A NEW CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE RUSCHEWEYH DERIVATIVES[†]

by

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ABSTRACT

The object of the present paper is to establish several interesting properties and characteristics of the class $\mathcal{A}_{n,p}(a,b)$ of analytic functions, which is introduced here by using the Ruscheweyh derivatives. A relevant problem associated with the general class $\mathcal{A}_{n,p}(a,b)$ is also proposed. This hitherto unresolved problem would generalize one of the results presented here.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (p \in \mathcal{N} = \{1,2,3,\ldots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. We denote by $f * g(z)$ the Hadamard product (or convolution) of two functions $f(z) \in \mathcal{A}(p)$ and $g(z) \in \mathcal{A}(p)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \qquad (p \in \mathcal{N}), \tag{1.2}$$

then

$$f * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \tag{1.3}$$

Following Goel and Sohi [7], we put

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \qquad (n > -p) \tag{1.4}$$

for $f(z) \in \mathcal{A}(p)$.

A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{K}(n,p)$ if and only if

$$\operatorname{Re}\left(\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)}\right) > \frac{n+p}{2(n+1)} \qquad (z \in \mathcal{U}) \tag{1.5}$$

for $n \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}$ and $p \in \mathcal{N}$. In particular, for $p = 1$, the class $\mathcal{K}(n,1)$ becomes the class $\mathcal{K}_n$ studied by Ruscheweyh [17] who, in fact, proved the basic property [17, p. 110, Theorem 1]:

$$\mathcal{K}_{n+1} \subset \mathcal{K}_n \qquad (n \in \mathcal{N}_0). \tag{1.6}$$

In the present paper, we introduce and study systematically the subclass $\mathcal{A}_{n,p}(a,b)$ of $\mathcal{A}(p)$, which is defined below by using the $(n+p-1)$th order Ruscheweyh derivative of $f(z)$.

DEFINITION. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}(p)$, and set

$$F_{n,p}(z) = \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - \frac{n+p}{2(n+1)} \tag{1.7}$$

for $n \in \mathcal{N}_0$ and $p \in \mathcal{N}$. Then we say that $f(z)$ is in the class $\mathcal{A}_{n,p}(a,b)$ if it satisfies the inequality

$$\operatorname{Re}\left\{(F_{n,p}(z))^a (F_{n+1,p}(z))^b\right\} > 0 \qquad (z \in \mathcal{U}) \tag{1.8}$$

for $n \in \mathcal{N}_0$ and $p \in \mathcal{N}$; here a and b are real numbers, and each of the power functions is interpreted as its principal value.

Clearly, we have [cf. Equation (1.6)]

$$\mathcal{A}_{n,1}(1,0) = \mathcal{K}(n,1) \equiv \mathcal{K}_n \tag{1.9}$$

and

$$\mathcal{A}_{n,1}(0,1) = \mathcal{K}(n+1,1) \equiv \mathcal{K}_{n+1}. \tag{1.10}$$

Several other classes of analytic functions defined by using the nth order Ruscheweyh derivatives of $f(z)$ have been studied in the literature by, for example, Ahuja [1], Al-Amiri ([2], [3]), Bulboaca [5], Fukui and Sakaguchi [6], Goel and Sohi ([8], [9]), Owa ([13], [14], [15], [16]), Kumar and Shukla [10], and Singh and Singh [18].

2. A PROPERTY OF THE CLASS $\mathcal{A}_{n,p}(a,b)$

We first state and prove an interesting property of the class $\mathcal{A}_{n,p}(a,b)$.

THEOREM 1. *Let* $n \in \mathcal{N}_0$, $p \in \mathcal{N}$, *and* $0 \leqq t \leqq 1$. *Then*

$$\mathcal{A}_{n,p}(a,b) \cap \mathcal{A}_{n,p}(1,0) \subset \mathcal{A}_{n,p}\big((a-1)t+1,\ bt\big). \tag{2.1}$$

PROOF. Following the technique used earlier by Owa [15], let the function $f(z)$ defined by (1.1) be in the class

$$\mathcal{A}_{n,p}(a,b) \cap \mathcal{A}_{n,p}(1,0).$$

Define $V_{n,p}(z)$ by

$$(2.2) \qquad V_{n,p}(z) = \left(F_{n,p}(z)\right)^{a}\left(F_{n+1,p}(z)\right)^{b},$$

where $F_{n,p}(z)$ is given by (1.7). Since $f(z) \in \mathcal{A}_{n,p}(a,b)$, we have

$$(2.3) \qquad \operatorname{Re}\left(V_{n,p}(z)\right) > 0 \qquad (z \in \mathcal{U}).$$

