Quasirandom forcing in Regular Tournaments.

by

Lina Maria Simbaqueba Marin
B.Sc., Universidad Nacional de Colombia, 2023

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

We acknowledge with respect the Lekwungen peoples on whose traditional territory the University of Victoria stands, and the Songhees, Esquimalt and WSÁNEĆ peoples whose historical relationships with the land continue to this day.

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ABSTRACT

The study of quasirandom forcing in various discrete structures has been a well-known problem in Extremal Combinatorics since 1987. In this work, we study quasirandom forcing in the case of tournaments. We say that a tournament H forces quasirandomness if in every quasirandom sequence $(T_n)_{n\in\mathbb{N}}$ of tournaments of increasing order, the density of H in T_n asymptotically equals its expected value. In contrast to the analogous problem in graphs, it was shown that there exists only one non-transitive tournament that forces quasirandomness. To obtain a richer family of tournaments with this property, we propose a variant of it, restricting the definition of quasirandom forcing to only nearly regular sequences of tournaments $(T_n)_{n\in\mathbb{N}}$. We characterize all tournaments on at most 5 vertices that force quasirandomness under this new setting, obtaining that 11 out of 16 tournaments on four or five vertices are quasirandom forcing in sequences of nearly regular tournaments.

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ACKNOWLEDGEMENTS

I would like to thank:

My supervisor, Jon Noel, for all the help and support I received from him during this time – for his patience, wise words and, all the fun math!

My supervisor, Jane Butterfield for her valuable feedback and wise advice on both theoretical mathematics and the skills needed to be a good mathematician.

Special thanks to Jackeline Suarez and José Luiz Ramírez, for introducing me to mathematics.

Finally, my parents, Jorge and Luz Mery, for their infinite love and support.

Chapter 1

Introduction

To understand our world and its rules, we have used various concepts and techniques to predict upcoming events. For instance, in science, Newton's Laws imply that the future state of an object is completely determined by its present state. Nevertheless, events like the weather, the evolution of a social network, the results of political consultations, or even the result of tossing a coin seem impossible to predict because they depend on many factors. Therefore, it seems that the result of those events is determined by chance rather than according to a plan, meaning that they behave randomly.

The study of probability has allowed us to create accurate models and algorithms and apply them in a wide variety of fields such as economics, sociology, and biology, allowing us to understand complex events better. We know what to expect even if we cannot predict the future. For instance, we cannot predict the result of tossing a coin, but we know that after many trials we will get approximately half of the time heads and half of the time tails. Otherwise, we can conclude that the coin is not fair.

Therefore, it becomes important to be able to simulate random events so that we can predict their expected value. This is one of the reasons why deterministic objects that behave like randomly generated ones, which are said to be *quasirandom*, are useful. For instance, pseudorandom numbers are used in cryptography, cybersecurity, and some Monte Carlo algorithms. Nevertheless, they are not easy to generate.

In our context, we aim to study large random networks, commonly called random graphs. Graphs are abstract objects with numerous applications in the real world. For instance, they can be used to study the internet, social or political networks, and the chemical structure of molecules, among others. A notable application is presented on page 51 of [21] by Caridi, which describes how researchers from the Argentine

Forensic Anthropology Team and Conicet used graphs to help locate individuals who disappeared during the country's military dictatorship (1976–1983).

One of the simplest models for a random graph is the $G(n,p)$ model, which represents a graph with n vertices where each unordered pair of vertices $u, v \in V(G)$ is connected by an edge with probability p independently of other pairs. While there are various other models for random graphs, as described in the survey [28], the $G(n,p)$ model provides a basic framework. Using classic concepts and tools such as expected values or the Chernoff bound, it is relatively straightforward to determine many properties that a graph generated by the $G(n,p)$ model will almost surely display. For instance, the number of edges is asymptotically equal to $n^2p/2 + o(n^2)$. Also, given two linear-sized disjoint sets $U, W \subseteq V(G)$, the number of edges between U and W equal to $(1 + o(1))|U||W|p$ with high probability. Moreover, almost every vertex has a degree of approximately np.

Imagine a graph representing a social network that grows over time as new people join. By recording the graph at each point in time, we obtain a sequence of graphs. As mentioned earlier, random graphs share many common characteristics, so we can predict the expected features of the sequence representing the social network, assuming it behaves like a random graph would. If the sequence deviates from these expectations, we can conclude that an external factor influences its behaviour.

Quasirandomness is also a key concept in mathematics. For instance, in Extremal Combinatorics, one of the most powerful tools to solve problems is the Regularity Lemma, which intuitively states that any dense enough graph can be approximated by a bounded number of "blocks" in which the edges between blocks behave quasirandomly.

This leads to the following question: how can we determine if a deterministic sequence of graphs is quasirandom? The study of this topic was first introduced by Thomason [41,42], who investigated *jumbled graphs*—graphs that share the same edge distribution as random graphs. In 1989, Chung, Graham, and Wilson [11] provided a list of properties that quasirandom graphs satisfy. Although these properties may seem different in nature, they are mathematically equivalent. This means that any sequence of graphs satisfying one of these properties will satisfy all of them, and thus, we can conclude that the sequence is quasirandom. Let's consider a sequence of graphs $\mathcal{G} = (G_n)_{n \in \mathbb{N}}$. One of the properties in the list by Chung, Graham, and Wilson states that $\mathcal{G}$ is quasirandom if it has approximately the expected number of copies of every subgraph H. However, another property in the list reveals that it is sufficient to check whether the sequence contains approximately the expected number of copies of C_4, the cycle with 4 vertices. This means that having the expected number of copies of C_4 implies the same for any graph H. Because of this powerful property, we say that C_4 *forces quasirandomness*. Moreover, C_4 is not the only graph with this property; Skokan and Thoma [40] showed that for $a, b \geq 2$, any complete bipartite graph $K_{a,b}$ also forces quasirandomness and one of the major open problems in extremal combinatorics is known as the Forcing Conjecture by Conlon, Fox and Sudakov [12], which states that quasirandomness in graphs can be characterized by the number of copies of any fixed bipartite graph with at least one cycle.

Quasirandomness has also been studied in other mathematical objects such as permutations [15,18,29], groups [19,25], hypergraphs [8,9,13,24,36], latin squares [14], subset of integers [11], among others.

In this work, we study the problem of quasirandomness in the context of tournaments, i.e., complete digraphs. The notion of quasirandomness here is similar to that in graphs; however, it turns out that, in contrast to graphs, the class of tournaments that force quasirandomness is much more restricted. To address this, we slightly modify the definition of quasirandom forcing, to focus only on sequences of *nearly regular* tournaments, in order to obtain a broader class of tournaments that force quasirandomness.

The structure of this work is as follows. In Chapter 1, we introduce the fundamental definitions and notation used throughout the document. Many of our proofs rely on the framework of Limit Theory, which associates an analytic limit object with a sequence of tournaments. Sections 1.1 and 1.2 provide definitions related to tournaments and an introduction to Limit Theory, respectively. In Section 1.3, we state our main results on forcing quasirandomness in regular tournaments.

Chapter 2 surveys previous results on quasirandom forcing in tournaments, primarily within the Limit Theory framework. This chapter is expository and serves to build intuition on the topic.

In Chapter 3, we focus on proving our main results by establishing necessary and sufficient conditions for a tournament to force quasirandomness in regular tournaments. We characterize tournaments on at most five vertices that force quasirandomness in regular tournaments. Some of these proofs utilize the Flag Algebra method, which we introduce in Chapter 4.

1.1 Preliminaries and notation

The objective of the first section of this work is to state the standard definitions and their respective notation related to digraphs, in particular, tournaments. We also describe some properties of randomness and quasirandomness and introduce the problem of determining the set of tournaments that force quasirandomness.

A *digraph* D is a pair $(V(D), E(D))$ where $V(D)$ is called the set of vertices and $E(D) \subseteq V(D) \times V(D)$ is the set of arcs or directed edges such that $(v, v) \notin E(D)$ and at most one (u, v) or (v, u) is in $E(D)$ for every $u, v \in V(D)$. We often write uv to denote the directed edge (u, v). Additionally, we let $v(D) = |V(D)|$ and $e(D) = |E(D)|$.

One of the main approaches in extremal combinatorics for studying digraphs is to analyze their local structures. Given a digraph H, an *homomorphism* from H to D is a map $f : V(H) \to V(D)$ such that $f(u)f(v) \in E(D)$ if $uv \in E(H)$. We denote by $\mathrm{Hom(H, D)}$ the set of such homomorphism and we let $\hom(H, D) = |\,\mathrm{Hom(H, D)}\,|$.

Also, it is natural to wonder how likely it is to find a homomorphic copy of the digraph H inside a larger digraph D. This is known as *homomorphism density*, denoted by $t(H, D)$ and is given by

$$t(H, D) := \frac{\hom(H, D)}{v(D)^{v(H)}}. \tag{1.1}$$

The *adjacency matrix* of a digraph D with n vertices is the $n \times n$ binary matrix $A = A[D]$ with columns and rows enumerated by the vertices of D, in which the (i, j)-entry, $A(i, j)$ equals 1 if ij is an arc of D and 0 otherwise. Given a vertex $v \in V(D)$ we define the *out-neighbourhood* of v as $N_D^+(v) = \{u \in V(D) : vu \in E(D)\}$ and the *in-neighbourhood* as $N_D^-(v) = \{u \in V(D) : uv \in E(D)\}$. The *out-degree* of a vertex $v \in V(D)$ is defined to be the number of out-neighbours of v, $d_D^+(v) = |N_D^+(v)|$. Similarly, the *in-degree* is $d_D^-(v) = |N_D^-(v)|$. In other words,

$$d_D^+(v) = \sum_{u \in V(D)} A(v, u)$$

and

$$d_D^-(v) = \sum_{u \in V(D)} A(u, v).$$

When the context is clear, we often let $d^+(v) = d_D^+(v)$. Note that each arc contributes to the out(resp.in)–neighbourhood of exactly one vertex $v \in V(D)$, therefore

$$\sum_{v \in V(D)} d^+(v) = \sum_{v \in V(D)} d^-(v) = e(D).$$

With the adjacency matrix, we can calculate the homomorphism density of a digraph H in the digraph D as follows. Let $A = A[D]$ and $f : V(H) \to V(D)$ be a homomorphism. By definition, if $ij \in E(H)$, then $f(i)f(j) \in E(D)$. Therefore, $A(f(i), f(j)) = 1$. Then, we have

$$t(H, D) = \frac{1}{v(D)^{v(H)}} \sum_{f: V(H) \to V(D)} \prod_{ij \in E(H)} A(f(i), f(j)).$$

We now focus on the specific notation and basic definitions for tournaments. A *tournament* T is a digraph with the additional condition that for every unordered pair $u, v \in V(T)$, exactly one uv or vu is in $E(T)$. Therefore, every tournament T satisfies $e(T) = \binom{v(T)}{2}$. Additionally, we say that a tournament T is *regular* if every vertex of T has out-degree $\frac{v(T)-1}{2}$.

Transitive tournaments are one of the most important classes of tournaments due to their properties and applications. A transitive tournament on h vertices, denoted by Tr_h has vertex set $V(Tr_h) = \{v_1, \dots, v_h\}$ and $E(Tr_h) = \{v_i v_j : i > j\}$.

Figure 1.1 shows all the possible tournaments on at most 5 vertices up to isomorphism. There are 20 such tournaments labelled $H_0, \dots, H_{19}$. The tournaments H_0, H_1, H_2, H_4 and H_8 are transitive. The tournament H_3 is the cyclic tournament on 3 vertices and is also referred to as C_3. Similarly, H_6 is the unique 4-vertex tournament containing a spanning cycle and is also referred to as C_4.

Let $\mathcal{T}_n$ denote the model of a uniformly random tournament on n vertices, where each arc is directed in one of the two possible directions with probability 1/2, independently of all other arcs.

Given any tournament H and a uniformly random tournament T_n with n vertices, the probability that an injective random function $f : V(H) \to V(T_n)$ is a homomorphism is $(1/2)^{\binom{k}{2}}$. Also, since the probability that a function is non-injective is $1 - \frac{n!/(n-v(H))!}{n^{v(H)}} = o(1)$. Therefore,

$$\mathbb{E}(t(H, T_n)) = (1 + o(1))(1/2)^{\binom{k}{2}}$$

where the asymptotics are as $n \to \infty$.

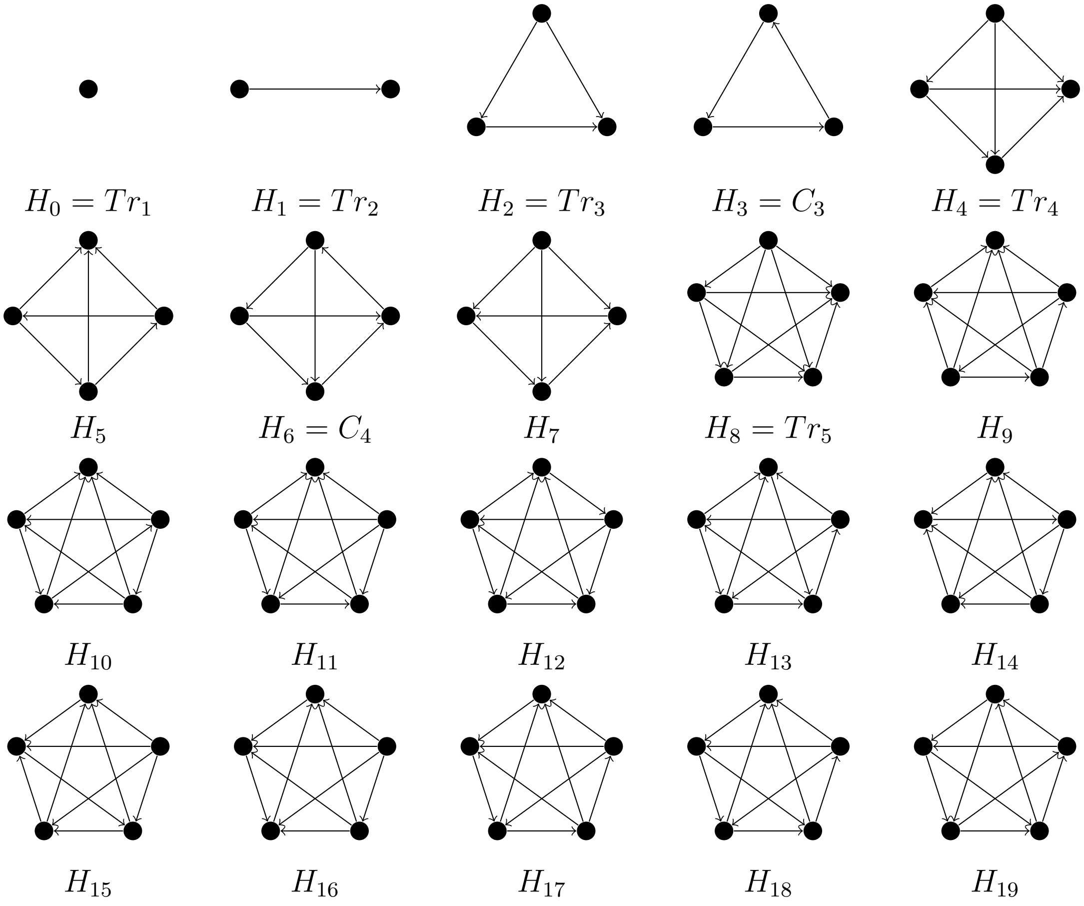


Figure 1.1: The tournaments on at most 5 vertices, up to isomorphism.

This last property characterizes when a sequence of tournaments with an unbounded number of vertices, $(T_n)_{n\in\mathbb{N}}$, behaves randomly, i.e., is quasirandom: the density of any tournament H in T_n, as $n \to \infty$, converges to its expected value in a random tournament.

Definition 1.1.1. A sequence of tournaments $(T_n)_{n\in\mathbb{N}}$ is said to be *quasirandom* if

$$\lim_{n\to\infty} t(H, T_n) = \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

for every tournament H with k vertices, for every $k \in \mathbb{N}$.

It is convenient to extend the definition of homomorphism density for a linear combination of digraphs. Let $\mathcal{D}$ be the set of all finite digraphs up to isomorphism and $\mathbb{R}[\mathcal{D}] := \{\sum_{i=1}^{t} \alpha_i H_i : \alpha_i \in \mathbb{R}, H_i \in \mathcal{D}, t \in \mathbb{N}\}$ be the set of finite formal linear combinations of digraphs. Given $D \in \mathcal{D}$ and $H = \sum_{i=1}^{t} \alpha_i H_i \in \mathbb{R}[\mathcal{D}]$, we can define the *homomorphism density* as follows

$$t(H, D) = \sum_{i=1}^{t} \alpha_i t(H_i, D).$$

Additionally, we let

$$t(H, 1/2) = \sum_{i=1}^{t} \alpha_i (1/2)^{\binom{v(H_i)}{2}}.$$

In 1991, Chung and Graham studied the problem of quasirandomness in various mathematical objects such as graphs [11], subsets of $\mathbb{Z}_n$ [11], hypergraphs [8], tournaments [10], among others. In the case of tournaments, they prove that there is a list of properties satisfied by quasirandom tournaments. Among these is the property we presented earlier, related to having the expected homomorphism density for every tournament H. Moreover, they proved that these properties are equivalent — if a sequence of tournaments satisfies one of them, it must satisfy all of them [10]. Below, we state some of these properties, but first, we need to define the linear combination $E4C \in \mathbb{R}[\mathcal{D}]$ that will be important when we state them. There are 4 possible orientations of the cycle with four vertices, as shown in Figure 1.2. Let $E4C = 2C_4^1 + 2C_4^4 + 4C_4^3$. Note that this linear combination represents a weighted count of the so-called *even* 4-cycles; i.e. orientations with an even number of arcs oriented clockwise.

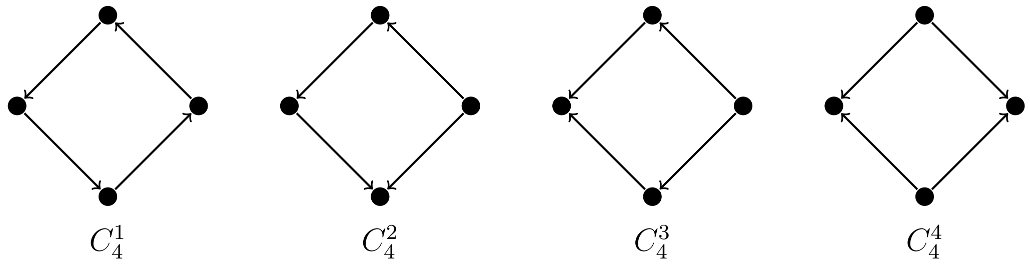

C_4^1 C_4^2 C_4^3 C_4^4

Figure 1.2: The four possible orientations of C_4.

