Holography and Causality in Einstein-Gauss-Bonnet Gravity

by

Michael Harder
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Supervisory Committee

Dr. Pavel Kovtun, Supervisor
(Department of Physics and Astronomy)

Dr. Adam Ritz, Departmental Member
(Department of Physics and Astronomy)

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Dr. Adam Ritz, Departmental Member
(Department of Physics and Astronomy)

ABSTRACT

Field theories with higher derivative gravity duals can violate the viscosity bound. However the extent of the violation is not arbitrary since it depends on the coupling of the higher derivative interactions, which can be constrained by requiring consistency of the boundary field theory. In particular, in Einstein-Gauss-Bonnet (EGB) gravity, the coupling λ can be constrained by requiring that the dual theory respect causality. We investigate the upper bound on λ by computing the quasinormal modes of an EGB black hole in order to explicitly find and interpret the causality violating excitations. We find that in the limit of infinite spatial momentum the imaginary part of these modes approaches 0, while the phase velocity approaches 1 from above. This behaviour at high momentum is confirmed by the existence of a lightlike pole in the stress-energy tensor two-point function. We therefore confirm that the requirements to interpret the poles of the two-point function as causality violating, propagating modes are met in the limit of infinite spatial momentum. The presence of such excitations not only constrains the viscosity bound but also limits the allowed couplings of EGB gravity.

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Notation and Conventions

The definitions of the curvature tensors used in appendix A follow those in Carroll [1], namely we define the Riemann tensor in the usual way

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

where

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

is the connection compatible with the metric $g_{\mu\nu}$ and we contract the first and third index to define the Ricci tensor

$$R_{\mu\nu} = g^{\sigma\rho} R_{\rho\mu\sigma\nu}.$$

g will denote the metric determinant, $\gamma_{\mu\nu}$ will be the induced metric with determinant γ and γ_{D-2} (or γ_3) will be used to denote the metric induced on the bifurcation surface (in our case the constant time metric at the horizon).

D will denote the spacetime dimension of the bulk while d will denote the spacetime dimension of the boundary.

k will denote the 4-momentum k_μ, with components $k_0 = \omega$ and $k_3 = q$ while $\mathbf{k}$ will denote the spatial momentum.

Curvature tensors with a hat, e.g. $\widehat{R}$, are evaluated using the induced metric.

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For Samantha, Jonathon, Mom and Dad

Chapter 1

Introduction

Our current understanding of the universe is based on two tremendous achievements of 20$^{\text{th}}$ century physics: quantum field theory and general relativity. Quantum field theory (QFT) applies the ideas of quantum mechanics to classical fields and provides a general theoretical framework underlying condensed matter and particle physics. Perhaps the most successful example of a QFT is the standard model of particle physics which describes the (non gravitational) interactions of all fundamental particles and has been tested to remarkable precision, the canonical experimental test being the gyromagnetic ratio of the electron [2]. General relativity on the other hand is a specific theory of gravitation describing the interaction between matter and spacetime. General relativity too has been precisely tested experimentally, for example in the Pound-Rebka experiment of the gravitational redshift [3, 4]. Yet despite the success of quantum field theory as an overarching theoretical framework and general relativity as a theory of gravity, the two have not been combined into a complete theory of quantum gravity. This apparent incompatibility between quantum theory and gravitation is a central problem in physics since we do not expect our fundamental understanding to be partially quantum and partially classical [5].

One candidate for a theory of quantum gravity is string theory, where point particles are replaced with one dimensional extended objects. The string spectrum contains a graviton, and therefore string theory contains gravity. However string theory is really more than just a theory of quantum gravity and is actually a candidate for a fully unified theory, although it is not yet complete. Despite this incompleteness, string theory has provided an important link between quantum field theory and gravitational physics: the AdS/CFT correspondence, which has allowed us to learn about the interaction of quantum field theories and gravity.

The general idea of AdS/CFT, proposed by Maldacena [6], is that a gravitational theory in a D dimensional asymptotically Anti de Sitter (AdS) spacetime is dual to a $D-1$ dimensional conformal field theory (CFT) which exists on the boundary of the AdS spacetime. This duality implies that in certain circumstances the same physics can be described by either a field theory, or a gravitational theory and in fact an explicit dictionary between the fields in the bulk (i.e. in the D dimensional spacetime) and the operators on the boundary (i.e. the $D-1$ dimensional hypersurface taken at the boundary) exists [7, 8], allowing for explicit computations on either side of the duality. The fact that AdS/CFT allows a dual description of gravitational physics in one less dimension makes it an explicit example of the holographic principle, which is expected to play a role in quantum gravity [9, 10, 11]. But perhaps the most useful aspect of the correspondence is the fact that it is a strong/weak duality. Most field theories and string theory are best understood only perturbatively, which makes calculations at strong coupling difficult. However in AdS/CFT exactly when the boundary field theory is strongly coupled, the gravitational theory is weakly coupled (i.e. classical). This provides a way to carry out difficult calculations in the boundary theory by performing a more tractable calculation in the bulk.

Transport coefficients, such as the viscosity, are an example of the field theory quantities which can be calculated through gauge/gravity duality. In fact the viscosity has been computed for various gravitational backgrounds in several ways [12, 13, 14, 15]. Interestingly enough the ratio of the shear viscosity to entropy density, has a universal value, $\eta/s = 1/4\pi$ (in natural units), across many different field theories which led to the conjecture that $1/4\pi$ was a universal lower bound for the viscosity of relativistic quantum field theories [16]. If the viscosity bound were truly universal it could provide a way to constrain allowable interactions in certain theories. From the perspective of effective field theories, where any interaction allowed by the symmetries is permissible, it would certainly be desirable to constrain the gravitational interactions based on the dual field theory, or vice versa. Furthermore it may be possible to search for fluids which violate the bound and such universal quantities may allow AdS/CFT results to extend beyond the supersymmetric field theories which have gravity duals to everyday theories such as QCD. Therefore several questions naturally arise: Can the bound be used as a constraint to identify unacceptable theories? When can the bound be violated, and if it is violated can other constraints be used to rule out certain theories?

It turns out that the viscosity bound can be violated by including the higher derivative interactions expected as quantum corrections to Einstein gravity [17] however it cannot be violated by an arbitrary amount. An analysis of the various gravitational perturbations shows that the coupling constant of the higher derivative interactions, λ, which controls the viscosity bound violation must be constrained in order for the boundary theory to preserve causality [18, 17, 19] which we expect for any well defined field theory. The bounds on λ translate into bounds on the maximum violation of the viscosity bound. These bounds also coincide with the constraint that the dual theory has positive energy one-point functions [20], which is equivalent to the absence of ghosts in the theory [21]. The fact that the bounds from these different approaches agree is indeed interesting, and it seems that viscosity bound violation, causality violation and energy flux positivity are tied together in some way.

In this thesis we will attempt to understand and further investigate the upper bound on λ, which is especially interesting because of its relevance to viscosity bound violation and its natural appearance in low energy string theory corrections. In order to do so we will calculate the two-point function of the stress-energy tensor using the AdS/CFT correspondence and search for propagating modes which violate causality. In the region in which we can explicitly calculate the dispersion relation of these modes, large imaginary frequencies damp the excitations within a distance on the order of a wavelength, making it difficult to interpret these excitations as propagating modes. However it seems that the imaginary component of the frequency tends toward 0 and that the phase velocity approaches 1 from above at large momenta. This result is confirmed by the presence of a lightlike pole in the stress-energy two-point function in the limit of infinite spatial momentum, indicating the presence of propagating excitations which we can identify with the causality violating modes.

The remainder of this thesis is organized into the following four chapters:

Chapter 2 provides a brief review of the subjects touched on in this thesis, grouped broadly into three categories: the AdS/CFT correspondence, higher derivative gravity, and recent results relating causality violation, the viscosity bound and constraints on central charges.

Chapter 3 contains the calculation of the two-point function for the scalar channel using the AdS/CFT prescription. A solution is found numerically, and the large momentum limit is studied. The quasinormal modes, which give the poles of the correlation function are also found numerically.

Chapter 4 contains a discussion of causality violation based on the two-point function and in particular on the dispersion relation of the quasinormal modes.

Chapter 5 concludes the thesis with a summary of the work performed

Chapter 2

AdS/CFT and Higher Derivative Gravity

In this chapter we will give a brief review of the theoretical background used in this thesis. This will include a section on the AdS/CFT correspondence, higher derivative gravity and the recent work constraining λ and the central charges.

2.1 The AdS/CFT Correspondence

Since Maldacena's original proposal of the AdS/CFT correspondence there has been a substantial amount of work both on better understanding and applying the duality and a plethora of reviews exist, e.g. [22, 23, 24, 25, 26, 27, 28]. In this section we will first summarize why such a statement could make sense following [25, 26] and then discuss the exact statement of the correspondence and how to perform calculations using AdS/CFT. We will also discuss the interpretation of Green's functions, the role of quasinormal modes and the viscosity bound.

2.1.1 Heuristic Motivations

The statement that the same physics can be described by a gravitational theory in D dimensions or a field theory in $D-1$ dimensions seems strange at first glance. Even worse is the fact that the D dimensions really comes after a compactification. For example, the most well known example of the correspondence is the duality between type IIB string theory in an $AdS_5\times$ S_5 background and $d=4$, $\mathcal{N}=4$ SU(N) supersymmetric Yang-Mills (SYM) theory, which means we are actually relating a 10 dimensional gravitational theory and a 4 dimensional field theory. There are however several motivations for the correspondence. First there is the Weinberg-Witten theorem [29, 30]. The Weinberg-Witten theorem contains two statements which limits the allowed massless particles in a quantum field theory. The first statement concerns the construction of a conserved current and massless particles with spin $j > 1/2$. However more interesting for the purposes of AdS/CFT is the second statement:

A theory which contains a conserved, Poincaré covariant stress-energy tensor cannot contain massless particles with spin $j > 1$ which have conserved, non-zero energy-momentum four vector.

In order to have a quantum theory of gravity which has a dynamical metric, we want to have a spin 2 massless graviton made up of some kind of degrees of freedom from the field theory. This means we need a way around the Weinberg-Witten theorem and one possibility is that the graviton is not in the same spacetime as the QFT. This requirement is met in AdS/CFT.

The second motivation is the holographic principle [9, 10, 11]. The entropy of a black hole (for theories without higher derivative interactions, as discussed in Sec. 2.2) is proportional to the area of its event horizon [31, 32, 33, 34]

$$S_{\mathrm{BH}} = \frac{A}{4G} \tag{2.1}$$

(with all units, $S_{\mathrm{BH}} = k_B A c^3/(4G\hbar)$) which means that the maximum entropy for a given region of spacetime grows as the area of its boundary, otherwise we could violate the second law of thermodynamics by adding matter into a volume of spacetime until a black hole forms and decreases the entropy. Therefore the degrees of freedom of a gravitating system can be encoded on its boundary. This is in contrast with the entropy of a regularized QFT where the number of states grows exponentially with the volume. This implies that the degrees of freedom for D dimensional gravitational physics scale in the same way as the degrees of freedom of a $D-1$ dimensional quantum field theory, which again fits nicely with the idea of AdS/CFT.

The final motivation gives an interpretation to the extra dimension of the gravitational theory. In the Wilsonian understanding of renormalization we can view a QFT as a set of effective theories defined at a series of energy scales parametrized by $\tilde{u}$. When we want to use the theory at some $\tilde{u}' \ll \tilde{u}$ we can integrate out high energy degrees of freedom to find an effective theory at $\tilde{u}'$. This coarse graining procedure defines the renormalization group flow which defines an effective theory at each energy scale $\tilde{u}$. Therefore we can view the d-dimensional theory in a $(d+1)$-dimensional spacetime with the extra dimension being the RG scale. The fact that the β function is local in the scale supports viewing $\tilde{u}$ as an extra dimension. As we move toward the boundary we are moving toward the UV in the QFT, however in the bulk we move toward the IR. In principle we could look somewhere in the middle, however when the bulk is in the IR (i.e. the gravity is classical) we know how to perform calculations and interpret the result in terms of the correlation functions of the field theory and so we set the field theory at the boundary.

This picture also leads to a simple argument for the bulk spacetime being AdS. Suppose we have $\beta = 0$. Then we would expect to have the dilatation symmetry $x^\mu \to \xi x^\mu$, which means the extra energy dimension scales as $\tilde{u} \to \tilde{u}/\xi$. The most general D dimensional spacetime with this scaling symmetry and Poincaré invariance is [26]

$$ds^2 = \left(\frac{\tilde{u}}{\tilde{L}}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{\tilde{u}^2} d\tilde{u}^2. \tag{2.2}$$

The coordinate $\tilde{u}$ and and the parameter $\tilde{L}$ have units of energy, but we can rescale to a coordinate r and parameter L with units of length $\tilde{u} = \frac{\tilde{L}}{L} r$ so that

$$ds^2 = \left(\frac{r}{L}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r^2} dr^2 \tag{2.3}$$

which is just the metric for AdS_D where L is the AdS radius and r runs from 0 to ∞ at the boundary. In the case that the spacetime contains a black hole, r will run from the horizon a to the boundary.

2.1.2 String Theory Argument and Calculations

With the three motivations from the previous subsection in mind, we will now briefly outline the original decoupling argument for the AdS/CFT correspondence [27, 28] before discussing the recipe to compute correlations functions in AdS/CFT. Recall that in string theory the fundamental constituent is a one dimensional extended object of length ℓ_s, which we call a string. The strings can be open or closed and we identify particles with the (massless or massive) modes of the strings. We can characterize the strings by their tension, T_s, which is related to the string length through the slope parameter $\alpha' = \ell_s^2$,

$$T_s = \frac{1}{2\pi\alpha'}, \tag{2.4}$$

and by the string coupling g_s, which controls how strings may join together or split apart, i.e. how they interact. However in addition to strings, string theory contains other massive dynamical objects such as D-branes. D-branes come in various dimensions and act as hypersurfaces where boundary conditions for the open strings can be imposed. For example a Dp-brane would be a p dimensional surface where the endpoints of the open strings would obey Dirichlet boundary conditions so that strings would be free to move on the surface of the brane, but not transverse to the brane. The D-branes themselves are dynamical and we can identify their fluctuations with the open string spectrum. It is also important to note that D-branes are non perturbative and their mass density, i.e. their tension, goes as $1/g_s$,

$$T_{Dp} = \frac{1}{(2\pi)^p g_s \ell_s^{p+1}} \tag{2.5}$$

so that they decouple in the limit $g_s \to 0$.

Now consider a stack of N D3-branes in type IIB string theory with N some large fixed number. To make our argument we will consider the low energy limit in two different regimes, $g_s N \ll 1$ and $g_s N \gg 1$. With a stack of N branes it is now possible to have open strings which start on one brane and end on another. The mass of these strings is related to the separation of the branes, and in particular when all of the branes lie on top of each other the mass is 0. This stack of branes sits in 10 dimensional Minkowski space and we have open strings on the branes and closed strings away from the branes. At low energies we can integrate out the massive modes, and in doing so we find that the dynamics of the interacting open strings is determined by $d = 4, \mathcal{N} = 4$ SU(N) SYM with coupling $g^2 = 4\pi g_s$. The coupling between the closed strings is governed by $G \sim g_s^2 \ell_s^8$ so at low energies (i.e. $\ell_s \to 0$) the closed strings are non-interacting. The same is true for the coupling between the closed and open strings. So at low energies we are left with $d = 4, \mathcal{N} = 4$ SU(N) SYM at the branes and non-interacting closed strings in Minkowski spacetime.

We can also consider the geometric effects of the D3-branes. Since the branes have mass they effect the flat metric around them, and in our configuration the metric for the N D3-branes which solves the supergravity equations is

$$ds^2 = H(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{2.6}$$

where

$$H(r) = 1 + \frac{L^4}{r^4}, \tag{2.7}$$

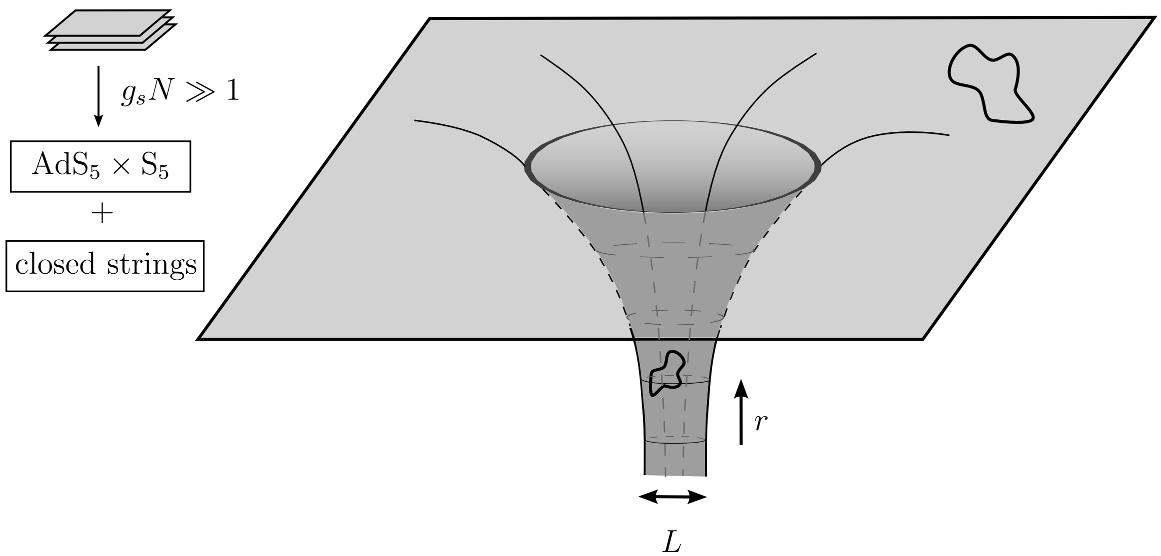
$$L^4 = 4\pi g_s N(\ell_s)^4, \tag{2.8}$$

and μ, ν run over the spacetime indices of the brane, $0, 1, 2, 3$. With respect to the dimensions transverse to the branes, the branes act like a point mass and the metric only depends on the radial distance from the mass. For $r \ll L$, $H(r) \to L^4/r^4$ and the metric takes the form $AdS_5 \times S_5$ (with AdS_5 written in the form of Eq. 2.3). On the other hand for $r \gg L$, $H(r) \to 1$ and the metric is just a 10 dimensional Minkowski spacetime. In this sense L characterizes the strength of the gravitational effects, and near the branes the spacetime develops a throat geometry as shown in Fig. 2.1a. Since we are still in the low energy limit, the same decoupling arguments hold as in the SYM picture of the branes and away from the throat we just have massless non-interacting closed strings. However even at low energies, in the throat we can have massive modes since to an observer far from the throat these modes appear red shifted due to the gravitational potential of the throat. At sufficiently low energies these modes are deep enough inside the throat that they decouple from the closed strings in the asymptotically Minkowski spacetime.

So far we have just taken the low energy limit, and the two brane descriptions we have hold at any value of $g_s N$. Now consider the SYM description and take $g_s N \ll 1$. In this limit the mass of the brane diverges as $1/g_s$, however the gravitational coupling goes to zero as g_s^2 so that the gravitational effects of the branes are negligible. We can see this effect in the geometric description as well since $L \ll \ell_s$ in this limit so that the range of the gravitational effects is small. In the large N limit, the gauge theory is controlled by the t'Hooft coupling $\lambda_t = g^2 N = 4\pi g_s N$ so for $g_s N \ll 1$ the gauge theory is weakly coupled and is tractable. So in the limit of small string coupling, we have the stack of branes sitting in 10-dimensional Minkowski spacetime with closed, non-interacting strings in the bulk and the interacting open strings governed by $\mathcal{N} = 4$, SU(N) SYM.

Going to the opposite extreme, with $g_s N \gg 1$ we see that the gravitational effects of the branes will now become important. Far from the branes we still have Minkowski

(a) Spacetime geometry of N D3-branes.



(b) Super Yang-Mills description of N D3-branes.

