i^{-1}\left(\mathcal{F}^{-n}\{y\}\right) \cap K.$$

Applying Lemma 2.1.8, we see that

$$\begin{aligned}
&\bigcup_{i=1}^{m} f_i^{-1}\left(\mathcal{F}^{-n}\{x\}\right) \cap K = \mathcal{F}^{-1}\left(\mathcal{F}^{-n}\{x\}\right), \text{ and} \\
&\bigcup_{i=1}^{m} f_i^{-1}\left(\mathcal{F}^{-n}\{y\}\right) \cap K = \mathcal{F}^{-1}\left(\mathcal{F}^{-n}\{y\}\right).
\end{aligned}$$

Therefore, $\mathcal{F}^{-(n+1)}\{x\} = \mathcal{F}^{-1}\left(\mathcal{F}^{-n}\{x\}\right) = \mathcal{F}^{-1}\left(\mathcal{F}^{-n}\{y\}\right) = \mathcal{F}^{-(n+1)}\{y\}$, and we conclude that $(x, y) \in R_{n+1}$. □

Definition 4.1.2. *Let $R = \bigcup_{n \geq 1} R_n$. By Proposition 4.1.1, this is also an equivalence relation on K.*

4.2 Building a local action

Definition 4.2.1 (Partial Homeomorphism)**.** *Let X and Y be topological spaces, and let $U \subseteq X$ and $V \subseteq Y$ be open. Then, we refer to a homeomorphism $\gamma : U \to V$ as a partial homeomorphism between X and Y.*

Remark 4.2.2. *In what follows, we will view partial homeomorphisms as ordered pairs; more specifically, if γ is a partial homeomorphism from $U \subseteq X$ to $V \subseteq Y$, then we view γ as a subset of $U \times V$, with $(x, y) \in \gamma$ meaning that γ maps x to y.*

Definition 4.2.3 (Local Action)**.** *Let (X, τ) be a topological space, and let Γ be a collection of partial homeomorphisms of X. For $U \subseteq X$, let Id_U denote the set $\{(x, x) \mid x \in U\}$. For $\gamma \in \Gamma$, let γ^{-1} denote the set of pairs (y, x) such that $(x, y) \in \gamma$. For γ_1, $\gamma_2 \in \Gamma$, let $\gamma_1 \circ \gamma_2$ denote the set of pairs (x, z) such that there exists a $y \in K$ for which $(x, y) \in \gamma_1$ and $(y, z) \in \gamma_2$. Then, we will call Γ a local action on X if the following four properties are satisfied:*

1. $\{U \subseteq X \mid Id_U \in \Gamma\}$ *forms a base for the topology* τ
2. $\gamma \in \Gamma \implies \gamma^{-1} \in \Gamma$
3. $\gamma_1, \gamma_2 \in \Gamma, (x, y) \in \gamma_1, (y, z) \in \gamma_2 \implies \exists\, \gamma \in \Gamma$ *such that* $(x, z) \in \gamma \subseteq \gamma_1 \circ \gamma_2$
4. $\gamma_1, \gamma_2 \in \Gamma,\ (x, y) \in \gamma_1 \cap \gamma_2 \implies \exists\, \gamma \in \Gamma$ *such that* $(x, y) \in \gamma \subseteq \gamma_1 \cap \gamma_2$.

In this section, for each $n \geq 1$, we construct a local action Γ_n on K such that $\bigcup \Gamma = R_n$. This is a standard way of defining a base for an étale topology on an equivalence relation, with the base being the local action itself. This idea is made rigorous in Section 4.3.

Definition 4.2.4. *For each $(x, y) \in R_n$, define the function $\gamma_{(x,y)}(z) = z + y - x$ for each $z \in \mathbb{R}^d$. Then, $\gamma_{(x,y)}$ is a translation on $\mathbb{R}^d$. Furthermore, by Lemma 2.3.8, we have that if ξ, $\eta \in \Sigma_n$ are such that $f_\xi^{-1}(x) = f_\eta^{-1}(y)$, then $f_\xi \circ f_\eta^{-1} = \gamma_{(x,y)}|_{f_\eta(K)}$.*

Remark 4.2.5. *Note that $\gamma_{(x,y)}$ is a continuous, injective map on $\mathbb{R}^d$. To build a local action for R_n, we will appropriately restrict the domain of $\gamma_{(x,y)}$ so that it becomes a partial homeomorphism mapping x to y. We want the partial homeomorphisms to map K into K, so it is clear that we will need to restrict these translations. Considering $f_\xi \circ f_\eta^{-1}$ does restrict our attention to a subset of K, but is still insufficient. For example, we want an open mapping, and $f_\eta(K)$ is not an open subset of K.*

Before proceeding, let us define a few sets which will play an important role in what is to come.

Definition 4.2.6. *For each $\xi \in \Sigma_n$, let $U_\xi = K \setminus f_\xi(K)$. Then, for each $x \in K$, and $n \geq 1$, define the following sets:*

$$S_n(x) := \{\xi \in \Sigma_n \mid x \notin f_\xi(K)\}$$
$$U_n(x) := \bigcap_{\xi \in S_n(x)} U_\xi.$$

Lemma 4.2.7. *Suppose that $x \in K$, and let $n \geq 1$. Then, $U_n(x)$ is an open subset of K with the following two properties:*

1. *For all $x' \in U_n(x)$, there is a $\xi \in \Sigma_n$ such that both x and x' are in $f_\xi(K)$.*

2. *For every $x' \in U_n(x)$ and $\xi \in \Sigma_n$, $x' \in f_\xi(K) \implies x \in f_\xi(K)$.*

Proof. Let $x \in K$, and $n \geq 1$. Because K is compact, and each $f \in \mathcal{F}$ is a continuous contraction from K into K, the set $U_\xi := K \setminus f_\xi(K)$ is an open subset of K for any $\xi \in \Sigma_n$. As $U_n(x)$ is a finite intersection of open subsets of K, it is also an open subset of K. Note that $x \in U_n(x)$.

To prove property 1, let $x' \in U_n(x)$. By Proposition 2.1.7, there exists an $\eta \in \Sigma_n$ such that $x' \in f_\eta(K)$. Suppose for contradiction that there is no $\xi \in \Sigma_n$ such that both x and x' are in $f_\xi(K)$. Then, $\eta \in S_n(x)$. However, $f_\eta(K)$ and U_η are disjoint, so $f_\eta(K)$ and $U_n(x) \subseteq U_\eta$ are also disjoint. This is a contradiction, because $x' \in f_\eta(K)$ and $x' \in U_n(x)$.

To prove property 2, suppose that $x' \in U_n(x)$. Then, $x' \in f_\xi(K)$ means that $\xi \notin S_n(x)$. In other words, $x \in f_\xi(K)$. So, $x' \in f_\xi(K) \implies$ and $x \in f_\xi(K)$. □

Lemma 4.2.8. *The sets $U_n(x)$ and $S_n(x)$ defined above satisfy the following:*

$$x' \in U_n(x) \implies S_n(x') \supseteq S_n(x) \implies U_n(x') \subseteq U_n(x).$$

Proof. One can see the first implication as follows. Suppose that $x' \in U_n(x)$. If $\xi \in S_n(x)$, then $x \notin f_\xi(K)$. Next, from Lemma 4.2.7, we know that $x' \in U_n(x)$ tells us that $x' \in f_\xi(K) \implies x \in f_\xi(K)$. Therefore, $x \notin f_\xi(K) \implies x' \notin f_\xi(K) \implies \xi \in S_n(x')$.

To see the second implication, suppose that $S_n(x') \supseteq S_n(x)$. From this, and the definition of $U_n(x')$,

$$U_n(x') = \bigcap_{\xi \in S_n(x')} U_\xi \subseteq \bigcap_{\xi \in S_n(x)} U_\xi = U_n(x)$$

as desired. □

Proposition 4.2.9. *Let $n \geq 1$ and $(x, y) \in R_n$. Define:*

$$U_n(x, y) = U_n(x) \cap \gamma_{(x,y)}^{-1}(U_n(y))$$

and

$$\mathcal{U}_n(x, y) = \{U \subseteq U_n(x, y) \mid U \text{ is open}\}.$$

Then, for each $U' \in \mathcal{U}_n(x, y)$ and $x' \in U'$, $(x', \gamma_{(x,y)}(x')) \in R_n$.

Proof. Let $U' \in \mathcal{U}_n(x, y)$ and $x' \in U'$. Let $y' = \gamma_{(x,y)}(x')$. We will show that $\mathcal{F}^{-n}\{x'\} = \mathcal{F}^{-n}\{y'\}$.

[$\subseteq$] Let $z' \in \mathcal{F}^{-n}\{x'\}$. Then, there exists an $\eta \in \Sigma_n$ such that $x' \in f_\eta(K)$. Then, because $x' \in U' \subset U_n(x)$, x must also be in $f_\eta(K)$. From this, we see that $f_\eta^{-1}(x) \in \mathcal{F}^{-n}\{x\} = \mathcal{F}^{-n}\{y\}$, because $(x, y) \in R_n$. Therefore, there exists a $\xi \in \Sigma_n$ such that $y = f_\xi \circ f_\eta^{-1}(x)$, and we can conclude that $\gamma_{(x,y)}|_{f_\eta(K)} = f_\xi \circ f_\eta^{-1}$. From here, $y' = \gamma_{(x,y)}(x') = f_\xi(z')$, so $z' \in \mathcal{F}^{-n}\{y'\}$.

[$\supseteq$] Let $z' \in \mathcal{F}^{-n}\{y'\}$. Then, there exists a $\xi \in \Sigma_n$ such that $y' \in f_\xi(K)$. By applying the above argument to the translation $\gamma_{(x,y)}^{-1}|_{f_\xi(K)}$, we can see that it is sufficient to prove that $y \in f_\xi(K)$, which follows from $y' \in f_\xi(K)$. □

Lemma 4.2.10. *Suppose that $n \geq 1$, $(x, y) \in R_n$, and $x' \in U_n(x, y)$. If $y' = \gamma_{(x,y)}(x')$, then $U_n(x', y') \subseteq U_n(x, y)$.*

Proof. As $U_n(x, y) \subseteq U_n(x)$, $x' \in U_n(x)$. Similarly, $y' \in U_n(y)$. So, $U_n(x') \subseteq U_n(x)$ and $U_n(y') \subseteq U_n(y)$. From this, and noting that $\gamma_{(x',y')} = \gamma_{(x,y)}$, we see that

$$U_n(x', y') = U_n(x') \cap \gamma_{(x',y')}^{-1}(U_n(y')) \subseteq U_n(x) \cap \gamma_{(x,y)}^{-1}(U_n(y)) = U_n(x, y)$$

as desired. □

Proposition 4.2.11. *For each $n \geq 1$, and $(x, y) \in R_n$, $\mathcal{U}_{n+1}(x, y) \subseteq \mathcal{U}_n(x, y)$.*

Proof. It is sufficient to show that for each $n \geq 1$, and $x \in K$, $U_{n+1}(x) \subseteq U_n(x)$. To see this, consider $x' \in U_{n+1}(x)$. Then, $x' \notin f_\xi(K)$ for all $\xi \in S_{n+1}(x)$. Suppose, by contradiction, that $x' \notin U_n(x)$. This means that there is an $\eta \in S_n(x)$ such that $x' \in f_\eta(K)$. Let $i \in \Sigma$ be such that $x \in f_i \circ f_\eta(K)$. Then the sequence $\xi = (i, \eta_1, \dots \eta_n)$ is such that $x \notin f_\xi(K) \subseteq f_\eta(K)$, because $x \notin f_\eta(K)$, which follows from $\eta \in S_n(x)$. Therefore, $\xi \in S_{n+1}(x)$, but $x' \in f_\xi(K)$, a contradiction. □

Remark 4.2.12. *It may be helpful for one's intuition to notice that*

$$U_n(x) \subseteq Int\left(\bigcup_{\xi \notin S_n(x)} f_\xi(K)\right).$$

To see this, suppose $x' \in U_n(x)$. In other words, x' is not in $f_\xi(K)$ whenever x is not in $f_\xi(K)$. Let $\eta \in \Sigma_n$ be such that $x' \in f_\eta(K)$. Then, $f_\eta(K)$ must contain x, and so $\eta \notin S_n(x)$. Therefore, $x' \in \bigcup_{\xi \notin S_n(x)} f_\xi(K)$. Containment of $U_n(x)$ in the interior of this set simply follows from $U_n(x)$ being open in K. The reverse inclusion also holds in the case where $Int(f_i(K)) \cap Int(f_j(K)) = \emptyset$ for all $i \neq j$. In other words, if the open set condition is satisfied (see Definition 2.1.13), then

$$U_n(x) = Int\left(\bigcup_{\xi \notin S_n(x)} f_\xi(K)\right).$$

This property is satisfied by all the examples given in Section 2.4.

Lemma 4.2.13. *Let $(x, y) \in R_n$, and let $U' \in \mathcal{U}_n(x, y)$. Then, $\gamma_{(x,y)}(U')$ is an open subset of K, and $\gamma_{(x,y)}\big|_{U'}$ is a homeomorphism (in the relative topology).*

Proof. Suppose $(x, y) \in R_n$, and let $U' \in \mathcal{U}_n(x, y)$. Then, $U' \subseteq U_n(x) \cap \gamma^{-1}_{(x,y)}(U_n(y))$. The set $\gamma_{(x,y)}(U')$ is open, as it is the translation of an open set. To see that $\gamma_{(x,y)}(U')$ is also a subset of K, it is sufficient to show that $\gamma_{(x,y)}(U_n(x))$ is an open subset of K, because $\gamma_{(x,y)}(\gamma^{-1}_{(x,y)}(U_n(y))) = U_n(y) \subseteq K$.

To this end, let $x' \in U_n(x)$, and let $\eta \in \Sigma_n$ be such that both x' and x are in $f_\eta(K)$. Then, as $(x, y) \in R_n$, we know that $f_\eta^{-1}(x) \in \mathcal{F}^{-n}\{x\} = \mathcal{F}^{-n}\{y\}$. This guarantees the existence of a $\xi \in \Sigma_n$ such that $f_\xi \circ f_\eta^{-1}(x) = y$. Then, as $f_\eta^{-1}(x') \in K$, we have $\gamma_{(x,y)}(x') = f_\xi \circ f_\eta^{-1}(x') \in K$. Therefore, $\gamma_{(x,y)}(U_n(x)) \subseteq K$.

Finally, as $\gamma_{(x,y)}\big|_{U'}$ is a translation, it is a homeomorphism from U' to its image. $\square$

Definition 4.2.14. *For each $n \geq 1$, each $(x, y) \in R_n$, and each $U' \in \mathcal{U}_n(x, y)$, define $\gamma_n(x, y, U')$ to be the set $\{(x', \gamma_{(x,y)}(x')) \mid x' \in U'\}$. Then, define Γ_n to be the collection*

$$\Gamma_n = \{\gamma_n(x, y, U') \mid (x, y) \in R_n,\ U' \in \mathcal{U}_n(x, y)\}.$$

Remark 4.2.15. *Suppose that $n \geq 1$, $(x, y) \in R_n$. Further, suppose $x' \in U_n(x, y)$ and $y' = \gamma_{(x,y)}(x')$. Then, $U_n(x', y') \subseteq U_n(x, y)$, and from this we see that*

$$(x', y') \in \gamma_n(x, y, U') \implies (x', y') \in \gamma_n(x', y', U' \cap U_n(x', y')) \subseteq \gamma_n(x, y, U').$$

Lemma 4.2.16. *For each $n \geq 1$, Γ_n is a local action on K, and $\bigcup \Gamma_n = R_n$.*

Proof. Fix $n \geq 1$. First, note from Lemma 4.2.13 that each $\gamma \in \Gamma_n$ is a partial homeomorphism. To show that Γ_n is a local action on K, the following properties must be verified:

1. $\{U \subseteq K \mid \mathrm{Id}_U \in \Gamma_n\}$ forms a base for the relative topology of K in $\mathbb{R}^d$

2. $\gamma \in \Gamma_n \implies \gamma^{-1} \in \Gamma_n$

3. $\gamma_1, \gamma_2 \in \Gamma_n, (x, y) \in \gamma_1, (y, z) \in \gamma_2 \implies \exists\, \gamma \in \Gamma_n$ such that $(x, z) \in \gamma \subseteq \gamma_1 \circ \gamma_2$

4. $\gamma_1, \gamma_2 \in \Gamma_n,\ (x, y) \in \gamma_1 \cap \gamma_2 \implies \exists\, \gamma \in \Gamma_n$ such that $(x, y) \in \gamma \subseteq \gamma_1 \cap \gamma_2$.

Proof of 1. First, we check that the sets $U \subseteq K$ with $\mathrm{Id}_U \in \Gamma_n$ cover K. Let $x \in K$. Then, $(x, x) \in R_n$ for all $n \geq 1$. Fix an integer $n \geq 1$, and pick any $U' \in \mathcal{U}_n(x, x)$. Then, $\mathrm{Id}_{U'} = \{(x', x') \mid x' \in U'\} = \gamma_n(x, x, U') \in \Gamma_n$; in other words, U' is a subset of K with $\mathrm{Id}_{U'}$ containing x.

Next, we check that if $x \in U_1 \cap U_2$, with $\mathrm{Id}_{U_1}, \mathrm{Id}_{U_2} \in \Gamma_n$, then there is a U_3 with $\mathrm{Id}_{U_3} \in \Gamma_n$ such that $x \in U_3 \subseteq U_1 \cap U_2$. To this end, notice that $\mathcal{U}_n(x, x) = \mathcal{U}_n(x)$ for all $n \geq 1$, $x \in K$. Furthermore, $\mathrm{Id}_U \in \Gamma_n$ means that there is an $(x, y) \in R_n$, and a $U' \in \mathcal{U}_n(x, y)$ such that

$$\{(x', x') \mid x' \in U\} = \{(x', \gamma_{(x,y)}(x')) \mid x' \in U'\}$$

which can only hold if $x = y$ and $U = U'$. Therefore,

$$\mathrm{Id}_U \in \Gamma_n \iff U \in \mathcal{U}_n(x, x) = \mathcal{U}_n(x)$$

for some $x \in K$. Now, suppose $U_1, U_2 \subseteq K$ are such that $\mathrm{Id}_{U_1}, \mathrm{Id}_{U_2} \in \Gamma_n$. Then, $U_1 \in \mathcal{U}_n(x_1)$ and $U_2 \in \mathcal{U}_n(x_2)$, for some $x_1, x_2 \in K$ and $n \geq 1$. Further, suppose that $x \in U_1 \cap U_2$. Then, there is an open set $U_3 \subseteq U_1 \cap U_2$ which contains x. Moreover, as U_3 is an open subset of $U_n(x_1)$, by definition, $U_3 \in \mathcal{U}_n(x_1)$. Therefore, $\mathrm{Id}_{U_3} \in \Gamma_n$.

Proof of 2. Suppose $\gamma_n(x, y, U') \in \Gamma_n$, where $(x, y) \in R_n$, $U' \in \mathcal{U}_n(x, y)$. We will show that $\gamma_n(x, y, U')^{-1} = \gamma_n(y, x, \gamma_{(x,y)}(U') \in \Gamma_n$. Let $V' = \gamma_{(x,y)}(U')$.