Theorem 1.1.2. *[Chung–Graham [10]] Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of tournaments. The following properties are equivalent for all $s \in \mathbb{N}$.*

$P_1(s)$: *For all tournaments H on s vertices,*

$$\hom(H, T_n) = (1 + o(1))v(T_n)^s 2^{-\binom{s}{2}}.$$

P_2: $t(E4C, T_n) = \frac{1}{2} + o(1)$

P_3: *Every $X \subset V(T)$, the induced subtournament $H = T[X]$ satisfies*

$$\sum_{v \in X} |d_H^+(v) - d_H^-(v)| = o(v(T_n)^2).$$

In this case, we say that H is almost balanced.

P_4: *Every subtournament H of T on $\lfloor v(T_n)/2 \rfloor$ nodes is almost balanced.*

P_5: *For every ordering π of $V(T)$,*

$$|\{uv \in E(D) : \pi(u) < \pi(v)\}| = (1 + o(1))v(T_n)^2/4.$$

Specifically, given that the second property of 1.1.2 is equivalent to the first property, we infer that if $t(E4C, T) = (1 + o(1))(1/2)$ then $t(H, T) = (1 + o(1))2^{-\binom{s}{2}}$ for every tournament H on s vertices. This means that if a sequence of tournaments has the expected density of $E4C$, then it has the expected density of every tournament H. Then, $E4C$ characterizes quasirandomness. The next definition formalizes this notion.

Definition 1.1.3. A set $S \subseteq \mathbb{R}[\mathcal{D}]$ *forces quasirandomness in tournaments* if every sequence of tournaments $(T_n)_{n\in\mathbb{N}}$ such that $v(T_n) \to \infty$ and $\lim_{n\to\infty} t(H, T_n) = t(H, 1/2)$ for all $H \in S$ is quasirandom. In particular, we say that a tournament H forces quasirandomness if $S = \{1 \cdot H\}$ does.

The definitions and most of the properties presented in this section can be described in the language of Limit Theory. We will present them in that language before stating the known results on quasirandom-forcing in tournaments.

1.2 Combinatorial limits preliminaries

In this section we introduce the standard definitions and notation related to quasirandomness in tournaments in the language of limit theory. This terminology associates analytic limit objects to sequences of combinatorial structures. Specifically, given a sequence of tournaments, the limit object, known as *tournamenton*, is a measurable function that can be understood to be a "continuous generalization" of the adjacency matrix of the limit tournament. These limit objects were first introduced by Diaconis-Janson in the case of graphs [20], and by Thörnblad in the case of tournaments [43]. For a deeper treatment of combinatorial limits, see Lovász [32].

Formally, a tournamenton is a measurable function $W : [0,1]^2 \to [0,1]$ with the property that $W(x,y) + W(y,x) = 1$ for all $x, y \in [0,1]$. This last property is analogous to the fact that in a tournament exactly one xy or yx is in the arc set for every $x \neq y$ in $V(T)$.

We can define the homomorphism density of a digraph in a tournamenton analogously as the homomorphism density between digraphs defined in (1.1).

Definition 1.2.1. Given a digraph D with k vertices and a tournamenton W, the *homomorphism density* of D in W is defined as

$$t(D,W) := \int_{[0,1]^k} \prod_{uv \in E(D)} W(x_u, x_v) \prod_{v \in V(D)} dx_v.$$

Moreover, given any measurable function $f : [0,1] \to [0,\infty)$, we let

$$t(D,W,f) = \int_{[0,1]^k} \prod_{uv \in E(D)} W(x_u, x_v) \prod_{v \in V(D)} f(x_v) dx_v.$$

Every tournament can be represented by its *step tournamenton.* Recall that a vector $w \in [0,1]^h$ is called a *stochastic vector*, if all its entries are non-negative and they sum up to one.

Definition 1.2.2. Let H be a tournament with h vertices and $w \in [0,1]^{V(H)}$ a stochastic vector. Let $(U_v)_{v \in V(H)}$ be a partition of $[0,1]$ into disjoint measurable sets such that the measure of U_v is w_v for all $v \in V(H)$. The *weighted step tournamenton* $W[H,w] : [0,1]^2 \to [0,1]$ is defined as follows

$$W[H,w](x,y) = \begin{cases} 1 & \text{if } x \in U_v,\ y \in U_z \text{ and } vz \in E(H) \\ 1/2 & \text{if } \{x,y\} \subseteq U_v \text{ for some } v \in V(H) \\ 0 & \text{otherwise.} \end{cases}$$

Additionally, the *blow-up* of H is the tournamenton $W[H] = W[H, \overrightarrow{h}]$ where $\overrightarrow{h} \in [0,1]^{V(H)}$ has all its entries equal to $1/h$.

The next is the definition of convergence for a sequence of tournaments.

Definition 1.2.3. A sequence of tournaments $(T_n)_{n\in\mathbb{N}}$ with $v(T_n) \to \infty$ is said to *converge* to a tournamenton W if

$$\lim_{n\to\infty} t(H, T_n) = t(H, W)$$

for every tournament H.

Remark 1.2.4. Note that a weighted step tournamenton $W = W[H,w]$ is the limit object of a sequence $H' = (H'_n)_{n\in\mathbb{N}}$ such that for each n, H'_n is a tournament on n vertices such that the vertex set can be partitioned into $(U_v)_{v\in V(H)}$ sets such that $\lfloor n \cdot w_v \rfloor \leq |U_v| \leq \lceil n \cdot w_v \rceil$ for every $v \in V(H)$ and with the condition that for $y \in U_v$ and $w \in U_z$, the arc $yw \in E(H'_n)$ with probability $1/2$ if $v = z$, probability 1 if $vz \in E(H)$ or probability 0 otherwise. This comes from the fact that for every tournament T,

$$t(T, W[H]) = \lim_{n\to\infty} t(T, H'_n)$$

with probability one.

The next two definitions are analogous to the out-degree and in-degree of a tournament. Given a tournamenton W and $S = \{x_1, \ldots, x_n\} \subseteq [0,1]$, we define the *common out-degree* of S to be

$$d^+_W(S) := \int_0^1 \prod_{i=1}^{n} W(x_i, y)dy$$

and the *common in-degree* to be

$$d^-_W(S) := \int_0^1 \prod_{i=1}^{n} W(y, x_i)dy.$$

If $S = \{x\}$ then we let $d^+(x) = d^+(S)$. Note that

$$\int_0^1 d^+(x)dx = \int_0^1 \int_0^1 W(x,y)dydx = \int_0^1 \int_0^1 (1 - W(y,x))\, dxdy = 1 - \int_0^1 d^+(y)dy$$

then, we get

$$2\int_0^1 d^+(x)dx = 1.$$

So we conclude $\int_0^1 d^+(x)dx = 1/2$ which is analogous to the property $\sum_{v\in V(T)} d^+(v) = \binom{v(T)}{2}$ for any tournament T.

Definition 1.2.5. A tournamenton W is *regular* if $d^+(x) = 1/2$ for almost every $x \in [0,1]$.

In the definition above, almost every means that the set $\{x \in [0,1] : d^+(x) \neq 1/2\}$ has measure zero. The next proposition relates the property of a tournament T being regular and the blow-up W_T being regular.

Proposition 1.2.6. *Let T be a tournament. Then T is regular if and only if the blow-up of T, W_T, is regular.*

Proof. Let T be a tournament on h vertices. Also, let $(U_v)_{v\in V(T)}$ be a partition of $[0,1]$ into disjoints sets with measure equal to $1/h$. Let $v \in V(T)$ and $x \in U_v$, we have

$$d^+_W(x) = \int_0^1 W(x,y)dy = \sum_{\{y\in U_w | vw\in E(T)\}} \int_{U_w} W(x,y)dy + \int_{U_v} W(x,y)dy = \frac{d^+_T(v)}{h} + \frac{1}{2h}.$$

First, assume T is regular, then for every $v \in V(T)$, $d^+_T(v) = (h-1)/2$ and by the last equation we conclude that for every $x \in U_v$,

$$d^+_W(x) = \frac{1}{h}\left(\frac{h-1}{2} + \frac{1}{2}\right) = \frac{1}{2}.$$

Conversely, if $d^+_W(x) = 1/2$, then

$$d^+_T(v) = h \cdot d^+_W(x) - \frac{1}{2} = \frac{h-1}{2}$$

as desired. □

Now, we rephrase the ideas related to randomness and quasirandomness in the language of limit theory. First, given a tournamenton W, we define the *W-random tournament* T as follows: pick k points $x_1, \ldots, x_k$ from $[0,1]$ uniformly at random and independently from one another and define $V(T) = \{x_1, \ldots, x_k\}$. Then add an arc from x_i to x_j with probability $W(x_i, x_j)$ and, otherwise, add an arc from x_j to x_i. Given a digraph D, the homomorphism density from D to W, $t(D, W)$ given in Definition 1.2.1 is precisely the probability that all arcs of D are contained in a W-random tournament T with $v(D)$ vertices.

Additionally, the following proposition is a standard extension of a fundamental theorem in graph limits of Lovász and Szegedy [33] and it will be useful to prove the main theorems of this work. It states that given a tournamenton W it is always possible to find a sequence $(T_n)_{n\in\mathbb{N}}$ that converges to W and conversely, every sequence of tournaments of unbounded order has a convergent subsequence.

Proposition 1.2.7. *For every sequence $(T_n)_{n\in\mathbb{N}}$ of finite tournaments with $v(T_n) \to \infty$, there is a subsequence of $(T_n)_{n\in A}$ for some $A \subseteq \mathbb{N}$ that converges to a tournamenton W. Conversely, given a tournamenton W, if $(T_n)_{n\in\mathbb{N}}$ is a sequence of W-random tournaments with $v(T_n) \to \infty$, then $(T_n)_{n\in\mathbb{N}}$ converges to W with probability* 1.

The next proposition follows from standard results in combinatorial limit theory; in particular, it is a slight extension of [26, Proposition 1].

Proposition 1.2.8. *Let $S \subseteq \mathbb{R}[\mathcal{T}]$. The set S forces quasirandomness if and only if every tournamenton W such that $t(H, W) = t(H, 1/2)$ for all $H \in S$ has the property that $W(x, y) = 1/2$ for almost all $(x, y) \in [0,1]^2$.*

The last proposition will be used several times in Chapters 2 and 3 to show positive results— i.e. when a tournament forces quasirandomness, and negative results — i.e. when a tournament does not force quasirandomness. The proof ideas are very similar; for the positive case, we usually show that given $H \in \mathbb{R}[\mathcal{T}]$, if $t(H, W) = t(H, 1/2)$ then $W = 1/2$ almost everywhere. On the other hand, for the negative case, we usually give an example of a tournamenton W such that $t(H, W) = t(H, 1/2)$ but the set $\{(x, y) \in [0,1]^2 | W(x, y) \neq 1/2\}$ has measure bigger than 0.

We have stated all the basic definitions necessary for developing this work. In the next section, we present the problem of forcing quasirandomness restricted to the set of regular tournaments and state our main results.

1.3 Quasirandom forcing in regular tournaments

The main results of this work are obtained by restricting the class of tournament sequences, considering only *nearly regular* sequences and studying quasirandom forcing there. As we summarize in Chapter 2, the class of tournaments that forces quasirandomness is very restricted. We obtain a wider class of tournaments forcing quasirandomness in the setting of nearly regular sequences. Most of the definitions in this section are given in the language of combinatorial limits but all of them have an analogue in the discrete language as stated in [37].

Definition 1.3.1. A sequence of tournaments $(T_n)_{n\in\mathbb{N}}$ is *nearly regular* if for every $\epsilon > 0$, there exists $\eta_0(\epsilon)$ such that, for all $n \geq \eta_0(\epsilon)$, all but at most $\epsilon \cdot v(T_n)$ vertices of T_n have out-degree between $(1/2-\epsilon)v(T_n)$ and $(1/2+\epsilon)v(T_n)$.

Note that if a sequence of tournaments $(T_n)_{n\in\mathbb{N}}$ is nearly regular and converges to a tournamenton W, then W is regular as in Definition 1.2.5.

Additionally, a sequence $(T_n)_{n\in\mathbb{N}}$ of uniformly random tournaments of increasing size is nearly regular with probability one. Indeed, given a vertex $v \in T_n$ let X_u^v be the random variable equal to 1 if $vu \in E(T_n)$ and 0 otherwise. Also, let $X^v = \sum_{u\in V(T_n)} X_u^v$ be the random variable counting the number of out-neighbours of v. We can calculate its expected value

$$\mathbb{E}[X^v] = \sum_{u\in V(T_n)} \mathbb{P}(X_u^v = 1) = \frac{n-1}{2}.$$

We want to show that all vertices have degree close to $(n-1)/2$ simultaneously. Then, let us use the union bound to estimate the probability that for some $v \in V(T_n)$ and every $\epsilon > 0$, $X^v > (1+\epsilon)\mathbb{E}[X^v]$.

$$\mathbb{P}\left[\bigcup_{v\in V(T_n)} \left(X^v > (1+\epsilon)\frac{n-1}{2}\right)\right] \leq \sum_{v\in V(T_n)} \mathbb{P}\left[X^v > (1+\epsilon)\frac{n-1}{2}\right].$$

By Chernoff bound A.0.5, we have

$$\mathbb{P}\left[X^v > (1+\epsilon)\frac{n-1}{2}\right] < e^{-\frac{\epsilon^2}{3}\cdot\frac{n-1}{2}} = e^{\frac{-(n-1)\epsilon^2}{6}}.$$

Putting the two last equations together we get

$$\mathbb{P}\left[\bigcup_{v\in V(T_n)}\left(X^v > (1+\epsilon)\frac{n-1}{2}\right)\right] \leq n\cdot e^{\frac{-(n-1)\epsilon^2}{6}}.$$

Since $\lim_{n\to\infty} n\cdot e^{\frac{-(n-1)\epsilon^2}{6}} = 0$, we conclude that almost every vertex on T_n have degree at most $(1+\epsilon)(n-1)/2$ with high probability when $n \to \infty$.

A similar argument but using the second part of the Chernoff bound A.0.5, allows us to conclude that, with high probability, almost every vertex on T_n has degree at least $(1-\epsilon)(n-1)/2$ when $n \to \infty$. Therefore, a random sequence of tournaments is nearly regular. For further reading on the probabilistic method in combinatorics, see the book of Molloy and Reed [34]or the book of Alon and Spencer [1].

Now, we restrict Definition 1.1.3 of quasirandom forcing by considering only sequences of nearly regular tournaments.

Definition 1.3.2. A tournament H on k vertices is said to *force quasirandomness in regular tournaments* if any regular tournamenton W such that $t(H,W) = (1/2)^{\binom{k}{2}}$ satisfies $W(x,y) = 1/2$ for almost every $(x,y) \in [0,1]^2$.

Next is our main result which characterizes tournaments on at most 5 vertices that force quasirandomness in regular tournaments, showing that this class of tournaments seems to be significantly broader than those that force quasirandomness in general. Some cases of the next theorem follow from earlier results [7, 17, 31].

Theorem 1.3.3. *Let $H_0, \dots, H_{19}$ be the list of all tournaments on at most 5 vertices up to isomorphism, as in Figure 1.1.*

1. *The tournaments H_i for $i \in \{0,1,2,3,9,12,16,18,19\}$ do not force quasirandomness in regular tournaments.*

2. *The tournaments H_i for $i \in \{4,5,6,7,8,10,11,13,14,15,17\}$ force quasirandomness in regular tournaments.*

In Section 3.2 we present the proof of Theorem 1.3.3. For this, we need the following well-known results.

This first result is a mild extension of a classical result of Kendall and Babington Smith [27], which states that the density of the transitive tournament on 3 vertices, Tr_3, is minimized by a regular tournamenton.

Proposition 1.3.4. *Every tournamenton W satisfies $t(Tr_3, W) \geq 1/8$ with equality if and only if W is regular.*

Proof. Let us write $t(Tr_3, W)$ in two different ways as follows

$$\begin{aligned} t(Tr_3, W) &= \int_{[0,1]^3} W(x,y)W(x,z)W(y,z)dxdydz \\ &= \int_{[0,1]^3} W(x,y)W(x,z)W(z,y)dxdydz. \end{aligned}$$

We can add the above expressions together. Since W is a tournamenton, $W(y,z) + W(z,y) = 1$, and then we have

$$\begin{aligned} 2t(Tr_3, W) &= \int_{[0,1]^3} W(x,y)W(x,z)\left(W(y,z) + W(z,y)\right)dxdydz \\ &= \int_{[0,1]^3} W(x,y)W(x,z)dxdydz \\ &= \int_0^1 \left(\int_0^1 W(x,y)dy\right)\left(\int_0^1 W(x,z)dz\right)dx \\ &= \int_0^1 \left(d^+(x)\right)^2 dx \\ &\geq \left(\int_0^1 d^+(x)dx\right)^2 = \frac{1}{4}. \end{aligned}$$

The last inequality above is a consequence of Jensen's Inequality A.0.4 since the function $f(z) = z^2$ is convex. Equality holds if and only if $d^+(x) = d^+(y)$ for almost every $x, y \in [0,1]$, which can occur only if $d^+(x) = 1/2$ for almost every $x \in [0,1]$. $\square$

The next result is also a well-known proposition of Beineke and Harary [3] on Turán density of a cycle C_3, which states that a regular tournamenton maximizes the density of the cyclic tournament on 3 vertices.