We note that $f(z) \in \mathcal{A}_{n,p}(1,0)$. This implies the inequality

$$(2.4) \qquad \operatorname{Re}\left(F_{n,p}(z)\right) > 0 \qquad (z \in \mathcal{U}).$$

Making use of (2.2), we have

$$(2.5) \qquad \left(F_{n,p}(z)\right)^{(a-1)t+1}\left(F_{n+1,p}(z)\right)^{bt} = \left(F_{n,p}(z)\right)^{1-t}\left(V_{n,p}(z)\right)^{t}.$$

Now we define a function $G(z)$ by

$$(2.6) \qquad G(z) = \left(F_{n,p}(z)\right)^{1-t}\left(V_{n,p}(z)\right)^{t}.$$

It is clear from (2.6) that $G(0) > 0$. Consequently, using (2.3) and (2.4), we prove that

$$(2.7) \qquad \left|\arg\left(G(z)\right)\right| \leqq (1-t)\left|\arg\left(F_{n,p}(z)\right)\right| + t\left|\arg\left(V_{n,p}(z)\right)\right| \leqq \frac{\pi}{2}.$$

This shows that $G(z)$ maps the unit disk $\mathcal{U}$ onto a domain which is contained in the right half-plane, that is, that

$$\mathrm{Re}\left(G(z)\right) > 0.$$

Thus we complete the proof of Theorem 1.

COROLLARY 1. Let $n \in \mathcal{N}_0$ and $0 \leqq t \leqq 1$. Then

$$\mathcal{K}(n+1,1) \subset \mathcal{A}_{n,1}(1-t,t). \tag{2.8}$$

PROOF. We note that

$$\begin{aligned} \mathcal{A}_{n,1}(0,1) \cap \mathcal{A}_{n,1}(1,0) &= \mathcal{K}(n+1,1) \cap \mathcal{K}(n,1) \\ &= \mathcal{K}(n+1,1) \end{aligned} \tag{2.9}$$

with the aid of (1.9) and (1.10). Taking $p = 1$, $a = 0$, and $b = 1$ in Theorem 1, we readily have the assertion (2.8).

We conclude this section by stating a problem which is closely related to our theorem.

PROBLEM. For $n \in \mathcal{N}_0$, $p \in \mathcal{N}$, and $0 \leqq t \leqq 1$, can we prove that

$$\mathcal{A}_{n,p}(a,b) \subset \mathcal{A}_{n,p}\left((a-1)t+1,\ bt\right)? \tag{2.10}$$

REMARK. In the special case when $p = 1$, we know that (2.10) holds true, that is, that

(2.11) $$\mathcal{A}_{n,1}(a,b) \subset \mathcal{A}_{n,1}\big((a-1)t+1,\ bt\big),$$

which is proved by Al-Amiri [2], and also by Kumar and Shukla [10].

3. THE INTEGRAL OPERATOR $\mathcal{J}_{n,p}$

For a function $f(z)$ belonging to the class $\mathcal{A}(p)$, we define the integral operator $\mathcal{J}_{n,p}$ by

(3.1) $$\mathcal{J}_{n,p}(f) = \frac{n+p}{z^n} \int_0^z t^{n-1} f(t)\,dt \qquad (n > -p;\ p \in \mathcal{N}).$$

The operator $\mathcal{J}_{n,p}$, when $n \in \mathcal{N}$ and $p = 1$, was introduced by Bernardi [4]. In particular, the operator $\mathcal{J}_{1,1}$ was studied by Libera [11] and Livingston [12]. For the general operator $\mathcal{J}_{n,p}$ defined by (3.1), we prove

THEOREM 2. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}_{n,p}(1,0)$ for $n > -p$ and $p \in \mathcal{N}$. Then

(3.2) $$\mathcal{J}_{n,p}(f) \in \mathcal{A}_{n+1,p}(1,0) \qquad (n > -p;\ p \in \mathcal{N}).$$