To show that $V' \in \mathcal{U}_n(y, x)$, we must verify that $\gamma_{(x,y)}(U')$ is an open subset of $U_n(y) \cap \gamma^{-1}_{(y,x)}(U_n(x))$. Recall that $U' \in \mathcal{U}_n(x, y)$ means that U' is an open subset of $U_n(x) \cap \gamma^{-1}_{(x,y)}(U_n(y))$. So, V' is clearly open, and:

$$V' = \gamma_{(x,y)}(U') \subseteq \gamma_{(x,y)}(U_n(x) \cap \gamma^{-1}_{(x,y)}(U_n(y))) = U_n(y) \cap \gamma^{-1}_{(y,x)}(U_n(x)).$$

This means that V' is in $\mathcal{U}_n(y, x)$. So, we conclude that $\gamma_n(y, x, V') \in \Gamma_n$.

Finally,

$$\begin{aligned}\gamma_n(x,y,U')^{-1} &= \{(y',x') \mid (x',y') \in \gamma_n(x,y,U')\} \\ &= \{(\gamma_{(x,y)}(x'),x') \mid x' \in U'\} \\ &= \{(y',\gamma_{(y,x)}(y')) \mid y' \in \gamma_{(x,y)}(U')\} \\ &= \gamma_n(y,x,V')\end{aligned}$$

so $\gamma_n(x,y,U')^{-1} \in \Gamma_n$.

Proof of 3. Let $\gamma_1 = \gamma_n(x_1,y_1,U_1'), \gamma_2 = \gamma_n(x_2,y_2,U_2') \in \Gamma_n$. Let $(x,y) \in \gamma_1$ and $(y,z) \in \gamma_2$. Then:

$$\begin{aligned}\gamma_1 \circ \gamma_2 &= \{(x',\gamma_{(x_2,y_2)}(y') \mid x' \in U_1',\ y' = \gamma_{(x_1,y_1)}(x') \in U_2'\} \\ &= \{(x',\gamma_{(x_2,y_2)} \circ \gamma_{(x_1,y_1)}(x') \mid x' \in U_1' \cap \gamma^{-1}_{(x_1,y_1)}(U_2')\} \\ &= \{(x',\gamma_{(x,z)}(x') \mid x' \in U_1' \cap \gamma^{-1}_{(x_1,y_1)}(U_2')\}.\end{aligned}$$

The last equality follows from $(x,y) \in \gamma_1 \implies \gamma_{(x_1,y_1)}(x) = y$ and $(y,z) \in \gamma_2 \implies \gamma_{(x_2,y_2)}(y) = z$. Next, notice that from Proposition 4.2.9, $(x,y),(y,z) \in R_n$, so transitivity tells us that $(x,z) \in R_n$ as well.

Let $U = U_n(x,z) \cap U_1' \cap \gamma^{-1}_{(x_1,y_1)}(U_2')$. U is clearly an open subset of $U_n(x,z)$, so $U \in \mathcal{U}_n(x,z)$. Moreover, $\gamma_n(x,z,U) \in \Gamma_n$ is a subset of $\gamma_1 \circ \gamma_2$, as desired.

Proof of 4. Let $\gamma_1 = \gamma_n(x_1,y_1,U_1'), \gamma_2 = \gamma_n(x_2,y_2,U_2') \in \Gamma_n$. Let $(x,y) \in \gamma_1 \cap \gamma_2$. Then, let $U = U_1' \cap U_2' \cap U_n(x,y)$. U is clearly an open subset of $U_n(x,y)$, and $(x,y) \in R_n$. Further, $\gamma = \gamma_n(x,y,U) \in \Gamma_n \subseteq \gamma_1 \cap \gamma_2$, as desired. □

4.3 Verifying the étale property

In this section, we verify that we have the correct definition of a local action, by showing that the maps r and s as defined in Section 3.3 are indeed local homeomorphisms.

Proposition 4.3.1. *Let (X, τ) be a topological space, and let Γ be a local action on X. Then, Γ is a base for an étale topology on $\bigcup\Gamma \subseteq X \times X$.*

Proof. First, we must show that Γ is indeed a base for a topology. As it is clear that the elements of Γ cover $\bigcup\Gamma$, we must verify that for any two $\gamma_1, \gamma_2 \in \Gamma$, if $(x, y) \in \gamma_1 \cap \gamma_2$, then there exists a $\gamma \in \Gamma$ containing (x, y) such that $\gamma \subseteq \gamma_1 \cap \gamma_2$; however, this is exactly property 4 of Definition 4.2.3, so there is nothing to prove.

It remains to show that the maps $s : (x, y) \mapsto (y, y)$ and $r : (x, y) \mapsto (x, x)$ are local homeomorphisms in this topology. By symmetry, it is enough to show that s is a local homeomorphism. Furthermore, property 1. of Definition 4.2.3 tells us that $\{(x, x) \in X \times X\} = \Delta_X$ is homeomorphic to X. Therefore, we can view s as a map from $\bigcup\Gamma$ to X, sending (x, y) to y.

Fix $\gamma \in \Gamma$. Then, s is surjective when viewed as a map from γ to $s(\gamma)$. Furthermore, it is injective because γ is a partial homeomorphism. Next, to show that s is continuous, we use that $\{U \subseteq X \mid \mathrm{Id}_U \in \Gamma\}$ is a basis for the topology of X. It is enough to show that $s^{-1}(U \cap s(\gamma))$ is an open subset of γ whenever $U \subseteq X$ has $\mathrm{Id}_U \in \Gamma$. In this setup, we can simplify $s^{-1}(U \cap s(\gamma))$ as follows:

$$\begin{aligned} s^{-1}(U \cap s(\gamma)) &= \{(x, y) \in \gamma \mid s(x, y) = y \in U\} \\ &= \{(x, y) \in \gamma \mid (y, y) \in \mathrm{Id}_U\} \\ &= \gamma \circ \mathrm{Id}_U. \end{aligned}$$

Then, by property 3 of Definition 4.2.3, we have that for any pair $(x, y) \in \gamma$ and

$(y, y) \in \mathrm{Id}_U$, there is a $\gamma' \in \Gamma$ such that $(x, y) \in \gamma' \subseteq \gamma \circ \mathrm{Id}_U$. Here, we use that both γ and Id_U are in Γ. Now, we have shown that Id_U is is open in γ, and so s is a continuous map from γ to $s(\gamma)$.

To show that $s|_\gamma$ has a continuous inverse, let $\gamma' \in \Gamma$, and consider $s(\gamma \cap \gamma')$. If $y_0 \in s(\gamma \cap \gamma')$, then there is an $x_0 \in X$ such that $(x_0, y_0) \in \gamma \cap \gamma'$. Then, by property 4 of Definition 4.2.3, we have that for any pair $(x, y) \in \gamma \cap \gamma'$, there is a $\gamma'' \in \Gamma$ such that $(x, y) \in \gamma'' \subseteq \gamma \cap \gamma'$. Let γ'' be such that this holds for (x_0, y_0). Finally, $s(\gamma'') \subseteq s(\gamma \cap \gamma')$ contains y_0, and is open because γ'' is a partial homeomorphism, and is therefore an open mapping. □

Proposition 4.3.2. *Let let $\{\Gamma_n\}_{n \geq 1}$ be the collection of local actions on K, from Definition 4.2.14. Then, $\Gamma = \bigcup_{n \geq 1} \Gamma_n$ is also a local action on K.*

Proof. First, we would like to verify that $\{U \subseteq K \mid \mathrm{Id}_U \in \Gamma\}$ forms a base for the topology of K. Recall that each $\{U \subseteq K \mid \mathrm{Id}_U \in \Gamma_n\}$ is a base for the topology of K. This follows from $\{U \subseteq K \mid \mathrm{Id}_U \in \Gamma\} = \bigcup_{n \geq 1} \{U \subseteq K \mid \mathrm{Id}_U \in \Gamma_n\}$.

Second, we would like to verify that the inverse of any $\gamma \in \Gamma$, γ^{-1}, is also in Γ, which follows from this property for each Γ_n.

Third, we would like to verify the composition property. To this end, let $(x, y) \in \gamma_1 \in \Gamma_n$ and $(y, z) \in \gamma_2 \in \Gamma_m$, and consider the point (x, z). Recall that given a pair in $\gamma \in \Gamma_n$, we can find a $\gamma' \in \Gamma_{n+1}$ containing that pair, and such that $\gamma' \subseteq \gamma$. Assume, without loss of generality, that $n \geq m$. With a simple induction, one can see that there exists a $\gamma' \in \Gamma_n$ such that $(y, z) \in \gamma' \subseteq \gamma_2$. Then, as Γ_n is a local action, there exists a $\gamma \in \Gamma_n$ containing (x, z) and such that $\gamma \subseteq \gamma_1 \cap \gamma'$.

Finally, we would like to verify the intersection property. Suppose that $\gamma_1 \in \Gamma_n$ and $\gamma_2 \in \Gamma_m$, with $(x, y) \in \gamma_1 \cap \gamma_2$. Again, assuming that $n \geq m$, we can find a $\gamma' \in \Gamma_n$ containing (x, y), and contained in γ_2. Then $\gamma_1 \cap \gamma' \subseteq \gamma_1 \cap \gamma_2$ contains (x, y), and because Γ_n is a local action, there is a $\gamma \in \Gamma_n$ containing (x, y) with $\gamma \subseteq \gamma_1 \cap \gamma'$.

Therefore, Γ is a local action for K. $\square$

Theorem 4.3.3. *For each $n \geq 1$, (R_n, τ_n) is an étale equivalence relation, where τ_n is the topology given by the base Γ_n. Moreover, $R = \bigcup_{n\geq 1} R_n$ is an étale equivalence relation in the topology τ given by the base $\Gamma = \bigcup_{n\geq 1} \Gamma_n$, and τ_n is exactly the relative topology of τ in R_n.*

Proof. To see that for each $n \geq 1$, $R_n = \bigcup \Gamma_n$, first notice that Γ_n covers R_n. Indeed, if $(x, y) \in R_n$, $\gamma_n(x, y, U_n(x, y)) \in \Gamma_n$ contains (x, y). On the other hand, by Proposition 4.2.9, every $(x, y) \in \Gamma_n$ is also in R_n.

Now, we know from Proposition 4.3.1 that Γ_n is indeed a base for a topology on R_n. Let τ_n be the topology on R_n generated by the base Γ_n. As we already know that the maps s and r are local homeomorphisms, to check that (R_n, τ_n) is an étale equivalence relation, it remains to verify the following:

(i) The set of pairs $R_n^2 = \{((x, y), (y, z)) \mid (x, y), (y, z) \in R_n\}$ is closed in the relative topology;

(ii) $((x, y), (y, z)) \mapsto (x, z)$ is a continuous map from R_n^2 into R_n; and

(iii) $(x, y) \mapsto (y, x)$ is a continuous map from R_n into R_n.

Proof of (i). We will show that the complement of R_n^2 is open. Let $((x, y), (w, z)) \in R_n \times R_n$ be such that $y \neq w$. We want to find a $\gamma_1, \gamma_2 \in \Gamma_n$ such that $((x, y), (w, z)) \in \gamma_1 \times \gamma_2 \subseteq (R_n \times R_n) \setminus R_n^2$. Choose $\gamma_1 = \gamma_n(x, y, U_1')$ and $\gamma_2 = \gamma_n(w, z, U_2')$, where $U_1' \in \mathcal{U}_n(x, y)$ and $U_2' \in \mathcal{U}_n(w, z)$ are such that

(a) $x \in U_1'$, $w \in U_2'$, and

(b) $\gamma_{(x,y)}(U_1') \bigcap U_2' = \emptyset$.

This can be accomplished as follows. Let V be an open subset of $\gamma_{(x,y)}(U_n(x,y))$ containing y but not w, and let U be an open subset of $U_n(w,z)$ containing w but not y. Note that here we make use of the fact that $y \neq w$, and the openness of $\gamma_{(x,y)}(U_n(x,y))$ and $U_n(w,z)$. Then, take U_1' to be $\gamma_{(x,y)}^{-1}(V')$, where V' is any open subset of $V \setminus \bar{U}$ containing y. Similarly, let U_2' be any open subset of $U \setminus \bar{V}$, and both properties (a) and (b) will be satisfied. With this choice of γ_1 and γ_2, $((x,y),(w,z)) \in \gamma_1 \times \gamma_2$, and if $(x',y') \in \gamma_1$, $y' = \gamma_{(x,y)}(x') \neq w'$ for any $(w',z') \in \gamma_2$, meaning $\gamma_1 \times \gamma_2 \subseteq (R_n \times R_n) \setminus R_n^2$, as desired. By construction, $((x,y),(w,z)) \in \gamma_1 \times \gamma_2$, and so we have shown that the compliment of R_n^2 is open, meaning R_n^2 is closed.

Proof of (ii). Consider the pre-image of $\gamma_n(x_0, y_0, U')$ under the map (ii):

$$S = \{((x,z),(z,y)) \in R_n^2 \mid (x,y) \in \gamma_n(x_0,y_0,U')\}.$$

We aim to show that this set is open in the relative topology of R_n^2 in $R_n \times R_n$. Let $((x,z),(z,y)) \in S$. We will show that $(\gamma_1 \times \gamma_2) \cap R_n^2$ is a subset of S containing $((x,z),(z,y))$, for the following choices of $\gamma_1, \gamma_2 \in \Gamma_n$:

$$\gamma_1 = \gamma_n(x_0, z_0, U'), \quad \gamma_2 = \gamma_n(z_0, y_0, \gamma_{(x_0,z_0)}(U'))$$

where $z_0 = z - y + y_0$.

If $((x',z'),(z',y')) \in (\gamma_1 \times \gamma_2) \cap R_n^2$, then:

$$y' = \gamma_{(z_0,y_0)}(\gamma_{(x_0,z_0)}(x')) = \gamma_{(x_0,y_0)}(x').$$

This along with $x' \in U'$ implies that $(x',y') \in \gamma_n(x_0,y_0,U')$. Therefore, $(\gamma_1 \times \gamma_2) \cap R_n^2 \subseteq S$. It remains to show that $((x,z),(z,y)) \in (\gamma_1 \times \gamma_2) \cap R_n^2$. We have that $\gamma_{(x_0,y_0)}(x) = y$, and $x \in U'$, because $((x,z),(z,x)) \in S$. From $\gamma_{(x_0,y_0)}(x) = y$, we get that $y = x + y_0 - x_0$, and so:

$$\gamma_{(x_0,z_0)}(x) = x + z_0 - x_0 = x + (z - y + y_0) - x_0 = z.$$

Therefore, $(x, z) \in \gamma_1$. Similarly, we have that:

$$\gamma_{(z_0,y_0)}(z) = z + y_0 - z_0 = z + y_0 - (z - y + y_0) = y$$

and $x \in U'$, $z = \gamma_{(x_0,z_0)}(x)$ tells us that $z \in \gamma_{(x_0,z_0)}(U')$. Therefore, $(z, y) \in \gamma_2$, and $((x, z), (z, y)) \in (\gamma_1 \times \gamma_2) \cap R_n^2$, and we can conclude that the map is continuous.

Proof of (iii). Consider the pre-image of $\gamma_n(x_0, y_0, U')$ under the map (iii), $\{(x, y) \in R_n \mid (y, x) \in \gamma_n(x_0, y_0, U')\}$. We aim to show that this set is open in the R_n; however, this is simply the set $\gamma_n(x_0, y_0, U')^{-1}$, which is open by the second property of Definition 4.2.3. Therefore, the map is continuous.

Therefore, for each $n \geq 1$, (R_n, τ_n) is an étale equivalence relation, where τ_n is the topology given by the base Γ_n. Next, let us verify that that R_n is an open subset of R_{n+1}, and that $\tau_n \subseteq \tau_{n+1}$. As we know $R_n \subseteq R_{n+1}$, showing the latter is sufficient.

To see that $\tau_n \subseteq \tau_{n+1}$, it is enough to show that whenever $(x, y) \in \gamma \in \Gamma_n$, there exists a $\gamma' \in \Gamma_{n+1}$ such that $(x, y) \in \gamma' \subseteq \gamma$. To this end, consider $(x', y') \in \gamma = \gamma_n(x, y, U') \in \Gamma_n$. Let $U'' = U' \cap U_{n+1}(x', y')$. From Lemma 4.2.10, we have that $U_{n+1}(x', y') \subseteq U_{n+1}(x, y)$. So, $U' \cap U_{n+1}(x', y')$ is an open subset of $U_{n+1}(x, y)$. Note that $R_n \subseteq R_{n+1}$ tells us that (x, y) is also in R_{n+1}. Let $\gamma' = \gamma_{n+1}(x, y, U'') \in \Gamma_{n+1}$. Both U' and $U_{n+1}(x', y')$, so γ' contains (x', y'). Furthermore, $U'' \subseteq U'$ tells us that $\gamma' \subseteq \gamma$, as desired.

Applying Proposition 3.2.4, we get that $R = \bigcup_{n\geq 1} R_n$ is an étale equivalence relation in the topology given by the base $\tau = \bigcup_{n\geq 1} \tau_n$. So, we have expressed an étale topology on R as the inductive limit of étale topologies on $\{R_n\}_{n\geq 1}$. However, it would be more helpful to view this topology as coming from a local action; namely, $\Gamma := \bigcup_{n\geq 1} \Gamma_n$.

Let us verify that Γ is indeed a base for the topology on R, given by τ. To this end, let U be open in the topology on R given by the base $\tau = \bigcup_{n\geq 1} \tau_n$. Then, U is an arbitrary union of sets in τ. For any set $V \in \tau$, there is an $n \geq 1$ for which $V \in \tau_n$, which is the topology on R_n given by the base Γ_n. So, such a V is an arbitrary union of elements of Γ_n, and U is then a union of elements of Γ.

Finally, to see why the relative topology of R_n in R is exactly τ_n, we will show that for each $U \in \tau_{n+1}$, $U \cap R_n \in \tau_n$. It is enough to show this for an arbitrary $U = \gamma_{n+1}(x, y, U') \in \Gamma_{n+1}$. In this case, $U \cap R_n = \{(x', \gamma_{(x,y)}(x')) \mid x' \in U',\ (x, y) \in R_n\}$. However, $U' \in \mathcal{U}_{n+1}(x, y) \subseteq \mathcal{U}_n(x, y)$, so $U \cap R_n$ is indeed in Γ_n. □

Remark 4.3.4. *Note that both r and s are continuous, and that this tells us the étale topology on R_n is finer than the relative topology of R_n in $K \times K$. Then, Lemma 4.2.10 shows that these topologies are equal. From this, we deduce that R_n is locally compact for each $n \geq 1$, and so R will also be locally compact in the inductive limit of the étale topologies. However, unlike R_n, the étale topology on R is not usually equal to the relative topology of R in $K \times K$.*

4.4 Isomorphism in the case of conjugate IFS

In this section, let $(X, \{f_i\}_{i=1}^m)$ and $(Y, \{g_i\}_{i=1}^m)$ be single-matrix affine IFS. Furthermore, for each $n \geq 1$, let (R_n, τ_n) and (R'_n, τ'_n) be the corresponding étale equivalence relations, respectively, as in Proposition 4.1.1 and Theorem 4.3.3. Let (R, τ) and (R, τ') be the associated inductive limits. In general, elements corresponding to $(Y, \{g_i\}_{i=1}^m)$ will be 'primed' to distinguish them from elements corresponding to $(X, \{f_i\}_{i=1}^m)$. We begin with homeomorphism of these as topological spaces, and work our way up to isomorphism as étale equivalence relations.