Proposition 1.3.5. *Every tournamenton W satisfies $t(C_3, W) \leq 1/8$ with equality if and only if W is regular.*

Proof. By the argument shown in the proof of Proposition 1.3.4, we know that $2t(Tr_3, W)$ equals $\int_0^1 \left(d^+(x)\right)^2 dx$ and an analogous argument allows us to conclude that $2t(Tr_3, W)$ equals $\int_0^1 \left(d^-(x)\right)^2 dx$. Also, we observe that

$$t(C_3, W) = \int_{[0,1]^3} W(y,x)W(x,z)W(z,y)dxdydz.$$

By using the definition of homomorphism density, we have

$$\begin{aligned}
t(C_3, W) + t(Tr_3, W) &= \int_0^1 \int_0^1 \int_0^1 W(y,x)W(x,z)dxdydz \\
&= \int_0^1 \left(\int_0^1 W(y,x)dy \right) \left(\int_0^1 W(x,z)dz \right) dx \\
&= \int_0^1 d^-(x)d^+(x)dx.
\end{aligned}$$

Putting it all together we get

$$\begin{aligned}
2t(C_3, W) + 6t(Tr_3, W) &= \int_0^1 \left(d^+(x)\right)^2 dx + 2\int_0^1 d^+(x)d^-(x)dx + \int_0^1 \left(d^-(x)\right)^2 dx \\
&= \int_0^1 \left(d^+(x) + d^-(x)\right)^2 dx \\
&= \int_0^1 \left(\int_0^1 W(x,y)dy + \int_0^1 W(y,x)dy \right)^2 dx = \int_0^1 dx = 1.
\end{aligned}$$

The above expression implies that

$$t(C_3, W) = \frac{1}{2} - 3t(Tr_3, W). \tag{1.2}$$

The result follows by Proposition 1.3.4. □

Chapter 2

Quasirandom forcing in tournaments

The objective of this chapter is to present some earlier well known results characterizing all possible tournaments that force quasirandomness. This is an expository chapter and the ideas are mostly taken from [4, 17, 26].

Proposition 2.0.1. *The following characterize all tournaments that force quasirandomness.*

1. *Transitive tournaments force quasirandomness for* $k \geq 4$ *[31, Exercise 10.44].*
2. H_{17} *from Figure 1.1 forces quasirandomness [17].*
3. *There are no non-transitive tournaments on at least* 7 *vertices forcing quasirandomness [4].*
4. *No other tournament forces quasirandomness with at most* 6 *vertices [26].*

The rest of the chapter is organized as follows: in Section 2.1 we prove Proposition 2.0.1(1) and 2.0.1(3) and in Section 2.2 we show some ideas that allow to conclude 2.0.1(4).

About 2.0.1(2), Coregliano, Parente and Sato [17] used the Flag Algebras method to show that H_{17} forces quasirandomness. We do not present the details of the proof in this survey chapter, but we will provide an introduction to Flag Algebras in Chapter 4 because this method is used to prove several of our main results.

2.1 Transitive tournaments

In this section, we prove Proposition 2.0.1(1) and 2.0.1(3), presenting the ideas of Bucić, Long, Shapira and Sudakov [4] in the language of Limit Theory. We need the following lemma, which gives us a sufficient condition for a tournamenton to be quasirandom. This Lemma is a slight extension of [31, Exercise 10.44].

Lemma 2.1.1. *Let W be a tournamenton. If $d^+(\{x,y\}) = 1/4$ for almost every $x, y \in [0,1]$, then $W(x,y) = 1/2$ for almost every $x, y \in [0,1]$.*

Proof. By definition of in-degree, we have

$$\begin{aligned}\int_0^1 d^-(x)^2 dx &= \int_0^1 \left(\int_0^1 W(y,x)dy\right)^2 dx\\ &= \int_0^1 \left(\int_0^1 W(y,x)dy\right)\left(\int_0^1 W(z,x)dz\right) dx\\ &= \int_0^1 \int_0^1 \int_0^1 W(y,x)W(z,x)dxdydz\\ &= \int_0^1 \int_0^1 d^+(\{y,z\})dydz = \frac{1}{4}.\end{aligned}$$

On the other hand, by Jensen's Inequality A.0.4, we have

$$\frac{1}{4} = \int_0^1 d^-(x)^2 dx \geq \left(\int_0^1 d^-(x)dx\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

So, we conclude $d^-(x) = 1/2$ for almost every x because the function $\phi(z) = z^2$ is strictly convex. Since $d^+(x) + d^-(x) = 1$, we conclude $d^+(x) = 1/2$ as well.

Now, let us show that this also implies that $W(x,y) = 1/2$ for almost every $x, y \in [0,1]$. For this, let us show that $\left(\int_{A\times B}(2W(x,y)-1)\,dxdy\right)^4 = 0$ for every pair of measurable sets $A, B \subseteq [0,1]$. Let $f(x) = \int_B (2W(x,y)-1)dy$, $\mathbb{1}_A(x)$ be the indicator function of $x \in A$ and $\lambda(A)$ the measure of A. The following is a consequence of Jensen's inequality A.0.4 using the convex function $\phi(x) = x^2$, which holds for every measurable set C.

$$\begin{aligned}\left(\int_C f(x)dx\right)^2 = \left(\int_0^1 f(x)\mathbb{1}_C(x)dx\right)^2 &\leq \int_0^1 f(x)^2dx \int_0^1 \mathbb{1}_C(x)^2dx\\ &= \int_0^1 f(x)^2dx \cdot \lambda(C) \leq \int_0^1 f(x)^2dx.\end{aligned}$$

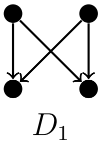
It gives us the following

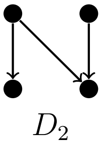
$$
\begin{aligned}
\left(\int_B\int_A (2W(x,y)-1)dxdy\right)^4 &\leq \left(\int_0^1 \left(\int_A (2W(x,y)-1)dx\right)^2 dy\right)^2\\
&= \left(\int_0^1 \left(\int_A (2W(x,y)-1)dx \int_A (2W(z,y)-1)dz\right) dy\right)^2\\
&= \left(\int_A\int_A\int_0^1 (2W(x,y)-1)(2W(z,y)-1)dydxdz\right)^2\\
&\leq \int_0^1\int_0^1 \left(\int_0^1 (2W(x,y)-1)(2W(z,y)-1)dy\right)^2 dxdz\\
&= \int_0^1\int_0^1\int_0^1\int_0^1 \prod_{\substack{m\in\{x,z\}\\ n\in\{y,w\}}} (2W(m,n)-1)dydwdxdz.
\end{aligned}
$$

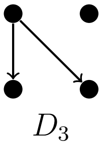
Now, let us expand on the last expression.

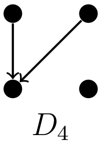
$$
\begin{aligned}
&\int_{[0,1]^4} \prod_{\substack{m\in\{x,z\}\\ n\in\{y,w\}}} (2W(m,n)-1)dxdydzdw\\
&\qquad = \int_{[0,1]^4} 16W(x,y)W(x,w)W(z,y)W(z,w)dydwdxdz\\
&\qquad - 4\int_{[0,1]^4} 8W(x,y)W(x,w)W(z,w)dxdydzdw\\
&\qquad + 2\int_{[0,1]^3} 4W(x,y)W(x,w)dxdydw + 2\int_{[0,1]^3} 4W(x,y)W(z,y)dxdydz\\
&\qquad + 2\int_{[0,1]^4} 4W(x,y)W(z,w)dxdy - 4\int_{[0,1]^2} 2W(x,y)dxdy + 1.
\end{aligned}
$$

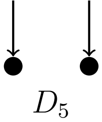
The last expression can be rephrased in terms of the homomorphism density of the following digraphs.


D_1


D_2


D_3


D_4


D_5

We can calculate its value given that $d^+(\{x,y\}) = 1/4$ and $d^-(x) = d^+(x) = 1/2$

as follows

$$16\int_{[0,1]^4} W(x,y)W(x,w)W(z,y)W(z,w)dydwdxdz = 16t(D_1,W) = 16\cdot\frac{1}{16} = 1.$$
$$32\int_{[0,1]^4} W(x,y)W(x,w)W(z,w)dxdydzdw = 32t(D_2,W) = 32\cdot\frac{1}{8} = 4.$$
$$8\int_{[0,1]^3} W(x,y)W(x,w)dxdydw = 8t(D_3,W) = 8\cdot\frac{1}{4} = 2.$$
$$8\int_{[0,1]^3} W(x,y)W(z,y)dxdydz = 8t(D_4,W) = 2.$$
$$8\int_{[0,1]^4} W(x,y)W(z,w)dxdy = 8t(D_5,W) = 2.$$
$$8\int_{[0,1]^2} W(x,y)dxdy = 8\cdot\frac{1}{2} = 4.$$

All together give us

$$\left(\int_B\int_A (2W(x,y)-1)dxdy\right)^4 \leq 1-4+6-4+1 = 0$$

where the equality holds if and only if $W(x,y) = 1/2$ for almost every $x, y \in [0,1]$. $\square$

Now, we are ready to prove that the transitive tournament Tr_h for $h \geq 4$ forces quasirandomness.

Proof of Proposition 2.0.1(1). We first show that for every $h \geq 2$ and any tournamenton W, $t(Tr_h, W) \geq t(H, 1/2)$. Moreover, we will show that equality holds if and only if $W(x,y) = 1/2$ for almost every $x, y \in [0,1]$, for every $h \geq 4$.

For the first part of the proof, we will prove a more general statement. Let $f : [0,1] \to [0,\infty)$ be a measurable function. We will show that

$$t(Tr_h, W, f) \geq \left(\frac{1}{2}\right)^{\binom{h}{2}} \left(\int_0^1 f(x)dx\right)^h$$

for every $h \geq 2$ by induction on h. For the base case take $h = 2$, we have

$$t(Tr_2, W, f) = \int_0^1\int_0^1 f(x)W(x,y)f(y)dxdy$$

but also,

$$t(Tr_2, W, f) = \int_0^1 \int_0^1 f(x)W(y,x)f(y)dxdy$$

adding those expressions together we get

$$\begin{aligned}
2t(Tr_2, W, f) &= \int_0^1 \int_0^1 f(x)(W(x,y) + W(y,x))f(y)dxdy \\
&= \int_0^1 \int_0^1 f(x)f(y)dxdy \\
&= \int_0^1 f(x)dx \int_0^1 f(y)dy \\
&= \left(\int_0^1 f(x)dx\right)^2.
\end{aligned}$$

Now suppose $h \geq 3$ and let D_h be the digraph obtained from Tr_h by deleting the arc uv where u and v are the vertices such that $d^+_{Tr_h}(u) = h-1$ and $d^+_{Tr_h}(v) = h-2$. Let x and y be variables associated with the vertices u and v respectively.

$$\begin{aligned}
&2t(Tr_h, W, f) = t(D_h, W, f) \\
&= \int_0^1 \int_0^1 f(x)f(y) \left(\int_{[0,1]^{h-2}} \prod_{zw \in E} W(x_z, x_w) \prod_{z \in V} f(x_z)W(x, x_z)W(y, x_z)dx_z \right) dxdy
\end{aligned}$$

where $E = E(Tr_{h-2})$ and $V = V(Tr_{h-2})$. Let $g_{x,y} : [0,1] \to [0,\infty)$ be $g_{x,y}(z) = W(x,z)W(y,z)f(z)$. Then, we get

$$\begin{aligned}
t(D_h, W, f) &= \int_0^1 \int_0^1 f(x)f(y) \left(\int_{[0,1]^{h-2}} \prod_{zw \in E} W(x_z, x_w) \prod_{z \in V} g_{x,y}(x_z)dx_z \right) dxdy \\
&= \int_0^1 \int_0^1 f(x)f(y)t(Tr_{h-2}, W, g_{x,y})dxdy \\
&\geq \int_0^1 \int_0^1 f(x)f(y) \left(\frac{1}{2}\right)^{\binom{h-2}{2}} \left(\int_0^1 g_{x,y}(z)dz \right)^{h-2} dxdy.
\end{aligned}$$

The last inequality is by inductive hypothesis. Now, we are going to apply Jensen's inequality A.0.3 to the function $\varphi(z) = z^{h-2}$, which is convex for $h \geq 3$ and strictly convex for $h \geq 4$, we have

$$\begin{aligned}
&t(D_h, W, f) \\
&\quad \geq \left(\frac{1}{2}\right)^{\binom{h-2}{2}} \left(\int_0^1 \int_0^1 f(x)f(y)dxdy\right) \left(\frac{\int_0^1 \int_0^1 f(x)f(y)\left(\int_0^1 g_{x,y}(z)dz\right)dxdy}{\int_0^1 \int_0^1 f(x)f(y)dxdy}\right)^{h-2} \\
&\quad = \left(\frac{1}{2}\right)^{\binom{h-2}{2}} \frac{\left(\int_0^1 \int_0^1 \int_0^1 f(x)f(y)W(x,z)W(y,z)f(z)dzdxdy\right)^{h-2}}{\left(\int_0^1 f(x)dx\right)^{2(h-3)}} \\
&\quad = \left(\frac{1}{2}\right)^{\binom{h-2}{2}} \frac{\left(\int_0^1 f(z)\left(\int_0^1 f(x)W(x,z)dx\right)^2 dz\right)^{h-2}}{\left(\int_0^1 f(x)dx\right)^{2(h-3)}}.
\end{aligned}$$

We can apply Jensen's inequality A.0.3 one more time to get

$$\begin{aligned}
t(D_h, W, f) &\geq \left(\frac{1}{2}\right)^{\binom{h-2}{2}} \frac{\left(\left(\int_0^1 f(z)dz\right)\left(\frac{\int_0^1 f(z)\int_0^1 W(x,z)f(x)dxdz}{\int_0^1 f(z)dz}\right)^2\right)^{h-2}}{\left(\int_0^1 f(x)dx\right)^{2(h-3)}} \\
&= \left(\frac{1}{2}\right)^{\binom{h-2}{2}} \frac{\left(\int_0^1 \int_0^1 f(z)W(x,z)f(x)dxdz\right)^{2h-2}}{\left(\int_0^1 f(x)dx\right)^{3h-8}} \\
&= \left(\frac{1}{2}\right)^{\binom{h-2}{2}} \frac{t(Tr_2, W, f)^{2(h-2)}}{\left(\int_0^1 f(x)dx\right)^{3h-8}} = \left(\frac{1}{2}\right)^{\binom{h}{2}-1} \left(\int_0^1 f(x)dx\right)^h.
\end{aligned}$$

Then,

$$t(Tr_h, W, f) = \frac{1}{2}t(D_h, W, f) \geq \left(\frac{1}{2}\right)^{\binom{h}{2}} \left(\int_0^1 f(x)dx\right)^h.$$

In particular, if we set $f = 1$, then the above inequality tells us that $t(Tr_h, W) \geq (1/2)^{\binom{h}{2}}$ for every graphon W. Now, suppose that $h \geq 4$ and that $t(Tr_h, W) = (1/2)^{\binom{h}{2}}$. Our goal is to prove that $W = 1/2$ almost everywhere. The only way that $t(Tr_h, W) = (1/2)^{\binom{h}{2}}$ can hold is if all of the inequalities that we have established while proving the lower bound $t(Tr_h, W) \geq (1/2)^{\binom{h}{2}}$ hold with equality. In particular, our first application of Jensen's inequality (applied to the case that $f = 1$) must hold with equality. Plugging in $f = 1$ into that inequality and doing some cancellation tells us that the following must be true:

$$\int_0^1 \int_0^1 \left(\int_0^1 g_{x,y}(z)dz \right)^{h-2} dxdy = \left(\int_0^1 \int_0^1 \int_0^1 g_{x,y}(z)dzdxdy \right)^{h-2}.$$

Since $h \geq 4$, this equality can only hold if $\int_0^1 g_{x,y}(z)dz$ is the same for almost every choice of x and y. In the case that $f = 1$, we have that $g_{x,y}(z)$ is simply $W(x,z)W(y,z)$, and so $\int_0^1 g_{x,y}(z)dz$ is precisely $d^+(x,y)$. Thus, if equality holds, then almost every pair $x, y \in [0,1]$ has the same value of $d^+(x,y)$.

We will be done if we can show that $d^+(x,y) = 1/4$ for almost every pair x, y. On our way to proving this, we show that W is regular. To see this, we note that our second application of Jensen's inequality must also hold with equality (in the case $f = 1$). Therefore, we have

$$\int_0^1 \left(\int_0^1 W(x,z)dx \right)^2 dz = \left(\int_0^1 \int_0^1 W(x,z)dxdz \right)^2.$$

By strict convexity, this can only hold if the function $d^-(z) = \int_0^1 W(x,z)dx$ is the same for almost every $z \in [0,1]$, which means that W is regular. Now, recall that $\int_0^1 d^-(z)^2 dz = \int_0^1 \int_0^1 d^+(\{x,y\})dxdy$ as in the beginning of the proof of Lemma 2.1.1. Since W is regular, the left side of this equality is $1/4$ and, since and $d^+(x,y)$ is the same for almost every pair $x, y \in [0,1]$, we must have that $d^+(x,y) = 1/4$ for almost all $x, y \in [0,1]$. Thus, we are done by Lemma 2.1.1. □

So far we have proved that transitive tournaments with at least 4 vertices force quasirandomness. The next result will be used to prove Proposition 2.0.1(3) which states that there is no other tournament on more than 7 vertices that forces quasirandomness. This proposition appears as a remark in [4] and as [26, Proposition 2].

Proposition 2.1.2. *Let H be a tournament on k vertices that is not transitive. If there exists a tournamenton W such that W is not equal to $1/2$ almost everywhere and*

$$t(H,W) \geq \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

then, H is not quasirandom-forcing.

Proof. Let H be a non-transitive tournament on k vertices. Consider $W_{Tr} : [0,1]^2 \to [0,1]$, $W_H = [0,1]^2 \to [0,1]$ and $U_\alpha = [0,1]^2 \to [0,1]$, defined as follows

$$W_{Tr}(x,y) = \begin{cases} 1/2 & \text{if } x = y \\ 1 & \text{if } x < y \\ 0 & \text{otherwise.} \end{cases}$$

Note that W_{Tr} is the limit of the sequence of transitive tournaments. Finally, let

$$U_\alpha(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in [0,\alpha]^2 \\ W_{Tr}(x,y) & \text{otherwise.} \end{cases}$$

We have $t(H, U_1) = t(H, W) \geq \left(\frac{1}{2}\right)^{\binom{k}{2}}$ and $t(H, U_0) = 0$ because H is non-transitive. Since $t(H, U_\alpha)$ is a continuous function on α, by the intermediate value property there exists α such that $t(H, U_\alpha) = \left(\frac{1}{2}\right)^{\binom{k}{2}} = t(H, 1/2)$. Nevertheless, by definition $U_\alpha \neq 1/2$ almost everywhere; therefore by Proposition 1.2.8, we conclude that T is not quasirandom forcing. □

Proof of Proposition 2.0.1(3). Let T be a non-transitive tournament with $V(T) = h \geq 7$. Let $W_T : [0,1]^2 \to [0,1]$ be the blow-up of T, which is not equal to 1/2 almost everywhere. To lower bound $t(T, W_T)$ we use Remark 1.2.4. We observe that every map $\phi : V(T) \to V(T')$ such that $\phi(v_i) \in V_i$ for every $v_i \in V(T)$ is a homomorphism. The probability that a random function satisfies this condition is $(1/h)^h$. Then, we have

$$t(T, W_T) \geq \left(\frac{1}{h}\right)^h \geq \left(\frac{1}{2}\right)^{\binom{h}{2}}$$

for every $h \geq 7$.