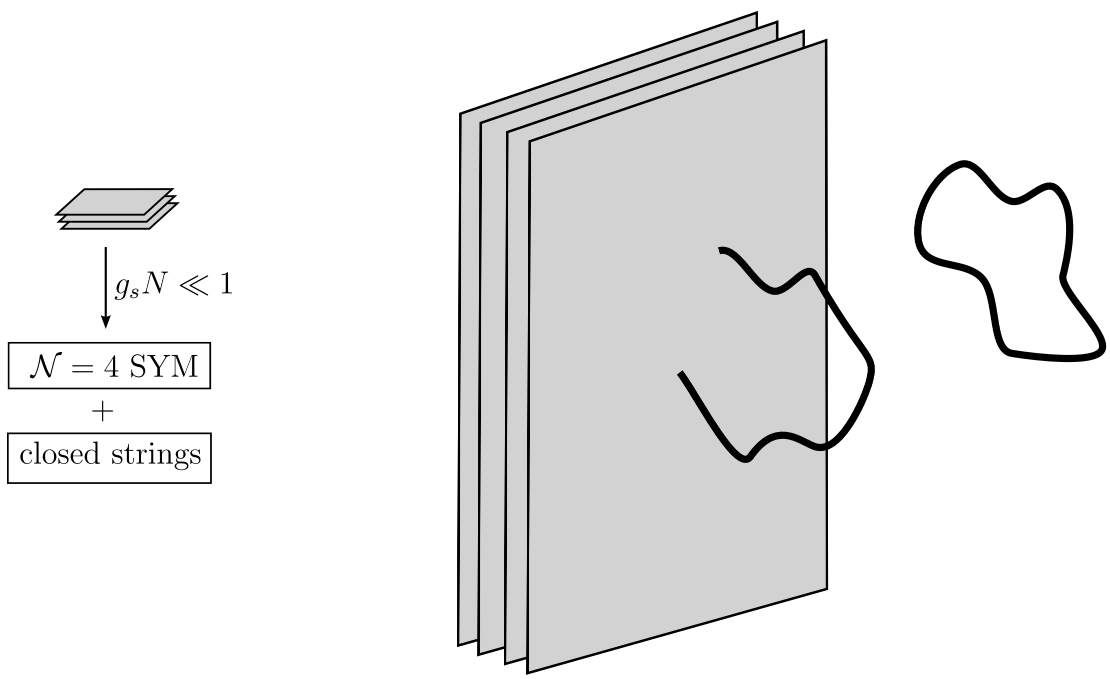


Figure 2.1: Two descriptions of the low energy limit of a stack of D3-branes in type IIB string theory. (a) At large coupling the gravitational effects of the branes are important and the spacetime geometry forms a throat near the branes where closed strings are trapped while remaining flat as $r \to \infty$ where non-interacting closed strings propagate. The circles in the throat correspond to 5-spheres of increasing (decreasing) radius as we move up (down) the throat. (b) For small coupling the dynamics of the open strings attached to the branes is governed by $\mathcal{N} = 4$ SU(N) SYM and we have non-interacting closed strings in a flat spacetime away from the branes. Figure inspired by [27].

spacetime and closed non-interacting strings. However as we move towards the branes a throat geometry, $AdS_5\times S_5$ forms. Inside the throat we have interacting, closed type IIB strings. At the same time, since the SYM is strongly coupled in the $g_sN \gg 1$ limit the geometric description forms the best, tractable viewpoint.

We can now identify the duality. At low energies, for $g_sN \ll 1$, we find massless, closed strings in a 10-dimensional Minkowski spactime and massless open strings interacting according to $\mathcal{N}=4$ SYM. On the other had, at low energies for $g_sN \gg 1$ we have massless, closed strings in a 10-dimensional Minkowski spactime and closed strings in $AdS_5\times S_5$. Both cases describe the same physics and only the coupling has changed, so since we still have massless closed strings in each description we can identify a duality between type IIB string theory in $AdS_5\times S_5$ and $\mathcal{N}=4$ SYM in 4-dimensions.

Returning to the geometric description of the branes, as we move up the throat ($r\to\infty$) the radius of S_5 becomes large and we approach the Minkowski spacetime. Therefore we say that the bulk fields in AdS_5 couple to operators in $\mathcal{N}=4$ SYM on the boundary. To make this statement concrete, let ϕ_i be a set of fields in AdS_5, $(\phi_i)_0$ be their boundary values, and $\mathcal{O}_i$ a set of operators on the boundary which couple to $(\phi_i)_0$. Then the statement of the duality is the equivalence of the generating functions for the quantum field theory and the quantum gravity

$$Z[(\phi_i)_0]_{QFT} = Z_{QG}[\phi_i]\Big|_{BC}. \tag{2.9}$$

By BC we mean that we must apply some appropriate boundary conditions to the gravitational generating function. Having this generating function, we can then compute correlation functions in the QFT in the usual way using $Z[(\phi_i)_0]=\exp(\int(\phi_i)_0\mathcal{O}_i)$,

$$\langle\prod_i \mathcal{O}_i\rangle = \prod_i \frac{\delta}{\delta(\phi_i)_0}\ln Z\Big|_{(\phi_i)_0=0}. \tag{2.10}$$

The equivalence of the generating functions extends beyond the specific example of type IIB strings and SYM and defines a general gauge/gravity duality, where we can identify a gauge theory at the boundary of some bulk gravitational theory. A simple example of this would be AdS_5 with a black hole. This bulk is still dual to $\mathcal{N}=4$ SYM, but the expectation value would now be taken in a thermal state at the Hawking temperature of the black hole. This is true in general.

The problem with using Eq. 2.9 is that we do not know the explicit expression for Z_{QG}. Fortunately in the large N limit we can make a saddle point approximation [26] and use a classical gravity theory

$$Z[(\phi_i)_0]_{QFT} \approx e^{-S_{\rm cl}}\Big|_{BC}. \tag{2.11}$$

Having Eq. 2.11, the final ingredient we need before being able to write down a recipe for computing correlation functions is some kind of dictionary between the bulk fields and the boundary operators. Since the stress-energy tensor is the response to metric variations

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{2.12}$$

$g_{\mu\nu}$ in the bulk will couple to $T_{\mu\nu}$ on the boundary. We would also expect gauge fields A^a_μ in the bulk to couple to currents J^a_μ and more generally the spin of the field corresponds to the spin of the operator while the mass of the field corresponds to the scaling dimension of the operator [7, 8].

Combining these results, given some bulk spacetime with or without a black hole we can compute correlation functions by performing the following steps:

1. First solve the linearized equations of motion for the bulk field. For example, if the bulk field is a massive scalar, this would mean solving the linearized Klein-Gordon equation in curved spacetime. For the metric this means we need to expand about the AdS background, $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ and solve the linearize Einstein equations. For now denote the boundary value of the field schematically as ϕ_0. Typically it will be easier to solve the Fourier transformed equations of motion so we will find $\phi_0(k)$.

2. Then we need to expand the action for the field to second order and write it in terms of the boundary value of the field, ϕ_0. In the case of the metric this means we need to include both the Gibbons-Hawking term and any necessary counter terms. Since we are expanding the action on shell (i.e. we are expanding around a classical solution and the perturbation satisfies the equation of motion) it should reduce to just boundary terms and will take the form

$$S = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\phi_0(-k)\mathcal{F}(k,r)\phi_0(k)\Big|_{r\to\infty} \tag{2.13}$$

where r is the extra dimension and we take it to the boundary ($r \to \infty$).

3. Now using Eq. 2.11 we can take derivatives to find the correlation function.

In step 1 we also need to decide which boundary conditions to apply. This choice is linked to whether we are in Euclidean or Minkowski signature and the type of Green's function we want to calculate. Originally the correspondence was formulated for Euclidean signature [7, 8] and in this case it is sufficient to demand that the solutions are regular at the horizon. However in the real-time case the solutions are oscillatory at the horizon, and we must use a different condition. One choice is that the solutions are ingoing at the horizon which will yield the retarded Green's function. This makes physical sense since we expect things to go into, but not out of the black hole. Alternatively we could choose the solution to be outgoing at the horizon, which would give the advanced Green's function however it is most common to use the ingoing condition and this is what we will use. Therefore once we have solved the linearized equation and have written the action in the form of Eq. 2.13 we can use the real-time formalism [35] and identify the retarded Green's function

$$G(k) = \lim_{r \to \infty} \mathcal{F}(k, r). \tag{2.14}$$

If we want to find the n-point function for $n > 2$ we can repeat the same procedure but we must now expand the equations of motion to higher order and also expand the action to higher order. In the Euclidean formalism we can just take more derivatives of the generating functional of Eq. 2.11, but for real-time correlation functions we have to use a generalization of the approach used to determine Eq. 2.14 [36].

Above we have taken two approaches to the AdS/CFT correspondence. In order to determine the relationship between type IIB string theory and $\mathcal{N} = 4$ SYM we used a top-down approach where we first constructed some brane configuration in order to determine its low energy behaviour in different regimes of the coupling. This determined what the bulk geometry was and in our case also told us what the dual field theory was. However in order to describe the method of computation we used a bottom-up approach (motivated by the insights of the original top-down example), where we picked a bulk spacetime, perturbed it by adding some field to the action, and then solved the linearized equations of motion and computed the action to quadratic order in the perturbing field. We will use this bottom-up approach when we look at Einstein-Gauss-Bonnet gravity.

2.1.3 Green's Functions and Quasinormal Modes

We outlined in Sec. 2.1.2 how we can compute n-point functions using the AdS/CFT correspondence. Of particular interest in this thesis will be retarded two-point functions which are physically important since they characterize the linear response of a system to a perturbation. To see this [37], suppose we have a QFT with a set of operators $\mathcal{O}_i(t, \mathbf{x})$. Now we perturb the system by adding a source $\phi_i(t, \mathbf{x})$ which will change the expectation value of $\mathcal{O}_i(t, \mathbf{x})$

$$\delta \langle \mathcal{O}_i(t, \mathbf{x}) \rangle = \langle \mathcal{O}_i(t, \mathbf{x}) \rangle - \langle \mathcal{O}_i(t, \mathbf{x}) \rangle_0 . \tag{2.15}$$

Here $\langle \mathcal{O}_i(t, \mathbf{x}) \rangle_0$ is the expectation value with the source turned off. This change in $\langle \mathcal{O}_i \rangle$ defines the response function G_{ij} as the linear change in the sources

$$\delta \langle \mathcal{O}_i(t, \mathbf{x}) \rangle = \int d^d x G_{ij}(t, \mathbf{x}; t', \mathbf{x}') \phi_j(t', \mathbf{x}'). \tag{2.16}$$

The sources couple to the operators, so the form of the perturbation to the Hamiltonian is

$$H_{\text{SRC}}(t) = \int d^{d-1}\mathbf{x} \phi_i(t, \mathbf{x}) \mathcal{O}_i(t, \mathbf{x}) \tag{2.17}$$

where we are in the Heisenberg picture. Using the time evolution operator to linear order in ϕ_i

$$U(t, t_0) = 1 - i \int_{t_0}^{t} H_{\text{SRC}}(t') dt' \tag{2.18}$$

we can calculate $\langle \mathcal{O}_i \rangle$

$$\begin{aligned} \langle \mathcal{O}_i(t, \mathbf{x}) \rangle &= \langle \mathcal{O}_i(t, \mathbf{x}) \rangle_0 - i \int_{t_0}^{t} dt' \langle [\mathcal{O}_i(t, \mathbf{x}), H_{\text{SRC}}(t')] \rangle \\ &= \langle \mathcal{O}_i(t, \mathbf{x}) \rangle_0 - i \int_{t_0}^{\infty} dt' \theta(t - t') \langle [\mathcal{O}_i(t, \mathbf{x}), H_{\text{SRC}}(t')] \rangle . \end{aligned} \tag{2.19}$$

So using Eq. 2.17

$$\langle \mathcal{O}_i(t, \mathbf{x}) \rangle = \langle \mathcal{O}_i(t, \mathbf{x}) \rangle_0 - i \int d^d x' \theta(t - t') \langle [\mathcal{O}_i(t, \mathbf{x}), \mathcal{O}_j(t', \mathbf{x}')] \rangle \phi_j(t', \mathbf{x}'). \tag{2.20}$$

Therefore

$$\delta \langle \mathcal{O}_i(t, \mathbf{x}) \rangle = -i \int d^d x' \theta(t - t') \langle [\mathcal{O}_i(t, \mathbf{x}), \mathcal{O}_j(t', \mathbf{x}')] \rangle \phi_j(t', \mathbf{x}') \tag{2.21}$$

and we can identify

$$G_{ij}(t,\mathbf{x};t',\mathbf{x}') = -i\theta(t-t')\left\langle [\mathcal{O}_i(t,\mathbf{x}),\mathcal{O}_j(t',\mathbf{x}')]\right\rangle \tag{2.22}$$

which is just the retarded Green's function. We therefore see that the retarded Green's function characterizes the linear response of a system to a perturbation. For a translationally invariant system, G_{ij} only depends on the differences, $t-t'$ and $\mathbf{x}-\mathbf{x}'$ and we can Fourier transform the Green's function so that

$$G_{ij}(x-x') = \int \frac{d^dk}{(2\pi)^d} e^{ik(x-x')} G_{ij}(k). \tag{2.23}$$

Taking $\mathcal{O}_i = T_{\mu\nu}$ we therefore have an expression for the two-point functions of the stress-energy tensor

$$G_{\mu\nu,\alpha\beta}(x-x') = -i\theta(t-t')\left\langle [T_{\mu\nu}(x),T_{\alpha\beta}(x')]\right\rangle. \tag{2.24}$$

Based solely on the number of indices, in d dimensions the two-point function would have d^4 components. However due to CPT invariance and the symmetry of $T_{\mu\nu}$ the two-point function for a thermal state has only 5 independent index structures and the addition of scale invariance reduces this number to 3 [38]. For momenta along the z direction (using spatial coordinates x, y, z, labelled by the subscripts 1, 2, 3 respectively, which define a right handed coordinate system in 3+1 dimensions) a convenient choice of the three independent components of $G_{\mu\nu,\alpha\beta}$ is $G_{12,12}, G_{13,13}$ and $G_{33,33}$ which each correspond to a gauge invariant combination of the perturbations $h_{\mu\nu}$ whose equations of motion decouple [38]. Two of these classes contain hydrodynamic modes, and we may classify the perturbations by their transformation under rotations. The components transforming as scalars contain a sound mode, the components transforming as vectors contain a shear mode, and the components transforming as tensors do not contain a hydrodynamic mode. Hence these three classifications are known as the sound channel, shear channel and scalar channel respectively (the scalar channel refers to the perturbations which transform as a tensor because the equations of motion for these perturbations are those of a massless scalar field).

The hydrodynamic modes we just discussed will correspond to poles in the two-point function, however it is also possible that the correlation function will contain other poles which could potentially correspond to propagating modes violating causality, making the location and dispersion relation of the poles important. It turns out that these poles are actually the quasinormal modes (QNM) of the black hole in our bulk spacetime [35]. To see this we return to the prescription from the previous section. The Green's function will generally arise from the $\phi\phi'$ term in the action so using our prescription $G(k) \propto \phi_0'(k)/\phi_0(k)$. The linearized equations of motion will be some linear, second order differential equation, so we can write the solution near the boundary as a sum of two Frobenius solutions

$$\phi(u) = A(k)u^{\Delta_-}(1 + \ldots) + B(k)u^{\Delta_+}(1 + \ldots) \tag{2.25}$$

where $\Delta_+ > \Delta_-$, the ellipses denote higher powers in u and we are using coordinates where $u = a^2/r^2$ so that u runs from 1 at the horizon to 0 at the boundary. Therefore ϕ' is given by

$$\begin{aligned}\phi'(u) = {} & A(k)\Delta_- u^{\Delta_- -1}(1 + \ldots) + A(k)u^{\Delta_-}(a_1 + 2a_2 u + \ldots) \\ & + B(k)\Delta_+ u^{\Delta_+ -1}(1 + \ldots) + B(k)u^{\Delta_+}(b_1 + 2b_2 u + \ldots).\end{aligned}$$

and the correlation function becomes (taking $u = \epsilon \to 0$)

$$\begin{aligned}G(k) \propto \frac{\phi_0'(k)}{\phi_0(k)} &= \frac{A(k)\Delta_-\epsilon^{\Delta_- -1}(1 + \mathcal{O}(\epsilon)) + B(k)\Delta_+\epsilon^{\Delta_+ -1}(1 + \mathcal{O}(\epsilon))}{A(k)\epsilon^{\Delta_-}\left[(1 + \mathcal{O}(\epsilon)) + \frac{B(k)}{A(k)}\epsilon^{\Delta_+ - \Delta_-}(1 + \mathcal{O}(\epsilon))\right]} \\ &= (\Delta_+ - \Delta_-)\frac{B(k)}{A(k)}\epsilon^{\Delta_+ - \Delta_- -1} + \mathcal{O}\left(\epsilon^{\Delta_+ - \Delta_-}\right) + \text{Contact Terms.}\end{aligned}$$

So the poles of G are given by the roots of A. But near the boundary ($u = 0$) the solution is $\phi(u) \sim A(k)u^{\Delta_-}$ so the condition that $A(k) = 0$ is the same as applying Dirichlet boundary conditions which we would use to find the quasinormal modes. So we see that the two-point function has a simple structure in terms of the coefficients of the solution to the linearized action, and the poles which determine the propagating modes are given by the quasinormal modes of the black hole. This method of finding the poles will be used in Ch. 3.

2.1.4 Viscosity Bound

Through the correlation functions, AdS/CFT provides a powerful method for calculating transport coefficients in the field theory, which, at strong coupling, is difficult to do using conventional field theory techniques. An important example is the shear viscosity which may be found from the pole structure of the shear and sound modes or through the Kubo formula [27]

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\lim_{\mathbf{k}\to 0}\mathrm{Im}G_{12,12}(k). \tag{2.26}$$

This shear viscosity has been calculated for a large number of bulk geometries [12, 14, 15] and interestingly, the ratio η/s was found to be universal over these examples, leading to the viscosity bound conjecture that $\eta/s \geq 1/4\pi$ for all relativistic quantum field theories [16]. The lower bound $\eta/s = 1/4\pi$ is much lower than any normal fluid, such as water, however for the quark gluon plasma, as measured through the elliptic flow at the Relativistic Heavy Ion Collider (RHIC), η/s is actually very close to the bound [39]. To an extent the smallness of the bound makes sense since our field theory is strongly coupled in the large N limit and from kinetic theory we can expect the viscosity to be proportional to the mean free path [40]. Arguments based on the uncertainty principle also favour a bound of order 1 [16].

The viscosity bound has withstood a number of tests including $1/\lambda_t$ corrections in SYM [41, 42]. However the bound can be violated by higher derivative gravity. Specifically in the case of Einstein-Gauss-Bonnet gravity it was found that

$$\eta/s = \frac{1}{4\pi}\left[1-4\lambda\right] \tag{2.27}$$

so that the bound is violated for $\lambda > 0$ [17]. Interestingly enough higher order Lovelock terms (see Sec. 2.2) do not explicitly violate the bound for planar horizons [43] although they enter implicitly through causality constraints [44] and therefore still influence a potential new lower bound. It is also important to keep in mind that although the viscosity bound is explicitly violated for EGB gravity, it is unclear what the exact UV completion is for EGB gravity, and there are examples of effective theories that do not have consistent UV completions [45, 46]. However, as discussed in Sec. 2.3 higher derivative theories such as EGB gravity allow non-equal “central charges” c and a and other field theories with non-equal central charges are known to have string theory embeddings [47, 48].

2.2 Higher Derivative Gravity

In discussing the viscosity bound above, we mentioned the role of higher derivative gravity theories. In such theories we go beyond the Ricci scalar and include higher curvature terms in the action. For example we may have an action of the form

$$S = \frac{1}{16\pi G}\int d^Dx\sqrt{-g}\left(R - 2\Lambda + a_1 R^2 + a_2 R^{\mu\nu}R_{\mu\nu} + a_3 R R^{\mu\nu}R_{\mu\nu} + \ldots\right). \quad (2.28)$$

There are several reasons to consider such modifications to general relativity. In the context of effective field theories [49, 50], we would naturally expect Einstein gravity to just be a low energy limit and in general all interactions consistent with the symmetries of GR (i.e. general covariance) would be present. This means we can build terms out of curvature invariants and suppress them by some appropriate mass. The curvature squared interactions also arise naturally in low energy limits of heterotic string theory [51]. This means if we want to consider corrections to the Einstein gravity dual, curvature squared terms would be a natural place to start. A final reason to consider higher derivative gravities is the importance of AdS_5 in the context of AdS/CFT (since it is dual to a 4-dimensional CFT). In $D > 4$ dimensions the principles which yield Einstein gravity give rise to more general higher derivative theories [52], so when we consider gravity in more than 4 dimensions it is natural to consider a more general gravitational theory.