Theorem 4.4.1. *If* $(X, \{f_i\}_{i=1}^m)$ *and* $(Y, \{g_i\}_{i=1}^m)$ *are conjugate as iterated function systems, then for all* $n \geq 1$, R_n *and* R'_n *are homeomorphic with respect to the associated étale topologies.*

Proof. Let K and K' be the attractors of $(X, \{f_i\}_{i=1}^m)$ and $(Y, \{g_i\}_{i=1}^m)$, respectively. Let $h : K \to K'$ be a homeomorphism such that $h \circ f_i = g_i \circ h$ for each $i = 1, \dots, m$. We aim to show that h leads to a homeomorphism between R_n and R'_n. Define $H : R_n \to R'_n$ by $H(x, y) = (h(x), h(y))$.

To see that H is a bijection between R_n and R'_n, let us first notice that injectivity of H follows directly from injectivity of h. To show that H is also surjective, let $(x', y') \in R'_n$. Suppose that $z \in \mathcal{F}^{-n}\{h^{-1}(x')\}$. Note that $z \in K$. Then, there exists a $\xi \in \Sigma_n$ such that $f_\xi(z) = h^{-1}(x')$. This tells us that $h \circ f_\xi(z) = x'$. It is a simple induction on n to show that the conjugacy property of h extends from Σ_1 to Σ_n. In other words, $h \circ f_\xi = g_\xi \circ h$ for all $\xi \in \Sigma_n$. Therefore, $g_\xi \circ h(z) = x'$, and $h(z) \in \mathcal{G}^{-n}\{x'\} = \mathcal{G}^{-n}\{y'\}$. So, there exists an $\eta \in \Sigma_n$ such that $g_\eta(h(z)) = y'$. As $g_\eta \circ h = h \circ f_\eta$, $y' = h \circ f_\eta(z)$; in other words, $h^{-1}(y') = f_\eta(z)$. Therefore, $z \in \mathcal{F}^{-n}\{h^{-1}(y')\}$. A symmetrical argument yields $\mathcal{F}^{-n}\{h^{-1}(y')\} \subseteq \mathcal{F}^{-n}\{h^{-1}(x')\}$, and we conclude that $(h^{-1}(x'), h^{-1}(y')) \in R_n$. Surjectivity of H follows, as $H(h^{-1}(x'), h^{-1}(y')) = (x', y')$.

Moreover, H is continuous with respect to the étale topologies on R_n and R'_n. Let $\gamma'_n(x', y', U')$ be an element of the local action on R'_n. Then, its preimage under H is given by:

$$H^{-1}(\gamma'_n(x', y', U')) = \{(h^{-1}(z), h^{-1}(\gamma_{(x',y')}(z)) \mid z \in U'\}.$$

Denote the relevant domain-specifying sets by $U_n(x)$, $U_n(x, y)$, and $\mathcal{U}_n(x, y)$ for R_n, and $U'_n(x')$, $U'_n(x', y')$, and $\mathcal{U}'_n(x', y')$ for R'_n, with $x = h^{-1}(x')$, $y = h^{-1}(y')$. We will show the following:

1. If $h(z) \in U_n'(x')$, then $z \in U_n(x)$.

2. $\gamma_{(x',y')}(h(z)) = h(\gamma_{(x,y)}(z))$ for all $z \in K$ such that $h(z) \in U_n'(x')$.

3. $h^{-1}(U')$ is an open subset of $U_n(x, y)$.

Once these three facts are verified, we can use 2. to simplify the pre-image:

$$2. \implies H^{-1}(\gamma_n'(x', y', U')) = \{(h^{-1}(z), \gamma_{(x,y)}(h^{-1}(z)) \mid h^{-1}(z) \in h^{-1}(U')\}$$

and then 3. implies that this is an element of the local action of R_n. Note that we will use 1. and 2. to prove 3.

Proof of 1. Let us recall the relevant definitions:

$$U_n'(x') = \bigcap_{\xi \in S_n(x')} g_\xi(K')^c, \quad S_n(x') = \{\xi \in \Sigma_n \mid x' \notin g_\xi(K')\}.$$

On the other hand,

$$U_n(x) = \bigcap_{\xi \in S_n(x)} f_\xi(K)^c, \quad S_n(x) = \{\xi \in \Sigma_n \mid x \notin f_\xi(K)\}.$$

Notice that $x' \notin g_\xi(K')$ if and only if $x = h^{-1}(x') \notin h^{-1}(g_\xi(K'))$. The conjugacy property tells us that $h^{-1} \circ g_\xi(K') = f_\xi \circ h^{-1}(K') = f_\xi(K)$, so this is equivalent to $x \notin f_\xi(K)$. Therefore, $S_n(x') = S_n(x)$. Furthermore, if $z \in K$ is such that $h(z) \in U_n'(x')$, then $h(z) \notin g_\xi(K')$ for all $\xi \in S_n(x')$, and so $z \notin h^{-1}(g_\xi)(K') = f_\xi(K)$ for all $\xi \in S_n(x') = S_n(x)$. Thus, $z \in U_n(x)$.

Proof of 2. Let $z \in K$ be such that $h(z) \in U_n'(x')$. Then, there exists an $\eta \in \Sigma_n$ such that both $h(z)$ and x' are in $g_\eta(K')$. So, there is a $z' \in K'$ such that $g_\eta(K') = x'$, and this z' is an element of $\mathcal{G}^{-n}\{x'\}$. Then, $(x', y') \in R_n'$ tells us that $z' \in \mathcal{G}^{-n}\{y'\}$, meaning that there exists a $\xi \in \Sigma_n$ such that $g_\xi(z') = y'$. Thus, $\gamma_{(x',y')}(h(z)) =$

$g_\xi \circ g_\eta^{-1}(h(z))$, recalling that we chose η so that $h(z)$ is indeed in $g_\eta(K')$. Using the conjugacy property, we see that

$$\gamma_{(x',y')}(h(z)) = g_\xi \circ g_\eta^{-1} \circ h(z) = h \circ f_\xi \circ f_\eta^{-1}(z)$$

so it remains to show that $f_\xi \circ f_\eta^{-1}(z) = \gamma_{(x,y)}(z)$. As $f_\xi \circ f_\eta^{-1}$ is a translation on its domain, it is enough to make sure that $f_\xi \circ f_\eta^{-1}(x) = y$; however, this also follows from the conjugacy:

$$f_\xi \circ f_\eta^{-1}(x) = f_\xi \circ f_\eta^{-1} \circ h^{-1}(x') = h^{-1} \circ g_\xi \circ g_\eta^{-1}(x') = h^{-1}(y') = y$$

as desired.

Proof of 3. $h^{-1}(U')$ is open because U' is open and h is continuous. To show that $U' \in \mathcal{U}'_n(x',y') \implies h^{-1}(U') \in \mathcal{U}_n(x,y)$, let $z \in h^{-1}(U')$. Then

$$z \in h^{-1}(U'_n(x',y')) = h^{-1}\left(U'_n(x') \cap \gamma^{-1}_{(x',y')}(U'_n(y'))\right) = h^{-1}\left(U'_n(x')\right) \cap h^{-1}\left(\gamma^{-1}_{(x',y')}(U'_n(y'))\right)$$

and

$$1. \implies h^{-1}\left(U'_n(x')\right) \subseteq U_n(x).$$

Next, consider

$$z \in h^{-1}\left(\gamma^{-1}_{(x',y')}(U'_n(y'))\right) \implies h(z) \in \gamma^{-1}_{(x',y')}(U'_n(y') \implies \gamma_{(x',y')}(h(z)) \in U'_n(y').$$

We can apply 2. here, because $z \in h^{-1}(U'_n(x')) \implies h(z) \in U'_n(x')$. Therefore

$$\gamma_{(x',y')}(h(z)) = h(\gamma_{(x,y)}(z) \in U'_n(y') \implies \gamma_{(x,y)}(z) \in h^{-1}(U'_n(y')) \subseteq U_n(y)$$

where we used 1. again in the last step. Therefore, $z \in U_n(x,y)$. As this holds for all $z \in h^{-1}(U')$, we conclude that $h^{-1}(U') \in \mathcal{U}_n(x,y)$. A symmetrical argument shows that H^{-1} is also continuous. □

Corollary 4.4.2. *If* $(X, \{f_i\}_{i=1}^m)$ *and* $(Y, \{g_i\}_{i=1}^m)$ *are conjugate as iterated function systems,* R *and* R' *are homeomorphic with respect to the associated étale topologies.*

Proof. We know that the map H from the proof of Theorem 4.4.1 is a bijection between R_n and R'_n for each $n \geq 1$, and so defines a bijection between R and R' as well. Furthermore, if U is an open set in the étale topology $\tau' = \bigcup_{n\geq 1} \tau'_n$ of R', $U = \bigcup_\alpha U_\alpha$ is a union of open sets U_α, each of which is in τ'_n for some n. We know that $H^{-1}(U_\alpha)$ is open in τ_n, and so is open in τ. Therefore, $H^{-1}(U)$ is a union of open sets in τ, and so is also open in τ. A symmetrical argument shows that the inverse of H is also continuous. □

Chapter 5

The associated C^*-algebras

Let R_n and R be étale equivalence relations as in Chapter 4, with topologies coming from the local actions Γ_n and Γ, respectively. We find ourselves in the case of Proposition 3.3.13, which tells us that for each $n \geq 1$, $C_r^*(R_n)$ is a C^*-subalgebra of $C_r^*(R)$, and that $C_r^*(R)$ coincides with the inductive limit C^*-algebra coming from the increasing sequence of C^*-subalgebras $C_r^*(R_1) \subseteq C_r^*(R_2) \subseteq C_r^*(R_3) \ldots$ In this Chapter, some properties of these C^*-algebras are explored. In Section 5.1, we will consider cases where the underlying IFS satisfies the open set condition and find a general expression for a chain of open invariant sets. We will look at these ideals in the C^*-algebras for the Siérpinski Gasket, and the Fudgeflake, as examples. Then, in Section 5.2, we will consider the ideals coming from a particular subset of one of these open invariant sets, in the case where the open set of the open set condition takes a certain form. Finally, in Section 5.3.2, for each $n \geq 1$, we find an isomorphism between $C_r^*(R_n)$ and a subalgebra of $C(K, \mathcal{M}_{3^n}(\mathbb{C}))$ in the case of the Siérpinski Gasket.

Before proceeding, let us first apply Remark 3.3.14 to this inductive limit of equivalence relation C^*-algebras, in order to conclude that the reduced C^*-algebras agree with the corresponding universal C^*-algebras. Consider an equivalence class $[x]_{R_n}$.

By injectivity of the functions f_ξ for $\xi \in \Sigma_n$, the pre-image $\mathcal{F}^{-n}\{x\}$ can consist of at most $|\Sigma_n| = m^n$ many points of K. This also means that there can be at most m^n many points y such that $\mathcal{F}^{-n}\{y\} = \mathcal{F}^{-n}\{x\}$, so this is a uniform upper bound on the number of elements in an R_n-equivalence class. Therefore, the étale equivalence relations are amenable by Remark 3.3.14, which means that the corresponding C^*-algebras are nuclear [3]. The same is true in the inductive limit. Therefore, we will use the notation $C^*(R_n)$, $C^*(R)$ for the associated C^*-algebras.

Furthermore, it should be noted that the C^*-algebras will be isomorphic when the underlying iterated function systems are conjugate, which follows directly from Corollary 4.4.2.

5.1 Open invariant subsets

Recall from Theorem 3.3.9 that open invariant subsets of an étale equivalence relation correspond to ideals in the associated C^*-algebra. In this section, we find a general expression for a collection of open invariant subsets of the étale equivalence relations defined in Chapter 4, which necessarily induces a chain of ideals in the associated C^*-algebras. However, it must be noted that these invariant subsets may be the entire attractor, meaning that the corresponding ideal is just the whole C^*-algebra. This is always the case for totally disconnected iterated function systems, but we will see some interesting ideal structures appear for both the Siérpinski Gasket and the Fudgeflake. The general expression is the following.

Definition 5.1.1. *For each $n \geq 1, k \geq 1$, define*

$$Y_{n,k} = \{x \in K \mid \#\mathcal{F}^{(-n)}\{x\} \leq k\}$$

where

$$\mathcal{F}^{(-n)}\{x\} = \{\xi \in \Sigma_n \mid x \in f_\xi(K)\}.$$

Remark 5.1.2. *For any $n \geq 1$, $k \geq 1$, we have:*

(i) $Y_{n,k} \subseteq Y_{n,k+1}$*,*

(ii) $Y_{n+1,k} \subseteq Y_{n,k}$*, and*

(iii) $Y_{n,m^n} = K$*.*

The first of these is clear, and the third is due to m^n being the number of elements in Σ_n. To see (ii), first notice that for any $\eta \in \mathcal{F}^{(-n)}\{x\}$, there exists a $z \in K$, and an $i \in \{1, 2, \ldots, m\}$ such that $f_\eta \circ f_i(z) = x$, which means that $(\eta, i) \in \mathcal{F}^{(-(n+1))}\{x\}$. Then, (ii) follows from (η, i) being distinct from (η', j) whenever $\eta \neq \eta'$, as this means that $\#\mathcal{F}^{(-n)}\{x\} \leq \#\mathcal{F}^{(-(n+1))}\{x\} \leq k$.

Consider an arbitrary point x in the attractor of a single-matrix affine IFS. The following lemma shows a nice consequence of the single-matrix property; namely, that for a given starting point z in the attractor, there is exactly one way to get from z to x using a sequence of functions in the system. This may not be true if one removes the single-matrix requirement.

Lemma 5.1.3. *Let $(X, \mathcal{F})$, $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ be a single-matrix affine IFS with attractor K. If $(X, \mathcal{F})$ satisfies the open set condition, then for each $x \in K$, there is a bijection between $\mathcal{F}^{-n}\{x\}$ and $\mathcal{F}^{(-n)}\{x\}$.*

Proof. Suppose that $z \in K$ is such that $f_\xi(z) = f_\eta(z)$ for some $\xi, \eta \in \Sigma_n$. By Lemma 2.3.6, if A is the common matrix of the functions in $\mathcal{F}$, then there are vectors $v_\xi, v_\eta \in \mathbb{R}^d$ such that $f_\xi(x) = A^n x + v_\xi$ and $f_\eta(x) = A^n x + v_\eta$ for all $x \in X$. Then, we see that $f_\xi(z) = f_\eta(z)$ implies that $v_\xi = v_\eta$, and $f_\xi = f_\eta$. Now, we will aim to show that $f_\xi = f_\eta \implies \xi = \eta$, for any pair $\xi, \eta \in \Sigma_n$. This can be done by induction on n, as follows.

To see that the statement holds when $n = 1$, recall that the open set condition means that there exists an open dense subset $V \subset K$ such that $V = \bigcup_{i=1}^m f_i(V)$, and $f_i(V) \cap f_j(V) = \emptyset$ for all $i \neq j$. So, if $f_i = f_j$, then $f_i(V) = f_j(V)$, which can only happen if $i = j$.

Next, assume that the statement holds for n. Let $\xi, \eta \in \Sigma_{n+1}$ be such that $f_\xi = f_\eta$. We aim to show that $f_\xi = f_\eta$. To use the assumption on n, let us write $f_\xi = f_i \circ f_{\xi'}$, $f_\eta = f_j \circ f_{\eta'}$, $\xi', \eta' \in \Sigma_n$. By our assumption, it is sufficient to show that $f_{\xi'} = f_{\eta'}$.

As in the base case, let $V \subseteq K$ be the open dense subset of the open set condition. Now, we have assumed that $f_i \circ f_{\xi'} = f_j \circ f_{\eta'}$, so clearly $f_i \circ f_{\xi'}(V) = f_j \circ f_{\eta'}(V)$. As V satisfies the self-similarity property $V = \bigcup_{i=1}^m f_i(V)$, an argument similar to that in the proof of Lemma 2.1.7 tells us that both $f_{\xi'}(V)$ and $f_{\eta'}(V)$ are both subsets of V. So, $f_i \circ f_{\xi'}(V)$ is a non-empty subset of $f_i(V)$, and $f_j \circ f_{\eta'}(V)$ is a non-empty subset of $f_j(V)$, which are disjoint unless $i = j$. Therefore, $i = j$, so $f_i \circ f_{\xi'} = f_j \circ f_{\eta'}$ implies that $f_{\xi'} = f_{\eta'}$, as desired. □

We are now equipped to prove that each $Y_{n,k}$ is an open invariant subset for R_n.

Proposition 5.1.4. *Let K be the attractor for a single-matrix iterated function system satisfying the open set condition. Then, for each $n \geq 1, k \geq 1$, $Y_{n,k}$ is*

(i) an open subset of K, and

(ii) an invariant set for R_n.

Proof. (i) We will show that $Y_{n,k}$ is open by considering its complement:

$$K - Y_{n,k} = \{x \in K \mid \#\mathcal{F}^{(-n)} \geq k+1\}$$
$$= \bigcup_{\substack{\xi_1,\xi_2,\ldots,\xi_{k+1}\in\Sigma_n \\ \xi_i\neq\xi_j \ \forall \ i\neq j}} \Big(f_{\xi_1}(K) \cap f_{\xi_2}(K) \cap \cdots \cap f_{\xi_{k+1}}(K) \Big)$$

which is a finite union of closed sets. Therefore, the complement of $Y_{n,k}$ is closed, and $Y_{n,k}$ is open.

(ii) Let $y \in Y_{n,k}$, and let $x \in [y]_{R_n}$. Then, $\mathcal{F}^{-n}\{x\} = \mathcal{F}^{-n}\{y\}$, and $\mathcal{F}^{(-n)}\{y\}$ contains at most k elements. Then, by Lemma 5.1.3,

$$\#\mathcal{F}^{(-n)}\{x\} = \#\mathcal{F}^{-n}\{x\} = \#\mathcal{F}^{-n}\{y\} = \#\mathcal{F}^{(-n)}\{y\} \leq k$$

as desired.

□

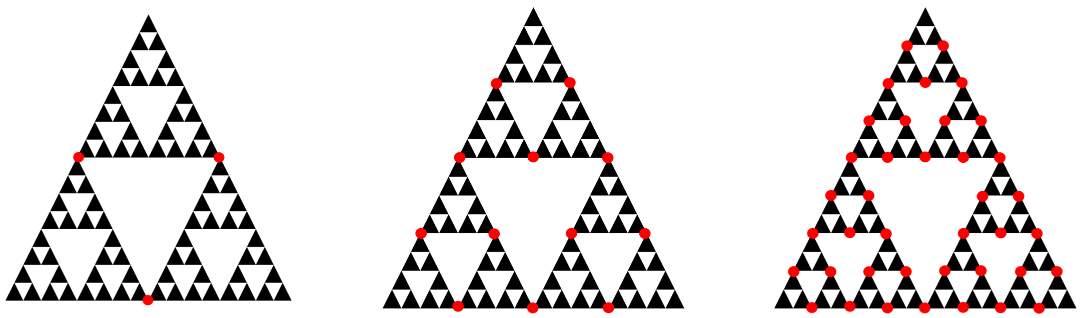
Let us first consider the case of totally disconnected single-matrix IFS, as in Definition 2.1.13, where the images of the maps are pairwise disjoint. Given $n \geq 1$, a point on the attractor of a totally disconnected IFS is in exactly one of the images $f_\xi(K)$ for $\xi \in \Sigma_n$. Therefore, for any $n \geq 1$, $k \geq 1$,

$$Y_{n,k} = \{x \in K \mid \#\mathcal{F}^{(-n)}\{x\} = 1 \leq k\} = K.$$

However, proper open invariant subsets appear when you have non-trivial intersections of the images of the attractor under the functions. Let us consider two just-touching examples: the Siérpinski Gasket, and the Fudgeflake.