Then, by Proposition 2.1.2 we conclude that no non-transitive tournament on at least 7 vertices forces quasirandomness. □

It remains to analyze the case of non-transitive tournaments on at most 6 vertices. We do this in the next section.

2.2 Tournaments with at most six vertices

So far, we have presented the earlier results that show that every transitive tournament on at least 4 vertices forces quasirandomness and that no non-transitive tournament on at least 7 vertices forces quasirandomness. In this section, we present some of the results from [26] to show that there are no more non-transitive quasirandom forcing tournaments besides H_{17}.

We first give some definitions to state two necessary conditions that a quasirandom forcing tournament has to display. We say that a tournament is *rigid* if it has no non-trivial automorphism. Also, two vertices u and v in a tournament are referred to as *twins* if $N^+(u)\backslash\{v\} = N^+(v)\backslash\{u\}$. A tournament with no twins is said to be *twin-free.*

The next two Propositions state that for H to be quasirandom forcing, it has to be strongly connected, or if it has 6 vertices, it has to be twin-free and rigid.

Proposition 2.2.1. *Let H be a non-transitive tournament. If H is not strongly connected, then H is not quasirandom-forcing.*

Proposition 2.2.2. *Let H be a non-transitive 6-vertex tournament. If H contains twins or has a non-trivial automorphism, then H is not quasirandom-forcing.*

Before presenting the proof of these Propositions, we analyze the small tournaments. We know that H_1 does not force quasirandomness because $t(H_1, W) = 1/2$ for every tournament W. The fact that $H_2 = Tr_3$ and $H_3 = C_3$ do not force quasirandomness follows from Proposition 1.3.4 and 1.3.5 respectively. Additionally, the results related to the Turán density of $H_6 = C_4$ imply that there exists a tournamenton W (known as the *carousel tournamenton*, which is not quasirandom, with $t(C_4, W) = 1/48 > 1/2^6$ [17], so by Proposition 2.1.2, C_4 is not quasirandom forcing, either. Finally, H_5 and H_7 are not quasirandom forcing by Proposition 2.2.1 because they are not strongly connected. Then, it remains to study the case of tournaments on 5 and 6 vertices.

For the case of tournaments with 5 vertices, Coregliano et al. [17] showed that all the strongly connected tournaments except H_{17} and H_{12} are not quasirandom forcing. Nevertheless, Hancock et al. [26] showed that H_{12} is not quasirandom forcing.

About tournaments with 6 vertices, there are 55 that are non-transitive, of which 20 are not strongly connected, 29 contain twins, and 15 have a non-trivial automorphism [26]. Then, by Propositions 2.2.1 and 2.2.2 it can be concluded that 41 out of 55 of them are not quasirandom forcing. The remaining 14 tournaments are shown to not be quasirandom forcing [26] by applying Proposition 2.1.2 to different constructions of step tournamentons. We omit the details for the remaining 14 tournaments.

In the rest of this section, we present the proof of Proposition 2.2.1 and Proposition 2.2.2 following the arguments from [26].

Proof of Proposition 2.2.1. Let H be a not strongly connected tournament with $k \geq 4$ vertices. We can partition the vertex set of H into two sets (X_1, X_2) such that all edges are oriented from X_1 to X_2. Let $k_1 = |X_1|$ and $k_2 = |X_2|$. Additionally, consider the following tournamenton defined for some $\alpha \in [0,1]$.

$$W_\alpha(x,y) = \begin{cases} 1 & \text{if } x < \alpha \ \text{ and } \ y > \alpha \\ 0 & \text{if } x > \alpha \ \text{ and } \ y < \alpha \\ 1/2 & \text{otherwise.} \end{cases}$$

Then, we find a lower bound for the density of H in W_α as follows. Let $A = [0,\alpha]$ and $B = (\alpha, 1]$.

$$\begin{aligned} t(H, W_\alpha) = &\int_{[0,1]^k} \prod_{uv \in E(H)} W(x_u, x_v) \prod_{u \in V(H)} dx_u \\ \geq &\int_{A^k} \prod_{uv \in E(H)} W(x_u, x_v) \prod_{u \in V(H)} dx_u + \int_{A^{k_1}} \int_{B^{k_2}} \prod_{uv \in E(H)} W(x_u, x_v) \prod_{u \in X_2} dx_u \prod_{u \in X_1} dx_u \\ &+ \int_{B^k} \prod_{uv \in E(H)} W(x_u, x_v) \prod_{u \in V(H)} dx_u \\ = &\alpha^k \left(\frac{1}{2}\right)^{\binom{k}{2}} + \alpha^{k_1}(1-\alpha)^{k_2} 2^{k_1 k_2} \left(\frac{1}{2}\right)^{\binom{k}{2}} + (1-\alpha)^k \left(\frac{1}{2}\right)^{\binom{k}{2}}. \end{aligned}$$

Where the inequality is strict if H has more than two strongly connected components. Now, we have to consider two different cases to lower bound this expression. First, assume that $2 \leq k_1 \leq k-2$, in this case, $k_1 k_2 \geq k_1 + k_2$. We can set $\alpha = 1/2$ to get that

$$t(H, W_\alpha) \geq 2^{-k+1} \left(\frac{1}{2}\right)^{\binom{k}{2}} + 2^{-k_1} 2^{-k_2} 2^{k_1 k_2} \left(\frac{1}{2}\right)^{\binom{k}{2}} \geq (1 + 2^{1-k}) \left(\frac{1}{2}\right)^{\binom{k}{2}} \geq \left(\frac{1}{2}\right)^{\binom{k}{2}}.$$

Then it remains to analyze the case $k_1 = 1$. We can find a lower bound for $t(H, W_\alpha)$

as follows.

$$\begin{aligned} t(H, W_\alpha) &\geq \alpha(1-\alpha)^{k-1}2^{k-1}\left(\frac{1}{2}\right)^{\binom{k}{2}} + (1-\alpha)^k\left(\frac{1}{2}\right)^{\binom{k}{2}} \\ &= \left(\frac{1}{2}\right)^{\binom{k}{2}} + \left(\frac{1}{2}\right)^{\binom{k}{2}}\alpha(2^{k-1}-k) + O(\alpha^2) \\ &> \left(\frac{1}{2}\right)^{\binom{k}{2}}. \end{aligned}$$

The last inequality holds because $2^{k-1} - k > 0$ since $k \geq 4$. Also, note that α is chosen to be small enough so that the $O(\alpha^2)$ is significantly small.

In both cases we showed that $t(H, W_\alpha) \geq \left(\frac{1}{2}\right)^{\binom{k}{2}}$ for some α and by Proposition 2.1.2, we conclude that H is not quasirandom forcing. □

Proof of Proposition 2.2.2. Let H be a non-transitive tournament with vertex set $V(H) = \{1, \ldots, 6\}$. Let $W_H = W[H]$ be the blow-up of H. Also, let $(U_1, \ldots, U_6)$ be the partition of $[0,1]$ as in Definition 1.2.2.

First, assume that H has a non-trivial automorphism $\phi : V(H) \to V(H)$. The following is a lower bound for $t(H, W_H)$.

$$\begin{aligned} &t(H, W_H) \\ &\quad\geq \int_{U_6\times\cdots\times U_1}\prod_{ij\in E(H)} W(x_i, x_j)\prod_{i=1}^{6} dx_i + \int_{U_{\phi(6)}\times\cdots\times U_{\phi(1)}}\prod_{ij\in E(H)} W(x_i, x_j)\prod_{i=1}^{6} dx_i \\ &\quad= \left(\frac{1}{6}\right)^6 + \left(\frac{1}{6}\right)^6 \\ &\quad= 2\cdot\left(\frac{1}{6}\right)^6 \geq \left(\frac{1}{2}\right)^{\binom{6}{2}}. \end{aligned}$$

On the other hand, if the vertices $\{1,2\} \subseteq V(H)$ are twins, then

$$\begin{aligned}
&t(H, W_H)\\
&\geq \int_{U_6\times\ldots\times U_1} \prod_{ij\in E(H)} W(x_i,x_j) \prod_{i=1}^{6} dx_i + \int_{U_1^2}\int_{U_3\times\ldots\times U_6} \prod_{ij\in E(H)} W(x_i,x_j) \prod_{3}^{6} dx_i dx_1 dx_2\\
&+ \int_{U_2^2}\int_{U_3\times\ldots\times U_6} \prod_{ij\in E(H)} W(x_i,x_j) \prod_{3}^{6} dx_i dx_1 dx_2\\
&= \left(\frac{1}{6}\right)^6 + \frac{1}{2}\left(\frac{1}{6}\right)^6 + \frac{1}{2}\left(\frac{1}{6}\right)^6\\
&=2 \cdot \left(\frac{1}{6}\right)^6 \geq \left(\frac{1}{2}\right)^{\binom{6}{2}}.
\end{aligned}$$

In both cases, we can conclude that H is not quasirandom forcing by Proposition 2.1.2. □

As a concluding remark, the known results about quasirandom forcing in tournaments yield that only transitive tournaments on more than 4 vertices and the tournament H_{17} forces quasirandomness.

Chapter 3

Quasirandomness in regular tournaments

In this chapter, we present the proof of Theorem 1.3.3. Recall that we are now "relaxing" the requirements for a tournament T to force quasirandomness, i.e., we only require that $t(T, W)$ equals the expected value for every regular tournamenton W (see Definition 1.3.2). Then, Theorem 1.3.3 states that 11 out of 16 tournaments on four or five vertices are quasirandom forcing in regular tournaments.

This chapter is organized as follows: in Section 3.1 we derive some conditions for forcing quasirandomness in regular tournaments. The results in this section are used in Section 3.2 to prove Theorem 1.3.3(1). On the other hand, in Section 3.3 we prove Theorem 1.3.3(2) using different techniques. In particular, we enounce four theorems derived from the Flag Algebras method which we will prove in Chapter 4.

3.1 Conditions for quasirandom forcing in regular tournaments

In this section, we derive some necessary and sufficient conditions for forcing quasirandomness in regular tournaments.

The next proposition presents three equivalent statements about sets that force quasirandomness if we consider that Tr_3 and C_3 are in the set. Recall Definition 1.1.3 and Proposition 1.2.8.

Proposition 3.1.1. *Let $S \subseteq \mathbb{R}[\mathcal{T}]$ be a set of finite formal linear combinations of tournaments. The following are equivalent:*

1. *S forces quasirandomness in regular tournaments,*
2. *$S \cup \{Tr_3\}$ forces quasirandomness,*
3. *$S \cup \{C_3\}$ forces quasirandomness.*

Proof. Let $S \subseteq \mathbb{R}[\mathcal{T}]$. First, assume that S forces quasirandomness in regular tournaments. Let W be a tournamenton, such that $t(H, W) = t(H, 1/2)$ for every $H \in S \cup \{Tr_3\}$. By Proposition 1.3.4, W is regular and since S forces quasirandomness in regular tournaments, we conclude $W(x, y) = 1/2$ for almost every $(x, y) \in [0, 1]^2$; therefore, $S \cup \{Tr_3\}$ forces quasirandomness. With a similar argument but using Proposition 1.3.5 instead, we can conclude that $S \cup \{C_3\}$ forces quasirandomness.

Now, assume that $S \cup \{Tr_3\}$ forces quasirandomness, and let W be a regular tournamenton such that $t(H, W) = t(H, 1/2)$ for every $H \in S$. Since W is regular, by Proposition 1.3.4 we conclude that $t(Tr_3, W) = t(H, 1/2) = 1/8$ and since $S \cup \{Tr_3\}$ forces quasirandomness, we conclude that $W = 1/2$ almost everywhere. Therefore, S forces quasirandomness. A similar argument using proposition 1.3.5 can be used to conclude that 3.1.1 (3) implies 3.1.1(1). $\square$

The next proposition gives us a sufficient condition to prove Theorem 1.3.3(2).

Proposition 3.1.2. *Let $H \in \mathcal{T}$ and $M \in \{Tr_3, C_3\}$. If there exists $\alpha, \beta \in \mathbb{R}$ such that the tournament $T = \alpha M + \beta H \in \mathbb{R}[\mathcal{T}]$ forces quasirandomness, then H forces quasirandomness in regular tournaments.*

Proof. First, assume that T forces quasirandomness. Then, $\{M, H\}$ forces quasirandomness. Indeed, let W be a graphon such that $t(M, W) = t(M, 1/2)$ and $t(H, W) = t(H, 1/2)$. By definition, W has to satisfy

$$t(T, W) = \alpha t(M, W) + \beta t(H, W) = \alpha t(M, 1/2) + \beta t(H, 1/2) = t(T, 1/2)$$

and since T forces quasirandomness, by Proposition 1.2.8, we conclude $W = 1/2$ almost everywhere; therefore $\{M, H\}$ forces quasirandomness. Finally, by Proposition 3.1.1, $\{H\}$ forces quasirandomness in regular tournaments. □

Finally, we show a necessary condition for quasirandom forcing in regular tournaments. Note that this Proposition is analogous to Proposition 2.1.2 but in this case, we deal exclusively with regular tournamentons.

Proposition 3.1.3. *Let $H \in \mathbb{R}[\mathcal{T}]$. If there exist regular tournamentons W_0 and W_1 such that*

$$t(H, W_0) < t(H, 1/2)$$

and

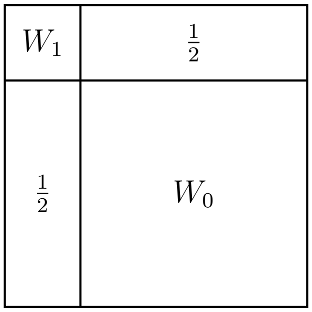
$$t(H, W_1) > t(H, 1/2),$$

then H does not force quasirandomness in regular tournaments.

Proof. For each $z \in (0, 1)$, let W_z be the tournamenton defined by

$$W_z(x, y) = \begin{cases} W_1(x/z, y/z) & \text{if } 0 \le x, y < z \\ W_0((x-z)/(1-z), (y-z)/(1-z)) & \text{if } z < x, y \le 1 \\ 1/2 & \text{otherwise.} \end{cases}$$

An illustration of the structure of this tournamenton in the case $z = 1/4$ is given below. Note that, in all pictures of tournamentons, the origin of $[0, 1]^2$ is in the top left corner, in analogy with adjacency matrices.



Since $t(H, W_z)$ is a continuous function such that $t(H, W_0) < t(H, 1/2)$ and $t(H, W_1) > t(H, 1/2)$, by the Intermediate Value Theorem, there exists $z \in (0, 1)$ such that $t(H, W_z) = t(H, 1/2)$. Since W_0 and W_1 are regular, then W_z is also regular. However, it is not the case that $W_z(x, y) = 1/2$ almost everywhere. Indeed, since $z > 0$, this would imply that $W_1(x, y) = 1/2$ almost everywhere as well but this would mean that $t(H, W_1) = t(H, 1/2)$ which contradicts the premises in the lemma. Then, by Proposition 1.2.8, H does not force quasirandomness in regular tournaments. $\square$

3.2 Negative results

The main objective in this section is to give a proof of Theorem 1.3.3(1). Given a tournament T, we let $W_T = W[T]$ be the blow-up of T as in Definition 1.2.2. Recall that T is regular if and only if W_T is regular by Proposition 1.2.6. We have to prove that the tournament H_i for $i \in \{0, 1, 2, 3, 9, 12, 16, 18, 19\}$ does not force quasirandomness in regular tournaments. The tournaments are depicted in Figure 1.1. Recall that for $i \in \{0, 1, 2, 3\}$, H_i is a tournament on at most 3 vertices and for $i \in \{9, 12, 16, 18, 19\}$, H_i is a tournament on 5 vertices.

Proof of Theorem 1.3.3(1). We separate this proof into three parts.

1. First, we prove that the tournaments H_i for $i \in \{0, 1, 2, 3\}$ does not force quasirandomness in regular tournaments. For this part, consider W_{C_3}, which is not equal to $1/2$ almost everywhere. We have $t(H_0, W_{C_3}) = 1$ and $t(H_1, W_{C_3}) = 1/2$. Then, by Proposition 1.2.8, H_0 and H_1 do not force quasirandomness in regular tournaments.

   For the case $H_2 = Tr_3$ and $H_3 = C_3$, by Propositions 1.3.4 and 1.3.5 we know that $t(H_2, W_{C_3}) = T(H_3, W_{C_3}) = 1/8$, because W_{C_3} is regular.

   Therefore, H_i for $i \in \{0, 1, 2, 3\}$ does not force quasirandomness in regular tournaments. Note that this proof can be rewritten using W_T, instead of W_{C_3}, for any regular tournament T for which $W_T \neq 1/2$ almost everywhere.

2. Now, we prove that the tournaments H_i for $i \in \{9, 16\}$ do not force quasirandomness in regular tournaments. For this part, we consider the tournamenton W_{C_3} which is regular but is not $1/2$ almost everywhere. The goal is to show that $t(H_9, W_{C_3}) = t(H_{16}, W_{C_3}) = 2^{-10}$.

   The first observation is that H_{16} is obtained from H_9 by reversing all the arcs, and so it is enough to show that $t(H_9, W_{C_3}) = 2^{-10}$. For this, consider the enumeration of the vertices of H_9 as shown in Figure 3.1.

   Let $\phi : V(H_9) \to \{1, 2, 3\}$. To calculate $t(H_9, W_{C_3})$ we refer to Remark 1.2.4 and estimate the probability that $\phi \in \mathrm{Hom}(H_9, C_3')$.

   Since $\{u_3, u_4, u_5\}$ induces a cycle in H_9, u_1 is a sink and the vertices u_1 and u_2 are twins, we conclude that $\phi(u_3) = \phi(u_4) = \phi(u_5)$. If $|\phi(V(H_9))| = 1$, then the probability that ϕ is a homomorphism is $3(1/3)^5(1/2)^{10}$. If $|\phi(V(H_9))| = 2$ then

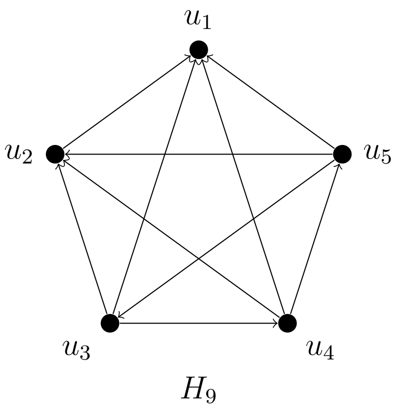


Figure 3.1: Enumeration of the vertices of H_9.

there are two cases to consider. First, if $\phi(u_1) = \phi(u_2) \neq \phi(u_3)$, the probability that ϕ is a homomorphism is $3(1/3)^5(1/2)^4$. Otherwise, $\phi(u_2) = \phi(u_3)$ and $\phi(u_1) \neq \phi(u_2)$, in this case the probability is $3(1/3)^5(1/2)^6$.