The Einstein tensor, $G_{\mu\nu}$ is a symmetric rank two tensor, which is divergence free (that is $\nabla_\mu G^{\mu\nu} = 0$) and which depends on the metric and its first two derivatives only. Note that even if we include a cosmological constant in the theory, the equations of motion still obey the divergence free condition since $\nabla_\mu g^{\mu\nu} = 0$ by metric compatibility. Assuming that these key characteristics hold in higher dimensions, general relativity naturally extends to Lovelock gravity [52]. It turns out that Lovelock theories are also the most general theories where Palatini and metric formulations are equivalent [53]. The Lagrangian for Lovelock theories depends on the dimension of the spacetime and is given by

$$\mathcal{L}_L = \sqrt{-g}\sum_{p=0}^{\lfloor\frac{D-1}{2}\rfloor}\alpha_p\mathcal{L}_p \quad (2.29)$$

where

$$\mathcal{L}_p = \frac{1}{2^p}\delta^{\mu_1\nu_1\ldots\mu_p\nu_p}_{\alpha_1\beta_1\ldots\alpha_p\beta_p}\prod_{r=1}^{p}R^{\alpha_r\beta_r}{}_{\mu_r\nu_r} \quad (2.30)$$

and $\delta^{\mu_1\nu_1...\mu_p\nu_p}_{\alpha_1\beta_1...\alpha_p\beta_p}$ is the antisymmetrized Kronecker delta

$$\delta^{\mu_1\nu_1...\mu_p\nu_p}_{\alpha_1\beta_1...\alpha_p\beta_p} = \begin{vmatrix} \delta^{\mu_1}_{\alpha_1} & \delta^{\mu_1}_{\beta_1} & \cdots & \delta^{\mu_1}_{\alpha_p} & \delta^{\mu_1}_{\beta_p} \\ \delta^{\nu_1}_{\alpha_1} & \delta^{\nu_1}_{\beta_1} & \cdots & \delta^{\nu_1}_{\alpha_p} & \delta^{\nu_1}_{\beta_p} \\ \vdots & & \ddots & & \vdots \\ \delta^{\mu_p}_{\alpha_1} & \delta^{\mu_p}_{\beta_1} & \cdots & \delta^{\mu_p}_{\alpha_p} & \delta^{\mu_p}_{\beta_p} \\ \delta^{\nu_p}_{\alpha_1} & \delta^{\nu_p}_{\beta_1} & \cdots & \delta^{\nu_p}_{\alpha_p} & \delta^{\nu_p}_{\beta_p} \end{vmatrix}. \tag{2.31}$$

The first few terms in Eq. 2.29 can easily be computed. $\mathcal{L}_0$ is just a constant, $\mathcal{L}_1$ is the Ricci scalar, $\mathcal{L}_2$ is the Gauss-Bonnet term

$$\mathcal{L}_2 = \mathcal{L}_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{2.32}$$

and an explicit form of the third order Lovelock term can be found in [54]. Therefore in 4-dimensions Lovelock gravity is just general relativity, and due to the Gauss-Bonnet theorem [55] $\mathcal{L}_{GB}$ is just a constant. In 5-dimensions however we have to include $\mathcal{L}_{GB}$ and we have Einstein-Gauss-Bonnet gravity.

The fact that the equations of motion depend only on the first two derivatives of the metric means that the theory will be ghost free when expanded around a flat spacetime, and even around some more general backgrounds [56, 57]. Since string theory is ghost free the quadratic curvature terms which appear at low energies should be organized into the Gauss-Bonnet term [57], making the EGB theory a natural gravitational theory to consider. Another important property of Lovelock theories is that despite their apparent complexity, many general results are available. For example, general spherically symmetric solutions exist [58, 59, 60] and a general form for the equations of motion can also be found [43, 61]. The general Gibbons-Hawking term was found in [62] and an explicit general form is given in [63, 64]. Explicit terms up to third order for the Lovelock tensor and the Gibbons-Hawking term can be found in [54].

In this thesis we will focus on the EGB theory with the action

$$S = \frac{1}{16\pi G}\int d^Dx\sqrt{-g}\left[R - 2\Lambda + \frac{\lambda L^2}{2}\mathcal{L}_{GB}\right] \tag{2.33}$$

where $\Lambda = -\frac{(D-2)(D-1)}{2L^2}$ and L is the AdS radius for $\lambda = 0$. The equations of motion for this action, denoted by $A_{\mu\nu}$, can be found by explicitly varying the action with respect to $g_{\mu\nu}$ and this calculation is performed in Appendix A. The result is

$$A^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + g^{\mu\nu}\Lambda - \frac{\lambda L^2}{4}g^{\mu\nu}\mathcal{L}_{GB} - \frac{\lambda L^2}{2}H^{\mu\nu} = 0 \tag{2.34}$$

where

$$H^{\mu\nu} = \left(-2RR^{\mu\nu} + 4R^{\mu\rho\nu\sigma}R_{\rho\sigma} - 2R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} + 4R^{\mu}{}_{\rho}R^{\nu\rho}\right). \tag{2.35}$$

To solve this equation we make the ansatz

$$ds^2 = -A^2 f(r)dt^2 + \frac{1}{f(r)}dr^2 + \frac{r^2}{L^2}d\mathbf{x}^2. \tag{2.36}$$

The non-zero components of $A^{\mu\nu}$ are then A^{rr}, A^{tt} and $A^{x_1x_1} = A^{x_2x_2} = A^{x_3x_3}$ which all have the same solution

$$f(r) = \frac{r^2}{2L^2\lambda}\left(1 - \sqrt{1 - 4\lambda\left(1 - \frac{a^4}{r^4}\right)}\right) \tag{2.37}$$

where a is the horizon radius ($f(a) = 0$). At the boundary, $r \to \infty$, $f(r) \to \left(1 - \sqrt{1-4\lambda}\right) r^2/2L^2\lambda$ so setting

$$A = \sqrt{\frac{1}{2}\left(1 + \sqrt{1-4\lambda}\right)} \tag{2.38}$$

sets the dt coefficient, and thus the speed of light at the boundary, to 1. We will use this definition of A from now on which allows us to write the AdS radius as $L' = AL$. This shows that for $\lambda > 1/4$ the solution is no longer AdS so we will only consider $\lambda \leq 1/4$. The thermodynamics of this solution are discussed in Appendix B, but it is useful to point out that this solution still follows the area law for the entropy [65] since it has a planar horizon. In general this is not true for higher derivative theories of gravity, where the entropy should be calculated using the Wald formula [66]. Even for EGB more general solutions with spherical or hyperbolic horizons exist and do not obey the area law [60, 65, 67]. However black holes with planar horizons will obey the area law in any Lovelock theory [60] and here we will only consider the planar solution described by Eq. 2.36 and Eq. 2.37.

The final aspect of higher derivative gravity we need to discuss is the Gibbons-Hawking boundary term. In order to have a well defined variational problem in the presence of a boundary, we must introduce the Gibbons-Hawking surfaceterm [68] so that we only need to constrain the metric and not its derivatives on the boundary. In general relativity this is given by [1]

$$S^b_{GR} = \frac{1}{8\pi G}\int d^dx\sqrt{-\gamma}K \tag{2.39}$$

where

$$\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \tag{2.40}$$

is the induced metric, which is the pullback of the metric onto the boundary, and n_μ is the normal to the boundary defined by

$$n^\mu = \frac{\xi^\mu}{|\xi_\nu\xi^\nu|^{1/2}} \tag{2.41}$$

with

$$\xi^\mu = g^{\mu\nu}\nabla_\nu f(x) \tag{2.42}$$

and $f(x) = C$ used to define the boundary. K is the trace of the extrinsic curvature

$$K_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu n_\nu + \nabla_\nu n_\mu\right). \tag{2.43}$$

Since the equations of motion for Lovelock theories contain only two derivatives of the metric, it is possible to generalize the Gibbons-Hawking term and for EGB we have [69]

$$S^b_{GB} = \frac{\lambda L^2}{8\pi G}\int d^dx\sqrt{-\gamma}\left(J - 2\widehat{G}^{\mu\nu}K_{\mu\nu}\right) \tag{2.44}$$

where $\widehat{G}_{\mu\nu}$ is the Einstein tensor of the induced metric and J is the trace of

$$J_{\mu\nu} = \frac{1}{3}(2KK_{\mu\sigma}K^\sigma{}_\nu + K_{\sigma\rho}K^{\sigma\rho}K_{\mu\nu} - 2K_{\mu\sigma}K^{\sigma\rho}K_{\rho\nu} - K^2K_{\mu\nu}). \tag{2.45}$$

In AdS/CFT we are interested in the action on the boundary and it is important to include the Gibbons-Hawking contribution in the action. We will see this explicitly when computing the correlation function of the stress-energy tensor.

2.3 Constraints on λ and Central Charges

In conformal theories the energy momentum tensor is traceless due to the scaling symmetry of the action. However in a curved background this symmetry is anomalous and for a 4-dimensional CFT the anomaly can be characterized by the two central charges, c and a [70]

$$\langle T^{\mu}{}_{\mu}\rangle = \frac{c}{16\pi^2}C_{\mu\nu\sigma\rho}C^{\mu\nu\sigma\rho} - \frac{a}{16\pi^2}\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{2.46}$$

Here $C_{\mu\nu\sigma\rho}$ is the Weyl tensor, which is the traceless component of the Riemann tensor. Requiring that the energy one-point function is positive leads to constraints on a and c. In particular with $\mathcal{N} = 1$ supersymmetry [20]

$$\frac{1}{2} \leq \frac{a}{c} \leq \frac{3}{2}. \tag{2.47}$$

Similar constraints exist for $\mathcal{N} = 2$ [20], while for $\mathcal{N} = 4$, $a = c$. c and a can also be written in terms of the EGB coupling [19, 71]

$$\begin{aligned} c &= \frac{\pi L^3}{8^{3/2}G}\left(1 + \sqrt{1-4\lambda}\right)^{3/2}\left(\sqrt{1-4\lambda}\right) \\ a &= \frac{\pi L^3}{8^{3/2}G}\left(1 + \sqrt{1-4\lambda}\right)^{3/2}\left(3\sqrt{1-4\lambda} - 2\right) \end{aligned} \tag{2.48}$$

so that

$$\frac{a}{c} = 3 - \frac{2}{\sqrt{1-4\lambda}}. \tag{2.49}$$

In the case of $\mathcal{N} = 1$ the bounds on a and c then translate to

$$-\frac{7}{36} \leq \lambda \leq \frac{9}{100}. \tag{2.50}$$

Returning to the AdS/CFT description of EGB gravity, the higher derivative interactions allow the dual CFT to have non-equal central charges, which is not the case for Einstein gravity duals. Furthermore requiring causality of the dual CFT allows restrictions to be placed on λ from the various perturbation channels. The scalar channel provides an upperbound, $\lambda \leq \frac{9}{100}$ [17, 18] and both the shear and sound channels provide lower bounds, $\lambda \geq -\frac{3}{4}$ from the shear channel and $\lambda \geq -\frac{7}{36}$ from the sound channel [19]. Therefore the causality bounds agree with the bounds on the central charges for the $\mathcal{N} = 1$ dual theory. The upper bound on λ has the effect of lowering the bound on the viscosity

We can see then that the viscosity bound, causality violation and the central charges are all interrelated. In the next chapter we will perform an explicit calculation of the two-point function of the stress-energy tensor with the goal of investigating and interpreting the upper bound on λ. Indeed the upperbound appears to be the most interesting since the low energy string theory corrections come with a positive coefficient and the viscosity bound will only be violated for $\lambda > 0$.

Chapter 3

Calculation of the Two-Point Function

In this chapter we will explicitly calculate the stress-energy tensor two-point function (numerically) and will determine the pole structure of the correlation function at large spatial momentum. We will also compute the quasinormal modes at several λ.

3.1 Series Solution to the Linearized Equation of Motion

Metric fluctuations will act as a source on the boundary for the stress-energy tensor, so in order to calculate the two-point function of $T_{\mu\nu}$ we need to perturb the metric. Since we are interested in the scalar channel and hence $G_{xy,xy}$, we will add an h_{xy} perturbation which we will write in terms of $h^x{}_y = \phi = L^2/r^2 h_{xy}$. Due to the rotation invariance we can write $\phi = \phi(r, z, t)$ and therefore the equation of motion for ϕ will be trivially invariant under the infinitesimal diffeomorphism $h_{\mu\nu} \to h_{\mu\nu} - \nabla_\mu \zeta_\nu - \nabla_\nu \zeta_\mu$, ($\zeta_\mu = \zeta_\mu(r) e^{-i\omega t + iqz}$) since $\phi \to \phi$ under this transformation. With this perturbation the metric becomes

$$ds^2 = -A^2 f(r) dt^2 + \frac{1}{f(r)} dr^2 + \frac{r^2}{L^2} \left[d\mathbf{x}^2 + 2\phi(r, z, t) dx dy \right] \tag{3.1}$$

with the horizon at $r = a$ and the boundary at $r = \infty$. Expanding the equations of motion, Eq. 2.35, to first order, the xy component gives

$$\frac{1}{2L^2}\left[rf'\left(-r+L^2\lambda f'\right)+f\left(-3r+L^2\lambda\left(2f'+rf''\right)\right)\right]\frac{\partial\phi}{\partial r}+\frac{rf}{2L^2}\left(-r+L^2\lambda f'\right)\frac{\partial^2\phi}{\partial r^2}$$
$$+\frac{r\left(r-L^2\lambda f'\right)}{2A^2L^2f}\frac{\partial^2\phi}{\partial t^2}+\frac{1}{2}\left(-1+L^2\lambda f''\right)\frac{\partial^2\phi}{\partial z^2}=0 \tag{3.2}$$

where $f = f(r)$ and primes are derivatives with respect to r. We now Fourier transform in z and t

$$\phi(r,z,t)=\int\frac{d\omega dq}{(2\pi)^2}\phi(r)e^{-i(\omega t-qz)} \qquad \text{using} \qquad k=(\omega,0,0,q) \tag{3.3}$$

and perform the rescalings

$$v=\frac{r}{a}, \qquad \tilde{\omega}=\frac{L^2}{a}\omega, \qquad \tilde{q}=\frac{L^2}{a}q,$$
$$\tilde{f}=\frac{L^2}{a^2}f=\frac{v^2}{2\lambda}\left[1-\sqrt{1-4\lambda\left(1-\frac{1}{v^4}\right)}\,\right]. \tag{3.4}$$

The horizon is now at $v = 1$ and the boundary at $v = \infty$ while the $\frac{\partial^2\phi}{\partial r^2}$ coefficient becomes $-\frac{a^4}{2L^4 v}K(v)$ with

$$K(v)=v^2\tilde{f}(v)\left(v-\lambda\tilde{f}'(v)\right), \qquad \tilde{f}(v)=\frac{L^2}{a^2}f(v)=\frac{v^2}{2\lambda}\left(1-\sqrt{1-4\lambda\left(1-\frac{1}{v^4}\right)}\right) \tag{3.5}$$

the $\frac{\partial\phi}{\partial r}$ coefficient is $-\frac{a^3}{2L^4v}K'(v)$ and the ϕ coefficient is $-\frac{a^2}{2L^4v}K_1(v)$ where

$$K_1(v)=\frac{\tilde{\omega}^2}{A^2\tilde{f}^2(v)}K(v)-\tilde{q}^2v\left(1-\lambda\tilde{f}''(v)\right). \tag{3.6}$$

Here primes denote the derivative with respect to the argument specified, e.g. $K'(v) = \frac{dK}{dv}$. The linear equation of motion is therefore

$$K(v)\phi''(v)+K'(v)\phi'(v)+K_1(v)\phi(v)=0. \tag{3.7}$$

In order to construct series solutions the coordinate $u = 1/v^2 = a^2/r^2$ is convenient since the coordinate change moves the horizon to $u = 1$ and the boundary to $u = 0$.

In terms of u the equation of motion is

$$4u^3K(u)\phi''(u) + \left[6u^2 + 4u^3K'(u)\right]\phi'(u) + K_1(u)\phi(u) = 0 \tag{3.8}$$

with

$$K(u) = u^{-3/2}\tilde{f}(u)\left(1 + 2u^2\lambda\tilde{f}'(u)\right), \qquad \tilde{f}(u) = \frac{1}{2\lambda u}\left(1 - \sqrt{1 - 4\lambda(1-u^2)}\right),$$
$$K_1(u) = K(u)\frac{\tilde{\omega}^2}{A^2\tilde{f}^2} - \tilde{q}^2u^{-1/2}\left[1 - \lambda\left(6u^2\tilde{f}'(u) + 4u^3\tilde{f}''(u)\right)\right].$$

Both the horizon and the boundary are regular singular points of Eq. 3.8 so we can construct series solutions using the Froebenius method [72] and since there are no other singularities between the boundary and horizon, the solution from the horizon should be convergent up to the boundary and the boundary solution should be convergent up to the horizon. At the boundary we will have two independent solutions and near the horizon we will choose the solution which is incoming. Rewriting the equation of motion

$$r(u)\phi''(u) + s(u)\phi^{'}(u) + t(u)\phi(u) = 0 \tag{3.9}$$

we can expand the solution and the coefficients near the horizon

$$\phi_h(u) = (1-u)^{\Delta}\sum_{n=0}^{\infty} a_n(\mathbf{k},\omega)(1-u)^n \tag{3.10}$$

$$r(u) = \sum_{k=0}^{\infty} r_k(1-u)^k, \qquad s(u) = \sum_{k=0}^{\infty} s_k(1-u)^k, \qquad t(u) = \sum_{k=0}^{\infty} t_{k-1}(1-u)^{k-1} \tag{3.11}$$

so that the linearized equation becomes

$$\sum_{k,n}\left[r_k(n+\Delta)(n+\Delta-1)a_n(1-u)^{n+k-2} - \left[s_k(n+\Delta) - t_{k-1}\right]a_n(1-u)^{n+k-1}\right] = 0. \tag{3.12}$$

The $(1-u)^{-2}$ coefficient is identically 0 since it is proportional to $r_0 = 0$ and the $(1-u)^{-1}$ coefficient gives the indicial equation,

$$r_1\Delta(\Delta-1) - s_0\Delta + t_{-1} = 0 \tag{3.13}$$

where

$$r_1 = -s_0 = 8 - 32\lambda, \qquad t_{-1} = \frac{\tilde{\omega}^2 (1 - 4\lambda)}{1 + \sqrt{1 - 4\lambda}}. \tag{3.14}$$

The indicial exponents are therefore $\Delta_\pm = \pm\frac{i\tilde{\omega}}{4A}$, with Δ_- representing the incoming solution. Writing the equation for $(1-u)^n$ the a_n coefficient is determined to be

$$a_n = \frac{\sum_{j=0}^{n-1} \left[r_{n+1-j}(j + \Delta_-)(j + \Delta_- - 1) - s_{n-j}(j + \Delta_-) + t_{n-j-1}\right] a_j}{s_0 n(n + 2\Delta_-)} \tag{3.15}$$

so the series solution at the horizon can be computed to any order in $(1-u)$. At the boundary we will generally need to keep both solutions. By expanding r, s and t as

$$r(u) = \sum_{k=0}^{\infty} r_k u^{k+1/2} \qquad s(u) = \sum_{k=0}^{\infty} s_{k-1} u^{k-1/2} \qquad t(u) = \sum_{k=0}^{\infty} t_{k-1} u^{k-1/2} \tag{3.16}$$

we find the indicial exponents $\Delta_+ = 2$ and $\Delta_- = 0$. The coefficients r_k, s_k and t_k in the expansion near the boundary in Eq. 3.16 are not the same as the coefficients in the expansion near the horizon in Eq. 3.11. The factors of $u^{1/2}$ in the expansions of $r(u)$, $s(u)$ and $t(u)$ arise due to the definition of $\Delta_\pm$ at the boundary, that is, we could absorb the factor of $1/2$ into $\Delta_\pm$. Since the indicial exponents differ by an integer one solution involves logarithms and we can write the solution at the boundary as $\phi_b = A\phi_1 + B\phi_2$ with

$$\phi_2 = \sum_{n=0}^{\infty} a_n u^{n+2} \qquad \phi_1 = C\phi_2 \ln(u) + \sum_{n=0}^{\infty} b_n u^n. \tag{3.17}$$

Note that the a_n in Eq. 3.17 for the expansion near the boundary are not the same as the a_n in Eq. 3.10 for the expansion near the horizon. The coefficients at the boundary are

$$a_n = \frac{-1}{r_0(n+2)n} \sum_{j=0}^{n-1} \left[r_{n-j}(j+2)(j+1) + s_{n-j-1}(j+2) + t_{n-j-2}\right] a_j \tag{3.18}$$

$$b_n = \frac{-1}{r_0 n(n-2)} \left[\sum_{j=0}^{n-1} \left(r_{n-j} j(j-1) + s_{n-j-1} j + t_{n-j-2} \right) b_j \right.$$
$$\left. - C \sum_{j=0}^{n-2} \left(r_{n-j-2}(2j+3) + s_{n-j-3} \right) a_j \right] \quad (3.19)$$

with the constant $C = -\frac{1}{2}\left(\frac{t_{-1}}{r_0}\right)^2$. Here we have set $a_0 = b_0 = 1$, and C is determined by choosing $b_2 = 0$ which we may verify from the b_j expression since $s_0 = t_0 = 0$ and

$$r_0 = -s_{-1} = \frac{2}{\lambda}\left(-1 + \sqrt{1-4\lambda} + 4\lambda\right)$$
$$t_{-1} = -4\sqrt{1-4\lambda}\tilde{q}^2 - \frac{16\sqrt{1-4\lambda}\lambda\tilde{\omega}^2}{\left(-1+\sqrt{1-4\lambda}\right)\left(1+\sqrt{1-4\lambda}\right)}. \quad (3.20)$$

With the series solutions constructed, we can immediately calculate the quasinormal mode spectrum but in order to use the solution to calculate the full two-point function we first need to expand the action to second order.