Example 5.1.5 (Siérpinski Gasket)**.** *Let K be the the Siérpinski Gasket, and let $x \in K$. Let $\mathcal{F} = \{f_1, f_2, f_3\}$ be as in* (2.1)*. Let $n \geq 0$. Any image $f_\xi(K)$, $\xi \in \Sigma_n$*

will have a non-trivial intersection with exactly one other image $f_\eta(K)$, $\eta \in \Sigma_n$, *with* $\eta \neq \xi$. *Moreover, this intersection will always consist of exactly one point of* K. *If* x *is not one of these intersection points, then* $\#\mathcal{F}^{(-n)}\{x\} = 1$. *On the other hand, if* x *is an intersection point, then* $\#\mathcal{F}^{(-n)}\{x\} = 2$. *Therefore,* $Y_{n,1}$ *will be the proper subset of* K *consisting of all the points which are not an intersection of images of maps* f_ξ, $\xi \in \Sigma_n$. *The complements of the sets* $Y_{1,1}$, $Y_{2,1}$, *and* $Y_{3,1}$ *are shown in Figure 5.1. However, for all* $k \geq 2$, *we find that* $Y_{n,k} = K$, *as the intersection of three or more*



(a) Complement of $Y_{1,1}$ (b) Complement of $Y_{2,1}$ (c) Complement of $Y_{3,1}$

Figure 5.1: Closed invariant sets for the Siérpinski Gasket IFS.

images of the maps is trivial. So, for each n, *the equivalence classes in* R_n *are split into two types: those which are in* $Y_{n,1}$, *and those which are not. Furthermore, aside from the three main corners of the gasket,* $Y_{n,1}$ *contains all the points with equivalence classes containing exactly* 3^n *many elements. On the other hand, the complement of* $Y_{n,1}$ *contains all the points with equivalence classes containing exactly* $\frac{1}{2}(3^n - 1)$ *many elements. The three main corners are anomalies, as these are the only points with trivial* R_n*-equivalence classes for all* $n \geq 1$.

Example 5.1.6 (Fudgeflake)**.** *Recall from Section 2.4.4 that the three functions in the Fudgeflake IFS map the attractor into three smaller, rotated copies of itself, and that these images overlap on the boundary of the attractor (see* (2.6) *and Figure 2.11). In particular, if* a_n *is number of line segments resulting from pairwise intersections of* *the images functions f_ξ for $\xi \in \Sigma_n$, then a_n satisfies the following recurrence relation.*

$$a_1 = 3,\ a_n = 2a_{n-1} + 3^n \tag{5.1}$$

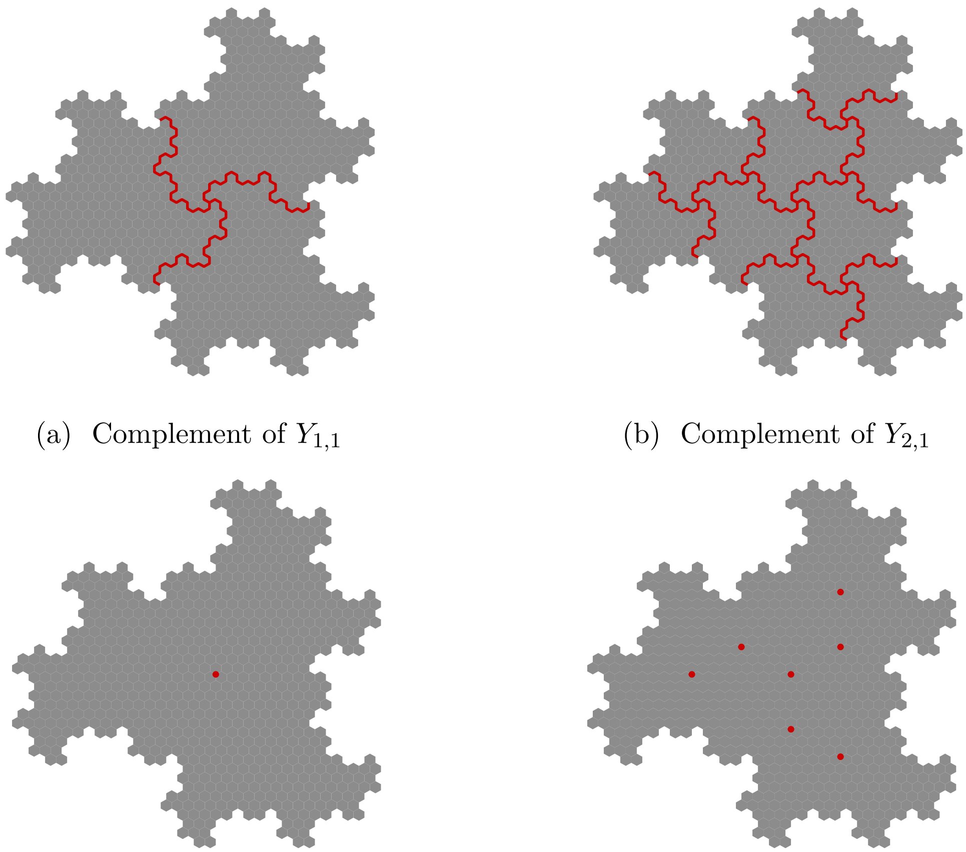
Moreover, some of these line segments will intersect in a point. These points are exactly the intersections of images for three distinct functions f_ξ for $\xi \in \Sigma_n$, and if b_n is the number of these intersections, then b_n can be calculated from the values of a_n using another recurrence relation, as follows.

$$b_1 = 1,\ b_n = b_{n-1} + a_{n-1} + 3^{n-1} \tag{5.2}$$

These a_n many line segments, when unioned together, form the complement of $Y_{n,1}$, and these b_n many intersection points form the complement of $Y_{n,2}$. As no four distinct images of functions f_ξ for $\xi \in \Sigma_n$ can have a non-empty intersection, $Y_{n,k} = K$ for all $k \geq 2$. Figure 5.2 shows these line segments, and intersection points for $n = 1$ and $n = 2$.

In contrast with the Siérpinski Gasket, the number of elements in an R_n-equivalence class depends on more than whether it is in $Y_{n,1}$ or not. In the limiting equivalence relation R, there are seven equivalence classes: six equivalence classes contained within the boundary of the Fudgeflake, and one equivalence class consisting of all interior points. In particular, boundary points behave differently than interior points.

For $n = 1$, the interior points consist of two different types of equivalence classes: classes with three members, and classes with 1 member. If an interior point also lies in $Y_{1,1}$, then its class has 3 members, one from each of the images shown in Figure 2.11. If, instead, an interior point lies in the complement of $Y_{1,1}$, it will have a trivial equivalence class. On the boundary, the situation is much more complicated. If ∂K is the boundary of the Fudgeflake, then a point $x \in \partial K$ will have a trivial equivalence



(a) Complement of $Y_{1,1}$ (b) Complement of $Y_{2,1}$

(c) Complement of $Y_{1,2}$ (d) Complement of $Y_{2,2}$

Figure 5.2: Closed invariant sets for the Fudgeflake IFS.

class if $\mathcal{F}^{-1}\{x\} \subseteq \mathcal{F}^{-1}(Y_{1,1}^c \cap \partial K)$. *Otherwise, the equivalence class of* x *will consist of two points.*

The situation in R_n *for* $n \geq 1$ *is even more complex, so let us simply point out the main similarity with the Siérpinski Gasket: if a point lies in both the interior of the Fudgeflake and* $Y_{n,1}$, *then its class has* 3^n *many members. We will consider this trend in more generality in Section 5.2.*

These examples are meant to provide some insight into the differences between the Siérpinski Gasket and the Fudgeflake IFS, while developing an appreciation for these open invariant subsets $Y_{n,k}$. Though the number of members of each equivalence class is fairly easy to understand for the Siérpinski Gasket, this may not be a reasonable approach to take with the Fudgeflake, due to the complexity of the equivalence class structure. Theorem 3.3.9 tells us that the non-trivial open invariant subsets $Y_{n,k}$ provide ideals in the corresponding C^*-algebras by restriction of the equivalence relation. In these two examples, the sets $Y_{n,k}$ have provided a first step in understanding the structure of $C^*(R_n)$.

5.2 Ideals related to the open set condition

In this section, we consider single-matrix affine iterated function systems which satisfy a stronger version of the open set condition. In particular, we show that such systems admit an open R_n-invariant set for which each point has an R_n-equivalence class of exactly m^n many members, where m is the number of functions in the system. These will be referred to as "typical" points. We begin with the following Lemma, which will help us to relate this open invariant set back to the open invariant sets $Y_{n,k}$ from Section 5.1.

Lemma 5.2.1. *Let $(X, \mathcal{F})$, $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ be an IFS with attractor K, satisfying the open set condition of Definition 2.1.13; that is, $f_i(V) \subseteq V$ for all $1 \leq i \leq m$, and $f_i(V) \cap f_j(V) = \emptyset$ for all $i \neq j$. Let $V \subseteq K$ be an open set satisfying the condition, and suppose that $f_i(V)$ is open for each $i \in \{1, 2, \ldots, m\}$. Then, if $x \in V$ and $y \in K$ are such that $f_i(x) = f_j(y)$ for some i, j, then $i = j$.*

Proof. If $f_j(y) = f_i(x)$, then $f_j(y)$ lies in $f_i(V)$, which is open by assumption. So, $y \in f_j^{-1}(f_i(V))$, which is also open by continuity of f_j. As V is satisfies the requirements of the open set condition, V is dense in K (see [7], p.141), so there exists some $z \in V \cap f_j^{-1}(f_i(V))$. Then, $f_j(z) \in f_i(V) \cap f_j(V)$. Finally, as a V satisfies the open set condition, its images are pairwise disjoint, and so $i = j$. $\square$

Before proceeding, let us consider a few examples of such open sets V for the Siérpinski Gasket. Consider the set $V = K - \{x_1, x_2, x_3\}$, where x_1, x_2, and x_3 are the three main corners. This set is open, and the images under the three functions f_1, f_2, and f_3 are pairwise disjoint. Further, the union of these images is contained in V, so V satisfies the requirements of the open set condition. The images $f_i(V)$ are also open, and so Lemma 5.2.1 applies. However, the set $V' = K - \{x_1, x_2\}$ is also open in K and satisfies the requirements of the open set condition, but the images $f_i(V')$ are not open in K, so the requirements of Lemma 5.2.1 are indeed stronger than the open set condition. However, this stronger condition does not guarantee uniqueness of the open set. Indeed, if we choose V'' to be the Siérpinski Gasket with the three line segments $\overline{x_1x_2}$, $\overline{x_2x_3}$, and $\overline{x_3x_1}$ removed, then V'' is another example of an open set satisfying the requirements of the open set condition, such that $f_1(V'')$, $f_2(V'')$, and $f_3(V'')$ are all open. Next, we realize a relationship between such open sets, and the intersections of the images $f_i(K)$.

Lemma 5.2.2. *Let the IFS $(X, \mathcal{F})$, its attractor K, and the open set $V \subseteq K$ be as in Lemma 5.2.1. Let us view the functions of $\mathcal{F}$ as maps on K. Then:*

$$V \subseteq \left(\bigcup_{\substack{\xi,\eta \in \Sigma_n \\ \xi \neq \eta}} f_\xi^{-1}(f_\xi(K) \cap f_\eta(K)) \right)^c$$

where the complement and pre-images are taken in K.

Proof. First, notice that for any point $x \in V$, if $f_i(x) \in f_j(K)$ then $i = j$. Indeed, if $f_i(x) \in f_j(K)$, then there exists a $y \in K$ such that $f_j(y) = f_i(x)$, and Lemma 5.2.1 tells us that $i = j$. So,

$$V \subseteq \left(\bigcup_{\substack{1 \leq i,j \leq m \\ i \neq j}} f_i^{-1}(f_i(K) \cap f_j(K)) \right)^c .$$

Let us proceed by induction on n. We have shown that the statement holds for $n = 1$. Now, suppose that for some k,

$$V \subseteq \left(\bigcup_{\substack{\xi,\eta \in \Sigma_k \\ \xi \neq \eta}} f_\xi^{-1}(f_\xi(K) \cap f_\eta(K)) \right)^c$$

and consider a point $x \in K$ such that $f_\xi \circ f_i(x) \in f_\eta \circ f_j(K)$, for $(\xi, i), (\eta_j) \in \Sigma_{k+1}$. By assumption, $f_i(V) \subseteq V$, which tells us that $f_i(x) \in V$. We also have that $f_j(K) \subseteq K$, so $f_\eta \circ f_j(K) \subseteq f_\eta(K)$. Therefore, by the inductive hypothesis, we must have $\xi = \eta$, and $f_\xi \circ f_i(x) \in f_\xi \circ f_j(K)$. Let $y \in K$ be such that $f_\xi \circ f_i(x) = f_\xi \circ f_j(y)$. By injectivity of ξ, we conclude that $f_i(x) = f_j(y)$. We are now in the case of Lemma 5.2.1, and conclude that $i = j$, and $(\xi, i) = (\eta, j)$. Therefore

$$V \subseteq \left(\bigcup_{\substack{\xi,\eta \in \Sigma_{k+1} \\ \xi \neq \eta}} f_\xi^{-1}(f_\xi(K) \cap f_\eta(K)) \right)^c$$

as desired. □

Remark 5.2.3. *From Lemma 5.2.2, it is clear that if $V \subseteq K$ is as in Lemma 5.2.1, then:*

$$f_\xi(V) \cap f_\eta(V) = \emptyset \text{ for all } \xi, \eta \in \Sigma_n.$$

We also get that

$$\bigcup_{\xi \in \Sigma_n} f_\xi(V) \subseteq V$$

though an induction on n similar to that found in the proof of Proposition 2.1.7, with the base case guaranteed by the property

$$\bigcup_{i=1}^{m} f_i(V) \subseteq V$$

of the open set condition.

Next, let us consider the situation of single-matrix affine iterated function systems satisfying this strengthened open set condition. In particular, we will use the open invariant sets from Definition 5.1.1 to discover "typical points" of the system. Let us clarify the relationship between such open sets V and $Y_{n,1}$, by looking at the union of images of functions f_ξ for $\xi \in \Sigma_n$, for each $n \geq 1$. The following lemma says that this union is contained in $Y_{n,1}$.

Lemma 5.2.4. *Let $(X, \mathcal{F})$, $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ be a single-matrix affine IFS with attractor K, satisfying the open set condition of Definition 2.1.13. Let $V \subseteq K$ be an open set satisfying the condition, and suppose that $f_i(V)$ is open for each $i \in \{1, 2, \ldots, m\}$. Then:*

$$\bigcup_{\xi \in \Sigma_n} f_\xi(V) \subseteq V \cap Y_{n,1}.$$

Proof. By Remark 5.2.3, we know that the union is contained in V, so it remains to show that it is also contained in $Y_{n,1}$. To this end, let $\xi \in \Sigma_n$, $z \in V$, and $x = f_\xi(z)$. We aim to show that $x \in Y_{n,1}$, so consider $\mathcal{F}^{(-n)}\{x\}$. Suppose by contradiction that there exists a $z' \in K$, and an $\eta \in \Sigma_n$ such that $x = f_\eta(z')$, and $\eta \neq \xi$. Then, $x \in f_\xi(K) \cap f_\eta(K)$, and so $z \in f_\xi^{-1}(f_\xi(K) \cap f_\eta(K))$. As $z \in V$, this contradicts Lemma 5.2.2. Thus, $\mathcal{F}^{(-n)}\{x\}$ contains at most one element, and $x \in Y_{n,1}$. □

In the following proposition, we show that this union of images of a set V satisfying the strengthened requirements of the open set condition will be an open invariant set. Moreover, the equivalence class of every point in this union will contain the same number of members as there are images.

Proposition 5.2.5. *Let $(X, \mathcal{F})$, $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ be a single-matrix affine IFS with attractor K, satisfying the open set condition of Definition 2.1.13. Let $V \subseteq K$ be an open set satisfying the condition, and suppose that $f_i(V)$ is open for each $i \in$*

$\{1, 2, \ldots, m\}$. Let $n \geq 1$ and $z \in V$. Then, if $x = f_\xi(z)$ for some $\xi \in \Sigma_n$, then the R_n-equivalence class of x will be given by

$$[x]_{R_n} = \{f_\eta(z) \mid \eta \in \Sigma_n\}.$$

In particular, $\bigcup_{\xi \in \Sigma_n} f_\xi(V)$ is an open invariant set for R_n such that each equivalence class has exactly m^n many members.

Proof. Let $n \geq 1$, and $x = f_\xi(z)$ for some $z \in V$, $\xi \in \Sigma_n$. We claim that $[x]_{R_n} = \{f_\eta(z) \mid \eta \in \Sigma_n\}$. Suppose $y \in [x]_{R_n}$. Then,

$$\#\mathcal{F}^{(-n)}\{y\} = \#\mathcal{F}^{-n}\{y\} = \#\mathcal{F}^{-n}\{x\} = 1$$

so there exists a unique $z' \in K, \eta \in \Sigma_n$ such that $f_\eta(z') = y$. Furthermore, $z' = z$ because $\mathcal{F}^{-n}\{y\} = \mathcal{F}^{-n}\{x\}$. Therefore, $y \in \{f_\eta(z) \mid \eta \in \Sigma_n\}$. This shows that $[x]_{R_n} \subseteq \{f_\eta(z) \mid \eta \in \Sigma_n\}$.

To show the reverse inclusion, let $\eta \in \Sigma_n$, and consider $\mathcal{F}^{-n}\{f_\eta(z)\}$. Suppose for contradiction that $\mathcal{F}^{-n}\{f_\eta(z)\}$ contains of at least one point $z' \neq z$. Then, there exists an $\eta' \in \Sigma_n$ such that $f_{\eta'}(z') = f_\eta(z)$. Notice that $\eta' \neq \eta$. This means that $f_\eta(z)$ is in the intersection $f_{\eta'}(K) \cap f_\eta(K)$. In particular,

$$z \in f_\eta^{-1}\left(f_\eta(K) \cap f_{\eta'}(K)\right).$$

However, because $z \in V$, we have that $f_\eta(z)$ will lie in $f_\eta(V)$, and cannot lie in $f_{\eta'}(K) \cap f_\eta(K)$ for $\eta' \neq \eta$, a contradiction. Therefore, $\mathcal{F}^{-n}\{f_\eta(z)\}$ contains a single element, which must be z, and $f_\eta(z) \in [x]_{R_n}$.

We have shown that $[x]_{R_n} = \{f_\eta(z) \mid \eta \in \Sigma_n\}$. From this, we can see that $[x]_{R_n}$ has m^n many elements, as this is the number of sequences in Σ_n. Furthermore, each of these points will lie in $\bigcup_{\xi\in\Sigma_n} f_\xi(V)$, as z lies in V. □

Next, let us investigate the ideals corresponding to such open invariant sets. In fact, we can specify exactly what this ideal is in terms of the open set V.