Therefore, we have

$$t(H_9, W_{C_3}) = 3\left(\frac{1}{3}\right)^5\left(\frac{1}{2}\right)^4\left(\left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^2 + 1\right) = 2^{-10}$$

as desired. Then, H_9 and H_{16} do not force quasirandomness in regular tournaments.

3. Finally, we prove that the tournaments H_i for $i \in \{12, 18, 19\}$ do not force quasirandomness in regular tournaments. For this part of the proof, we will use Proposition 3.1.3. Then, let $z \in [0, 1/2]$ and consider the tournamenton defined by

$$U_z(x, y) = \frac{1}{2} + z(2 \cdot W_{C_3}(x, y) - 1)$$

for every $(x, y) \in [0, 1]^2$.

Visually, U_z is the function obtained by dividing $[0, 1]^2$ into a regular 3×3 grid and assigning values to points in each square of the grid according to the following diagram.

$\frac{1}{2}$	$\frac{1}{2}+z$	$\frac{1}{2}-z$
$\frac{1}{2}-z$	$\frac{1}{2}$	$\frac{1}{2}+z$
$\frac{1}{2}+z$	$\frac{1}{2}-z$	$\frac{1}{2}$

Note that $U_0 = 1/2$ and $U_{1/2} = W_{C_3}$. Then, using a computer, we calculate $t(H_i, U_z)$ and we get that

$$t(H_{12}, U_z) = \frac{1}{1024} - \frac{1}{96}z^4 + \frac{7}{108}z^6 + \frac{1}{18}z^8,$$
$$t(H_{18}, U_z) = \frac{1}{1024} - \frac{1}{96}z^4 - \frac{1}{108}z^6 - \frac{7}{162}z^8, \text{ and}$$
$$t(H_{19}, U_z) = \frac{1}{1024} + \frac{5}{288}z^4 - \frac{5}{36}z^6 + \frac{5}{162}z^8.$$

From this, we see that

$$t(H_{12}, U_{1/2}) = \frac{43}{27648} > \frac{1}{1024},$$
$$t(H_{12}, U_{1/3}) = \frac{57161}{60466176} < \frac{1}{1024},$$
$$t(H_{18}, U_{1/2}) = \frac{1}{82944} < \frac{1}{1024},$$
$$t(H_{19}, U_{1/3}) = \frac{546961}{544195584} > \frac{1}{1024},$$
$$t(H_{19}, U_{1/2}) = \frac{1}{82944} < \frac{1}{1024}.$$

Then, by Proposition 3.1.3 H_{12} and H_{19} do not force quasirandomness in regular tournaments. Additionally, to complete the proof for H_{18}, let T be the 7–vertex tournament, found by an exhaustive computer search through all tournaments on a small number of vertices, with adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

By computer, we have found that

$$t(H_{18}, W_T) \approx 0.001249886 > \frac{1}{1024}.$$

Then, H_{18} does not force quasirandomness in regular tournaments.

□

Note that to prove Theorem 1.3.3(1) we mainly rely on Proposition 3.1.3 and Definition 1.3.2. In the next section, we provide a proof for the cases of tournaments that force quasirandomness in regular tournaments.

3.3 Positive results

In this section, we present some propositions used in the proof of Theorem 1.3.3(2).

Proposition 3.3.1. *The tournaments H_i for $i \in \{4, 8, 17\}$ force quasirandomness in regular tournaments.*

Proof. Note that $H_4 = Tr_4$, $H_8 = Tr_5$. By Theorem 2.0.1, these 3 tournaments force quasirandomness so they also force quasirandomness in regular tournaments. $\square$

Proposition 3.3.2. *The tournaments H_i for $i \in \{5, 7\}$ force quasirandomness in regular tournaments.*

Proof. First, we notice that H_5 is a cycle on three vertices with a sink vertex and H_7 is a cycle on three vertices with a source. Therefore, an analogous argument to the one we show below for H_5 also works for H_7. Let $V(H_5) = \{1, 2, 3, 4\}$ where 1 is the sink vertex.

Let W be a regular tournamenton and $f : [0,1] \to [0, \infty)$ be a measurable function. Also, let $x = x_1$, $y = x_2$, $z = x_3$, and $w = x_4$, we have

$$t(H_5, W, f) = \int_0^1 \left(\int_{[0,1]^3} W(y,w)W(w,z)W(z,y)f(x) \prod_{u \in \{w,z,y\}} W(x,u)f(u)du \right) dx.$$

Then, we define the measurable function $g_x : [0,1] \to [0, \infty)$ by $g_x(s) = f(s)W(x,s)$. Therefore

$$\begin{aligned} t(H_5, W, f) &= \int_0^1 f(x) \left(\int_{[0,1]^3} W(y,w)W(w,z)W(z,y)g_x(y)g_x(z)g_x(w)dwdydz \right) dx \\ &= \int_0^1 f(x)t(C_3, W, g_x)dx. \end{aligned}$$

Now, we are going to find an expression that relates $t(C_3, W, g_x)$ with $\int_0^1 g_x(y)dy$ and $t(Tr_3, W, g_x)$. Let $d^+_{g_x}(s) := \int_0^1 W(s,y)g_x(y)dy$ and $d^-_{g_x}(s) := \int_0^1 W(y,s)g_x(y)dy$. We use a similar argument as in the proof of Proposition 1.3.5. Let us compute the sum of $t(C_3, W, g_x)$ and $t(Tr_3, W, g_x)$.

$$\begin{aligned}
t(C_3, W, g_x)+&t(Tr_3, W, g_x)\\
&=\int_{[0,1]^3} W(y,w)W(w,z)(W(z,y)+W(y,z))g_x(w)g_x(y)g_x(z)dydzdw\\
&=\int_{[0,1]^3} W(y,w)W(w,z)g_x(w)g_x(y)g_x(z)dydzdw\\
&=\int_0^1 g_x(w)\left(\int_0^1 W(w,z)g_x(z)dz\int_0^1 W(y,w)g_x(y)dy\right)dw\\
&=\int_0^1 g_x(w)d^+_{g_x}(w)d^-_{g_x}(w)dw.
\end{aligned}$$

Note that $d^+_{g_x}(s)+d^-_{g_x}(s)=\int_0^1(W(s,y)+W(y,s))g_x(y)dy=\int_0^1 g_x(y)dy$. Also, recall that

$$\begin{aligned}
2t(Tr_3, W, g_x) &= \int_{[0,1]^3} W(y,w)W(y,z)g_x(w)g_x(y)g_x(z)\\
&=\int_{[0,1]^3} g_x(y)(d^+_{g_x}(y))^2dy
\end{aligned}$$

and $2t(Tr_3, W, g_x)=\int_{[0,1]^3} g_x(y)(d^-_{g_x}(y))^2dy$. This is an analogous calculation as the one made for the proof of Proposition 1.3.4. The next equation gives us a relation between $t(C_3, W, g_x)$ and $t(Tr_3, W, g_x)$.

$$\begin{aligned}
\left(\int_0^1 g_x(y)dy\right)^3 &= \int_0^1 g_x(z)\left(\int_0^1 g_x(y)dy\right)^2 dz\\
&=\int_0^1 g_x(z)\left(d^+_{g_x}(z)+d^-_{g_x}(z)\right)^2dz\\
&=\int_0^1 g_x(z)(d^+_{g_x}(z))^2dz+2\int_0^1 g_x(z)d^+_{g_x}(z)d^-_{g_x}(z)dz\\
&\quad+\int_0^1 g_x(z)(d^-_{g_x}(z))^2dz\\
&=2t(Tr_3, W, g_x)+2(t(Tr_3, W, g_x)+t(C_3, W, g_x))+2t(Tr_3, W, g_x)\\
&=2t(C_3, W, g_x)+6t(Tr_3, W, g_x).
\end{aligned}$$

From the last equation, we conclude that

$$t(C_3, W, g_x) = \frac{1}{2}\left(\int_0^1 g_x(y)dy\right)^3 - 3t(Tr_3, W, g_x).$$

Putting all this together, we have

$$t(H_5, W, f) = \int_0^1 f(x)\left(\frac{1}{2}\left(\int_0^1 g_x(y)dy\right)^3 - 3t(Tr_3, W, g_x)\right)dx.$$

Finally, let us find equivalences to each of the terms in the last expression. First, by definition of $g_x(s)$, we have

$$\begin{aligned}\int_0^1 f(x)\left(\int_0^1 g_x(y)dy\right)^3 =& \int_{[0,1]^4} W(x,y)W(x,z)W(x,w)f(x)f(y)f(z)f(w)dxdydzdw \\ =& t(S_3^+, W, f)\end{aligned}$$

where S_3^+ is the digraph on 4 vertices, 3 arcs and a single source vertex. On the other hand,

$$\begin{aligned}&\int_0^1 f(x)t(Tr_3, W, g_x)dx \\ =& \int_{[0,1]^4} W(y,z)W(y,w)W(z,w)g_x(y)g_x(z)g_x(w)f(x)dxdydzdw \\ =& \int_{[0,1]^4} W(y,z)W(y,w)W(z,w)W(x,y)W(x,z)W(x,w)f(x)f(y)f(z)f(w)dxdydzdw \\ =& t(Tr_4, W, f).\end{aligned}$$

Then, we can write $t(H_5, W, f)$ as follows

$$t(H_5, W, f) = \frac{1}{2}t(S_3^+, W, f) - 3t(Tr_4, W, f).$$

If we let $f(z) = 1$ for every $z \in [0,1]$, we obtain

$$t(H_5, W) = \frac{1}{2}t(S_3^+, W) - 3t(Tr_4, W).$$

Now, assume $t(H_5, W) = (1/2)^6$. Since W is regular we have

$$\frac{1}{2^6} = \frac{1}{2}\frac{1}{2^3} - 3t(Tr_4, W)$$

then,

$$t(Tr_4, W) = \left(\frac{1}{2}\right)^6.$$

Since Tr_4 forces quasirandomness, we conclude that $W = 1/2$ almost everywhere. Therefore, H_5 forces quasirandomness in regular tournaments. $\square$

The next tournament to analyze is $H_6 = C_4$. To show that this tournament forces quasirandomness in regular tournaments, we use the result [7, Corollary 6], stated as follows.

Lemma 3.3.3. *If W is a tournamenton and $1/2 \leq z \leq 1$ such that*

$$t(C_3, W) = \frac{1}{8}\left(z^3 + (1-z)^3\right)$$

then

$$t(C_4, W) \geq \frac{1}{64}\left(z^4 + (1-z)^4\right)$$

where equality holds if and only if there exists a measurable function $f : [0,1] \to [0,1/2]$ such that $W(x,y) = 1/2 + f(x) - f(y)$ for almost every $(x,y) \in [0,1]^2$.

Proposition 3.3.4. *H_6 forces quasirandomness in regular tournaments.*

Proof. Let W be a regular tournamenton. By Proposition 1.3.5 we have $t(C_3, W) = 1/8$. Then, by Lemma 3.3.3 with $z = 1$ we conclude $t(C_4, W) \geq 1/64$. Now, if we assume $t(C_4, W) = 1/64$, Lemma 3.3.3 tell us that there exists a measurable funtion $f : [0,1] \to [0,1/2]$ such that $W(x,y) = 1/2 + f(x) - f(y)$ for almost every $(x,y) \in [0,1]^2$. Since W is regular, we have the following for almost all $x \in [0,1]$

$$\frac{1}{2} = d^+(x) = \int_0^1 W(x,y)dy = \int_0^1 \left(\frac{1}{2} + f(x) - f(y)\right) dy.$$

The last equation implies that $f(x) = \int_0^1 f(y)dy$, meaning that f is constant almost everywhere. Therefore, $W(x,y) = 1/2$ for almost every $(x,y) \in [0,1]^2$. Then, $H_6 = C_4$ forces quasirandomness in regular tournaments. $\square$

Now we state four theorems which together with Proposition 3.1.2 imply that H_i for $i \in \{10, 11, 13, 14\}$ force quasirandomness in regular tournaments. The proofs of these theorems are given in the next Chapter using the flag algebra method.

Theorem 3.3.5. *For every tournamenton W,*

$$8 \cdot t(C_3, W) + \frac{1}{4} \cdot 1024 \cdot t(H_{10}, W) \leq \frac{5}{4}$$

where equality holds if and only if $W(x, y) = 1/2$ for almost all $(x, y) \in [0, 1]^2$.

Theorem 3.3.6. *For every tournamenton W,*

$$8 \cdot t(C_3, W) + \frac{1}{4} \cdot 1024 \cdot t(H_{11}, W) \leq \frac{5}{4}$$

where equality holds if and only if $W(x, y) = 1/2$ for almost all $(x, y) \in [0, 1]^2$.

Theorem 3.3.7. *For every tournamenton W,*

$$8 \cdot t(Tr_3, W) + \frac{1}{7} \cdot 1024 \cdot t(H_{13}, W) \geq \frac{8}{7}$$

where equality holds if and only if $W(x, y) = 1/2$ for almost all $(x, y) \in [0, 1]^2$.

Theorem 3.3.8. *For every tournamenton W,*

$$8 \cdot t(Tr_3, W) + \frac{1}{5} \cdot 1024 \cdot t(H_{14}, W) \geq \frac{6}{5}$$

where equality holds if and only if $W(x, y) = 1/2$ for almost all $(x, y) \in [0, 1]^2$.

Proof of Theorem 1.3.3(2). Note that H_{15} is obtained from H_{10} by reversing all the arcs. Then, by Theorems 3.3.5, 3.3.6, 3.3.7 and 3.3.8 and Proposition 3.1.2, together with Propositions 3.3.1, 3.3.2, and 3.3.4 we have that H_i for $i \in \{4, 5, 6, 7, 8, 10, 11, 13, 14, 15, 17\}$ forces quasirandomness in regular tournaments. $\square$

Then, it remains to prove Theorems 3.3.5–3.3.8. For this, we use the flag algebra method, introduced in the next chapter.

Chapter 4

Flag Algebras

The *flag algebra* method was introduced in 2007 by Razborov [38]. Since this method can be applied to any combinatorial structure described by a universal theory in a finite first-order language with equality and without constant and function symbols, this method was introduced in the language of model theory. Note that graphs, hypergraphs, digraphs and tournaments are examples of such theories. The main objective of this method is as follows: in a theory T, every set of elements induces a model. Therefore, given two finite models M and N, we let $d(M, N)$ be the density with which M appears as a sub-model of N, which is analogous to the homomorphism densities in the case of graphs or digraphs. Then, the objective is to find relations between $d(M_1, N), \dots, d(M_h, N)$ for some finite models $M_1, \dots, M_h$ in the limit, i.e. when N grows to infinity.

The first application of this method [38] was to present a new proof of a result by Fisher which found the minimal possible density of triangles in a graph with fixed edge density [23]. After this, the method has been used to solve various open problems. For instance, in 2008, Razborov solved the Erdős-Rademacher problem [39] which asks how many triangles are forced in an n-vertex graph with m edges. Other applications of this method includes problems involving tournaments [5,16,17,30] and graphs [2,35].

In Section 4.1 we introduce the method in the language of tournaments along the lines of the treatment in [6,35]. Then, in Section 4.2 we focus on proving Theorems 3.3.5, 3.3.6, 3.3.7, and 3.3.8 using Flag Algebras.

4.1 Introduction to the method

Given a tournament H and a tournamenton W, we define the *induced density* of H in W, as follows

$$d(H,W) = \frac{v(H)!}{\mathrm{aut}(H)} \cdot t(H,W)$$

where $\mathrm{aut}(H)$ is the number of automorphisms of H. Note that $d(H,W)$ is the probability that a W-random tournament with $v(H)$ vertices is isomorphic to H.

The first lemma in this section is just a consequence of the fact that the sum of the probabilities of all possible events equals 1.

Lemma 4.1.1. *For any $m \geq 1$ and a tournamenton W,*

$$\sum_{J:v(J)=m} d(J,W) = 1.$$

In Lemma 4.1.1, the sum is over all tournaments on m vertices up to isomorphism. The following is the definition of a *flag* restricted to the theory of digraphs. Note that in this document we focus mostly only in the case of tournaments, but this more general version will be convenient for some of the proofs presented in this section.

Definition 4.1.2. A *flag* is a pair (D,r) where D is a digraph such that $V(D) = \{1,\dots,v(D)\}$ and $r \leq v(D)$. The vertices $\{1,\dots,r\}$ are known as the *roots* of the flag.

The following definition is the homomorphism density of a tournament H in a tournamenton W integrating only over the variables x_i for $i \in \{r+1,\dots,v(H)\}$; i.e. integrating over the variables associated to the non-root vertices.