3.2 Expansion of the Action and the Two-Point Function

Since we must obtain the equations of motion in Eq. 3.7 from the Euler-Lagrange equations, we know that the general form of the second order action is

$$\mathcal{L} = N\left[K\left(\phi'\right)^2 - K_1\phi^2 + F(\phi, \phi', v)\right] \quad (3.21)$$

where $F(\phi, \phi', v)$ must satisfy the Euler-Lagrange equation and N is an overall constant. Since we expect the action to reduce to boundary terms, we know $F(\phi, \phi', v) = \partial_v g(\phi, \phi', v)$ assuming that the appropriate integration by parts has been performed i.e. we need to use the equations of motion to have zero bulk term, this means that the first two terms in Eq. 3.21 must cancel by themselves in the bulk, so that aside from an integration by parts which must then cancel, F must be a total derivative. Furthermore since the Riemann tensors and its contractions all have two derivatives, ϕ will only come in with up to two derivatives and since we only expand to second order, the most general form of g will be

$$g = K_2\phi^2 + K_3\phi\phi' + K_4\left(\phi'\right)^2. \tag{3.22}$$

The ϕ^2 term will automatically satisfy the Euler-Lagrange equations for an arbitrary K_2, but to determine the general form of F and the normalization constant N we need to explicitly expand the action to second order.

Using Eq. 3.1 the second order EGB action has the form

$$S = \int drdzdt\left[P_1(r) + \frac{1}{2}\phi(\text{EOM}) + \partial_r P_2(r,z,t) + \partial_z P_3(r,z,t) + \partial_t P_4(r,z,t)\right] \tag{3.23}$$

after integrating by parts as discussed in Appendix C. Here EOM denotes the equations of motion and P_2, P_3 and P_4 are explicit functions of r, z, t and also depend on r, z, t through a dependence on ϕ. The equations of motion here agree with the linearized equations of motion as expected, verifying the correctness of Eq. 3.7. The explicit forms of the P functions are not important, except to note that $P_1(r)$ can be explicitly integrated and diverges as the area of the boundary at large r. Since we are interested in the boundary at large r the total z and t derivatives will not contribute. The P_2 term contains ϕ' contributions which we can remove with the Gibbons-Hawking term, however we also have ϕ independent terms which diverge at the boundary and which cannot be removed by the surface term, for example the area divergence in $P_1(r)$. In general this divergent behaviour is expected from a gravitational stress-energy tensor [73]. However since the gravitational divergence occurs on the boundary, through AdS/CFT it can be interpreted as a UV divergence of the boundary QFT [74] and therefore we can add counterterms at the boundary to remove the divergence, just as we would add local counterterms in the QFT. To linear order in λ, the counterterm can be computed by determining the EGB Hamiltonian and writing the Hamilton-Jacobi equation as a function of a general counterterm action consisting of curvature invariants ordered by their number of derivatives. The Hamilton-Jacobi equation then determines the coefficients of each term in the counterterm action perturbatively in λ [75]. Alternatively the counterterm can be computed by explicitly calculating the divergence and adding terms to the action with the correct power law divergence at the boundary. In the case of EGB gravity the appropriate counterterms found with this method are [76]

$$S^b_{CT} = \frac{1}{8\pi G}\int d^4x\sqrt{-\gamma}\left(c_1 - \frac{c_2}{2}\widehat{R}\right),$$

$$c_1 = \frac{-1-4\lambda+\sqrt{1-4\lambda}}{\sqrt{2\lambda L^2}\sqrt{1-\sqrt{1-4\lambda}}}, \qquad c_2 = \frac{\sqrt{2\lambda L^2}\left(3-4\lambda-3\sqrt{1-4\lambda}\right)}{2\left(1-\sqrt{1-4\lambda}\right)^{\frac{3}{2}}}. \tag{3.24}$$

The full boundary term will then contain the Einstein and Gauss-Bonnet, Gibbons-Hawking contributions and the counterterm. When the full boundary term is expanded we find a contribution which is the antiderivative of P_1, P_1^A, so that the r^4 divergence is removed and we are left with a constant, which, in the $\lambda \to 0$ limit where $A \to 1$, is the same as the constant for Einstein gravity

$$\int_a^\infty dr P_1(r) - P_1^A = \frac{a^4 A}{16GL^5\pi}. \tag{3.25}$$

The action then reduces to only surface terms as expected and after Fourier transforming we have

$$S = -\frac{a^4 A}{32GL^5\pi}\int \frac{d\omega dq}{(2\pi)^2}\left[K_2\phi^2 + K\phi\partial_v\phi\right], \tag{3.26}$$

where

$$K_2 = -\frac{2v^2\lambda\tilde{\omega}^2}{A^2} + 4v^2\tilde{f}(v) - \frac{\sqrt{2}v^3\sqrt{\lambda\tilde{f}(v)}}{\lambda\sqrt{1-\sqrt{1-4\lambda}}}\left[1+4\lambda-\sqrt{1-4\lambda}\right] + v\tilde{f}'(v)\left[v^2+\lambda\tilde{q}^2-2v\lambda\tilde{f}(v)\right]. \tag{3.27}$$

Therefore the bulk action can be written as

$$S = -\frac{a^4 A}{32GL^5\pi}\int dv\frac{d\omega dq}{(2\pi)^2}\left[K\left(\partial_v\phi\right)^2 - K_1\phi^2 + \partial_v\left(K_2\phi^2\right)\right]. \tag{3.28}$$

Ignoring contact terms, i.e. the K_1 and K_2 terms, the boundary action in Eq. 3.26 allows us to identify the retarded two-point function as

$$G_{xy,xy}(k) = \frac{a^4 A}{16GL^5\pi}\frac{K(v)\phi'(v)}{\phi(v)}\bigg|_{\text{boundary}} \tag{3.29}$$

where ϕ is the solution to Eq. 3.8 which can be determined numerically with boundary conditions from the series solution Eq. 3.10 and 3.15. In terms of u or r we have respectively

$$G_{xy,xy}(k) = \frac{a^4 A}{16GL^5\pi}\frac{u^{3/2}K(u)\phi'(u)}{\phi(u)}\bigg|_{\text{boundary}} \qquad G_{xy,xy}(k) = \frac{A}{16GL^5\pi}\frac{\tilde{K}(r)\phi'(r)}{\phi(r)}\bigg|_{\text{boundary}} \tag{3.30}$$

with

$$\tilde{K}(r) = r^2L^2f(r)\left(r - \lambda L^2 f'(r)\right). \tag{3.31}$$

The numerical plots of the two-point function in Eq. 3.29 are dominated by a large zero temperature component which diverges as ω^4. In order to observe the effects of non-zero temperature we need to subtract this background. The $T = 0$ solution corresponds to the spacetime without a black hole, so we can use our result at non zero temperature and set the horizon radius $a = 0$. This requires a coordinate change back to r and the two-point function is given by Eq. 3.30. ϕ will now be the solution to Eq. 3.8, written in terms of r, in the $a \to 0$ limit. In that case the equation of motion becomes the same as Eq. 3.7 but written as a function of r and with

$$\begin{aligned} K(r) &= r^2\tilde{f}(r)\left(r - \lambda\tilde{f}'(r)\right), \qquad \tilde{f}(r) = L^2 f(r) = \frac{r^2}{2\lambda}\left(1 - \sqrt{1-4\lambda}\right) \\ K_1(r) &= K(r)\frac{4\mathfrak{w}}{A^2\tilde{f}^2(r)} - 4\mathfrak{q}^2 r\left(1 - \lambda\tilde{f}''(r)\right), \qquad \tilde{\omega} = 2\mathfrak{w}, \qquad \tilde{q} = 2\mathfrak{q}. \end{aligned} \tag{3.32}$$

Then letting $\phi(r) \to \Phi(r)/r^2$ and $r \to \sqrt{D}/x$ with

$$D = 2(\mathfrak{w}^2 - \mathfrak{q}^2)(1 + \sqrt{1-4\lambda}) \tag{3.33}$$

the equation of motion becomes

$$x^2\Phi''(x) + x\Phi'(x) + (x^2 - 4)\Phi(x) \tag{3.34}$$

which is just Bessel's equation. The oscillatory solutions to Bessel's equation are the Hankel functions, of which the second Hankel function will be incoming at the horizon. Therefore the $T = 0$ solution can be written in terms of the spherical Hankel function of the second kind

$$\phi(r) = C\frac{h_2^{(2)}\left(\frac{\sqrt{D}}{r}\right)}{r^2}. \tag{3.35}$$

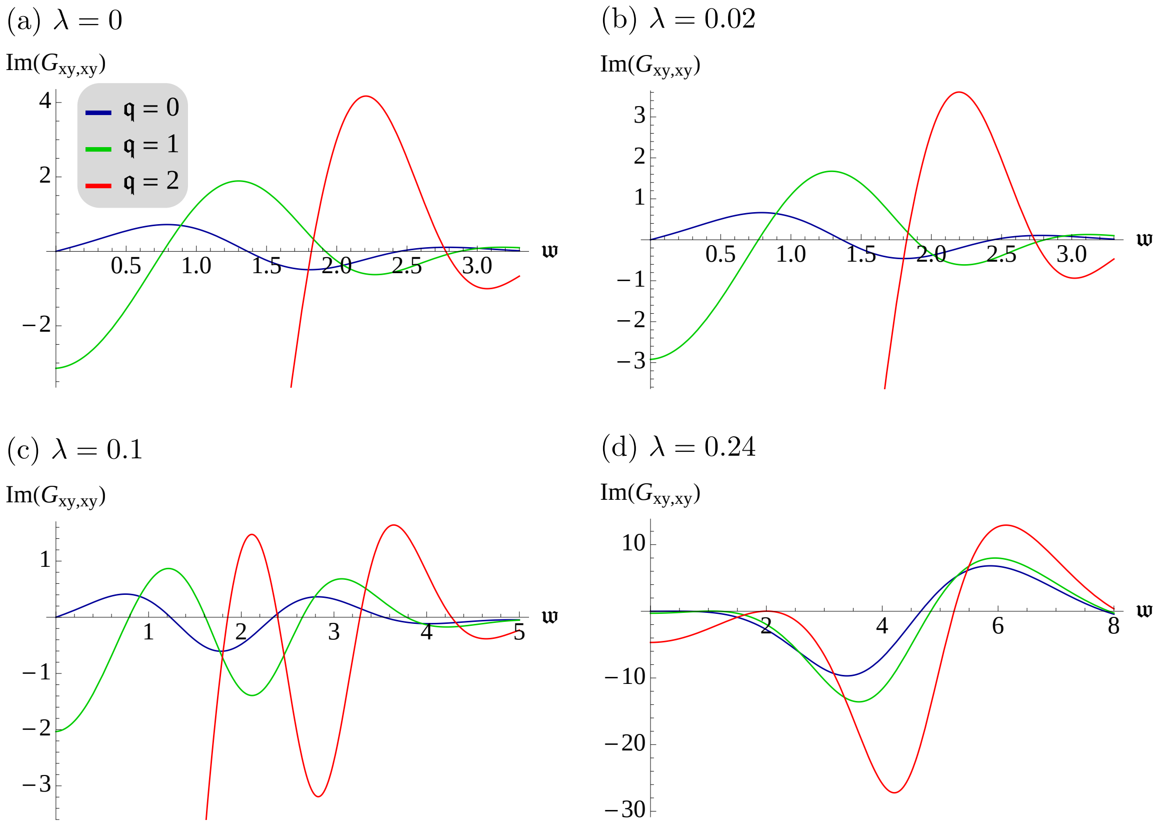


Figure 3.1: The imaginary part of the two-point function is shown for (a) $\lambda = 0$, (b) $\lambda = 0.02$, (c) $\lambda = 0.1$ and (d) $\lambda = 0.24$. In each figure blue represents $\mathfrak{q} = 0$, green corresponds to $\mathfrak{q} = 1$ and red is $\mathfrak{q} = 2$. Since the zero temperature contribution from Eq. 3.36 has been subtracted these plots represent the thermal excitations. The $\mathfrak{q} = 2$ curve has a finite negative value at small $\mathfrak{w}$ which is not shown in (a) - (c) since its amplitude is much larger than for $\mathfrak{q} = 0$ or $\mathfrak{q} = 1$. The presence of poles is indicated by the oscillatory behaviour of the function and the shift of the poles to higher frequencies with increasing λ and increasing $\mathfrak{q}$ is also evident.

Taking $r \to \infty$ the zero temperature two-point function is therefore

$$G^{T=0}_{xy,xy}(k) = \frac{1}{16GL^5\pi}\left(2A^2 - 1\right) A^3 k^4 \ln(|k^2|). \tag{3.36}$$

The coefficient of this result can be written in terms of the central charge c in agreement with [71]. The difference between the full two-point function in Eq. 3.29 and the zero temperature result in Eq. 3.36 is the thermal excitation which is shown in

Fig. 3.1 where the imaginary part of Eq. 3.36

$$\mathrm{Im}\left(G^{T=0}_{xy,xy}\right) = \frac{\pi\left(2A^2-1\right)A^3}{16GL^5\pi}\left(\mathfrak{w}^2-\mathfrak{q}^2\right)^2 \tag{3.37}$$

has been subtracted and we have set $16GL^5\pi = 1$. Several values of λ are used in Fig. 3.1 and we can see that the location of the poles in $G_{xy,xy}$, which are indicated by the oscillatory behaviour, change as λ is changed. In order to determine the exact location of the poles we must determine the quasinormal modes numerically, which is done in Sec. 3.3, however in the limit of large $\mathfrak{q}$ we can determine the pole structure by rewriting the equation of motion in Schrödinger form and using the WKB approximation. To do so let $\phi(v) = \frac{1}{\sqrt{K(v)}}\psi(v)$ in the linearized equation of motion (Eq. 3.7) which yields

$$\frac{1}{\mathfrak{q}^2}\psi'' = Q(v)\psi, \qquad Q(v) = \left[\frac{1}{\mathfrak{q}^2}\left(\frac{K''}{2K} - \frac{1}{4}\frac{K'^2}{K^2}\right) - \frac{1}{\mathfrak{q}^2}\frac{K_1}{K}\right]. \tag{3.38}$$

As $\mathfrak{q} \to \infty$ the second term in $Q(v)$ (which involves $\mathfrak{q}$ and $\mathfrak{w}$) dominates and we may drop the first term which leaves

$$Q(v) = \frac{1}{A^2\tilde{f}^2}(c^2(v) - \alpha^2), \qquad \alpha = \frac{\mathfrak{w}}{\mathfrak{q}}, \qquad c^2(v) = \frac{A^2\tilde{f}}{v^2}\frac{(1-\lambda\tilde{f}'')}{1-\frac{\lambda\tilde{f}}{v}}. \tag{3.39}$$

Asymptotically $Q(v)$ is

$$Q(v) = \frac{8\lambda^2\left(1-\alpha^2\right)}{(-1+\sqrt{1-4\lambda})^2(1+\sqrt{1-4\lambda})}\frac{1}{v^4} + \mathcal{O}\left(\frac{1}{v^8}\right) \tag{3.40}$$

so that $Q(v)$ approaches 0 as $v \to \infty$, and whether it is positive or negative at large v depends on the magnitude of α. We will see in Sec. 3.3 that α approaches 1 from above in the $\mathfrak{q} \to \infty$ limit so that $Q(v)$ is negative for large v. On the other hand at the horizon $Q(v) \to -\infty$. Therefore we will have a maximum in $Q(v)$ if at some v, $c^2(v) > \alpha^2 > 1$. Expanding $c^2(v)$ near the boundary as in [17]

$$c^2(v) = 1 - \left(\frac{5}{2} - \frac{2}{1-4\lambda} + \frac{1}{2\sqrt{1-4\lambda}}\right)\frac{1}{v^4} + \mathcal{O}\left(\frac{1}{v^8}\right) \tag{3.41}$$

we see that $c^2(v) > 1$ for $\lambda > \frac{9}{100}$. This means that $Q(v)$ can have a maximum for $\lambda > \frac{9}{100}$ if $\alpha > 1$. The location and height of the maximum will depend on the specific values of λ and α. The maximum in $Q(v)$ can be interpreted as a potential barrier which the excitation must propagate through to reach the horizon. The α^2 term then acts like the "energy" of the Schrödinger equation, although this term has a v dependence from $\tilde{f}(v)$. In Ch. 4 we will see that it is possible to change coordinates so that the v dependence is removed from the α term. In the limit of large $\mathfrak{q}$, ψ can

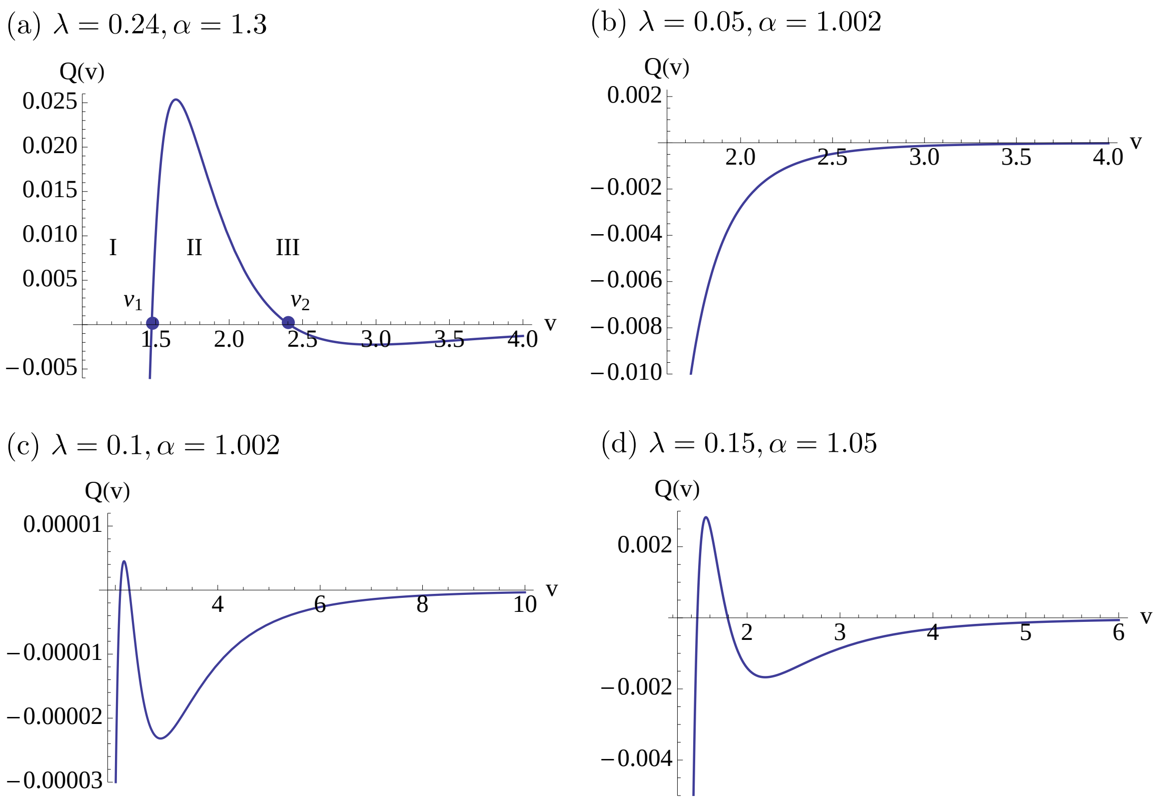


Figure 3.2: $Q(v)$ from Eq. 3.39 in the $\mathfrak{q} \to \infty$ limit. For $\lambda > \frac{9}{100}$ $Q(v)$ develops a maximum near the horizon with a height which is dependent on λ and α. This maximum may be interpreted as a potential barrier which has implications for a causality interpretation, although with this definition of $Q(v)$ the "energy" would be dependent on v. At the horizon, $v = 1$, $Q(v)$ is divergent. (a) For $\lambda = 0.24$ and $\alpha = 1.3$ the real zeros of $Q(v)$ are located at $v_1 = 1.47268$ and $v_2 = 2.41052$. Near these zeros $Q(v)$ has a first order zero. The three regions, I, II and III used for the WKB approximation are also shown. (b) For $\lambda = 0.05$ and $\alpha = 1.002$ there are no real zeros and $Q(v)$ is a monotonically increasing negative function. (c) For $\lambda = 0.1$ and $\alpha = 1.002$ the real zeros are located at $v_1 = 2.09745$ and $v_2 = 2.27405$ and (d) for $\lambda = 0.15$ and $\alpha = 1.05$ the real zeros are $v_1 = 1.46349$ and $v_2 = 1.79076$.

be found using a WKB approximation [77]

$$\psi(v) \sim \exp\left[\mathfrak{q}\sum_{n=0}^{\infty}\left(\frac{1}{\mathfrak{q}}\right)^n S_n(v)\right]. \tag{3.42}$$

Making a physical optics approximation (i.e. keeping the S_0 and S_1 terms in the WKB approximation) the solution in region I ($v < v_1$), II ($v_1 < v < v_2$), and III ($v_2 < v$) (see Fig. 3.2) will have the form

$$\psi(v) = C|Q(v)|^{-\frac{1}{4}}\exp\left[\mathfrak{q}\int_{v_0}^{v}\sqrt{Q(t)}dt\right] + D|Q(v)|^{-\frac{1}{4}}\exp\left[-\mathfrak{q}\int_{v_0}^{v}\sqrt{Q(t)}dt\right] \tag{3.43}$$

where v_0 is an arbitrary constant located in the appropriate region. The behaviour of the solution depends on the sign of $Q(v)$. For $Q(v) < 0$ the solution should be oscillatory (ingoing and outgoing solutions), while for $Q(v) > 0$ we expect an exponential solution (increasing and decreasing).