Theorem 5.2.6. *Let $(X, \mathcal{F})$, $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ be a single-matrix affine IFS with attractor K, satisfying the open set condition of Definition 2.1.13. Let $V \subseteq K$ be an open set satisfying the condition, and suppose that $f_i(V)$ is open for each $i \in \{1, 2, \ldots, m\}$. Then, for each $n \geq 1$, the ideal of $C^*(R_n)$ corresponding to the open invariant set $\bigcup_{\xi\in\Sigma_n} f_\xi(V)$ is isomorphic to $C_0(V) \otimes \mathcal{M}_{m^n}(\mathbb{C})$.*

Proof. Let us denote the open invariant set $\bigcup_{\xi\in\Sigma_n} f_\xi(V)$ by V_n. By Theorem 3.3.9, the ideal of $C^*(R_n)$ corresponding to V_n is $C^*(R_n|_{V_n})$, where the restriction of R_n to V_n consists of exactly the following equivalence classes:

$$\{f_\xi(z) \mid \xi \in \Sigma_n\},\ z \in V$$

by Proposition 5.2.5. The first observation we make is that V_n is homeomorphic to $V \times \Sigma_n$. The following map is a well-defined bijection from $V \times \Sigma_n$ to V_n:

$$h(z, \xi) = f_\xi(z).$$

Moreover, h is continuous. To see this, notice that any open set in V_n can be written as a union of open sets, each contained in a distinct $f_\xi(V)$, and so preimage of this set under h will be a union of open sets in V, along with the associated indices. Similarly, h^{-1} is continuous. Consider $h(U \times \{\xi_1, \ldots, \xi_\ell\})$ for an open subset $U \subseteq V$. This image will be exactly the union of ℓ many smaller copies of U, one lying in each $f_\xi(V)$, which will be open in V_n. Therefore, h is a homeomorphism.

Next, let us extend this homeomorphism to a homeomorphism on $R_n|_{V_n}$. Let $\tilde{R}_n$

be the following equivalence relation on $V \times \Sigma_n$:

$$\tilde{R_n} = \{((z_1, \xi_1), (z_2, \xi_2)) \in V \times \Sigma_n \mid z_1 = z_2\}.$$

Then, $h \times h$ will be a bijection mapping $\tilde{R_n}$ to $R_n|_{V_n}$, where

$$h \times h((z, \xi_1), (z, \xi_2)) = (f_{\xi_1}(z), f_{\xi_2}(z)).$$

Furthermore, it can be seen that $h \times h$ is a homeomorphism, by noticing what h does to equivalence classes:

$$h([(z, \xi)]_{\tilde{R_n}}) = h(\{z\} \times \Sigma_n) = [f_\xi(z)]_{R_n}.$$

Further, $\tilde{R_n}$ is also homeomorphic to $\Delta_V \times \Sigma_n^2$, where $\Delta_V = \{(x, x) \mid x \in V\}$ is equality on V. Therefore, we have:

$$C^*(\tilde{R_n}) \cong C^*(\Delta_V) \otimes C^*(\Sigma_n^2) \cong C_0(V) \otimes M_{m^n}(\mathbb{C}),$$

as desired. □

Remark 5.2.7. *Although these non-trivial ideals of $C^*(R_n)$ exist for IFS satisfying this variation of the open set condition, it should be noted that the inductive limit C^*-algebra may or may not be simple. In the case of the Siérpinski Gasket, all R-equivalence classes are dense in K, so $C^*(R)$ is simple by Theorem 3.3.12. The Fudgeflake, on the other hand, has R yielding a single equivalence class that is dense: the interior points. The other equivalence classes are contained in the boundary. Noting that the inclusion of $Y_{n+1,1} \subseteq Y_{n,1}$ of Remark 5.1.2 (ii) may be proper, these ideals may not compatible with the inductive limit structure.*

5.3 The C^*-algebra of the Siérpinski Gasket

In this section, the C^*-algebra associated to the standard Siérpinski Gasket iterated function system is concretely characterized in terms of matrix algebras and continuous functions on the attractor. In particular, we restrict our attention to the equivalence relations R_n of Chapter 4, after the equivalence classes of the three main corners of the attractor are excluded. These three equivalence classes are excluded because each is trivial, which we demonstrate in the following lemma.

Lemma 5.3.1. *Let x_1, x_2, and x_3 be the three main corners of the Siérpinski Gasket, as given by the IFS in* (2.1)*. For each $n \geq 1$, let R_n denote the étale equivalence relation associated to this system, as defined in Proposition 4.1.1. Then for each $i = 1, 2, 3$, the R_n-equivalence class of the corner x_i is trivial, meaning $[x_i]_{R_n} = \{x_i\}$. Moreover, we have the following short exact sequence, involving the associated C^*-algebra $C^*(R_n)$ and its restriction to $R'_n = R_n \setminus \{(x_i, x_i) \mid i = 1, 2, 3\}$:*

$$0 \to C^*(R'_n) \to C^*(R_n) \to \mathbb{C}^3 \to 0 \tag{5.3}$$

coming from evaluation at the three corners. The same result holds when R_n is replaced by R, and R'_n is replaced by $R' = R \setminus \{(x_i, x_i) \mid i = 1, 2, 3\}$.

Proof. Let K denote the Siérpinski Gasket, as the attractor of the IFS in (2.1). First, observe that $\mathcal{F}^{-n}\{x_i\} = \{x_i\}$ for all $n \geq 1$. Now, let us use induction on n to show that for each $i = 1, 2, 3$, the R_n-equivalence class of x_i is trivial. For $n = 1$, notice that $f_j(x_i) = f_i(x_j)$ for any $1 \leq i, j \leq 3$. This tells us that if $y \neq x_i$, then $x_i \notin \mathcal{F}^{-1}\{y\}$, as if $f_j(x_i) = y$ for some $j \neq i$, then $\{x_i, x_j\} \subseteq \mathcal{F}^{-1}\{y\}$. We conclude that $\mathcal{F}^{-1}\{x_i\} = \mathcal{F}^{-1}\{y\} \implies y = x_i$, which shows that $[x_i]_{R_1} = \{x_i\}$.

Next, assume that the R_k-equivalence class of x_i is trivial for some $k \geq 1$. Suppose

$y \in K$ is such that $\mathcal{F}^{-(k+1)}\{y\} = \mathcal{F}^{-(k+1)}\{x_i\} = \{x_i\}$ for some $1 \leq i \leq 3$. Then, there exists a $\xi \in \Sigma_k$, $j \in \{1,2,3\}$ such that $f_\xi \circ f_j(x_i) = y$, and $f_j(x_i) \in \mathcal{F}^{-k}\{y\}$. If $j \neq i$, then $f_j(x_i) = f_i(x_j)$, and x_j would also be in $\mathcal{F}^{-(k+1)}$, violating $\mathcal{F}^{-(k+1)}\{y\} = \{x_i\}$. Therefore, $j = i$, $y = f_\xi(x_i)$, and $\{x_i\} \subseteq \mathcal{F}^{-k}\{y\}$. It remains to show the reverse inclusion. Suppose $z \in \mathcal{F}^{-k}\{y\}$. Then, there exists an $\eta \in \Sigma_k$ such that $f_\eta(z) = y$. As $z \in K$, there exists an ℓ such that $z = f_\ell(z')$. Then, $z' \in \mathcal{F}^{-(k+1)}\{y\} = \{x_i\}$, so $z' = x_i$. However, if $z' = x_i$, then $y = f_\eta \circ f_\ell(x_i)$, and as before, this can only hold for $\ell = i$. We conclude that $z = f_\ell(x_i) = x_i$, and so $\mathcal{F}^{-k}\{y\} \subseteq \{x_i\}$, as desired. Finally, as $\mathcal{F}^{-k}\{y\} = \{x_i\} = \mathcal{F}^{-k}\{x_i\}$, the inductive hypothesis confirms that y must equal x_i, and so the R_{k+1}-equivalence class of x_i is trivial.

Next, let us consider $C^*(R_n)$ restricted to R'_n, which we will denote by $C^*(R'_n)$, as R'_n is clearly also an amenable, étale equivalence relation. $C^*(R'_n)$ is included in $C^*(R_n)$ as an ideal, and the quotient of $C^*(R_n)$ by $C^*(R'_n)$ will be $C(\{(x_i, x_i) \mid i = 1,2,3\}) \cong \mathbb{C}^3$. This yields the short exact sequence (5.3). Finally, as the equivalence class of each x_i in R is the union $\bigcup_{n\geq 1}[x_i]_{R_n}$, we also have $[x_i]_R = \{x_i\}$, and so the same short exact sequence holds with R and $R' = R \setminus \{(x_i, x_i) \mid i = 1,2,3\}$ in place of R_n and R'_n. □

In what follows, an explicit isomorphism of $C^*(R'_1)$ with such an algebra is shown in detail, and the method for generalizing to $C^*(R'_n)$ is discussed.

5.3.1 Notation and Definitions

Let us establish some notation and important definitions, summarized in the points below.

- Let X denote the solid equilateral triangle in $\mathbb{R}^2$ with vertices at $x_1 = (0,0)$, $x_2 = (1,0)$, and $x_3 = \left(1/2, \sqrt{3}/2\right)$.

- Recall from Section 2.4.1 the standard iterated function system with the Siérpinski Gasket as its attractor. Let $f_1, f_2, f_3 : X \to X$ be the maps from (2.1), which have the Siérpinski Gasket as the attractor of the corresponding iterated function system, as shown in Figure 2.1. Denote the Gasket by K. Note that each x_i is the fixed point of f_i, $i = 1, 2, 3$.

- Let $X_0 = X - \{x_1, x_2, x_3\}$ and $K_0 = K - \{x_1, x_2, x_3\}$.

- For each $n \geq 1$, let R'_n be the étale equivalence relation $\{(x, y) \in K_0 \times K_0 \mid \mathcal{F}^{-n}\{x\} = \mathcal{F}^{-n}\{y\}\}$. In other words, R'_n is the equivalence relation of Lemma 5.3.1.

- Let A be the set of all continuous functions $g : K \to \mathcal{M}_3(\mathbb{C})$ with the additional property that there are complex numbers a, b, c such that

$$g(x_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{bmatrix}, \quad g(x_2) = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{bmatrix}, \quad g(x_3) = \begin{bmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.4}$$

- Let $B = C_0(K_0, \mathcal{M}_3(\mathbb{C})) = \{g \in A \mid g(x_1) = g(x_2) = g(x_3) = 0\}$.

Note that A is a closed $*$-subalgebra of the C^*-algebra $C(K, \mathcal{M}^3(\mathbb{C}))$, equipped with the norm $||g|| = \sup\{||g(x)||_{\mathcal{M}_3(\mathbb{C})} \mid x \in K\}$, where $||\cdot||_{\mathcal{M}_3(\mathbb{C})}$ is the standard norm on $\mathcal{M}_3(\mathbb{C})$.

5.3.2 An isomorphism for $C^*(R'_1)$

We will define an isometric $*$-homomorphism mapping $C_c(R'_1)$ onto a dense subset of A. We will need the following lemma.

Lemma 5.3.2. *If $x \in K_0$, then $(f_i(x), f_j(x)) \in R'_1$ for each $i, j \in \{1, 2, 3\}$.*

Proof. First, note that $f_i(x) \in K_0$ for all $x \in K_0$, $i \in \{1,2,3\}$. Next, we will show that $\mathcal{F}^{-1}\{f_i(x)\} = \{x\}$. Let $z \in \mathcal{F}^{-1}\{f_i(x)\}$. Then, for some k, $f_k(z) = f_i(x)$. This means that $f_i(x) \in f_k(K) \cap f_i(K)$. The only points in K with $f_i(x) \in f_k(K) \cap f_i(K)$ for $k \neq i$ are x_1, x_2, x_3, and $x \in K_0$, so we can conclude that $k = i$. This tells us that $f_i(z) = f_i(x)$, and because f_i is injective, we get that $z = x$. As z was arbitrary, we see that $\mathcal{F}^{-1}\{f_i(x)\} \subseteq \{x\}$. As x is clearly in $\mathcal{F}^{-1}\{f_i(x)\}$, we have that $\mathcal{F}^{-1}\{f_i(x)\} = \{x\}$, as desired. So, for any pair $i, j \in \{1,2,3\}$, $\mathcal{F}^{-1}\{f_i(x)\} = \mathcal{F}^{-1}\{f_j(x)\} = \{x\}$, and $(f_i(x), f_j(x)) \in R_1'$. □

We can now define the map from $C_c(R_1')$ to A.

Definition 5.3.3. *Define $\alpha : C_c(R_1') \to A$ as follows. For each $g \in C_c(R_1')$, and $x \in K$, let $\alpha(g)(x)$ be the 3×3 matrix with entries given by:*

$$\alpha(g)(x)_{i,j} = \begin{cases} g(f_i(x), f_j(x)) & (f_i(x), f_j(x)) \in R_1' \\ 0 & (f_i(x), f_j(x)) \notin R_1'. \end{cases}$$

Note that Lemma 5.3.2 ensures that α is well-defined. Furthermore, $\alpha(g)$ is in $C(K, \mathcal{M}_3(\mathbb{C}))$ for each $g \in C_c(R_1')$, because f_1, f_2, f_3 are all continuous, and by noting that R_1' is an open subset of $K \times K$, so that compact support of g on R_1' tells us that extending g to be zero on $(K \times K) \setminus R_1'$ will give a continuous map on $K \times K$. Here, we are using the fact that the étale topology is equal to the relative product topology in the case of R_1'. Finally, notice that if $y_1 = f_1(K) \cap f_2(K)$, $y_2 = f_2(K) \cap f_3(K)$,

$y_3 = f_3(K) \cap f_1(K)$, *then*

$$\alpha(g)(x_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & g(y_1, y_1) & 0 \\ 0 & 0 & g(y_3, y_3) \end{bmatrix},$$
$$\alpha(g)(x_2) = \begin{bmatrix} g(y_1, y_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g(y_2, y_2) \end{bmatrix},$$
$$\alpha(g)(x_3) = \begin{bmatrix} g(y_3, y_3) & 0 & 0 \\ 0 & g(y_2, y_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, indeed, $\alpha(g)(x) \in A$ *for each* $g \in C_c(R_1')$, $x \in K$.

Proposition 5.3.4. α *is a* $*$*-homomorphism from* $C_c(R_1')$ *to* A.

Proof. Let us first verify that α preserves products. Consider $\alpha(g_1 g_2)$ for two functions $g_1, g_2 \in C_c(R_1')$:

$$\alpha(g_1 g_2)(x)_{i,j} = \begin{cases} \sum\limits_{z \in [f_i(x)]_{R_1'}} g_1(f_i(x), z) g_2(z, f_j(x)) & (f_i(x), f_j(x)) \in R_1' \\ 0 & (f_i(x), f_j(x)) \notin R_1'. \end{cases}$$

If $x \in K_0$, then by Lemma 5.3.2, $[f_i(x)]_{R_1'} = \{f_1(x), f_2(x), f_3(x)\}$. If $x \in \{x_1, x_2, x_3\}$, then $[f_i(x)]_{R_1'} = \{f_i(x)\}$, and $(f_i(x), f_j(x)) \in R_1'$ if and only if $i = j$. Let us consider these two cases separately.

Case 1. If $x \in K_0$, then $(f_i(x), f_j(x)) \in R_1'$ for all $i, j \in \{1, 2, 3\}$, and:

$$\alpha(g_1 g_2)(x)_{i,j} = \sum_{i=1}^{3} g_1(f_i(x), f_k(x)) g_2(f_k(x), f_j(x)) = ((\alpha(g_1)(x))(\alpha(g_2)(x)))_{i,j}.$$

Case 2. If $x \in \{x_1, x_2, x_3\}$, then $(f_i(x), f_j(x)) \in R_1'$ if and only if $i = j$, and:

$$\alpha(g_1g_2)(x)_{i,j} = \begin{cases} g_1(f_i(x), f_i(x))g_2(f_i(x), f_i(x)) & i = j \\ 0 & i \neq j. \end{cases}$$

Similarly,

$$\alpha(g_1)(x)_{i,j} = \begin{cases} g_1(f_i(x), f_i(x)) & i = j \\ 0 & i \neq j \end{cases}$$

and

$$\alpha(g_2)(x)_{i,j} = \begin{cases} g_2(f_i(x), f_i(x)) & i = j \\ 0 & i \neq j \end{cases}$$

so the product of these diagonal matrices will be exactly $\alpha(g_1g_2)(x)$, as desired.

Next, let us verify that α preserves conjugation. Consider $\alpha(g^*)$, $g \in C_c(R_1')$:

$$\alpha(g^*)(x)_{i,j} = \begin{cases} \overline{g(f_j(x), f_i(x))} & (f_i(x), f_j(x)) \in R_1' \\ 0 & (f_i(x), f_j(x)) \notin R_1'. \end{cases}$$

Case 1. If $x \in K_0$, then $(f_i(x), f_j(x)) \in R_1'$ for all $i, j \in \{1, 2, 3\}$, and:

$$\alpha(g^*)(x)_{i,j} = \overline{g(f_j(x), f_i(x))}$$

so $\alpha(g^*)(x)$ is the complex conjugate transpose of $\alpha(g)(x)$, and so $\alpha(g^*)$ is just point-wise conjugation, and $\alpha(g^*) = \alpha(g)^*$.

Case 2. If $x \in \{x_1, x_2, x_3\}$, then $(f_i(x), f_j(x)) \in R_1'$ if and only if $i = j$, and:

$$\alpha(g^*)(x)_{i,j} = \begin{cases} \overline{g(f_i(x), f_i(x))} & i = j \\ 0 & i \neq j \end{cases}$$

and because this is a diagonal matrix, this is again just point-wise matrix conjugation, as desired. □

Proposition 5.3.5. *$\alpha(C_c(R_1'))$ is dense in A.*

Proof. We will show that there is a subset of $\alpha(C_c(R_1'))$ which is dense in A, but first, let us establish a useful notation. For each function $h \in C(K, M_3(\mathbb{C}))$, define $h_{i,j} : K \to \mathbb{C}$ by taking $h_{i,j}(x)$ to be the (i,j)-th entry of $h(x)$, for each $x \in K$. Then, for every $h \in A$, we have:

$$h_{i,j}(x_k) = 0 \text{ for all } i \neq j,\ k = 1, 2, 3$$
$$h_{i,i}(x_i) = 0 \text{ for all } i = 1, 2, 3$$
$$h_{2,2}(x_1) = h_{1,1}(x_2),\ h_{3,3}(x_1) = h_{1,1}(x_3), \text{ and } h_{3,3}(x_2) = h_{2,2}(x_3).$$

Let $\delta > 0$, and define B_δ to be the set of functions in A which satisfy the following two conditions:

1. $h_{i,j}(K \cap B_\delta(x_k)) = \{0\}$ for all $i \neq j,\ k = 1, 2, 3$

2. $h_{i,i}(K \cap B_\delta(x_i)) = \{0\}$ for all $i = 1, 2, 3$

where $B_\delta(x)$ denotes all the values of the functions in B_δ when evaluated at the point x.

Claim 1: $B_\delta \subseteq \alpha(C_c(R_1'))$ for all $\delta > 0$.