Definition 4.1.3. Let W be a tournamenton, D be a digraph and (D,r) be a flag. Define $t_r(D,W) : [0,1]^r \to [0,1]$ by

$$t_r(D,W)(x_1,\dots,x_r) = \int_{[0,1]^{v(D)-r}} \prod_{ij\in E(D)} W(x_i,x_j)dx_{r+1}\dots dx_{v(D)}.$$

Now, given a tournament R with $V(R) = \{1,\dots,r\}$ and a graph G with vertex set $\{1,\dots,m\}$ for $r \leq m$ such that G contains a clique on $\{1,\dots,r\}$, we define $\mathcal{F}_G(R)$ to be the set of all flags (D,r) where D is obtained from orienting the edges of G so that the subdigraph induced by $\{1,\dots,r\}$ equals R (not just isomorphic to R, but exactly the same arc set). The following result gives us a relationship between $t(R,W)$ and $t_r(D,W)$ for $(D,r) \in \mathcal{F}_G(R)$

Lemma 4.1.4. *Let R be a digraph with $V(R) = \{1,\dots,r\}$ and let G be a graph with $V(G) = \{1,\dots,m\}$ such that $m \geq r$. Then, for any tournamenton W we have*

$$\sum_{(D,r)\in\mathcal{F}_G(R)} t_r(D,W)(x_1,\dots,x_r) = t_r(R,W)(x_1,\dots,x_r).$$

Proof. The proof is by induction on $e(G)$. First, if $e(G) = \binom{r}{2}$, then $\mathcal{F}_G(R) = \{R\}$ and the equation is clearly satisfied. Now, assume that $e(G) > \binom{r}{2}$ and let $uv \in E(G)\backslash E(R)$. Note that for any $(D,r) \in \mathcal{F}_G(R)$ such that $uv \in e(D)$, there exists D' containing all the same arcs as D except that the edge uv is oriented in the other direction. Let D'' be obtained from D by deleting uv. Then, since $W(x,y) + W(y,x) = 1$ for almost every $x, y \in [0,1]$, we have

$$\begin{aligned} t_r(D,W)(x_1,\dots,x_r) + t_r(D',W)(x_1,\dots,x_r) \\ = & t_r(D'',W)(x_1,\dots,x_r)(W(x_u,x_v) + W(x_v,x_u)) \\ = & t_r(D'',W)(x_1,\dots,x_r). \end{aligned}$$

Therefore, we can pair up all digraphs in $\mathcal{F}_G(R)$ to get

$$\sum_{(D,r)\in\mathcal{F}_G(R)} t_r(D,W)(x_1,\dots,x_r) = \sum_{(D'',r)\in\mathcal{F}_{G\backslash\{uv\}}(R)} t_r(D'',W)(x_1,\dots,x_r).$$

By induction, the right-hand side of the last equality is $t_r(R,W)(x_1,\dots,x_r)$. This completes the proof. □

Now, given $r \leq k$ and a tournament R with $V(R) = \{1,\dots,r\}$, let $\mathcal{F}_k(R) := \mathcal{F}_{K_k}(R)$, where K_k is the complete graph on k vertices. Additionally, two flags $(F_1,r),(F_2,r) \in \mathcal{F}_k(R)$ are said to be *R-compatible.*

Corollary 4.1.5. *Let R be a tournament with $V(R) = \{1,\dots,r\}$ and let $k \geq r$. For any tournament F and tournamenton W, we have*

$$\sum_{(F,r)\in\mathcal{F}_k(R)} t_r(F,W)(x_1,\dots,x_r) = t_r(R,W)(x_1,\dots,x_r).$$

Proof. Since $\mathcal{F}_k(R) = \mathcal{F}_{K_k}(R)$, We are done by Lemma 4.1.4. □

Let us compute the cardinality of $\mathcal{F}_k(R)$ because we are going to use it to prove the next result. Note that the vertex set of any $F \in \mathcal{F}_k(R)$ can be partitioned into the sets A, B such that $A = \{1, \ldots, r\}$ and $B = \{r+1, \ldots, k\}$. The edges between vertices of A are the same for every $F \in \mathcal{F}_k(R)$. Therefore, the number of tournaments in $F \in \mathcal{F}_k(R)$ equals the number of different orientations of the edges in B, which is $2^{\binom{k-r}{2}}$, times the number of possible orientation of the edges among the sets A and B, which equals to $2^{r(k-r)}$. Therefore, $|\mathcal{F}_k(R)| = 2^{\binom{k-r}{2}+r(k-r)}$.

For the next result, we also need the next definition: given tournaments H and J with $v(H) \leq v(J)$, define the *injective homomorphism density* $t_{inj}(H, J)$ to be the probability that a uniformly random injective function from $V(H)$ to $V(J)$ is a homomorphism. That is,

$$t_{inj}(H, J) = \frac{|\{f : V(H) \to V(J)|\ f \text{ is an injective homomorphism}\}|}{|\{f : V(H) \to V(J)|\ f \text{ is an injective function}\}|}.$$

Finally, let $\mathcal{T}_{\leq m}$ be the set of all tournaments on at most m vertices up to isomorphism and $\mathbb{R}[\mathcal{T}_{\leq m}]$ be the set of all formal linear combinations of elements of $\mathcal{T}_{\leq m}$. If $H = \sum_{i=1}^{s} \alpha_i H_i \in \mathbb{R}[\mathcal{T}_{\leq m}]$ and J is a digraph, we let

$$t_{inj}(H, J) = \sum_{i=1}^{s} \alpha_i t_{inj}(H_i, J).$$

Now we are ready to state the next result, which allows us to estimate $t(H, W)$ in terms of $d(J, W)$ for a tournament H and all possible tournaments J containing an induced copy of H.

Lemma 4.1.6. *For any $m \geq 1$, $H \in \mathbb{R}[\mathcal{T}_{\leq m}]$ and a tournamenton W,*

$$t(H, W) = \sum_{J:v(J)=m} t_{inj}(H, J) \cdot d(J, W).$$

where the sum is over all m-vertex tournaments J up to isomorphism.

Proof. We may assume that H is a single tournament since the result in the general case $H \in \mathbb{R}[\mathcal{T}_{\leq m}]$ will follow from linearity. Let $V(H) = \{1, \ldots, r\}$. By Lemma 4.1.5 we have that

$$t_r(H, W)(x_1, \ldots, x_r) = \sum_{(J,r) \in \mathcal{F}_m(H)} t_r(F, W)(x_1, \ldots, x_r)$$

for almost every $(x_1, \ldots, x_r) \in [0,1]^r$. Integrating this over all $(x_1, \ldots, x_r) \in [0,1]^r$, we get

$$t(H, W) = \int_{[0,1]^r} t_r(H, W)(x_1, \ldots, x_r) dx_1 \ldots dx_r = \sum_{(J,r) \in \mathcal{F}_m(H)} t(J, W).$$

Note that there are $2^{r(m-r)+\binom{m-r}{2}}$ terms on the right side of the last equation, each of which is of the form $t(J, W)$ for some tournament J on m vertices such that $V(J) = \{1, \ldots, m\}$ and $\{1, \ldots, r\}$ induces H. To compute the coefficient of $t(J, W)$ we observe that the group Γ of bijections on $\{1, \ldots, m\}$ whose restriction to $\{1, \ldots, r\}$ is a homomorphism of H to J is a subgroup of the symmetric group S_m. Moreover, $\mathrm{Aut}(J)$ is a subgroup of Γ. Then, the coefficient of $t(J, W)$ equals to the number of cosets of $\mathrm{Aut}(J)$ in Γ which, equals to $\frac{\mathrm{hom}_{\mathrm{inj}}(H,J)(m-r)!}{\mathrm{aut}(J)}$. All together gives us

$$\begin{aligned} t(H, W) &= \sum_{F:v(J)=m} \frac{\mathrm{hom}_{inj}(H, J)(m-r)!}{\mathrm{aut}(J)} t(J, W) \\ &= \sum_{J:v(J)=m} \left(\frac{\mathrm{hom}_{inj}(H, J)(m-r)!}{m!} \right) \cdot \left(\frac{m!}{\mathrm{aut}(J)} t(J, W) \right) \\ &= \sum_{J:v(J)=m} t_{inj}(H, J) d(J, W) \end{aligned}$$

as desired. □

The next is a key definition for the Flag Algebras method and gives us the notion of multiplication on pairs of flags.

Definition 4.1.7. Let F be a tournament with vertex set $\{1, \ldots, r\}$. Given two R-compatible flags (F_1, r) and (F_2, r) and a tournamenton W, define $t_r(F_1 \cdot F_2, W) : [0,1]^r \to [0,1]$ by

$$t_r(F_1 \cdot F_2, W)(\vec{x}) = \begin{cases} \frac{t_r(F_1, W)(\vec{x}) \cdot t_r(F_2, W)(\vec{x})}{t_r(R, W)(\vec{x})} & \text{if } t_r(R, W)(\vec{x}) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\vec{x} = (x_1, \ldots, x_r)$.

The last definition is important because states that for two compatible R-flags the integral of $t_r(F_1 \cdot F_2, W)(x_1, \ldots, x_r)$ over $[0,1]^r$ can be expressed as a linear combination of $d(J, W)$, over all tournaments J with $v(F_1) + v(F_2) - v(R) = 2k - r$ vertices, as stated in the following lemma.

Lemma 4.1.8. *Let R be a tournament with vertex set $\{1, \ldots, r\}$ and (F_1, r) and (F_2, r) be elements of $\mathcal{F}_k(R)$ and let $m \geq 2k - r$. Then, there exist constants $b_r(F_1, F_2; J)$ for each tournament J on m vertices such that*

$$\int_{[0,1]^r} t_r(F_1 \cdot F_2, W)(x_1, \ldots, x_r) dx_1 \ldots dx_r = \sum_{J: v(J)=m} b_r(F_1, F_2; J) \cdot d(J, W)$$

for every tournamenton W, where the sum on the right side is over all m-vertex tournaments up to isomorphism.

We provide an example to illustrate how to compute the coefficients $b_r(F_1, F_2; J)$ in Lemma 4.1.8. For this, consider the flags $F_1^1 = (Tr_3, 2)$ and $F_3^1 = (C_3, 2)$, both elements of $\mathcal{F}_3(Tr_2)$ as depicted in Figure 4.1. We have

$$t_2(Tr_3, W)(x, y) = W(y, x) \int_0^1 W(z, y) W(z, x) dz,$$

$$t_2(C_3, W)(x, y) = W(y, x) \int_0^1 W(x, z) W(z, y) dz$$

and

$$t_2(Tr_2, W)(x, y) = W(y, x).$$

All together this gives us

$$\begin{aligned}
&\int_0^1 \int_0^1 t_2(Tr_3 \cdot C_3, W)(x, y) dx dy \\
&= \int_0^1 \int_0^1 W(y, x) \left(\int_0^1 W(z, x) W(z, y) dz \right) \left(\int_0^1 W(z, y) W(x, z) dz \right) dx dy \\
&= \int_0^1 \int_0^1 W(y, x) \left(\int_0^1 W(z, x) W(z, y) dz \right) \left(\int_0^1 W(w, y) W(x, w) dw \right) dx dy \\
&= \int_{[0,1]^4} W(y, x) W(z, x) W(z, y) W(w, y) W(x, w) dx dy dz dw.
\end{aligned}$$

Now, since $W(w, z) + W(z, w) = 1$, multiplying the last expression by this term, we have

$$\begin{aligned}
&\int_0^1\int_0^1 t_2(Tr_3\cdot C_3,W)(x,y)dxdy\\
&\qquad+\int_{[0,1]^4} W(y,x)W(z,x)W(z,y)W(w,y)W(x,w)W(w,z)dxdydzdw\\
&\qquad+\int_{[0,1]^4} W(y,x)W(z,x)W(z,y)W(w,y)W(x,w)W(z,w)dxdydzdw\\
&\qquad=t(H_6,W)+t(H_7,W)\\
&\qquad=\frac{1}{24}d(H_6,W)+\frac{3}{24}d(H_7,W).
\end{aligned}$$

Therefore, we have $b_2(F_1^1,F_3^1;H_5)=0$, $b_2(F_1^1,F_3^1;H_6)=\frac{1}{24}$, $b_2(F_1^1,F_3^1;Tr_4)=0$, and $b_2(F_1^1,F_3^1;H_7)=\frac{3}{24}$. A similar procedure can be done for any flags (F_1,r) and (F_2,r) in $\mathcal{F}_k(R)$ and tournaments J with at least $2k-r$ vertices.

Finally, the next lemma will be used in the proofs of Theorems 3.3.5–3.3.8, this is an important result to find a lower bound of the homomorphism density $t(H,W)$ using the coefficients of Lemma 4.1.8. For the definition of positive semidefinite matrices, see Definition A.0.6.

Lemma 4.1.9. *Let m be an integer and $H\in\mathbb{R}[\mathcal{T}_{\le m}]$. Let $\ell,t_1,\dots,t_l,r_1,\dots,r_l$, and $k_1,\dots k_l$ be positive integers such that $k_q\ge r_q+1$ and $2k_q-r_q\le m$ for all $1\le q\le l$. For each $1\le q\le l$, let $(F_1^q,r_q),\dots,(F_{t_q}^q,r_q)$ be pairwise compatible flags on vertex set $\{1,\dots,k_q\}$ and A_q be a positive semidefinite $t_q\times t_q$ matrix. Then, every tournamenton W satisfies*

$$t(H,W)\ge\min_{J:v(J)=m}\left\{t_{inj}(H,J)-\sum_{q=1}^{\ell}\sum_{i=1}^{t_q}\sum_{j=1}^{t_q}b_{r_q}(F_i^q\cdot F_j^q;J)\cdot(A_q)_{i,j}\right\}.$$

Moreover, if W is a tournamenton such that equality holds, then, for every $1\le q\le l$ and almost every $(x_1,\dots,x_{r_q})\in[0,1]^{r_q}$, the vector

$$\left(t_{r_q}(F_1^q,W)(x_1,\dots,x_{r_q}),\dots,t_{r_q}(F_{t_q}^q,W)(x_1,\dots,x_{r_q})\right)$$

is in the kernel of A_q.

Proof. Let W be a tournamenton. Then, by Lemma 4.1.6,

$$t(H,W) = \sum_{J:v(J)=m} t_{inj}(H,J)d(J,W). \tag{4.1}$$

Since A_q is positive semidefinite, for any $1 \le q \le l$ and $x_1,\dots,x_{r_q} \in [0,1]^{r_q}$, we have

$$\sum_{i=1}^{t_q}\sum_{j=1}^{t_q}(A_q)_{i,j} \cdot t(F_i^q \cdot F_j^q, W)(x_1,\dots,x_{r_q}) \ge 0 \tag{4.2}$$

Then, by integrating over the variables and applying Lemma 4.1.8, we get

$$\begin{aligned}\sum_{q=1}^{l}\int_{[0,1]^{r_q}}\sum_{i=1}^{t_q}\sum_{j=1}^{t_q}&(A_q)_{i,j} \cdot t(F_i^q \cdot F_j^q, W)(x_1,\dots,x_{r_q})dx_1\dots dx_{r_q}\\ &=\sum_{q=1}^{l}\sum_{i=1}^{t_q}\sum_{j=1}^{t_q}(A_q)_{i,j} \cdot \int_{[0,1]^{r_q}} t(F_i^q \cdot F_j^q, W)(x_1,\dots,x_{r_q})dx_1\dots dx_{r_q}\\ &= \sum_{J:v(J)=m}\sum_{q=1}^{l}\sum_{i=1}^{t_q}\sum_{j=1}^{t_q}(A_q)_{i,j} \cdot b_{r_q}(F_i^q \cdot F_j^q; J) \cdot d(J,W) \ge 0.\end{aligned}$$

Which together with Equation (4.1) gives us

$$t(H,W) \ge \sum_{J:v(J)=m}\left(t_{inj}(H,J) - \sum_{q=1}^{\ell}\sum_{i=1}^{t_q}\sum_{j=1}^{t_q}(A_q)_{i,j} \cdot b_{r_q}(F_i^q \cdot F_j^q; J)\right) \cdot d(J,W).$$

Recall that $\sum_{J:v(J)=m} d(J,W) = 1$, so the quantity on the right side of the above inequality is bounded below by the minimum of the coefficient of $d(J,W)$ over all J. This is precisely the bound in the lemma.

Moreover, Equation 4.2 is strictly positive if and only if the vector

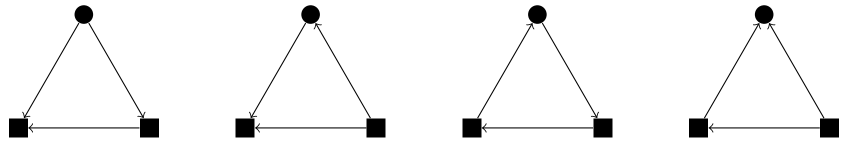
$$\big(t_{r_q}(F_1^q, W)(x_1,\dots,x_{r_q}),\dots,t_{r_q}(F_{t_q}^q, W)(x_1,\dots,x_{r_q})\big)$$

is not in the kernel of A_q. This fact translates to a gap in the final bound. This completes the proof of the theorem. □

We have presented all the theorems we need to prove Theorems 3.3.5–3.3.8 using the Flag Algebras method. The proofs are shown in the next section.

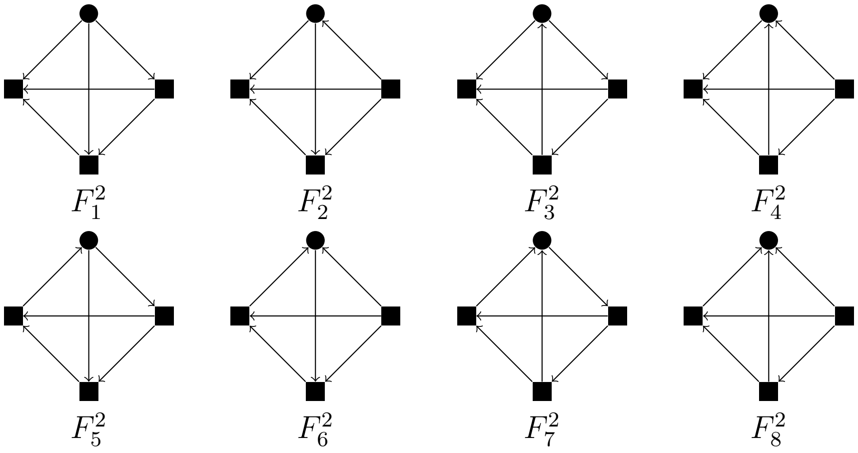
4.2 Flag algebras and quasirandom forcing

We now present the flags that we will use in the proofs of Theorems 3.3.5–3.3.8. In each depiction of a flag (F, r), the root vertices $1, \ldots, r$ are depicted as square nodes, listed left to right in increasing order. Each flag will have exactly one non-root vertex which is depicted as a circular node. We use the flags in $\mathcal{F}_3(Tr_2)$, labelled by F_i^1 for $1 \leq i \leq 4$, as in Figure 4.1. We also use the flags in $\mathcal{F}_4(Tr_3)$, which we label by F_i^2 for $1 \leq i \leq 8$, as in Figure 4.2. Finally, we have the elements of $\mathcal{F}_4(C_3)$, labelled F_i^3 for $1 \leq i \leq 8$, as in Figure 4.3.