For certain values of α as shown in Fig. 3.2, $Q(v)$ may have two zeros, $v_1 < v_2$. Near the zeros the WKB approximation can no longer be used, but near v_1 and v_2 we can instead approximate $Q(v)$ by a Taylor series

$$Q(v) \sim (v - v_{1,2})\left.\frac{dQ}{dv}\right|_{v_{1,2}}. \tag{3.44}$$

Although we can only compute the turning points numerically for fixed α, for $v \neq 0$ $Q(v)$ has a first order 0. At $v = 0$ however $Q(v)$ is divergent, although this does not effect the expansion around the turning point since $v_1 > 0$. Using the series expansion near the zeros the equation for ψ becomes

$$\psi''(v) = \mathfrak{q}^2(v - v_{1,2})b_{1,2}\psi, \qquad b_{1,2} = \left.\frac{dQ}{dv}\right|_{v_{1,2}}. \tag{3.45}$$

Changing to coordinate $z = b_{1,2}^{\frac{1}{3}}\mathfrak{q}^{\frac{2}{3}}(v - v_{1,2})$ the ψ equation becomes $\psi''(z) = z\psi(z)$ and the solutions are Airy functions, $\psi(z) = AAi(z) + BB_i(z)$. At v_1, b_1 is positive so for $v < v_1$ z is negative and for $v_1 < v$ z is positive. Near v_2, b_2 is negative, so for $v < v_2$ z is positive and for $v_2 < v$ z is negative. So near the turning points, as $\mathfrak{q} \to \infty$, the argument of the Airy function approaches ∞ or $-\infty$ and we may use the asymptotic expansions for $z \to \infty$

$$A_i(z) \sim \frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}}\exp\left[-\frac{2}{3}z^{\frac{3}{2}}\right] = \frac{b_{1,2}^{-\frac{1}{12}}}{2\sqrt{\pi}}\mathfrak{q}^{-\frac{1}{6}}(v-v_{1,2})^{-\frac{1}{4}}\exp\left[-\frac{2}{3}b_{1,2}^{\frac{1}{2}}\mathfrak{q}(v-v_{1,2})^{\frac{3}{2}}\right],$$
$$B_i(z) \sim \frac{z^{-\frac{1}{4}}}{\sqrt{\pi}}\exp\left[-\frac{2}{3}z^{\frac{3}{2}}\right] = \frac{b_{1,2}^{-\frac{1}{12}}}{\sqrt{\pi}}\mathfrak{q}^{-\frac{1}{6}}(v-v_{1,2})^{-\frac{1}{4}}\exp\left[\frac{2}{3}b_{1,2}^{\frac{1}{2}}\mathfrak{q}(v-v_{1,2})^{\frac{3}{2}}\right] \tag{3.46}$$

and for $z \to -\infty$

$$\begin{aligned}A_i(z) &\sim \frac{1}{\sqrt{\pi}}(-z)^{-\frac{1}{4}}\sin\left[\frac{2}{3}(-z)^{\frac{3}{2}}+\frac{\pi}{4}\right]\\ &= \frac{1}{\sqrt{\pi}}|b_{1,2}|^{-\frac{1}{12}}\mathfrak{q}^{-\frac{1}{6}}|v-v_{1,2}|^{-\frac{1}{4}}\sin\left[\frac{2}{3}|b_{1,2}|^{\frac{1}{2}}\mathfrak{q}|v-v_{1,2}|^{\frac{3}{2}}+\frac{\pi}{4}\right],\\ B_i(z) &\sim \frac{1}{\sqrt{\pi}}(-z)^{-\frac{1}{4}}\cos\left[\frac{2}{3}(-z)^{\frac{3}{2}}+\frac{\pi}{4}\right]\\ &= \frac{1}{\sqrt{\pi}}|b_{1,2}|^{-\frac{1}{12}}\mathfrak{q}^{-\frac{1}{6}}|v-v_{1,2}|^{-\frac{1}{4}}\cos\left[\frac{2}{3}|b_{1,2}|^{\frac{1}{2}}\mathfrak{q}|v-v_{1,2}|^{\frac{3}{2}}+\frac{\pi}{4}\right].\end{aligned} \tag{3.47}$$

Expanding the WKB solution for linear $Q(v)$ the form of the solution is

$$\begin{aligned}\psi(v) = C|b_{1,2}|^{-\frac{1}{4}}|v-v_{1,2}|^{-\frac{1}{4}}&\exp\left[\frac{2}{3}\mathfrak{q}b_{1,2}^{\frac{1}{2}}(v-v_{1,2})^{\frac{3}{2}}\right]\\ &+ D|b_{1,2}|^{-\frac{1}{4}}|v-v_{1,2}|^{-\frac{1}{4}}\exp\left[-\frac{2}{3}\mathfrak{q}b_{1,2}^{\frac{1}{2}}(v-v_{1,2})^{\frac{3}{2}}\right].\end{aligned} \tag{3.48}$$

At each turning point there is an overlapping region of validity of the WKB and the Airy function solution [77], and we can match the coefficients. We will denote the C, D coefficients in each region using subscripts I, II and III and the A, B coefficients with subscripts $1, 2$ depending on which zero we are expanding around. Between the horizon and the first zero, v_1, the solution should be incoming, corresponding to the incoming condition on ϕ so we set $C_{\mathrm{I}} = 0$. Therefore near v_1 for $v < v_1$

$$\psi_{\mathrm{I}} = D_{\mathrm{I}}|b_1|^{-\frac{1}{4}}|v-v_1|^{-\frac{1}{4}}\exp\left[i\frac{2}{3}\mathfrak{q}|v-v_1|^{\frac{3}{2}}\right] \tag{3.49}$$

and

$$\psi_{\mathrm{I}} = A_1 \frac{|b_1|^{-\frac{1}{12}}}{\sqrt{\pi}} \mathfrak{q}^{-\frac{1}{6}} |v - v_1|^{-\frac{1}{4}} \sin\left[\frac{2}{3}|b|^{\frac{1}{2}}\mathfrak{q}|v - v_1|^{\frac{3}{2}} + \frac{\pi}{4}\right]$$
$$+ B_1 \frac{|b_1|^{-\frac{1}{12}}}{\sqrt{\pi}} \mathfrak{q}^{-\frac{1}{6}} |v - v_1|^{-\frac{1}{4}} \cos\left[\frac{2}{3}|b|^{\frac{1}{2}}\mathfrak{q}|v - v_1|^{\frac{3}{2}} + \frac{\pi}{4}\right] \tag{3.50}$$

so

$$A_1 = iB_1, \qquad B_1 = \left(\frac{\mathfrak{q}}{|b_1|}\right)^{\frac{1}{6}} \sqrt{\pi} e^{-i\frac{\pi}{4}} D_{\mathrm{I}}. \tag{3.51}$$

Near v_1 for $v > v_1$

$$\psi_{\mathrm{II}1} = C_{\mathrm{II}} |b_1|^{-\frac{1}{4}} |v - v_1|^{-\frac{1}{4}} \exp\left[\frac{2}{3}\mathfrak{q} b_1^{\frac{1}{2}} (v - v_1)^{\frac{3}{2}}\right]$$
$$+ D_{\mathrm{II}} |b_1|^{-\frac{1}{4}} |v - v_1|^{-\frac{1}{4}} \exp\left[-\frac{2}{3}\mathfrak{q} b_1^{\frac{1}{2}} (v - v_1)^{\frac{3}{2}}\right]$$
$$\psi_{\mathrm{II}1} = \frac{A_1}{2\sqrt{\pi}} |b_1|^{-\frac{1}{12}} \mathfrak{q}^{-\frac{1}{6}} |v - v_1|^{-\frac{1}{4}} \exp\left[-\frac{2}{3} b_1^{\frac{1}{2}} \mathfrak{q} (v - v_1)^{\frac{3}{2}}\right]$$
$$+ \frac{B_1}{\sqrt{\pi}} |b_1|^{-\frac{1}{12}} \mathfrak{q}^{-\frac{1}{6}} |v - v_1|^{-\frac{1}{4}} \exp\left[\frac{2}{3} b_1^{\frac{1}{2}} \mathfrak{q} (v - v_1)^{\frac{3}{2}}\right] \tag{3.52}$$

which gives

$$C_{\mathrm{II}} = \frac{B_1}{\sqrt{\pi}} \left(\frac{|b_1|}{\mathfrak{q}}\right)^{\frac{1}{6}}, \qquad D_{\mathrm{II}} = \frac{A_1}{2\sqrt{\pi}} \left(\frac{|b_1|}{\mathfrak{q}}\right)^{\frac{1}{6}}. \tag{3.53}$$

Near v_2 for $v < v_2$ the expansion has the same form as for $v > v_1$ since $z > 0$ so

$$C_{\mathrm{II}} = \frac{B_2}{\sqrt{\pi}} \left(\frac{|b_2|}{\mathfrak{q}}\right)^{\frac{1}{6}}, \qquad D_{\mathrm{II}} = \frac{A_2}{2\sqrt{\pi}} \left(\frac{|b_2|}{\mathfrak{q}}\right)^{\frac{1}{6}} \tag{3.54}$$

and combining Eq. 3.54 with Eq. 3.53 and Eq. 3.51 we have

$$B_2 = \left(\frac{\mathfrak{q}}{|b_2|}\right)^{\frac{1}{6}} \sqrt{\pi} e^{-i\frac{\pi}{4}} D_{\mathrm{I}}, \qquad A_2 = i\left(\frac{\mathfrak{q}}{|b_2|}\right)^{\frac{1}{6}} \sqrt{\pi} e^{-i\frac{\pi}{4}} D_{\mathrm{I}}. \tag{3.55}$$

Now for $v > v_2$

$$\psi_{\mathrm{III}} = C_{\mathrm{III}} |b_2|^{-\frac{1}{4}} |v - v_2|^{-\frac{1}{4}} \exp\left[-\frac{2i}{3}\mathfrak{q}|b_2|^{\frac{1}{2}} (v - v_2)^{\frac{3}{2}}\right]$$
$$+ D_{\mathrm{III}} |b_2|^{-\frac{1}{4}} |v - v_2|^{-\frac{1}{4}} \exp\left[\frac{2i}{3}\mathfrak{q}|b_2|^{\frac{1}{2}} (v - v_2)^{\frac{3}{2}}\right] \tag{3.56}$$

and using Eq. 3.55

$$\psi_{\rm III} = D_{\rm I}|b_2|^{-\frac{1}{4}}|v-v_2|^{-\frac{1}{4}}\exp\left[\frac{2i}{3}|b_2|^{\frac{1}{2}}\mathfrak{q}|v-v_2|^{\frac{3}{2}}\right] \tag{3.57}$$

therefore

$$C_{\rm III} = 0, \qquad D_{\rm III} = D_{\rm I} \tag{3.58}$$

and the WKB solution near the boundary has the form

$$\psi = D|Q(v)|^{-\frac{1}{4}}\exp\left[-i\mathfrak{q}\int_{v_2}^{v}\sqrt{|Q(t)|}dt\right]. \tag{3.59}$$

In terms of ψ the two-point function becomes

$$G_{xy,xy}(k) = \frac{a^4 A}{16GL^5\pi}\left[-\frac{1}{2}K' + K\frac{\partial}{\partial v}\ln(\psi)\right]\Bigg|_{\rm boundary} \tag{3.60}$$

and therefore expanding ψ near the boundary and dropping contact terms,

$$G_{xy,xy}(k) = \frac{a^4 A}{16GL^5\pi}\left(\frac{d_1(\lambda)\mathfrak{w}^2 + d_2(\lambda)\mathfrak{q}^2}{\mathfrak{w}^2-\mathfrak{q}^2}\right), \tag{3.61}$$

$$d_1(\lambda) = \frac{4-\sqrt{1-4\lambda}-16\lambda}{2(1-4\lambda)}, \qquad d_2(\lambda) = \frac{6-7\sqrt{1-4\lambda}-24\lambda}{2(1-4\lambda)}. \tag{3.62}$$

We see that the pole structure is $(\mathfrak{w}^2-\mathfrak{q}^2)^{-1}$, so that at large momenta there is a lightlike pole. This result agrees with the large momentum behaviour found on the field theory side in [21]. This behaviour is also seen in the Einstein gravity case either from the asymptotic behaviour of the quasinormal modes using a Bohr-Sommerfeld quantization [78] or from numerical calculation of the quasinormal modes at large momenta [79].

3.3 Quasinormal Modes

In order to explicitly determine if there are any causality violating modes for $\lambda > \frac{9}{100}$ we would like to find the dispersion relations for the poles we see in the two-point function by finding the quasinormal modes. To find these modes we could use our series solution at the horizon and require $\phi(0) = 0$ by setting $\sum_{n=0}^{\infty} a_n = 0$ with a_n given in Eq. 3.15. However we can obtain a better numerical result if we set the boundary series to zero at $u = 0$ and find the frequencies which allow us to match the two solutions at some $0 < u_0 < 1$. Since the boundary solution has the form $\phi_b = A\phi_1 + B\phi_2$ with ϕ_1 and ϕ_2 given in Eq. 3.17 this means we should set $A = 0$ and require that the quasinormal modes satisfy

$$\phi_h(u_0)\phi_2'(u_0) - \phi_2(u_0)\phi_h'(u_0) = 0. \tag{3.63}$$

where ϕ_h is the series solution at the horizon. Solving Eq. 3.63 numerically we find the quasinormal frequencies shown in Table 3.1. The accuracy of these solutions is limited by the number of terms kept in the series solutions for ϕ. Therefore to determine the quasinormal frequencies a fixed number of terms m in the series solutions was kept and the modes were computed. m was then incremented and the number of digits which did not change are reported in Table 3.1 for the 4 lowest quasinormal modes for several values of λ. From a practical viewpoint, $m = 23$ provides reasonable accuracy within a reasonable time frame, however for $\mathrm{Re}(\mathfrak{w}) \sim \mathrm{Im}(\mathfrak{w}) \sim 5$ the solutions become less accurate as indicated by only one or two decimal places being kept for the 4th quasinormal mode. Near the limit $\lambda = \frac{1}{4}$ the quasinormal modes seem to increase rapidly resulting in fewer modes being found within the $\mathrm{Re}(\mathfrak{w}) < 5$ window. In principle we could keep more coefficients in the series to obtain arbitrary accuracy, however the oscillatory nature of the solutions makes accurate numerical computation more difficult.

Table 3.1: The 4 lowest quasinormal modes of the planar EGB black hole for several values of $\mathfrak{q}$ and λ.

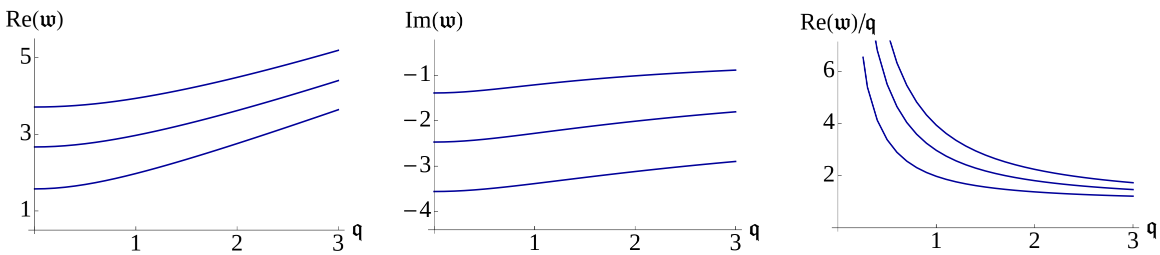
$\mathfrak{q} = 0$						
	$\lambda = 0$		$\lambda = 0.02$		$\lambda = 0.05$	
n	$\mathrm{Re}(\mathfrak{w})$	$\mathrm{Im}(\mathfrak{w})$	$\mathrm{Re}(\mathfrak{w})$	$\mathrm{Im}(\mathfrak{w})$	$\mathrm{Re}(\mathfrak{w})$	$\mathrm{Im}(\mathfrak{w})$
1	±1.559722	-1.37334	±1.557912	-1.378782	±1.575096	-1.389473
2	±2.5847	-2.3817	±2.5942	-2.4031	±2.6726	-2.4690
3	±3.593	-3.383	±3.62	-3.43	±3.79	-3.57
4	±4.603	-4.3	±4.67	-4.4	±4.8	-4.5
	$\lambda = 0.1$		$\lambda = 0.15$		$\lambda = 0.24$	
n	$\mathrm{Re}(\mathfrak{w})$	$\mathrm{Im}(\mathfrak{w})$	$\mathrm{Re}(\mathfrak{w})$	$\mathrm{Im}(\mathfrak{w})$	$\mathrm{Re}(\mathfrak{w})$	$\mathrm{Im}(\mathfrak{w})$
1	±1.664800	-1.428213	±1.865027	-1.505988	±2.90	-3.72
2	±2.944	-2.6655	±3.376	-2.981		
3	±4.24	-3.93	±4.84	-4.44		
4	±5.4	-5.0	±5.3	-5.8		

$\mathfrak{q} = 1$						
	$\lambda = 0$		$\lambda = 0.02$		$\lambda = 0.05$	
n	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)
1	±1.954326	-1.267316	±1.952732	-1.24403	±1.976412	-1.21054
2	±2.8802	-2.2979	±2.8942	-2.2723	±2.9752	-2.2752
3	±3.836	-3.313	±3.87	-3.29	±4.015	-3.37
4	±4.80	-4.32	±4.87	-4.3	±4.9	-4.4
	$\lambda = 0.1$		$\lambda = 0.15$		$\lambda = 0.24$	
n	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)
1	±2.07525	-1.17787	±2.258812	-1.192735	±3.14	-3.47
2	±3.1927	-2.3973	±3.532	-2.703		
3	±4.36	-3.70	±4.9	-4.28		
4	±5.4	-4.9	±5.4	-5.7		

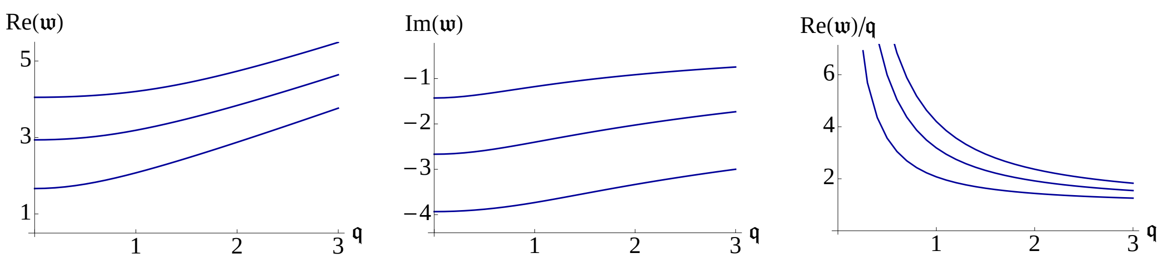
$\mathfrak{q} = 2$						
	$\lambda = 0$		$\lambda = 0.02$		$\lambda = 0.05$	
n	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)
1	±2.741904	-1.1204	±2.73460	-1.08087	±2.757559	-1.012625
2	±3.5314	-2.1331	±3.5365	-2.068	±3.6196	-2.0060
3	±4.39	-3.15	±4.42	-3.078	±4.55	-3.06
4	±5.2	-4.1	±5.3	-4.0	±5.4	-4.1
	$\lambda = 0.1$		$\lambda = 0.15$		$\lambda = 0.24$	
n	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)	Re($\mathfrak{w}$)	Im($\mathfrak{w}$)
1	±2.87494	-0.915188	±3.08	-0.859937	±3.20	-3.22
2	±3.8358	-2.016	±4.12	-2.176		
3	±4.84	-3.25	±5.1	-3.72		
4	±5.7	-4.5	±5.8	-5.1		

The dispersion relations found using the matching condition in Eq. 3.63 are shown in Fig. 3.3. As $\mathfrak{q}$ increases, $\mathrm{Re}(\mathfrak{w})$ seems to become linear in $\mathfrak{q}$ so that $\mathrm{Re}(\mathfrak{w})/\mathfrak{q}$ approaches 1. $\mathrm{Im}(\mathfrak{w})$ also increases in this limit, although up to $\mathfrak{q} = 3$ where the computation is most reliable, $\mathrm{Im}(\mathfrak{w})$ is still a negative number of order 1. Since the scalar channel does not contain a hydrodynamic mode, $\mathrm{Re}(\mathfrak{w})/\mathfrak{q}$ diverges in the limit $\mathfrak{q} \to 0$.