Let $h \in B_\delta$. The goal is to find a $g \in C_c(R_1')$ such that $\alpha(g) = b$. Define the function $g : R_1' \to \mathbb{C}$ by:

$$g(x, y) = h_{i,j}(z)$$

where i, j are such that there is a $z \in K$ with $(x, y) = (f_i(z), f_j(z))$. To see that g

has compact support, we have:

$$\{(x,y) \in R_1' \mid g(x,y) \neq 0\} = \{(f_i(z), f_j(z)) \in R_1' \mid h_{i,j}(z) \neq 0\}$$
$$\subseteq \left\{(f_i(z), f_j(z)) \in R_1' \;\middle|\; z \in K - \bigcup_{k=1}^{3} B_\delta(x_k)\right\} =: (*).$$

It is clear that the set $S := K - \bigcup_{k=1}^{3} B_\delta(x_k)$ is compact in K. To see that g has compact support, it is enough to show that the above set $(*)$ is also compact. We know that $f_i(z) \notin \{y_1, y_2, y_3, x_1, x_2, x_3\}$ because the set S does not contain the corners $\{x_1, x_2, x_3\}$. So, for every $z \in S$ and each pair $1 \leq i, j \leq 3$, $(f_i(z), f_j(z)) \in R_1'$. Moreover, if $f_i(x) = Ax + b_i$ for $i \in \{1,2,3\}$, then the equivalence class of each $f_i(z)$ can be written as $[f_i(z)]_{R_1'} = \{f_i(z) + b_j - b_i \mid 1 \leq j \leq 3\}$. Let $b_{i,j} = b_j - b_i$. Then, we can rewrite $(*)$ as follows:

$$(*) = \left\{(f_i(z), f_j(z)) \in R_1' \;\middle|\; z \in S\right\} = \bigcup_{i=1}^{3} \bigcup_{j=1}^{3} \{(f_i(z), f_i(z) + b_{i,j}) \mid z \in S\}.$$

Let $\mathcal{O} \subseteq \Gamma_1$ be an open cover of $(*)$. For each $z \in S$, there is a $\gamma_{i,j}(z) = \gamma_1(x, x + b_{i,j}, U) \in \mathcal{O}$ such that $(f_i(z), f_i(z) + b_{i,j}) \in \gamma_{i,j}(z)$. Recall that the set U is some open subset of $U_1(x, x + b_{i,j})$. Let us refer to these domains as $U_{i,j}(z)$. The goal is to find a way to choose a finite subset $F \subseteq S$ for which the sets $\{\gamma_{i,j}(z) \mid z \in F\}$ still cover $(*)$. As each f_i is continuous, $f_i(S)$ is also compact. The sets $\{U_{i,j}(z)\ 1 \leq j \leq 3, z \in S\}$ form an open cover of $f_i(S)$ for each $i \in \{1,2,3\}$, so compactness of $f_i(S)$ gives us a finite subcover for each i. Therefore, there exists a finite subset $F \subseteq S$ for which $\{U_{i,j}(z)\ 1 \leq i, j \leq 3, z \in F\}$ covers $\bigcup_{i=1}^{3} f_i(S)$. Using this subset, we find that $\{\gamma_{i,j}(z) \mid 1 \leq i, j \leq 3, z \in F\}$ is a finite subcover of $\mathcal{O}$.

Finally, note that α sends g to the function which, when evaluated at $z \in K$, gives the matrix with entries $h_{i,j}(z)$. Indeed, $(f_i(z), f_j(z)) \in R_1'$ for all $z \in K_0$, and if

$z = x_k$ for some $k \in \{1, 2, 3\}$, then:

$$(\alpha(g)(x_k)) = \begin{bmatrix} h_{1,1}(x_k) & 0 & 0 \\ 0 & h_{2,2}(x_k) & 0 \\ 0 & 0 & h_{3,3}(x_k) \end{bmatrix} = h(x_k).$$

Claim 2: $B = \bigcup_{\delta>0} B_\delta$ is dense in A.

As the operator norm on $\mathcal{M}_3(\mathbb{C})$ is equivalent to the uniform norm (which takes the maximum of the moduli of the entries), it is sufficient to show that B is dense in A with respect to the following norm:

$$||h|| = \max_{1\leq i,j\leq 3} \{||h_{i,j}||_\infty\}.$$

Let $a \in A$ and let $\epsilon > 0$. Consider the neighborhood of a

$$B_\epsilon(a) = \left\{ h \in A \;\middle|\; \max_{1\leq i,j\leq 3} \{||h_{i,j} - a_{i,j}||_\infty\} < \epsilon \right\}.$$

Given $\delta > 0$, there exists a continuous function $h_\delta : K \to \mathcal{M}_3(\mathbb{C})$ which satisfies

1. $h_{i,j}\left(K \cap B_{\frac{\delta}{2}}(x_k)\right) = \{0\}$ for all $i \neq j,\ k = 1, 2, 3$
2. $h_{i,i}\left(K \cap B_{\frac{\delta}{2}}(x_i)\right) = \{0\}$ for all $i = 1, 2, 3$

and is equal to $a(x)$ for $x \in K \setminus \bigcup_{k=1}^{3} B_\delta(x_k)$. This function will be an element of B. Furthermore, it is clear by the definition that h_δ approaches a (entry-wise) as $\delta \to 0$, so there must be a $\delta > 0$ for which $h_\delta \in B_\epsilon(a)$. □

Proposition 5.3.6. *α is isometric.*

Proof. Let $g \in C_c(R_1')$. The goal is to show that $||g||_r = ||\alpha(g)||$. First, let us separate out the suprema using the points $\{y_1, y_2, y_3\}$ defined above. On one side, we have

$$||g||_r = \max\left\{ \sup_{y\in K_0-\{y_1,y_2,y_3\}} \{||\pi_\lambda^y(g)||\},\ \max_{y\in\{y_1,y_2,y_3\}} \{||\pi_\lambda^y(g)||\} \right\}$$

and on the other side, we have

$$||\alpha(g)|| = \max\left\{\sup_{x\in K_0}\{||\alpha(g)(x)||\}, \max_{x\in\{x_1,x_2,x_3\}}\{||\alpha(g)(x)||\}\right\}.$$

Claim 1:

$$\sup_{y\in K_0-\{y_1,y_2,y_3\}}\{||\pi_\lambda^y(g)||\} = \sup_{x\in K_0}\{||\alpha(g)(x)||\}$$

Consider the representation $\pi_\lambda^x : C_c(R_1') \to \mathcal{B}(\ell^2\{f_1(x), f_2(x), f_3(x)\})$. Define the map $U : \ell^2\{f_1(x), f_2(x), f_3(x)\} \to \mathbb{C}^3$ sending ξ to $(\xi(f_1(x)), \xi(f_2(x)), \xi(f_3(x)))$. Note that U is invertible with $(U^{-1}\eta)(f_i(x)) = (U^*\eta)(f_i(x)) = \eta_i$ for $i = 1, 2, 3$, $\eta \in \mathbb{C}$. So, U is an inner-product preserving bijection. As it is clearly linear, this means it is an isomorphism of the Hilbert spaces. Then, we see that for $i = 1, 2, 3$:

$$\begin{aligned}
(U^*(\alpha(g)(x))U\xi)(f_i(x)) &= (\alpha(g)(x)U\xi)_i \\
&= ((\alpha(g)(x))(\xi(f_1(x)), \xi(f_2(x)), \xi(f_3(x))))_i \\
&= \sum_{j=1}^{3} g(f_i(x), f_j(x))\xi(f_j(x)) \\
&= \sum_{z\in[f_i(x)]_{R_1'}} g(f_i(x), z)\xi(z) \\
&= \pi_\lambda^{f_i(x)}(g)(f_i(x))\xi(f_i(x))
\end{aligned}$$

where we used that $x \in K_0$ when evaluating $\alpha(g)(x)$. From this, we conclude that $U^*\alpha(g)U = \pi_\lambda^{f_i(x)}(g)$, and so for each $i = 1, 2, 3$:

$$||U^*\alpha(g)U|| = ||\alpha(g)|| = ||\pi_\lambda^{f_i(x)}(g)||$$

which will be equal to the supremum for at least one value of i.

Claim 2:

$$\max_{y\in\{y_1,y_2,y_3\}}\{||\pi_\lambda^y(g)||\} = \max_{x\in\{x_1,x_2,x_3\}}\{||\alpha(g)(x)||\}$$

First, notice that

(a) $||\alpha(g)(x_1)|| = ||\mathrm{diag}(0, g(y_1,y_1), g(y_3,y_3))|| = \max\{|g(y_1,y_1)|, |g(y_3,y_3)|\}$,

(b) $||\alpha(g)(x_2)|| = ||\mathrm{diag}(g(y_1,y_1), 0, g(y_2,y_2))|| = \max\{|g(y_1,y_1)|, |g(y_2,y_2)|\}$,

(c) $||\alpha(g)(x_3)|| = ||\mathrm{diag}(g(y_3,y_3), g(y_2,y_2), 0)|| = \max\{|g(y_2,y_3)|, |g(y_3,y_3)|\}$.

On the other hand, $[y_i]_{R_1'} = \{y_i\}$, and so the representation of g on $\ell^2(\{y_i\})$ simplifies to:

$$\pi_\lambda^{y_i}(g)\xi = g(y_i,y_i)\xi(y_i,y_i)$$

which tells us that $||\pi_\lambda^{y_i}(g)|| = |g(y_i,y_i)|$. Therefore, the left-hand side is equal to the whichever is the largest of $|g(y_i,y_i)|$, $i = 1,2,3$. The same is true of the right-hand side. □

Extending α to $C^*(R_1')$ then gives an isomorphism to A.

5.3.3 Regarding $C^*(R_n')$ for $n > 1$

One can find an isomorphism for $C^*(R_n')$ similar to that of $C^*(R_1')$ with A, as described in Section 5.3.2. For the rest of this section, we will refer to A as A_1, and the images of the corresponding isomorphisms for $C^*(R_n')$ as A_n, for each $n \geq 1$. Let us first consider what A_2 should be. Define the map $\beta : C_c(R_2') \to C(K, \mathcal{M}_9(\mathbb{C}))$ as follows:

$$\beta(g)(x)_{(i,j),(k,\ell)} = \begin{cases} g(f_i\circ f_j(x), f_k\circ f_\ell(x)) & (f_i\circ f_j(x), f_k\circ f_\ell(x)) \in R_2' \\ 0 & (f_i\circ f_j(x), f_k\circ f_\ell(x)) \notin R_2' \end{cases} \tag{5.5}$$

where the ordered pairs (i, j), $1 \leq i, j \leq 3$ are interpreted in a fixed one-to-one correspondence with the 9 indices describing $\mathcal{M}_9(\mathbb{C})$.

We do not include x_1, x_2, x_3 in R'_n for each $n \geq 1$, just as we did for R'_1. Thus, equivalence classes in R'_2 consist of either 9 members, or 4 members. The relationships between the equivalence classes with 4 members translates to the following relationship between $\beta(g)(x_1)$, $\beta(g)(x_2)$, and $\beta(g)(x_3)$ (up to a permutation of the matrix indices):

$$\beta(g)(x_1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{1,1} & 0 & a_{1,2} & a_{1,3} & a_{1,4} & 0 & 0 & 0 \\ 0 & 0 & b_{1,1} & 0 & 0 & 0 & b_{1,2} & b_{1,3} & b_{1,4} \\ 0 & a_{2,1} & 0 & a_{2,2} & a_{2,3} & a_{2,4} & 0 & 0 & 0 \\ 0 & a_{3,1} & 0 & a_{3,2} & a_{3,3} & a_{3,4} & 0 & 0 & 0 \\ 0 & a_{4,1} & 0 & a_{4,2} & a_{4,3} & a_{4,4} & 0 & 0 & 0 \\ 0 & 0 & b_{2,1} & 0 & 0 & 0 & b_{2,2} & b_{2,3} & b_{2,4} \\ 0 & 0 & b_{3,1} & 0 & 0 & 0 & b_{3,2} & b_{3,3} & b_{3,4} \\ 0 & 0 & b_{4,1} & 0 & 0 & 0 & b_{4,2} & b_{4,3} & b_{4,4} \end{bmatrix} \tag{5.6}$$

$$\beta(g)(x_2) = \begin{bmatrix} a_{2,2} & a_{2,3} & a_{2,4} & a_{2,1} & 0 & 0 & 0 & 0 & 0 \\ a_{3,2} & a_{3,3} & a_{3,4} & a_{3,1} & 0 & 0 & 0 & 0 & 0 \\ a_{4,2} & a_{4,3} & a_{4,4} & a_{4,1} & 0 & 0 & 0 & 0 & 0 \\ a_{1,2} & a_{1,3} & a_{1,4} & a_{1,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ 0 & 0 & 0 & 0 & 0 & c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ 0 & 0 & 0 & 0 & 0 & c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ 0 & 0 & 0 & 0 & 0 & c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{bmatrix} \tag{5.7}$$

$$\beta(g)(x_3) = \begin{bmatrix} b_{3,3} & b_{3,4} & b_{3,2} & 0 & 0 & 0 & 0 & b_{3,1} & 0 \\ b_{4,3} & b_{4,4} & b_{4,2} & 0 & 0 & 0 & 0 & b_{2,1} & 0 \\ b_{2,3} & b_{2,4} & b_{2,2} & 0 & 0 & 0 & 0 & b_{4,1} & 0 \\ 0 & 0 & 0 & c_{3,3} & c_{3,4} & c_{3,2} & c_{3,1} & 0 & 0 \\ 0 & 0 & 0 & c_{4,3} & c_{4,4} & c_{4,2} & c_{2,1} & 0 & 0 \\ 0 & 0 & 0 & c_{2,3} & c_{2,4} & c_{2,2} & c_{4,1} & 0 & 0 \\ 0 & 0 & 0 & c_{1,3} & c_{1,2} & c_{1,4} & c_{1,1} & 0 & 0 \\ b_{1,3} & b_{1,2} & b_{1,4} & 0 & 0 & 0 & 0 & b_{1,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.8}$$

where $(a_{i,j}), (b_{i,j}), (c_{i,j}) \in \mathcal{M}_4(\mathbb{C})$. Though the explicit isomorphisms are challenging to write down, in general, we can at least formulate a general picture for $C^*(R'_n)$.

Theorem 5.3.7. *Let x_1, x_2, x_3 be the three main corners of Siérpinski Gasket, K. For each $n \geq 1$, let $C^*(R_n)$ be the n-th C^*-algebra associated to the Siérpinski Gasket IFS of* (2.1)*. Then, x_1, x_2, x_3 each has a trivial equivalence class. If*

$$R_n' = R_n \setminus \{(x_1, x_1), (x_2, x_2), (x_3, x_3)\},$$

then $C^*(R_n')$ is a subalgebra A_n of $C(K, \mathcal{M}_{3^n}(\mathbb{C}))$. Moreover, A_n is such that there exists a surjective $*$-homomorphism mapping A_n onto $\bigoplus_1^3 \mathcal{M}_{k_n}(\mathbb{C})$, where $k_n = \frac{1}{2}(3^n - 1)$, yielding the following short exact sequence:

$$0 \to C_0\left(K_0, \mathcal{M}_{3^n}(\mathbb{C})\right) \to A_n \to \bigoplus_1^3 \mathcal{M}_{k_n}(\mathbb{C}) \to 0 \tag{5.9}$$

where $K_0 = K \setminus \{x_1, x_2, x_3\}$.

Chapter 6

K-theory for the Siérpinski Gasket

In this chapter, we consider the K-theory of the C^*-algebra associated to the standard Siérpinski Gasket iterated function system. We restrict our attention to $C^*(R'_n)$ for $n \geq 1$ and $C^*(R')$, as defined in Lemma 5.3.1, where we exclude the three trivial equivalence classes corresponding to the three corners of the Siérpinski Gasket. Some standard results on short exact sequences and K-groups are used to calculate $K_0(C^*(R'_1))$ and $K_1(C^*(R'_1))$. Then, we find an explicit generator of $K_0(C^*(R'_1))$, and use it to find the K_0-group of the inductive limit C^*-algebra $C^*(R')$. To do this, we look at the inclusion of $K_0(C^*(R'_1))$ in $K_0(C^*(R'_2))$. This idea is extended to form an inductive limit of abelian groups that yields $K_0(C^*(R'))$. Throughout this Chapter, we will use the notation and definitions of Section 5.3.1 for notation and definitions related to the C^*-algebra of the Siérpinski Gasket.

6.1 Preliminary results on the K-theory of $C_0(X_0)$

Recall from Section 5.3.1 that X denotes the solid equilateral triangle, and X_0 denotes X with its three corners removed.

Lemma 6.1.1.

$$K_*(C_0(X_0)) \cong \begin{cases} 0 & * = 0 \\ \mathbb{Z}^2 & * = 1 \end{cases}$$

Proof. Consider the short exact sequence

$$0 \to C_0(X_0) \to C(X) \to \mathbb{C}^3 \to 0 \tag{6.1}$$

where $C_0(X_0) \to C(X)$ is inclusion, and $C(X) \to \mathbb{C}^3$ is given by evaluation at the vertices of X; namely, $g \mapsto (g(x_1), g(x_2), g(x_3))$. This induces the following 6-term exact sequence.

$$\begin{array}{ccccc} K_0(C_0(X_0)) & \longrightarrow & K_0(C(X)) & \longrightarrow & K_0(\mathbb{C}^3) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{C}^3) & \longleftarrow & K_1(C(X)) & \longleftarrow & K_1(C_0(X_0)) \end{array}$$

We can use the following standard results to make some simplifications (see, for example, [26]). First, recall the K-theory of the complex numbers:

$$K_*(\mathbb{C}) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1. \end{cases} \tag{6.2}$$

Then, as $K_*(A \oplus B) = K_*(A) \oplus K_*(B)$, we get:

$$K_*(\mathbb{C}^3) \cong \begin{cases} \mathbb{Z}^3 & * = 0 \\ 0 & * = 1. \end{cases} \tag{6.3}$$

Furthermore, as X is contractible compact Hausdorff space, we get that:

$$K_*(C(X)) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \end{cases} \tag{6.4}$$

where the generator of $K_0(C(X))$ is $[1]_0$, where 1 denotes the constant function taking a value of 1 at every point of X [26]. Applying these simplifications gives the following.

$$\begin{array}{ccccc} K_0(C_0(X_0)) & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^3 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & K_1(C_0(X_0)) \end{array}$$

To see that $K_0(C_0(X_0)) \cong 0$, notice that the map $K_0(C_0(X_0)) \to \mathbb{Z}$ is injective. So, it is sufficient to show that the image of this map is $\{0\}$. By exactness, this image is equal to the kernel of the map $\mathbb{Z} \to \mathbb{Z}^3$. To see that the map $\mathbb{Z} \to \mathbb{Z}^3$ is injective, consider the identity function $1 \in C(X)$. Under evaluation at the corners x_1, x_2, x_3, $1 \mapsto (1,1,1) \in \mathbb{C}^3$. We also have that $1 \in C(X) \mapsto 1 \in \mathbb{Z}$ and $(1,1,1) \in \mathbb{C}^3 \mapsto (1,1,1) \in \mathbb{Z}^3$ under the functor K_0. Then, commutativity of the following diagram tells us that $1 \in \mathbb{Z} \mapsto (1,1,1) \in \mathbb{Z}^3$.

$$\begin{array}{ccc} C(X) & \longrightarrow & \mathbb{C}^3 \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}^3 \end{array}$$

As 1 is a generator for $\mathbb{Z}$, we get that $n \mapsto (n,n,n)$ for all $n \in \mathbb{Z}$, which is indeed an injective map.