PROOF. We note that, for $f(z) \in \mathcal{A}(p)$,

(3.3) $$\mathcal{J}_{n,p}(f) = \frac{n+p}{z^n} \int_0^z t^{n-1}\left(t^p + \sum_{k=1}^{\infty} a_{p+k} t^{p+k}\right) dt$$

$$= z^p + \sum_{k=1}^{\infty} \left(\frac{n+p}{n+p+k}\right) a_{p+k} z^{p+k}$$

$$= \left\{z^p + \sum_{k=1}^{\infty} \frac{(n+p)_k}{(n+p+1)_k} z^{p+k}\right\} * f(z)$$

and

(3.4) $$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z)$$

$$= \left\{z^p + \sum_{k=1}^{\infty} \frac{(n+p)_k}{(1)_k} z^{p+k}\right\} * f(z),$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

(3.5) $$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1) \ldots (\lambda+n-1), & \text{if } n \in \mathcal{N}. \end{cases}$$

By using (3.3) and (3.4), we observe that

(3.6) $$D^{n+p} \mathcal{J}_{n,p}(f) = \left\{z^p + \sum_{k=1}^{\infty} \frac{(n+p)_k}{(1)_k} z^{p+k}\right\} * f$$

$$= D^{n+p-1} f(z)$$

and

(3.7) $$(n+p+1) D^{n+p+1} \mathcal{J}_{n,p}(f) - D^{n+p} \mathcal{J}_{n,p}(f)$$

$$= (n+p)\left\{\left\{z^p + \sum_{k=1}^{\infty} \frac{(n+p+1)_k}{(1)_k} z^{p+k}\right\} * f(z)\right\}$$

$$= (n+p) D^{n+p} f(z).$$

Consequently, we have

$$(3.8) \qquad \mathrm{Re}\left\{\frac{(n+p+1)D^{n+p+1}\mathcal{J}_{n,p}(f) - D^{n+p}\mathcal{J}_{n,p}(f)}{(n+p)D^{n+p}\mathcal{J}_{n,p}(f)}\right\}$$

$$= \mathrm{Re}\left\{\frac{D^{n+p}\ f(z)}{D^{n+p-1}\ f(z)}\right\} > \frac{n+p}{2(n+1)},$$

or

$$(3.9) \qquad \mathrm{Re}\left\{\frac{D^{n+p+1}\mathcal{J}_{n,p}(f)}{D^{n+p}\mathcal{J}_{n,p}(f)}\right\} > \frac{n+p}{n+p+1}\left(\frac{1}{n+p} + \frac{n+p}{2(n+1)}\right).$$

In view of (3.9), we only need to show that

$$(3.10) \qquad \frac{n+p}{n+p+1}\left(\frac{1}{n+p} + \frac{n+p}{2(n+1)}\right) \geqq \frac{n+p+1}{2(n+2)},$$

that is, that

$$(3.11) \qquad \Phi(n,p) \equiv (n+2)\left\{2(n+1) + (n+p)^2\right\} - (n+1)(n+p+1)^2 \geqq 0$$

for $n > -p$ and $p \in \mathcal{N}$. Observe that

$$(3.12) \qquad \begin{aligned} \Phi(n,p) &= n^2 + 3n + (p-1)^2 + 2 \\ &\geqq \Phi(n,1) \\ &= n^2 + 3n + 2 \\ &\geqq \Phi(-1,1) = 0. \end{aligned}$$

This implies the inequality

$$\text{(3.13)} \qquad \operatorname{Re}\left\{\frac{D^{n+p+1}\mathcal{J}_{n,p}(f)}{D^{n+p}\mathcal{J}_{n,p}(f)}\right\} > \frac{n+p+1}{2(n+2)} \qquad (n>-p;\ p\in\mathcal{N}),$$

which completes the proof of Theorem 2.

Next we deduce

COROLLARY 2. *Let the function* $f(z)$ *be in the class* $\mathcal{K}(n,1)$ *for* $n>-1$. *Then*

$$\text{(3.14)} \qquad \mathcal{J}_{n,1}(f) \in \mathcal{K}(n+1,1).$$

PROOF. It follows from (1.8) and (1.10) that

$$\text{(3.15)} \qquad \mathcal{A}_{n+1,1}(1,0) = \mathcal{A}_{n,1}(0,1) = \mathcal{K}(n+1,1).$$

Thus, in view of (1.9) and (3.15), the assertion (3.14) results immediately upon setting $p = 1$ in Theorem 2.

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