(a) $\lambda = 0.05$



(b) $\lambda = 0.1$



(c) $\lambda = 0.24$

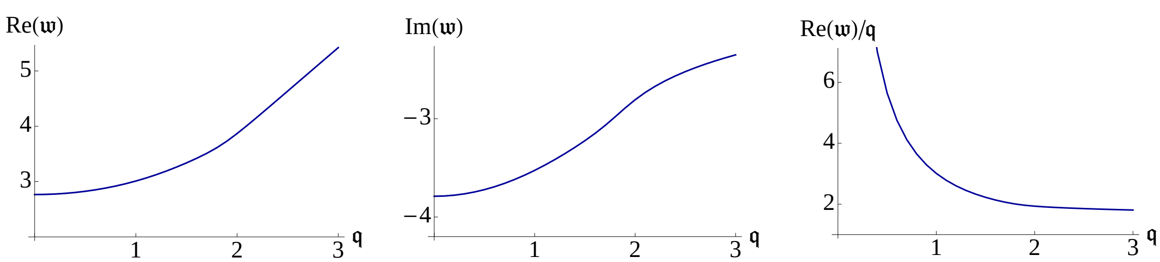


Figure 3.3: Dispersion relation for $\mathrm{Re}(\mathfrak{w})$, $\mathrm{Im}(\mathfrak{w})$ and the phase velocity $\mathrm{Re}(\mathfrak{w})/\mathfrak{q}$ calculated using the matching condition in Eq. 3.63 for (a) $\lambda = 0.05$, (b) $\lambda = 0.1$ and (c) $\lambda = 0.24$. Although we refer to $\mathrm{Re}(\mathfrak{w})/\mathfrak{q}$ as the phase velocity, this interpretation is really valid only as $\mathrm{Im}(\mathfrak{w}) \to 0$. (a) and (b) show the three lowest modes, however in (c) for $\lambda = 0.24$ the quasinormal modes have increased and spread out enough that only the lowest mode can be computed reliably using the matching condition. In each case $\mathrm{Im}(\mathfrak{w})$ is approaching 0 and $\mathrm{Re}(\mathfrak{w})$ is approaching $\mathfrak{q}$ at large $\mathfrak{q}$.

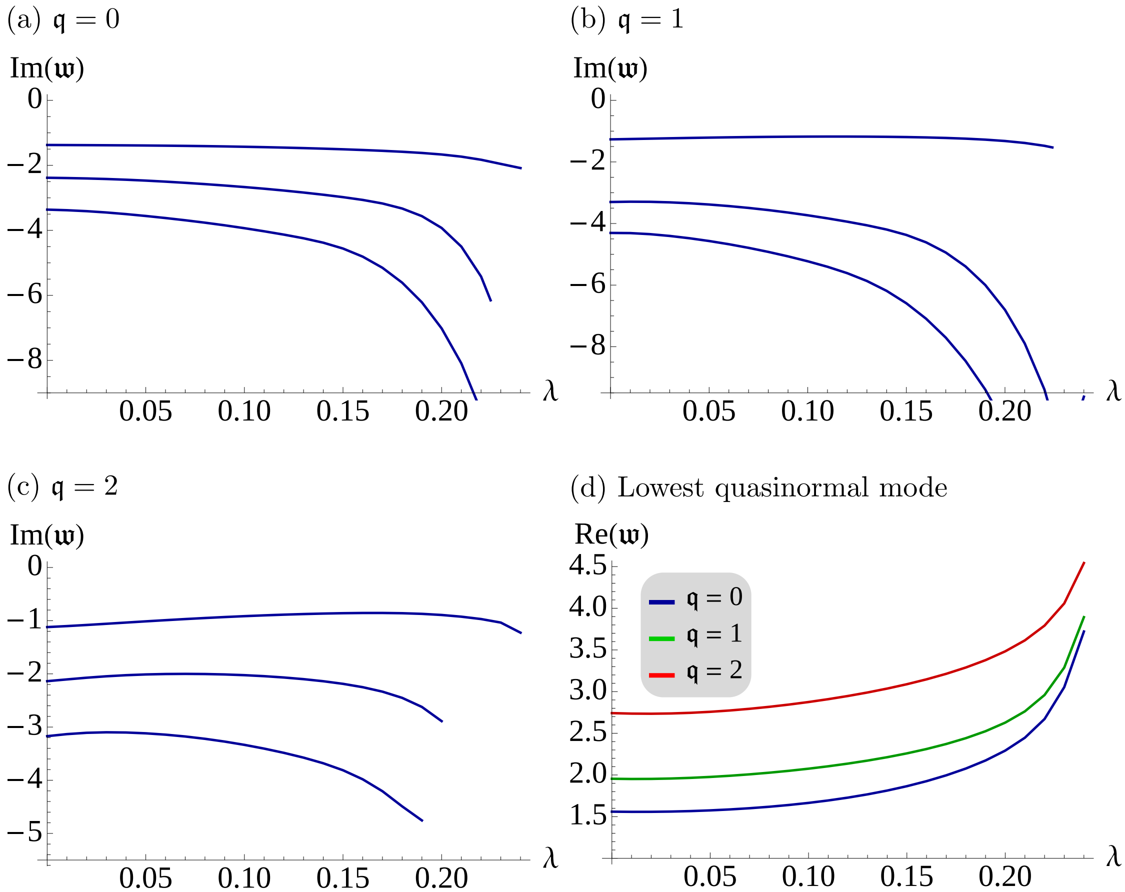


Figure 3.4: λ dependence of 3 lowest quasinormal modes. The imaginary part of $\mathfrak{w}$ is shown for (a) $\mathfrak{q} = 0$, (b) $\mathfrak{q} = 1$ and (c) $\mathfrak{q} = 2$. Im($\mathfrak{w}$) generally decreases with λ although as $\mathfrak{q}$ increases, there is a region at small λ where Im($\mathfrak{w}$) increases. (d) Shows the real part of $\mathfrak{w}$ for $\mathfrak{q} = 0$ (blue), $\mathfrak{q} = 1$ (green) and $\mathfrak{q} = 2$ (red). Re($\mathfrak{w}$) increases monotonically with increasing λ and with increasing $\mathfrak{q}$.

Using the same method of solution, the dependence of the 3 lowest quasinormal modes on λ is shown in Fig. 3.4. The imaginary part of $\mathfrak{w}$ generally decreases with increasing λ although it seems that for larger $\mathfrak{q}$ there is a region of small λ where Im($\mathfrak{w}$) increases slightly. Im($\mathfrak{w}$) is also increasing with $\mathfrak{q}$. The real part of $\mathfrak{w}$ is increasing with λ and with $\mathfrak{q}$. We also see that the real (imaginary) part of $\mathfrak{w}$ increases (decreases) rapidly as we approach the limit $\lambda = \frac{1}{4}$.

In this chapter we have numerically computed the two-point function of the stress-energy tensor as well as several of the quasinormal modes and have examined the dependence of the quasinormal modes on λ and $\mathfrak{q}$. We have also used the WKB approximation to determine that there is a lightlike pole in the two-point function in the $\mathfrak{q} \to \infty$ limit. The behaviour of the quasinormal modes will be used in the next chapter to analyze the causality violation of these modes on the boundary.

Chapter 4

Causality Violation

In this chapter we turn to the question of causality violation with the goal of explicitly finding causality violating modes in the scalar channel analogous to those in the sound channel [19] and clarifying the arguments for these modes made in [18, 19]. To determine how to characterizes these modes we can follow [18] and again rewrite the linearized equation of motion in Schrödinger form using a transformation to coordinate y and field ψ

$$\frac{dy}{dv} = \frac{1}{A\tilde{f}(v)} \qquad \phi(v) = B(v)\psi(v) \qquad B(v) = \sqrt{\frac{\tilde{f}(v)}{K(v)}}. \tag{4.1}$$

The equation of motion can then be written as

$$-\psi''(y) + V(y)\psi(y) = 4\mathfrak{w}^2\psi(y) \tag{4.2}$$

with

$$V(v) = 4\mathfrak{q}^2 c^2(v) - V_1(v) \qquad V_1(v) = Af(v)\frac{d}{dv}\left(\frac{Af(v)B'(v)}{B(v)}\right) - \left(\frac{Af(v)B'(v)}{B(v)}\right)^2 \tag{4.3}$$

$$c^2(v) = \frac{A^2\tilde{f}(v)}{v^2}\frac{1-\lambda\tilde{f}''(v)}{1-\frac{\lambda\tilde{f}'(v)}{v}}. \tag{4.4}$$

In chapter 3 we also considered the equation of motion in Schrödinger form, but there are key differences between this equation and the one considered earlier. Previously we had an "energy" term α^2 which would depend on the position v. By transforming to the new coordinate y, this v dependence is removed, which allows us to explicitly separate the "energy" from the potential $V(y)$. However in this form, y is defined in terms of its v derivative and can only be determined numerically at fixed v. Therefore using this new equation we could not compute the integral in the exponential of the WKB approximation in order to determine the pole structure at large momenta as we did in the previous chapter. However the ability to determine the potential explicitly, which is shown in Fig. 4.1 for different values of $\mathfrak{q}$ and λ, allows conditions for causality violation to be determined. At large $\mathfrak{q}$ the V_1 term becomes unimportant and the potential develops a maximum near the horizon when $\lambda > \frac{9}{100}$ due to the behaviour of $c^2(v)$ as discussed earlier. It is possible then to have a disturbance on the boundary which propagates through the bulk and is unable to tunnel through the barrier to fall into the black hole. If the velocity of the excitation exceeds $c = 1$ it may then be possible for the disturbance to violate causality by travelling through the bulk. The phase velocity of this disturbance, determined from the dispersion relation, is $\mathrm{Re}(\mathfrak{w})/\mathfrak{q}$, however we can only interpret this as a velocity if $\mathrm{Im}(\mathfrak{w}) \ll \mathrm{Re}(\mathfrak{w})$. If this inequality is not satisfied the mode is damped over a distance on the order of a wavelength and we cannot say that we have a propagating mode at all. The small imaginary part is also important so that the tunnelling rate is sufficiently small to allow the excitation to spend enough time in the region where $c > 1$. Therefore to interpret the existence of causality violating modes in this way we must satisfy the three conditions: $V(v)$ has a potential barrier, $\mathrm{Im}(\mathfrak{w}) \ll \mathrm{Re}(\mathfrak{w})$ and $\mathrm{Re}(\mathfrak{w})/\mathfrak{q} > 1$.

The dispersion relations computed numerically are shown in Fig. 3.3. We see there that in the large $\mathfrak{q}$ limit, the phase velocity approaches 1 as required, however since we can only reliably calculate up to $\mathfrak{q}$ of order 1, we do not have $\mathrm{Im}(\mathfrak{w}) \ll \mathrm{Re}(\mathfrak{w})$, although in the range of $\mathfrak{q}$ shown in Fig. 3.3, the imaginary part is increasing as $\mathfrak{q}$ increases as we require. Therefore it seems that in the $\mathfrak{q} \to \infty$ limit the phase velocity approaches 1 from above and $\mathrm{Im}(\mathfrak{w}) \to 0$. This agrees with the lightlike pole found as $\mathfrak{q} \to \infty$ and with the asymptotic results for $\lambda = 0$ [78, 79]. It seems then that all three conditions are met for $\lambda > \frac{9}{100}$, although to explicitly verify that $\mathrm{Im}(\mathfrak{w}) \ll \mathrm{Re}(\mathfrak{w})$ the quasinormal modes still need to be calculated using a different approach. It is important to note that from the quasinormal mode calculation, even for $\lambda \leq \frac{9}{100}$ the requirements of the phase velocity and $\mathrm{Im}(\mathfrak{w})$ may be met, however the potential will not develop a maximum unless $\lambda > \frac{9}{100}$.

It is interesting to contrast the behaviour of the quasinormal modes in the scalar channel with their behaviour in the sound channel studied in [19]. In the sound channel there is a hydrodynamic mode so that for the lowest quasinormal mode the

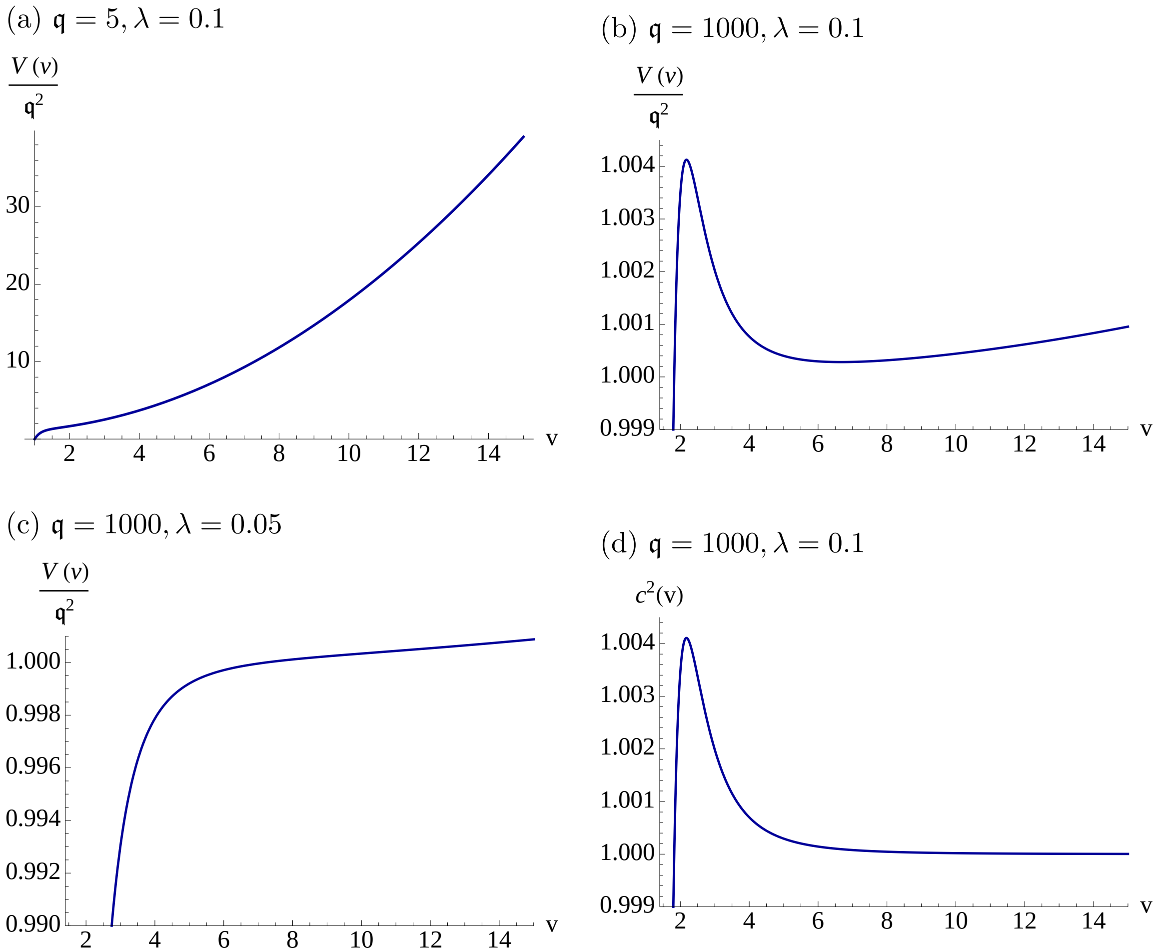


Figure 4.1: The effective Schrödinger potential, $\left[\mathfrak{q}^2 c^2(v) + V_1(v)\right]/\mathfrak{q}^2$ from the equation of motion for the metric perturbation. (a) Using $\mathfrak{q} = 5$ and $\lambda = 0.1$ as an example we see that when $\mathfrak{q}$ is small there is no maximum that develops, even for $\lambda > \frac{9}{100}$ because the $V_1(v)$ term dominates. (b) For $\mathfrak{q} = 1000$ the $V_1(v)$ term is negligible and a potential barrier develops near the horizon when $\lambda \geq \frac{9}{100}$ but (c) is not present for small λ. (d) The potential without $V_1(v)$. Contrasting with (b) we see that $V_1(v)$ will dominate at large v. $V_1(v)$ in fact diverges at the boundary, however $V_1(v)$ is suppressed by $\frac{1}{\mathfrak{q}^2}$ so in the limit of $\mathfrak{q} \to \infty$, this term will only contributes at the boundary.

velocity is finite as $\mathfrak{q} \to 0$. As $\mathfrak{q}$ increases so does the velocity, reaching a maximum near $\mathfrak{q} \sim 0.3$ and then approaching 1 from above as $\mathfrak{q} \to \infty$. The imaginary part of the frequency also goes to 0 at small $\mathfrak{q}$ and reaches a minimum before again approaching 0 in the limit $\mathfrak{q} \to \infty$. However in the sound channel the imaginary part of the frequency becomes smaller much faster than in the scalar channel, and at $\mathfrak{q} \sim 1$, $\mathrm{Re}(\mathfrak{w}) \sim 100$

Im($\mathfrak{w}$). This seems to indicate that we already have causality violated modes in the sound channel at very small momenta. However we also need the potential to have the correct form. To see what this requires we can examine the general form of the equation of motion for any of the channels, which is

$$\phi_s''(v) + C_1(v, \mathfrak{w}, \mathfrak{q})\phi_s'(v) + C_2(v, \mathfrak{w}, \mathfrak{q})\phi_s(v) = 0. \tag{4.5}$$

In order to write this equation in Schrödinger form we can change to a new coordinate y and make a change of field $\phi = B\psi$. The equation of motion then becomes

$$\frac{1}{\mathfrak{q}^2}\frac{d^2\psi(y)}{dy^2} + \frac{1}{\mathfrak{q}^2}\left[\frac{d}{dy}\left(\frac{B'}{B}\right) - \left(\frac{B'}{B}\right)^2 + \frac{C_2}{\left(dy/dv\right)^2}\right]\psi(y) = 0 \tag{4.6}$$

where primes are now derivatives with respect to y and B must be chosen to satisfy

$$\frac{B'}{B} = -\frac{1}{2}\left[\frac{d^2y/dv^2}{\left(dy/dv\right)^2} + \frac{C_1}{dy/dv}\right] \tag{4.7}$$

so that the ψ' coefficient is 0. In general C_2 and C_1 are functions of $\mathfrak{q}$ and $\mathfrak{w}$ which we can write as functions of α since we have divided by an overall factor of $\mathfrak{q}$. Therefore in order to have an "energy" which is independent of v we must choose dy/dv to cancel the v dependence of any term with α. In the case of the scalar channel equation of motion, only C_2 depends on $\mathfrak{q}$ and $\mathfrak{w}$ which allows us to separate the $\mathfrak{w}$ dependence from the potential at any momenta. The remaining potential then has a $1/\mathfrak{q}^2$ term which in negligible for large momenta. This means that even at finite momenta, the potential is independent of $\mathfrak{w}$ allowing us to interpret the mode as moving in a potential depending only on λ and v and possibly $\mathfrak{q}$. On the other hand, for both the shear and sound channels, C_1 and C_2 are complicated functions of $\mathfrak{w}$ and $\mathfrak{q}$ and in general the v dependence of the $\mathfrak{w}$ dependent terms cannot be removed. This means that the potential depends on both $\mathfrak{w}$ and $\mathfrak{q}$ and we cannot easily make the interpretation that the excitation is moving in some potential. Only in the $\mathfrak{q} \to \infty$ limit can the $\mathfrak{w}$ dependence be removed, so that the potential depends only on λ and v, and for sufficiently small λ ($\lambda < -7/36$ for the sound channel and $\lambda < -3/4$ for the shear channel [19]) the potential develops a maximum. Therefore even though we have $\text{Im}(\mathfrak{w}) \ll \text{Re}(\mathfrak{w})$ at $\mathfrak{q} \sim 1$ we still cannot say that we have causality violation until $\mathfrak{q} \to \infty$. In the scalar channel while we are also required to take $\mathfrak{q} \to \infty$ we must do so in order to obtain a small imaginary frequency as well.