To see that $K_1(C_0(X_0)) \cong \mathbb{Z}^2$, notice that the map $\mathbb{Z}^3 \to K_1(C_0(X_0))$ is surjective. So, by the fundamental isomorphism theorem for groups, $K_1(C_0(X_0))$ is isomorphic to the quotient group of $\mathbb{Z}^3$ by the kernel of this map. By exactness, this kernel is equal to the image of the map $\mathbb{Z} \to \mathbb{Z}^3$, which we have already determined is given by $n \mapsto (n, n, n)$. So, we get that

$$K_1(C_0(X_0)) \cong \frac{\mathbb{Z}^3}{\langle 1, 1, 1 \rangle}.$$

To see that this is also isomorphic to $\mathbb{Z}^2$, consider the group homomorphism $\mathbb{Z}^3 \to \mathbb{Z}^2$ given by

$$(a, b, c) \mapsto (a - c, b - c).$$

The kernel of this map is then $\{(a, b, c) \in \mathbb{Z}^3 \mid a = b = c\}$, which is exactly $\langle 1, 1, 1 \rangle$. □

It will be helpful in future computations to find generators of $K_1(C_0(X_0))$. In other words, can we find unitary elements $u, v \in C_0(X)^\sim$ for which $[u]_1, [v]_1 \in K_1(C_0(X))$ correspond to $(1, 0), (0, 1) \in \mathbb{Z}^2$? Such a u and v can by found by computing the exponential map $\exp_X : \mathbb{Z}^3 \to K_1(C_0(X_0))$ for the three generators $(1, 0, 0), (0, 1, 0), (0, 0, 1) \in \mathbb{Z}^3$.

Proposition 6.1.2. *Let $s_1, s_2, s_3 \in C(X)$ be $[0, 1]$-valued continuous functions on the triangle X such that $s_i(x_i) = 1$ and $s_i(\overline{f_i(X)^c}) = \{0\}$. Then*

$$\left[e^{2\pi i s_3}\right]_1 = -\left[e^{2\pi i s_1}\right]_1 - \left[e^{2\pi i s_2}\right]_1$$

and

$$K_1(C_0(X_0)) = \left\langle \left[e^{2\pi i s_1}\right]_1, \left[e^{2\pi i s_2}\right]_1 \right\rangle.$$

Proof. Note that each s_i is real-valued, and is therefore self-adjoint. Then, the elements $e^{2\pi i s_i}$ are unitaries in $C(X)$. Furthermore, under the evaluation map $C(X) \to$

$\mathbb{C}^3$, we have:

$$s_1 \mapsto (1,0,0)$$
$$s_2 \mapsto (0,1,0)$$
$$s_3 \mapsto (0,0,1)$$

which means that $e^{2\pi i s_i} \mapsto (1,1,1) \in \mathbb{C}^3$ for each i, because $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ are projections in $\mathbb{C}^3$. This then tells us that each $e^{2\pi i s_i}$ is actually in the unitization of $C_0(X_0)$. Indeed, $e^{2\pi i s_i} = (e^{2\pi i s_i} - 1) + 1$ and $e^{2\pi i s_i} - 1 \in C_0(X_0)$, since it must be mapped to $(0,0,0)$ under evaluation at the three corners. Now that we know these are unitaries in $C_0(X_0)$, we can consider the corresponding K_1-classes. By the definition of the exponential map, we have:

$$\begin{aligned}
\exp_X(1,0,0) &= \left[e^{2\pi i s_1}\right]_1 \\
\exp_X(0,1,0) &= \left[e^{2\pi i s_2}\right]_1 \\
\exp_X(0,0,1) &= \left[e^{2\pi i s_3}\right]_1 .
\end{aligned}$$

Recall that the kernel of $\exp_X$ is $\langle (1,1,1) \rangle$. So, calculating the sum of the classes gives us:

$$\begin{aligned}
\left[e^{2\pi i s_1}\right]_1 + \left[e^{2\pi i s_2}\right]_1 + \left[e^{2\pi i s_3}\right]_1 &= \exp_X(1,0,0) + \exp_X(0,1,0) + \exp_X(0,0,1) \\
&= \exp_X(1,1,1) \\
&= 0
\end{aligned}$$

revealing that

$$\left[e^{2\pi i s_3}\right]_1 = -\left[e^{2\pi i s_1}\right]_1 - \left[e^{2\pi i s_2}\right]_1$$

and

$$\left\langle \left[e^{2\pi i s_1}\right]_1, \left[e^{2\pi i s_2}\right]_1, \left[e^{2\pi i s_3}\right]_1 \right\rangle = \left\langle \left[e^{2\pi i s_1}\right]_1, \left[e^{2\pi i s_2}\right]_1 \right\rangle.$$

Finally, as $\exp_X$ is surjective,

$$\begin{aligned} K_1(C_0(X_0)) &= \exp_X(K_0(\mathbb{C}^3)) \\ &= \exp_X\left(\langle [(1,0,0)]_0, [(0,1,0)]_0, [(0,0,1)]_0 \rangle\right) \\ &= \left\langle \left[e^{2\pi i s_1}\right]_1, \left[e^{2\pi i s_2}\right]_1, \left[e^{2\pi i s_3}\right]_1 \right\rangle \\ &= \left\langle \left[e^{2\pi i s_1}\right]_1, \left[e^{2\pi i s_2}\right]_1 \right\rangle \end{aligned}$$

as desired. □

6.2 Preliminary results on the K-theory of $C_0(K_0)$

In what follows, let us denote $\bigoplus_{1}^{\infty} \mathbb{Z}$ by $\mathbb{Z}^\infty$. Recall from Section 5.3.1 that K denotes the Siérpinski Gasket, and K_0 denotes K with the same three corners removed.

Lemma 6.2.1.

$$K_*(C_0(K_0)) \cong \begin{cases} 0 & * = 0 \\ \mathbb{Z}^2 \oplus \mathbb{Z}^\infty & * = 1 \end{cases}$$

We will need the following result on exact sequence of abelian groups:

Proposition 6.2.2. *Let H and G be abelian groups. If $0 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} \mathbb{Z}^\infty \to 0$, is a short exact sequence, then $G \cong H \oplus \mathbb{Z}^\infty$.*

Proof. As the sequence is exact, β is surjective. So, we can pick a sequence of elements $(g_i)_{i \geq 1}$ in G such that $\beta(g_i)$ is the element of $\mathbb{Z}^\infty$ consisting of all zeros, and a 1 in the i-th place. Then, the following map splits the exact sequence:

$$\gamma : \mathbb{Z}^\infty \to G,\ (n_i)_{i\geq 1} \mapsto \sum_{i\geq 1} n_i g_i,$$

as $\beta \circ \gamma = \mathrm{Id}_{\mathbb{Z}^\infty}$.

Next, let us show that $g - \gamma \circ \beta(g) \in \alpha(H)$ for all $g \in G$. By exactness, we have that $\alpha(H) = \ker(\beta)$. So, it is enough to verify that this element is mapped to zero under β. This follows from γ splitting the exact sequence, and β being a group homomorphism:

$$\beta(g - \gamma \circ \beta(g)) = \beta(g) - (\beta \circ \gamma) \circ \beta(g) = \beta(g) - \mathrm{Id}_{\mathbb{Z}^\infty}(\beta(g)) = \beta(g) - \beta(g) = 0.$$

Let us define the map $\eta : G \to \alpha(H)$ by

$$\eta(g) = g - \gamma \circ \beta(g)$$

and check that it is a group homomorphism. Let $g, g' \in G$, $\beta(g) = (n_i)_{i\geq 1}$, $\beta(g') = (m_i)_{i\geq 1}$. We have:

$$\begin{aligned}
\eta(g + g') &= g + g' - \gamma \circ \beta(g + g') \\
&= g + g' - \gamma(\beta(g_1) + \beta(g_2)) \\
&= g + g' - \gamma((n_i + m_i)_{i\geq 1}) \\
&= g + g' - \sum_{i\geq 1} (n_i + m_i) g_i \\
&= g - \left(\sum_{i\geq 1} n_i g_i\right) + g' - \left(\sum_{i\geq 1} m_i g_i\right) \\
&= (g - \gamma \circ \beta(g)) + (g' - \gamma \circ \beta(g')) \\
&= \eta(g) + \eta(g').
\end{aligned}$$

We can now define a homomorphism $\phi : G \to H \oplus \mathbb{Z}^\infty$ by

$$\phi(g) = (\alpha^{-1} \circ \eta(g), \beta(g))$$

noting that α is injective by exactness. It remains to check that ϕ is a bijection.

To see that ϕ is injective, suppose that $\phi(g) = 0$. Then, $\alpha^{-1}(g - \gamma \circ \beta(g)) = 0$ and $\beta(g) = 0$. Then, $\gamma \circ \beta(g) = 0$ because $\beta(g) = 0$, which gives $\alpha^{-1}(g) = 0$. As α is in bijection with its image $\alpha(H)$, we conclude that $g = 0$.

To see that ϕ is surjective, it is enough to show that $H \oplus \{0\}$ and $\{0\} \oplus \mathbb{Z}^\infty$ are contained in $\phi(G)$. To show the former, consider $\phi(\alpha(H))$. For any $g \in \alpha(H)$, $\beta(g) = 0$ by exactness. So, we have:

$$\begin{aligned}
\phi(\alpha(H)) &= \{(\alpha^{-1}(\alpha(h) - \gamma \circ \beta(\alpha(h))), \beta(\alpha(h))) \mid h \in H\} \\
&= \{(\alpha^{-1}(\alpha(h) - \gamma(0)), 0) \mid h \in H\} \\
&= \{(\alpha^{-1}(\alpha(h)), 0) \mid h \in H\} \\
&= \{(h, 0) \mid h \in H\} \\
&= H \oplus \{0\}.
\end{aligned}$$

To show the latter, consider $\phi\left(\gamma\left(\mathbb{Z}^\infty\right)\right)$. We have:

$$\begin{aligned}
\phi\left(\gamma\left(\mathbb{Z}^\infty\right)\right) &= \left\{(\alpha^{-1}(\gamma(n) - \gamma \circ \beta \circ \gamma(n)), \beta \circ \gamma(n) \;\middle|\; n \in \mathbb{Z}^\infty\right\} \\
&= \left\{(\alpha^{-1}(\gamma(n) - \gamma(n)), n) \;\middle|\; n \in \mathbb{Z}^\infty\right\} \\
&= \{(\alpha^{-1}(0), n) \mid n \in \mathbb{Z}^\infty\} \\
&= \{(0, n) \mid n \in \mathbb{Z}^\infty\} \\
&= \{0\} \oplus \mathbb{Z}^\infty.
\end{aligned}$$

Therefore, ϕ is a group isomorphism between G and $H \oplus \mathbb{Z}^\infty$. $\square$

We are now equipped to begin the proof of Lemma 6.1.1.

Proof. Define T_1 to be the interior of the innermost (upside-down) triangle formed by the three images $f_1(X)$, $f_2(X)$, and $f_3(X)$:

$$T_1 = X \setminus (f_1(X) \cup f_2(X) \cup f_3(X)).$$

Then, let $(T_n)_{n\geq 1}$ be an enumeration of $\{T_1\} \cup \{f_\xi(T_1) \mid \xi \in \bigcup_{k\geq 1} \Sigma_k\}$. Note that each T_n is an open subset of X_0. Let

$$T_\infty = \bigcup_{n\geq 1} T_n. \tag{6.5}$$

Then T_∞ is an open subset of X_0, and $K_0 = X_0 \setminus T_0$. Moreover, T_∞ is homeomorphic to an infinite, disjoint union of open disks. In other words,

$$T_\infty \cong \bigsqcup_{k=1}^{\infty} \mathbb{R}^2.$$

Consider the short exact sequence

$$0 \to C_0(T_\infty) \to C_0(X_0) \to C_0(K_0) \to 0 \tag{6.6}$$

where $C_0(T_\infty) \to C_0(X_0)$ is inclusion, and $C_0(X_0) \to C_0(K_0)$ is given by restriction to $K_0 \subseteq X_0$. This induces the following 6-term exact sequence:

$$\begin{array}{ccccc} K_0(C_0(T_\infty)) & \longrightarrow & K_0(C_0(X_0)) & \longrightarrow & K_0(C_0(K_0)) \\ \uparrow & & & & \downarrow \\ K_1(C_0(K_0)) & \longleftarrow & K_1(C_0(X_0)) & \longleftarrow & K_1(C_0(T_\infty)) \end{array}$$

which we can simplify, using Lemma 6.1.1, to the following.

$$\begin{array}{ccccc} K_0(C_0(T_\infty)) & \longrightarrow & 0 & \longrightarrow & K_0(C_0(K_0)) \\ \uparrow & & & & \downarrow \\ K_1(C_0(K_0)) & \longleftarrow & \mathbb{Z}^2 & \longleftarrow & K_1(C_0(T_\infty)) \end{array}$$

It is also well known that $K_0(\mathbb{R}^2) \cong \mathbb{Z}$, and $K_1(\mathbb{R}^2) \cong 0$. In light of the former, and by viewing $C_0(T_\infty)$ as $\bigoplus_{n=1}^{\infty} C_0(\mathbb{R}^2)$, we can simplify one step further, as follows.

$$\begin{array}{ccccc} \mathbb{Z}^\infty & \longrightarrow & 0 & \longrightarrow & K_0(C_0(K_0)) \\ \uparrow & & & & \downarrow \\ K_1(C_0(K_0)) & \longleftarrow & \mathbb{Z}^2 & \longleftarrow & 0 \end{array}$$

Notice that exactness at

$$0 \to K_0(C_0(K_0)) \to 0$$

forces $K_0(C_0(K_0)) \cong 0$. We are left with

$$0 \to \mathbb{Z}^2 \to K_1(C_0(K_0)) \to \mathbb{Z}^\infty \to 0.$$

By Proposition 6.2.2, we conclude that $K_1(C_0(K_0)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^\infty$. $\square$

6.3 The K-theory of $C^*(R_1')$

Recall from Section 5.3.1 the definition of the C^*-algebras $A \cong C^*(R_1')$, and B:

- A is the set of all continuous functions $g : K \to \mathcal{M}_3(\mathbb{C})$ with the additional property that there are complex numbers a, b, c such that

$$g(x_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{bmatrix}, \quad g(x_2) = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{bmatrix}, \quad g(x_3) = \begin{bmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

- $B = C_0(K_0, \mathcal{M}_3(\mathbb{C})) = \{g \in A \mid g(x_1) = g(x_2) = g(x_3) = 0\}$.

Lemma 6.3.1. *Consider the short exact sequence*

$$0 \to B \to A \to \mathbb{C}^3 \to 0 \tag{6.7}$$

where $B \to A$ is inclusion, and $A \to \mathbb{C}^3$ is given by $g \mapsto (a,b,c)$, where (a,b,c) are as in (6.3)*. Let $\exp_A : K_0(\mathbb{C}^3) \to K_1(B)$ be the corresponding exponential map. Then* $\ker(\exp_A) \cong \mathbb{Z}$ *and* $\operatorname{Im}(\exp_A) = \operatorname{Im}(\exp_X) \cong \mathbb{Z}^2$.

Proof. We have the following six-term exact sequence.

$$\begin{array}{ccccc} K_0(B) & \longrightarrow & K_0(A) & \longrightarrow & K_0(\mathbb{C}^3) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{C}^3) & \longleftarrow & K_1(A) & \longleftarrow & K_1(B) \end{array}$$

Once again, we can use (6.3) to reduce this to the following.

$$\begin{array}{ccccc} K_0(B) & \longrightarrow & K_0(A) & \longrightarrow & \mathbb{Z}^3 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(A) & \longleftarrow & K_1(B) \end{array}$$

Furthermore, using the result that $K_*(\mathcal{M}_n(A)) \cong K_*(A)$ for any C^*-algebra A, we can use Lemma 6.2.1 along with the identification $B = C_0(K_0, \mathcal{M}_3(\mathbb{C})) = \mathcal{M}_3(C_0(K_0))$, as follows:

$$K_*(B) = K_*(\mathcal{M}_3(C_0(K_0))) \cong K_*(C_0(K_0)) \cong \begin{cases} 0 & * = 0 \\ \mathbb{Z}^2 \oplus \mathbb{Z}^\infty & * = 1. \end{cases}$$

More specifically, it will be helpful to recall further that we had the isomorphism $K_1(C_0(K_0)) \cong K_1(X_0) \oplus \mathbb{Z}^\infty = \operatorname{Im}(\exp_X) \oplus \mathbb{Z}^\infty$.

We are interested in computing $\exp_A : \mathbb{Z}^3 \to K_1(B)$. To this end, let us make use of the lifts $s_1, s_2, s_3 \in C(X)$ defined in the proof of Lemma 6.1.2. Define the maps

$g_1, g_2, g_3 : K \to \mathcal{M}_3(\mathbb{C})$ by

$$g_1(x) = \mathrm{diag}(s_2(x), s_1(x), 0)$$
$$g_2(x) = \mathrm{diag}(0, s_3(x), s_2(x))$$
$$g_3(x) = \mathrm{diag}(s_3(x), 0, s_1(x))$$

for each $x \in K$. Each s_i is continuous, so each g_i is continuous as well. Furthermore, a straight-forward calculation shows that

$$g_1(x) \mapsto (1, 0, 0)$$
$$g_2(x) \mapsto (0, 1, 0)$$
$$g_3(x) \mapsto (0, 0, 1)$$

under $A \mapsto \mathbb{C}^3$. As before, this tells us how to compute $\exp_A$:

$$\begin{aligned}\exp_A([(1,0,0)]_0) &= \left[e^{2\pi i g_1}\right]_1 \\ &= \left[\mathrm{diag}\left(e^{2\pi i s_2}, e^{2\pi i s_1}, 1\right)\right]_1 \\ &= \left[e^{2\pi i s_2}\right]_1 + \left[e^{2\pi i s_1}\right]_1\end{aligned}$$

$$\begin{aligned}\exp_A([(0,1,0)]_0) &= \left[e^{2\pi i g_2}\right]_1 \\ &= \left[\mathrm{diag}\left(1, e^{2\pi i s_3}, e^{2\pi i s_2}\right)\right]_1 \\ &= \left[e^{2\pi i s_3}\right]_1 + \left[e^{2\pi i s_2}\right]_1 \\ &= -\left[e^{2\pi i s_1}\right]_1\end{aligned}$$

$$\begin{aligned}\exp_A([(0,0,1)]_0) &= \left[e^{2\pi i g_3}\right]_1 \\ &= \left[\operatorname{diag}\left(e^{2\pi i s_3}, 1, e^{2\pi i s_1}\right)\right]_1 \\ &= \left[e^{2\pi i s_3}\right]_1 + \left[e^{2\pi i s_1}\right]_1 \\ &= -\left[e^{2\pi i s_2}\right]_1\end{aligned}$$

where we are using Proposition 6.1.2. Notice that these computations mean and that $K_1(C_0(X_0)) \subseteq K_1(B)$, as $K_1(B)$ contains the generators of $K_1(C_0(X_0))$. Moreover, as $[(1,0,0)]_0$, $[(0,1,0)]_0$, and $[(0,0,1)]_0$ generate $K_0(\mathbb{C}^3)$, the image of $\exp_A$ is exactly $K_1(C_0(X_0))$:

$$\begin{aligned}\operatorname{Im}(\exp_A) &= \left\langle \left[e^{2\pi i s_2}\right]_1 + \left[e^{2\pi i s_1}\right]_1, -\left[e^{2\pi i s_1}\right]_1, -\left[e^{2\pi i s_2}\right]_1 \right\rangle \\ &= \left\langle \left[e^{2\pi i s_1}\right]_1, \left[e^{2\pi i s_2}\right]_1 \right\rangle \\ &= \operatorname{Im}(\exp_X) \cong \mathbb{Z}^2.\end{aligned}$$

Next, to find $\ker(\exp_A)$, let $k[(1,0,0)]_0 + \ell[(0,1,0)]_0 + m[(0,0,1)]_0 \in K_0(\mathbb{C}^3)$ be mapped to 0 under $\exp_A$. Then,

$$k\left[e^{2\pi i s_2}\right]_1 + k\left[e^{2\pi i s_1}\right]_1 - \ell\left[e^{2\pi i s_1}\right]_1 - m\left[e^{2\pi i s_2}\right]_1 = 0$$

and as the two generators are distinct, we must have $k - m = 0$ and $k - \ell = 0$. In other words, $k = \ell = m$. Therefore, $\ker(\exp) = \{(k,\ell,m) \in \mathbb{Z}^3 \mid k = \ell = m\} \cong \mathbb{Z}$, as desired. □

Proposition 6.3.2.

$$K_*(A) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^\infty & * = 1 \end{cases}$$

Proof. Recall from Lemma 6.3.1 that $\ker(\exp_A) \cong \mathbb{Z}$, $\operatorname{Im}(\exp_A) = \operatorname{Im}(\exp_X) \cong \mathbb{Z}^2$, and that we have the following commutative diagram.