F_1^1 F_2^1 F_3^1 F_4^1

Figure 4.1: Flags in $\mathcal{F}_3(Tr_2)$



F_1^2 F_2^2 F_3^2 F_4^2

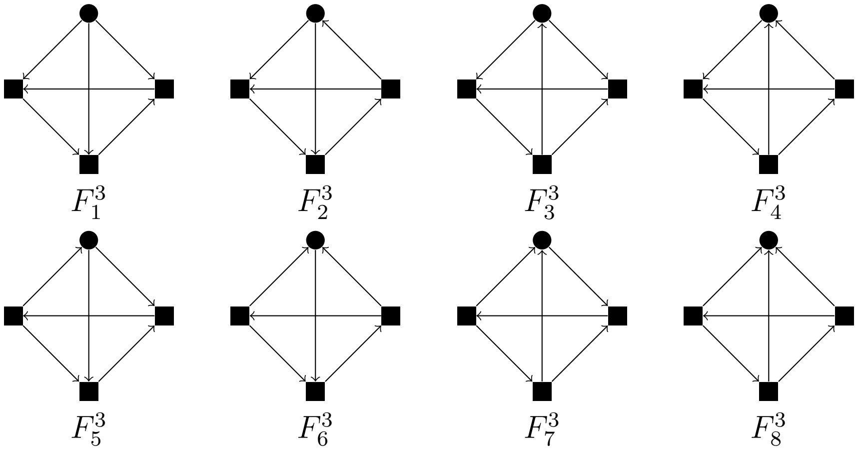
F_5^2 F_6^2 F_7^2 F_8^2

Figure 4.2: Flags in $\mathcal{F}_4(Tr_3)$.

The next result will be very useful in characterizing the equality cases for the proof in Theorems 3.3.5–3.3.8.

Lemma 4.2.1. *If W is a tournamenton such that $8t(C_4, W) = t(Tr_4, W)$, and $8t(Tr_4, W) = t(Tr_3, W)$, then $W = 1/2$ almost everywhere.*

Proof. There are four tournaments on four vertices: Tr_4, C_4, H_5, and H_7. By Lemma



F_1^3 F_2^3 F_3^3 F_4^3

F_5^3 F_6^3 F_7^3 F_8^3

Figure 4.3: Flags in $\mathcal{F}_4(C_3)$.

4.1.6, we have

$$t(Tr_3, W) = \frac{4}{24}d(Tr_4, W) + \frac{2}{24}d(C_4, W) + \frac{3}{24}d(H_5, W) + \frac{3}{24}d(H_7, W).$$

Now, by Lemma 4.1.1, we conclude

$$d(H_5, W) + d(H_7, W) = 1 - d(Tr_4, W) - d(C_4, W).$$

Using the last two equations we get

$$\begin{aligned} t(Tr_3, W) =& \frac{4}{24}d(Tr_4, W) + \frac{2}{24}d(C_4, W) + \frac{3}{24} - \frac{3}{24}d(Tr_4, W) - \frac{3}{24}d(C_4, W) \\ =& \frac{1}{24}d(Tr_4, W) - \frac{1}{24}d(C_4, W) + \frac{3}{24}. \end{aligned}$$

Also, since $\mathrm{aut}(Tr_3) = \mathrm{aut}(C_4) = 1$, the last equation is equivalent to

$$t(Tr_3, W) - \frac{1}{8} = t(Tr_4, W) - t(C_4, W).$$

Using the hypothesis of the lemma, $t(Tr_4, W) = t(C_4, W)$. Therefore, $t(Tr_3, W) = 1/8$ and applying the hypothesis once more, $t(Tr_4, W) = 1/64$. Since Tr_4 forces quasirandomness by Proposition 2.0.1, we conclude that $W = 1/2$ almost everywhere.

□

Now we are ready to prove Theorems 3.3.5–3.3.8 to complete the proof of Theorem 1.3.3.

All these proofs follows a very similar argument, so we are going to present an overview of it. We first let $H = \alpha_1 M + \alpha_2 N \in \mathbb{R}[\mathcal{T}_{\leq 5}]$ where $M \in \{Tr_3, C_3\}$ and $N \in \{H_{10}, H_{11}, H_{13}, H_{14}\}$, depending on each case. We also define some positive semidefinite matrices $\{A_1, \ldots, A_l\}$ for $1 \leq l \leq 3$ and prove that for every tournament J on 5 vertices,

$$t_{inj}(H, J) - \sum_{q=1}^{\ell} \sum_{i=1}^{t_q} \sum_{j=1}^{t_q} b_{r_q}(F_i^q \cdot F_j^q; J) \cdot (A_q)_{i,j} = c.$$

To show that the last equality holds for every tournament J on 5 vertices, we calculate all coefficients $b_2(F_i^q \cdot F_j^q; J)$ for every $1 \leq q \leq 3$. The values of the coefficients can be found in the Appendix B. In each proof, we show that the equality holds for one or two examples of J, but all the equations were checked by computer.

Therefore, by Lemma 4.1.9 we conclude

$$t(H, W) = \alpha_1 t(M, W) + \alpha_2 t(N, W) \geq c$$

which is the general form of the inequality that we want to show in each of the Theorems 3.3.5–3.3.8.

Then, it remains to show that the equality holds if and only if $W(x, y) = 1/2$ for almost every $x, y \in [0, 1]$. For this, we use Lemma 4.1.9 once more; specifically, the "moreover" part of the lemma, which states that the equality holds if and only if a certain vector is in the kernel of the matrices A_q for $1 \leq q \leq l$. In each case, we get one of the following two cases: the hypothesis of Lemma 4.2.1 must be satisfied if the vector is in the kernel of the matrices, and as a consequence of this lemma, we can conclude $W = 1/2$ almost everywhere; or $t(Tr_4, W) = 1/64$ and since Tr_4 forces quasirandomness, we conclude $W = 1/2$ almost everywhere.

Proof of Theorem 3.3.5. In this first case, we let $H = -8C_3 - \frac{1024}{4} H_{10}$. In this proof, we use the flags F_i^2 and F_i^3 for $1 \leq i \leq 8$ as in Figures 4.2 and 4.3. Therefore, we define the following positive semidefinite matrices:

$$A_2 = \frac{1}{245}\begin{bmatrix} 4724 & -1883 & 1081 & -4598 & 3827 & -293 & 1390 & -4248 \\ -1883 & 6512 & -4787 & 3150 & 39 & 378 & -2965 & -444 \\ 1081 & -4787 & 3856 & -2274 & -371 & -313 & 2010 & 798 \\ -4598 & 3150 & -2274 & 5490 & -3658 & 156 & -1734 & 3468 \\ 3827 & 39 & -371 & -3658 & 4600 & -420 & 945 & -4962 \\ -293 & 378 & -313 & 156 & -420 & 606 & -660 & 546 \\ 1390 & -2965 & 2010 & -1734 & 945 & -660 & 2376 & -1362 \\ -4248 & -444 & 798 & 3468 & -4962 & 546 & -1362 & 6204 \end{bmatrix}$$

$$A_3 = \frac{1}{245}\begin{bmatrix} 612 & 162 & 162 & -162 & 162 & -162 & -162 & -612 \\ 162 & 2082 & 342 & 252 & 342 & -24 & -1050 & -2106 \\ 162 & 342 & 2082 & -24 & 342 & -1050 & 252 & -2106 \\ -162 & 252 & -24 & 1554 & -1050 & -186 & -186 & -198 \\ 162 & 342 & 342 & -1050 & 2082 & 252 & -24 & -2106 \\ -162 & -24 & -1050 & -186 & 252 & 1554 & -186 & -198 \\ -162 & -1050 & 252 & -186 & -24 & -186 & 1554 & -198 \\ -612 & -2106 & -2106 & -198 & -2106 & -198 & -198 & 7524 \end{bmatrix}$$

In this case, we set $c = -5/4$. Therefore, we want to show that

$$-8 \cdot t_{inj}(C_3, J) - \frac{1}{4} \cdot 1024 {\cdot} t_{inj}(H_{10}, J) - \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^2 \cdot F_j^2; J) \cdot (A_2)_{i,j}$$
$$- \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^3 \cdot F_j^3; J) \cdot (A_3)_{i,j} = -\frac{5}{4}.$$

As we mentioned earlier, all the coefficients can be found in the Appendix B.

Now, we give some examples of tournaments that satisfy the last equation. Recall that all tournaments satisfy it. First, in the case that $J = H_8$, it becomes

$$-8 \cdot 0 - \frac{1}{4} \cdot 1024 \cdot 0 - \frac{2}{120}(A_2)_{1,1} - \frac{1}{120}(A_2)_{1,2} - \frac{1}{120}(A_2)_{1,4} - \frac{1}{120}(A_2)_{1,8} - \frac{1}{120}(A_2)_{2,1}$$
$$-\frac{2}{120}(A_2)_{2,2} - \frac{1}{120}(A_2)_{2,4} - \frac{1}{120}(A_2)_{2,8} - \frac{1}{120}(A_2)_{4,1} - \frac{1}{120}(A_2)_{4,2} - \frac{2}{120}(A_2)_{4,4}$$
$$-\frac{1}{120}(A_2)_{4,8} - \frac{1}{120}(A_2)_{8,1} - \frac{1}{120}(A_2)_{8,2} - \frac{1}{120}(A_2)_{8,4} - \frac{2}{120}(A_2)_{8,8}$$

which, upon plugging the appropriate values from A_2, becomes

$$-8{\cdot}0-\frac{1}{4}{\cdot}1024{\cdot}0-\frac{2}{120}\cdot\frac{4724}{245}+\frac{1}{120}\cdot\frac{1883}{245}+\frac{1}{120}\cdot\frac{4598}{245}+\frac{1}{120}\cdot\frac{4248}{245}+\frac{1}{120}\cdot\frac{1883}{245}-\frac{2}{120}\cdot\frac{6512}{245}$$

$$-\frac{1}{120}\cdot\frac{3150}{245}+\frac{1}{120}\cdot\frac{444}{245}+\frac{1}{120}\cdot\frac{4598}{245}-\frac{1}{120}\cdot\frac{3150}{245}-\frac{2}{120}\cdot\frac{5490}{245}-\frac{1}{120}\cdot\frac{3468}{245}+\frac{1}{120}\cdot\frac{4248}{245}$$

$$+\frac{1}{120}\cdot\frac{444}{245}-\frac{1}{120}\cdot\frac{3468}{245}-\frac{2}{120}\cdot\frac{6204}{245}=-\frac{5}{4}$$

As another example, consider $J = H_{12}$. The expression becomes

$$-8\cdot\frac{12}{60}-\frac{1}{4}\cdot 1024\cdot 0-\frac{3}{120}(A_2)_{3,5}-\frac{3}{120}(A_2)_{5,3}-\frac{3}{120}(A_2)_{6,7}-\frac{3}{120}(A_2)_{7,6}-\frac{3}{120}(A_3)_{1,8}$$

$$-\frac{3}{120}(A_3)_{2,6}-\frac{3}{120}(A_3)_{3,4}-\frac{3}{120}(A_3)_{4,3}-\frac{3}{120}(A_3)_{5,7}-\frac{3}{120}(A_3)_{6,2}-\frac{3}{120}(A_3)_{7,5}$$

$$-\frac{3}{120}(A_3)_{8,1}$$

which evaluates to

$$-8\cdot\frac{12}{60}-\frac{1}{4}\cdot 1024\cdot 0+\frac{3}{120}\cdot\frac{371}{245}+\frac{3}{120}\cdot\frac{371}{245}+\frac{3}{120}\cdot\frac{660}{245}+\frac{3}{120}\cdot\frac{660}{245}+\frac{3}{120}\cdot\frac{612}{245}$$

$$+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{24}{245}+\frac{3}{120}\cdot\frac{612}{245}=-\frac{5}{4}.$$

Now, assume that W is a regular tournamenton such that

$$8t(C_3,W)+\frac{1024}{4}t(H_{10},W)=\frac{5}{4}.$$

The kernel of A_2 is spanned by $(1,1,1,1,1,1,1,1)^T$. Therefore, by Lemma 4.1.9 we have

$$t_3(F_1^2,W)(x_1,x_2,x_3)=t_3(F_2^2,W)(x_1,x_2,x_3)=\ldots=t_3(F_8^2,W)(x_1,x_2,x_3)$$

for almost all $x_1,x_2,x_3\in[0,1]$. By Corollary 4.1.5, we get that for almost all $(x_1,x_2,x_3)\in[0,1]^3$,

$$\sum_{i=1}^{8}t(F_i^2,W)(x_1,x_2,x_3)=t(Tr_3,W)(x_1,x_2,x_3).$$

Then, we conclude

$$t_3(F_1^2, W)(x_1, x_2, x_3) = \frac{1}{8} t(Tr_3, W)(x_1, x_2, x_3)$$

and

$$t_3(F_7^2, W)(x_1, x_2, x_3) = \frac{1}{8} t(Tr_3, W)(x_1, x_2, x_3).$$

Now, we integrate over all $(x_1, x_2, x_3) \in [0,1]^3$ to get that $t(Tr_4, W) = \frac{1}{8} t(Tr_3, W)$ and $t(C_4, W) = \frac{1}{8} t(Tr_3, W)$. Then, by Lemma 4.2.1 we conclude that $W = 1/2$ almost everywhere. □

Proof of Theorem 3.3.6. In this case, we let $H = -8C_3 - \frac{1024}{4} H_{11}$. In this proof, we use the flags F_i^2 and F_i^3 for $1 \leq i \leq 8$ as in Figures 4.2 and 4.3. We define the following positive semidefinite matrices:

$$A_2 = \frac{1}{7} \begin{bmatrix} 257 & -130 & -214 & 127 & -57 & 201 & 40 & -224 \\ -130 & 199 & 123 & -179 & 72 & -130 & -86 & 131 \\ -214 & 123 & 210 & -134 & 45 & -190 & -48 & 208 \\ 127 & -179 & -134 & 204 & -79 & 120 & 77 & -136 \\ -57 & 72 & 45 & -79 & 58 & -44 & -30 & 35 \\ 201 & -130 & -190 & 120 & -44 & 219 & 51 & -227 \\ 40 & -86 & -48 & 77 & -30 & 51 & 59 & -63 \\ -224 & 131 & 208 & -136 & 35 & -227 & -63 & 276 \end{bmatrix}$$

$$A_3 = \frac{1}{7} \begin{bmatrix} 184 & -31 & -31 & 22 & -31 & 22 & 22 & -157 \\ -31 & 62 & -10 & -26 & -10 & 4 & -11 & 22 \\ -31 & -10 & 62 & 4 & -10 & -11 & -26 & 22 \\ 22 & -26 & 4 & 61 & -11 & -11 & -11 & -28 \\ -31 & -10 & -10 & -11 & 62 & -26 & 4 & 22 \\ 22 & 4 & -11 & -11 & -26 & 61 & -11 & -28 \\ 22 & -11 & -26 & -11 & 4 & -11 & 61 & -28 \\ -157 & 22 & 22 & -28 & 22 & -28 & -28 & 175 \end{bmatrix}$$

In this case, we set $c = -5/4$. Therefore, we want to show that

$$-8 \cdot t_{inj}(C_3, J) - \frac{1}{4} \cdot 1024 \cdot t_{inj}(H_{10}, J) - \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^2 \cdot F_j^2; J) \cdot (A_2)_{i,j}$$
$$- \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^3 \cdot F_j^3; J) \cdot (A_3)_{i,j} = -\frac{5}{4}.$$

As an example, let $J = H_{19}$. It becomes

$$-8 \cdot \frac{15}{60} - \frac{1}{4} \cdot 1024 \cdot 0 - \frac{5}{120}(A_2)_{5,7} - \frac{5}{120}(A_2)_{7,5} - \frac{5}{120}(A_3)_{2,7} - \frac{5}{120}(A_3)_{3,6} - \frac{5}{120}(A_3)_{4,5}$$
$$-\frac{5}{120}(A_3)_{5,4} - \frac{5}{120}(A_3)_{6,3} - \frac{5}{120}(A_3)_{7,2}$$

which is

$$-8 \cdot \frac{15}{60} - \frac{1}{4} \cdot 1024 \cdot 0 + \frac{5}{120} \cdot \frac{30}{7} + \frac{5}{120} \cdot \frac{30}{7} + \frac{5}{120} \cdot \frac{11}{7} + \frac{5}{120} \cdot \frac{11}{7} + \frac{5}{120} \cdot \frac{11}{7} + \frac{5}{120} \cdot \frac{11}{7}$$
$$+\frac{5}{120} \cdot \frac{11}{7} + \frac{5}{120} \cdot \frac{11}{7} = -\frac{5}{4}.$$

Now, suppose that W is a regular tournamenton such that

$$8 \cdot t(C_3, W) + \frac{1}{4} \cdot 1024 \cdot t(H_{11}, W) = \frac{5}{4}.$$

The kernel of A_2 is spanned by $(1, 1, 1, 1, 1, 1, 1, 1)^T$. Then, a similar argument as the one shown in the proof of Theorem 3.3.5 works in this case to conclude that $W = 1/2$ almost everywhere. $\square$

Proof of Theorem 3.3.7. In this case, we let $H = 8Tr_3 + \frac{1024}{7}H_{10}$. In this proof, we use the flags F_i^2 for $1 \leq i \leq 8$ as in Figure 4.2. Therefore, we define the following positive semidefinite matrix:

$$A_2 = \frac{1}{945}\begin{bmatrix} 3680 & 1296 & 1080 & -2376 & -6376 & 1400 & 2808 & -1512 \\ 1296 & 4528 & -1080 & -1512 & -4424 & -1400 & 5832 & -3240 \\ 1080 & -1080 & 5616 & -2160 & -3240 & 3240 & -5400 & 1944 \\ -2376 & -1512 & -2160 & 4320 & 5400 & -1512 & -3240 & 1080 \\ -6376 & -4424 & -3240 & 5400 & 16200 & -5400 & -5400 & 3240 \\ 1400 & -1400 & 3240 & -1512 & -5400 & 5400 & -3240 & 1512 \\ 2808 & 5832 & -5400 & -3240 & -5400 & -3240 & 16200 & -7560 \\ -1512 & -3240 & 1944 & 1080 & 3240 & 1512 & -7560 & 4536 \end{bmatrix}.$$

In this case, we set $c = 8/7$. Therefore, we want to show that

$$8 \cdot t_{inj}(Tr_3, J) + \frac{1}{7} \cdot 1024 \cdot t_{inj}(H_{13}, J) - \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^2 \cdot F_j^2; J) \cdot (A_2)_{i,j}$$

is equal to $8/7$.