In this chapter we have seen that three conditions are required in order to have causality violating modes. We require a maximum to develop in the potential when the equation of motion is written in Schrödinger form, that $\mathrm{Re}(\mathfrak{w})/\mathfrak{q} > 1$ and that $\mathrm{Re}(\mathfrak{w}) \gg \mathrm{Im}(\mathfrak{w})$. Based on the quasinormal mode dispersion relation and the form of the potential, these conditions are met for $\lambda > \frac{9}{100}$ and $\mathfrak{q} \to \infty$ and therefore we explicitly see causality violating modes.

Chapter 5

Conclusions

By studying the causality of field theories dual to higher derivative gravities it is possible to constrain the higher derivative couplings providing a means other than symmetry considerations to restrict allowed interactions. In the case of Einstein-Gauss-Bonnet gravity it has been argued that causality violation exists for a gravitational coupling, $\lambda > \frac{9}{100}$, which also places a limit on the maximum viscosity bound violation. From the viewpoint of effective theories, where the list of interactions we could consider is endless, additional constraints are certainly desirable. However the dispersion relations of the modes which lead to this causality violation had not been explicitly computed.

The goal of this project was to verify that causality violating modes exist and can be appropriately interpreted in the scalar channel of metric perturbations. The scalar channel places an upper bound on λ, unlike the shear and sound channels which provide lower bounds. To find the causality violating modes we used the AdS/CFT correspondence to numerically compute the two-point function of the stress-energy tensor in a thermal state dual to a planar Einstein-Gauss-Bonnet black hole. The pole structure of the two-point function was found in the $\mathfrak{q} \to \infty$ limit from the gravitational side of the correspondence for the first time using a WKB approach. We also found the quasinormal mode spectrum of the planar black hole in Einstein-Gauss-Bonnet gravity, which according to the gauge/gravity duality corresponds to the poles of the two-point function and therefore characterizes the excitations. These quasinormal modes were found by applying Dirchlet boundary conditions at the AdS boundary and matching the Froebenius solutions from the boundary and the horizon. Using this method the dependence of the quasinormal modes on both the spatial momentum $\mathfrak{q}$ and the Gauss-Bonnet coupling λ was computed.

We have also clarified the three necessary conditions in order to interpret the excitations of the two-point function as causality violating modes: the Schrödinger potential must have a maximum near the horizon, the phase velocity must exceed 1, $(\mathrm{Re}(\mathfrak{w})/\mathfrak{q}) > 1)$, and $\mathrm{Re}(\mathfrak{w}) \gg \mathrm{Im}(\mathfrak{w})$ so that the excitation can be interpreted as a propagating mode. The quasinormal modes that were calculated seem to support the condition $\mathrm{Re}(\mathfrak{w}) \gg \mathrm{Im}(\mathfrak{w})$ in the limit $\mathfrak{q} \to \infty$ since $\mathrm{Im}(\mathfrak{w})$ tends toward 0 and $\mathrm{Re}(\mathfrak{w})$ toward $\mathfrak{q}$ in this limit. This result is also supported by the presence of a lightlike pole in the two-point function as $\mathfrak{q} \to \infty$. However despite the evidence supporting the behaviour of the quasinormal modes, in order to numerically compute the quasinormal modes in the limit of $\mathfrak{q} \to \infty$ further work must be done employing an alternative to the Froebenius method.

In addition to computing the quasinormal modes and the asymptotic behaviour of the two-point function, we also contrasted the behaviour of the Schrödinger potential in the scalar channel with the sound channel, again clarifying the necessary conditions for causality violating modes. In the sound channel the condition $\mathrm{Re}(\mathfrak{w}) \gg \mathrm{Im}(\mathfrak{w})$ is met already for a $\mathfrak{q}$ of order 1 however there the potential cannot be reliably interpreted outside of the $\mathfrak{q} \to \infty$ limit due to its dependence on $\mathfrak{w}$. Therefore we conclude that in the $\mathfrak{q} \to \infty$ limit, for certain values of λ, we can explicitly find causality violating modes by determining the quasinormal mode spectrum. This seems to provide a way to constrain allowable gravitational interactions for theories with field theory duals based on consistency of the dual theory.

Appendix A

EGB Equations of Motion

In this appendix we will derive the full Einstein-Gauss-Bonnet equations of motion by explicitly varying the action in Eq. 2.33. The variations we will find for the various curvature squared terms differ between several references [80, 81, 82] however the total equations of motion are agreed upon.

In order to perform the variation we will need to first derive the following identities

$$\nabla^\beta\nabla^\alpha R_{\mu\alpha\nu\beta} = \Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R + R^{\eta\lambda}R_{\mu\eta\nu\lambda} - R^\beta{}_\mu R_{\beta\nu} \tag{A.1}$$

$$\nabla_\rho\nabla_\sigma R^{\rho\sigma} = \frac{1}{2}\Box R \tag{A.2}$$

$$\nabla^\rho\nabla_\mu R_{\rho\nu} + \nabla^\rho\nabla_\nu R_{\rho\mu} = \frac{1}{2}\left(\nabla_\mu\nabla_\nu R + \nabla_\nu\nabla_\mu R\right) - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} + 2R_{\mu\rho}R_\nu{}^\rho. \tag{A.3}$$

To arrive at Eq. A.1 we start with the second Bianchi identity

$$0 = \nabla_\eta R_{\lambda\mu\nu\kappa} + \nabla_\nu R_{\lambda\mu\kappa\eta} + \nabla_\kappa R_{\lambda\mu\eta\nu}. \tag{A.4}$$

Taking the ∇_ρ derivative and contracting μ with η and κ with ρ we have

$$0 = \nabla_\kappa\nabla_\nu R_{\lambda\mu\nu\kappa} + \nabla^\kappa\nabla_\nu R_{\lambda\kappa} - \Box R_{\lambda\nu}. \tag{A.5}$$

Now starting again with the second Bianchi identity, taking the covariant derivative, ∇_ρ and contracting λ with ν and μ with κ we have

$$0 = \nabla_\rho\nabla_\eta R - 2\nabla_\rho\nabla_\lambda R^\lambda{}_\eta. \tag{A.6}$$

Applying the commutator of covariant derivatives to the Ricci tensor gives

$$\nabla_\rho \nabla_\lambda R^\lambda{}_\eta = \nabla_\lambda \nabla_\rho R^\lambda{}_\eta - R_{\sigma\rho} R^\sigma{}_\eta + R^{\lambda\sigma} R_{\sigma\eta\lambda\rho} \tag{A.7}$$

so that using Eq. A.7 in Eq. A.6 and relabelling indices we have

$$\nabla_\nu \nabla_\lambda R = 2\nabla^\kappa \nabla_\nu R_{\lambda\kappa} - 2R_{\kappa\nu} R^\kappa{}_\lambda + 2R^{\kappa\sigma} R_{\sigma\lambda\kappa\nu}. \tag{A.8}$$

Finally using Eq. A.8 in Eq. A.5 gives

$$0 = 2\nabla^\beta \nabla^\alpha R_{\mu\alpha\nu\beta} + \nabla_\nu \nabla_\mu R + 2R_{\beta\nu} R^\beta{}_\mu - 2R^{\eta\lambda} R_{\lambda\mu\eta\nu} - 2\Box R_{\mu\nu}$$

which is just the identity in Eq. A.1 since $\nabla_\nu \nabla_\mu R = \nabla_\mu \nabla_\nu R$ due to the symmetry of the Christoffel symbols. Eq. A.2 now follows by taking the trace of Eq. A.1. In order to establish Eq. A.3 we need to return to the second Bianchi identity and now take the ∇^η derivative and then contract λ with ν and relabel indices giving

$$\nabla_\rho \nabla^\mu R^{\rho\nu} = \Box R^{\mu\nu} - \nabla_\rho \nabla_\sigma R^{\mu\rho\nu\sigma}. \tag{A.9}$$

Now using Eq. A.1 in Eq. A.9 we have

$$\nabla_\rho \nabla^\mu R^{\rho\nu} = \frac{1}{2} \nabla^\mu \nabla^\nu R - R^{\mu\rho\nu\sigma} R_{\rho\sigma} + R^\mu{}_\sigma R^{\nu\rho}. \tag{A.10}$$

Adding Eq. A.10 to its transpose gives the identity in Eq. A.3. With the three identities established we must recall a few standard general relativity results [1]. First, the variation of $\sqrt{-g}$ is given by

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu}. \tag{A.11}$$

Second, taking the variation of $\delta^\nu{}_\mu = g_{\mu\rho} g^{\rho\nu}$ we find

$$\begin{aligned} \delta g^{\nu\rho} &= -g^{\alpha\nu} g^{\beta\rho} \delta g_{\alpha\beta} \\ \delta g_{\mu\nu} &= -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta}. \end{aligned}$$

This allows us to raise and lower the indices on $\delta g_{\mu\nu}$ by simply taking account of the extra negative sign. Finally the variation of the Riemann tensor is

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \nabla_{\mu}\left(\delta\Gamma^{\rho}_{\nu\sigma}\right) - \nabla_{\nu}\left(\delta\Gamma^{\rho}_{\mu\sigma}\right) \tag{A.12}$$

where

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\nabla_{\mu}\delta g_{\sigma\nu} + \nabla_{\nu}\delta g_{\sigma\mu} - \nabla_{\sigma}\delta g_{\mu\nu}\right) \tag{A.13}$$

follows from the definition of the connection. Therefore

$$\begin{aligned}\delta R^{\gamma}{}_{\alpha\sigma\beta} = \frac{1}{2}g^{\gamma\mu}\,[&\nabla_{\sigma}\nabla_{\beta}\delta g_{\mu\alpha} + \nabla_{\sigma}\nabla_{\alpha}\delta g_{\mu\beta} - \nabla_{\sigma}\nabla_{\mu}\delta g_{\beta\alpha} \\ &-\nabla_{\beta}\nabla_{\sigma}\delta g_{\mu\alpha} - \nabla_{\beta}\nabla_{\alpha}\delta g_{\mu\sigma} + \nabla_{\beta}\nabla_{\mu}\delta g_{\sigma\alpha}]\,.\end{aligned} \tag{A.14}$$

To vary the Einstein-Gauss-Bonnet action, we first need to compute $\delta R_{\rho\alpha\sigma\beta}$

$$\delta R_{\rho\alpha\sigma\beta} = \delta\left(g_{\rho\gamma}R^{\gamma}{}_{\alpha\sigma\beta}\right) = R^{\gamma}{}_{\alpha\sigma\beta}\delta g_{\rho\gamma} + g_{\rho\gamma}\delta R^{\gamma}{}_{\alpha\sigma\beta}$$

so using Eq. A.14

$$\begin{aligned}\delta R_{\rho\alpha\sigma\beta} = R^{\gamma}{}_{\alpha\sigma\beta}\delta g_{\rho\gamma} + \frac{1}{2}\,[&\nabla_{\sigma}\nabla_{\beta}\delta g_{\rho\alpha} + \nabla_{\sigma}\nabla_{\alpha}\delta g_{\rho\beta} - \nabla_{\sigma}\nabla_{\rho}\delta g_{\beta\alpha} \\ &-\nabla_{\beta}\nabla_{\sigma}\delta g_{\rho\alpha} - \nabla_{\beta}\nabla_{\alpha}\delta g_{\rho\sigma} + \nabla_{\beta}\nabla_{\rho}\delta g_{\sigma\alpha}]\end{aligned} \tag{A.15}$$

Variation of R^2

Since

$$R = g^{\mu\nu}g^{\rho\sigma}R_{\rho\mu\sigma\nu}, \tag{A.16}$$

$$\begin{aligned}\delta(R^2) &= 2R\left[\delta g^{\mu\nu}R_{\mu\nu} + \delta g^{\rho\sigma}R_{\rho\sigma} + \delta g^{\rho\sigma}g^{\mu\nu}R_{\rho\mu\sigma\nu} + g^{\mu\nu}g^{\rho\sigma}\delta R_{\rho\mu\sigma\nu}\right] \\ &= 4RR_{\mu\nu}\delta g^{\mu\nu} + 2Rg^{\mu\nu}g^{\rho\sigma}\delta R_{\rho\mu\sigma\nu}.\end{aligned}$$

So using Eq. A.15

$$\begin{aligned}\delta(R^2) = 4RR_{\mu\nu}\delta g^{\mu\nu} + 2Rg^{\mu\nu}g^{\rho\sigma}\Bigg[R^{\alpha}_{\mu\sigma\nu}\delta g_{\rho\alpha} + \frac{1}{2}\Big(&\nabla_{\sigma}\nabla_{\nu}\delta g_{\rho\mu} + \nabla_{\sigma}\nabla_{\mu}\delta g_{\rho\nu} - \nabla_{\sigma}\nabla_{\rho}\delta g_{\nu\mu} \\ &- \nabla_{\delta}\nabla_{\gamma}\delta g_{\alpha\beta} - \nabla_{\delta}\nabla_{\beta}\delta g_{\alpha\gamma} + \nabla_{\delta}\nabla_{\alpha}\delta g_{\gamma\beta}\Big)\Bigg]\end{aligned}$$

$$\delta(R^2) = -2RR^{\mu\nu}\delta g_{\mu\nu} + R\nabla^{\nu}\nabla^{\mu}\delta g_{\mu\nu} + R\nabla^{\mu}\nabla^{\nu}\delta g_{\mu\nu} - 2Rg^{\mu\nu}\nabla^{\rho}\nabla_{\rho}\delta g_{\mu\nu}. \tag{A.17}$$

Therefore the contribution to the equations of motion from the variation of R^2 is

$$\delta\left(R^2\right) \rightarrow -2RR^{\mu\nu} + \nabla^\mu\nabla^\nu R + \nabla^\nu\nabla^\mu R - 2g^{\mu\nu}\Box R. \tag{A.18}$$

Variation of $R_{\mu\nu}R^{\mu\nu}$

$$\begin{aligned}\delta\left(R^{\mu\nu}R_{\mu\nu}\right) &= \delta R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu}\delta R_{\mu\nu} = R_{\mu\nu}\delta\left(g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta}\right) + R^{\mu\nu}\delta R_{\mu\nu} \\ &= R_{\mu\nu}R_\alpha{}^\nu\delta g^{\alpha\mu} + R_{\mu\nu}R^\mu{}_\beta\delta g^{\beta\nu} + R_{\mu\nu}R_\rho{}^\mu{}_\sigma{}^\nu\delta g^{\sigma\rho} + R^{\alpha\beta}g^{\sigma\rho}\delta R_{\rho\alpha\sigma\beta} \\ &\quad + R^{\mu\nu}R_{\rho\mu\sigma\nu}\delta g^{\sigma\rho} + R^{\alpha\beta}g^{\sigma\rho}\delta R_{\rho\alpha\sigma\beta} \\ &= 2R_{\mu\nu}R_\alpha{}^\nu\delta g^{\alpha\mu} + 2R^{\mu\nu}R_{\rho\mu\sigma\nu}\delta g^{\sigma\rho} + 2R^{\alpha\beta}g^{\sigma\rho}\delta R_{\rho\alpha\sigma\beta}.\end{aligned}$$

Using Eq. A.15

$$\begin{aligned}\delta\left(R^{\mu\nu}R_{\mu\nu}\right) &= 2R_{\mu\nu}R_\alpha{}^\nu\delta g^{\alpha\mu} + 2R^{\mu\nu}R_{\rho\mu\sigma\nu}\delta g^{\sigma\rho} + 2R^{\alpha\beta}g^{\sigma\rho}\Big\{R^\gamma{}_{\alpha\sigma\beta}\delta g_{\rho\gamma} + \frac{1}{2}\Big[\nabla_\sigma\nabla_\beta\delta g_{\rho\alpha} \\ &\quad + \nabla_\sigma\nabla_\alpha\delta g_{\rho\beta} - \nabla_\sigma\nabla_\rho\delta g_{\beta\alpha} - \nabla_\beta\nabla_\sigma\delta g_{\rho\sigma} - \nabla_\beta\nabla_\alpha\delta g_{\rho\sigma} + \nabla_\beta\nabla_\mu\delta g_{\sigma\alpha}\Big]\Big\} \\ &= -2R^{\mu\beta}R^\nu{}_\beta\delta g_{\mu\nu} - 2R_{\alpha\beta}R^{\nu\alpha\mu\beta}\delta g_{\mu\nu} + 2R_{\alpha\beta}R^{\nu\alpha\mu\beta}\delta g_{\mu\nu} + R^{\nu\beta}\nabla^\mu\nabla_\beta\delta g_{\mu\nu} \\ &\quad + R^{\mu\beta}\nabla^\nu\nabla_\beta\delta g_{\mu\nu} - R^{\mu\nu}\nabla^\rho\nabla_\rho\delta g_{\mu\nu} - R^{\mu\beta}\nabla_\beta\nabla^\nu\delta g_{\mu\nu} - R^{\alpha\beta}g^{\mu\nu}\nabla_\beta\nabla_\alpha\delta g_{\mu\nu} \\ &\quad + R^{\mu\beta}\nabla_\beta\nabla^\nu\delta g_{\mu\nu}\end{aligned}$$

$$\begin{aligned}\delta\left(R^{\mu\nu}R_{\mu\nu}\right) = -2R^{\mu\beta}R^\nu{}_\beta\delta g_{\mu\nu} + R^{\nu\beta}\nabla^\mu\nabla_\beta\delta g_{\mu\nu} + R^{\mu\beta}\nabla^\nu\nabla_\beta\delta g_{\mu\nu} \\ - R^{\mu\nu}\nabla^\rho\nabla_\rho\delta g_{\mu\nu} - R^{\alpha\beta}g^{\mu\nu}\nabla_\beta\nabla_\alpha\delta g_{\mu\nu}.\end{aligned} \tag{A.19}$$

So the contribution to the equations of motion from the variation of $R_{\mu\nu}R^{\mu\nu}$ is

$$\delta\left(R_{\mu\nu}R^{\mu\nu}\right) \rightarrow -2R^{\mu\beta}R^\nu{}_\beta + \nabla_\beta\nabla^\mu R^{\nu\beta} + \nabla_\beta\nabla^\nu R^{\mu\beta} - \Box R^{\mu\nu} - g^{\mu\nu}\nabla_\alpha\nabla_\beta R^{\alpha\beta}. \tag{A.20}$$

Using the identities in Eq. A.2 and Eq. A.3 we may rewrite Eq. A.20 as

$$\delta\left(R_{\mu\nu}R^{\mu\nu}\right) \rightarrow \frac{1}{2}\left(\nabla^\mu\nabla^\nu R + \nabla^\nu\nabla^\mu R\right) - 2R^{\mu\rho\nu\sigma}R_{\rho\sigma} - \Box R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\Box R. \tag{A.21}$$

Variation of $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$

$$\delta\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right) = R_{\alpha\beta\gamma\delta}\left[\delta g^{\alpha\mu}R_{\mu}{}^{\beta\gamma\delta} + \delta g^{\beta\nu}R^{\alpha}{}_{\nu}{}^{\gamma\delta} + \delta g^{\gamma\rho}R^{\alpha\beta}{}_{\rho}{}^{\delta} + \delta g^{\delta\sigma}R^{\alpha\beta\gamma}{}_{\sigma} + g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}\delta R_{\mu\nu\rho\sigma}\right] + R^{\alpha\beta\gamma\delta}\delta R_{\alpha\beta\gamma\delta}$$
$$= 2R_{\alpha\beta\gamma\delta}R_{\mu}{}^{\beta\gamma\delta}\delta g^{\alpha\mu} + R_{\alpha\beta\gamma\delta}R^{\gamma\delta}{}_{\mu}{}^{\beta}\delta g^{\alpha\mu} + R_{\alpha\gamma\delta\beta}R_{\mu}{}^{\gamma\delta\beta}\delta g^{\alpha\mu} + 2R^{\alpha\beta\gamma\delta}\delta R_{\alpha\beta\gamma\delta}$$
$$= -4R^{\mu\beta\gamma\delta}R^{\alpha}{}_{\beta\gamma\delta}\delta g_{\alpha\mu} + 2R^{\alpha\beta\gamma\delta}\delta R_{\alpha\beta\gamma\delta}. \tag{A.22}$$