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & \mathbb{Z}^3 \\ \big\uparrow & & & & \big\downarrow{\scriptstyle \exp_A} \\ 0 & \longleftarrow & K_1(A) & \longleftarrow & \mathrm{Im}(\exp_X) \oplus \mathbb{Z}^\infty \end{array}$$

To summarize, we have an injective map from $K_0(A)$ into $\mathbb{Z}^3$, and a surjective map from $\mathrm{Im}(\exp_X) \oplus \mathbb{Z}^\infty$ onto $K_1(A)$. By exactness, the image of $K_0(A)$ in $\mathbb{Z}^3$ the kernel of $\exp_A$, which is isomorphic to $\mathbb{Z}$. Then, because the map from $K_0(A)$ into $\mathbb{Z}^3$ is injective, $K_0(A)$ will be isomorphic to its image; in other words, we see that $K_0(A) \cong \ker(\exp) \cong \mathbb{Z}$.

To see that $K_1(A) \cong \mathbb{Z}^\infty$, we use the surjective map from $\mathrm{Im}(\exp_X) \oplus \mathbb{Z}^\infty$ onto $K_1(A)$, the image of $\exp_A$, and the fundamental isomorphism theorem for groups. Indeed, by exactness, this surjection has kernel equal to $\mathrm{Im}(\exp_A) = \mathrm{Im}(\exp_X)$, so

$$K_1(A) \cong \frac{\mathrm{Im}(\exp_X) \oplus \mathbb{Z}^\infty}{\mathrm{Im}(\exp_X)} \cong \mathbb{Z}^\infty.$$

□

6.3.1 A generator for $K_0(A) \cong \mathbb{Z}$

We will find a generator of the K_0-group of the algebra A by first defining three maps from K to $\mathbb{C}^3$, which form an orthonormal basis at every $x \in K$. Let c be the center point of the equilateral triangle X. For each $i = 1, 2, 3$, and each $x \in f_i(X)$, let the parameter $t_i(x)$ be the component of $x - x_i$ in the direction of $c - x_i$, normalized to range between 0 and $\frac{\pi}{2}$. In other words, if you draw a line from x_i to c, then $t_i(x)$ is the distance from x_i to x along this line, scaled appropriately. Note that $t_i(x) = 0$ at $x = x_i$, and $t_i(x) = \frac{\pi}{2}$ for all x on the edge of $f_i(X)$ closest to c. We now define three functions on K in terms of the parameters t_i:

$$\xi_1(x) = \begin{cases} (\cos(t_1(x)), \sin(t_1(x)), 0) & x \in f_1(K) \\ (0, 1, 0) & x \in f_2(K) \\ (0, \sin(t_3(x)), -\cos(t_3(x))) & x \in f_3(K) \end{cases}$$

$$\xi_2(x) = \begin{cases} (0, 0, 1) & x \in f_1(K) \\ (-\cos(t_2(x)), 0, \sin(t_2(x))) & x \in f_2(K) \\ (0, \cos(t_3(x)), \sin(t_3(x)) & x \in f_3(K) \end{cases}$$

$$\xi_3(x) = \begin{cases} (\sin(t_1(x)), -\cos(t_1(x)), 0) & x \in f_1(K) \\ (\sin(t_2(x)), 0, \cos(t_2(x))) & x \in f_2(K) \\ (1, 0, 0) & x \in f_3(K). \end{cases}$$

Note that each ξ_i is well-defined at the intersections of the images $f_1(K), f_2(K), f_3(K)$, and that $\{\xi_i(x)\}_{i=1}^3$ is an orthonormal basis for $\mathbb{C}^3$ for every $x \in K$. Moreover, if $\xi : K \to \mathcal{M}_3(\mathbb{C})$ is given by $\xi(x) = (\xi_1(x)\ \xi_2(x)\ \xi_3(x))$, then $\xi(x)$ is an orthogonal matrix for every $x \in K$. Now, we construct another map $f : K \to \mathcal{M}_3(\mathbb{C})$ given by:

$$f(x) = \xi(x)^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi(x).$$

f is continuous because each ξ_i is. Furthermore, f is an element of A:

$$f(x_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(x_2) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(x_3) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Recall from Section 5.3.3 that we get an isomorphism between $C^*(R'_n)$ and a subalgebra A_n of the continuous functions from K into $\mathcal{M}_{3^n}(\mathbb{C})$, where a certain condition is satisfied at the three corners x_1, x_2, and x_3. These conditions lead to the following short exact sequence, which can be used to find the K-theory of A_n in the same way as done for Proposition 6.3.2:

$$0 \to C_0(K_0, \mathcal{M}_{3^n}(\mathbb{C})) \to A_n \to \bigoplus_1^3 \mathcal{M}_{k_n}(\mathbb{C}) \to 0$$

where $k_n = \frac{1}{2}(3^n - 1)$ is the number of points in the closed invariant set $Y_{n,1}^c$ from Section 5.1. By inspection of the approach for A_1, we can see that the K_0 and K_1 groups for A_n will be isomorphic to those of $A_1 = A$.

Theorem 6.3.3. *Let $\{R_n\}_{n\geq 1}$ be the étale equivalence relations associated to the Siérpinski Gasket iterated function system* (2.1). *For each $n \geq 1$, let $R'_n = R_n \setminus \{(x_1, x_1), (x_2, x_2), (x_3, x_3)\}$ be R_n with the three trivial equivalence classes removed, as in Lemma 5.3.1. Then, for each $n \geq 1$,*

$$K_*(C^*(R'_n)) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^\infty & * = 1. \end{cases}$$

6.4 The inductive limit for $K_0(C^*(R'))$

Let R'_n and R' be as in Lemma 5.3.1. We have that $K_0(C^*(R'_n))$ is the integers for each $n \geq 1$. As the K-theory of an inductive limit of C^*-algebras is the inductive limit of the K-groups, we can compute $K_*(C^*(R'))$ once the map $K_0(A_n) \to K_0(A_{n+1})$ is determined. Let us discover this map for $K_0(A_1) \mapsto K_0(A_2)$. Then, we will see that the map will be the same for each $n \geq 1$. To do this, consider the function $f \in A_1$ from Section 6.3.1, and K_0 of the inclusion map $\iota_{1,2}$. $[f]_0$ is a generator for $K_0(A_1)$, so we can determine the map $K_0(\iota_{1,2}) : K_0(A_1) \mapsto K_0(A_2)$ by finding the element of $K_0(A_2)$ which $[f]_0$ is mapped to. Using the definition of K_0 of a map, and noting that f is a projection A_1, we see that

$$K_0(\iota_{1,2})([f]_0) = [\iota_{1,2}(f)]_0$$

despite A_1 being non-unital. In other words, we aim to find the integer corresponding to the K_0-equivalence class of f when included as an element of A_2.

First, let us determine a method for finding the K_0-equivalence class of a projection $p \in A_n$. Let p be a projection in A_n such that p is mapped to $\left(\frac{1}{k_n} I_{k_n}, \frac{1}{k_n} I_{k_n}, \frac{1}{k_n} I_{k_n}\right)$ under $A_n \mapsto \bigoplus_1^3 \mathcal{M}_{k_n}(\mathbb{C})$. Then, $[p]_0$ will be mapped to $(1,1,1) \in \mathbb{Z}^3 \cong K_0\left(\bigoplus_1^3 \mathcal{M}_{k_n}(\mathbb{C})\right)$, and so is a generator for $K_0(A_n)$. Notice that the trace of $p(x)$ must be 2 for all values of $x \in K$. To see this, recall that a projection in A_n must have an integer-valued trace when evaluated at a point, as a continuous map from K into the integers must be constant. So, the trace of $p(x)$ does not depend on x. In particular, if we evaluate p at, for example, x_1, the trace will be $0 + \mathrm{Tr}\left(\frac{1}{k_n} I_{k_n}\right) + \mathrm{Tr}\left(\frac{1}{k_n} I_{k_n}\right) = 2$. Therefore, the map sending a projection in A_n to half of its trace when evaluated at a point sends a generator of $K_0(A_n)$ to the integer 1. Let us call this map $\frac{1}{2}\mathrm{Tr}_n : \mathcal{P}(A_n) \to \mathbb{Z}$, where $\mathcal{P}(A_n)$ denotes the set of projections in A_n. This map will extend to a group isomorphism between $K_0(A_n)$ and the integers.

To determine the map between the integers resulting from inclusion of $K_0(A_1)$ in $K_0(A_2)$, it is enough to calculate half of the trace of $\iota_{1,2}(f)(x)$ for some $x \in K$. To do this, we can make use of the explicit formula for f as determined in Section 6.3.1. Let us first consider $\iota_{1,2}(f)$, the inclusion of f in A_2. First, there exists a function g in $C_c(R_1')$ such that $\alpha(g) = f$, though we will not need its explicit form at every pair $(x, y) \in C_c(R_1')$. We can extend g to be included in $C_c(R_2')$ by defining $g(x, y)$ to be zero when (x, y) is not in R_1'. The recall the isomorphism of $C^*(R_2')$ with A_2 coming from (5.5), where we had $\beta(g)(x)_{(i,j),(k,l)} = g(f_i \circ f_j(x), f_k \circ f_l(x))$ when this is in R_2, and 0 otherwise. Here, we index a 9 by 9 matrix by labeling the rows and columns by ordered pairs (i, j) for $i, j = 1, 2, 3$. We aim to evaluate $\frac{1}{2}\mathrm{Tr}(\beta(g)(x))$ for some $x \in K$. To evaluate $\mathrm{Tr}(\beta(g)(x))$, we only need to determine the values of $\beta(g)(x)_{(i,j),(k,l)}$ where $(k, l) = (i, j)$.

Let us choose a point $x \in K$ such that $x \in f_1(K)$ with $t_1(x) = \frac{\pi}{2}$, and $x \notin f_2(K)$, $x \notin f_3(K)$. In particular, choose $x = f_1(y_2)$. Then, $\xi_1(x) = (0, 1, 0)$, $\xi_2(x) = (0, 0, 1)$, and $\xi_3(x) = (1, 0, 0)$. From this, we calculate $f(x)$ to be:

$$f(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This tells us that $g(f_1(x), f_1(x)) = g(f_2(x), f_2(x)) = 1$, and $g(f_3(x), f_3(x)) = 0$. Let us write these points in terms of compositions of two functions. We have:

$$f_1(x) = f_1 \circ f_1(y_2)$$

$$f_2(x) = f_2 \circ f_1(y_2)$$

$$f_3(x) = f_3 \circ f_1(y_2)$$

so

$$g(f_1(x), f_1(x)) = g(f_1 \circ f_1(y_2), f_1 \circ f_1(y_2)) = 1$$
$$g(f_2(x), f_2(x)) = g(f_2 \circ f_1(y_2), f_2 \circ f_1(y_2)) = 1$$
$$g(f_3(x), f_3(x)) = g(f_3 \circ f_1(y_2), f_3 \circ f_1(y_2)) = 0.$$

Next, consider the points $(f_i \circ f_2(y_2), f_i \circ f_2(y_2))$ and $(f_i \circ f_3(y_2), f_i \circ f_3(y_2))$ for $i = 1, 2, 3$, all of which are in R_1'. We can find the values of g for these points by calculating $f(f_2(y_2))$ and $f(f_3(y_2))$. We have that $f_2(y_2)$ lies in $f_2(K)$, but not $f_1(K)$ or $f_3(K)$. Furthermore, $t_2(f_2(y_2)) = \frac{\pi}{4}$. Then, $\xi_1(f_2(y_2)) = (0, 1, 0)$, $\xi_2(x) = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, and $\xi_3(x) = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. From this, we calculate $f(f_2(y_2))$ to be:

$$f(f_2(y_2)) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

so

$$g(f_1 \circ f_2(y_2), f_1 \circ f_2(y_2)) = 1$$
$$g(f_2 \circ f_2(y_2), f_2 \circ f_2(y_2)) = \frac{1}{2}$$
$$g(f_3 \circ f_2(y_2), f_3 \circ f_2(y_2)) = \frac{1}{2}.$$

Finally, we have that $f_3(y_2)$ lies in $f_3(K)$, but not $f_1(K)$ or $f_2(K)$. Furthermore, $t_3(f_3(y_2)) = \frac{\pi}{4}$. Then, $\xi_1(f_2(y_2)) = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $\xi_2(x) = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $\xi_3(x) = (1, 0, 0)$. From this, we calculate $f(f_3(y_2))$ to be:

$$f(f_3(y_2)) = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$g(f_1 \circ f_3(y_2), f_1 \circ f_3(y_2)) = 1$$
$$g(f_2 \circ f_3(y_2), f_2 \circ f_3(y_2)) = 1$$
$$g(f_3 \circ f_3(y_2), f_3 \circ f_3(y_2)) = 0.$$

Therefore, $\frac{1}{2}\mathrm{Tr}(\beta(g)(y_2)) = \frac{1}{2}(1+1+0+1+\frac{1}{2}+\frac{1}{2}+1+1+0) = 3$. We conclude that the map between the integers coming from the inclusion of A_1 in A_2 is multiplication by 3. In fact, for each $n \geq 1$, the map between $K_0(A_n)$ and $K_0(A_{n+1})$ will still be multiplication by 3, the idea being that each equivalence class is the union of three copies of an equivalence class in the previous step. Let us summarize these results in the following lemma.

Lemma 6.4.1. *For each $n \geq 1$, there is a group isomorphism $\frac{1}{2}Tr_n : K_0(A_n) \to \mathbb{Z}$ such that for any $p \in \mathcal{P}(A_n)$, $\frac{1}{2}Tr_n([p]_0) = \frac{1}{2}Tr(p(x))$, for any $x \in K$. Furthermore, the following diagram commutes.*

$$\begin{array}{ccc} K_0(A_n) & \xrightarrow{K_0(\iota_{n,n+1})} & K_0(A_{n+1}) \\ \downarrow{\scriptstyle \frac{1}{2}Tr_n} & & \downarrow{\scriptstyle \frac{1}{2}Tr_{n+1}} \\ \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} \end{array}$$

This allows us to determine the inductive limit for $K_0(C^*(R'))$.

Theorem 6.4.2. *Let $\{R'_n\}_{n\geq 1}$ be the étale equivalence relations associated to the Siérpinski Gasket with the trivial equivalence classes of the three corners x_1, x_2, and x_3 excluded, as in Lemma 5.3.1. Let $C^*(R')$ be the inductive limit of the associated C^*-algebras $C^*(R'_n)$. Then $K_0(C^*(R')) \cong \mathbb{Z}[\frac{1}{3}]$, as the result of the following inductive limit of abelian groups:*

$$\mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{3} \dots$$

Chapter 7

Conclusions

In associating an equivalence relation C^*-algebra to a broad class of iterated function systems, we have provided a new approach to applying the theory of C^*-algebras to the study of topological dynamics generating self-similar sets. The class of systems treated includes many classic examples, such as the Siérpinski Gasket, for which current constructions cannot be directly applied due to a requirement for the functions in the system to be inverse branches of a continuous map. In comparison, we release this requirement, and instead consider only affine iterated function systems for which the functions share the same invertible, linear transformation, and differ only by a translation. That being said, it appears that the results can be generalized to the case where the composition of a function with its inverse is an isometry. This would include the cases where the functions differ only by rotations, but would still exclude the cases where the functions have different contraction factors. Other potential generalizations to be investigated include graph iterated function systems, and a groupoid structure that takes into consideration the re-scaling that happens when functions are composed. The latter may result in C^*-algebras similar to those constructed under this inverse branch condition.

In particular, for each $n \geq 1$, we considered pre-images of compositions of n

many functions in the system to define an étale equivalence relation $R_n = \{(x,y) \in K\times K \mid \mathcal{F}^{-n}\{x\} = \mathcal{F}^{-n}\{y\}\}$, where K is the attractor of the system. We showed that these form an increasing sequence of étale equivalence relations, $R_1 \subseteq R_2 \subseteq R_3 \subseteq \dots,$ creating an inductive limit structure for an étale topology on $R = \bigcup_{n\geq 1} R_n$. To show the étale property, a local action was constructed for each R_n, and it was verified that this definition of local action yields a basis for an étale topology on R_n.

The inductive limit structure of the étale equivalence relation R means that the associated C^*-algebra $C^*(R)$ is the inductive limit of the increasing sequence of subalgebras $C^*(R_n)$ for $n \geq 1$. The subalgebras $C^*(R_n)$ are, in some sense, more tractable than the inductive limit. For example, the equivalence classes of each R_n are always finite, and in many cases can be used to investigate the ideal structure of $C^*(R_n)$.

The investigation of open invariant sets in Chapter 5 revealed a connection to the open set condition. If an open set, satisfying the requirements of this condition, also has an open image under each of the functions in the system, then there is a corresponding ideal of $C^*(R_n)$, taking the form of a matrix algebra tensored with the continuous functions vanishing off of this open set. These preliminary results on ideal structure also showed a natural use of pre-images $\mathcal{F}^{(-n)}\{x\}$, so it would be interesting to compare the results with a construction based on the equivalence relation $R_{(n)} = \{(x,y) \in K \times K \mid \mathcal{F}^{(-n)}\{x\} = \mathcal{F}^{(-n)}\{y\}\}$.

We also used the Siérpinski Gasket to illustrate how the ideal structure of $C^*(R_n)$ can be used to find the K-theory of $C^*(R_n)$, and this can be followed through the inductive limit structure to find the K-theory of $C^*(R)$. In particular, we found an isomorphism of of each $C^*(R_n)$ to a C^*-subalgebra of $C(K, \mathcal{M}_{3^n}(\mathbb{C}))$, where K is the Siérpinski Gasket. We used this, along with calculations of the K-theory of the continuous functions on the Siérpinski Gasket itself, to calculate $K_0(C^*(R))$. Although these calculations were only done for the Siérpinski Gasket, the demonstrated techniques will work for many other examples as well; however, a more complicated ideal structure will lead to more complicated calculations.

Lastly, there are many interesting questions that may be asked about dynamical properties appearing in the C^*-algebras. We verified that the C^*-algebras are isomorphic in the case of conjugate systems, but it remains unclear when the non-conjugate systems have non-isomorphic C^*-algebras. A next step would be to calculate the K_1-group for the Siérpinski Gasket, and generalize the results to the Siérpinski n-gons. Based on the calculations we have seen for the Gasket, this would likely result in a K_0-group depending on n, and may provide some examples of non-isomorphic C^*-algebras. The Fudgeflake also provides a nice example with a connected attractor, with a system involving rotations. The number of functions is equal to that for the Siérpinski Gasket, and a comparison of the ideal structures and K-theory could provide some insights into the diversity found in this new class of C^*-algebras.

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