As an example, let $J = H_{13}$. We get

$$8 \cdot \frac{6}{60} + \frac{1}{7} \cdot 1024 \cdot \frac{1}{120} - \frac{1}{120}(A_2)_{1,7} - \frac{1}{120}(A_2)_{2,7} - \frac{1}{120}(A_2)_{4,5} - \frac{1}{120}(A_2)_{5,4}$$

$$-\frac{2}{120}(A_2)_{5,5} - \frac{1}{120}(A_2)_{5,8} - \frac{1}{120}(A_2)_{7,1} - \frac{1}{120}(A_2)_{7,2} - \frac{2}{120}(A_2)_{7,7} - \frac{1}{120}(A_2)_{8,5}$$

and plugging in yields

$$8 \cdot \frac{6}{60} + \frac{1}{7} \cdot 1024 \cdot \frac{1}{120} - \frac{1}{120} \cdot \frac{2808}{945} - \frac{1}{120} \cdot \frac{5832}{945} - \frac{1}{120} \cdot \frac{5400}{945} - \frac{1}{120} \cdot \frac{5400}{945}$$

$$-\frac{2}{120} \cdot \frac{16200}{945} - \frac{1}{120} \cdot \frac{3240}{945} - \frac{1}{120} \cdot \frac{2808}{945} - \frac{1}{120} \cdot \frac{5832}{945} - \frac{2}{120} \cdot \frac{16200}{945} - \frac{1}{120} \cdot \frac{3240}{945} = \frac{8}{7}.$$

Let W be a tournamenton such that the last equality holds. In this case, the kernel of A_2 is spanned by $(1,1,1,1,1,1,1,1)^T$ and $(1,1,-1,-1,1,1,-1,-1)^T$. By Lemma 4.1.5 we know that for almost every $(x_1, x_2, x_3) \in [0,1]^3$,

$$\sum_{i=1}^{8} t(F_i^2, W)(x_1, x_2, x_3) = t(Tr_3, W)(x_1, x_2, x_3)$$

and since $\big(t_{r_q}(F_1^q, W)(x_1, \ldots, x_{r_q}), \ldots, t_{r_q}(F_{t_q}^q, W)(x_1, \ldots, x_{r_q})\big)$ is in the kernel of A_2, we get that there exist $a, b \geq 0$ such that $t_3(F_i^2, W)(x_1, x_2, x_3) = a$ for every $i \in \{1, 2, 5, 6\}$ and $t_3(F_i^2, W)(x_1, x_2, x_3) = b$ for $i \in \{3, 4, 7, 8\}$. Therefore, for any $(i, j) \in \{1, 2, 5, 6\} \times \{3, 4, 7, 8\}$,

$$t(F_i^2, W)(x_1, x_2, x_3) + t(F_j^2, W)(x_1, x_2, x_3) = \frac{1}{4} t(Tr_3, W)(x_1, x_2, x_3).$$

Taking $i = 5$ and $j = 7$ and integrating over all (x_1, x_2, x_3) we get

$$t(C_4, W) = \frac{1}{8} t(Tr_3, W).$$

Also, setting $i = 1$ and $j = 4$, integrating over all (x_1, x_2, x_3) gives us

$$t(Tr_4, W) = \frac{1}{8} t(Tr_3, W).$$

This concludes the proof by Lemma 4.2.1. $\square$

Proof of Theorem 3.3.8. In this case, we let $H = 8Tr_3 - \frac{1024}{5} H_{14}$. In this proof, we use the flags F_i^1 and F_i^3 for $1 \leq i \leq 8$ as in Figures 4.1 and 4.3. Therefore, we define the following positive semidefinite matrices:

$$A_1 = \frac{4}{5} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

and

$$A_3 = \frac{8}{15} \begin{bmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & 3 \\ -1 & 27 & -5 & -5 & -5 & -5 & -5 & -1 \\ -1 & -5 & 27 & -5 & -5 & -5 & -5 & -1 \\ -1 & -5 & -5 & 27 & -5 & -5 & -5 & -1 \\ -1 & -5 & -5 & -5 & 27 & -5 & -5 & -1 \\ -1 & -5 & -5 & -5 & -5 & 27 & -5 & -1 \\ -1 & -5 & -5 & -5 & -5 & -5 & 27 & -1 \\ 3 & -1 & -1 & -1 & -1 & -1 & -1 & 3 \end{bmatrix}.$$

In this case, we set $c = 6/5$. Therefore, we want to show that

$$8 \cdot t_{inj}(Tr_3, J) + \frac{1}{5} \cdot 1024 \cdot t_{inj}(H_{14}, J) - \sum_{i=1}^{4}\sum_{j=1}^{4} b_2(F_i^1 \cdot F_j^1; J) \cdot (A_1)_{i,j} - \sum_{i=1}^{8}\sum_{j=1}^{8} b_2(F_i^3 \cdot F_j^3; J) \cdot (A_3)_{i,j} = \frac{6}{5}.$$

For example, let $J = H_8$. The expression becomes

$$\begin{aligned} 8 \cdot \frac{10}{60} + \frac{1}{15} \cdot 1024 \cdot 0 - \frac{10}{120} \cdot (A_1)_{1,1} - \frac{10}{120} \cdot (A_1)_{1,2} - \frac{10}{120} \cdot (A_1)_{1,4} - \frac{10}{120} \cdot (A_1)_{2,2} \\ - \frac{10}{120} \cdot (A_1)_{2,4} - \frac{10}{120} \cdot (A_1)_{4,4} \\ = \frac{4}{3} + 0 - \frac{1}{15} + \frac{1}{15} - \frac{1}{15} - \frac{1}{15} + \frac{1}{15} - \frac{1}{15} = \frac{6}{5}. \end{aligned}$$

Also, if $J = H_9$, we get

$$\begin{aligned} 8 \cdot \frac{9}{60} + \frac{1}{15} \cdot 1024 \cdot 0 - \frac{6}{120}A_1(1,1) - \frac{18}{120}A_1(1,2) - \frac{6}{120}A_1(1,4) - \frac{6}{120}A_1(2,2) \\ - \frac{6}{120}A_1(2,4) - \frac{12}{120}A_1(3,4) - \frac{6}{120}A_1(4,4) - \frac{6}{120}A_3(8,8) \\ = \frac{6}{5} + 0 - \frac{1}{25} + \frac{3}{25} - \frac{1}{25} - \frac{1}{25} + \frac{1}{25} + \frac{2}{25} - \frac{1}{25} - \frac{2}{25} = \frac{6}{5}. \end{aligned}$$

Now, let W be a tournamenton such that

$$8 \cdot t(Tr_3, W) + \frac{1}{5} \cdot 1024 \cdot t(H_{14}, W) = \frac{6}{5}.$$

First, the kernel of A_1 is spanned by $(1,1,0,0)^T$, $(1,0,1,0)^T$, and $(0,1,0,1)^T$. This implies that

$$t_2(F_2^1, W)(x_1, x_2) + t_2(F_3^1, W)(x_1, x_2) = \frac{1}{2}t(H_1, W)(x_1, x_2) = \frac{1}{2}W(x_1, x_2)$$

for almost all $(x_1, x_2) \in [0,1]^2$. Integrating this out over all x_1 and x_2 yields

$$t(TT_3, W) + t(C_3, W) = \frac{1}{4}$$

which, when combined with (1.2), tells us that $t(C_3, W) = 1/8$, which implies that

W is regular.

Now, the kernel of A_3 is spanned by $(1,1,1,1,1,1,1,1)^T$ and $(1,0,0,0,0,0,0,-1)^T$. Then, we conclude that $t(F_2^3, W)(x_1, x_2, x_3) = \frac{1}{8}t(C_3, W)(x_1, x_2, x_3)$ and also that $t(F_1^3, W)(x_1, x_2, x_3) + t(F_8^3, W)(x_1, x_2, x_3) = \frac{1}{4}t(C_3, W)(x_1, x_2, x_3)$ for almost all ordered pairs $(x_1, x_2, x_3) \in [0,1]^3$. Integrating over all x_1, x_2 and x_3 and using the fact that $t(C_3, W) = 1/8$, we get

$$t(C_4, W) = \frac{1}{8}t(C_3, W) = \frac{1}{64}$$

and

$$t(H_5, W) + t(H_7, W) = \frac{1}{4}t(C_3, W) = \frac{1}{32}.$$

Since $\sum_{J:v(J)=m} d(J, W) = 1$ and $\mathrm{aut}(Tr_4) = \mathrm{aut}(C_4) = 1$ and $\mathrm{aut}(H_5) = \mathrm{aut}(H_7) = 3$, we have

$$24t(Tr_4, W) + 24t(C_4, W) + 8t(H_5, W) + 8t(H_7, W) = 1.$$

This yields $t(Tr_4, W) = 1/64$ and since Tr_4 forces quasirandomness we conclude $W = 1/2$ almost everywhere. $\square$

Chapter 5

Conclusion

The class of tournaments that force quasirandomness in regular tournaments is broader than the class of tournaments that force quasirandomness in general. We have observed that if a tournament T forces quasirandomness, it also forces quasirandomness in regular tournaments. This is because requiring T to have its expected density in only regular tournamentons, implies that the tournamenton W behaves randomly, i.e., it equals $1/2$ almost everywhere, is a mild requirement as the same statement for every tournamenton W.

We characterize all tournaments with at most five vertices that force quasirandomness in regular tournaments. Among the main techniques used to disprove regular quasirandom forcing for a specific tournament T, we typically identify a regular tournament W such that $t(T, W)$ equals its expected value, but W is not equal to $1/2$ almost everywhere. Also, Proposition 3.1.3 was of main importance for this purpose. Indeed, it provides a sufficient condition to conclude that a tournament does not force quasirandomness, and we could find the tournamentons satisfying this condition through an exhaustive computer search. On the other hand, to prove that a tournament T forces quasirandomness in regular tournaments, we primarily rely on equations that relate the density of T in W to the density of some other small tournament in W. For instance, C_3 and Tr_3 are examples of tournaments used in these proofs. Finally, using the definition or, in the most challenging cases, the flag algebra method, we show that these equations imply that the tournament T forces quasirandomness in regular tournaments. See, for instance, Proposition 3.1.1 or 3.3.5.

Although 11 out of 16 tournaments on four or five vertices are quasirandom forcing in regular tournaments, no known property characterizes quasirandom forcing for regular tournaments in general. For instance, note that being strongly connected was a necessary condition for forcing quasirandomness, but it seems not to be a necessary condition for forcing quasirandomness in regular tournaments. For instance, H_5 is not strongly connected and is regular quasirandom forcing. This observation leads to the following related open problem.

Problem 5.0.1. Classify tournaments H that force quasirandomness in regular tournaments.

The solution to the last problem could provide insight into whether there are infinitely many non-transitive tournaments that force quasirandomness in regular tournaments.

Appendix A

Standard results

In this appendix, we are going to enounce some standard results used in the document.

The first one is about convexity. We first define a convex function.

Definition A.0.1. Let $I \subseteq \mathbb{R}$ be an interval (of possible infinite length). A function $\phi : I \to \mathbb{R}$ is said to be *convex* if, for all $x, y \in I$ and $t \in [0, 1]$,

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y).$$

Remark A.0.2. A twice-differentiable function $\phi :\to \mathbb{R}$ is convex if and only if its second derivative is non-negative everywhere on I.

Now we state Jensen's Inequality which is a fundamental fact about convex functions and we use in many of the proofs presented in this document.

Theorem A.0.3 (Jensen's Inequality)**.** *If f is a convex function on a convex set C, and $p_1, \cdots, p_n$ are positive numbers with $p_1 + \cdots + p_n = 1$, then*

$$f(p_1 x_1 + \cdots + p_n x_n) \leq p_1 f(x_1) + \cdots + p_n f(x_n).$$

If f is strictly convex, then equality holds if and only if $x_1 = \cdots = x_n$.

Moreover, we can express Jensen's Inequality in a more general form.

Theorem A.0.4. *If f is a convex function, then for any $z_1, \cdots, z_n$ in the domain of f and any positive numbers $a_1, \cdots, a_n$,*

$$f\left(\frac{\sum_{i=1}^{n} a_i z_i}{\sum_{i=1}^{n} a_i}\right) \leq \frac{\sum_{i=1}^{n} a_i f(z_i)}{\sum_{i=1}^{n} a_i}$$

Proof. Let $a_1, \cdots, a_n$ be any positive numbers. Define $A = a_1 + \cdots + a_n$ and $p_i = a_i/A$ for every $1 \leq i \leq n$. The numbers $p_1, \cdots, p_n$ satisfies $p_1 + \cdots + p_n = 1$ and by Jensen's Inequality A.0.3, we have

$$f\left(\frac{\sum_{i=1}^n a_i z_i}{\sum_{i=1}^n a_i}\right) = f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i) = \frac{\sum_{i=1}^n a_i f(z_i)}{\sum_{i=1}^n a_i}$$

□

The next result is an important tool used to estimate the probability that the value of a random variable X substantially deviates from its expected value.

Theorem A.0.5 (The Chernoff Bound [22])**.** *Let* $X := \sum_{i=1}^n X_i$ *where* X_i *or* $i \in \{1, \cdots, n\}$ *are indicator variables independently distributed. Then, for every* $0 \leq \epsilon \leq 1$ *we have*

$$\mathbb{P}[X > (1+\epsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\epsilon^2}{3}\mathbb{E}[X]\right)$$

and

$$\mathbb{P}[X < (1-\epsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\epsilon^2}{2}\mathbb{E}[X]\right)$$

Finally, we give a brief introduction to positive semidefinite matrices [44].

Definition A.0.6. Let A be a real symmetric $n \times n$ matrix. A is positive semidefinite, denoted by $A \succeq 0$, if

$$x^T A x \geq 0$$

for every $x \in \mathbb{R}^n$.

Lemma A.0.7. *A symmetric matrix* A *is positive semidefinite if and only if all of its eigenvalues are non-negative.*

Appendix B

Flag Algebra coefficients

The purpose of this appendix is to list all of the coefficients $b_2(F_i^1, F_j^1; J)$ for $1 \le i, j \le 4$ and $b_3(F_i^q, F_j^q; J)$ for $1 \le i, j \le 8$ and $q \in \{2, 3\}$, where J is a tournament on 5 vertices and the flags F_i^q are as in Figures 4.1–4.3. All of these coefficients were computed by computer but can be easily checked by hand. It will be convenient to record these coefficients in matrices. For every such J, let $B_2^1(J)$ be the 4×4 matrix in which the entry on the ith row and jth column is $b_2(F_i^1, F_j^1; J)$. Similarly, for every such J and $q \in \{2, 3\}$, let $B_3^q(J)$ be the 8×8 matrix in which the entry on the ith row and jth column is $b_3(F_i^q, F_j^q; J)$. The matrices are as follows.

For $J = H_8$, we have

$$B_2^1(H_8) = \frac{1}{120}\begin{bmatrix} 10 & 5 & 0 & 5 \\ 5 & 10 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 5 & 5 & 0 & 10 \end{bmatrix}$$

$$B_3^2(H_8) = \frac{1}{120}\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$B_3^3(H_8) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_9$, we have

$$B_2^1(H_9) = \frac{1}{120}\begin{bmatrix} 6 & 9 & 0 & 3 \\ 9 & 6 & 0 & 3 \\ 0 & 0 & 0 & 6 \\ 3 & 3 & 6 & 6 \end{bmatrix}$$

$$B_3^2(H_9) = \frac{1}{120}\begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_9) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

For $J = H_{10}$, we have

$$B_2^1(H_{10}) = \frac{1}{120}\begin{bmatrix} 4 & 8 & 1 & 3 \\ 8 & 4 & 2 & 2 \\ 1 & 2 & 2 & 7 \\ 3 & 2 & 7 & 4 \end{bmatrix}$$

$$B_3^2(H_{10}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$B_3^3(H_{10}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

For $J = H_{11}$, we have

$$B_2^1(H_{11}) = \frac{1}{120}\begin{bmatrix} 6 & 6 & 3 & 3 \\ 6 & 6 & 0 & 6 \\ 3 & 0 & 0 & 3 \\ 3 & 6 & 3 & 6 \end{bmatrix}$$

$$B_3^2(H_{11}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{11}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{12}$, we have

$$B_2^1(H_{12}) = \frac{1}{120}\begin{bmatrix} 0 & 3 & 6 & 3 \\ 3 & 0 & 6 & 3 \\ 6 & 6 & 6 & 6 \\ 3 & 3 & 6 & 0 \end{bmatrix}$$

$$B_3^2(H_{12}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{12}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{13}$, we have

$$B_2^1(H_{13}) = \frac{1}{120}\begin{bmatrix} 2 & 1 & 4 & 5 \\ 1 & 2 & 8 & 1 \\ 4 & 8 & 8 & 4 \\ 5 & 1 & 4 & 2 \end{bmatrix}$$

$$B_3^2(H_{13}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{13}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{14}$, we have

$$B_2^1(H_{14}) = \frac{1}{120}\begin{bmatrix} 4 & 2 & 3 & 5 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 6 & 3 \\ 5 & 2 & 3 & 4 \end{bmatrix}$$

$$B_3^2(H_{14}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{14}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{15}$, we have

$$B_2^1(H_{15}) = \frac{1}{120}\begin{bmatrix} 4 & 2 & 7 & 3 \\ 2 & 4 & 2 & 8 \\ 7 & 2 & 2 & 1 \\ 3 & 8 & 1 & 4 \end{bmatrix}$$

$$B_3^2(H_{15}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{15}) = \frac{1}{120}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{16}$, we have

$$B_2^1(H_{16}) = \frac{1}{120}\begin{bmatrix} 6 & 3 & 6 & 3 \\ 3 & 6 & 0 & 9 \\ 6 & 0 & 0 & 0 \\ 3 & 9 & 0 & 6 \end{bmatrix}$$

$$B_3^2(H_{16}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{16}) = \frac{1}{120}\begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{17}$, we have

$$B_2^1(H_{17}) = \frac{1}{120}\begin{bmatrix} 2 & 4 & 5 & 3 \\ 4 & 2 & 4 & 4 \\ 5 & 4 & 4 & 5 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

$$B_3^2(H_{17}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{17}) = \frac{1}{120}\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

For $J = H_{18}$, we have

$$B_2^1(H_{18}) = \frac{1}{120}\begin{bmatrix} 0 & 3 & 6 & 3 \\ 3 & 0 & 6 & 3 \\ 6 & 6 & 6 & 6 \\ 3 & 3 & 6 & 0 \end{bmatrix}$$

$$B_3^2(H_{18}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{18}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

For $J = H_{19}$, we have

$$B_2^1(H_{19}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 0 \\ 5 & 10 & 10 & 5 \\ 5 & 0 & 5 & 0 \end{bmatrix}$$

$$B_3^2(H_{19}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3^3(H_{19}) = \frac{1}{120}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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