Using Eq. A.15 we have

$$2R^{\alpha\beta\gamma\delta}\delta R_{\alpha\beta\gamma\delta} = -2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma}\delta g^{\mu\nu} + R^{\alpha\beta\gamma\delta}\left[\nabla_\gamma\nabla_\delta\delta g_{\alpha\beta} + \nabla_\gamma\nabla_\beta\delta g_{\alpha\delta} - \nabla_\gamma\nabla_\alpha\delta g_{\delta\beta} - \nabla_\delta\nabla_\gamma\delta g_{\alpha\beta} - \nabla_\delta\nabla_\beta\delta g_{\alpha\gamma} + \nabla_\delta\nabla_\alpha\delta g_{\gamma\beta}\right]$$
$$= -2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma}\delta g^{\mu\nu} + 2R^{\mu\nu\gamma\delta}\nabla_\gamma\nabla_\delta\delta g_{\mu\nu} - 2R^{\mu\beta\nu\gamma}\nabla_\gamma\nabla_\beta\delta g_{\mu\nu} - 2R^{\alpha\gamma\nu\alpha}\nabla_\gamma\nabla_\alpha\delta g_{\mu\nu}.$$

The second term is 0 since $R^{\mu\nu\gamma\delta}$ is antisymmetric under $\mu \leftrightarrow \nu$ whereas $\delta g_{\mu\nu}$ is symmetric. This leaves

$$2R^{\alpha\beta\gamma\delta}\delta R_{\alpha\beta\gamma\delta} = -2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma}\delta g^{\mu\nu} - 2R^{\mu\beta\nu\gamma}\nabla_\gamma\nabla_\beta\delta g_{\mu\nu} - 2R^{\mu\gamma\nu\alpha}\nabla_\gamma\nabla_\alpha\delta g_{\mu\nu} \tag{A.23}$$

so using Eq. A.22 we have

$$\delta\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right) = -2R^{\mu\alpha\beta\gamma}R^{\nu}{}_{\alpha\beta\gamma}\delta g_{\mu\nu} - 2R^{\mu\beta\nu\gamma}\nabla_\gamma\nabla_\beta\delta g_{\mu\nu} - 2R^{\mu\gamma\nu\alpha}\nabla_\gamma\nabla_\alpha\delta g_{\mu\nu}. \tag{A.24}$$

Therefore the contribution to the equations of motion from the variation of $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is

$$\delta\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right) \to -2R^{\mu\alpha\beta\gamma}R^{\nu}{}_{\alpha\beta\gamma} - 2\nabla_\beta\nabla_\gamma R^{\mu\beta\nu\gamma} - 2\nabla_\alpha\nabla_\gamma R^{\mu\gamma\nu\alpha}. \tag{A.25}$$

Using Eq. A.1 we can write Eq. A.25 as

$$\delta\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right) \to -2R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} - 4\Box R^{\mu\nu} + \nabla^\mu\nabla^\nu R + \nabla^\nu\nabla^\mu R - 4R^{\mu\rho\nu\sigma}R_{\rho\sigma} + 4R^{\mu}{}_{\rho}R^{\nu\rho}. \tag{A.26}$$

We can now combine Eq. A.11, Eq. A.18, Eq. A.21 and Eq. A.26 to find the equations of motion from the Gauss-Bonnet term in the action

$$\begin{aligned}\delta\left(\sqrt{-g}\mathcal{L}_{GB}\right) \to \frac{1}{2}g^{\mu\nu}\mathcal{L}_{GB} &- 2RR^{\mu\nu} + \nabla^\mu\nabla^\nu R + \nabla^\nu\nabla^\mu R \\ &- 2g^{\mu\nu}\Box R - 2\left(\nabla^\mu\nabla^\nu R + \nabla^\nu\nabla^\mu R\right) + 8R^{\mu\rho\nu\sigma}R_{\rho\sigma} + 4\Box R^{\mu\nu} \\ &+ 2g^{\mu\nu}\Box R - 2R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} - 4\Box R^{\mu\nu} + \nabla^\mu\nabla^\nu R + \nabla^\nu\nabla^\mu R \\ &- 4R^{\mu\rho\nu\sigma}R_{\rho\sigma} + 4R^{\mu}{}_{\rho}R^{\nu\rho} \\ = \frac{1}{2}g^{\mu\nu}\mathcal{L}_{GB} &- 2RR^{\mu\nu} + 4R^{\mu\rho\nu\sigma}R_{\rho\sigma} - 2R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} + 4R^{\mu}{}_{\rho}R^{\nu\rho}.\end{aligned}$$

Therefore the equations of motion for the full Einstein-Gauss-Bonnet action are

$$\begin{aligned}R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + g^{\mu\nu}\Lambda - \frac{\lambda L^2}{4}g^{\mu\nu}\left(R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\sigma\rho}R^{\alpha\beta\sigma\rho}\right)& \\ - \frac{\lambda L^2}{2}\left(-2RR^{\mu\nu} + 4R^{\mu\rho\nu\sigma}R_{\rho\sigma} - 2R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} + 4R^{\mu}{}_{\rho}R^{\nu\rho}\right) = 0.& \qquad \text{(A.27)}\end{aligned}$$

In the presence of matter fields the right hand side would be supplemented by an energy momentum tensor

$$T^{\mu\nu}_{\text{matter}} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}}. \qquad \text{(A.28)}$$

Appendix B

Thermodynamics of EGB Black-Holes with Planar Horizons

In this appendix we will compute the Hawking temperature and the entropy of the EGB black hole considered in the main text. Rotating to Euclidean signature, $t \to i\tau$, our EGB metric in Eq. 2.36 has the form

$$ds^2 = A^2 f(r) d\tau^2 + g(r) d\mathbf{x}^2 + \frac{1}{f(r)} dr^2 \tag{B.1}$$

where $g(r) = r^2/L^2$. Near the horizon $f(r) \sim (r-a)f'(a)$ so letting $\rho^2 = \frac{4}{f'(a)}(r-a)$ the metric becomes

$$ds^2 = \rho^2 d\left(\frac{A\tau}{2} f'(a)\right)^2 + d\rho^2 + g(a) d\mathbf{x}^2. \tag{B.2}$$

Therefore to avoid a conical singularity at the ρ origin the τ coordinate must have period J and we can identify

$$\frac{A\tau f'(a)}{2} + 2\pi = \frac{Af'(a)}{2}(\tau + J). \tag{B.3}$$

$f'(a) = 4a/L^2$ so that the period is given by

$$J = \frac{4\pi}{Af'(a)} = \frac{\pi L^2}{aA}. \tag{B.4}$$

Therefore the Hawking temperature is

$$T = \frac{1}{J} = \frac{aA}{\pi L^2}. \tag{B.5}$$

Turning to the calculation of the entropy, there are several approaches we could take, all of which yield the same result. Knowing before hand that Lovelock black holes with planar horizons obey the area law for the entropy [60], the easiest way to calculate the entropy is to find the area of the horizon (at a fixed time)

$$A = \int \sqrt{-\gamma_{D-2}} d^{D-2}x = \left(\frac{a}{L}\right)^{D-2} \int d^{D-2}x. \tag{B.6}$$

Therefore the entropy density for the $D = 5$ solution is

$$s = \frac{1}{4G}\left(\frac{a}{L}\right)^3. \tag{B.7}$$

A more general approach to finding the entropy, which does not require the assumption that the area law holds, is to use the Wald entropy [47, 66]

$$S = -2\pi \int d^{D-2}x \sqrt{\gamma_{D-2}} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}. \tag{B.8}$$

Here the integral is taken over the bifurcation surface, which is the spacelike cross-section of the Killing horizon where the Killing field vanishes [83, 84]. For our geometry this is the horizon, $r = a$ with t constant. $\epsilon_{\mu\nu} = \nabla_\mu \chi_\nu$ is the binormal vector to the bifurcation surface with χ_ν the Killing field which vanishes on the horizon, normalized such that $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$ [83, 84]. This normalization is equivalent to normalizing the surface gravity, $\kappa^2 = -\frac{1}{2}\nabla_\mu\chi_\nu\nabla^\mu\chi^\nu$ to 1. The binormal vector is antisymmetric under $\mu \leftrightarrow \nu$. In our case $\chi_\mu = \partial_t$ so $\epsilon_{tr} = 1$ and $\epsilon_{rt} = -1$ are the only non-zero components. The functional derivative of $\mathcal{L}$ is performed by treating the metric and the Riemann tensor independently, so using the EGB Lagrangian in Eq. 2.33

$$\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} = \frac{1}{16\pi G}\left(g^{\nu\sigma}g^{\mu\sigma} + \lambda L^2 g^{\nu\sigma}g^{\mu\sigma} - 4\lambda L^2 g^{\mu\rho}R^{\nu\sigma} + \lambda L^2 R^{\mu\nu\rho\sigma}\right). \tag{B.9}$$

$\sqrt{\gamma_3} = a^3/L^3$ so the entropy becomes

$$\begin{aligned} S &= \frac{1}{4G}\frac{a^3}{L^3}V\left(1 + \lambda L^2 R + 2\lambda L^2 \epsilon^{\sigma}{}_{\mu}\epsilon_{\sigma\nu}R^{\mu\nu} - \frac{\lambda L^2}{2}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}R^{\mu\nu\rho\sigma}\right)\Bigg|_{r=a} \\ &= \frac{1}{4G}\frac{a^3}{L^3}V\left(1 + \lambda L^2 R + 4\lambda L^2 g^{tt}R^{rr} - 2\lambda L^2 R^{rtrt}\right)\Bigg|_{r=a} \end{aligned} \tag{B.10}$$

where $V = \int d^3x$ and we have used the symmetry of $\epsilon_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$. For our metric

$$R\big|_{r=a} = -\frac{4\,(5+8\lambda)}{L^2}, \qquad g^{tt}R^{rr}\big|_{r=a} = \frac{4+16\lambda}{L^2}, \qquad R^{rtrt}\big|_{r=a} = \frac{-2+16\lambda}{L^2} \tag{B.11}$$

so the entropy density is

$$s = \frac{1}{4G}\frac{a^3}{L^3} \tag{B.12}$$

in agreement with the area formula for the entropy.

For higher derivative black holes, the area law does not generally hold and the method of calculating the area cannot be used. In the case of EGB gravity the entropy generally depends on the curvature of the bifurcation surface, obeying the area law only for the planar case [65, 67]. The easiest way to find the entropy in these more general cases is using the Wald formula employed here. However it is also possible to determine the entropy using the free energy, F, identified from the Euclidean action, S_E, using the relationship $-F/T = S_E$ and then using the thermodynamic relation $S = \partial F/\partial T$ so that $S = -\left(S_E + T\frac{\partial S_E}{\partial T}\right)$. The result using this method agrees with the Wald entropy calculation we have performed [17]. Finally, the entropy has also been found by determining the ADM mass in terms of the horizon radius and using the first law of black hole thermodynamics, $S = \int dM/T = \int_0^a (\partial M/\partial a)T^{-1}da$ [65].

Appendix C

Expansion of the EGB Action

In this appendix we will show how to arrive at Eq. 3.23. We just need to calculate the EGB action in Eq. 2.33 using the perturbed metric, Eq. 3.1, however due to the complexity of the expansion it is important to do this in a systematic way. Expanding the EGB action to quadratic order in ϕ we find

$$\begin{aligned}\mathcal{L}_{\mathrm{EGB}} = F_0 &+ F_1\phi^2 + F_2\left(\partial_z\phi\right)^2 + F_3\phi\partial_z^2\phi + F_4\left(\partial_t\phi\right)^2 + F_5\left(\partial_t\partial_z\phi\right)^2 + F_6\phi\partial_t^2\phi + F_7\partial_t^2\phi\partial_z^2\phi \\ &+ F_8\phi\partial_r\phi + F_9\partial_z^2\phi\partial_r\phi + F_{10}\partial_t^2\phi\partial_r\phi + F_{11}\left(\partial_r\phi\right)^2 + F_{12}\left(\partial_r\partial_z\phi\right)^2 + F_{13}\partial_t\phi\partial_r\partial_t\phi \\ &+ F_{14}\left(\partial_r\partial_t\phi\right)^2 + F_{15}\phi\partial_r^2\phi + F_{16}\partial_z^2\phi\partial_r^2\phi + F_{17}\partial_t^2\phi\partial_r^2\phi + F_{18}\partial_r\phi\partial_r^2\phi \qquad \text{(C.1)}\end{aligned}$$

where $\phi = \phi(r,t,z)$ and all F's are functions of r. We should be able to integrate all the terms by parts in order to write the Lagrangian in terms of the equation of motion, plus total derivatives, and a ϕ independent term which is removed by the counterterm. In order to do this systematically we can rewrite each term as a total derivative plus additional contributions. For example the F_5 term can be written as

$$F_5\left(\partial_z\partial_t\phi\right)^2 = \partial_z\left(F_5\partial_t\phi\partial_z\partial_t\phi\right) - \partial_t\left(F_5\partial_t\phi\partial_z^2\phi\right) + F_5\partial_t^2\phi\partial_z^2\phi. \qquad \text{(C.2)}$$

Doing this for each term in the Lagrangian we find

$$\begin{aligned}\mathcal{L}_{\text{EGB}} = {} & F_0 + F_1\phi^2 + F_2\left(\partial_z\phi\right)^2 + \partial_z\left(F_3\phi\partial_z\phi\right) - F_3\left(\partial_z\phi\right)^2 + F_4\left(\partial_t\phi\right)^2 + \partial_z\left(F_5\partial_t\phi\partial_z\partial_t\phi\right) \\ & - \partial_t\left(F_5\partial_t\phi\partial_z^2\phi\right) + \left[F_5 + F_7\right]\partial_t^2\phi\partial_z^2\phi + \partial_t\left(F_6\phi\partial_t\phi\right) - F_6\left(\partial_t\phi\right)^2 + F_8\phi\partial_r\phi \\ & + \partial_z\left(F_9\partial_z\phi\partial_r\phi\right) - \frac{1}{2}\partial_r\left(F_9\left(\partial_z\phi\right)^2\right) + \frac{1}{2}\partial_r F_9\left(\partial_z\phi\right)^2 + \partial_t\left(F_{10}\partial_t\phi\partial_r\phi\right) \\ & - \frac{1}{2}\partial_r\left(F_{10}\left(\partial_t\phi\right)^2\right) + \frac{1}{2}\partial_r F_{10}\left(\partial_t\phi\right)^2 + F_{11}\left(\partial_r\phi\right)^2 + \partial_z\left(F_{12}\partial_r\partial_z\phi\partial_r\phi\right) \\ & - \partial_r\left(F_{12}\partial_z^2\phi\partial_r\phi\right) + \partial_z\left(\partial_r F_{12}\partial_z\phi\partial_r\phi\right) - \frac{1}{2}\partial_r\left(\partial_r F_{12}\left(\partial_z\phi\right)^2\right) + \frac{1}{2}\partial_r^2 F_{12}\left(\partial_z\phi\right)^2 \\ & + \left[F_{12} + F_{16}\right]\partial_z^2\phi\partial_r^2\phi + \frac{1}{2}\partial_r\left(F_{13}\left(\partial_t\phi\right)^2\right) - \frac{1}{2}\partial_r F_{13}\left(\partial_t\phi\right)^2 + \partial_t\left(F_{14}\partial_r\partial_t\phi\partial_r\phi\right) \\ & - \partial_r\left(F_{14}\partial_t^2\phi\partial_r\phi\right) + \partial_t\left(\partial_r F_{14}\partial_t\phi\partial_r\phi\right) - \frac{1}{2}\partial_r\left(\partial_r F_{14}\left(\partial_t\phi\right)^2\right) + \frac{1}{2}\partial_r^2 F_{14}\left(\partial_t\phi\right)^2 \\ & + \left[F_{14} + F_{17}\right]\partial_t^2\phi\partial_r^2\phi + \partial_r\left(F_{15}\phi\partial_r\phi\right) - \partial_r F_{15}\phi\partial_r\phi - F_{15}\left(\partial_r\phi\right)^2 + \frac{1}{2}\partial_r\left(F_{18}\left(\partial_r\phi\right)^2\right) \\ & - \frac{1}{2}\partial_r F_{18}\left(\partial_r\phi\right)^2 \end{aligned} \tag{C.3}$$

where $F_{12} + F_{16} = F_{14} + F_{17} = 0$. In order to determine the linearized equation of motion we can drop the total boundary terms, so that defining

$$\begin{aligned} G_1 &= F_2 - F_3 + \frac{1}{2}\partial_r F_9 + \frac{1}{2}\partial_r^2 F_{12} \\ G_2 &= F_4 - F_6 + \frac{1}{2}\partial_r F_{10} - \frac{1}{2}\partial_r F_{13} + \frac{1}{2}\partial_r^2 F_{14} \\ G_3 &= F_8 - \partial_r F_{15} \\ G_4 &= F_{11} - F_{15} - \frac{1}{2}\partial_r F_{18} \end{aligned}$$

the quadratic EGB Lagrangian becomes

$$\mathcal{L}_{\text{EGB}}^{\text{nb}} = F_0 + F_1\phi^2 + G_1\left(\partial_z\phi\right)^2 + G_2\left(\partial_t\phi\right)^2 + G_3\phi\partial_r\phi + G_4\left(\partial_r\phi\right)^2 \tag{C.4}$$

from which the equation of motion can be determined to be

$$\left(2F_1 - \partial_r G_3\right)\phi - 2\partial_r G_4\partial_r\phi - 2G_4\partial_r^2\phi - 2G_2\partial_t^2\phi - 2G_1\partial_z^2\phi = 0. \tag{C.5}$$

This equation of motion agrees with the result in Eq. 3.7 obtained by directly linearizing the full EGB equation of motion. Returning to the full quadratic EGB Lagrangian with total r, z and t derivatives, we have

$$\mathcal{L}_{\mathrm{EGB}} = F_0 + F_1\phi^2 + G_1\left(\partial_z\phi\right)^2 + G_2\left(\partial_t\phi\right)^2 + G_3\phi\partial_r\phi + G_4\left(\partial_r\phi\right)^2 + \partial_r T_0(r,z,t) + \partial_z T_1(r,z,t) + \partial_t T_2(r,z,t) \tag{C.6}$$

where

$$\begin{aligned} T_0(r,z,t) &= -\frac{1}{2}\left[F_9\left(\partial_z\phi\right)^2 + F_{10}\left(\partial_t\phi\right)^2 + 2F_{12}\partial_z^2\phi\partial_r\phi + \partial_r F_{12}\left(\partial_z\phi\right)^2 - F_{13}\left(\partial_t\phi\right)^2 \right. \\ &\quad \left. +2F_{14}\partial_t^2\phi\partial_r\phi + \partial_r F_{14}\left(\partial_t\phi\right)^2 - 2F_{15}\phi\partial_r\phi - F_{18}\left(\partial_r\phi\right)^2\right] \\ T_1(r,z,t) &= F_5\partial_t\phi\partial_z\partial_t\phi + F_9\partial_z\phi\partial_r\phi + F_{12}\partial_r\partial_z\phi\partial_r\phi + \partial_r F_{12}\partial_z\phi\partial_r\phi \\ T_2(r,z,t) &= -F_5\partial_t\phi\partial_z^2\phi + F_6\phi\partial_t\phi + F_{10}\partial_t\phi\partial_r\phi + F_{14}\partial_r\partial_t\phi\partial_r\phi + \partial_r F_{14}\partial_t\phi\partial_r\phi. \end{aligned} \tag{C.7}$$

Rewriting the G terms using

$$\begin{aligned} G_1\left(\partial_z\phi\right)^2 &= \partial_z\left(G_1\left(\partial_z\phi\right)\phi\right) - G_1\left(\partial_z^2\phi\right)\phi \\ G_2\left(\partial_t\phi\right)^2 &= \partial_t\left(G_2\left(\partial_t\phi\right)\phi\right) - G_2\left(\partial_t^2\phi\right)\phi \\ G_3\phi\partial_r\phi &= \frac{1}{2}\partial_r\left(G_3\phi^2\right) - \frac{1}{2}\partial_r G_3\phi^2 \\ G_4\left(\partial_r\phi\right)^2 &= \partial_r\left(G_4\left(\partial_r\phi\right)\phi\right) - \partial_r G_4\partial_r\phi\phi - G_4\left(\partial_r^2\phi\right)\phi \end{aligned} \tag{C.8}$$

the Lagrangian becomes

$$\mathcal{L}_{\mathrm{EGB}} = P_1(r) + \frac{1}{2}\phi\left\{2F_1\phi - 2G_1\partial_z^2\phi - 2G_2\partial_t^2\phi - \partial_r G_3\phi - 2\partial_r G_4\partial_r\phi - 2G_4\partial_r^2\phi\right\} + \partial_r P_2(r,z,t) + \partial_z P_3(r,z,t) + P_4(r,z,t) \tag{C.9}$$

where we have renamed $F_0 = P_1(r)$ and

$$\begin{aligned} P_2(r,z,t) &= T_0(r,z,t) + \frac{1}{2}G_3\phi^2 + G_4\partial_r\phi\phi \\ P_3(r,z,t) &= T_1(r,z,t) + G_1\partial_z\phi\phi \\ P_4(r,z,t) &= T_2(r,z,t) + G_2\partial_t\phi\phi. \end{aligned}$$

Therefore the quadratic EGB action has the form in Eq. 3.23.

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